Number 15 WINTER 2019

R M M

ROMANIAN MATHEMATICAL MAGAZINE

SOLUTIONS

Founding Editor
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Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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JP.211. Prove that there are infinitely many triples (a, b, c) of positive integers satisfying:

$$\frac{a^3 + b^3 + c^3}{3} - abc = a + b + c$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Andrew Okukura-Romania

We will assume at least one of a, b or c is a non-zero integer

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

That means: $\frac{a^3+b^3+c^3}{3} - abc = a + b + c \Leftrightarrow$

$$\Leftrightarrow \frac{1}{3}(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a + b + c \Leftrightarrow$$

$$\Leftrightarrow a^2 + b^2 + c^2 - ab - bc - ca = 3| \cdot 2 \Leftrightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 6$$

For $a = x + 2, b = x + 1$ ***and*** $c = x$***, where*** $x \in \mathbb{N}$

As such any triplet $(x + 2, x + 1, x)$ satisfies the equation, meaning that we have infinitely many triplets which satisfy the equation.

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$a^2 + b^2 + c^2 - ab - bc - ca = 3$$

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 3(ab + bc + ca + 1)$$

$$(a + b + c)^2 = 3(ab + bc + ca + 1)$$

Let $a = k - l, b = k, c = k + l; 9k^2 = 3(k^2 - 4l + k^2 + 4l + k^2 - l^2 + 1)$

$9k^2 = 3(3k^2 - l^2 + 1)$***. For*** $l = 1. 9k^2 = 3 \cdot 3k^2. 9k^2 = 9k^2$.

So, $(k - 1, k, k + 1), k > 1$.

JP.212. Find all real roots of the following equation:

$$(x^3 - 2)^3 + (x^2 - 2)^2 = 0$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Orlando Irahola Ortega-Bolivia

$$(2-x^3)^3=(x^2-2)^2$$

$$a=2-x^3\Rightarrow a-2=-x^3\ \dots(\ )^2\Rightarrow(a-2)^2=x^6\quad \textbf{(2)}$$

$$b=x^2-2\Rightarrow b+2=x^2\ \dots(\ )^3\Rightarrow(b+2)^3=x^6\quad \textbf{(3)}$$

$$a^3=b^2\quad \textbf{(1)}$$

$$\textbf{(2)}=\textbf{(3)}\Rightarrow\begin{cases}(b+2)^3=(a-2)^2\ \dots(4)\\ a^3=b^2\ \dots(1)\end{cases}$$

$$\textbf{(4)}-\textbf{(1)}:$$

$$\Rightarrow(b+2)^3-a^3=(a-2)^2-b^2\Rightarrow(b+2-a)(a^2+b^2+ab+4b+2a+4)=$$

$$=(b+2-a)(2-a-b)$$

$$(\underbrace{b+2-a=0}_{(A)}\wedge\underbrace{a^2+ab+b^2+3a+5b+2=0}_{(B)})$$

(A) $b+2=a\Rightarrow x^3+x^2-2=0\to(x-1)(x^2+2x+2)=0\Rightarrow x-1=0\Rightarrow x_1=1$

$$x^2+2x+2=0$$

$$x_{2,3}=-1\pm i$$

(B) $a^2+ab+b^2+3a+5b+2=0\Rightarrow x^6-\underbrace{\underbrace{x^5}_{+}+x^4-\underbrace{3x^3}_{+}}_{(\forall)x\in\mathbb{R}}+3x^2+2=0$

$$x^6-x^5+x^4-3x^3+3x^2+2>0\Rightarrow x\notin\mathbb{R}$$

$$\text{C.S.}=\{1\}$$

JP.213. Prove that in any ABC triangle the following inequality holds:

$$\frac{r}{4R}(7R^2-4r^2)\le\sum m_a^2\tan^2\frac{B}{2}\tan^2\frac{C}{2}\le 4R^2-13r^2$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum m_a^2\tan^2\frac{B}{2}\tan^2\frac{C}{2}=\left(\frac{\prod\sin\frac{A}{2}}{\prod\cos\frac{A}{2}}\right)^2\sum m_a^2\cot^2\frac{A}{2}=\left(\frac{\frac{r}{4R}}{\frac{s}{4R}}\right)^2\sum m_a^2\left(\csc^2\frac{A}{2}-1\right)$$

$$=\frac{r^2}{s^2}\left(\sum m_a^2\frac{bc(s-a)}{r^2s}-\sum m_a^2\right)=\frac{r^2\cdot4Rrs}{s^2\cdot r^2s}\sum\frac{m_a^2(s-a)}{a}-\frac{r^2}{s^2}\cdot\frac{3}{4}\sum a^2$$

$$= \frac{r^2 \cdot 4Rrs^2}{s^2r^2s}\sum\frac{m_a^2}{a} - \frac{4Rr}{s^2}\cdot\frac{3}{4}\sum a^2 - \frac{3r^2}{4s^2}\sum a^2$$

$$= \frac{Rr}{s}\sum\frac{2b^2+2c^2+2a^2-3a^2}{a} - \frac{3\sum a^2}{4s^2}(4Rr+r^2)$$

$$= \frac{2Rr}{s}\sum a^2\cdot\frac{\sum ab}{4Rrs} - \frac{3Rr}{s}(2s) - \frac{3\sum a^2}{4s^2}(4Rr+r^2)$$

$$= \frac{(s^2-4Rr-r^2)(s^2+4Rr+r^2)}{s^2} - 6Rr - \frac{3(4Rr+r^2)(s^2-4Rr-r^2)}{2s^2}$$

$$= \frac{2(s^2-4Rr-r^2)(s^2+4Rr+r^2) - 3(4Rr+r^2)(s^2-4Rr-r^2) - 12Rrs^2}{2s^2}$$

$$\overset{(1)}{=} \frac{2s^4 - s^2(24Rr+3r^2) + r^2(4R+r)^2}{2s^2} \ge \frac{r}{4R}(7R^2-4r^2)$$

$$\Leftrightarrow 4Rs^4 - s^2(55R^2r + 6Rr^2 - 4r^3) + 32R^3r^2 + 16R^2r^3 + 2Rr^4 \overset{(a)}{\ge} 0$$

Now, LHS of (a) $\overset{Gerretsen}{\ge} 4Rs^2(16Rr-5r^2) -$

$$-s^2(55R^2r + 6Rr^2 - 4r^3) + 32R^3r^2 + 16R^2r^3 + 2Rr^4$$

$$= s^2(9R^2r - 26Rr^2 + 4r^3) + 2Rr^2(4R+r)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow s^2(R-2r)(9R-8r) + 2Rr(4R+r)^2 \underset{(b)}{\overset{?}{\ge}} 12r^2s^2$$

