Number 14 — Autumn 2019

R M M

ROMANIAN MATHEMATICAL MAGAZINE

SOLUTIONS

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

SOLUTIONS

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Proposed by

Daniel Sitaru – Romania

Hoang Le Nhat Tung – Hanoi – Vietnam

Marian Ursărescu – Romania

D.M. Bătinețu-Giurgiu – Romania

Neculai Stanciu – Romania

George Apostolopoulos – Messolonghi – Greece

Marin Chirciu – Romania

Nguyen Viet Hung – Hanoi – Vietnam

Vasile Mircea Popa – Romania

Solutions by

Daniel Sitaru – Romania,Srinivasa Raghava-AIRMC-India

Tran Hong-Dong Thap-Vietnam,Soumava Chakraborty-Kolkata-India

Amit Dutta-Jamshedpur-India, Minh Tam Le-Vietnam

Marian Ursărescu – Romania,Le Ngo Duc-Vietnam

Sanong Huayrerai-Nakon Pathom-Thailand,Michael Sterghiou-Greece

Soumitra Mandal-Chandar Nagore-India,Bogdan Fustei-Romania

Mustafa Tarek-Cairo-Egypt,Ravi Prakash-New Delhi-India

Abdul Hafeez Ayinde-Nigeria,Pedro Nagasava-Brazil

Dawid Bialek-Poland,Remus Florin Stanca-Romania

Tobi Joshua-Nigeria,D.M. Bătinețu-Giurgiu – Romania

Neculai Stanciu – Romania,Serban George Florin-Romania

Sudhir Jha-Kolkata-India

JP.196. Let a, b, c be the sides in a triangle such that $abc = 1$. Find the minimum value of:

$$T = \frac{a^3}{\sqrt[3]{b^3+c^3-1}} + \frac{b^3}{\sqrt[3]{c^3+a^3-1}} + \frac{c^3}{\sqrt[3]{a^3+b^3-1}} + \frac{3(ab+bc+ca)}{a^2+b^2+c^2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\because \text{ (}a, b, c\text{ – the sides of a triangle)} \Rightarrow$$

$$\Rightarrow \sqrt[3]{1\cdot 1\cdot (b^3+c^3-1)} \overset{AM-GM}{\leq} \frac{b^3+c^3-1+1+1}{3} = \frac{b^3+c^3+1}{3} \text{ (etc)}$$

$$\Rightarrow \Omega = \sum \frac{a^3}{\sqrt[3]{b^3+c^3-1}} \geq 3\sum \frac{a^3}{b^3+c^3+1} = 3\sum \frac{a^4}{ab^3+ac^3+a} \overset{Schwarz}{\geq}$$

$$\geq 3\cdot \frac{(a^2+b^2+c^2)^2}{(ab^3+ba^3)+(ac^3+ca^3)+(bc^3+cb^3)+a+b+c}$$

$$= 3\cdot \frac{(a^2+b^2+c^2)^2}{ab(a^2+b^2)+ac(b^2+c^2)+bc(b^2+c^2)+a+b+c} = \Upsilon$$

We need to prove: $\Upsilon = 3\cdot \frac{a^2+b^2+c^2}{ab+bc+ca}$ $(\because abc = 1)$

$$\Leftrightarrow (ab+bc+ca)(a^2+b^2+c^2) = [ab(a^2+b^2)+ac(a^2+c^2)+bc(b^2+c^2)+a+b+c]$$

$$\Leftrightarrow ab(a^2+b^2)+bc(b^2+c^2)+ca(a^2+c^2)+abc(a+b+c) =$$

$$[ab(a^2+b^2)+ac(a^2+c^2)+bc(b^2+c^2)+a+b+c] \Leftrightarrow$$

$$\Leftrightarrow abc(a+b+c) = a+b+c \overset{abc=1}{\Leftrightarrow} a+b+c = a+b+c \text{ (true)}$$

$$\Rightarrow \Omega \geq \Upsilon = 3\cdot \frac{a^2+b^2+c^2}{ab+bc+ca}$$

$$\Rightarrow T \geq 3\cdot \frac{a^2+b^2+c^2}{ab+bc+ca} + 3\cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \overset{AM-GM}{\geq} 2\sqrt{3\cdot 3} = 6$$

$$\Rightarrow T_{\min} = 6 \Leftrightarrow \begin{cases} abc = 1 \\ a = b = c > 0 \end{cases} \Leftrightarrow a = b = c = 1$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$T = \sum \frac{a^3}{\sqrt[3]{b^3+c^3-abc}} + \frac{3\sum ab}{\sum a^2} \quad (\because 1 = abc)$$

$$b^3+c^3-abc \geq bc(b+c)-abc = bc(b+c-a) > 0 \Rightarrow b^3+c^3-abc > 0$$

Similarly, $c^3+a^3-abc>0, a^3+b^3-abc>0$

$$\sum\frac{a^3}{\sqrt[3]{b^3+c^3-abc}}=\sum\frac{a^3}{\sqrt[3]{abc\cdot abc(b^3+c^3-abc)}}\overset{G\le A}{\le}\sum\frac{3a^3}{b^3+c^3+abc}$$

$$=3\sum\frac{a^4}{ab^3+ac^3+a^2bc}\overset{Bergström}{\ge}3\frac{\left(\sum a^2\right)^2}{\sum(ab^3+ac^3+a^2bc)}=3\frac{\left(\sum a^2\right)^2}{\sum ab(a^2+b^2)+\sum a^2bc}$$

$$=3\frac{(\sum a^2)^2}{\sum ab(\sum a^2-c^2)+\sum a^2bc}=\frac{3(\sum a^2)^2}{(\sum ab)(\sum a^2)-\sum abc^2+\sum a^2bc}=\frac{3\sum a^2}{\sum ab}$$

$$\Rightarrow T=\sum\frac{a^3}{\sqrt[3]{b^3+c^3-abc}}+\frac{3\sum ab}{\sum a^2}\ge\frac{3\sum a^2}{\sum ab}+\frac{3\sum ab}{\sum a^2}\overset{A-G}{\ge}6$$

$\therefore T_{\min}=6$ ***(equality at*** $a=b=c=1$***)***

JP.197. Solve for real numbers:

$$6\sqrt[3]{2x^2-2x+1}+4\sqrt[4]{3x^3-2x^4}=2x^5-5x+13$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Amit Dutta-Jamshedpur-India

$$2x^2-2x+1>0\ \{\because\Delta<0\}$$

$$3x^3-2x^4\ge 0\ \text{\{Domain\}}$$

$$2x^4-3x^3\le 0;\ x^3(2x-3)\le 0;\ x(2x-3)\le 0;$$

$$x\in\left[0,\frac{3}{2}\right]\ \ (1)$$

Now, using GM$\le AM$***:*** $\sqrt[3]{(2x^2-2x+1)\cdot 1\cdot 1}\le\frac{(2x^2-2x+1)+1+1}{3}$

$$6\sqrt[3]{(2x^2-2x+1)}\le 4x^2-4x+6\ \ (2)$$

Equality holds when $(2x^2-2x+1)=1$ ***(a)***

Again, using GM≤AM: $\sqrt[4]{(3x^3-2x^4)\cdot 1\cdot 1\cdot 1}\le\left(\frac{3x^3-2x^4+3}{4}\right)$

$$\Rightarrow 4\sqrt[4]{(3x^3-2x^4)}\le(3x^3-2x^4+3)\ \ (3)$$

Equality holds when $3x^3-2x^4=1$ ***(b)***

Now, adding (2) and (3):

$$6\sqrt[3]{2x^2-2x+1}+4\sqrt[4]{3x^3-2x^4}\le 4x^2-4x+3x^3-2x^4+9$$

$$2x^5-5x+13\le 4x^2-4x+3x^3-2x^4+9$$

$$2x^5+2x^4-3x^3-4x^2-4x+4\le 0;\ (x-1)^2(2x^3+6x^2+7x+4)\le 0$$

From (1), $x\in\left[0,\frac{3}{2}\right]\Rightarrow(2x^3+6x^2+7x+4)>0\Rightarrow(x-1)^2\le 0\Rightarrow(x-1)^2=0$

$x=1$ *(c). From (a), (b), (c): we have only one real solution i.e.* $x=1$.

Solution 2 by Minh Tam Le-Vietnam

$$\text{Let } \begin{matrix}\sqrt{x}=a\\ \sqrt{y}=b\end{matrix}\ (a,b\ge 0).$$

$$\left.\begin{matrix}We\ have\ a^6+b^6=ab(a^4+b^4)\\ But\begin{cases}5a^6+b^6\overset{AM-GM}{\ge}6a^5b\\ 5b^6+a^5\overset{AM-GM}{\ge}6ab^5\end{cases}\Rightarrow a^6+b^6\ge ab(a^4+b^4)\end{matrix}\right\}\Rightarrow a=b \text{ or } x=y$$

$$\text{If } x=y,\ 6\sqrt[3]{2x^2-2y+1}+4\sqrt[4]{3x^2y-2x^4}=2y^5-5\sqrt{xy}+13$$

$$\Leftrightarrow 6\sqrt[3]{2x^2-2x+1}+4\sqrt[4]{3x^2-2x^4}=2x^5-5x+13$$

$$LHS=2\cdot 3\sqrt[3]{2x^2-2x+1}+4\sqrt[4]{x^2\cdot x(3-2x)}\overset{AM-GM}{\le}2(2x^2-2x+1+1+1)+$$

$$=x^2+x+1+3-2x=5x^2-5x+10$$

$$RHS=x^5+x^5+1+1+1-5x+10\overset{AM-GM}{\ge}5x^2-4x+10$$

So, the equality holds if $x=1\Rightarrow y=1$. *Hence,* $x=1$ *and* $y=1$.

JP.198. Prove that in any ΔABC the following inequality holds:

$$\min(a^2,b^2,c^2)\le 4r(R+r)\le\max(a^2,b^2,c^2)$$

Proposed by Marian Ursărescu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\min(a^2,b^2,c^2)\overset{(1)}{\le}4r(R+r)\overset{(2)}{\le}\max(a^2,b^2,c^2)$$

$$\max(a^2,b^2,c^2)\ge\frac{\sum a^2}{3}\overset{?}{\ge}4r(R+r)\Leftrightarrow s^2-4Rr-r^2\ge 6r(R+r)$$

$$\Leftrightarrow s^2\ge 10Rr+7r^2\Leftrightarrow(s^2-16Rr+5r^2)+6r(R-2r)\ge 0\to \textit{true}$$

$$\because s^2-16Rr+5r^2\overset{Gerretsen}{\ge}\text{ and } R-2r\overset{Euler}{\ge}0\ \therefore\ \max(a^2,b^2,c^2)\ge 4r(R+r)$$

Now, $4r(R+r) = 4\frac{abc}{4\Delta}\left(\frac{\Delta}{s}\right) + 4\frac{\Delta^2}{s^2} = \frac{abc}{s} + \frac{4(s-a)(s-b)(s-c)}{s} \overset{(i)}{=} \frac{(x+y)(y+z)(z+x)+4xyz}{x+y+z}$

Letting $s-a = x, s-b = y, s-c = z \therefore s = \sum x$ ***and*** $\therefore a = y+z, b = z+x, c = x+y.$

Case 1: $\min(a^2, b^2, c^2) = a^2 \therefore$ ***(1)*** $\Leftrightarrow (y+z)^2 \le \frac{4xyz+\prod(x+y)}{\sum x}$ ***(by (i))***

$$\Leftrightarrow x^2y + x^2z + 4xyz \overset{(1a)}{\ge} y^3 + z^3 + 2y^2z + 2yz^2$$

$$\because a^2 \le b^2 \therefore y+z \le z+x \Rightarrow x \ge y \text{ and } \because a^2 \le c^2 \therefore y+z \le x+y \Rightarrow x \ge z$$

$$\therefore x^2y \ge y^3 (\because x \ge y), x^2z \ge z^3 \ (\because x \ge z)$$

$$2xyz \ge 2y^2z (\because x \ge y) \text{ and } 2yzx \ge 2yz^2 \ (\because x \ge z)$$

Adding the last 4 inequalities, (1a)⇒ (1) is true.

Case 2: $\min(a^2, b^2, c^2) = b^2 \therefore$ ***(1)*** $\Leftrightarrow (z+x)^2 \le \frac{4xyz+\prod(x+y)}{\sum x}$

$$\Leftrightarrow y^2x + y^2z + 4xyz \overset{(1b)}{\ge} 2x^2z + 2xz^2 + x^3 + z^3$$

$$\because b^2 \le a^2 \therefore z+x \le y+z \Rightarrow y \ge x \text{ and } \because b^2 \le c^2 \therefore z+x \le x+y \Rightarrow y \ge z$$

$$\therefore y^2x \ge x^3 (\because y \ge x), y^2z \ge z^3 \ (\because y \ge x)$$

$$2xyz \ge 2x^2z (\because y \ge x), 2xyz \ge 2xz^2 (\because y \ge z)$$

Adding the last 4 inequalities, (1b)⇒ (1) is true.

Case 3: $\min(a^2, b^2, c^2) = c^2 \therefore$ ***(1)***$\Leftrightarrow (x+y)^2 \le \frac{4xyz+\prod(x+y)}{\sum x}$

$$\Leftrightarrow xz^2 + yz^2 + 4xyz \overset{(1c)}{\ge} 2x^2y + 2xy^2 + x^3 + y^3$$

$$\because c^2 \le a^2 \therefore x+y \le y+z \Rightarrow z \ge x \text{ and } \because c^2 \le b^2 \therefore x+y \le z+x \Rightarrow z \ge y$$

$$\therefore xz^2 \ge x^3 (\because z \ge x), yz^2 \ge y^3 (\because z \ge y)$$

$$2xyz \ge 2x^2y (\because z \ge x) \text{ and } 2xyz \ge 2xy^2 (\because z \ge y)$$

Adding the last 4 inequalities, (1c)⇒(1) is true.

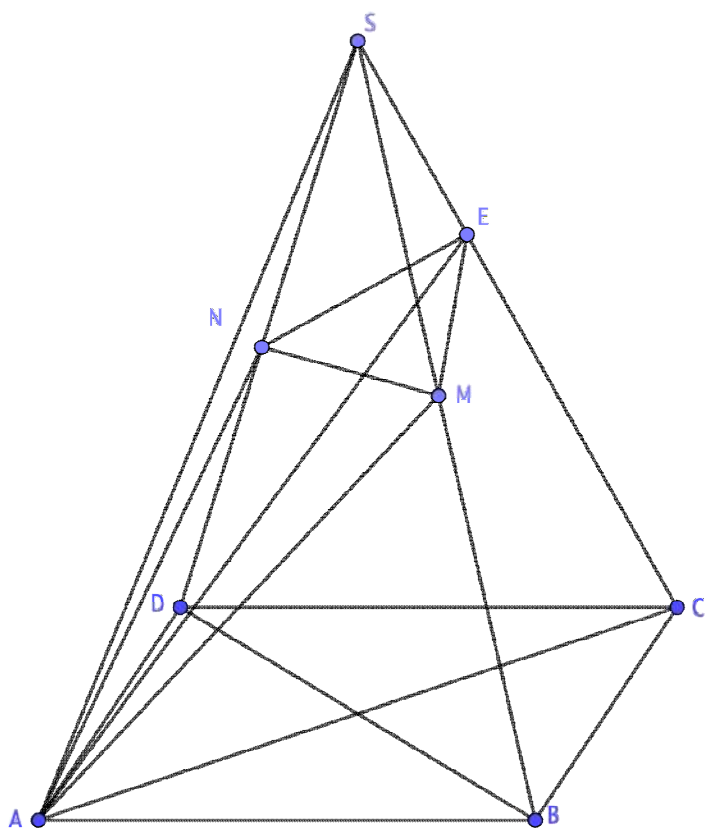
Combining the 3 cases, $\min(a^2, b^2, c^2) \le 4r(R+r)$ ***(Proved)***

JP.199. Let $SABCD$ be a pyramid with the base $ABCD$ parallelogram and E any point which belongs to the side SC such that: $\frac{SE}{SC}=k$. Through the vertex A and the point E we consider a variable plane which intersects the segment SB in M and the segment SD in N. Prove that:

$$\frac{V_{SAEMN}}{V_{SABCD}}\geq\frac{2k^2}{k+1}$$

Proposed by Marian Ursărescu – Romania

Solution by Marian Ursărescu – Romania



$$V_{SABCD}=V$$

$$V_{SABCD}=V_{SBDC}=V_{SACD}=V_{SABC}=\frac{V}{2}$$

$$\frac{V_{SANE}}{\frac{V}{2}}=k\cdot\frac{SN}{SD}\Rightarrow\frac{V_{SANE}}{V}=\frac{k}{2}\cdot\frac{SN}{SD}\quad (1)$$

$$\frac{V_{SAME}}{\frac{V}{2}}=k\cdot\frac{SM}{SB}\Rightarrow\frac{V_{SAME}}{V}=\frac{k}{2}\cdot\frac{SM}{SB}\quad (2)$$

From (1)+(2)$\Rightarrow\frac{V_{SAEMN}}{V}=\frac{k}{2}\left(\frac{SM}{SB}+\frac{SN}{SD}\right)$ *(3)*

$$\frac{V_{SANM}}{\frac{V}{2}}=\frac{SM}{SB}\cdot\frac{SN}{SD}\quad (4)$$

$$\frac{S_{SENM}}{\frac{V}{2}}=k\cdot\frac{SM}{SB}\cdot\frac{SN}{SD}\quad (5)$$

From (4)+(5)$\Rightarrow\frac{V_{SAEMN}}{V}=\frac{k+1}{2}\cdot\frac{SM}{SB}\cdot\frac{SN}{SD}$ *(6)*

From (3)+(6)$\Rightarrow\frac{V_{SAEMN}}{V}=\frac{k}{2}\left(\frac{SM}{SB}+\frac{SN}{SD}\right)\ge\frac{k}{2}\cdot 2\sqrt{\frac{SM}{SB}\cdot\frac{SN}{SD}}=k\sqrt{\frac{V_{SAEMN}}{V}\cdot\frac{2}{k+1}}\Rightarrow\frac{V_{SAEMN}}{V}\ge\frac{2k^2}{k+1}$

JP.200. Let be $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x)+f(y)\ge 2f\left(\sqrt{\frac{x^2+y^2}{2}}\right);(\forall)x,y\in\mathbb{R}$$

Prove that:

$$f(x_1)+f(x_2)+\dots+f(x_n)\ge nf\left(\sqrt{\frac{x_1^2+x_2^2+\dots+x_n^2}{n}}\right);(\forall)n\ge 2$$

$$(\forall)x_1,x_2,\dots,x_n\in\mathbb{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

We prove by induction: I. $P(2):f(x_1)+f(x_2)\ge 2f\left(\sqrt{\frac{x_1^2+x_2^2}{2}}\right);(\forall)x_1,x_2\in\mathbb{R}$ *(true)*

II. Let $P(2),P(3),\dots,P(n-1)$ *true.*

$$P(n):f(x_1)+f(x_2)+\dots+f(x_n)\ge nf\left(\sqrt{\frac{x_1^2+x_2^2+\dots+x_n^2}{n}}\right)$$

Case I. If $n = 2k$

$$f(x_1)+f(x_2)+\cdots+f(x_k)\ge kf\left(\sqrt{\frac{x_1^2+x_2^2+\cdots+x_k^2}{k}}\right)\ \left(P\left(\frac{n}{2}\right)=P(k)\ true\right)$$

$$f(x_{k+1})+\cdots+f(x_{2k})\ge kf\left(\sqrt{\frac{x_{k+1}^2+\cdots+x_{2k}^2}{k}}\right)\ \left(P\left(\frac{n}{2}\right)=P(k)\ true\right)$$

$$\Rightarrow f(x_1)+\cdots+f(x_{2k})\ge k\left[f\left(\sqrt{\frac{x_1^2+\cdots+x_k^2}{k}}\right)+f\left(\sqrt{\frac{x_{k+1}^2+\cdots+x_{2k}^2}{k}}\right)\right]\quad \textbf{(1)}$$

From $P(2)\Rightarrow f\left(\sqrt{\frac{x_1^2+\cdots+x_k^2}{2}}\right)+f\left(\sqrt{\frac{x_{k+1}^2+\cdots+x_{2k}^2}{2}}\right)\ge 2f\left(\sqrt{\frac{x_1^2+\cdots+x_{2k}^2}{2k}}\right)$ ***(2)***

From (1)+(2)$\Rightarrow f(x_1)+\cdots+f(x_{2k})\ge 2kf\left(\sqrt{\frac{x_1^2+x_2^2+\cdots+x_{2k}^2}{2k}}\right)$

Case II. If $n = 2k-1 \Rightarrow$ ***we prove the relation for*** $n = 2k \Rightarrow$

$$\left.\begin{array}{c} f(x_1)+f(x_2)+\cdots+f(x_{2k-1})+f(x_{2k})\ge 2kf\left(\sqrt{\frac{x_1^2+x_2^2+\cdots+x_{2k}^2}{2k}}\right)\\ Let\ x_{2k}=\sqrt{\frac{x_1^2+x_2^2+\cdots+x_{2k-1}^2}{2k-1}}\ in\ relation\ (3)\end{array}\right\}\Rightarrow$$

$$f(x_1)+f(x_2)+\cdots+f(x_{2k-1})+f\left(\sqrt{\frac{x_1^2+\cdots+x_{2k-1}^2}{2k-1}}\right)\ge$$

$$\ge 2kf\left(\sqrt{\frac{x_1^2+x_2^2+\cdots+x_{2k-1}^2+\frac{x_1^2+x_2^2+\cdots+x_{2k-1}^2}{2k-1}}{2k}}\right)\Rightarrow$$

$$f(x_1)+\cdots+f(x_{2k-1})+f\left(\sqrt{\frac{x_1^2+\cdots+x_{2k-1}^2}{2k-1}}\right)\ge 2kf\left(\sqrt{\frac{x_1^2+\cdots+x_{2k-1}^2}{2k-1}}\right)\Rightarrow$$

$$f(x_1)+\cdots+f(x_{2k-1})\ge(2k-1)f\left(\sqrt{\frac{x_1^2+x_2^2+\cdots+x_{2k-1}^2}{2k-1}}\right)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

JP.201. If $x,y,z>0$ then:

$$\frac{(x+y)^3}{x+y+2z}+\frac{(y+z)^3}{y+z+2x}+\frac{(z+x)^3}{z+x+2y}\geq 2\sqrt{3xyz(x+y+z)}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

Let $P=\sum_{cyc(x,y,z)}\frac{(x+y)^3}{(x+z)+(y+z)}$. *Now, let* $(x+y)=p,(y+z)=q,(x+z)=r$

$$p=\sum_{cyc(p,q,r)}\frac{p^3}{q+r}=\sum_{cyc(p,q,r)}\frac{p^4}{pq+pr};\ p\overset{Bergstrom}{\geq}\frac{(p^2+q^2+r^2)^2}{2(pq+qr+pr)}$$

$$p\geq\left(\frac{p^2+q^2+r^2}{2}\right)\{\because p^2+q^2+r^2\geq pq+qr+pr,\forall p,q,r\in\mathbb{R}\}$$

$$p\geq\frac{1}{2}\{(x+y)^2+(y+z)^2+(x+z)^2\};\ p\geq(x^2+y^2+z^2+xy+yz+xz)$$

$$p\geq 2(xy+yz+xz)\ \ \{\because x^2+y^2+z^2\geq xy+yz+xz,\forall x,y,z\in\mathbb{R}\}$$

Now, since we know that: $a^2+b^2+c^2\geq ab+bc+ac,\forall a,b,c\in\mathbb{R}$

$(a+b+c)^2\geq 3(ab+bc+ac)$. *Put* $a=xy,b=yz,c=xz$

$$(xy+yz+xz)^2\geq 3xyz(x+y+z)$$

$$(xy+yz+xz)\geq\sqrt{3xyz(x+y+z)}\quad \textit{(1)}$$

$\because p\geq 2(xy+yz+xz)$. *Using (i):* $p\geq 2\sqrt{3xyz(x+y+z)}$ *(Proved)*

Solution 2 by Le Ngo Duc-Vietnam

$$\sum_{cyc}\frac{(x+y)^3}{x+y+2z}\overset{Holder}{\geq}\frac{8(x+y+z)^3}{3\cdot 4(x+y+z)}=\frac{2}{3}(x+y+z)^2$$

We need to prove $\frac{2}{3}(x+y+z)^2\geq 2\sqrt{3xyz(x+y+z)}$

$$\Leftrightarrow\frac{(x+y+z)^4}{9}\geq 3xyz(x+y+z)\Leftrightarrow(x+y+z)^3\geq 27xyz$$

Which is correct by AM-GM. Inequality holds when $x=y=z$.

Solution 3 by Soumava Chakraborty-Kolkata-India

Let $x+y=a,y+z=b,z+x=c\ \therefore a+b>c,b+c>a,c+a>b\Rightarrow a,b,c$

are sides of a triangle with semi – perimeter, circumradius, inradius $= s, R, r$

respectively (say). Now, $2\sum x = \sum a = 2s \Rightarrow \sum x = s \Rightarrow z = s - a, x = s - b, y = s - c$

Using this substitution, the given inequality becomes: $\sum \frac{a^3}{b+c} \overset{(1)}{\geq} 2\sqrt{3r^2s(s)} = 2\sqrt{3}\Delta$

WLOG, we may assume $a \geq b \geq c$***. Then,*** $a^2 \geq b^2 \geq c^2$ ***and*** $\frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b}$

$$\therefore \sum \frac{a^3}{b+c} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a^2\right)\left(\sum \frac{a}{b+c}\right)$$

$\overset{Nesbitt}{\geq} \frac{1}{3} \cdot \frac{3}{2} \sum a^2 = \frac{\sum a^2}{2} \overset{\substack{Ionescu \\ Weitzenbock}}{\geq} \frac{4\sqrt{3}\Delta}{2} = 2\sqrt{3}\Delta \Rightarrow$ ***(1) is true (Proved)***

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0$***, we get as follows:*** $\frac{(x+y)^3}{(x+y+2z)} + \frac{(y+z)^3}{(y+z+2x)} + \frac{(z+x)^3}{(z+x+2y)} =$

$$= \frac{(x+y)^4}{(x+y)(x+y+2z)} + \frac{(y+z)^4}{(y+z)(y+z+2x)} + \frac{(z+x)^4}{(z+x)(z+x+2y)}$$

$$\geq \frac{[(x+y)^2 + (y+z)^2 + (z+x)^2]^2}{(x+y)(x+y+2z) + (y+z)(y+z+x) + (z+x)(z+x+2y)} =$$

$$= \frac{[(x+y+z)^2 + x^2 + y^2 + z^2]^2}{2[(x+y+z)^2 + (xy+yz+zx)]}$$

$$\geq \frac{(x+y+z)^2 + (x^2+y^2+z^2)}{2} \geq \frac{2(x+y+z)^2}{3} \geq 2\sqrt{3xyz(x+y+z)}$$

If $\frac{4}{9}(x+y+z)^4 \geq 4\big(3xyz(x+y+z)\big)$***. If*** $\frac{(x+y+z)^3}{9} \geq 3xyz$***. If*** $\frac{27xyz}{9} = 3xyz$ ***ok.***

Therefore, it is true.

Solution 5 by Tran Hong-Dong Thap-Vietnam

By Holder's inequality: $\frac{(x+y)^3}{x+y+2z} + \frac{(y+z)^3}{y+z+2x} + \frac{(x+z)^3}{x+z+2y} \geq \frac{1}{3}\left[\frac{(2x+2y+2z)^3}{4(x+y+z)}\right]$

$= \frac{2(x+y+z)^3}{3(x+y+z)} = \frac{2}{3}(x+y+z)^2$***. We must show that:*** $\frac{2}{3}(x+y+z)^2 \geq 2\sqrt{3xyz(x+y+z)}$

$$\Leftrightarrow (x+y+z)^2 \geq 3\sqrt{3xyz(x+y+z)} \Leftrightarrow (x+y+z)^4 \geq 27xyz(x+y+z)$$

$\Leftrightarrow (x+y+z)^3 \geq 27xyz$ ***(true by AM-GM)***

Solution 6 by Michael Sterghiou-Greece

$$x,y,z>0\rightarrow \sum_{cyc}\frac{(x+y)^3}{x+y+2z}\geq 2\sqrt{3xyz(x+y+z)}\ \textit{(1)}$$

(1) homogeneous so, WLOG, let $x+y+z=3$. *Consider* $f(t)=\frac{(3-t)^3}{3+t}, 0<t<3$

$$f''(t)=\frac{2(3-t)(t^2+12t+63)}{(t+3)^2}>0 \text{ for } 0<t<3. \text{ By Jensen:}$$

$LHS\ (1)\geq 3\cdot\frac{(3-1)^3}{3+1}=6\geq 2\sqrt{9xyz}\rightarrow xyz\leq 1$ *which is true by AM-GM as long as*

$$x+y+z=3$$

JP.202. Let a,b,c be positive real numbers such that

$a^2+b^2+c^2+2abc=1$. Prove that:

$$\frac{a^3}{\sqrt{2b^2+16bc+7c^2}}+\frac{b^3}{\sqrt{2c^2+16ca+7a^2}}+\frac{c^3}{\sqrt{2a^2+16ab+7b^2}}\geq\frac{3}{20}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$\sqrt{25a^2}\cdot\sqrt{2b^2+16bc+7c^2}\overset{AM-GM}{\leq}\frac{25a^2+2b^2+16bc+7c^2}{2}$$

$$\Leftrightarrow\sqrt{a^2}\cdot\sqrt{2b^2+16bc+7c^2}\leq\frac{25a^2+2b^2+16bc+7c^2}{10}$$

$$\Rightarrow\frac{a^3}{\sqrt{2b^2+16bc+7c^2}}=\frac{a^4}{\sqrt{a^2}\sqrt{2b^2+16bc+7c^2}}\geq\frac{10a^4}{25a^2+2b^2+7c^2+16bc}\quad\text{(etc)}$$

$$\Rightarrow LHS=\sum\frac{a^3}{\sqrt{2b^2+16bc+7c^2}}\geq 10\sum\frac{a^4}{25a^2+2b^2+7c^2+16bc}\overset{Schwarz}{\geq}$$

$$10\cdot\frac{(a^2+b^2+c^2)^2}{34(a^2+b^2+c^2)+16(ab+bc+ca)}=\frac{5(a^2+b^2+c^2)^2}{17(a^2+b^2+c^2)+8(ab+bc+ca)}\geq$$

$$\overset{(\sum ab\leq\sum a^2)}{\geq}\frac{5(a^2+b^2+c^2)^2}{17(a^2+b^2+c^2)+8(a^2+b^2+c^2)}=\frac{5(a^2+b^2+c^2)^2}{25(a^2+b^2+c^2)}=\frac{a^2+b^2+c^2}{5}=\Omega$$

$$\because \text{Because: } a^2+b^2+c^2+2abc=1$$

$$\Rightarrow\exists\Delta XYZ \text{ such that: } a=\cos X; b=\cos Y; c=\cos Z$$

$$\Rightarrow a^2+b^2+c^2=\cos^2 X+\cos^2 Y+\cos^2 Z$$

$$= 3 - (\sin^2 X + \sin^2 Y + \sin^2 Z) \geq 3 - \frac{9}{4} = \frac{3}{4}\left(\because \sum \sin^2 X \leq \frac{9}{4}\right) \Rightarrow LHS \geq \Omega \geq \frac{3}{4\cdot 5} = \frac{3}{20}.$$

Proved. Equality $\Leftrightarrow a = b = c = \frac{1}{2}$.

JP.203. If $a, b, c > 0$; $a^{b^2} \cdot b^{c^2} \cdot c^{a^2} = 1$ then:

$$b^2\left(\sqrt[3]{a^{a+b+c}} - a^{\sqrt[3]{abc}}\right) + c^2\left(\sqrt[3]{b^{a+b+c}} - b^{\sqrt[3]{abc}}\right) + a^2\left(\sqrt[3]{c^{a+b+c}} - c^{\sqrt[3]{abc}}\right) \geq 0$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$\sum\left(\frac{b^2}{a^2+b^2+c^2} \cdot a^{\frac{a+b+c}{3}}\right) = \sum\left(\frac{b^2}{a^2+b^2+c^2} \cdot a^{\frac{a+b+c}{3} - \sqrt[3]{abc} + \sqrt[3]{abc}}\right) \geq$$

$$\overset{CEBYSHEV}{\geq} \frac{1}{(a^2+b^2+c^2)^2}\left(\sum b^2 \cdot a^{\sqrt[3]{abc}}\right)\left(\sum b^2 \cdot a^{\frac{a+b+c}{3} - \sqrt[3]{abc}}\right)$$

$$\sum\left(b^2 \cdot a^{\frac{a+b+c}{3}}\right) \geq \frac{1}{a^2+b^2+c^2}\left(\sum b^2 \cdot a^{\sqrt[3]{abc}}\right)\left(\sum b^2\, a^{\frac{a+b+c}{3} - \sqrt[3]{abc}}\right) \geq$$

$$\overset{AM-GM}{\geq} \left(\sum b^2 \cdot a^{\sqrt[3]{abc}}\right) \cdot \sqrt[a^2+b^2+c^2]{\left(a^{b^2} \cdot b^{c^2} \cdot c^{a^2}\right)^{\frac{a+b+c}{3} - \sqrt[3]{abc}}} =$$

$$= \left(\sum b^2 \cdot a^{\sqrt[3]{abc}}\right) \cdot \sqrt[a^2+b^2+c^2]{1^{\frac{a+b+c}{3} - \sqrt[3]{abc}}} = \sum\left(b^2 \cdot a^{\sqrt[3]{abc}}\right)$$

$$\sum\left(b^2 \cdot a^{\frac{a+b+c}{3}}\right) \geq \sum\left(b^2 \cdot a^{\sqrt[3]{abc}}\right)$$

$$\sum b^2\left(a^{\frac{a+b+c}{3}} - a^{\sqrt[3]{abc}}\right) \geq 0$$

$$b^2\left(\sqrt[3]{a^{a+b+c}} - a^{\sqrt[3]{abc}}\right) + c^2\left(\sqrt[3]{b^{a+b+c}} - b^{\sqrt[3]{abc}}\right) + a^2\left(\sqrt[3]{c^{a+b+c}} - c^{\sqrt[3]{abc}}\right) \geq 0$$

JP.204. In ΔABC the following relationship holds:

$$\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\tan\frac{C}{2}} + \frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\tan\frac{A}{2}} + \frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\tan\frac{B}{2}} > \pi$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{ab}{(s-a)(s-b)}}=\sum\frac{s}{c}=s\sum\frac{1}{a}\overset{Bergström}{\geq}\frac{9s}{\sum a}=\frac{9}{2}>\pi$$

(Proved)

JP.205. Let a,b,c be positive real numbers. Prove that:

$$\left(\frac{a^4+b^4}{c^4}+\frac{2}{3}\right)\left(\frac{b^4+c^4}{a^4}+\frac{2}{3}\right)\left(\frac{c^4+a^4}{b^4}+\frac{2}{3}\right)\geq\left(\frac{8}{3}\right)^3$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Marian Ursărescu-Romania

We must show: $\frac{(3a^4+3b^4+2c^4)(3b^4+3c^4+2a^4)(3c^4+3a^4+2b^4)}{3^3\cdot a^4b^4c^4}\geq\left(\frac{8}{3}\right)^3\Leftrightarrow$

$$(3a^4+3b^4+2c^4)(3a^4+3c^4+2b^4)(3b^4+3c^4+3a^4)\geq 2^9a^4b^4c^4 \quad (1)$$

$$3a^4+3b^4+2c^4=a^4+b^4+2(a^4+b^4+c^4)\geq 2a^2b^2+2(a^2b^2+b^2c^2+c^2a^2)\geq$$

$$\geq 2a^2b^2+2abc(a+b+c)=2ab(ab+ac+bc+c^2) \quad (2)$$

From (1)+(2) we must show:

$$2^3a^2b^2c^2(ab+ac+bc+a^2)(ab+ac+bc+b^2)(ab+ac+bc+c^2)\geq 2^9a^4b^4c^4$$

$$\Leftrightarrow (ab+ac+bc+a^2)(ab+ac+bc+b^2)(ab+ac+bc+c^2)\geq 2^6a^2b^2c^2 \quad (3)$$

But
$$\left.\begin{aligned}ab+ac+bc+a^2\geq 4\sqrt[4]{a^2b^2c^2}\\ ab+ac+bc+b^2\geq 4\sqrt[4]{a^2b^4c^2}\\ ab+ac+bc+c^2\geq 4\sqrt[4]{a^2b^2c^4}\end{aligned}\right\}\Rightarrow$$

$\Rightarrow (ab+ac+bc+a^2)(ab+ac+bc+b^2)(ab+ac+bc+c^2)\geq 2^6a^2b^2c^2\Rightarrow$ **(3) it is true.**

Solution 2 by Amit Dutta-Jamshedpur-India

$\because a,b,c>0$. Using $AM\geq GM$, $\frac{a^4+b^4}{c^4}\geq\frac{2a^2b^2}{c^4}$

$$\left(\frac{a^4+b^4}{c^4}+\frac{2}{3}\right)\geq\left(\frac{2a^2b^2}{c^4}+\frac{2}{3}\right)$$

$$\left(\frac{a^4+b^4}{c^4}+\frac{2}{3}\right)\geq\left(\frac{2a^2b^2}{c^4}+\frac{2}{3}\right)\geq\frac{2}{3}\cdot\frac{a^2b^2}{c^4}+\frac{2a^2b^2}{3c^4}+\frac{2a^2b^2}{3c^4}+\frac{2}{3}$$

$$\geq \frac{2}{3}\left\{\frac{a^2b^2}{c^4}+\frac{a^2b^2}{c^4}+\frac{a^2b^2}{c^4}+1\right\} \overset{AM-GM}{\geq} \frac{2}{3}\times 4\left\{\frac{(ab)^{\frac{3}{2}}}{c^3}\right\}$$

$$\left(\frac{a^4+b^4}{c^4}+\frac{2}{3}\right) \geq \frac{8}{3}\cdot\frac{(ab)^{\frac{3}{2}}}{c^3} \quad \textbf{(1)}$$

Similarly, $\left(\frac{b^4+c^4}{a^4}+\frac{2}{3}\right) \geq \frac{8}{3}\cdot\frac{(bc)^{\frac{3}{2}}}{a^3}$ **(2)**

$$\left(\frac{c^4+a^4}{b^4}+\frac{2}{3}\right) \geq \frac{8}{3}\cdot\frac{(ac)^{\frac{3}{2}}}{b^3} \quad \textbf{(3)}$$

Multiplying (1), (2), (3): $\left(\frac{a^4+b^4}{c^4}+\frac{2}{3}\right)\left(\frac{b^4+c^4}{a^4}+\frac{2}{3}\right)\left(\frac{c^4+a^4}{b^4}+\frac{2}{3}\right) \geq \left(\frac{8}{3}\right)^3\left\{\frac{(abc)^3}{(abc)^3}\right\} \geq \left(\frac{8}{3}\right)^3$

Proved. Equality when $a=b=c$.

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\prod_{cyc}\left(\frac{a^4+b^4}{c^4}+\frac{2}{3}\right) \overset{\substack{HOLDER'S\\INEQUALITY}}{\geq} \left(\sqrt[3]{\prod_{cyc}\left(\frac{a^4+b^4}{c^4}\right)}+\frac{2}{3}\right)^3 \geq \left(\sqrt[3]{8}+\frac{2}{3}\right)^3 = \left(\frac{8}{3}\right)^3$$

Proved.

Solution 4 by Tran Hong-Dong Thap-Vietnam

Inequality $\Leftrightarrow \frac{[3(a^4+b^4)+2c^4][3(b^4+c^4)+2a^4][3(a^4+c^4)+b^4]}{3^3(abc)^4} \geq \frac{8^3}{3^3}$

$$\Leftrightarrow \prod_{cyc}[3(a^4+b^4)+2c^4] \geq 8^3(abc)^4$$

$$\because 3(a^4+b^4)+2c^4 = 3a^4+3b^4+2c^4 = a^4+a^4+a^4+b^4+b^4+b^4+c^4+c^4$$

$$\overset{AM-GM}{\geq} 8\sqrt[8]{(a^4)^3(b^4)^3(c^4)^2} \quad \text{(etc)}$$

$$\Rightarrow \prod_{cyc}[3(a^4+b^4)+2c^4] \geq 8\cdot 8\cdot 8\sqrt[8]{(a^4)^8(b^4)^8(c^4)^8} = 8^3(abc)^4$$

Proved. Equality $\Leftrightarrow a=b=c$.

Solution 5 by Soumava Chakraborty-Kolkata-India

Let $a^4+b^4=x, b^4+c^4=y, c^4+a^4=z$. *Then,* $x+y>z, y+z>x, z+x>y \Rightarrow$

$\Rightarrow x,y,z$ *are sides of a triangle with semi-perimeter, circumradius, inradius* $= s,R,r$ *respectively (say). Now,* $2\sum a^4 = \sum x = 2s \Rightarrow \sum a^4 = s \Rightarrow c^4 = s-x, a^4 = s-y,$

$b^4 = s - z$. ***Using this substitution, given inequality becomes:***

$$\prod\left(\frac{x}{s-x}+\frac{2}{3}\right)\geq\left(\frac{8}{3}\right)^3\Leftrightarrow\prod\left[\frac{3x+2(s-x)}{3(s-x)}\right]\geq\left(\frac{8}{3}\right)^3\Leftrightarrow\prod\left(\frac{2s+x}{s-x}\right)\geq 512$$

$$\Leftrightarrow(2s+x)(2s+y)(2s+z)\geq 512r^2s\Leftrightarrow$$

$$\Leftrightarrow 8s^3+4s^2\left(\sum x\right)+2s\left(\sum xy\right)+xyz\geq 512r^2s$$

$$\Leftrightarrow 8s^3+4s^2(2s)+2s(s^2+4Rr+r^2)+4Rrs\geq 512r^2s\Leftrightarrow 18s^3+12Rrs\geq 510r^2s$$

$\Leftrightarrow 3s^2+2Rr\overset{(1)}{\geq}85r^2$. ***But, LHS of (1)*** $\overset{Gerretsen}{\geq} 3(16Rr-5r^2)+2Rr\overset{?}{\geq}85r^2$

$\Leftrightarrow 50Rr\overset{?}{\geq}100r^2\Leftrightarrow R\overset{?}{\geq}2r\to$ ***true (Euler)*** $\Rightarrow$ ***(1)*** $\Rightarrow$ ***given inequality is true (Proved)***

Solution 6 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$, ***we have:*** $\left(\frac{a^4+b^4}{c^4}+\frac{2}{3}\right)\left(\frac{b^4+c^4}{a^4}+\frac{2}{3}\right)\left(\frac{c^4+a^4}{b^4}+\frac{2}{3}\right)=$

$$=\left(\frac{a^4}{c^4}+\frac{b^4}{c^4}+\frac{2}{3}\right)\left(\frac{b^4}{a^4}+\frac{c^4}{a^4}+\frac{2}{3}\right)\left(\frac{c^4}{b^4}+\frac{a^4}{b^4}+\frac{2}{3}\right)\geq\left(1+1+\frac{2}{3}\right)^3=\left(2+\frac{2}{3}\right)^3=\left(\frac{8}{3}\right)^3$$

Because: $\left(\frac{a^4}{c^4}\right)\left(\frac{c^4}{b^4}\right)\left(\frac{b^4}{a^4}\right)=1$, $\left(\frac{b^4}{c^4}\right)\left(\frac{c^4}{a^4}\right)\left(\frac{a^4}{b^4}\right)=1$. ***Therefore, it is true.***

Solution 7 by Michael Sterghiou-Greece

$$\prod_{cyc}\left(\frac{a^4+b^4}{c^4}+\frac{2}{3}\right)\geq\left(\frac{8}{3}\right)^3\quad (1)$$

Let $x=a^4, y=b^4, z=c^4, x,y,z>0$

$$(1)\to\prod_{cyc}\left(\frac{x+y}{z}+\frac{2}{3}\right)\geq\left(\frac{8}{3}\right)^3\quad (2)$$

(2) is homogeneous so, we can assume $x+y+z=3$

The function $f(t)=\ln\left(\frac{3-t}{t}+\frac{2}{3}\right)$ ***with*** $f''(t)=\frac{9(9-2t)}{(t-9)^2t^2}>0$ ***is convex on*** $(0,3)$

$$(2)\to\ln\prod_{cyc}\left(\frac{3-z}{z}+\frac{2}{3}\right)=\sum_{cyc}\ln\left(\frac{3-z}{z}+\frac{2}{3}\right)\geq 3\cdot\ln\left(\frac{3}{\frac{x+y+z}{x}}-1+\frac{2}{3}\right)=\ln\left(\frac{8}{3}\right)^3$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

JP.206. Let ABC be a triangle with inradius r and circumradius R. Let h_a, h_b, h_c the altitudes to sides BC, CA, AB respectively and let r_a, r_b, r_c the exradii to A, B, C respectively. Prove that:

$$\frac{4r}{R^2} \le \frac{h_a}{r_b \cdot r_c} + \frac{h_b}{r_c \cdot r_a} + \frac{h_c}{r_a \cdot r_b} \le \frac{R}{2r^2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Marian Ursărescu-Romania

$$\sum \frac{h_a}{r_b r_c} = \sum \frac{\frac{2S}{a}}{\frac{S}{s-b} \cdot \frac{S}{s-c}} = \frac{2}{s} \cdot \sum \frac{(s-b)(s-c)}{a} = \frac{2}{sr} \sum \frac{(s-b)(s-c)}{a} \quad (1)$$

But, $\sum \frac{(s-b)(s-c)}{a} = \frac{r[s^2+(4R+r)^2]}{4sR}$ *(2)*

From (1)+(2) we must show: $\frac{2}{sr} \cdot \frac{r[s^2+(4R+r)^2]}{4sR} \ge \frac{4r}{R^2} \Leftrightarrow s^2 + (4R+r)^2 \ge \frac{8s^2 r}{R}$ *(3)*

But $R \ge 2r \Rightarrow \frac{1}{R} \le \frac{1}{2r}$ *(4)*

From (3)+(4) we must show:

$s^2 + (4R+r)^2 \ge 4s^2 \Leftrightarrow (4R+r)^2 \ge 3s^2$*, true because it is Doucet's inequality.*

Now, $\sqrt{(s-b)(s-c)} \le \frac{s-b+s-c}{2} \Rightarrow (s-b)(s-c) \le \frac{a^2}{4} \Rightarrow$

$$\Rightarrow \frac{2}{sr} \sum \frac{(s-b)(s-c)}{a} \le \frac{2}{sr} \sum \frac{a}{4} = \frac{2}{sr} \cdot \frac{2s}{4} = \frac{1}{r} \Rightarrow$$

we must show: $\frac{1}{r} \le \frac{R}{2r^2} \Leftrightarrow 2r \le R$ *true Euler's inequality.*

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\frac{h_a}{r_b r_c} + \frac{h_b}{r_c r_a} + \frac{h_c}{r_a r_b} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{h_a h_b h_c}{(r_a r_b r_c)^2}} = 3\sqrt[3]{\frac{2s^2 r^2}{R} \cdot \frac{1}{(s^2 r)^2}} = 3\sqrt[3]{\frac{2}{s^2 R}}$$

We must show that: $3\sqrt[3]{\frac{2}{s^2 R}} \ge \frac{4r}{R^2} \Leftrightarrow 27 \cdot \frac{2}{s^2 R} \ge \frac{4^3 r^3}{(R^2)^3} \Leftrightarrow \frac{27}{s^2} \ge \frac{32 r^3}{R^5} \Leftrightarrow 27R^5 \ge 32 r^3 s^2$

It is true because: $\begin{cases} s^2 \le \frac{27}{4} R^2 \\ r^3 \le 8R^3 \end{cases} \Rightarrow s^2 r^3 \le \frac{27}{32} R^5 \Rightarrow 32 r^3 s^2 \le 27 R^5$

$$\frac{h_a}{r_b r_c} + \frac{h_b}{r_c r_a} + \frac{h_c}{r_a r_b} = \sum \frac{h_a}{r_b r_c} \overset{h_a^2 \le r_b r_c}{\le} \sum \frac{h_a}{h_a^2} = \sum \frac{1}{h_a} = \frac{1}{r} \overset{(2)}{\le} \frac{R}{2r^2}$$

(2) $\Leftrightarrow \frac{1}{r} \le \frac{R}{2r^2} \Leftrightarrow R \ge 2r$ ***(Euler) (proved)***

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{h_a}{r_b r_c} = \sum \frac{h_a r_a}{rs^2} = \frac{1}{rs^2}\sum\left(\frac{2rs}{4R\sin\frac{A}{2}\cos\frac{A}{2}}\right)\left(\frac{s\sin\frac{A}{2}}{\cos\frac{A}{2}}\right) = \frac{1}{2R}\sum\frac{bc}{s(s-a)}$$

$$= \left(\frac{1}{2Rr^2s^2}\right)\left\{\sum bc(s-b)(s-c)\right\} = \left(\frac{1}{2Rr^2s^2}\right)\left\{\sum (bc(s^2-s(2s-a)+bc)\right\}$$

$$= \left(\frac{1}{2Rr^2s^2}\right)\left\{s^2\left(\sum ab\right) - 2s^2\left(\sum ab\right) + 3sabc + \left(\sum ab\right)^2 - 2abc(2s)\right\}$$

$$= \left(\frac{1}{2Rr^2s^2}\right)\{(4Rr+r^2)(s^2+4Rr+r^2) - 4Rrs^2\}$$

$$= \left(\frac{1}{2Rr^2s^2}\right)(s^2r^2 + r^2(4R+r)^2) = \frac{s^2+(4R+r)^2}{2Rs^2} \therefore \sum\frac{h_a}{r_b r_c} \overset{(1)}{=} \frac{s^2+(4R+r)^2}{2Rs^2}$$

$$\therefore \sum\frac{h_a}{r_b r_c} \le \frac{R}{2r^2} \overset{by\,(1)}{\Leftrightarrow} \frac{s^2+(4Rr+r)^2}{2Rs^2} \le \frac{R}{2r^2} \Leftrightarrow (R^2-r^2)s^2 \overset{(a)}{\ge} r^2(4R+r)^2$$

Now, $\because s^2 \ge 27r^2 \therefore$ ***LHS of (a)*** $\ge 27r^2(R^2-r^2) \overset{?}{\ge} r^2(4R+r)^2$

$$\Leftrightarrow 11R^2 - 8Rr - 28r^2 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)(11R+14r) \overset{?}{\ge} 0 \to \textit{true} \because R \overset{Euler}{\ge} 2r$$

$\Rightarrow$ ***(a) is true*** $\Rightarrow \sum\frac{h_a}{r_b r_c} \le \frac{R}{2r^2}$

Again, $\frac{4r}{R^2} \le \sum\frac{h_a}{r_b r_c} \overset{by\,(1)}{\Leftrightarrow} \frac{s^2+(4R+r)^2}{2Rrs^2} \ge \frac{4r}{R^2} \Leftrightarrow Rs^2 + R(4R+r)^2 \ge 8rs^2$

$$\Leftrightarrow (R-2r)s^2 + R(4R+r)^2 \overset{(b)}{\ge} 6rs^2$$

Now, LHS of (b) $\underset{(i)}{\overset{Gerretsen}{\ge}} (R-2r)(16Rr-5r^2) + R(4R+r)^2$

and, RHS of (b) $\underset{(ii)}{\overset{Gerretsen}{\ge}} 6r(4R^2+4Rr+3r^2)$

(i), (ii) $\Rightarrow$ ***in order to prove (b), it suffices to prove:***

$$(R-2r)(16Rr-5r^2) + R(4R+r)^2 \ge 6r(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 4t^3 - 15t - 2 \ge 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(4t^2+8t+1) \ge 0 \to \textit{true} \because t \overset{Euler}{\ge} 2$$

$\Rightarrow$ ***(b) is true*** $\Rightarrow \frac{4r}{R^2} \le \sum\frac{h_a}{r_b r_c}$ ***(Proved)***

JP.207. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r and circumradius R, and let r_a, r_b, r_c the exradii to A, B, C respectively. Prove that:

$$6r \le \frac{a^2}{r_b + r_c} + \frac{b^2}{r_c + r_a} + \frac{c^2}{r_a + r_b} \le \frac{2R^2 - Rr}{r}$$

Proposed by George Apostolopulos-Messolonghi-Greece

Solution 1 by Marian Ursărescu-Romania

$$\frac{a^2}{r_b + r_c} = \frac{a^2}{\frac{S}{s-b} + \frac{S}{s-c}} = \frac{a^2}{S\left(\frac{s-c+s-b}{(s-b)(s-c)}\right)} = \frac{a^2(s-b)(s-c)}{sa} = \frac{a(s-b)(s-c)}{s}$$

$$\Rightarrow \sum \frac{a^2}{r_b+r_c} = \frac{1}{s}\sum a(s-b)(s-c) \ \textbf{(1)}$$

But $\sum a(s-b)(s-c) = 2S(2R-r)$ ***(2)***

From ***(1)+(2)***$\Rightarrow \sum \frac{a^2}{r_b+r_c} = 2(2R-r)$ ***(3)***

First, we must show: $6r \le 2(2R-r) \Leftrightarrow 3r \le 2R - r \Leftrightarrow 4r \le 2R \Leftrightarrow 2r \le R$ ***(true)***

Second, we must show: $2(2R-r) \le \frac{R(2R-r)}{r} \Leftrightarrow 2r \le R$ ***true.***

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\frac{a^2}{r_b + r_c} + \frac{b^2}{r_c + r_a} + \frac{c^2}{r_a + r_b} \overset{Chebyshev}{\ge} \frac{(a+b+c)^2}{2(r_a + r_b + r_c)} = \frac{4s^2}{2(4R+r)} = \frac{2s^2}{4R+r}$$

We must show that: $\frac{2s^2}{4R+r} \ge 6r \Leftrightarrow s^2 \ge 3r(4R+r) \Leftrightarrow s^2 \ge 12Rr + 3r^2$

$s^2 \ge 16Rr - 5r^2 \ge 12Rr + 3r^2 \Leftrightarrow 4Rr \ge 8r^2 \Leftrightarrow R \ge 2r$ ***(Euler)***

$$\frac{a^2}{r_b + r_c} + \frac{b^2}{r_c + r_a} + \frac{c^2}{r_a + r_b} = \sum \frac{a^2}{r_b + r_c} =$$

$$= \sum \frac{(2R\sin A)^2}{4R\cos^2\frac{A}{2}} = R\sum \frac{\sin^2 A}{\cos^2\frac{A}{2}} = 4R\sum \frac{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{\cos^2\frac{A}{2}}$$

$$= 4R\sum \sin^2\frac{A}{2} = 4R\left(\frac{2R-r}{2R}\right) = 2(2R+r) \overset{(2)}{\le} \frac{2R^2 - Rr}{r}$$