Now, LHS of (b) $\underset{(i)}{\overset{Gerretsen}{\ge}} (16Rr-5r^2)(R-2r)(9R-8r) +$

$+2R(4R+r)^2$ ***and RHS of (b)*** $\underset{(ii)}{\overset{Gerretsen}{\ge}} 12r^2(4R^2+4Rr+3r^2)$

(i), (ii) $\Rightarrow$ ***in order to prove (b), it suffices to prove:***

$$(16R-5r)(R-2r)(9R-8r) + 2R(4R+r)^2 \ge 12r(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 176t^3 - 493t^2 + 340t - 116 \ge 0 \quad \left(t = \frac{R}{r}\right)$$

$\Leftrightarrow (t-2)\{176(t-2)+211t+58\} \ge 0 \rightarrow$ ***true*** $\because t \overset{Euler}{\ge} 2 \Rightarrow$ ***(b)***$\Rightarrow$ ***(a) is true.***

$$\therefore \sum m_a^2\tan^2\frac{B}{2}\tan^2\frac{C}{2} \ge \frac{r}{4R}(7R^2-4r^2)$$

Again (1)$\Rightarrow \sum m_a^2\tan^2\frac{B}{2}\tan^2\frac{C}{2} \le 4R^2 - 13r^2$

$$\Leftrightarrow \frac{2s^4 - s^2(24Rr+3r^2) + r^2(4R+r)^2}{2s^2} \le 4R^2 - 13r^2$$

$$\Leftrightarrow 2s^4 - s^2(8R^2 + 24Rr - 23r^2) + r^2(4R+r)^2 \overset{(c)}{\le} 0$$

Now, Rouche $\Rightarrow s^2 \ge m - n \Rightarrow s^2 - m + n \overset{(iii)}{\ge} 0$ *and* $s^2 \le m + n \Rightarrow s^2 - m - n \overset{(iv)}{\le} 0$,

where $m = 2R^2 + 10Rr - r^2$ *and* $n = 2(R-2r)\sqrt{R^2 - 2Rr}$

(iii), (iv)$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + (2R^2 + 10Rr - r^2)^2 - 4(R-2r)^2(R^2 - 2Rr) \le 0$$

$$\Rightarrow 2s^4 - s^2(8R^2 + 40Rr - 4r^2) + 128R^3r + 96R^2r^2 + 24Rr^3 + 2r^4 \overset{(d)}{\le} 0$$

(d)$\Rightarrow$ in order to prove (c), it suffices to prove:

$$2s^4 - s^2(8R^2 + 24Rr - 23r^2) + r^2(4Rr + r)^2 \le$$

$$\le 2s^4 - s^2(8R^2 + 40Rr - 4r^2) + 128R^3r + 96R^2r^2 + 24Rr^3 + 2r^4$$

$$\Leftrightarrow s^2(16Rr + 19r^2) + r^2(4Rr + r)^2 - 2r(4R + r)^3 \overset{(e)}{\le} 0$$

Now, LHS of (e) $\overset{Gerretsen}{\le} (4R^2 + 4Rr + 3r^2)(16Rr + 19r^2) +$

$$+r^2(4Rr + r)^2 - 2r(4Rr + r)^3 \overset{?}{\le} 0 \Leftrightarrow 16t^3 - 15t^2 - 27t - 14 \overset{?}{\ge} 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(16t^2 + 17t + 7) \overset{?}{\ge} 0 \to \textit{true} \because t \overset{Euler}{\ge} 2$$

$\Rightarrow$ *(e)$\Rightarrow$(c) is true* $\therefore \sum m_a^2 \tan^2\frac{B}{2}\tan^2\frac{C}{2} \le 4R^2 - 13r^2$ *(proved)*

JP.214. Prove that in any ABC triangle the following inequality holds:

$$\frac{27r^3}{2R} \le \sum m_a^2 \sin^2\frac{A}{2} \le \frac{27R^2}{16}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum m_a^2 \sin^2\frac{A}{2} = \sum \frac{2b^2 + 2c^2 + 2a^2 - 3a^2}{4}\sin^2\frac{A}{2}$$

$$= \frac{\sum a^2}{4}\sum(1 - \cos A) - \frac{3}{4}\sum \frac{a^2(s-b)(s-c)}{bc}$$

$$= \frac{\sum a^2}{4}\sum\left(3 - 1 - \frac{r}{R}\right) - \frac{3}{16Rrs}\sum a^3\,(s^2 - s(b+c) + bc)$$

$$=\left(\frac{2R-r}{4R}\right)\sum a^2-\frac{3}{16Rrs}\left(s^2\sum a^3-s\sum(a^3b+ab^3)+4rs\cdot\sum a^2\right)$$

$$=\left(\frac{2R-r}{4R}\right)\sum a^2-\frac{3}{16Rrs}\left(s^2\sum a^3-s\sum ab\left(\sum a^2-c^2\right)+4Rrs\cdot\sum a^2\right)$$

$$=\left(\frac{2R-r}{4R}\right)\sum a^2-\frac{3}{16Rrs}\left[\left(-s\sum ab+4Rrs\right)\cdot\sum a^2+s^2\sum a^3+8Rrs^3\right]$$

$$=\left(\frac{2R-r}{4R}\right)\sum a^2-\frac{3}{16Rrs}\left[-2s(s^2+r^2)(s^2-4Rr-r^2)+2s^2(s^2-6Rr-3r^2)+8Rrs^3\right]$$

$$=\frac{(2R-r)(s^2-4Rr-r^2)}{2R}-\frac{3}{8R}(2Rs^2-3rs^2+4Rr^2+r^3)$$

$$=\frac{4(2R-r)(s^2-4Rr-r^2)-3[(2R-3r)s^2+4Rr^2+r^3]}{8R}$$

$$\overset{(1)}{=}\frac{s^2(2R+5r)-32R^2r-4Rr^2+r^3}{8R}\le\frac{27R^2}{16}$$

$$\Leftrightarrow s^2(4R+10r)-64R^2r-8Rr^2+2r^3\overset{(a)}{\le}27R^3$$

Now, LHS of (a) $\overset{Gerretsen}{\le}(4R^2+4Rr+3r^2)(4R+10r)-$

$$-64R^2r-8Rr^2+2r^3\overset{?}{\le}27R^3\Leftrightarrow 11t^3+8t^2-44t-32\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(11t^2+30t+16)\overset{?}{\ge}0\to \textit{true}\because t\overset{Euler}{\ge}2$$