(2) $\Leftrightarrow 2r(2R-r) \le 2R^2 - Rr$

$$\Leftrightarrow 2R^2-5Rr+2r^2\ge 0 \Leftrightarrow (R-2r)(2R-r)\ge 0 \text{ (True: } R\ge 2r\text{) (Proved)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\textit{Firstly, } \sum ar_a = \sum\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)s\tan\frac{A}{2} = 2Rs\sum(1-\cos A) \overset{(1)}{=} 2s(2R-r)$$

$$\sum\frac{a^2}{r_b+r_c} = \sum\frac{a^3}{a(r_b+r_c)} \overset{Holder}{\ge} \frac{8s^3}{3\sum a(r_b+r_c)} = \frac{8s^3}{3\sum a(\sum r_a - r_a)}$$

$$\overset{by\ (1)}{=} \frac{8s^3}{3\{(4R+r)(2s)-2s(2R-r)\}} = \frac{2s^2}{3(R+r)} \overset{?}{\ge} 6r \Leftrightarrow s^2 \overset{?}{\ge} 9r(R+r)$$

$$\Leftrightarrow (s^2-16Rr+5r^2)+7r(R-2r) \overset{?}{\ge} 0 \to \textit{true} \because s^2-16Rr+5r^2 \overset{Gerretsen}{\ge} 0$$

$$\textit{and, } R-2r \overset{Euler}{\ge} 0 \therefore 6r \le \sum\frac{a^2}{r_b+r_c}$$

$$\textit{Now, Bogdan Fustei} \Rightarrow \frac{b+c}{2} \le \sqrt{R(r_b+r_c)} \Rightarrow r_b+r_c \ge \frac{(b+c)^2}{4R}, \textit{ etc}$$

$$\therefore \sum\frac{a^2}{r_b+r_c} \le 4R\sum\frac{a^2}{(b+c)^2} \overset{A-G}{\le} 4R\sum\frac{a^2}{4bc} = \frac{R}{4Rrs}\sum a^3 = \frac{2s(s^2-6Rr-3r^2)}{4rs}$$

$$= \frac{s^2-6Rr-3r^2}{2r} \overset{?}{\le} \frac{2R^2-Rr}{r} \Leftrightarrow s^2 \overset{?}{\le} 4R^2+4Rr+3r^2 \to \textit{true (Gerretsen)}$$

$$\Rightarrow \sum\frac{a^2}{r_b+r_c} \le \frac{2R^2-Rr}{r} \quad \textit{(proved)}$$

Solution 4 by Bogdan Fustei-Romania

$$\textit{We know that: } r_a+r_b+r_c = 4R+r$$

$$a^2 = (r_a-r)(r_b+r_c) \textit{ (and analogs)}$$

$$\frac{a^2}{r_b+r_c} = r_a - r \textit{ (and analogs)} \Rightarrow \sum\frac{a^2}{r_b+r_c} = r_a-r+r_b-r+r_c-r =$$

$$= 4R+r-3r = 4R-2r = 2(2R-r)$$

$$\textit{We will prove that: } 6r \le 2(2R-r) \le \frac{2R^2-Rr}{r}$$

$$6r \le 2(2R-r) \Rightarrow 3r \le 2R-r \Rightarrow 4R \le 2R \Rightarrow 2r \le R \textit{ (Euler's inequality)}$$

$$2(2R-r) \le \frac{2R^2-Rr}{r} = \frac{R}{r}(2R-r)\ (2R-r \Rightarrow 2R-r>0) \Rightarrow$$

$$\Rightarrow 2 \le \frac{R}{r} \Rightarrow 2r \le R \textit{ (Euler's inequality). So, the inequality from enunciation is proved.}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

JP.208. Prove that in any ABC triangle the following inequality holds:

$$\sum \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{m_a^2} \le \frac{R}{Sr}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Marian Ursărescu-Romania

We have in any ΔABC: $m_a \ge \sqrt{s(s-a)} \Rightarrow ma^2 \ge s(s-a) \Rightarrow$

$$\Rightarrow \sum \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{m_a^2} \le \sum \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{s(s-a)} \Rightarrow \text{we must show: } \sum \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{s(s-a)} \le \frac{R}{sr^2} \Leftrightarrow \sum \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{s-a} \le \frac{R}{r^2} \Leftrightarrow$$

$$\sum \frac{\tan\frac{B}{2}}{s-a} + \sum \frac{\tan\frac{C}{2}}{s-a} \le \frac{R}{r^2} \quad (1)$$

$$\sum \frac{\tan\frac{B}{2}}{s-a} = \sum \frac{\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}{s-a} = \sum \sqrt{\frac{s-c}{s(s-a)(s-c)}} = \sum \frac{s-c}{S} =$$

$$= \frac{s}{S} = \frac{1}{r} \quad (2)$$

$$\sum \frac{\tan\frac{C}{2}}{s-a} = \sum \frac{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}{s-a} = \sum \sqrt{\frac{(s-b)}{s(s-a)(s-c)}} = \sum \frac{s-b}{S} =$$

$$= \frac{s}{S} = \frac{1}{r} \quad (3)$$

From (1)+(2)+(3) we must show: $\frac{2}{r} \le \frac{R}{r^2} \Leftrightarrow 2r \le R$ *true.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{m_a^2} \le \frac{R}{Sr} \Leftrightarrow \sum \frac{s\tan\frac{B}{2}+s\tan\frac{C}{2}}{m_a^2} \le \frac{R}{r^2} \Leftrightarrow \sum \frac{r_b+r_c}{m_a^2} \overset{(1)}{\le} \frac{R}{r^2}$$

WLOG, we may assume $a \ge b \ge c \therefore r_b + r_c \le r_c + r_a \le r_a + r_b$, *and,*

$$\frac{1}{m_a^2} \ge \frac{1}{m_b^2} \ge \frac{1}{m_c^2}$$

$$\therefore \sum \frac{r_b+r_c}{m_a^2} \overset{Chebyshev}{\le} \frac{\sum(r_b+r_c)}{3} \sum \frac{1}{m_a^2} \overset{m_a^2 \ge s(s-a)}{\le} \frac{2(4R+r)}{3} \sum \frac{1}{s(s-a)}$$

$$=\frac{2(4r+r)}{3s}\left\{\frac{\sum(s-b)(s-c)}{r^2s}\right\}=\frac{2(4R+r)}{3r^2s^2}\sum(s^2-s(b+c)+bc)$$

$$=\frac{2(4R+r)}{3r^2s^2}(3s^2-4s^2+s^2+4Rr+r^2)=\frac{2(4R+r)^2}{3rs^2}\overset{?}{\leq}\frac{R}{r^2}$$

$$\Leftrightarrow 3Rs^2\underset{(2)}{\overset{?}{\geq}}2r(16R^2+8Rr+r^2)$$

Now, LHS of (2) $\overset{Gerretsen}{\geq} 3R(16Rr-5r^2)\overset{?}{\geq}2r(16R^2+8Rr+r^2)$

$$\Leftrightarrow 16R^2-31Rr-2r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(16R+r)\overset{?}{\geq}0\rightarrow \textit{true} \because R\overset{Euler}{\geq}2r$$

$\Rightarrow$ ***(2)⇒(1)⇒ given inequality is true (Proved)***

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$\sum\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{m_a^2}\leq\frac{R}{Sr}\quad (1)$$

$$(1)\Leftrightarrow\sum\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{m_a^2}\leq\frac{R}{sr^2}$$

$$\Leftrightarrow\sum\frac{\frac{r_b}{s}+\frac{r_c}{s}}{m_a^2}\leq\frac{R}{sr^2}\Leftrightarrow\sum\frac{r_b+r_c}{m_a^2}\leq\frac{R}{r^2}$$

$$\left(r_b+r_c=4R\cos^2\frac{A}{2}\right);m_a\geq\frac{b+c}{2}\cos\frac{A}{2};\textit{ etc}$$

$$\Rightarrow\sum\frac{r_b+r_c}{m_a^2}=\sum\frac{4R\cos^2\frac{A}{2}}{m_a^2}\leq\sum\frac{4R\cos^2\frac{A}{2}}{\left(\frac{b+c}{2}\cos\frac{A}{2}\right)^2}=$$

$$=16R\sum\frac{1}{(b+c)^2}\overset{AM-GM}{\leq}16R\sum\frac{1}{4bc}=4R\sum\frac{1}{bc}=4R\left(\frac{a+b+c}{abc}\right)=4R\cdot\frac{2s}{4Rrs}=\frac{2}{r}$$

We must show that: $\frac{2}{r}\leq\frac{R}{r^2}\Leftrightarrow R\geq 2r$ ***(Euler) (Proved)***

Solution 4 by Bogdan Fustei-Romania

$$\sum\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{m_a^2}\leq\frac{R}{Sr}$$

$r_a=s\tan\frac{A}{2}$ ***(and the analogs);*** $s(s-a)=r_br_c$ ***(and analogs);*** $s=sr$

$$\sum\frac{s\tan\frac{B}{2}+s\tan\frac{C}{2}}{m_a^2}\leq\frac{R}{r^2}\Leftrightarrow\sum\frac{r_b+r_c}{m_a^2}\leq\frac{R}{r^2}$$

$m_a^2 \ge r_b r_c = s(s-a)$ *(and analogs)* $\Rightarrow \sum \frac{r_b+r_c}{m_a^2} \le \sum \frac{r_b+r_c}{r_b r_c} =$

$$= \sum\left(\frac{1}{r_c}+\frac{1}{r_b}\right) = 2\sum\frac{1}{r_a}; \frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{1}{r}$$

$\Rightarrow \sum \frac{r_b+r_c}{m_a^2} \le \frac{2}{r}$. *We will prove that:* $\frac{2}{r} \le \frac{R}{r^2} \Rightarrow 2 \le \frac{R}{r} \Rightarrow 2r \le R$ *(Euler's inequality)*

So, the inequality from the enunciation is proved.

Solution 5 by Mustafa Tarek-Cairo-Egypt

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{bc}{s(s-a)}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{(s-b)(s-c)}{\Delta} = \frac{(a^2-(b-c)^2)}{4\Delta} \le \frac{a^2}{4\Delta}$$

Similarly, $\tan\frac{B}{2} \le \frac{b^2}{4\Delta}, \tan\frac{C}{2} \le \frac{c^2}{4\Delta}$. *Now,* $\sum \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{m_a^2} \le \sum \frac{b^2+c^2}{4\Delta m_a^2} \overset{Tereshin}{\le} \sum \frac{4R\cdot m_a}{4\Delta\cdot m_a^2}$

$$= \frac{R}{\Delta}\sum\frac{1}{m_a} \overset{??}{\le} \frac{R}{\Delta r} \Leftrightarrow \sum\frac{1}{m_a} \le \frac{1}{r} \quad \textit{(1)}$$

But $m_a \ge h_a$, *etc,* $\therefore \frac{1}{m_a} \le \frac{1}{h_a}$, *etc.* $\therefore \sum\frac{1}{m_a} \le \sum\frac{1}{h_a} = \frac{1}{r} \therefore$ *(1) true (Proved)*

JP.209. If $a, b, c, d \in \mathbb{R}$ then:

$$ac + bd + |ad - bc| \le \sqrt{2(a^2+b^2)(c^2+d^2)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have: $ac + bd + |ad - bc| \le |ac + bd| + |ad - bc|$

We must show that: $|ac + bd| + |ad - bc| \le \sqrt{2(a^2+b^2)(c^2+d^2)}$

$$\Leftrightarrow \{|ac+bd| + |(ad-bc)|\}^2 \le 2(a^2+b^2)(c^2+d^2) \quad (*)$$

$$\because \{|ac+bd| + |ad-bc|\}^2 \overset{BCS}{\le} 2\{(ac+bd)^2 + (ad-bc)^2\}$$

$= 2\{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\} = 2(a^2+b^2)(c^2+d^2) \Rightarrow$ *(*) true. Proved.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(ac+bd)^2+(ad-bc)^2\overset{(1)}{=}(a^2+b^2)(c^2+d^2)$$

$$LHS\overset{(2)}{\leq}|ac+bd|+|ad-bc|$$

Case 1: $ad-bc=0$***. Then, we have to prove:***

$$(ac+bd)^2\leq 2(ac+bd)^2+2(ad-bc)^2 \quad \textbf{(by (1))}$$

$$\Leftrightarrow (ac+bd)^2+2(ad-bc)^2\geq 0$$

$\Leftrightarrow (ac+bd)^2\geq 0\to$ ***true*** $\Rightarrow$ ***the given inequality is true.***

Case 2: $ac+bd=0$***. Then we have to prove:***

$$(ad-bc)^2\leq 2(ac+bd)^2+2(ad-bc)^2 \quad \textbf{(by(1))}$$

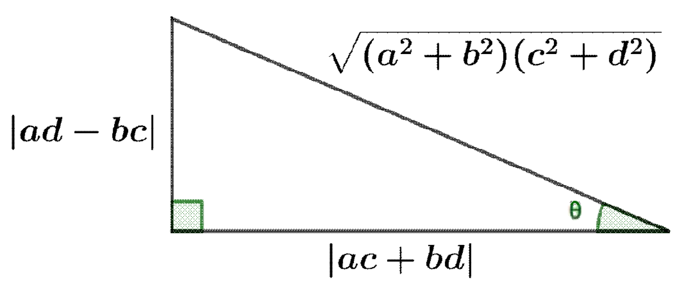
$\Leftrightarrow (ad-bc)^2\geq 0\to$ ***true*** $\Rightarrow$ ***the given inequality is true.***

Case 3: $ad-bc=ac+bd=0$***.***

Then, RHS $\sqrt{2[(ac+bd)^2+(ad-bc)^2]}=0$ ***and of course,***

$LHS=0\Rightarrow LHS=RHS\Rightarrow$ ***the given inequality is true.***

Case 4:



$$ad-bc,ac+bd\neq 0\Rightarrow |ad-bc|,|ac+bd|>0$$

$$\because (ad-bc)^2,(ac+bd)^2>0$$

$\therefore (ad-bc)^2+(ac+bd)^2>0\Rightarrow (a^2+b^2)(c^2+d^2)>0$ ***(by (1))***

Let, $\sqrt{(a^2+b^2)(c^2+d^2)}=p>0$

$\therefore |ad-bc|=p\sin\theta$ ***and*** $|ac+bd|=p\cos\theta$

$$\therefore LHS\overset{by\ (2)}{\leq}p(\cos\theta+\sin\theta)\leq RHS=\sqrt{2}p$$

$\Leftrightarrow p^2(1+\sin 2\theta)\leq 2p^2\Leftrightarrow \sin 2\theta\leq 1\to$ ***true*** $\Rightarrow$ ***the given inequality is true.***

Solution 3 by Ravi Prakash-New Delhi-India

Let $a = r\cos\theta$, $b = r\sin\theta$ where $r = \sqrt{a^2+b^2}$. Now, LHS $= ac + bd + |ad - bc| =$

$$= r(c\cos\theta + d\sin\theta) = r|d\cos\theta - c\sin\theta|$$

If $d\cos\theta - c\sin\theta \ge 0$

$$LHS = r[(c+d)\cos\theta + (d-c)\sin\theta] \le r\sqrt{(c+d)^2 + (d-c)^2}$$

$$\left[\because |a\cos\theta + b\sin\theta| \le \sqrt{a^2+b^2}\right]$$

$$\Rightarrow LHS \le r\sqrt{2(c^2+d^2)} = \sqrt{2(a^2+b^2)(c^2+d^2)} = RHS$$

If $d\cos\theta - c\sin\theta < 0$,

$$LHS = r(c\cos\theta + d\sin\theta) + r(c\sin\theta - d\cos\theta)$$

$$= r[(c-d)\cos\theta + (c+d)\sin\theta] \le r\sqrt{(c-d)^2 + (c+d)^2}$$

$$= r\sqrt{2(c^2+d^2)} = \sqrt{(a^2+b^2)(c^2+d^2)}$$

JP.210. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that:

$$\frac{a+b+c}{9} \le \frac{1}{a^3+(b+c)^3} + \frac{1}{b^3+(c+a)^3} + \frac{1}{c^3+(a+b)^3} \le \frac{1}{3abc}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ and $a^2 + b^2 + c^3 = 3$, we have: $a + b + c \le 3$. Hence:

$$(a^2+b^2+c^2)(a+b+c) = a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b \le 9$$

$$(a^2+b^2+c^2)(a+b+c) = a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b \le 9$$

$$(a^2+b^2+c^2)(a+b+c) = a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b \le 9$$

Find then:

$$a^3+b^3+c^3+3(a^2b+b^2a)+a^3+b^3+c^3+3(b^2c+c^2b)+a^3+b^3+$$

$$+c^3+3(a^2c+c^2a) = c^3+(a+b)^3+a^3+(b+c)^3+b^3+(c+a)^3 \le 27$$

$$\Rightarrow \frac{1}{a^3+(b+c)^3} + \frac{1}{b^3+(c+a)^3} + \frac{1}{c^3+(a+b)^3} \ge \frac{1}{3} \ge \frac{a+b+c}{9} : a+b+c \le 3$$

Next, from this fact $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{a+b+c}{abc}\Rightarrow\frac{1}{ab(a+b+c)}+\frac{1}{bc(a+b+c)}+\frac{1}{ca(a+b+c)}=\frac{1}{abc}$

$$\Rightarrow\frac{1}{3ab(a+b+c)}+\frac{1}{3bc(a+b+c)}+\frac{1}{3ca(a+b+c)}=\frac{1}{3abc}$$

$$\Rightarrow\frac{1}{3abc+3(a^2b+ab^2)}+\frac{1}{3abc+3(b^2c+bc^2)}+\frac{1}{3abc+3(c^2a+ca^2)}=\frac{1}{3abc}$$

$$\Rightarrow\frac{1}{a^3+b^3+c^3+3(a^2b+ab^2)}+\frac{1}{a^2+b^3+c^3+3(b^2c+bc^2)}+$$

$$+\frac{1}{a^3+b^3+c^3+3(c^2a+ca^2)}\le\frac{1}{3abc}\Rightarrow\frac{1}{c^3+(a+b)^3}+\frac{1}{a^3+(b+c)^3}+\frac{1}{b^3+(c+a)^3}\le\frac{1}{3abc}\ \textit{ok}$$

Therefore, it is true.

SP.196. Find:

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n}\sqrt[p]{\frac{n^3}{ijk}},p\in\mathbb{N}^*,p\ge 2$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

Next, $\sum_{i=1}^{n}a_i^3$ ***contains*** n ***terms and*** $n\le\sum_{i=1}^{n}a_i^3\le(n)\left(n^{\frac{1}{\infty}}\right)\Rightarrow\frac{1}{n^2}\le\frac{1}{n^3}\sum_{i=1}^{n}a_i^3\le\frac{1}{n^{\frac{2-1}{p}}}$

Taking limit as $n\to\infty$***, we get:*** $\frac{1}{n^3}\sum_{i=1}^{n}a_i^3=0$***. Also,***

$$\frac{1}{n}\sum_{i=1}^{n}a_i=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(\frac{i}{n}\right)^{\frac{1}{p}}}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(\frac{i}{n}\right)^{\frac{1}{p}}}=\int_0^1\frac{dx}{x^{\frac{1}{p}}}=\left.\frac{x^{1-\frac{1}{p}}}{1-\frac{1}{p}}\right]_0^1=\frac{p}{p-1}\quad(p\ge 2)$$

Now,

$$\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^3=\frac{1}{n^3}\sum_{i=1}^{n}a_i^3+6\frac{1}{n^3}\sum_{1\le i<j}a_i^2a_j+6\frac{1}{n^3}\sum_{1\le i<j<k\le n}a_ia_ja_k$$

Taking limit as $n \to \infty$*, we get:* $\left(\frac{p}{p-1}\right)^3 = 0 + 0 + 6\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le k\le n} a_i a_j a_k$

Thus, $\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n} a_i a_j a_k = \frac{1}{6}\left(\frac{p}{p-1}\right)^3$*. Let* $a_i = \left(\frac{n}{i}\right)^{\frac{1}{p}}, 1 \le i \le n, p \ge 2$

$\sum_{1\le i<j\le n} a_i^2 a_j$ *contains* $\frac{n(n-1)}{2}$ *terms. Also,* $1 \le a_i \le n^{\frac{1}{p}}\ \forall i$

$$\therefore \frac{n(n-1)}{2n^3} \le \frac{1}{n^3}\sum_{1\le i<j\le n} a_i^2 a_j \le \frac{n(n-1)}{2n^3} n^{\frac{1}{p}}$$

$$\Rightarrow \frac{1}{2n}\left(1-\frac{1}{n}\right) \le \frac{1}{n^3}\sum_{1\le i<j\le n} a_i^2 a_j \le \frac{1}{2n^{1-p}}\left(1-\frac{1}{2n}\right)$$

Taking limit, we get: $\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n} a_i^2 a_j = 0$

SP.197. If $x, y, z \ge 0$ then:

$$\frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}} + \frac{7^{\sqrt{yz}}}{5^{\sqrt{yz}}+3^{\sqrt{yz}}} + \frac{7^{\sqrt{zx}}}{5^{\sqrt{zx}}+3^{\sqrt{zx}}} \le$$

$$\le \frac{\sqrt{7^{x+y}}}{\sqrt{5^{x+y}}+\sqrt{3^{x+y}}} + \frac{\sqrt{7^{y+z}}}{\sqrt{5^{y+z}}+\sqrt{3^{y+z}}} + \frac{\sqrt{7^{z+x}}}{\sqrt{5^{z+x}}+\sqrt{3^{z+x}}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b \ge 0$*, we have:* $\left(\frac{7}{5}\right)^{(ab)^{\frac{1}{2}}} \le \left(\frac{7}{5}\right)^{\frac{a+b}{2}} \leftrightarrow 7^{(ab)^{\frac{1}{2}}} 5^{\frac{(a+b)}{2}} \le 7^{(a+b)} \cdot 5^{(ab)^{\frac{1}{2}}}$

$$\left(\frac{7}{3}\right)^{(ab)^{\frac{1}{2}}} \le \left(\frac{7}{3}\right)^{\frac{a+b}{2}} \leftrightarrow 7^{(ab)^{\frac{1}{2}}} \cdot 3^{\frac{a+b}{2}} \le 7^{\frac{a+b}{2}} \cdot 3^{(ab)^{\frac{1}{2}}}$$

$$\Rightarrow 7^{(ab)^{\frac{1}{2}}} \cdot 5^{\frac{a+b}{2}} + 7^{(ab)^{\frac{1}{2}}} \cdot 3^{\frac{a+b}{2}} \le 7^{\frac{a+b}{2}} \cdot 5^{(ab)^{\frac{1}{2}}} + 7^{\frac{a+b}{2}} \cdot 3^{(ab)^{\frac{1}{2}}}$$

$$\Rightarrow 7^{(ab)^{\frac{1}{2}}}\left(5^{\frac{a+b}{2}} + 3^{\frac{a+b}{2}}\right) \le 7^{\frac{a+b}{2}}\left(5^{(ab)^{\frac{1}{2}}} + 3^{(ab)^{\frac{1}{2}}}\right) \Rightarrow \frac{7^{(ab)^{\frac{1}{2}}}}{5^{(ab)^{\frac{1}{2}}} + 3^{(ab)^{\frac{1}{2}}}} \le \frac{7^{\frac{a+b}{2}}}{5^{\frac{a+b}{2}} + 3^{\frac{a+b}{2}}}$$

Hence for $x, y, z \ge 0$*, we get that:*

$$\frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}}+\frac{7^{\sqrt{yz}}}{5^{\sqrt{yz}}+3^{\sqrt{yz}}}+\frac{7^{\sqrt{zx}}}{5^{\sqrt{zx}}+3^{\sqrt{zx}}}\le\frac{7^{\frac{x+y}{2}}}{5^{\frac{x+y}{2}}+3^{\frac{x+y}{2}}}+\frac{7^{\frac{y+z}{2}}}{5^{\frac{y+x}{2}}+3^{\frac{y+z}{2}}}+\frac{7^{\frac{z+x}{2}}}{5^{\frac{z+x}{2}}+3^{\frac{z+x}{2}}}$$

Therefore, it is true.