$\therefore\sum m_a^2\sin^2\frac{A}{2}\le\frac{27R^2}{16}$. *Again, (1)*$\Rightarrow\sum m_a^2\sin^2\frac{A}{2}\ge\frac{27r^3}{2R}$

$$\Leftrightarrow\frac{s^2(2R+5r)-32R^2r-4Rr^2+r^3}{8R}\ge\frac{27r^3}{2R}$$

$$\Leftrightarrow s^2(2R+5r)-32R^2r-4Rr^2-107r^3\overset{(b)}{\ge}0$$

Now, LHS of (b) $\overset{Gerretsen}{\ge}(16Rr-5r^2)(2R+5r)-32R^2r-4Rr^2-107r^3\overset{?}{\ge}0$

$$\Leftrightarrow 66Rr-132r^2\overset{?}{\ge}0\Leftrightarrow R-2r\overset{?}{\ge}0\to \textit{true (Euler)}$$

$\Rightarrow$ *(b) is true* $\therefore\sum m_a^2\sin^2\frac{A}{2}\ge\frac{27r^3}{2R}$ *(Proved)*

Solution 2 by Mustafa Tarek-Cairo-Egypt

$$\frac{27r^3}{2R}\overset{(a)}{\le}\sum m_a^2\sin^2\frac{A}{2}\overset{(b)}{\le}\frac{27R^2}{16}$$

First, we will prove (a): $\sum m_a^2 \sin^2\frac{A}{2} \overset{m_a \geq \sqrt{s(s-a)}}{\geq} \sum s(s-a)\sin^2\frac{A}{2} = \sum bc\cos^2\frac{A}{2}\sin^2\frac{A}{2}$

$= \sum \frac{bc\sin^2 A}{4} = \sum \frac{2\Delta \sin A}{4} = \frac{rs}{2} = \frac{\sum a}{2R} = \frac{s^2 r}{2R} \overset{Mitrinovic}{\geq} \frac{27r^3}{2R}$, so (a) is true.

Now, we will prove (b): $\sum m_a^2 \sin^2\frac{A}{2} \overset{m_a \leq h_a \cdot \frac{R}{2r}}{\leq} \sum \frac{h_a^2 R^2}{4r^2}\sin^2\frac{A}{2} = \sum \frac{b^2c^2R^2}{4r^2 \cdot 4R^2}\sin^2\frac{A}{2}$

$$= \sum \frac{4\Delta^2}{4\sin^2\frac{A}{2}\cos^2\frac{A}{2}} \cdot \frac{\sin^2\frac{A}{2}}{16r^2} = \sum \frac{r^2 s^2}{16r^2\cos^2\frac{A}{2}} = \sum \frac{s^2}{16\cos^2\frac{A}{2}} = \sum \frac{s^2}{16\frac{s^2}{AI_a^2}} = \frac{\sum AI_a^2}{16}$$

[where I_a, I_b, I_c are the excenters of ΔABC]. So, we must prove that $\sum AI_a^2 \leq 27R^2$

But: AI_a, BI_b, CI_c are the altitudes of the excentral triangle $\Delta I_a I_b I_c$ of ΔABC (1)

and m'_a, m'_b, m'_c are the medians of $\Delta I_a I_b I_c$ and also, a', b', c' are the sides of $\Delta I_a I_b I_c$

(2) and $\because R'$ (the circumradius of $\Delta I_a I_b I_c$) $= 2R$ (3)

From (1)+(2)+(3)$\Rightarrow \sum AI_a^2 \overset{(h_a \leq m_a)}{\leq} \sum m_a'^2 = \frac{3}{4}\sum a'^2 \overset{Leibniz}{\leq} \frac{3}{4}\cdot 9R'^2$

$= \frac{3}{4}\cdot 9 \cdot 4R^2 = 27R^2 \therefore \sum AI_a^2 \leq 27R^2 \therefore$ (b) is true (Proved)

Equality holds in each side (b) and (a) randomly if ΔABC is equilateral.

Solution 3 by Tran Hong-Dong Thap-Vietnam

Using inequality: $m_a^2 \cdot m_b^2 \cdot m_c^2 \geq 3\sqrt{3}S^3$ (1)

We have: $\sum m_a^2 \sin^2\frac{A}{2} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod m_a^2 \sin^2\frac{A}{2}} \overset{(1)}{\geq} 3\sqrt[3]{3\sqrt{3}\cdot S^3\left(\prod \sin^2\frac{A}{2}\right)}$

$$= 3\sqrt[3]{3\sqrt{3}\cdot S^3 \cdot \left(\frac{r}{4R}\right)^2} = 3\sqrt[3]{3\sqrt{3}\cdot s^3 r^3 \cdot \left(\frac{r}{4R}\right)^2}$$

We must show that: $3\sqrt[3]{3\sqrt{3}\cdot \frac{s^3 \cdot r^5}{16R^2}} \geq \frac{27r^3}{2R} \leftrightarrow 3\sqrt{3}\cdot \frac{s^3\cdot r^5}{16R^2} \geq \left(\frac{9}{2}\right)^3 \cdot \frac{r^9}{R^3} \leftrightarrow Rs^3 \geq 162\sqrt{3}r^4$

It is true because: $\begin{cases} R \geq 2r \\ s \geq 3\sqrt{3}r \end{cases} \rightarrow Rs^3 \geq 2r\left(3\sqrt{3}r\right)^3 = 162\sqrt{3}r^4$

Suppose: $A \leq B \leq C \rightarrow \begin{cases} \sin^2\frac{A}{2} \leq \sin^2\frac{B}{2} \leq \sin^2\frac{C}{2} \\ m_a^2 \geq m_b^2 \geq m_c^2 \end{cases}$

We have: $\sum m_a^2 \sin^2\frac{A}{2} \overset{Chebyshev}{\leq} \frac{1}{3}\cdot\left(\sum m_a^2\right)\left(\sum \sin^2\frac{A}{2}\right)$

$$=\frac{1}{3}\cdot\frac{3}{4}\cdot\left(\sum a^2\right)\cdot\left(\sum \sin^2\frac{A}{2}\right)\overset{Leibniz}{\leq}\frac{1}{3}\cdot\frac{3}{4}\cdot 9\cdot R^2\cdot\left(\sum\sin^2\frac{A}{2}\right)\overset{\sum\sin^2\frac{A}{2}\leq\frac{3}{4}}{\leq}$$

$$\leq\frac{1}{3}\cdot\frac{3}{4}\cdot 9\cdot R^2\cdot\frac{3}{4}=\frac{27}{16}R^2.$$ ***Proved.***

JP.215. Prove that in any ABC triangle the following inequality holds:

$$(4R+r)^2\cdot\frac{r}{2R}\leq\sum m_a^2\cos^2\frac{A}{2}\leq(4R+r)^2\cdot\frac{1}{16}\left(5-\frac{2r}{R}\right)$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r(4R+r)^2}{2R}\leq\sum m_a^2\cos^2\frac{A}{2}\leq\frac{(4R+r)^2}{16}\left(5-\frac{2r}{R}\right)$$

$$\sum m_a^2\cos^2\frac{A}{2}=\sum m_a^2\left(1-\sin^2\frac{A}{2}\right)=\sum m_a^2-\sum\left[\left(\frac{2\sum a^2-3a^2}{4}\right)\sin^2\frac{A}{2}\right]$$

$$=\frac{3\sum a^2}{4}-\frac{\sum a^2}{4}\sum(1-\cos A)+\frac{3}{4}\sum\left[a^2\frac{(s-b)(s-c)(s-a)}{bc(s-a)}\right]$$

$$=\frac{3\sum a^2}{4}-\frac{\sum a^2}{4}\left(3-1-\frac{r}{R}\right)+\frac{3r^2s}{4}\sum\frac{a^2}{bc(s-a)}$$

$$\overset{(1)}{=}\frac{\sum a^2}{4}\left(\frac{R+r}{R}\right)+\frac{3r^2s}{4}\sum\frac{a^2}{bc(s-a)}$$

Now, $\sum\frac{a^2}{bc(s-a)}=\sum\frac{a^2-s^2+s^2}{bc(s-a)}=-\sum\frac{(s-a)(s+a)}{bc(s-a)}+\frac{s^2}{4Rrs}\sum\frac{a}{s-a}$

$$=-s\sum\frac{1}{bc}-\sum\frac{a^2}{4Rrs}+\frac{s^2}{4Rrs}\sum\frac{a-s+s}{s-a}$$

$$=-\frac{s(2s)}{4Rrs}-\frac{2(s^2-4Rr-r^2)}{4Rrs}+\frac{s^2}{4Rrs}\sum\left(-1+\frac{s\sum(s-b)(s-c)}{r^2s}\right)$$

$$=\frac{-2s^2+4Rr+r^2}{2Rrs}+\frac{s^2}{4Rrs}\left(-3+\frac{4R+r}{r}\right)=\frac{-2s^2+4Rr+r^2}{2Rrs}+\frac{s^2(2R-r)}{2Rr^2s}$$

$$\overset{(2)}{=}\frac{s^2(2R-3r)+r^2(4R+r)}{2Rr^2s}$$

(1), (2) $\Rightarrow\sum m_a^2\cos^2\frac{A}{2}=\frac{\sum a^2}{4}\left(\frac{R+r}{R}\right)+\frac{3r^2s}{4}\left[\frac{s^2(2R-3r)+r^2(4R+r)}{2Rr^2s}\right]$

$$=\frac{(R+r)(s^2-4Rr-r^2)}{2R}+\frac{3s^2(2R-3r)+3r^2(4R+r)}{8R}$$

$$=\frac{4(R+r)(s^2-4Rr-r^2)+3(2R-2r)s^2+3r^2(4R+r)}{8R}$$

$$=\frac{(10R-5r)s^2-4r(R+r)(4R+r)+3r^2(4R+r)}{8R}$$

$$\overset{(3)}{=}\frac{(10R-5r)s^2-r(4R+r)^2}{8R}$$

(3)$\Rightarrow \sum m_a^2\cos^2\frac{A}{2}\le\frac{(4R+r)^2}{16}\left(5-\frac{2r}{R}\right)$

$$\Leftrightarrow\frac{(10R-5r)s^2-s(4R+r)^2}{8R}\le\frac{(5R-2r)(4R+r)^2}{16R}$$

$\Leftrightarrow (4R-2r)s^2\overset{(i)}{\le}R(4R+r)^2$***. Now, Rouche*** $\Rightarrow$ ***LHS of (i)*** $\le$

$$(4R-2r)\left\{2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\right\}\overset{(?)}{\le}R(4R+r)^2$$

$$\Leftrightarrow(8R^3-28R^2r+25Rr^2-2r^3)\overset{?}{\ge}2(R-2r)(4R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow(R-2r)(8R^2-12Rr+r^2)\overset{?}{\ge}2(R-2r)(4R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow 8R^2-12Rr+r^2\overset{?}{>}2(4R-2r)\sqrt{R^2-2Rr}\quad\left(\because R-2r\overset{Euler}{\ge}0\right)$$

$$\Leftrightarrow(8R^2-12Rr-r^2)^2\overset{?}{>}4(R^2-2Rr)(4R-2r)^2\ (\because 8R^2-12Rr+r^2>0)$$

$\Leftrightarrow r^2(4R+r)^2\overset{?}{>}0\to$ ***true*** $\Rightarrow$ ***(i) is true*** $\therefore\sum m_a^2\cos^2\frac{A}{2}\le\frac{(4R+r)^2}{16}\left(5-\frac{2r}{R}\right)$

Again, (3)$\Rightarrow\frac{r(4R+r)^2}{2R}\le\sum m_a^2\cos^2\frac{A}{2}$

$$\Leftrightarrow\frac{(10R-5r)s^2-r(4R+r)^2}{8R}\ge\frac{r(4R+r)^2}{2R}$$

$$\Leftrightarrow(10R-5r)s^2\ge 5r(4R+r)^2\Leftrightarrow(2R-r)s^2\overset{(ii)}{\ge}r(4R+r)^2$$

Now, LHS of (ii) $\overset{Gerretsen}{\ge}(2R-r)(16Rr-5r^2)\overset{?}{\ge}r(4R+r)^2$

$\Leftrightarrow 8R^2-17Rr+2r^2\overset{?}{\ge}0\Leftrightarrow(R-2r)(8R-r)\overset{?}{\ge}0\to$ ***true*** $\because R\overset{Euler}{\ge}2r$

$\Rightarrow$ ***(ii) is true*** $\therefore\frac{r(4R+r)^2}{2R}\le\sum m_a^2\cos^2\frac{A}{2}$ ***(Proved)***