Solution 2 by Marian Ursărescu-Romania

Inequality $\Leftrightarrow \frac{1}{\left(\frac{5}{7}\right)^{\sqrt{xy}}+\left(\frac{3}{7}\right)^{\sqrt{xy}}}+\frac{1}{\left(\frac{5}{7}\right)^{\sqrt{yz}}+\left(\frac{3}{7}\right)^{\sqrt{xz}}}+\frac{1}{\left(\frac{5}{7}\right)^{\sqrt{xz}}+\left(\frac{3}{7}\right)^{\sqrt{xz}}}\le$

$$\le\frac{1}{\sqrt{\left(\frac{5}{7}\right)^{x+y}}+\sqrt{\left(\frac{3}{7}\right)^{x+y}}}+\frac{1}{\sqrt{\left(\frac{5}{7}\right)^{y+z}}+\sqrt{\left(\frac{3}{7}\right)^{y+z}}}+\frac{1}{\sqrt{\left(\frac{5}{7}\right)^{z+x}}+\sqrt{\left(\frac{3}{7}\right)^{z+x}}}\quad (1)$$

$\sqrt{xy}\le\frac{x+y}{2}$ ***and*** $\frac{5}{7}$ ***and*** $\frac{3}{7}\in(0,1)\Rightarrow$

$$\left.\begin{array}{l}\left(\frac{5}{7}\right)^{\sqrt{xy}}\ge\left(\frac{5}{7}\right)^{\frac{x+y}{2}}\\ \left(\frac{3}{7}\right)^{\sqrt{xy}}\ge\left(\frac{3}{7}\right)^{\frac{x+y}{2}}\end{array}\right\}\Rightarrow\left(\frac{5}{7}\right)^{\sqrt{xy}}+\left(\frac{3}{7}\right)^{\sqrt{xy}}\ge\sqrt{\left(\frac{5}{7}\right)^{x+y}}+\sqrt{\left(\frac{3}{7}\right)^{x+y}}\Rightarrow$$

$\frac{1}{\left(\frac{5}{7}\right)^{\sqrt{xy}}+\left(\frac{3}{7}\right)^{\sqrt{xy}}}\le\frac{1}{\sqrt{\left(\frac{5}{7}\right)^{x+y}}+\sqrt{\left(\frac{3}{7}\right)^{x+y}}}$ ***and two similar relationship, and by summing*** $\Rightarrow$ ***(1) is true.***

Solution 3 by Amit Dutta-Jamshedpur-India

Let $F(t)=\frac{7^t}{5^t+3^t};\ F'(t)=\frac{(3^t+5^t)7^t\ln 7-7^t(5^t\ln 5+3^t\ln 3)}{(5^t+3^t)^2}$

$$F'(t)=\frac{1}{(5^t+3^t)^2}[(35)^t\ln 7+(21)^t\ln 7-(35)^t\ln 5-(21)^t\ln 3]$$

$F'(t)=\frac{1}{(5^t+3^t)^2}\left\{(35)^t\ln\left(\frac{7}{5}\right)+(21)^t\ln\left(\frac{7}{3}\right)\right\}$, ***clearly,*** $F'(t)>0$

$F(t)$ ***is an increasing function. By AM ≥ GM,*** $\frac{x+y}{2}\ge\sqrt{xy}$

$$F\left(\frac{x+y}{2}\right)\ge F(\sqrt{xy})$$

$$\frac{7^{\frac{x+y}{2}}}{5^{\frac{x+y}{2}}+3^{\frac{x+y}{2}}}\ge\frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}}\quad (1)$$

Again, $\frac{y+z}{2}\ge\sqrt{yz}$ ***{AM ≥ GM};*** $F\left(\frac{y+z}{2}\right)\ge F(\sqrt{yz})$

$$\frac{7^{\frac{y+z}{2}}}{5^{\frac{y+z}{2}}+3^{\frac{y+z}{2}}}\ge\frac{7^{\sqrt{yz}}}{5^{\sqrt{yz}}+3^{\sqrt{yz}}}\quad (2)$$

Also, again, by AM≥GM: $\frac{x+z}{2} \geq \sqrt{xz}$; $F\left(\frac{x+z}{2}\right) \geq F(\sqrt{xz})$

$$\frac{7^{\frac{x+z}{2}}}{5^{\frac{x+z}{2}}+3^{\frac{x+z}{2}}} \geq \frac{7^{\sqrt{xz}}}{5^{\sqrt{xz}}+3^{\sqrt{xz}}} \quad (3)$$

Adding (1), (2), (3), we have the desired inequality: $\frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}} + \frac{7^{\sqrt{yz}}}{5^{\sqrt{yz}}+3^{\sqrt{yz}}} + \frac{7^{\sqrt{xz}}}{5^{\sqrt{xz}}+3^{\sqrt{xz}}} \leq$

$$\leq \frac{\sqrt{7^{x+y}}}{\sqrt{5^{x+y}}+\sqrt{3^{x+y}}} + \frac{\sqrt{7^{y+z}}}{\sqrt{5^{y+z}}+\sqrt{3^{y+z}}} + \frac{\sqrt{7^{x+z}}}{\sqrt{5^{x+z}}+\sqrt{3^{x+z}}} \text{ (Proved)}$$

Solution 4 by Michael Sterghiou-Greece

$$\sum_{cyc} \frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}} \leq \sum_{cyc} \frac{\sqrt{7^{x+y}}}{\sqrt{5^{x+y}}+\sqrt{3^{x+y}}} \quad (1)$$

RHS of (1) $\rightarrow \sum_{cyc} \frac{7^{\frac{x+y}{2}}}{5^{\frac{x+y}{2}}+3^{\frac{x+y}{2}}}$*. Consider the function*

$$\frac{7^t}{5^t+3^t} = f(t), t \geq 0, f'(t) = \frac{21t \ln\frac{7}{3} + 35^t \cdot \ln\frac{42}{30}}{(3^t+5^t)^2} > 0$$

So, $f(t) \uparrow$ *on* $[0, +\infty]$*, But* $\sqrt{xy} \leq \frac{x+y}{2}$ *and same in a cyclical manner so,*

$$\sum_{cyc} f(\sqrt{xy}) \leq \sum_{cyc} f\left(\frac{x+y}{2}\right) \rightarrow \text{(1) is true.}$$

SP.198. If $x, y, z, t \in \mathbb{R}; x^2 + y^2 = z^2 + t^2 = 10$ **then:**

$$(10 - x - 3y)(10 - xz - yt)(10 - z - 3t) < 10125$$

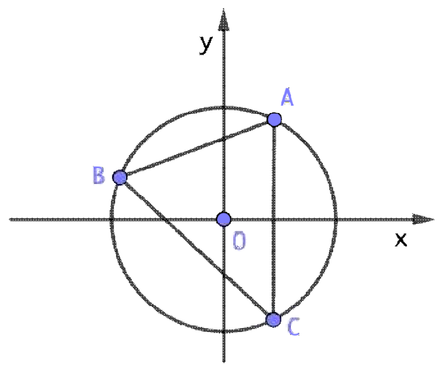
Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\text{LHS} = (10 - x - 3y)(10 - xz - yt)(10 - z - 3t)$$
$$\leq |(10 - x - 3y)(10 - xz - yt)(10 - z - 3t)|$$
$$= |(x + 3y - 10)| \cdot |(xz + yt - 10)| \cdot |(z + 3t - 10)|$$
$$\leq (|x + 3y| + 10) \cdot (|xz + yt| + 10) \cdot (|z + 3t| + 10)$$
$$\overset{BCS}{\leq} \left(\sqrt{1^2+3^2}\sqrt{x^2+y^2} + 10\right)\left(\sqrt{x^2+y^2}\sqrt{z^2+t^2} + 10\right)\left(\sqrt{1^2+3^2}\sqrt{t^2+z^2} + 10\right)$$
$$= (\sqrt{10} \cdot \sqrt{10} + 10)(\sqrt{10} \cdot \sqrt{10} + 10)(\sqrt{10} \cdot \sqrt{10} + 10)$$
$$= 20 \cdot 20 \cdot 20 = 8000 < 10125. \text{ Proved.}$$

Solution 2 by proposer

Let be $A(1,3); B(x,y); C(z,t)$



$$R = OA = \sqrt{1^2+3^2} = \sqrt{10}\ \textbf{(1)},\ A,B,C \in \mathcal{C}(O,R);\ \mathcal{C}: x^2+y^2 = 10$$

$$AB = \sqrt{(x-1)^2+(y-3)^2} = \sqrt{x^2-2x+1+y^2-6y+9} =$$

$$= \sqrt{10-2x-6y+10} = \sqrt{20-2x-6y}$$

$$AC = \sqrt{(z-1)^2+(t-3)^2} = \sqrt{z^2-2z+1+t^2-6t+9} =$$

$$= \sqrt{10-2z-6t+10} = \sqrt{20-2z-6t}$$

$$BC = \sqrt{(x-z)^2+(y-t)^2} = \sqrt{x^2-2xz+z^2+y^2-2yt+t^2} =$$

$$= \sqrt{20-2xz-2yt}$$

The maximum of area of ΔABC ***is obtained when*** ΔABC ***is an equilateral one.***

The side AB ***can be obtained by:***

$$\frac{2}{3}\cdot\frac{AB\sqrt{3}}{2} = R \Rightarrow AB = \frac{3R}{\sqrt{3}} = R\sqrt{3} \overset{(1)}{=} \sqrt{30}$$

$$S_{\max}[ABC] = \frac{\left(\sqrt{30}\right)^2\cdot\sqrt{3}}{4} = \frac{30\sqrt{3}}{4} = \frac{15\sqrt{3}}{2} \to \frac{AB\cdot AC\cdot BC}{4\cdot R} < \frac{15\sqrt{3}}{2}$$

$$AB\cdot AC\cdot BC < \frac{15\sqrt{3}\cdot 4\cdot\sqrt{30}}{2} = 30\sqrt{90} = 90\sqrt{10}$$

$$\sqrt{20-2x-6y}\cdot\sqrt{20-2z-6t}\cdot\sqrt{20-2xz-2yt} < 90\sqrt{10}$$

$$\sqrt{(10-x-3y)(10-xz-yt)(10-z-3t)} < 45\sqrt{5}$$

$$(10-x-3y)(10-xz-yt)(10-z-3t) < \left(45\sqrt{5}\right)^2 = 10125$$

SP.199. If $n > 1$ then:

$$\frac{1}{\log 2}\left(\frac{2^n-1}{n}\right)^{2n+1} < \frac{1\cdot 3\cdot 7\cdot \ldots \cdot (2^{2n}-1)}{(2n)!}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\text{Let be } I_n = \int_0^1 2^{nx}dx\,;\, n \geq 1$$

$$I_n^2 = \left(\int_0^1 2^{nx}\,dx\right)^2 = \left(\int_0^1 \left(\sqrt{2^{(n-k)x}}\cdot\sqrt{2^{(n+k)x}}\right)dx\right)^2 \leq$$

$$\leq \left(\int_0^1 2^{(n-k)x}dx\right)\left(\int_0^1 2^{(n+k)x}\,dx\right) = I_{n-k}\cdot I_{n+k}$$

$$I_n^2 \leq I_{n-k}\cdot I_{n+k}; 0 \leq k \leq n$$

$$I_n^2 \leq I_{n-1}\cdot I_{n+1}; I_n^2 \leq I_{n-2}\cdot I_{n+2}; \ldots; I_n^2 \leq I_0\cdot I_{2n}$$

$$I_n^{2n} < I_0\cdot I_1\cdot I_2\cdot \ldots \cdot I_{n-1}\cdot I_{n+1}\cdot \ldots \cdot I_{2n}$$

$$I_n^{2n+1} < I_0\cdot I_1\cdot I_2\cdot \ldots \cdot I_{2n}$$

$$\left(\frac{2^{nx}}{n\log 2}\Big|_0^1\right)^{2n+1} < \left(\frac{2^x}{\log 2}\Big|_0^1\right)\cdot\left(\frac{2^{2x}}{2\log 2}\Big|_0^1\right)\cdot\left(\frac{2^{3x}}{3\log 2}\Big|_0^1\right)\cdot \ldots \cdot\left(\frac{2^{2nx}}{2n\log 2}\Big|_0^1\right)$$

$$\left(\frac{2^n-1}{n\log 2}\right)^{2n+1} < \frac{(2-1)(2^2-1)(2^3-1)\cdot \ldots \cdot (2^{2n}-1)}{(2n)!\cdot(\log 2)^{2n}}$$

$$\frac{1}{\log 2}\left(\frac{2^n-1}{n}\right)^{2n+1} < \frac{1\cdot 3\cdot 7\cdot \ldots \cdot (2^{2n}-1)}{(2n)!}$$

SP.200. If $a, b, c, d \in \mathbb{R}$ then:

$$2|ad-bc|(ac+bd)+(ac+bd)^2 \leq (ad-bc)^2+(a^2+b^2)(c^2+d^2)\sqrt{2}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$\text{Let be } \vec{u} = a\vec{i}+b\vec{j}; \vec{v} = c\vec{i}+d\vec{j}$$

$$\cos\left(\widehat{\vec{u},\vec{v}}\right)=\frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}$$

$$\sin^2\left(\widehat{\vec{u},\vec{v}}\right)=1-\cos^2\left(\widehat{\vec{u},\vec{v}}\right)=1-\frac{(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}=$$

$$=\frac{a^2c^2+a^2d^2+b^2c^2+b^2d^2-a^2c^2-b^2d^2-2abcd}{(a^2+b^2)(c^2+d^2)}=$$

$$=\frac{a^2d^2-2abcd+b^2c^2}{(a^2+b^2)(c^2+d^2)}=\frac{(ad-bc)^2}{(a^2+b^2)(c^2+d^2)}$$

$$\sin\left(\widehat{\vec{u},\vec{v}}\right)=\frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

$$\sin 2\left(\widehat{\vec{u},\vec{v}}\right)=2\sin\left(\widehat{\vec{u},\vec{v}}\right)\cos\left(\widehat{\vec{u},\vec{v}}\right)=2\cdot\frac{|ad-bc|(ac+bd)}{(a^2+b^2)(c^2+d^2)}$$

$$\cos 2\left(\widehat{\vec{u},\vec{v}}\right)=2\cos^2\left(\widehat{\vec{u},\vec{v}}\right)-1=2\cdot\frac{(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}-1=$$

$$=\frac{2(a^2c^2+b^2c^2+2abcd)-(a^2+b^2)(c^2+d^2)}{(a^2+b^2)(c^2+d^2)}=$$

$$=\frac{2a^2c^2+2b^2d^2+4abcd-a^2c^2-a^2d^2-b^2c^2-b^2d^2}{(a^2+b^2)(c^2+d^2)}=$$

$$=\frac{a^2c^2+b^2d^2+4abcd-a^2d^2-b^2c^2}{(a^2+b^2)(c^2+d^2)}=\frac{(ac+bd)^2-(ad-bc)^2}{(a^2+b^2)(c^2+d^2)}$$

$$\sin 2x+\cos 2x=\sin 2x+\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}\cos 2x=$$

$$=\frac{\sin 2x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos 2x}{\frac{\sqrt{2}}{2}}=\frac{\sin\left(2x+\frac{\pi}{4}\right)}{\frac{1}{\sqrt{2}}}=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)\le\sqrt{2}$$

$$\sin 2x+\cos 2x\le\sqrt{2}$$

$$\sin 2\left(\widehat{\vec{u},\vec{v}}\right)+\cos 2\left(\widehat{\vec{u},\vec{v}}\right)\le\sqrt{2}$$

$$\frac{2|ad-bc|(ac+bd)}{(a^2+b^2)(c^2+d^2)}+\frac{(ac+bd)^2-(ad-bc)^2}{(a^2+b^2)(c^2+d^2)}\le\sqrt{2}$$

$$2(ad-bc)(ac+bd)+(ac+bd)^2-(ad+bc)^2\le\sqrt{2}(a^2+b^2)(c^2+d^2)$$

$$2(ad-bc)(ac+bd)+(ac+bd)^2\le(ad-bc)^2+(a^2+b^2)(c^2+d^2)\sqrt{2}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

SP.201. Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)\tan^{-1}\left(\frac{2k^2+4k+1}{2(k+1)}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\tan^{-1}\left(\frac{k+2}{k+1}\right)-\tan^{-1}\left(\frac{k+1}{k}\right)=\tan^{-1}\left(\frac{\frac{k+2}{k+1}-\frac{k+1}{k}}{1+\frac{k+2}{k+1}\cdot\frac{k+1}{k}}\right)=$$

$$=\tan^{-1}\left(\frac{k^2+2k-k^2-2k-1}{k(k+1)}\cdot\frac{k(k+1)}{k^2+k+k^2+3k+2}\right)=$$

$$=\tan^{-1}\left(-\frac{1}{2k^2+4k+1}\right)=-\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$$

$$\tan^{-1}\left(\frac{k+2}{k+1}\right)+\tan^{-1}\left(\frac{k+1}{k}\right)=\tan^{-1}\left(\frac{\frac{k+2}{k+1}+\frac{k+1}{k}}{1-\frac{k+2}{k+1}\cdot\frac{k+1}{k}}\right)=$$

$$=\tan^{-1}\left(\frac{k^2+2k+k^2+2k+1}{k(k+1)}\cdot\frac{k(k+1)}{k^2+k-k^2-3k-2}\right)$$

$$=\tan^{-1}\left(\frac{2k^2+4k+1}{-2k-2}\right)=-\tan^{-1}\left(\frac{2k^2+4k+1}{2(k+1)}\right)$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{k+2}{k+1}\right)-\tan^{-1}\left(\frac{k+1}{k}\right)\right)\cdot\left(\tan^{-1}\left(\frac{k+2}{k+2}\right)+\tan^{-1}\left(\frac{k+1}{k}\right)\right)\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\left(\tan^{-1}\left(\frac{k+2}{k+1}\right)\right)^2-\left(\tan^{-1}\left(\frac{k+1}{k}\right)\right)^2\right)\right)=$$

$$=\lim_{n\to\infty}\left(\left(\tan^{-1}\left(\frac{n+2}{n+1}\right)\right)^2-\left(\tan^{-1}\left(\frac{1+1}{1}\right)\right)^2\right)=$$

$$=(\tan^{-1}1)^2-(\tan^{-1}2)^2$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

SP.202. Prove that in any triangle ABC, the following relationship holds:

$$\frac{m_a}{l_a}+\frac{m_b}{l_b}+\frac{m_c}{l_c}\ge 3+\left(\frac{b-c}{b+c}\right)^2+\left(\frac{c-a}{c+a}\right)^2+\left(\frac{a-b}{a+b}\right)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Mustafa Tarek-Cairo-Egypt

$$\sum\frac{m_a}{w_a}\ge 3+\sum\frac{(b-c)^2}{(b+c)^2}\quad (1)$$

$$\because m_a\ge\frac{b+c}{2}\cos\frac{A}{2}=\frac{2bc}{b+c}\cos\frac{A}{2}\cdot\frac{(b+c)^2}{4bc}=w_a\cdot\frac{(b+c)^2}{4bc}$$

Similarly, $m_b\ge w_b\frac{(a+c)^2}{4ac}, m_c\ge w_c\frac{(a+b)^2}{4ab}$

$\therefore\sum\frac{m_a}{w_a}\ge\sum\frac{(b+c)^2}{4bc}$, RHS of (1) $=\sum\left(\frac{(b+c)^2+(b-c)^2}{(b+c)^2}\right)=\sum\frac{2(b^2+c^2)}{(b+c)^2}$, so, we must prove that:

$$\frac{(b+c)^2}{4bc}\ge\frac{2(b^2+c^2)}{(b+c)^2}\Leftrightarrow(b+c)^4\ge 8b^3c+8c^3b$$

$$\Leftrightarrow(b^2+c^2+2bc)^2=b^4+c^4+2b^2c^2+4b^2c^2+4b^3c+4c^3b\ge 8b^3c+8c^3b$$

$$\Leftrightarrow b^4+c^4+8b^2c^2-2b^2c^2-4b^3c-9c^3b\ge 0$$

$$\Leftrightarrow(b^2-c^2)^2-4bc(b^2-c^2-2bc)\ge 0$$

$\Leftrightarrow(b-c)^2((b+c)^2-4bc)\ge 0\Leftrightarrow(b-c)^4\ge 0\Leftrightarrow$ true, similarly, $\frac{(a+c)^2}{4ac}\ge\frac{(a^2+c^2)}{(a+c)^2}$

$\frac{(a+b)^3}{4ab}\ge\frac{2(a^2+b^2)}{(a+b)^2}\therefore\sum\frac{(b+c)^3}{4bc}\ge\sum\frac{2(b^2+c^2)}{(b+c)^2}$ and $\sum\frac{m_a}{w_a}\ge\sum\frac{(b+c)^2}{4bc}\therefore\sum\frac{m_a}{w_b}\ge\sum\frac{2(b^2+c^2)}{(b+c)^2}=RHS$

Proved

Solution 2 by Marian Ursărescu-Romania

In any ΔABC we have: $m_a\ge\frac{b+c}{2}\cos\frac{A}{2}$ and $l_a=\frac{2bc}{b+c}\cos\frac{A}{2}\Rightarrow\frac{m_a}{l_a}\ge\frac{(b+c)^2}{4bc}\Rightarrow$ we must

show: $\frac{1}{4}\sum\frac{(b+c)^2}{bc}\ge 3+\sum\left(\frac{b-c}{b+c}\right)^2\quad (1)$

But $(b+c)^2\ge 4bc\Rightarrow\frac{1}{(b+c)^2}\le\frac{1}{4bc}\Rightarrow\left(\frac{b-c}{b+c}\right)^2\le\frac{(b-c)^2}{4bc}\quad (2)$

From (1)+(2) we must show: $\frac{1}{4}\sum\frac{(b+c)^2}{bc}\ge 3+\frac{1}{4}\sum\frac{(b-c)^2}{bc}\Leftrightarrow\frac{1}{4}\sum\frac{(b+c)^2-(b-c)^2}{bc}\ge 3\Leftrightarrow$

$\Leftrightarrow\frac{1}{4}\sum\frac{4bc}{bc}\ge 3\Leftrightarrow 3\ge 3$ true.

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$m_a^2=\frac{2(b^2+c^2)-a^2}{4};l_a^2=\frac{4bcs(s-a)}{(b+c)^2}$$

We must show that: $\because \frac{m_a}{l_a}\geq 1+\left(\frac{b-c}{b+c}\right)^2=\frac{2(b^2+c^2)}{(b+c)^2}\Leftrightarrow \frac{m_a^2}{l_a^2}\geq \frac{4(b^2+c^2)^2}{(b+c)^4}$

$$\Leftrightarrow\left[\frac{2(b^2+c^2)-a^2}{4}\right]\left[\frac{(b+c)^2}{4bcs(s-a)}\right]\geq \frac{4(b^2+c^2)^2}{(b+c)^4}$$

$$\Leftrightarrow [2(b^2+c^2)-a^2][b+c]^6\geq 64bcs(s-a)(b^2+c^2)^2$$

$$\Leftrightarrow [2(b^2+c^2)-a^2](b+c)^6\geq 16bc(a+b+c)(b+c-a)(b^2+c^2)^2$$

$$\Leftrightarrow (b-c)^2\left[\frac{a^2}{2}(b^4+c^4)+\{b^6+c^6-b^2c^2(b^2+c^2)\}+4a^2bc(b^2+bc+c^2)\right]\geq 0$$

It is true because: $(b-c)^2\geq 0$

$$b^6+c^6-b^2c^2(b^2+c^2)\geq 0\Leftrightarrow (b^2-c^2)^2(b^2+c^2)\geq 0$$

$$4a^2bc(b^2+bc+c^2)+\frac{(b^4+c^4)a^2}{2}>0\ (a,b,c>0)$$

Similarly: $\frac{m_b}{l_b}\geq 1+\left(\frac{a-c}{a+c}\right)^2;\frac{m_c}{l_c}\geq 1+\left(\frac{a-b}{a+b}\right)^2\Rightarrow \sum\frac{m_a}{l_a}\geq 3+\sum\left(\frac{b-c}{b+c}\right)^2$ ***Proved.***

Equality $\Leftrightarrow a=b=c.$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$m_a^2-w_a^2=\frac{2b^2+2c^2-a^2}{4}-\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}$$

$$=\frac{2b^2+2c^2-a^2}{4}-\frac{bc[(b+c)^2-a^2]}{(b+c)^2}=\left(\frac{2b^2+2c^2}{4}-bc\right)+a^2\left[\frac{bc}{(b+c)^2}-\frac{1}{4}\right]$$

$$=\frac{2(b-c)^2}{4}-\frac{a^2(b-c)^2}{4(b+c)^2}=\frac{(b-c)^2}{4}\left\{2-\frac{a^2}{(b+c)^2}\right\}\overset{(1)}{=}\frac{(b-c)^2}{4}\left[\frac{2(b+c)^2-a^2}{(b+c)^2}\right]$$

Now, $\frac{m_a}{w_a}\geq 1+\frac{(b-c)^2}{(b+c)^2}=\frac{2(b^2+c^2)}{(b+c)^2}\Leftrightarrow \frac{m_a^2}{w_a^2}\geq\frac{4(b^2+c^2)^2}{(b+c)^4}$

$$\Leftrightarrow \frac{m_a^2-w_a^2}{w_a^2}\geq\frac{\{2(b^2+c^2)+(b+c)^2\}\{2(b^2+c^2)-(b+c)^2\}}{(b+c)^4}$$

$$\overset{by\ (1)}{\Leftrightarrow}\frac{(b-c)^2}{4}\left[\frac{2(b+c)^2-a^2}{(b+c)^2}\right]\frac{(b+c)^2}{bc\{(b+c)^2-a^2\}}\geq\frac{2(b^2+c^2)+(b+c)^2}{(b+c)^4}(b-c)^2$$

$$\Leftrightarrow \frac{(b-c)^2}{4}\left[\frac{2(b+c)^2-a^2}{\{(b+c)^2-a^2\}bc}\right] \geq (b-c)^2\left[\frac{2(b^2+c^2)+(b+c)^2}{(b+c)^4}\right]$$

$\because (b-c)^2 \geq 0 \therefore$ *it suffices to prove: (in order to prove:* $\frac{m_a}{w_a} \geq 1 + \frac{(b-c)^2}{(b+c)^2}$*)*

$$\frac{2(b+c)^2-a^2}{4bc\{(b+c)^2-a^2\}} > \frac{2(b^2+c^2)+(b+c)^2}{(b+c)^4}$$

$$\Leftrightarrow \frac{\{(b+c)^2-a^2\}+(b+c)^2}{4bc\{(b+c)^2-a^2\}} > \frac{1}{(b+c)^2} + \frac{2(b^2+c^2)}{(b+c)^4}$$

$$\Leftrightarrow \left[\frac{1}{4bc} - \frac{1}{(b+c)^2}\right] + \frac{(b+c)^2}{4bc\{(b+c)^2-a^2\}} > \frac{2(b^2+c^2)}{(b+c)^4}$$

$$\Leftrightarrow \frac{(b-c)^2}{4bc(b+c)^2} + \frac{(b+c)^2}{4bc\{(b+c)^2-a^2\}} > \frac{(b+c)^2}{(b+c)^4} + \frac{(b-c)^2}{(b+c)^4}$$

$$\Leftrightarrow (b-c)^2\left[\frac{1}{4bc(b+c)^2} - \frac{1}{(b+c)^4}\right] + (b+c)^2\left[\frac{1}{4bc\{(b+c)^2-a^2\}} - \frac{1}{(b+c)^4}\right] > 0$$

$$\Leftrightarrow (b-c)^2\frac{(b-c)^2}{4bc(b+c)^4} + (b+c)^2\left[\frac{(b+c)^4-4bc\{(b+c)^2-a^2\}}{4bc(b+c)^4\{(b+c)^2-a^2\}}\right] > 0$$

$\Leftrightarrow \frac{(b-c)^4}{4bc(b+c)^4} + (b+c)^2\left[\frac{(b+c)^2(b-c)^2+4a^2bc}{4bc(b+c)^4\{(b+c)^2-a^2\}}\right] > 0 \to$ *true* $\therefore \frac{m_a}{w_a} \overset{(a)}{\geq} 1 + \frac{(b-c)^2}{(b+c)^2}$

Similarly, $\frac{m_b}{w_b} \overset{(b)}{\geq} 1 + \frac{(c-a)^2}{(c+a)^2}$ *and* $\frac{m_c}{w_c} \overset{(c)}{\geq} 1 + \frac{(a-b)^2}{(a+b)^2}$

(a)+(b)+(c)$\Rightarrow \sum \frac{m_a}{w_a} \geq 3 + \sum \frac{(b-c)^2}{(b+c)^2}$ *(Proved)*

SP.203. Let a, b, c be positive real numbers such that:

$(a+b)(b+c)(c+a) = 8$. Prove that:

$$\frac{1}{a+b+c} + \frac{1}{ab+bc+ca} \geq \frac{2}{3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let $p = a+b+c; q = ab+bc+ca; r = abc \Rightarrow (a+b)(b+c)(c+a) = pq - r = 8$

$$\Rightarrow pq = 8 + r; pq \geq 9r \Rightarrow 8 + r \geq 9r \Rightarrow 0 < r \leq 1$$

$$\frac{1}{a+b+c}+\frac{1}{ab+bc+ca}\ge\frac{2}{3}\Leftrightarrow 3(p+q)\ge 2pq\Leftrightarrow 3(p+q)\ge 2(8+r)$$

$$\Leftrightarrow 3(p+q)-2r-16\ge 0 \because \textit{But: } 3(p+q)\overset{Cauchy}{\ge} 6\sqrt{pq}=6\sqrt{8+r}$$

We must show that: $6\sqrt{8+r}-2r-16\ge 0\Leftrightarrow 3\sqrt{8+r}\ge r+8\Leftrightarrow$

$\Leftrightarrow 9(8+r)\ge(r+8)^2\Leftrightarrow r+8\le 9\Leftrightarrow r\le 1$ ***(true) Proved.***

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ ***and*** $(a+b)(b+c)(c+a)=a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+$

$+2abc=8\Rightarrow a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+3abc\le 9: abc\le 1$

$$\Rightarrow (a+b+c)(ab+bc+ca)\le 9\Rightarrow\frac{1}{(ab+bc+ca)(a+b+c)}\ge\frac{1}{9}$$

$$\Rightarrow\sqrt{\frac{1}{(a+b+c)(ab+bc+ca)}}\ge\frac{1}{3}\Rightarrow 2\sqrt{\frac{1}{(a+b+c)(ab+bc+ca)}}\ge\frac{2}{3}$$

$\Rightarrow\frac{1}{(a+b+c)}+\frac{1}{ab+bc+ca}\ge\frac{2}{3}$ ***ok. Therefore, it is true.***

SP.204. Let x,y,z be positive real numbers such that $x+y+z=3$. Prove that:

$$\frac{x}{y^2+2z}+\frac{y}{z^2+2x}+\frac{z}{x^2+2y}\ge 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

$$\frac{x}{y^2+2z}+\frac{y}{z^2+2x}+\frac{z}{x^2+2y}=\frac{x^2}{xy^2+2zx}+\frac{y^2}{yz^2+2xy}+\frac{z^2}{x^2z+2yz}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(x+y+z)^2}{xy^2+yz^2+zx^2+2(xy+yz+xz)}=\frac{9}{xy^2+yz^2+zx^2+2(xy+yz+xz)}\Rightarrow$$

We must show: $\frac{9}{xy^2+yz^2+zx^2+2(xy+yz+yz)}\ge 1\Leftrightarrow$

$\Leftrightarrow xy^2+yz^2+zx^2+2(xy+yz+yz)\le 9$ ***(1)***

Because $x+y+z=3$ $(x,y,z>0)\Rightarrow\exists\, a,b,c>0$ ***such that:***

$$x=\frac{3a}{a+b+c},\ y=\frac{3b}{a+b+c},\ z=\frac{3c}{a+b+c}\quad \textbf{(2)}$$

From (1)+(2)⇒ we must show: $\frac{27(ab^2+bc^2+ca^2)}{(a+b+c)^3}+\frac{2\cdot 9(ab+bc+ac)}{(a+b+c)^2}\le 9\Leftrightarrow$

$$\Leftrightarrow (a+b+c)^3\ge 3(ab^2+bc^2+ca^2)+2(a+b+c)(ab+bc+ac)\Leftrightarrow$$

$$\Leftrightarrow a^3+b^3+c^3+3a^2b+3ab^2+3a^2c+3ac^2+3b^2c+3bc^2+6abc\ge$$

$$\ge 3ab^2+3bc^2+3ca^2+2a^2b+2abc+2a^2c+2ab^2+2b^2c+$$

$$+2abc+2abc+2bc^2+2ac^2\Leftrightarrow$$

$$\Leftrightarrow a^2+b^3+c^3+a^2b+ac^2+b^2c\ge 2ab^2+2a^2c+2bc^2\ \textbf{(3)}$$

$$\text{But}\ \left.\begin{array}{c} a^3+ac^2=a(a^2+c^2)\ge 2a^2c\\ b^3+a^2b=b(b^2+a^2)\ge 2ab^2\\ c^3+b^2c=c(c^2+b^2)\ge 2bc^2\end{array}\right\}\Rightarrow$$

$\Rightarrow a^3+b^3+c^3+ac^2+a^2b+b^2c\ge 2(a^2c+ab^2+bc^2)\Rightarrow$ ***(3) is true.***

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x+y+z=3, x,y,z>0$ ***we have:*** $x^2+y^2+z^2\ge xy^2+yz^2+zx^2$

$$\Rightarrow (x+y+z)^2\ge xy^2+yz^2+zx^2+2(xy+yz+zx)$$

$$\Rightarrow \frac{(x+y+z)^2}{(xy^2+2xz)+(yz^2+2xy)+(zx^2+2yz)}\ge 1\Rightarrow \frac{x}{y^2+2z}+\frac{y}{z^2+2x}+\frac{z}{x^2+2y}\ge 1$$

Therefore, it is true. Remark: Because $(x+y+z)(x-zx)+(y-xy)+(z-yz)=$

$$=3[(x-zx)+(y-xy)+(z-yz)]\ge 0.$$

Hence $x(x-zx)+y(y-xy)+z(z-yz)\ge 0$. ***That is*** $x^2+y^2+z^2\ge x^2z+z^2y+y^2x$

Prove that: $x^2+y^2+z^2\ge xy^2+yz^2+zx^2, x+y+z=3, x,y,z>0$

Proof: give $x=\frac{3a}{a+b+c}, y=\frac{3b}{a+b+c}, z=\frac{3c}{a+b+c}$

$$x^2+y^2+z^2\ge xy^2+yz^2+zx^2\leftrightarrow \frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\ge \frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

$$\leftrightarrow \frac{a(a+b+c)}{3bc}+\frac{b(a+b+c)}{3ca}+\frac{c(a+b+c)}{3ab}\ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

$$\leftrightarrow \frac{1}{3}\left[\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right]\ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

$$\leftrightarrow \frac{1}{3}\left[\left(\frac{a^2}{bc}+\frac{c}{b}\right)+\left(\frac{b^2}{ca}+\frac{a}{c}\right)+\left(\frac{c^2}{ab}+\frac{b}{a}\right)+\frac{a}{b}+\frac{b}{c}+\frac{c}{b}\right]\ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

$$\leftrightarrow \frac{1}{3}\left[2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right] \ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \leftrightarrow \frac{1}{3}\left[3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\right] = \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \text{ ok}$$

Therefore, it is true.