JP.216. Prove that in any ABC triangle the following inequality holds:

$$\frac{(4R+r)^2}{r(R+r)}(-2R^2+17r^2)\le\sum m_a^2\cot^2\frac{B}{2}\cot^2\frac{C}{2}\le\frac{3(4R+r)^2}{r^2(2R-r)}(R^3-5r^3)$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(4R+r)^2}{r(R+r)}(-2R^2+17r^2)\le\sum m_a^2\cot^2\frac{B}{2}\cot^2\frac{C}{2}\le\frac{3(4R+r)^2}{r^2(2R-r)}(R^3-5r^3)$$

Firstly, $\sum\sec^2\frac{A}{2}=\sum\frac{bc}{s(s-a)}=\frac{\sum bc(s-b)(s-c)}{r^2s^2}$

$$=\frac{\sum bc\,(s^2-s(b+c)+bc)}{r^2s^2}=\frac{s^2\sum ab-s\sum bc\,(2s-a)+(\sum ab)^2-2abc(2s)}{r^2s^2}$$

$$=\frac{-s^2\sum ab+(\sum ab)^2-4Rs^2}{r^2s^2}=\frac{(4R+r^2)(s^2+4Rr+r)-4Rrs^2}{r^2s^2}\overset{(1)}{=}\frac{s^2+(4R+r)^2}{s^2}$$

Secondly, $\sum(s-b)(s-c)=\sum\left(s^2-s(b+c)+bc\right)=3s^2-4s^2+s^2+4Rr+r^2\overset{(2)}{=}4Rr+r^2$

Now, $\sum m_a^2\cot^2\frac{B}{2}\cot^2\frac{C}{2}=\left(\frac{\prod\cos\frac{A}{2}}{\prod\sin\frac{A}{2}}\right)^2\sum m_a^2\tan^2\frac{A}{2}$

$$=\left(\frac{\frac{s}{4R}}{\frac{r}{4R}}\right)^2\sum m_a^2\left(\sec^2\frac{A}{2}-1\right)=\frac{s^2}{r^2}\left[\sum\left(\frac{2\sum a^2-3a^2}{4}\right)\sec^2\frac{A}{2}-\frac{3}{4}\sum a^2\right]$$

$$=\frac{s^2}{r^2}\left[\left(\frac{\sum a^2}{2}\right)\sum\sec^2\frac{A}{2}-\frac{3}{4}\sum a^2-\frac{3}{4}\sum\frac{a^2bc}{s(s-a)}\right]$$

$$\overset{by\,(1)}{=}\frac{s^2}{r^2}\left[\left(\frac{\sum a^2}{2}\right)\left(\frac{s^2+(4R+r)^2}{s^2}\right)-\frac{3}{4}\sum a^2-\left(\frac{3\cdot 4Rrs}{4s}\right)\sum\frac{a-s+s}{s-a}\right]$$

$$=\frac{s^2}{r^2}\left[\left(\frac{\sum a^2}{2}\right)\left(\frac{s^2+(4R+r)^2}{s^2}\right)-\frac{3}{4}\sum a^2-3Rr\sum\left(-1+\frac{s}{r^2s}\sum(s-b)(s-c)\right)\right]$$

$$\overset{by\,(2)}{=}\frac{s^2}{r^2}\left[\left(\frac{\sum a^2}{2}\right)\left(\frac{s^2+(4R+r)^2}{s^2}\right)-\frac{3}{4}\sum a^2-3Rr\left(-3+\frac{4R+r}{r}\right)\right]$$

$$=\frac{s^2}{r^2}\left[\left(\frac{\sum a^2}{2}\right)\left(\frac{s^2+(4R+r)^2}{s^2}\right)-\frac{3}{4}\sum a^2-3R(4R-2r)\right]$$

$$=\frac{s^2}{r^2}\left[(s^2-4Rr-r^2)\left(\frac{s^2+(4R+r)^2}{s^2}\right)-\frac{3}{2}(s^2-4Rr-r^2)-3R(4R-2r)\right]$$

$$= \frac{1}{2r^2}[2(s^2-4Rr-r^2)(s^2+(4R+r)^2-3s^2(s^2-4Rr-r^2)-6Rs^2(4R-2r)]$$

$$\overset{(3)}{=} \frac{-s^4+s^2(8R^2+32Rr+3r^2)-r(128R^3+96R^2r+24Rr^2+2r^3)}{2r^2}$$

$$\therefore \frac{(4R+r)^2}{r(R+r)}(-2R^2+17r^2) \le \sum m_a^2 \cot^2\frac{B}{2}\cot^2\frac{C}{2}$$

$$\overset{by\ (3)}{\Leftrightarrow} \frac{-s^4+s^2(8R^2+32Rr+3r^2)-r(128R^3+96R^2r+24Rr^2+2r^3)}{2r^2}+$$

$$+\frac{(4R+r)^2}{r(R+r)}(2R^2-17r^2) \ge 0$$

$$\Leftrightarrow \frac{1}{2r^2(R+r)}\begin{bmatrix}(R+r)\{-s^4+s^2(8R^2+32Rr+3r^2)-r(128R^3+96R^2r+24Rr^2+2r^3)\}\\+2r(2R^2-17r^2)(4R+r)^2\end{bmatrix} \ge 0$$

$$\Leftrightarrow (R+r)s^4+r(64R^4+192R^3r+660R^2r^2+298Rr^3+36r^4)$$

$$\overset{(i)}{\le} s^2(8R^3+40R^2r+35Rr^2+3r^3)$$

Now, LHS of (i) $\overset{Gerretsen}{\le} (R+r)(4R^2+4Rr+3r^2)s^2+$

$$+r(64R^4+192R^3r+660R^2r^2+298Rr^3+36r^4) \le$$

$$\overset{(i)}{\le} s^2(8R^3+40R^2r+35Rr^2+3r^3)$$

$$\Leftrightarrow s^2(4R^3+32R^2r+28Rr^2) \underset{(ii)}{\overset{?}{\ge}} r(64R^4+192R^3r+660R^2r^2+298Rr^3+36r^4)$$

Now, LHS of (ii) $\overset{Gerretsen}{\ge} (16Rr-5r^2)(4R^3+32R^2r+28Rr^2) \ge$

$$\overset{?}{\ge} r(64R^4+192R^3r+660R^2r^2+298Rr^3+36r^4)$$

$\Leftrightarrow 50t^3-62t^2-73t-6 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(50t^2+38t+3) \overset{?}{\ge} 0 \to$ ***true***

$\because t \overset{Euler}{\ge} 2 \Rightarrow$ ***(ii)*** $\Rightarrow$ ***(i) is true***

$$\therefore \frac{(4R+r)^2}{r(R+r)}(-2R^2+17r^2) \le \sum m_a^2 \cot^2\frac{B}{2}\cot^2\frac{C}{2}$$

Again, $\sum m_a^2 \cot^2\frac{B}{2}\cot^2\frac{C}{2} \le \frac{3(4R+r)^2}{r^2(2R-r)}(R^3-5r^3) \overset{by\ (3)}{\Leftrightarrow}$

$$\frac{-s^4+s^2(8R^2+32Rr+3r^2)-r(128R^3+96R^2r+24Rr^2+2r^3)}{2r^2}-$$

$$-\frac{3(4R+r)^2}{r^2(2R-r)}(R^3-5r^3)\le 0$$

$$\Leftrightarrow (2R-r)\left(-s^4+s^2(8R^2+32Rr+3r^2)-r(128R^3+96R^2r+24Rr^2+2r^3)\right)$$

$$-6(R^3-5r^3)(4R+r)^2\le 0 \Leftrightarrow 6(R^3-5r^3)(4R+r)^2+(2R-r)s^4$$

$$+r(2R-r)(128R^3+96R^2r+24Rr^2+2r^3)\overset{(iii)}{\ge}(2R-r)(8R^2+32Rr+3r^2)s^2$$

Now, LHS of (iii) $\overset{Gerretsen}{\ge} 6(R^3-5r^3)(4R+r)^2+(2R-r)(16Rr-5r^2)s^2$

$$+r(2R-r)(128R^3+96R^2r+24Rr^2+2r^3)\overset{?}{\ge}(2R-r)(8R^2+32Rr+3r^2)s^2$$

$$\Leftrightarrow s^2(2R-r)(8R^2+16Rr+8r^2)\underset{(iv)}{\overset{?}{\le}}6(R^3-5r^3)(4R+r)^2+$$

$$+r(2R-r)(128R^3+96R^2r+24Rr^2+2r^3)$$

Now, LHS of (iv) $\overset{Gerretsen}{\le}(4R^2+4Rr+3r^2)(2R-r)(8R^2+16Rr+8r^2)$

$$\overset{?}{\le}6(R^3-5r^3)(4R+r)^2+r(2R-r)(128R^3+96R^2r+24Rr^2+2r^3)$$

$$\Leftrightarrow 16t^5+72t^4-37t^3-284t^2-114t-4\overset{?}{\ge}0$$

$\Leftrightarrow (t-2)(16t^4+104t^3+171t^2+58t+2)\overset{?}{\ge}0 \to$ ***true*** $\because t\overset{Euler}{\ge}2 \Rightarrow$***(iv)***$\Rightarrow$ ***(iii)***

is true $\therefore \sum m_a^2\cot^2\frac{B}{2}\cot^2\frac{C}{2}\le\frac{3(4R+r)^2}{r^2(2R-r)}(R^3-5r^3)$ ***(Proved)***

JP.217. Prove that in any ABC triangle the following inequality holds:

$$n\sum\sin^2 A-k\sum\cos^3 A\le\frac{3}{8}(6n-k),\ \text{where } n,k\ge 0$$

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$n\sum\sin^2 A-k\sum\cos^3 A=(n-k)\sum\sin^2 A+k\sum\sin^2 A-k\sum\cos A(1-\sin^2 A)$$

$$=(n-k)\sum\sin^2 A-k\sum\cos A+k\sum\sin^2 A(1+\cos A)$$

$$=n\sum\sin^2 A-k\sum\sin^2 A-k\sum\cos A+2k\sum\sin^2 A\cos^2\frac{A}{2}$$

$$= n\left(\frac{\sum a^2}{4R^2}\right) - k\sum \sin^2 A - k\sum \cos A + 2k\sum \sin^2 A\cos^2\frac{A}{2}$$

$$\overset{Leibnitz}{\leq} \left(\frac{9R^2}{4R^2}\right)n - k\sum \sin^2 A - k\sum \cos A + 2k\sum \sin^2 A\cos^2\frac{A}{2}\ \ (\because n \geq 0)$$

$$= \frac{9n}{4} - k\sum \sin^2 A - k\sum \cos A + 2k\sum \sin^2 A\cos^2\frac{A}{2} \overset{?}{\leq} \frac{3}{8}(6n-k) = \frac{9n}{4} - \frac{3k}{8}$$

$$\Leftrightarrow k\left\{2\sum \sin^2 A\cos^2\frac{A}{2} - \left(\sum \sin^2 A + \sum \cos A - \frac{3}{8}\right)\right\} \overset{?}{\leq} 0$$

$$\Leftrightarrow 2\sum \sin^2 A\cos^2\frac{A}{2} \underset{(1)}{\overset{?}{\leq}} \sum \sin^2 A + \sum \cos A - \frac{3}{8}\ (\because k \geq 0)$$

Now, $2\sum \sin^2 A\cos^2\frac{A}{2} = 2\sum\frac{a^2}{4R^2}\cdot\frac{s(s-a)}{bc} = \frac{2s}{4R^2\cdot 4Rrs}\sum a^3(s-a) \overset{(a)}{\leq} \frac{s\sum a^3 - \sum a^4}{8R^3 r}$

Now, $(\sum a^3)(\sum a) = \sum a^4 + \sum a^3 b + \sum ab^3$

$$\Rightarrow -\sum a^4 = -2s\cdot\sum a^3 + \sum ab\left(\sum a^2 - c^2\right)$$

$$\Rightarrow s\sum a^3 - \sum a^4 = -2s^2(s^2 - 6Rr - 3r^2) + \sum ab\cdot\sum a^2 - 4Rrs(2s)$$

$$= 2r\{s^2(2R+3r) - r(4R+r)^2\}$$

$\Rightarrow \frac{s\sum a^3 - \sum a^4}{8R^3 r} \overset{(b)}{=} \frac{s^2(2R+3r) - r(4R+r)^2}{4R^3}$ $\Rightarrow$ *LHS of (1)* $\overset{(i)}{\leq} \frac{s^2(2R+3r) - r(4R+r)^2}{4R^3}$

(using (a), (b))