SP.205. In $\Delta ABC, n_a, n_b, n_c$ are lenght's of Nagel's cevians. Prove that:

$$n_a n_b n_c \ge r_a r_b r_c$$

r_a, r_b, r_c – exradii of triangle.

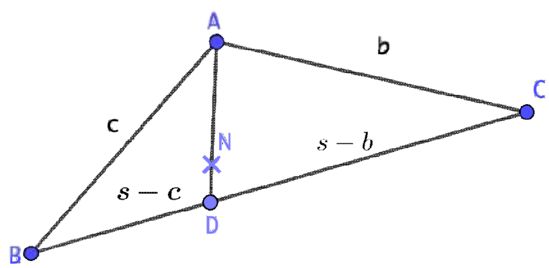
Proposed by Daniel Sitaru-Romania

Solution by proposer

Lemma 1 (Tarek's lemma)

In $\Delta ABC: n_a \ge m_a$ (1)

Proof:



Let AD be the Nagel's cevian of A; $AD = n_a$

By Stewart's theorem in ΔABC:

$$a \cdot n_a^2 = c^2(s-b) + b^2(s-c) - a(s-b)(s-c)$$

$$n_a^2 = \frac{c^2(s-b) + b^2(s-c)}{a} - (s-b)(s-c)$$

$$n_a \ge m_a \Leftrightarrow n_a^2 \ge m_a^2$$

$$\frac{c^2(s-b) + b^2(s-c)}{a} - (s-b)(s-c) \ge \frac{2(b^2+c^2) - a^2}{4}$$

$$\frac{c^2(a+c-b) + b^2(a+b-c)}{2a} \ge \frac{(a+b-c)(a+c-b) + 2(b^2+c^2) - a^2}{4}$$

$$\frac{c^2(a+c-b) + b^2(a+b-c)}{2a} \ge \frac{b^2+c^2+2bc}{4}$$

$$2(c^2a + c^3 - bc^2 + b^2a + b^3 - b^2c) \ge a(b^2+c^2+2bc)$$

$$2c^2a + 2c^3 - 2bc^2 + 2b^2a + 2b^3 - 2b^2c - ab^2 - ac^2 - 2abc \ge 0$$

$$ab^2+ac^2+2c^3+2b^3-2bc^2-2b^2c-2abc\ge 0$$

$$2c^2(c-b)-2b^2(c-b)+ac(c-b)-ab(c-b)\ge 0$$

$$(c-b)[2c^2-2b^2+a(c-b)]\ge 0$$

$(c-b)^2(2c+2b+a)\ge 0$ ***which is true.***

Lemma 2.

In $\Delta ABC: m_a\ge\sqrt{s(s-a)}$

Proof:

$$m_a\ge\sqrt{s(s-a)}\Leftrightarrow m_a^2\ge s(s-a)$$

$$\frac{2(b^2+c^2)-a^2}{4}\ge\frac{(a+b+c)(b+c-a)}{4}$$

$$2b^2+2c^2-a^2\ge(b+c)^2-a^2$$

$$b^2+c^2-2bc\ge 0\Leftrightarrow(b-c)^2\ge 0$$

Back to the problem:

$$n_an_bn_c\overset{Lemma\ 1}{\ge}m_am_bm_c\overset{Lemma\ 2}{\ge}\sqrt{s(s-a)}\cdot\sqrt{s(s-b)}\cdot\sqrt{s(s-c)}=$$

$$=s\sqrt{s(s-b)(s-c)(s-a)}=sS=\frac{sS^3}{s^2}=\frac{s}{s(s-a)(s-b)(s-c)}\cdot S^3=$$

$$=\frac{S^3}{(s-a)(s-b)(s-c)}=\frac{S}{s-a}\cdot\frac{S}{s-b}\cdot\frac{S}{s-c}=r_ar_br_c$$

SP.206. Prove that in any ABC triangle the following inequality holds:

$$-2R^2+17r^2\le\sum m_a^2\tan^2\frac{A}{2}\le\frac{6}{R}(R^3-5r^3)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Marian Ursărescu-Romania

We have: $m_a\ge\sqrt{s(s-a)}\Rightarrow$

$\sum m_a^2\tan^2\frac{A}{2}\ge\sum s(s-a)\cdot\frac{(s-b)(s-c)}{s(s-a)}=\sum(s-b)(s-c)=4Rr+r^2\Rightarrow$ ***we must show:***

$4Rr+r^2\ge-2R^2+17r^2\Leftrightarrow 2R^2+4Rr\ge 16r^2\Leftrightarrow R^2+2Rr\ge 8r^2$***, which is true,***

because: $R^2\ge 4r^2$ ***and*** $2Rr\ge 4r^2\Rightarrow R^2+2Rr\ge 8r^2$***. We have:*** $m_a\le 2R\cos^2\frac{A}{2}\Rightarrow$

$$\sum m_a^2\tan^2\frac{A}{2}\le\sum 4R^2\cos^4\frac{A}{2}\cdot\frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}=\sum R^2\cdot 4\sin^2\frac{A}{2}\cdot\cos^2\frac{A}{2}=$$

$$=R^2\cdot\sum\sin^2 A=R^2\sum\frac{a^2}{4R^2}=\frac{1}{4}(a^2+b^2+c^2)\quad \textbf{(1)}$$

But $a^2+b^2+c^2\le 9R^2$ ***(2)***

From (1)+(2)$\Rightarrow\sum m_a^2\tan^2\frac{A}{2}\le\frac{9}{4}R^2\Rightarrow$ ***we must show:***

$$\frac{9}{4}R^2\le\frac{6}{R}(R^3-5r^3)\Leftrightarrow 3R^3\le 8R^3-40r^3\Leftrightarrow 40r^3\le 5R^3\Leftrightarrow 8r^3\le R^3\Leftrightarrow 2r\le R \text{ true}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Firstly, $\sum\sec^2\frac{A}{2}=\sum\frac{bc(s-b)(s-c)}{s(s-a)(s-b)(s-c)}=\frac{\sum bc(s^2-s(b+c)+bc)}{r^2s^2}=$

$$=\frac{s^2\sum ab-s\sum ab(2s-c)+(\sum ab)^2-2abc(2s)}{r^2s^2}$$

$$=\frac{-s^2(\sum ab)+(\sum ab)^2-4Rrs^2}{r^2s^2}=\frac{(s^2+4Rr+r^2)(4Rr+r^2)-4Rrs^2}{r^2s^2}$$

$=\frac{s^2r^2+r^2(4R+r)^2}{r^2s^2}\overset{(i)}{=}1+\frac{(4R+r)^2}{s^2}$***. Now,*** $\sum m_a^2\sec^2\frac{A}{2}=\frac{1}{4}\sum(2b^2+2c^2+2a^2-3a^2)\sec^2\frac{A}{2}$

$$=\left(\frac{2\sum a^2}{4}\right)\left(\sum\sec^2\frac{A}{2}\right)-\frac{3}{4}\sum a^2\frac{bc}{s(s-a)}$$

$$\overset{by\,(i)}{=}(s^2-4Rr-r^2)\left\{1+\frac{(4R+r)^2}{s^2}\right\}-\frac{3}{4s}\cdot 4Rrs\sum\frac{a}{s-a}$$

$$=s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4R+r)^2}{s^2}-3Rr\sum\left(\frac{a-s}{s-a}+\frac{s(s-b)(s-c)}{sr^2}\right)$$

$$=s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4R+r)^2}{s^2}-3Rr\left(-3+\frac{\sum(s^2-s(b+c)+bc)}{r^2}\right)$$

$$=s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4R+r)^2}{s^2}-3Rr\left(-3+\frac{3s^2-4s^2+s^2+4Rr+r^2}{r^2}\right)$$

$$\overset{(ii)}{=}s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4R+r)^2}{s^2}-3Rr\left(\frac{4R-2r}{r}\right)$$

Now, $\sum m_a^2\tan^2\frac{A}{2}=\sum m_a^2\sec^2\frac{A}{2}-\sum m_a^2=s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4Rr+r)^2}{s^2}-$

$$-3R(4R-2r)-\frac{3}{4}\cdot 2(s^2-4Rr-r^2)\le\frac{6}{R}(R^3-5r^3)\Leftrightarrow$$

$$\Leftrightarrow \frac{s^2-4Rr-r^2}{2}+3R(4R-2r)+\frac{6}{R}(R^3-5r^2)\ge\frac{(s^2-4Rr-r^2)(4Rr+r)^2}{s^2}\Leftrightarrow$$

$$\Leftrightarrow Rs^4-Rs^2(4Rr+r^2)+s^2(12R^3-60r^3+6R^2(4R-2r)-2R(4R+r)^2)+$$

$$+2Rr(4R+r)^3\overset{(1)}{\ge}0.$$ ***Now, LHS of (1)*** $\overset{Gerretsen}{\ge} Rs^2(12Rr-6r^2)+$

$$+s^2(12R^3-60r^3+6R^2(4R-2r)-2R(4R+r)^2)+2Rr(4R+r)^3=$$

$$=s^2(4R^3-16R^2r-8Rr^2-60r^3)+2Rr(4R+r)^3\overset{?}{\ge}0$$

$$\Leftrightarrow s^2(R-2r)(4R^2-8Rr)+2Rr(4R+r)^3\underset{(2)}{\overset{?}{\ge}}s^2(24Rr^2+60r^3)$$

Now, LHS of (2) $\underset{(a)}{\overset{Gerretsen}{\ge}}(16Rr-5r^2)(R-2r)(4R^2-8Rr)+2Rr(4R+r)^3$ ***and***

RHS of (2) $\underset{(b)}{\overset{Gerretsen}{\le}}(4R^2+4Rr+3r^2)(24Rr^2+60r^3)$

(a), (b)⇒ in order to prove (2), it suffices to prove:

$$(16R-5r)(R-2r)(4R^2-8Rr)+2R(4R+r)^3\ge(4R^2+4Rr+3r^2)(24Rr+60r^2)$$

$$\Leftrightarrow 96t^4-138t^3+12t^2-195t-90\ge0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(96t^3+54t^2+120t+45)\ge0\to \textit{true} \because t\overset{Euler}{\ge}2$$

$\Rightarrow$ ***(2)*** $\Rightarrow$ ***(1)*** $\Rightarrow \sum m_a^2\tan^2\frac{A}{2}\le\frac{6}{R}(R^3-5r^3)$ ***is true.***

Again, $\sum m_a^2\tan^2\frac{A}{2}\overset{m_a^2\ge s(s-a),etc}{\ge}\sum s(s-a)\frac{(s-b)(s-c)}{s(s-a)}$

$$=\sum\{s^2-s(b+c)+bc\}=3s^2-4s^2+s^2+4Rr+r^2\ge-2R^2+17r^2$$

$$\Leftrightarrow 2R^2+4Rr-16r^2\ge0\Leftrightarrow(R-2r)(R+4r)\ge0\to \textit{true}\because t\overset{Euler}{\ge}2r\Rightarrow$$

$\Rightarrow \sum m_a^2\tan^2\frac{A}{2}\ge-2R^2+17r^2$ ***(proved)***

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$m_a\ge\frac{b+c}{2}\cos\frac{A}{2}\Rightarrow m_a^2\ge\frac{(b+c)^2}{4}\cdot\cos^2\frac{A}{2}\ge bc\cos^2\frac{A}{2}$$

$\Rightarrow m_a^2\tan^2\frac{A}{2}\ge bc\cdot\frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}\cdot\cos^2\frac{A}{2}=bc\sin^2\frac{A}{2}$ ***(etc)***

$$\Rightarrow \sum\left(m_a^2\tan^2\frac{A}{2}\right) \overset{AM-GM}{\geq} 3\sqrt[3]{(abc)^2\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2}$$

$$= 3\sqrt[3]{(4Rrs)^2\left(\frac{r}{4R}\right)^2} = 3\sqrt[3]{s^2r^4} \geq 3\sqrt[3]{\left(3\sqrt{3}r\right)^2 r^4} = 9r^2$$

We must show that: $9r^2 \geq -2R^2 + 17r^2 \Leftrightarrow 2R^2 \geq 8r^2 \Leftrightarrow R \geq 2r$ ***(true).***

Suppose: $A \leq B \leq C \Rightarrow a \leq b \leq c \Rightarrow \begin{cases} m_a \geq m_b \geq m_c \\ \tan^2\frac{A}{2} \leq \tan^2\frac{B}{2} \leq \tan^2\frac{C}{2} \end{cases}$

$$\Rightarrow \sum m_a^2\tan^2\frac{A}{2} \leq \frac{1}{3}\left(m_a^2 + m_b^2 + m_c^2\right)\left(\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}\right)$$

$$= \frac{1}{3}\cdot\frac{3}{4}\cdot\left(a^2 + b^2 + c^2\right)\left(\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}\right) \leq \frac{9R^2}{4}\cdot\frac{(4R + r)^2 - 2s^2}{s^2} \overset{(2)}{\leq} \frac{6}{R}\left(R^3 - 5r^3\right)$$

(2)$\Leftrightarrow \frac{3[(4+t)^2-2s^2]}{4s^2} \leq 2[1 - 5t^3]\ \left(t = \frac{r}{R}, \frac{2}{5} \overset{(*)}{\leq} t \leq \frac{1}{2}\right) \Leftrightarrow 3(4 + t)^2 \leq \frac{s^2}{R^2}\left(14 - 40t^3\right)$

$\because s^2 \geq 16Rr - 5r^2 \Rightarrow \frac{s^2}{R^2} \geq 16\cdot\frac{r}{R} - 5\left(\frac{r}{R}\right)^2 = 16t - 5t^2$***. So, we must show that:***

$$3(4 + t)^2 \leq \left(16t - 5t^2\right)\left(14 - 40t^3\right) \Leftrightarrow \left(t - \frac{1}{2}\right)\left(\frac{12}{25} + \frac{26}{25}x + \frac{27}{20}x^2 + \frac{27}{10}x^3 - x^4\right) \leq 0$$

It is true because: $\frac{2}{5} \leq t \leq \frac{1}{2} \Rightarrow \begin{cases} t - \frac{1}{2} \leq 0 \\ \frac{12}{25} + \frac{26}{25}x + \frac{27}{20}x^2 + \frac{27}{10}x^3 - x^4 \geq \frac{803}{625} - \frac{1}{16} > 1 > 0 \end{cases}$

$\Rightarrow$ ***(2) true. Proved.***

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$s^2 \leq 4R^2 + 4Rr + 3r^2, ab + bc + ca = s^2 + r^2 + 4Rr$$

and $4m_a^2 = 2b^2 + 2c^2 - a^2 = 2bc + b^2 + c^2$ ***where*** $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$$\therefore \frac{b^2 + c^2}{2}\cos^2\frac{A}{2} \geq m_a^2 \geq bc\cos^2\frac{A}{2} \Rightarrow \frac{b^2 + c^2}{2}\sin^2\frac{A}{2} \geq m_a^2 \geq bc\sin^2\frac{A}{2}$$

$$\Rightarrow \sum_{cyc}\frac{b^2 + c^2}{2}\sin^2\frac{A}{2} \geq \sum_{cyc} m_a^2\tan^2\frac{A}{2} \geq \sum_{cyc} bc\sin^2\frac{A}{2}$$

$$\Rightarrow \frac{1}{3}\left(\sum_{cyc}\frac{b^2 + c^2}{2}\right)\left(\sum_{cyc}\sin^2\frac{A}{2}\right) \overset{\substack{CHEBYSHEV'S \\ INEQUALITY}}{\geq} \sum_{cyc}\frac{b^2 + c^2}{2}\sin^2\frac{A}{2} \geq \sum_{cyc} m_a^2\tan^2\frac{A}{2} \geq \sum_{cyc} bc\sin^2\frac{A}{2}$$

[let $b^2+c^2\ge c^2+a^2\ge a^2+b^2$ *then* $\sin^2\frac{A}{2}\le\sin^2\frac{B}{2}\le\sin^2\frac{C}{2}$*]*

$$\Rightarrow\frac{2}{3}(s^2-r^2-4Rr)\left(\sum_{cyc}\frac{(s-b)(s-c)}{bc}\right)\ge\sum_{cyc}m_a^2\tan^2\frac{A}{2}\ge\sum_{cyc}(s-a)(s-b)$$

$$\Rightarrow\frac{2}{3}(s^2-r^2-4Rr)\cdot\frac{1}{4Rrs}\cdot\left(\sum_{cyc}a(s-b)(s-c)\right)\ge\sum_{cyc}m_a^2\tan^2\frac{A}{2}\ge r^2+4Rr$$

$$\Rightarrow\frac{2}{3}(4R^2+2r^2)\frac{1}{4Rrs}(4Rrs-2sr^2)\ge\sum_{cyc}m_a^2\tan^2\frac{A}{2}\ge r^2+4Rr$$

$\Leftrightarrow\frac{(4R^2+2r^2)}{3R}(2R-r)\ge\sum_{cyc}m_a^2\tan^2\frac{A}{2}\ge r^2+4Rr$ *we need to prove,*

$\frac{6}{R}(R^3-5r^3)\ge\frac{2}{3R}(2R^2+r^2)(2R-r)$ *and* $r^2+4Rr\ge 17r^2-2R^2$

$\Rightarrow 5R^3+2R^2r-2Rr^2-44r^3\ge 0$ *and* $R^2+2Rr-8r^2\ge 0$

$\Rightarrow 5t^3+2t^2-2t-44\ge 0$, *where* $t=\frac{R}{r}\ge 2$ *and* $(R-2r)(R+4r)\ge 0$

$\Rightarrow(t-2)(5t^2+12t+22)\ge 0$ *and* $(R-2r)(R+4r)\ge 0$, *which are both true*

$\therefore 17r^2-2R^2\le\sum_{cyc}m_a^2\tan^2\frac{A}{2}\le\frac{6}{R}(R^3-5r^3)$ *(proved)*

SP.207. Prove that in any ABC triangle the following inequality holds:

$$9(8R^2-23r^2)\le\sum m_a^2\cot^2\frac{A}{2}\le\frac{81R}{32r^2}(13R^3-88r^3)$$

Proposed by Marin Chirciu – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sum m_a^2\csc^2\frac{A}{2}\le\frac{4}{r^2}(4R^4-37r^4)\Leftrightarrow\sum m_a^2\left(\csc^2\frac{A}{2}-1\right)\le\frac{4}{r^2}(4R^4-37r^4)-\sum m_a^2$$

$$\sum m_a^2\cdot\cot^2\frac{A}{2}\le\frac{4}{r^2}(4R^4-37r^4)-\frac{3}{4}\sum a^2$$

$$\Leftrightarrow\sum m_a^2\cot^2\frac{A}{2}\le\frac{4}{r^2}(4R^4-37r^4)-\frac{3}{4}(2s^2-8Rr-2r^2)=$$

$$=\frac{16R^4}{r^2}-\frac{3}{2}s^2+6Rr-\frac{293}{2}r^2\overset{(1)}{\le}\frac{81R}{32r^2}(13R^3-88r^3)$$

$$\text{(1)}\Leftrightarrow \frac{915}{4}Rr-\frac{293}{2}r^2\le\frac{541}{32}\cdot\frac{R^4}{r^2}+\frac{3}{2}s^2\Leftrightarrow\frac{915}{4}\cdot\frac{R}{r}-\frac{293}{2}\le\frac{541}{32}\left(\frac{R}{r}\right)^4+\frac{3}{2}\cdot\frac{s^2}{r^2}$$

$$\because s^2\ge 16Rr-5r^2\Rightarrow\frac{s^2}{r^2}\ge 16\cdot\frac{R}{r}-5.\ \textit{Let } t=\frac{R}{r}\ (t\ge 2)$$

We show that: $\frac{915}{4}t-\frac{293}{2}\le\frac{541}{32}t^4+\frac{3}{2}(16t-5)$

$$\Leftrightarrow\frac{541}{32}t^4-\frac{819}{4}t+154\ge 0\Leftrightarrow\frac{1}{32}(541t^4-6552t+4928)\ge 0$$

(It is true because: Let $f(t)=541t^4-655t+4928$

$$\Rightarrow f'(t)=4\cdot 541t^3-655=0\Leftrightarrow t=\sqrt[3]{\frac{655}{4\cdot 541}}$$

$$\Rightarrow f'(t)>0\ \forall t>\sqrt[3]{\frac{655}{4\cdot541}}\Rightarrow f(t)\ge f(2)=12274>0.$$ ***Hence, (1) true.***

$$\Rightarrow\sum m_a^2\cot^2\frac{A}{2}=\sum m_a^2\csc^2\frac{A}{2}-\sum m_a^2=\Omega$$

$$\sum m_a^2\frac{bc(s-a)}{r^2s}=\sum\frac{bcm_a^2}{r^2}-\frac{4Rrs}{r^2s}\cdot\frac{3}{4}\cdot 2(s^2-4Rr-r^2)$$

$$=\frac{\sum bc(2b^2+2c^2+2a^2-3a^2)}{4r^2}-\frac{6Rr(s^2-4Rr-r^2)}{r^2}$$

$$=\frac{2(\sum a^2)(\sum ab)-3\cdot 4Rrs(2s)}{4r^2}-\frac{6Rr(s^2-4Rr-r^2)}{r^2}$$

$$=\frac{4(s^2+4Rr+r^2)(s^2-4Rr-r^2)-24Rrs^2}{4r^2}-\frac{6Rr(s^2-4Rr-r^2)}{r^2}$$

$$=\frac{s^4-12Rrs^2+r^2(4R+r)(2R-r)}{r^2}$$

$$\sum m_a^2=\frac{3}{4}\sum a^2=\frac{3}{4}\cdot 2(s^2-4Rr-r^2)=\frac{3}{2}(s^2-4Rr-r^2)$$

$$\Rightarrow\Omega=\frac{s^4-12Rrs^2+r^2(4Rr+r)(2R-r)}{r^2}-\frac{3}{2}(s^2-4Rr-r^2)$$

$$=\frac{2s^4-24Rrs^2+2r^2(4R+r)(2R-r)-3r^2s^2+12Rr^3+3r^4}{2r^2}$$

We must show that:

$$2s^4-24Rrs^2+2r^2(4R+r)(2R-r)-3r^2s^2+12Rr^3+3r^4\ge 2r^2(72R^2-207r^2)$$

$$\Leftrightarrow s^2(2s^2-24Rr-3r^2)+2r^2(8R^2-2Rr-r^2)+12Rr^3+$$

$$+3r^4 \geq 144R^2r^2 - 414r^4 \Leftrightarrow s^2(2s^2 - 24Rr - 3r^2) + 8Rr^3 \geq 128R^2r^2 - 415r^4$$

$$\therefore s^2(2s^2 - 24Rr - 3r^2) \geq (16Rr - 5r^2)(8Rr - 13r^2) = r^2(16R - 5r)(8R - 13r)$$

We need to prove: $128R^2 - 415r^2 \geq 128R^2 - 248Rr + 65r^2 \Leftrightarrow 248Rr \geq 480r^2$

$\Leftrightarrow R > \frac{60}{31}r$. ***(True because:*** $R \geq 2r > \frac{60}{31}r$***). Proved.***

SP.208. Prove that in any ABC triangle the following inequality holds:

$$36r^2 \leq \sum m_a^2 \sec^2\frac{A}{2} \leq 9R^2$$

Proposed by Marin Chirciu – Romania

Solution 1 by Tran Hong - Dong Thap – Vietnam

Let $\Omega = \sum m_a^2 \sec^2\frac{A}{2} = \sum \frac{m_a^2}{\cos^2\frac{A}{2}}$

$$m_a \geq \frac{b+c}{2}\cdot\cos\frac{A}{2} \textit{ (etc)} \Rightarrow \Omega \geq \sum\frac{(b+c)^2}{4} \overset{AM-GM}{\geq} \sum(bc) = \sum ab = s^2 + 4Rr + r^2 \overset{(1)}{\geq} 36r^2$$

$$\textit{(1)} \Leftrightarrow s^2 \geq 35r^2 - 4Rr$$

$\because s^2 \geq 16Rr - 5r^2 \geq 35r^2 - 4Rr \Leftrightarrow 20Rr \geq 40r^2 \Leftrightarrow R \geq 2r$ ***(Euler)*** $\Rightarrow$ ***(1) true.***

$$m_a \leq 2R\cos^2\frac{A}{2} \textit{ (etc)} \Rightarrow \Omega \leq \sum\left\{(4R^2)\cdot\cos^4\frac{A}{2}\cdot\frac{1}{\cos^2\frac{A}{2}}\right\} =$$

$$= 4R^2\sum\cos^2\frac{A}{2} = 4R^2\cdot\frac{4R+r}{2R} = 8R^2 + 2Rr \overset{(2)}{\leq} 9R^2$$