Again, RHS of (1) $= \frac{\sum a^2}{4R^2} + \frac{R+r}{R} - \frac{3}{8} = \frac{4(s^2-4Rr-r^2)+8R(R+r)-3R^2}{8R^2}$

$$\overset{(ii)}{=} \frac{4s^2 + 5R^2 + 8Rr - 4(4Rr + r^2)}{8R^2}$$

(i), (ii)⇒ in order to prove (1), it is equivalent to proving:

$$\frac{s^2(2R+3r)}{4R^3} + \frac{4(4Rr+r^2)}{8R^2} \leq \frac{r(4R+r)^2}{4R^3} + \frac{4s^2 + 5R^2 + 8Rr}{8R^2}$$

$$\Leftrightarrow 2s^2(2R+3r) + 4R(4Rr + r^2) \leq 2r(4R+r)^2 + R(4s^2 + 5R^2 + 8Rr)$$

$$\Leftrightarrow s^2\cdot 6r \overset{(2)}{\leq} 5R^3 + 8R^2 r + 2r(4R+r)^2 - 4R(4Rr + r^2)$$

Now, LHS of (2) $\overset{Gerretsen}{\leq} 6r(4R^2 + 4Rr + 3r^2)$

$$\overset{?}{\leq} 5R^3 + 8R^2 r + 2r(4R+r)^2 - 4R(4Rr + r^2) \Leftrightarrow 5t^3 - 12t - 16 \overset{?}{\geq} 0\ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow 5t^3 - 20t + 8t - 16 \overset{?}{\geq} 0$$

$$\Leftrightarrow 5t(t+2)(t-2) + 8(t-2) \overset{?}{\geq} 0 \rightarrow \textit{true} \because t \overset{Euler}{\geq} 2$$

$\Rightarrow$ ***(2)*** $\Rightarrow$ ***(1)*** $\Rightarrow$ ***given inequality is true (Proved)***

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$n(\sin^2 A + \sin^2 B + \sin^2 C) - k(\cos^3 A + \cos^3 B + \cos^3 C) \overset{?}{\leq} \frac{3}{8}(6n-k)$$

$$n(\sin^2 A + \sin^2 B + \sin^2 C) + k(-\cos^3 A - \cos^3 B - \cos^3 C) \overset{?}{\leq} \frac{3}{8}(6n-k)$$

Let be the function: $f(x) = -\cos^3 x, f'(x) = 3\sin x\cos^2 x \geq 0$

So, f ***is a convex function and hence by using Popoviciu's inequality:***

$$\frac{1}{3}(-\cos^3 A - \cos^3 B - \cos^3 C) - \cos^3\left(\frac{A+B+C}{3}\right) \geq$$

$$\geq \frac{2}{3}\left(-\cos^3\left(\frac{A+B}{2}\right) - \cos^3\left(\frac{B+C}{2}\right) - \cos^3\left(\frac{A+C}{2}\right)\right)$$

$$-\frac{1}{3}(\cos^3 A + \cos^3 B + \cos^3 C) - \frac{1}{8} \geq -\frac{2}{3}\left(\sin^3\frac{C}{2} + \sin^3\frac{A}{2} + \sin^3\frac{B}{2}\right)$$

$$\frac{1}{3}(\cos^3 A + \cos^3 B + \cos^3 C) + \frac{1}{8} \geq \frac{2}{3}\left(\sin^3\frac{C}{2} + \sin^3\frac{A}{2} + \sin^3\frac{B}{2}\right)$$

$$\cos^3 A + \cos^3 B + \cos^3 C + \frac{3}{8} \geq 2\left(\sin^3\frac{C}{2} + \sin^3\frac{A}{2} + \sin^3\frac{B}{2}\right)$$

$$\cos^3 A + \cos^3 B + \cos^3 C \geq -\frac{3}{8} + 2\left(\sin^3\frac{C}{2} + \sin^3\frac{A}{2} + \sin^3\frac{B}{2}\right)$$

but: $\sin^3\frac{C}{2} + \sin^3\frac{A}{2} + \sin^3\frac{B}{2} \geq \frac{3}{8}$

$$2\left(\sin^3\frac{C}{2} + \sin^3\frac{A}{2} + \sin^3\frac{B}{2}\right) \geq \frac{6}{8}$$

$$\left(\begin{array}{l}\sin^3\frac{A}{2} + \sin^3\frac{C}{2} + \sin^3\frac{B}{2} \geq 3\sqrt[3]{\sin^3\frac{A}{2}\sin^3\frac{B}{2}\sin^3\frac{C}{2}} \\ \sin^3\frac{A}{2} + \sin^3\frac{C}{2} + \sin^3\frac{B}{2} \geq 3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \geq \frac{3}{8}\end{array}\right)$$

$$\cos^3 A + \cos^3 B + \cos^3 C \geq \frac{3}{8}$$

$$-k(\cos^3 A + \cos^3 B + \cos^3 C) \leq -\frac{3k}{8} \quad \textbf{\textit{(1)}}$$

$$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{3}\le\left(\frac{\sin A+\sin B+\sin C}{3}\right)^2 \quad \textit{AM-GM}$$

$$\sin^2 A+\sin^2 B+\sin^2 C\le\frac{1}{3}(\sin A+\sin B+\sin C)^2$$

As you know: $\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}$. *So:* $(\sin A+\sin B+\sin C)\le\frac{27}{4}$

$$\sin^2 A+\sin^2 B+\sin^2 C\le\frac{1}{3}\cdot\frac{27}{4}\Rightarrow\sin^2 A+\sin^2 B+\sin^2 C\le\frac{9}{4}$$

$$n(\sin^2 A+\sin^2 B+\sin^2 C)\le\frac{9n}{4} \quad (2)$$

From (1) and (2):

$$n(\sin^2 A+\sin^2 B+\sin^2 C)-k(\cos^3 A+\cos^3 B+\cos^3 C)\le\frac{9n}{4}-\frac{3k}{8}$$

$$n(\sin^2 A+\sin^2 B+\sin^2 C)-k(\cos^3 A+\cos^3 B+\cos^3 C)\le\frac{3}{8}(6n-k)$$

JP.218. Let a, b and c be positive real numbers. Prove that:

(a) $\frac{a^4+b^4}{(a^2-ab+b^2)^2}+\frac{b^4+c^4}{(b^2-bc+c^2)^2}+\frac{c^4+a^4}{(c^2-ca+a^2)^2}\le 6$

(b) $\sqrt{\frac{a^5+b^5}{a^2+b^2}}+\sqrt{\frac{b^5+c^5}{b^2+c^2}}+\sqrt{\frac{c^5+a^5}{c^2+a^2}}\ge 3\sqrt{abc}$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Ravi Prakash-New Delhi-India

$$2-\frac{a^4+b^4}{(a^2-ab+b^2)^2}=\frac{2(a^4+a^2b^2+b^4-2a^3b-2ab^3+2a^2b^2)-(a^4+b^4)}{(a^2-ab+b^2)^2}=$$

$$=\frac{a^4-4a^3b+6a^2b^2-4ab^3+b^4}{(a^2-ab+b^2)^2}=\frac{(a-b)^4}{(a^2-ab+b^2)^2}\ge 0\Rightarrow\frac{a^4+b^4}{(a^2-ab+b^2)^2}\le 2$$

Similarly, for other two expressions. Thus: $\sum\frac{a^4+b^4}{(a^2-ab+b^2)^2}\le 6$

Equality when $a=b=c$.

Consider $\frac{a^5+b^5}{a^2+b^2}-\frac{1}{2}(a^3+b^3)=\frac{2a^5+2b^5-(a^5+a^2b^3+a^3b^2+b^5)}{2(a^2+b^2)}$

$$=\frac{a^2(a^3-b^3)+b^2(b^3-a^3)}{2(a^2+b^2)}=\frac{(a^2-b^2)(a^3-b^3)}{2(a^2+b^2)}\ge 0$$

$$\Rightarrow \sqrt{\frac{a^5+b^5}{a^2+b^2}} \geq \sqrt{\frac{a^3+b^3}{2}} \geq (ab)^{\frac{3}{2}} \Rightarrow \sum_{cyc}\sqrt{\frac{a^5+b^5}{a^2+b^2}} \geq \sum_{cyc}(ab)^{\frac{3}{2}} \geq 3\sqrt{abc}$$

Equality when $a=b=c$.

Solution 2 by Marian Ursărescu-Romania

(a) First, we show: $\frac{a^4+b^4}{(a^2-ab+b^2)^2} \leq 2$ ***(1)***

$$\Leftrightarrow a^4+b^4 \leq 2(a^2-ab+b^2)^2 \Leftrightarrow a^4+b^4 \leq 2(a^2+b^2)^2-4ab(a^2+b^2)+2a^2b^2 \Leftrightarrow$$

$$\Leftrightarrow a^4+b^4+6a^2b^2-4ab(a^2+b^2) \geq 0$$

$$\Leftrightarrow (a^2+b^2)^2-4ab(a^2+b^2)+4a^2b^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (a^2+b^2-2ab)^2 \geq 0 \Leftrightarrow (a-b)^4 \geq 0$ ***true. From (1)*** $\Rightarrow \sum \frac{a^4+b^4}{(a^2-ab+b^2)^2} \leq 6$

(b) We show this: $\sqrt{\frac{a^5+b^5}{a^2+b^2}} \geq \sqrt{ab\sqrt{ab}}$ ***(2)***

$\Leftrightarrow \frac{a^5+b^5}{a^2+b^2} \geq ab\sqrt{ab}$ ***(3)***

But $a^5+b^5 \geq ab(a^3+b^3)$ ***(4) (because*** $\Leftrightarrow a^5-a^4b+b^5-ab^4 \geq 0$***)***

$$\Leftrightarrow a^4(a-b)-b^4(a-b) \geq 0 \Leftrightarrow (a-b)(a^4-b^4) \geq 0 \Leftrightarrow$$

$\Leftrightarrow (a-b)^2(a+b)(a^2+b^2) \geq 0$ ***which it is true.***

From (3) and (4) we must show: $\frac{ab(a^3+b^3)}{a^2+b^2} \geq ab\sqrt{ab} \Leftrightarrow$

$\Leftrightarrow a^3+b^3 \geq \sqrt{ab}(a^2+b^2)$ ***(5)***

$a^3+b^3=(a+b)(a^2-ab+b^2) \geq 2\sqrt{ab}(a^2-ab+b^2)$ ***(6)***

From (5)+(6)$\Rightarrow 2\sqrt{ab}(a^2-ab+b^2) \geq \sqrt{ab}(a^2+b^2) \Leftrightarrow$

$\Leftrightarrow 2(a^2-ab+b^2) \geq a^2+b^2 \Leftrightarrow a^2-2ab+b^2 \geq 0 \Leftrightarrow (a-b)^2 \geq 0$ ***true.***

From (2)$\Rightarrow \sum\sqrt{\frac{a^5+b^5}{a^2+b^2}} \geq \sqrt{ab\sqrt{ab}}+\sqrt{bc\sqrt{bc}}+\sqrt{ac\sqrt{ac}} \geq$

$$\geq 3\sqrt[3]{\sqrt{a^2b^2c^2\sqrt{a^2b^2c^2}}} = 3\sqrt{abc}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

1) For $x,y>0$, we get:

$$(x-y)^2(x^2+x+y^2)\ge 3xy(x-y)^2\Rightarrow (x-y)(x^3-y^3)\ge 3xy(x-y)^2$$

$$\Rightarrow x^3(x-y)+y^3(y-x)\ge 3xy(x-y)^2$$

$$\Rightarrow x^4-x^3y+y^4-y^3x\ge 3(x^3y+y^3x-2x^2y^2)$$

$$\Rightarrow x^4+y^4+6x^2y^2\ge 4(x^3y+y^3x)$$

$$\Rightarrow 2(x^4+y^4)+6x^2y^2-4(x^3y+y^2x)\ge x^6+y^4\Rightarrow \frac{x^4+y^4}{(x^2-xy+y^2)^2}\le 2$$

Hence for $a,b,c>0$ ***we have:*** $\frac{a^4+b^4}{(a^2-ab+b^2)^2}+\frac{b^4+c^4}{(b^2-bc+c^2)^2}+\frac{c^4+a^4}{(c^2-ca+a^2)^2}=2+2+2=6$ ***ok***

2) For $a,b,c>0$, ***we know:*** $\sqrt{\frac{a^5+b^5}{a^2+b^2}}+\sqrt{\frac{b^5+c^5}{b^2+c^2}}+\sqrt{\frac{c^5+a^5}{c^2+a^2}}\ge$

$$\ge\sqrt{\frac{(a^3+b^3)(a^2+b^2)}{2(a^2+b^2)}}+\sqrt{\frac{(b^3+c^3)(b^2+c^2)}{2(b^2+c^2)}}+\sqrt{\frac{(c^3+a^3)(c^2+a^2)}{2(c^2+a^2)}}$$

$$=\sqrt{\frac{a^3+b^3}{2}}+\sqrt{\frac{b^3+c^3}{2}}+\sqrt{\frac{c^3+a^3}{2}}\ge 3\sqrt[6]{\frac{(a^3+b^3)(b^3+c^3)(c^3+a^3)}{8}}\ge 3\sqrt{abc}$$

Iff $\