(2) $\Leftrightarrow 2Rr \leq R^2 \Leftrightarrow 2r \leq R$ ***(Euler) Proved.***

Solution 2 by Soumava Chakraborty – Kolkata – India

$$36r^2 \overset{(1)}{\leq} \sum m_a^2 \sec^2\frac{A}{2} \overset{(2)}{\leq} 9R^2$$

Firstly, $\sum\sec^2\frac{A}{2} = \sum\frac{bc(s-b)(s-c)}{s(s-a)(s-b)(s-c)} = \frac{\sum bc(s^2 - s(b+c) + bc)}{r^2s^2} =$

$$= \frac{s^2\sum ab - s\sum ab\,(2s - c) + (\sum ab)^2 - 2abc(2s)}{r^2s^2} =$$

$$= \frac{-s^2(\sum ab) + (\sum ab)^2 - 4Rrs^2}{r^2s^2} = \frac{(s^2 + 4Rr + r^2)(4Rr + r^2) - 4Rrs^2}{r^2s^2} =$$

$$=\frac{s^2r^2+r^2(4R+r)^2}{r^2s^2}\overset{(i)}{=}1+\frac{(4R+r)^2}{s^2}$$

Now, $\sum m_a^2\sec^2\frac{A}{2}=\frac{1}{4}\sum(2b^2+2c^2+2a^2-3a^2)\sec^2\frac{A}{2}=$

$$=\left(\frac{2\sum a^2}{4}\right)\left(\sum\sec^2\frac{A}{2}\right)-\frac{3}{4}\sum a^2\frac{bc}{s(s-a)}=$$

$$\overset{by\ (i)}{=}(s^2-4Rr-r^2)\left\{1+\frac{(4R+r)^2}{s^2}\right\}-\frac{3}{4s}\cdot 4Rrs\sum\frac{a}{s-a}=$$

$$=s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4R+r)^2}{s^2}-3Rr\sum\left(\frac{a-s}{s-a}+\frac{s(s-b)(s-c)}{sr^2}\right)$$

$$=s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4Rr+r)^2}{s^2}-3Rr\left(-3+\frac{\sum(s^2-s(bc)+bc)}{r^2}\right)$$

$$=s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4Rr+r)^2}{s^2}-3Rr\left(-3+\frac{3s^2-4s^2+s^2+4Rr+r^2}{r^2}\right)$$

$$=s^2-4Rr-r^2+\frac{(s^2-4Rr-r^2)(4Rr+r)^2}{s^2}-3Rr\left(\frac{4R-2r}{r}\right)\le 9R^2\Leftrightarrow$$

$$\Leftrightarrow s^2+\frac{(s^2-4Rr-r^2)(4Rr+r)^2}{s^2}\le 21R^2-2Rr+r^2\Leftrightarrow$$

$$\Leftrightarrow s^4+s^2(4Rr+r)^2-r(4R+r)^3\overset{(2a)}{\le}s^2(21R^2-2Rr+r^2)$$

Now, LHS of (2a) $\le s^2\left(4R^2+4Rr+3r^2+(4R+r)^2\right)-r(4R+r)^3\overset{?}{\le}s^2\left(21R^2-2Rr+r^2\right)$

$$\Leftrightarrow s^2(R^2-14Rr-3r^2)+r(4R+r)^3\overset{?}{\ge}0\Leftrightarrow$$

$$\Leftrightarrow s^2(R-2r)^2+r(4R+r)^3\underset{(2b)}{\overset{?}{\ge}}s^2(10Rr+7r^2)$$

Now, LHS of (2b) $\underset{(m)}{\overset{Gerretsen}{\ge}}(16Rr-5r^2)(R-2r)^2+r(4R+r)^3$ ***& RHS of (2b)***

$$\overset{(n)}{\le}(4R^2+4Rr+3r^2)(10Rr+7r^2)$$

(m), (n)⇒ in order to prove (2b), it suffices to prove:

$$(16R-5r)(R-2r)^2+(4R+r)^3\ge(4R^2+4Rr+3r^2)(10Rr+7r^2)\Leftrightarrow$$

$\Leftrightarrow 40t^3-89t^2+38t-40\ge 0\Leftrightarrow(t-2)\{40t(t-2)+71t+20\}\ge 0\rightarrow$ ***true*** $\because t\overset{Euler}{\ge}2$

⇒ (2b) ⇒(2a) ⇒(2) is true.

Again, $\sum m_a^2 \sec^2\frac{A}{2} \ge \frac{1}{3}\left(\sum m_a \sec\frac{A}{2}\right)^2 \overset{Ioscu}{\ge} \frac{1}{3}\left(\sum \frac{b+c}{2}\right)^2 =$

$$= \frac{4s^2}{3} \overset{s^2\ge 27r^2}{\ge} \frac{108r^2}{3} = 36r^2 \Rightarrow \textit{(1) is true. Proved.}$$

Solution 3 by Soumitra Mandal-Chandar Nagore – India

$$ab + bc + ca = 4\sqrt{3}\Delta,\ s \ge 3\sqrt{3}r,\ 4m_a^2 = 2b^2 + 2c^2 - a^2 = 2bc\cos A + b^2 + c^2$$

$$(b^2 + c^2)(1 + \cos A) \overset{AM-GM}{\ge} 4m_a^2 \overset{AM-GM}{\ge} 2bc(1 + \cos A)$$

$$\Rightarrow \frac{b^2 + c^2}{2}\cos^2\frac{A}{2} \ge m_a^2 \ge bc\cos^2\frac{A}{2} \Rightarrow \frac{b^2 + c^2}{2} \ge m_a^2 \sec^2\frac{A}{2} \ge bc$$

$$\Rightarrow \sum_{cyc}\frac{b^2 + c^2}{2} \ge \sum_{cyc} m_a^2 \sec^2\frac{A}{2} \ge ab + bc + ca \Rightarrow \sum_{cyc} a^2 \ge \sum_{cyc} m_a^2 \ge \sum_{cyc} ab$$

$$\Rightarrow 9R^2 \ge \sum_{cyc} m_a^2 \sec^2\frac{A}{2} \ge 4\sqrt{3}\Delta = 4\sqrt{3}sr \ge 4\sqrt{3}r \cdot 3\sqrt{3}r = 36r^2 \ \textit{Proved}$$

SP.209. Prove that in any ABC triangle the following inequality holds:

$$27R^2 \le \sum m_a^2 \csc^2\frac{A}{2} \le \frac{4}{r^2}(4R^4 - 37r^4)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

In any $\Delta ABC,\ 27R^2 \overset{(1)}{\le} \sum m_a^2 \csc^2\frac{A}{2} \overset{(2)}{\le} \frac{4}{r^2}(4R^4 - 37r^4)$

$$\sum m_a^2 \csc^2\frac{A}{2} = \sum m_a^2 \frac{bc(s-a)}{r^2 S} = \frac{\sum bcm_a^2}{r^2} - \frac{4Rrs}{r^2 S} \cdot \frac{3}{4} \cdot 2(s^2 - 4Rr - r^2)$$

$$= \frac{\sum bc\,(2b^2 + 2c^2 + 2a^2 - 3a^2)}{4r^2} - \frac{6Rr(s^2 - 4Rr - r^2)}{r^2}$$

$$= \frac{2(\sum a^2)(\sum ab) - 3 \cdot 4Rrs(2S)}{4r^2} - \frac{6Rr(s^2 - 4Rr - r^2)}{r^2}$$

$$= \frac{4(s^2 + 4Rr + r^2)(s^2 - 4Rr - r^2) - 24Rrs^2}{4r^2} - \frac{6Rr(s^2 - 4Rr - r^2)}{r^2}$$

$$= \frac{S^4 - (4Rr + r^2)^2 - 6Rrs^2 - 6Rr(s^2 - 4Rr - r^2)}{r^2}$$

$$=\frac{S^4-12Rrs^2+(4Rr+r^2)(6Rr-4Rr-r^2)}{r^2}\overset{(i)}{=}\frac{S^4-12Rrs^2+r^2(4Rr+r)(2R-r)}{r^2}$$

(i)⇒(2)⇔ $S^4-12Rrs^2+r^2(4R+r)(2R-r)\overset{(2a)}{\leq}16R^4-148r^4$

Now, LHS of (2a) $\overset{Gerretsen}{\leq}s^2(4R^2-8Rr+3r^2)+r^2(4R+r)(2R-r)$

$$\overset{Gerretsen}{\leq}(4R^2+4Rr+3r^2)(4R^2-8Rr+3r^2)+r^2(4R+r)(2R-r)$$

$$\left(\because 4R^2-8Rr+3r^2=4R(R-2r)+3r^2\overset{Euler}{\geq}3r^2>0\right)$$

$$\overset{?}{\leq}16R^4-148r^4\Leftrightarrow 8t^3+7t-78\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow 8(t-2)(t^2+2t+4)+7(t-2)\overset{?}{\geq}0$$

Which is true $\because t\overset{Euler}{\geq}2\Rightarrow$ ***(2a)⇒(2) is true. Again, (i)⇒(1)⇔***

$$S^4-12Rrs^2+r^2(4R+r)(2R-r)\overset{(1a)}{\geq}27R^2r^2$$

Now, LHS of (1a) $\overset{Gerretsen}{\geq}S^2(4Rr-5r^2)+r^2(4R+r)(2R-r)$

$$\overset{Gerretsen}{\geq}r^2(16R-5r)(4R-5r)+r^2(4R+r)(2R-r)$$

$$\left(\because 4Rr-5r^2=4r(R-2r)+3r^2\overset{Euler}{\geq}3r^2>0\right)$$

$\overset{?}{\geq}27R^2r^2\Leftrightarrow 15t^2-34t+8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(15t-4)\overset{?}{\geq}0\to$ ***true*** $\because t\overset{Euler}{\geq}2$

⇒(1a)⇒(1) is true (Proved)

Solution 2 by Tran Hong-Dong Thap-Vietnam

Let $\Omega=\sum m_a^2\csc^2\frac{A}{2}=\sum\frac{m_a^2}{\sin^2\frac{A}{2}}$

$$m_a\leq 2R\cos^2\frac{A}{2}\Rightarrow m_a^2\leq 4R^2\left(\cos^2\frac{A}{2}\right)^2=$$

$$=4R^2\left(1-\sin^2\frac{A}{2}\right)^2=4R^2\left(1-2\sin^2\frac{A}{2}+\sin^4\frac{A}{2}\right)$$

$$\Rightarrow\Omega=4R^2\sum\frac{1-2\sin^2\frac{A}{2}+\sin^4\frac{A}{2}}{\sin^2\frac{A}{2}}=4R^2\left\{\sum\frac{1}{\sin^2\frac{A}{2}}-6+\sum\sin^2\frac{A}{2}\right\}$$

$$= 4R^2\left\{\frac{s^2+r^2-8Rr}{r^2}-6+\frac{2R-r}{2R}\right\} \le 4R^2\left\{\frac{4R^2+4Rr+3r^2+r^2-8Rr}{r^2}-5-\frac{r}{2R}\right\}$$

$$= 4R^2\left\{4\left(\frac{R}{r}\right)^2-4\left(\frac{R}{r}\right)-1-\frac{1}{2}\left(\frac{r}{R}\right)\right\}$$

We must show that: $4\left(\frac{R}{r}\right)^2-4\left(\frac{R}{r}\right)-\frac{1}{2}\left(\frac{r}{R}\right)-1 \le 4\left(\frac{R}{r}\right)^2-37\cdot\left(\frac{r}{R}\right)^2$

$$\Leftrightarrow 37t^2-\frac{t}{2}-\frac{4}{t}-1 \le 0\left(\because 0 < t \le \frac{1}{2}\right)$$

$\Leftrightarrow 74t^3-t^2-2t-8 \le 0 \Leftrightarrow \left(t-\frac{1}{2}\right)\left(t^2+\frac{18t}{37}+\frac{8}{37}\right) \le 0$ ***(true because*** $0 < t \le \frac{1}{2}$***)***

$$\Rightarrow \Omega \le \frac{4}{r^2}(4R^2-37r^4)$$

$m_a \ge \sqrt{s(s-a)} \Rightarrow m_a^2 \ge s(s-a)$ ***(etc)***$\Rightarrow \Omega \ge \sum\frac{s(s-a)}{\sin^2\frac{A}{2}} = s\sum\frac{s-a}{\sin^2\frac{A}{2}}$

$$s\left\{s\sum\frac{1}{\sin^2\frac{A}{2}}-\sum\frac{a}{\sin^2\frac{A}{2}}\right\} = s\left\{s\sum\frac{1}{\sin^2\frac{A}{2}}-\sum\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{\sin^2\frac{A}{2}}\right\}$$

$$= s\left\{s\cdot\frac{s^2+r^2-8Rr}{r^2}-4R\cdot\frac{s}{r}\right\} = s^2\left(\frac{s^2+r^2-8Rr}{r^2}-\frac{4R}{r}\right)$$

$$= s^2\left(\frac{s^2+r^2-12Rr}{r^2}\right) \overset{(2)}{\ge} 27R^2$$

(2)$\Leftrightarrow s^2(s^2+r^2-12Rr) \ge 27R^2r^2$

$$\because s^2 \ge 16Rr-5r^2 \Rightarrow s^2+r^2-12Rr \ge 4Rr-4r^2$$

$$\Rightarrow s^2(s^2+r^2-12Rr) \ge (16Rr-5r^2)(4Rr-4r^2) = 64R^2r^2-84Rr^3+20r^4$$

We must show: $64R^2r^2-84Rr^3+20r^4 \ge 27R^2r^2 \Leftrightarrow 37R^2r^2-84Rr^3+20r^4 \ge 0$

$\Leftrightarrow 37R^2-84Rr+20r^2 \ge 0 \Leftrightarrow (37R-10r)(R-2r) \ge 0$ ***(∵ true*** $R \ge 2r$***) Proved.***

SP.210. Let ABC be an acute-angled triangle. If $a+b+c=\pi$ and $A\cos a+B\cos b+C\cos c=\frac{\pi}{2}$; ($A, B, C$ – the measures in radians), then ΔABC is equilateral.

Proposed by Marian Ursărescu – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$a+b+c=\pi;\ (a,b,c>0)$$

$$\because\ a+b>c\Rightarrow a+b+c>2c\Rightarrow 0<c<\frac{\pi}{2}.\ \textit{Similarly: } 0<a,b<\frac{\pi}{2}.$$

$$\textit{Let } f(x)=\cos x\left(0<x<\frac{\pi}{2}\right)\Rightarrow f'(x)=-\sin x\Rightarrow f''(x)=-\cos x<0\ \left(0<x<\frac{\pi}{2}\right)$$

$$\textit{Suppose: } A\le B\le C\Rightarrow a\le b\le c\Rightarrow \cos a\ge\cos b\ge\cos c\ \left(\because f(x)=\cos x\searrow\left(0;\frac{\pi}{2}\right)\right)$$

$$\Rightarrow LHS=A\cos a+B\cos b+C\cos c\le\frac{1}{3}(A+B+C)(\cos a+\cos b+\cos c)$$

$$=\frac{\pi}{3}\cdot(\cos a+\cos b+\cos c)\overset{Jensen}{\le}\frac{\pi}{3}\cdot 3\cos\left(\frac{a+b+c}{3}\right)=\pi\cdot\cos\left(\frac{\pi}{3}\right)=\frac{\pi}{2}$$

$$\textit{Hence, } LHS=\frac{\pi}{2}\Leftrightarrow\begin{cases}A=B=C\\ a=b=c\end{cases}.\ \textit{Proved.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\textit{If } a\ge\frac{\pi}{2},\textit{ then } b+c\le\frac{\pi}{2}\ (\because \textstyle\sum a=\pi)$$

$$\Rightarrow b+c\le a\Rightarrow \textit{violation of triangle inequality}\Rightarrow a<\frac{\pi}{2}.\ \textit{Similar argument}\Rightarrow b,c<\frac{\pi}{2}$$

$$\textit{Let } f(x)=\sin^2\frac{x}{2},\forall x\in\left(0,\frac{\pi}{2}\right).\ \textit{Then, } f''(x)=\frac{\cos x}{2}>0\Rightarrow f(x)\textit{ is strictly convex.}$$

$$\sum A\cos a=\sum A\left(1-2\sin^2\frac{a}{2}\right)=\sum A-2\pi\sum\left(\frac{A}{\pi}\sin^2\frac{a}{2}\right)=\pi-2\pi\sum\left(\frac{A}{\pi}\sin^2\frac{a}{2}\right)$$

$$\underset{(1)}{\overset{Jensen}{\le}}\pi-2\pi\sin^2\left(\frac{\sum\left(\frac{A}{\pi}a\right)}{2}\right)\ \left(\because \sum\frac{A}{\pi}=1\textit{ and }\sin^2\frac{x}{2}\forall x\in\left(0,\frac{\pi}{2}\right)\textit{ is strictly convex}\right)$$

$$\textit{Now, WLOG we may assume } a\ge b\ge c$$

$$\therefore A\ge B\ge C\ \therefore\frac{1}{2}\sum\left(\frac{A}{\pi}a\right)\overset{Chebyshev}{\ge}\frac{1}{2\pi}\cdot\frac{1}{3}\left(\sum A\right)\left(\sum a\right)=\frac{\pi^2}{6\pi}=\frac{\pi}{6}\Rightarrow\frac{1}{2}\sum\left(\frac{A}{\pi}a\right)\overset{(i)}{\ge}\frac{\pi}{6}$$

$$\because A,B,C<\frac{\pi}{2}\ \&\ a,b,c\textit{ also }<\frac{\pi}{2}\ \therefore\frac{1}{2}\sum\left(\frac{A}{2}a\right)<\frac{1}{2\pi}\left(\frac{3\pi^2}{4}\right)=\frac{3\pi}{8}\Rightarrow\frac{1}{2}\left(\sum\frac{A}{\pi}a\right)\overset{(ii)}{<}\frac{3\pi}{8}$$

$$(i),(ii)\Rightarrow\frac{\pi}{6}\le\frac{1}{2}\sum\left(\frac{A}{\pi}a\right)<\frac{3\pi}{8}\Rightarrow\sin\left(\frac{\sum\left(\frac{A}{\pi}a\right)}{2}\right)\overset{(2)}{\ge}\sin\frac{\pi}{6}=\frac{1}{2}$$

$$(1),(2)\Rightarrow\sum A\cos a\le\pi-2\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{2},\textit{ equality when } a=b=c,$$

($\because$ *the equality of Chebyshev's inequality holds at* $A=B=C$ ($\&a=b=c$) *and the equality of Jensen's inequality holds at* $a=b=c$, *as* $f(x)=\sin^2\frac{x}{2}\forall x\in\left(0,\frac{\pi}{2}\right)$ *is* *strictly convex) and ∴ equality relation holds (as* $\sum A\cos a=\frac{\pi}{2}$*), ∴* $a=b=c\Rightarrow\Delta ABC$

is equilateral (proved)

UP.196. Let be $x_n, y_n > 0, x_n \neq y_n$ such that:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = p, p\in\mathbb{N}^*. \text{ Find:}$$

$$\lim_{n\to\infty}\frac{x_n^{y_n}-y_n^{x_n}}{\sqrt[p]{x_n}-\sqrt[p]{y_n}}$$

Proposed by Marian Ursărescu – Romania

Solution by proposer

$$\Omega=\lim_{n\to\infty}\frac{x_n^{y_n}-y_n^{x_n}}{\sqrt[p]{x_n}-\sqrt[p]{y_n}}=\lim_{n\to\infty}\frac{\left(x_n^{y_n}-y_n^{x_n}\right)\left(\sqrt[p]{x_n^{p-1}}+\cdots+\sqrt[p]{y_n^{p-1}}\right)}{x_n-y_n}$$

$$=p\sqrt[p]{p^{p-1}}\lim_{n\to\infty}\frac{x_n^{y_n}-y_n^{x_n}}{x_n-y_n}=p\sqrt[p]{p^{p-1}}\lim_{n\to\infty}\frac{x_n^{y_n}-y_n^{y_n}+y_n^{y_n}-y_n^{x_n}}{x_n-y_n}=$$

$$=p\sqrt[p]{p^{p-1}}\left(\lim_{n\to\infty}\frac{x_n^{y_n}-y_n^{y_n}}{x_n-y_n}+\lim_{n\to\infty}\frac{y_n^{y_n}-y_n^{x_n}}{x_n-y_n}\right)$$

$$=p\sqrt[p]{p^{p-1}}\left(\lim_{n\to\infty}\frac{e^{y_n\ln x_n}-e^{y_n-\ln y_n}}{x_n-y_n}+\lim_{n\to\infty}\frac{y_n\ln y_n-e^{x_n\ln y_n}}{x_n-y_n}\right)$$

$$=p\sqrt[p]{p^{p-1}}\left(\lim_{n\to\infty}\frac{e^{y_n\ln y_n}\left(e^{y_n\ln\frac{x_n}{y_n}-1}\right)}{y_n\ln\frac{x_n}{y_n}}\cdot\frac{y_n\ln\left(\frac{x_n}{y_n}\right)}{x_n-y_n}+\lim_{n\to\infty}\frac{e^{x_n\ln y_n}\left(e^{\ln y_n(y_n-x_n)}-1\right)}{\ln y_n\,(x_n-y_n)}\ln y_n\right)$$

$$=p\sqrt[p]{p^{p-1}}\left(p^p\lim_{n\to\infty}y_n\frac{\ln\left(1+\frac{x_n-y_n}{y_n}\right)}{x_n-y_n}-p^p\ln p\right)=$$

$$=p\sqrt[p]{p^{p-1}}\left(p^p-p^p\ln p\right)=p^{p+1}\sqrt[p]{p^{p-1}}(1-\ln p)$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

UP.197. Let be $f:\mathbb{R}\to(0,\infty)$ continuous such that for $a,b,c>0$ fixed values:

$$a^3f(x)+b^3f(y)+c^3f(z)=f(x)f(y)f(z),\forall x,y,z\in\mathbb{R}$$

Prove that:

$$\int_\alpha^\beta f(x)\,dx\ge\frac{(\beta-\alpha)(a+b+c)\sqrt{a+b+c}}{3};(\forall)0<\alpha\le\beta$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}a^3f(x)=f(x)f(y)f(z)\Rightarrow\sum_{cyc}\frac{a^3}{f(y)f(z)}=1$$

$$\Rightarrow 1\overset{HOLDER'S\ INEQUALITY}{\ge}\frac{(a+b+c)^3}{3\sum_{cyc}f(x)f(y)}\Rightarrow\sum_{cyc}f(x)f(y)\ge(a+b+c)^3$$

$$\Rightarrow\left(\sum_{cyc}f(x)\right)^2\ge3\sum_{cyc}f(x)f(y)\ge(a+b+c)^3\Rightarrow\sum_{cyc}f(x)\ge(a+b+c)^{\frac{3}{2}}$$

$$\sum_{cyc}\int_\alpha^\beta f(x)\,dx\ge(a+b+c)^{\frac{3}{2}}\int_\alpha^\beta dx=(\beta-\alpha)(a+b+c)\sqrt{a+b+c}$$

$$\Rightarrow 3\int_\alpha^\beta f(x)\,dx\ge(\beta-\alpha)(a+b+c)\sqrt{a+b+c}$$

$$\therefore\int_\alpha^\beta f(x)\,dx\ge\frac{(\beta-\alpha)(a+b+c)\sqrt{a+b+c}}{3}$$ ***(Proved)***

Solution 2 by Srinivasa Raghava-AIRMC-India

$\int_\alpha^\beta f(x)\,dx=(\beta-\alpha)\sqrt{a^3+b^3+c^3}$ ***(if** $x=y=z\Rightarrow f(x)=\sqrt{a^3+b^3+c^3},f(x)>0$**)***

We know that:

$$\frac{1}{3}(a^3+b^3+c^3)\ge\left(\frac{1}{3}(a+b+c)\right)^3\Rightarrow\sqrt{a^3+b^3+c^3}\ge\frac{1}{3}(a+b+c)\sqrt{a+b+c}$$

Hence from above, $\int_\alpha^\beta f(x)\,dx\ge\frac{(\beta-\alpha)}{3}(a+b+c)\sqrt{a+b+c}$

Solution 3 by Ravi Prakash-New Delhi-India

Taking $x = y = z$, we get: $(a^3 + b^3 + c^3)f(x) = f(x)^3$

As $f(x) \neq 0$, we get: $f(x)^2 = a^3 + b^3 + c^3 \Rightarrow f(x) = \sqrt{a^3 + b^3 + c^3}$ as $f(x) > 0$

Now,

$$\int_\alpha^\beta f(x)\,dx = \sqrt{a^3 + b^3 + c^3}\int_\alpha^\beta dx = (\beta - \alpha)\sqrt{a^3 + b^3 + c^3} \quad (1)$$

$$\text{But, } \frac{a^3+b^3+c^3}{3} \geq \left(\frac{a+b+c}{3}\right)^3$$

$$\Rightarrow \sqrt{a^3 + b^3 + c^3} \geq \frac{(a+b+c)\sqrt{a+b+c}}{3} \quad (2)$$

From (1), (2), we get: $\int_\alpha^\beta f(x)\,dx \geq \frac{(\beta-\alpha)(a+b+c)\sqrt{a+b+c}}{3}$

UP.198. Let n be a positive integer. Evaluate:

$$\lim_{x\to 0}\frac{1 - (\cos x)^n \cos(nx)}{x^2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

$$\lim_{x\to 0}\frac{1 - (\cos x)^n \cdot \cos nx}{x^2} = \lim_{x\to 0}\frac{1 - (\cos x)^n + (\cos x)^n - (\cos x)^n \cdot \cos nx}{x^2}$$

$$= \lim_{x\to 0}\frac{1 - (\cos x)^n}{x^2} + \lim_{x\to 0}\frac{(\cos x)^n(1 - \cos nx)}{x^2} =$$

$$= \lim_{x\to 0}\frac{(1 - \cos x)(1 + \cos x + \cdots + (\cos x)^{n-1})}{x^2} + \lim_{x\to 0}\frac{(\cos x)^n - 2\sin^2\frac{nx}{2}}{x^2} =$$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}(1 + \cos x + \cdots + (\cos x)^{n-1})}{x^2} + \lim_{x\to 0}\frac{(\cos x)^n \cdot 2\sin^2\frac{nx}{2}}{x^2}$$

$$= 2\cdot\frac{1}{4}\cdot n + 2\cdot\frac{n^2}{4} = \frac{n}{2} + \frac{n^2}{2} = \frac{n(n+1)}{2}$$

Solution 2 by Abdul Hafeez Ayinde-Nigeria

$$\Omega = \lim_{x\to 0}\left(\frac{1 - (\cos x)^n \cos(nx)}{x^2}\right)$$

$\Omega = \frac{0}{0}$; *indeterminate. Applying L'Hospital's rule.*

$$\Omega = \lim_{x\to 0}\left(\frac{(\cos x)^n(n\sin nx) + n(\cos x)^{n-1}\sin x\cdot\cos(nx)}{2x}\right)$$

$\Omega = \frac{0}{0}$. *Applying L'Hospital's rule again.*

$$\Omega = \lim_{x\to 0}\left(\frac{\begin{array}{c}-n(\cos x)^{n-1}\sin x\cdot(n\sin nx) + n^2(\cos x)^n(\cos nx) + \\ +n((\cos x)^{n-1}\cos x\cdot\cos(nx)) + \\ \sin x\,(-\sin x\,(n-1)\cos(nx)\,(\cos x)^{n-2}) - \\ -n\sin(nx)\cdot(\cos x)^{n-1}\end{array}}{2}\right)$$

$$\Omega = \left(\frac{n^2 + n(1+0)}{2}\right);\ \Omega = \frac{n^2+n}{2}$$

UP.199. Given the triangle ABC. The internal angle bisectors from A, B, C meet sides BC, CA, AB at A_1, B_1, C_1 respectively. Prove that:

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} +$$
$$+\frac{\cos\left(\overrightarrow{BB_1}, \overrightarrow{CC_1}\right)}{\cos\frac{A}{2}} + \frac{\cos\left(\overrightarrow{CC_1}, \overrightarrow{AA_1}\right)}{\cos\frac{B}{2}} + \frac{\cos\left(\overrightarrow{AA_1}, \overrightarrow{BB_1}\right)}{\cos\frac{C}{2}} = 0$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Marian Ursărescu – Romania

We have $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{4R+r}{s}$ *(1)*

$AA_1 = \frac{2bc}{4+c}\cdot\cos\frac{A}{2} \Rightarrow \cos\frac{A}{2} = \frac{(b+c)AA_1}{2bc}$ *and similarly* $\Rightarrow$

$$\frac{\cos\left(\overrightarrow{AA_1}, \overrightarrow{BB_1}\right)}{\cos\frac{C}{2}} = \frac{\cos\left(\overrightarrow{AA_1}, \overrightarrow{BB_1}\right)}{\frac{(a+b)CC_1}{2ab}} = \frac{2ab\cos\left(\overrightarrow{AA_1}, \overrightarrow{BB_1}\right)}{(a+b)CC_1} =$$

$$= \frac{2ab}{(a+b)CC_1}\cdot\frac{\overrightarrow{AA_1}\cdot\overrightarrow{BB_1}}{AA_1\cdot BB_1} = \frac{2ab}{AA_1\cdot BB_1\cdot CC_1(a+b)}\cdot\frac{\left(b\overrightarrow{AB} + c\overrightarrow{AC}\right)}{b+c}\cdot\frac{\left(a\overrightarrow{BA} + c\overrightarrow{BC}\right)}{a+c}$$

$$=\frac{1}{(a+b)(a+c)(b+c)AA_1BB_1CC_1}\left(-2a^2b^2c^2-2ab^2c\overrightarrow{BA}\cdot\overrightarrow{BC}-2a^2bc\overrightarrow{AC}\cdot\overrightarrow{AB}+2abc^2\overrightarrow{CA}\cdot\overrightarrow{CB}\right)=$$

$$=-2abc\left(abc+a\overrightarrow{AC}\cdot\overrightarrow{AB}+b\overrightarrow{BA}\cdot\overrightarrow{BC}-c\overrightarrow{CA}\cdot\overrightarrow{CB}\right)\quad (2)$$

From (2) $\Rightarrow \frac{\cos(\overrightarrow{BB_1},\overrightarrow{CC_1})}{\cos\frac{A}{2}}+\frac{\cos(\overrightarrow{CC_1},\overrightarrow{AA_1})}{\cos\frac{B}{2}}+\frac{\cos(\overrightarrow{AA_1},\overrightarrow{BB_1})}{\cos\frac{C}{2}}=$

$$=\frac{-2abc\left(3abc+a\overrightarrow{AB}\cdot\overrightarrow{AC}+b\overrightarrow{BA}\cdot\overrightarrow{BC}+c\overrightarrow{CB}\cdot\overrightarrow{CA}\right)}{(a+b)(a+c)(b+c)AA_1BB_1CC_1}=$$

$$=\frac{-2abc\left(3abc+a\left(\frac{b^2+c^2-a^2}{2}\right)+b\left(\frac{a^2+c^2-b^2}{2}\right)+c\left(\frac{a^2+b^2+c^2}{2}\right)\right.}{(a+b)(b+c)(a+c)AA_1BB_1CC_1}=$$

$$=\frac{-2abc\left(6abc+\sum bc(b+c)-\sum a^3\right)}{(a+b)(a+c)(b+c)AA_1BB_1CC_1}\quad (3)$$

But $abc=4sRr$ (4)

$$\sum bc(b+c)=2s(s^2+r^2-2Rr)\ (5)$$

$$\sum a^3=2s(s^2-3r^2-6Rr)\quad (6)$$

and $(a+b)(a+c)(b+c)AA_1BB_1CC_1=$

$$=(a+b)(a+c)(b+c)\cdot\frac{2bc}{b+c}\cdot\cos\frac{A}{2}\cdot\frac{2ac}{a+c}\cdot\cos\frac{B}{2}\cdot\frac{2ab}{a+b}\cdot\cos\frac{C}{2}=$$

$$=8a^2b^2c^2\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}=8a^2b^2c^2\cdot\frac{s}{4R}=2a^2b^2c^2\cdot\frac{s}{R}\overset{(4)}{=}$$

$$=32s^2R^2r^2\frac{s}{R}=32s^3Rr^2\quad (7)$$

From (3)+(4)+(5)+(6)+(7)$\Rightarrow \frac{\cos(BB_1,CC_1)}{\cos\frac{A}{2}}+\frac{\cos(\overrightarrow{AA_1},\overrightarrow{CC_1})}{\cos\frac{B}{2}}+\frac{\cos(\overrightarrow{AA_1},\overrightarrow{BB_1})}{\cos\frac{C}{2}}=-\frac{4R+r}{s}$ (8)

From (1)+(8)$\Rightarrow$ the relationship is true.

UP.200. If $0<a\le b$ then:

$$\int_a^b\int_a^b\int_a^b\int_a^b\frac{(x+y+z+t)dxdydzdt}{\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}}\le\frac{(b+a)^2(b-a)^4}{4ab}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$x,y\in[a,b]$. By Schweitzer inequality:

$$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)\le\frac{(a+b)^2}{ab}$$

$$\frac{(x+y)^2}{xy}\le\frac{(a+b)^2}{ab}$$

$$ab(x+y)^2\le xy(a+b)^2$$

$$\sqrt{ab}(x+y)\le\sqrt{xy}(a+b)\quad (1)$$

Analogous:

$$\sqrt{ab}(y+z)\le\sqrt{yz}(a+b)\quad (2)$$

$$\sqrt{ab}(z+t)\le\sqrt{zt}(a+b)\quad (3)$$

$$\sqrt{ab}(t+x)\le\sqrt{tx}(a+b)\quad (4)$$

By adding (1); (2); (3); (4):

$$2\sqrt{ab}(x+y+z+t)\le(a+b)\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}\right)$$

$$\frac{x+y+z+t}{\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}}\le\frac{a+b}{2\sqrt{ab}}=$$

$$=\frac{a+b}{2}\cdot\frac{1}{\sqrt{ab}}\overset{GM-HM}{\le}\frac{a+b}{2}\cdot\frac{1}{\frac{2}{\frac{1}{a}+\frac{1}{b}}}=\frac{a+b}{2}\cdot\frac{1}{\frac{2ab}{a+b}}=\frac{(a+b)^2}{4ab}$$

$$\frac{x+y+z+t}{\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}}\le\frac{(a+b)^2}{4ab}$$

$$\int_a^b\int_a^b\int_a^b\int_a^b\frac{(x+y+z+t)dx\,dy\,dz\,dt}{\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}}\le$$

$$\le\int_a^b\int_a^b\int_a^b\int_a^b\frac{(a+b)^2}{4ab}dx\,dy\,dz\,dt=\frac{(b+a)^2(b-a)^4}{4ab}$$

UP.201. Calculate the integral: $\int_0^\infty\frac{\arctan x}{x^4-x^2+1}dx$. It is required to express the integral value with the usual mathematical constants and $\psi_1\left(\frac{1}{3}\right)$, where $\psi_1(x)$ is the trigamma function.

Proposed by Vasile Mircea Popa – Romania

Solution by Pedro Nagasava-Brazil

$\int_0^\infty \frac{\arctan(x)}{x^4-x^2+1}dx$. *Rewriting the integral:* $I = \int_0^\infty \int_0^1 \frac{x}{(x^4-x^2+1)(1+y^2x^2)}dy\,dx$

Using Fubini-Tonelli Theorem, it is possible to switch the order of integration:

$$I = \int_0^1 \int_0^\infty \frac{x}{(x^4-x^2+1)(1+y^2x^2)}dx\,dy$$

Let $x^2 = z$: $I = \frac{1}{2}\int_0^1 \int_0^\infty \frac{1}{(z^2-z+1)(1+zy^2)}dz\,dy =$

$$= \frac{1}{2}\int_0^1 \frac{1}{y^4+y^2+1}\int_0^\infty \left[\frac{-y^2z}{z^2-z+1} + \frac{y^2+1}{z^2-z+1} + \frac{y^4}{1+zy^2}\right]dz\,dy$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{y^4+y^2+1}\int_0^\infty \left[-\frac{y^2}{2}\left(\frac{2z-1}{z^2-z+1}\right) + \frac{\frac{y^2}{2}+1}{z^2-z+1} + \frac{y^4}{1+zy^2}\right]dz\,dy$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{y^4+y^2+1}{}_0^\infty\left[y^2\log\left(\frac{1+zy^2}{\sqrt{z^2-z+1}}\right) + \frac{2}{\sqrt{3}}\left(\frac{y^2}{2}+1\right)\arctan\left(\frac{2z-1}{\sqrt{3}}\right)\right]dy$$

$$I = \int_0^1 \frac{y^2\log(y)}{y^4+y^2+1}dy + \frac{\pi}{3\sqrt{3}}\int_0^1 \frac{y^2+2}{y^4+y^2+1}dy$$

Considering the following function to evaluate the first integral:

$$f(n) = \int_0^1 \frac{y^n}{y^4+y^2+1}\cdot\frac{1-y^2}{1-y^2}dy = \int_0^1 \frac{y^n-y^{n+2}}{1-y^6}dy = \sum_{k=0}^\infty \int_0^1 \left(y^{n+6k}-y^{n+6k+2}\right)dy$$

$f(n) = \sum_{k=0}^\infty \left(\frac{1}{n+6k+1} - \frac{1}{n+6k+3}\right)$. *Therefore:*

$$f'(2) = \int_0^1 \frac{y^2\log(y)}{y^4+y^2+1}dy = \sum_{k=0}^\infty \left(-\frac{1}{(6k+3)^2} + \frac{1}{(6k+5)^2}\right) =$$

$$= -\frac{\left(1-\frac{1}{4}\right)\zeta(2)}{9} + \frac{\psi^{(1)}\left(\frac{5}{6}\right)}{36}$$

For the second integral, notice that it can be rewritten as:

$$\frac{\pi}{3\sqrt{3}}\int_0^1 \frac{y^2+2}{y^4+y^2+1}dy = \frac{\pi}{3\sqrt{3}}\left[\int_0^\infty \frac{dy}{y^4+y^2+1} + \int_0^1 \frac{dy}{y^4+y^2+1}\right]$$

Evaluating the indefinite integral:

$$\int \frac{dy}{y^4+y^2+1} = \frac{1}{2}\left[\int \frac{1+\frac{1}{y^2}}{\left(y-\frac{1}{y}\right)^2+3}dy - \int \frac{1-\frac{1}{y^2}}{\left(y+\frac{1}{y}\right)^2-1}dy\right]$$

$$\int \frac{dy}{y^4+y^2+1} = \frac{1}{2}\left[\frac{1}{\sqrt{3}}\arctan\left[\frac{\left(y-\frac{1}{y}\right)}{\sqrt{3}}\right] + \frac{1}{2}\log\left|\frac{y^2+y+1}{y^2+y-1}\right|\right]$$

Therefore: $\frac{\pi}{3\sqrt{3}}\int_0^1 \frac{y^2+2}{y^4+y^2+1}dy = \frac{\pi}{6\sqrt{3}}\left[\frac{\pi\sqrt{3}}{2} + \frac{\log(3)}{2}\right]$

Gathering all results: $\int_0^\infty \frac{\arctan(x)}{x^4-x^2+1}dx = \frac{5\pi^2}{72} + \frac{\pi}{12\sqrt{3}}\log(3) + \frac{\psi^{(1)}\left(\frac{5}{6}\right)}{36}$

UP.202. Prove that:

$$\Psi_1\left(\frac{5}{12}\right) = \frac{32-6\sqrt{3}}{3}\pi^2 + 40G - 10\Psi_1\left(\frac{1}{3}\right)$$

$$\Psi_1\left(\frac{11}{12}\right) = \frac{32+6\sqrt{3}}{3}\pi^2 - 40G - 10\Psi_1\left(\frac{1}{3}\right)$$

where $\Psi_1(x)$ is the trigamma function and G is the Catalan's constant.

Proposed by Vasile Mircea Popa – Romania

Solution by Dawid Bialek-Poland

$$\Psi^{(1)}\left(\frac{11}{12}\right) = \frac{32}{3}\pi^2 + 2\sqrt{3}\pi^2 - 40G - 10\Psi^{(1)}\left(\frac{1}{3}\right)$$

$$-\Psi^{(1)}\left(\frac{5}{12}\right) = \frac{32}{3}\pi^2 - 2\sqrt{3}\pi^2 + 40G - 10\Psi^{(1)}\left(\frac{1}{3}\right)$$

— —

$$\Psi^{(1)}\left(\frac{11}{12}\right) - \Psi^{(1)}\left(\frac{5}{12}\right) = 4\sqrt{3}\pi^2 - 80G \quad (1)$$

To prove (1), we consider the known values of trigamma:

$$\Psi^{(1)}\left(\frac{1}{4}\right) = \pi^2 + 8G \quad \Psi^{(1)}\left(\frac{3}{4}\right) = \pi^2 - 8G \quad (2)$$

Let's apply the following triplication formula for trigamma function $\Psi^{(1)}\left(\frac{1}{4}\right)$:

$$9\Psi^{(1)}(3x) = \Psi^{(1)}(x) + \Psi^{(1)}\left(x+\frac{1}{3}\right) + \Psi^{(1)}\left(x+\frac{2}{3}\right)$$

Then, we get: $9\Psi^{(1)}\left(\frac{1}{4}\right) = \Psi^{(1)}\left(\frac{1}{12}\right) + \Psi^{(1)}\left(\frac{5}{12}\right) + \Psi^{(1)}\left(\frac{9}{12}\right)$

$$\Psi^{(1)}\left(\frac{1}{12}\right) + \Psi^{(1)}\left(\frac{5}{12}\right) = 9\Psi^{(1)}\left(\frac{1}{4}\right) - \Psi^{(1)}\left(\frac{3}{4}\right) \quad (3)$$

Using the reflection formula for $\Psi^{(1)}\left(\frac{1}{12}\right)$, we get:

$$\Psi^{(1)}\left(\frac{1}{12}\right) = \frac{\pi^2}{\sin^2\left(\frac{\pi}{12}\right)} - \Psi^{(1)}\left(\frac{11}{12}\right) = 8\pi^2 + 4\sqrt{3}\pi^2 - \Psi^{(1)}\left(\frac{11}{12}\right) \quad (4)$$

Rewriting (3) with (2), (4), we get:

$$8\pi^2 + 4\sqrt{3}\pi^2 - \Psi^{(1)}\left(\frac{11}{12}\right) + \Psi^{(1)}\left(\frac{5}{12}\right) = 9[\pi^2 + 8G] - \pi^2 + 8G$$

$$\Psi^{(1)}\left(\frac{11}{12}\right) - \Psi^{(1)}\left(\frac{5}{12}\right) = -9\pi^2 - 80G + \pi^2 + 8\pi^2 + 4\sqrt{3}\pi^2 = 4\sqrt{3}\pi^2 - 80G$$

Where G – Catalan's constant.

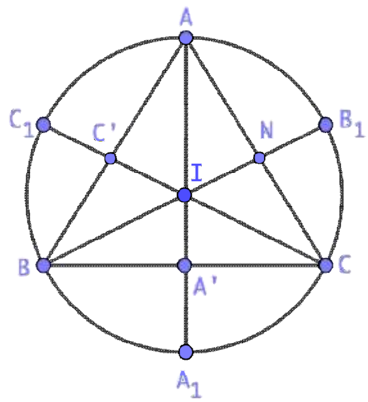
UP.203. Given a triangle ABC with incenter I. The lines AI, BI, CI meet the sides BC, CA, AB at A', B', C' and meet the circumcircle at the second points A_1, B_1, C_1 respectively. Prove that:

(a) $\frac{AI}{AA'} + \frac{BI}{BB'} + \frac{CI}{CC'} = 2,$

(b) $\frac{A_1I}{AI} + \frac{B_1I}{BI} + \frac{C_1I}{CI} = \frac{2R}{r} - 1$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania



a) In ΔABC from bisector theorem $\Rightarrow \frac{BA'}{AC} = \frac{c}{b} \Rightarrow \frac{BA'}{a} = \frac{c}{b+c} \Rightarrow BA' = \frac{ac}{b+c}$

In $\Delta BAA' \Rightarrow \frac{AI}{IA'} = \frac{c}{BA'} = \frac{b+c}{a} \Rightarrow \frac{AI}{IA'} = \frac{b+c}{a+b+c}$ and similarly $\Rightarrow$

$$\Rightarrow \frac{AI}{AA'} + \frac{BI}{BB'} + \frac{CI}{CC'} = \frac{b+c+a+c+a+b}{a+b+c} = 2$$

b) $\mu(I) = -AI \cdot AI = OI^2 - R^2 \Rightarrow A_1I = \frac{R^2-OI^2}{AI} = \frac{R^2-R^2+2Rr}{\frac{r}{\sin\frac{A}{2}}} \Rightarrow$

$\Rightarrow A_1I = 2R\sin\frac{A}{2}$ and similarly $\Rightarrow$

$\Rightarrow \frac{A_1I}{AI} + \frac{B_1I}{BI} + \frac{C_1I}{CI} = \frac{2R}{r}\left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}\right)$ (1). But $\sum \sin^2\frac{A}{2} = 1 - \frac{r}{2R}$ (2)

From (1)+(2) $\Rightarrow \frac{A_1I}{AI} + \frac{B_1I}{BI} + \frac{C_1I}{CI} = \frac{2R}{r}\left(1 - \frac{r}{2R}\right) = \frac{2R}{r} - 1$

Solution 2 by Tran Hong-Dong Thap-Vietnam

a) $\frac{BA'}{A'C'} = \frac{c}{b}; BA' + A'C = a$

$$BA' = \frac{ac}{b+c}; A'C = \frac{bc}{b+c} \because \frac{BA'}{AI} = \frac{BA'}{BA} = \frac{ac}{c(b+c)} = \frac{a}{b+c}$$

$$\frac{AA'}{AI} = \frac{AI + IA'}{AI} = 1 + \frac{a}{b+c} = \frac{a+b+c}{b+c}$$

$\frac{AI}{AA'} = \frac{b+c}{a+b+c}$ (etc)

$$\Rightarrow \frac{AI}{AA'} + \frac{BI}{BB'} + \frac{CI}{CC'} = \frac{(b+c)+(a+c)+(a+b)}{a+b+c} = \frac{2(a+b+c)}{a+b+c} = 2$$

b) $AA' \cdot A'A_1 = BA' \cdot A'C \Rightarrow A'A_1 = \frac{BA' \cdot A'C}{AA'}$

$$A_1I=A'I+A'A_1=\frac{a}{a+b+c}\cdot AA'+\frac{b+cb+c}{AA'}=\frac{a}{a+b+c}\cdot AA'+\frac{bca^2}{(b+c)^2}\cdot\frac{1}{AA'}$$

$$\Rightarrow\frac{A_1I}{AI}=\frac{a}{a+b+c}\cdot\frac{AA'}{AI}+\frac{(bca^3)}{(b+c)^2}\cdot\frac{1}{AA'\cdot AI}$$

$$=\frac{a}{a+b+c}\cdot\frac{a+b+c}{b+c}+\frac{(bca^2)}{(b+c)^2}\cdot\frac{1}{\frac{b+c}{a+b+c}\cdot AB^2}$$

$$=\frac{a}{b+c}+\frac{(bca^2)(a+b+c)}{(b+c)^3AB^2}=\frac{a}{b+c}+\frac{(bca^2)(a+b+c)}{(b+c)^3}\cdot\frac{(b+c)^2}{2bc(a+b+c)(b-a)}$$

$$=\frac{a}{b+c}+\frac{a^2}{2(b+c)(s-a)}=\frac{a}{b+c}+\frac{a}{2(s-a)}=\frac{a}{b+c}\cdot\frac{2s-a}{2(s-a)}$$

$$=\frac{a}{b+c}\cdot\frac{b+c}{2(s-a)}=\frac{a}{2(s-a)}.\ \textit{Similarly: }\frac{B_1I}{BI}=\frac{b}{2(s-b)};\frac{C_1I}{CI}=\frac{c}{2(s-c)}$$

$$\Rightarrow\frac{A_1I}{AI}+\frac{B_1I}{BI}+\frac{C_1I}{CI}=\frac{1}{2}\cdot\left[\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right]=\frac{1}{2}\left[\frac{4R-2r}{r}\right]=\frac{2R}{r}-1\ \textit{Proved}$$

UP.204. Let $(a_n)_{n\ge1}$ be a positive real sequence such that $\lim_{n\to\infty}\frac{a_{n+1}}{n^{t+1}a_n}=a\in\mathbb{R}_+^*$, where t is a positive integer. Compute:

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n}}\sum_{k=1}^{n}[k^t\cdot b]$$

where $b\in\mathbb{R}$; we denote by $[x]$ the integer part of x.

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$L=\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n}}\cdot\sum_{k=1}^{n}[k^tb]=\lim_{n\to\infty}\frac{n^{t+1}}{\sqrt[n]{a_n}}\cdot\frac{1}{n^{t+1}}\cdot\sum_{k=1}^{n}[k^tb]\quad\textit{(1)}$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}[k^tb]}{n^{t+1}}\overset{C.S.}{=}\lim_{n\to\infty}\frac{[(n+1)^tb]}{(n+1)^{t+1}-n^{t+1}}=\lim_{n\to\infty}\frac{[(n+1)^tb]}{C_{t+1}^1n^t+\dots+C_{t+1}^{t+1}}=\frac{b}{t+1}\quad\textit{(2), because}$$

$$(n+1)^tb-1<[(n+1)^tb]\le(n+1)^tb\Rightarrow\frac{(n+1)^tb-1}{C_{t+1}^bn^t+\cdots}<\frac{[(n+1)^tb]}{C_{t+1}^bn^t+\cdots}\le\frac{(n+1)^tb}{C_{t+1}^bn^t+\cdots}$$

$$\lim_{n\to\infty}\frac{n^{t+1}}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^{n(t+1)}}{a_n}}\overset{C.D.}{=}\lim_{n\to\infty}\frac{(n+1)^{(n+1)(t+1)}}{a_{n+1}}\cdot\frac{a_n}{n^{n(t+1)}}=$$

$$=\lim_{n\to\infty}\frac{(n+1)^{n(t+1)}}{n^{n(t+1)}}\cdot\frac{(n+1)^{t+1}a_n}{a_{n+1}}=\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n\right)^{t+1}\cdot\frac{(n+1)^{t+1}}{n^{t+1}}\cdot\frac{n^{t+1}a_n}{a_{n+1}}=$$

$$=e^{t+1}\cdot 1\cdot\frac{1}{a}=\frac{e^{t+1}}{a}\quad (3)$$

$$\text{From } (1)+(2)+(3)\Rightarrow L=\frac{b}{a}\cdot\frac{e^{t+1}}{t+1}$$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega=\lim_{n\to\infty}\frac{n^{t+1}}{\sqrt[n]{a_n}}\cdot\frac{1}{n^{t+1}}\sum_{k=1}^{n}[k^tb]=\lim_{n\to\infty}\frac{n^{t+1}}{\sqrt[n]{a_n}}\cdot\lim_{n\to\infty}\frac{\sum_{k=1}^{n}[k^tb]}{n^{t+1}}=$$

$$=\lim_{n\to\infty}\left(\frac{n^{(t+1)n}}{a_n}\right)^{\frac{1}{n}}\cdot\lim_{n\to\infty}\frac{\sum_{k=1}^{n}[k^tb]}{n^{t+1}}=\lim_{n\to\infty}e^{\frac{\ln\left(\frac{n^{(t+1)n}}{a_n}\right)}{n}}\lim_{n\to\infty}\frac{\sum_{k=1}^{n}[k^tb]}{n^{t+1}}=$$

$$\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{n(t+1)}\cdot\frac{1}{\frac{a_{n+1}}{n^{t+1}a_n}}\right)\cdot\lim_{n\to\infty}\frac{\sum_{k=1}^{n}[k^tb]}{n^{t+1}}=e^{t+1}\cdot\frac{1}{a}\cdot\lim_{n\to\infty}\frac{\sum_{k=1}^{n}[k^tb]}{n^{t+1}}=$$

$$\overset{Stolz\ Cesaro}{=}\frac{e^{t+1}}{a}\cdot\lim_{n\to\infty}\frac{[(n+1)^tb]}{(n+1)^{t+1}-n^{t+1}}=\frac{e^{t+1}}{a}\cdot\lim_{n\to\infty}\frac{[(n+1)^tb]}{n^{t+1}\left(\left(\frac{n+1}{n}\right)^{t+1}-1\right)}=$$

$$=\frac{e^{t+1}}{a}\cdot\lim_{n\to\infty}\frac{[(n+1)^tb]}{n^t\left(\left(\frac{n+1}{n}\right)^0+\dots+\left(\frac{n+1}{n}\right)^t\right)}=\frac{e^{t+1}}{a(t+1)}\cdot\lim_{n\to\infty}\frac{[(n+1)^tb]}{n^t}\quad (1)$$

$$\frac{(n+1)^tb-1}{n^t}<\frac{[(n+1)^tb]}{n^t}<\frac{(n+1)^tb}{n^t};\lim_{n\to\infty}\frac{(n+1)^tb-1}{n^t}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^tb=b$$

$$\lim_{n\to\infty}\frac{(n+1)^tb}{n^t}=b\Rightarrow\lim_{n\to\infty}\frac{[(n+1)^tb]}{n^t}=b\overset{(1)}{\Rightarrow}\Omega=\frac{b}{a}\cdot\frac{e^{t+1}}{t+1}$$

UP.205. Compute:

$$\lim_{n\to\infty}\left(\lim_{n\to\infty}\left((\Gamma(x+2))^{\frac{F_n}{(x+1)F_{n+1}}}-(\Gamma(x+1))^{\frac{F_n}{xF_{n+1}}}\right)x^{\frac{F_{n-1}}{F_{n+1}}}\right)$$

where $(F_n)_{n\geq 0}, F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n, \forall n\in\mathbb{N}$ is the Fibonacci sequence.

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Srinivasa Raghava-AIRMC-India

Let $a(n)=\left(\frac{\Gamma(n+2)^{\frac{1}{n+1}}}{\Gamma(n+1)^{\frac{1}{n}}}\right)^{\frac{F_n}{F_{n+1}}}$ *for* $n=1,2,3\dots$ *then we see that* $\lim_{n\to\infty}a(n)=1\Rightarrow$

$$\Rightarrow\lim_{n\to\infty}\frac{a(n)-1}{\log(a(n))}=1$$

$\lim_{n\to\infty}a(n)^n=\lim_{n\to\infty}\left(\frac{n}{(n+1)}\cdot\frac{(n+1)}{(n+1)!^{\frac{1}{n+1}}}\right)^{\frac{F_n}{F_{n+1}}}=e^{\frac{1}{\phi}}\to\left(\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\phi\right)$ *(ϕ = Golden Ratio)*

Hence, we have:

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}x^{\frac{F_{n-1}}{F_{n+1}}}\left(\Gamma(x+2)^{\frac{F_n}{(x+1)F_{n+1}}}-\Gamma(x+1)^{\frac{F_n}{xF_{n+1}}}\right)\right)=e^{\frac{1}{\phi}}\log\left(e^{\frac{1}{\phi}}\right)=\frac{e^{\frac{1}{\phi}}}{\phi}=1.14662\dots$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$\lim_{x\to\infty}\frac{\sqrt[x]{\Gamma(x+1)}}{x}=\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$*. Let* $u_n=\left(\frac{\sqrt[n+1]{\Gamma(n+2)}}{\sqrt[n]{\Gamma(n+1)}}\right)^{\frac{F_n}{F_{n+1}}}$ *for all* $n\in\mathbb{N}$

Now, $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\frac{\sqrt[n+1]{\Gamma(n+2)}}{n+1}}{\frac{\sqrt[n]{\Gamma(n+1)}}{n}}\left(1+\frac{1}{n}\right)\right)^{\frac{F_n}{F_{n+1}}}=1$*, then* $\frac{u_n-1}{\ln u_n}\to1$ *as* $n\to\infty$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{\Gamma(n+2)}{\Gamma(n+1)}\cdot\frac{1}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{\Gamma(n+2)}}\right)^{\frac{F_n}{F_{n+1}}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{\frac{F_n}{F_{n+1}}}$$

$$=e^{\frac{1}{\varphi}}\text{ where }\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\varphi$$

$$\therefore\lim_{n\to\infty}\left(\left(\lim_{x\to\infty}\left((\Gamma(x+2))^{\frac{F_n}{(x+1)F_{n+1}}}-(\Gamma(x+1))^{\frac{F_n}{xF_{n+1}}}\right)x^{\frac{F_{n-1}}{F_{n+1}}}\right)\right.$$

$$=\lim_{n\to\infty}\left(\frac{\Gamma(n+1)}{n}\right)^{\frac{F_n}{F_{n+1}}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=e^{\frac{1}{\varphi}}\cdot1\cdot\ln e^{\frac{1}{\varphi}}=\frac{e^{\frac{1}{\varphi}}}{\varphi}$$

Solution 3 by Tobi Joshua-Nigeria

$$I=\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\left(\left(\Gamma(x+2)\right)^{\frac{F_n}{(x+1)F_{n+1}}}\right)-\left(\left(\Gamma(x+1)\right)^{\frac{F_n}{(x)F_{n+1}}}\right)\right)\left(x^{\frac{F_{n-1}}{F_{n+1}}}\right)\right)$$

Consider $F_{n+2}-F_{n+1}-F_n=0\Rightarrow F_n=A\lambda_1^n+B\lambda_2^n+\cdots\forall n\geq 0\Rightarrow \lambda^2-\lambda-1=0$

$$\Rightarrow\left(\lambda-\frac{1+\sqrt{5}}{2}\right)\left(\lambda+\frac{1-\sqrt{5}}{2}\right)=0\Rightarrow \lambda_1=\frac{\sqrt{5}-1}{2},\lambda_2=\frac{\sqrt{5}+1}{2}$$

$$F_n=A\left(\frac{\sqrt{5}-1}{2}\right)^n+B\left(\frac{\sqrt{5}+1}{2}\right)^n,\text{ using } F_0=0,F_1=1$$

$$A=-1,B=1\Rightarrow F_n=-\left(\frac{\sqrt{5}-1}{2}\right)^n+\left(\frac{\sqrt{5}+1}{2}\right)^n \text{ then}\Rightarrow F_{n+1}=-\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}+\left(\frac{\sqrt{5}+1}{2}\right)^{n+1}$$

$$\text{and}\Rightarrow F_{n-1}=-\left(\frac{\sqrt{5}-1}{2}\right)^{n-1}+\left(\frac{\sqrt{5}+1}{2}\right)^{n-1}.\text{ Now,}$$

$$\lim_{n\to\infty}\frac{F_n}{F_{n+1}}=\frac{-\left(\frac{\sqrt{5}-1}{2}\right)^n+\left(\frac{\sqrt{5}+1}{2}\right)^n}{-\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}+\left(\frac{\sqrt{5}+1}{2}\right)^{n+1}}=\left(\frac{\sqrt{5}+1}{2}\right)=\varphi$$

Then

$$I=\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\left(\left(\Gamma(x+2)\right)^{\frac{F_n}{(x+1)F_{n+1}}}\right)-\left(\left(\Gamma(x+1)\right)^{\frac{F_n}{(x)F_{n+1}}}\right)\right)\left(x^{\frac{F_{n-1}}{F_{n+1}}}\right)\right)$$

since $F_{n+1}-F_n=F_{n-1}\ \oplus$

$$I=\lim_{x\to\infty}\left(\lim_{n\to\infty}\left(\left(\frac{\sqrt[x+1]{\Gamma(x+2)}}{\sqrt[x]{\Gamma(x+1)}}\right)^{\frac{F_n}{F_{n+1}}}-1\right)\right)\times\lim_{x\to\infty}\left(\lim_{n\to\infty}\left(\left(\frac{\sqrt[x+1]{\Gamma(x+1)}}{x}\right)^{\frac{F_n}{F_{n+1}}}\right)\right)\times x$$

$$I=\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\frac{\left(\frac{\sqrt[x+1]{\Gamma(x+2)}}{\sqrt[x]{\Gamma(x+1)}}\right)^{\frac{F_n}{F_{n+1}}}-1}{\log\left(\left(\frac{\sqrt[x+1]{\Gamma(x+2)}}{\sqrt[x]{\Gamma(x+1)}}\right)^{\frac{F_n}{F_{n+1}}}\right)}\right)\right)\times\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\left(\frac{\sqrt[x+1]{\Gamma(x+1)}}{x}\right)^{\frac{F_n}{F_{n+1}}}\right)\right)\times$$

$$\times \lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\log\left(\left(\frac{\sqrt[x+1]{\Gamma(x+2)}}{\sqrt[x]{\Gamma(x+1)}}\right)^{\frac{F_n}{F_{n+1}}}\right)\right)^x\right)$$

$$I = 1\times\lim_{n\to\infty}\left(e^{\frac{F_n}{F_{n+1}}}\right)\times\log\lim_{n\to\infty}\left(\left(e^{\frac{F_n}{F_{n+1}}}\right)\right);\ I = \left(e^{\frac{1}{\varphi}}\right)\times\log\left(e^{\frac{1}{\varphi}}\right) = \frac{e^{\frac{1}{\varphi}}}{\varphi};\ I = \frac{2e^{\frac{\sqrt{5}-1}{2}}}{\sqrt{5}+1}$$

Compute:

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{F_{n+1}^2}{(x+1)F_{2n+1}}}-\left(\Gamma(x+1)\right)^{\frac{F_{n+1}^2}{xF_{2n+1}}}\right)x^{\frac{F_n^2}{F_{2n+1}}}\right)$$

where $(F_n)_{n\geq 0}, F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1}+F_n, \forall n\in\mathbb{N}$ is the Fibonacci sequence.

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by proposers

We denote $u_n = \frac{F_{n+1}^2}{F_{2n+1}}$, *we have* $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{5}}\cdot\frac{\left(\alpha^{n+1}-\beta^{n+1}\right)^2}{\alpha^{2n+1}-\beta^{n2+1}} = \frac{1}{\alpha\sqrt{5}}$, *where*

$\alpha = \frac{\sqrt{5}+1}{2}, \beta = \frac{1-\sqrt{5}}{2}, F_n = \frac{1}{\sqrt{5}}(\alpha^n-\beta^n)$. *Also, we have*

$$\lim_{n\to\infty}\frac{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}{x} = \lim_{n\to\infty}\frac{\left(\Gamma(n+1)\right)^{\frac{1}{n}}}{n} = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

We denote $v(x) = \left(\frac{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}\right)^{u_n}$, *we have* $\lim_{n\to\infty} v(x) = 1$, *so* $\lim_{n\to\infty}\frac{v(x)-1}{\ln v(x)} = 1$ *and*

$$\lim_{x\to\infty}\left(v(x)\right)^x = \lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}\right)^{u_n} = \lim_{x\to\infty}\left(\frac{x+1}{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}\right)^{u_n} = e^{u_n}$$

therefore $\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(v(x)\right)^x\right) = e^{\frac{1}{\alpha\sqrt{5}}}$. *Hence:*

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{F_{n+1}^2}{(x+1)F_{2n+1}}}-\left(\Gamma(x+1)\right)^{\frac{F_{n+1}^2}{xF_{2n+1}}}\right)x^{\frac{F_n^2}{F_{2n+1}}}\right)$$

$$= \lim_{n\to\infty}\left(\left(\lim_{x\to\infty}(\Gamma(x+2))^{\frac{u_n}{x+1}}-(\Gamma(x+1))^{\frac{u_n}{x}}\right)x^{\frac{F_n^2}{F_{2n+1}}}\right)=$$

$$= \lim_{n\to\infty}\left(\lim_{x\to\infty}\left((\Gamma(x+2))^{\frac{u_n}{x+1}}-(\Gamma(x+1))^{\frac{u_n}{x}}\right)x^{\frac{F_{2n+1}-F_{n+1}^2}{F_{2n+1}}}\right)=$$

$$= \lim_{n\to\infty}\left(\lim_{x\to\infty}\left((\Gamma(x+1))^{\frac{u_n}{x}}\right)(v(x)-1)x^{1-u_n}\right)=$$

$$= \lim_{n\to\infty}\left(\lim_{x\to\infty}\left((\Gamma(x+1))^{\frac{u_n}{x}}\right)\frac{v(x)-1}{\ln v(x)}x^{1-u_n}\ln v(x)\right)=$$

$$= \lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\right)^{u_n}\frac{v(x)-1}{\ln v(x)}\ln(v(x))^x\right)=$$

$$\lim_{n\to\infty}\left(\left(\frac{1}{e}\right)^{u_n}\cdot 1\cdot\ln e^{u_n}\right)=\left(\frac{1}{e}\right)^{\frac{1}{\alpha\sqrt{5}}}\ln e^{\frac{1}{\alpha\sqrt{5}}}=\frac{1}{\alpha\sqrt{5}e^{\frac{1}{\alpha\sqrt{5}}}}$$

UP.207. Let be $A\in M_3(\mathbb{R})$ such that $\det A=-1$. Prove that:

$$(\operatorname{Tr}A+\operatorname{Tr}A^{-1}+1)^2\geq 3(\operatorname{Tr}A\cdot\operatorname{Tr}A^{-1}-1)$$

Proposed by Marian Ursărescu – Romania

Solution by proposer

$$\left.\begin{array}{c}p_A(x)=x^3-\operatorname{Tr}A\,x^2+\operatorname{Tr}A^*\,x-\det A,\ with\ \lambda_1,\lambda_2,\lambda_3\in\mathbb{C}\\ \operatorname{Tr}A^*=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=\lambda_1\lambda_2\lambda_3(\lambda_1^{-1}+\lambda_2^{-1}+\lambda_3^{-1})=\det A\cdot\operatorname{Tr}A^{-1}=-\operatorname{Tr}A^{-1}\end{array}\right\}\Rightarrow$$

$$p_A(x)=x^3-\operatorname{Tr}A\,x^2-\operatorname{Tr}A^{-1}\,x+1$$

We have $\det(A^2+A+I_3)\geq 0$ ***(1). Let be*** $f(x)=x^2+x+1\Rightarrow$

$$\det(A^2+A+I_3)=f(\lambda_1)\cdot f(\lambda_2)\cdot f(\lambda_3)\overset{(1)}{\geq}0\quad (2)$$

$p_A(x)=(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$***. Let ε be a root of 3rd degree of the unit.***

$$\varepsilon^2+\varepsilon+1=0,\varepsilon^3=1$$

$$\left.\begin{array}{c}p_A(\varepsilon)=(\varepsilon-\lambda_1)(\varepsilon-\lambda_2)(\varepsilon-\lambda_3)\\ p_A(\varepsilon^2)=(\varepsilon^2-\lambda_1)(\varepsilon^2-\lambda_2)(\varepsilon^2-\lambda_3)\end{array}\right\}\Rightarrow$$

$$p_A(\varepsilon)p_A(\varepsilon^2) = (\lambda_1^2 + \lambda_1 + 1)(\lambda_1^2 + \lambda_2 + 1)(\lambda_3^2 + \lambda_3 + 1) = f(\lambda_1)f(\lambda_2)f(\lambda_3) \overset{(2)}{\geq} 0 \quad \textbf{\textit{(3)}}$$

$$\textit{But } \left.\begin{aligned} p_A(\varepsilon) &= 2 - \operatorname{Tr}\varepsilon^2 - \operatorname{Tr}A^{-1}\varepsilon \\ p_A(\varepsilon^2) &= 2\operatorname{Tr}A\,\varepsilon - \operatorname{Tr}A^{-1}\varepsilon^2 \end{aligned}\right\} \Rightarrow$$

$$p_A(\varepsilon)\cdot p_A(\varepsilon^2) = 4 + (\operatorname{Tr}A)^2 + (\operatorname{Tr}A^{-1})^2 + 2\operatorname{Tr}A + 2\operatorname{Tr}A^{-1} - \operatorname{Tr}A\operatorname{Tr}A^{-1} \quad \textbf{\textit{(4)}}$$

From (3)+(4)$\Rightarrow (\operatorname{Tr}A)^2 + (\operatorname{Tr}A^{-1})^2 + 2\operatorname{Tr}A + 2\operatorname{Tr}A^{-1} - \operatorname{Tr}A\operatorname{Tr}A^{-1} + 3 \geq 0$

$$(\operatorname{Tr}A + \operatorname{Tr}A^{-1} + 1)^2 - 3\operatorname{Tr}A\operatorname{Tr}A^{-1} + 3 \geq 0 \Rightarrow$$

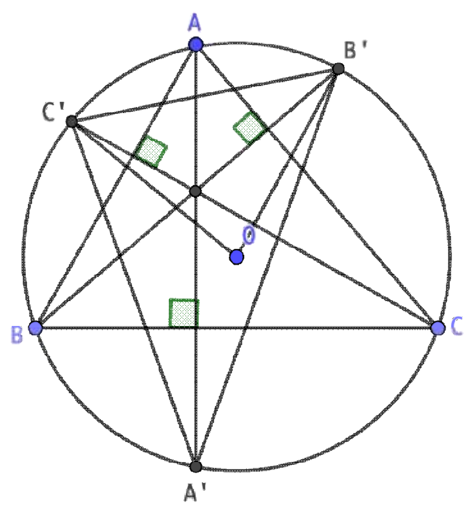
$$(\operatorname{Tr}A + \operatorname{Tr}A^{-1} + 1)^2 \geq 3(\operatorname{Tr}A\operatorname{Tr}A^{-1} - 1)$$

UP.208. Let ABC be an acute-angled triangle and A', B', C', the points in which the heights of the triangle intersect the circumcircle of ΔABC. Prove that:

$$\frac{S_{A'B'C'}}{S_{ABC}} \leq \left(\frac{2r}{R}\right)^2$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam



We have: $\angle C'B'B = \angle C'CB = 90° - B$; $\angle BB'A' = \angle BAA' = 90° - B$

$$\Rightarrow B' = \angle C'B'B + \angle BB'A' = 180° - 2B$$

Similarly: $A' = 180° - 2A$; $B' = 180° - 2C$

$$\because B'C'^2 = OC'^2 + OB'^2 - 2\cdot OC'\cdot OB'\cdot\cos(C'OB')$$

$$= R^2 + R^2 - 2 \cdot R \cdot R \cdot \cos 2A' = 2R^2 - 2R^2 \cos(360° - 4A)$$

$$= 2R^2(1 - \cos 4A) = 4R^2 \sin^2 2A \Rightarrow B'C' = 2R \sin 2A \; (\because A, B, C\text{: acute})$$

Similarly: $A'B' = 2R \sin 2C$; $A'C' = 2R \sin 2B$. *Hence:*

$$S_{A'B'C'} = \frac{A'B' \cdot B'C' \cdot A'C'}{4R} = \frac{8R^2 \cdot \sin 2A \cdot \sin 2B \cdot \sin 2C}{4R} = 2R \cdot \sin 2A \cdot \sin 2B \cdot 2C$$

$$S_{ABC} = \frac{AB \cdot BC \cdot CA}{4R} = \frac{8R^2 \cdot \sin A \cdot \sin B \cdot \sin C}{4R} = 2R \cdot \sin A \cdot \sin B \cdot \sin C$$

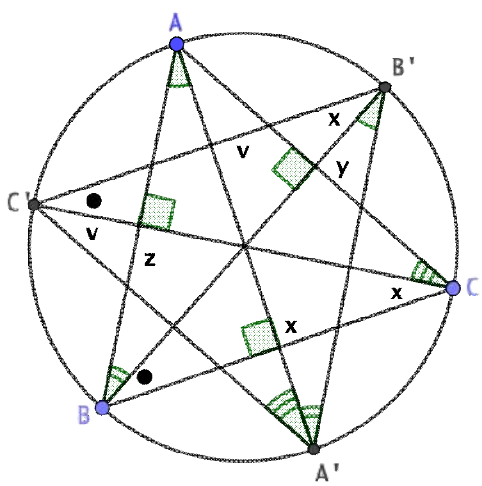
$$\Rightarrow \frac{S_{A'B'C'}}{S_{ABc}} = \frac{2R \prod \sin 2A}{2R \prod \sin A} = 8 \cos A \cdot \cos B \cdot \cos C$$

$$= 8 \cdot \frac{s^2 - (2R + r)^2}{4R^2} = 2 \cdot \frac{s^2 - (2R + r)^2}{R^2}$$

We need to prove: $2 \cdot \frac{s^2 - (2R+r)^2}{R^2} \le \frac{4r^2}{R^2} \Leftrightarrow s^2 - (2R + r)^2 \le 2r^2$

$$\Leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2$$ *(true). Proved.*

Solution 2 by Soumava Chakraborty-Kolkata-India



$$\angle A' = \angle ABY + \angle ACZ = (90° - A) + (90° - A) = 180° - 2A$$

Similarly, $\angle B' = 180° - 2B$ *and* $\angle C' = 180° - 2C$

$$\therefore S_{A'B'C'} = \frac{1}{2}(A'C')(A'B') \sin(180° - 2A)$$

$$= \frac{1}{2} 2R \sin(180° - 2B) \cdot 2R \sin(180° - 2C) \cdot \sin 2A$$

$$= 2R^2 \sin 2A \sin 2B \sin 2C$$

$$= (2R^2 \sin A \sin B \sin C) 8 \cos A \cos B \cos C = S_{ABC} \frac{2\{s^2 - (2R + r)^2\}}{R^2}$$

$$\Rightarrow \frac{S_{A'B'C'}}{S_{ABC}} = \frac{2\{s^2 - 4R^2 - 4Rr - r^2\}}{R^2} \le \frac{4r^2}{R^2}$$

$$\Leftrightarrow s^2 \le 4R^2 + 4Rr + r^2 + 2r^2 = 4R^2 + 4Rr + 3r^2 \to \textit{true (Gerretsen)}$$

(Proved)

UP.209. Demonstrate the following inequality:

$$\frac{x_1}{x_1 + n} + \frac{x_2}{x_2 + n} + \cdots + \frac{x_n}{x_n + n} \le \frac{n}{n + 1}$$

where $x_1, x_2, \dots, x_n$ are strictly positive real numbers which satisfy the relationship: $x_1^2 + x_2^2 + \cdots + x_n^2 = n$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Serban George Florin-Romania

$$\sum_{k=1}^{n} \frac{x_k}{x_{k+n}} = \sum_{k=1}^{n} \left(1 - \frac{n}{x_k + n}\right) = n - n \sum_{k=1}^{n} \frac{1}{x_k + n} \le \frac{n}{n + 1} \Rightarrow 1 - \sum_{k=1}^{n} \frac{1}{x_k + n} \le \frac{1}{n + 1}$$

$$\sum_{k=1}^{n} \frac{1}{x_k + n} \ge 1 - \frac{1}{n + 1}, \sum_{k=1}^{n} \frac{1}{x_k + n} \ge \frac{n}{n + 1}$$

$$\sum_{k=1}^{n} \frac{1}{x_k + n} \overset{Bergstrom}{\ge} \frac{(1 + 1 + \cdots + 1)^2}{\sum_{k=1}^{n} (x_k + n)} = \frac{n^2}{\sum_{k=1}^{n} x_k + n^2} \ge \frac{n}{n + 1}$$

$$\Rightarrow \sum_{k=1}^{n} x_k + n^2 \le \frac{n^2 (n + 1)}{n}, \sum_{k=1}^{n} x_k \le n^2 + n - n^2$$

$$\sum_{k=1}^{n} x_k \le n, \left(\sum_{k=1}^{n} x_k\right)^2 \overset{CBS}{\le} n \sum_{k=1}^{n} x_k^2 \Rightarrow \left(\sum_{k=1}^{n} x_k\right)^2 \le n \cdot n \Rightarrow \left(\sum_{k=1}^{n} x_k\right)^2 \le n^2$$

$$\Rightarrow \sum_{k=1}^{n} x_k \le n \textit{ true.}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$x_1, x_2, \dots, x_n > 0$$

$$\frac{x_1}{x_1+n}+\frac{x_2}{x_2+n}+\cdots+\frac{x_n}{x_n+n}\le\frac{n}{n+1}\quad (*)$$

$$\left(\sum_{i=1}^{n}x_i^2=n\right)$$

$$(*)\Leftrightarrow\left(\frac{x_1}{x_1+n}-1\right)+\left(\frac{x_2}{x_2+n}-1\right)+\cdots+\left(\frac{x_n}{x_n+n}-1\right)\le\frac{n}{n+1}-n$$

$$\Leftrightarrow\frac{1}{x_1+n}+\frac{1}{x_2+n}+\cdots+\frac{1}{x_n+n}\ge\frac{n}{n+1}\quad (1)$$

$$LHS_1\overset{Schwarz}{\ge}\frac{(1+1+\cdots+1)^2}{\sum_{i=1}^{n}x_i+n^2}=\frac{n^2}{n^2+\sum_{i=1}^{n}x_i}=\Omega$$

But $\sum_{i=1}^{n}x_i\overset{BCS}{\le}\sqrt{1^2+1^2+\cdots+1^2}\sqrt{x_1^2+x_2^2+\cdots+x_n^2}=\sqrt{n}\cdot\sqrt{n}=n$

$$\Rightarrow\Omega\ge\frac{n^2}{n^2+n}=\frac{n}{1+n}\text{ (Proved)}$$

Solution 3 by Sudhir Jha-Kolkata-India

$$\frac{x_1}{x_1+n}=1+\frac{x_1}{x_1+n}-1=1+\frac{x_1-x_1-n}{x_1+n}=1-\frac{n}{x_1+n}$$

Similarly, $\frac{x_2}{x_2+n}=1-\frac{n}{x_2+n}\dots..$

… … … … … … … … … … … … … … … …

$$\dots\frac{x_n}{x_n+n}=1-\frac{n}{x_n+n}$$

Adding

$$\Rightarrow\frac{x_1}{x_1+n}+\frac{x_2}{x_2+n}+\cdots+\frac{x_n}{x_n+n}=n-n\left[\frac{1}{x_1+n}+\frac{1}{x_2+n}+\cdots+\frac{1}{x_n+n}\right]\quad (1)$$

Considering $(x_1+n),(x_2+n),\dots,(x_n+n)$ applying m^{th} power theorem by taking $m=-1$, we get: $\frac{(x_1+n)^{-1}+(x_2+n)^{-1}+\cdots+(x_n+n)^{-1}}{n}\ge\left(\frac{x_1+x_2+\cdots+x_n+n\cdot n}{n}\right)^{-1}$

$$\Rightarrow\frac{1}{x_1+n}+\frac{1}{x_2+n}+\cdots+\frac{1}{x_n+n}\ge\frac{n^2}{x_1+x_2+\cdots+x_n+n^2}\quad (2)$$

Again, considering $x_1,x_2,\dots,x_n$ and applying m^{th} power theorem by taking $m=2$, we get: $\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}\ge\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^2$

$$\Rightarrow\frac{n}{n}\ge\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^2\because\left(x_1^2+x_2^2+\cdots+x_n^2=n\right)$$

$$\Rightarrow(x_1+x_2+\cdots+x_n)^2\le n^2$$

$$\Rightarrow x_1 + x_2 + \cdots + x_n \le n \quad (3)$$

Now, from (2), we get: $\frac{1}{x_1+n} + \frac{1}{x_2+n} + \cdots + \frac{1}{x_n+n} \ge \frac{n^2}{n+n^2} = \frac{n}{n+1}$

Then, from (1), we get: $\frac{x_1}{x_1+n} + \frac{x_2}{x_2+n} + \cdots + \frac{x_n}{x_n+n} \le n - n\left(\frac{n}{n+1}\right) = \frac{n^2+n-n^2}{n+1}$

$\Rightarrow \frac{x_1}{x_1+n} + \frac{x_2}{x_2+n} + \cdots + \frac{x_n}{x_n+n} \le \frac{n}{n+1}$ *(The equality holds for* $x_1 = x_2 = \cdots = x_n = 1$*)*

(proved)

Solution 4 by Michael Sterghiou-Greece

$$x_i > 0, i = \overline{1,n}: \sum_1^n x_i^2 = n$$

Prove that: $\sum_1^n \frac{x_i}{x_i+n} \le \frac{n}{n+1} \quad$ *(1)*

The function $f(t) = t^2$ *is convex on* $(0, +\infty)$ *hence by Jensen*

$$n = \sum_1^n x_i^2 \ge n\left(\frac{\sum x_i}{n}\right)^2 \to \sum_1^n x_i \le n$$

The function $f(t) = \frac{t}{t+n}$ *is concave (*) on* $(0, +\infty)$ *hence by Jensen*

$$\sum_1^n \frac{x_i}{x_i+n} \le n \cdot \frac{\sum_1^n \frac{x_i}{n}}{\frac{\sum x_i}{n}+n} \overset{?}{\le} \frac{n}{n+1} \to \sum_1^n x_i \le n \text{ which holds.}$$

$$(*)\ f''(t) = -\frac{2n}{(n+t)^3} < 0$$

UP.210. Prove that for any acute triangle ABC the following inequality holds:

$$\cot A + \cot B + \cot C + \sqrt{3} \ge 2\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sum \cot A + \sqrt{3} \ge 2\sum \tan\frac{A}{2} \quad (1)$$

$$(1) \Leftrightarrow \sum \cot A - 2\sum \tan\frac{A}{2} \ge -\sqrt{3} \quad (2)$$

Let $f(x) = \cot x - 2\tan\frac{x}{2} \quad \left(0 < x < \frac{\pi}{2}\right)$

$$\Rightarrow f'(x) = -\frac{1}{\sin^2 x} - \frac{1}{\cos^2 \frac{x}{2}} \Rightarrow f''(x) = \frac{2\cos x}{\sin^3 x} + \frac{\sin \frac{x}{2}}{\cos^3 \frac{x}{2}} > 0 \quad \left(0 < x < \frac{\pi}{2}\right)$$

Using Jensen's inequality: $f(A) + f(B) + f(C) \geq 3f\left(\frac{A+B+C}{3}\right) = 3f\left(\frac{\pi}{3}\right) = 3 \cdot \frac{-\sqrt{3}}{3} = -\sqrt{3}$

$$\Leftrightarrow \sum \cot A - 2\sum \tan \frac{A}{2} \geq -\sqrt{3} \Leftrightarrow \sum \cot A + \sqrt{3} \geq 2\sum \tan \frac{A}{2}$$

(proved). Equality $\Leftrightarrow A = B = C = \frac{\pi}{3}$.

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru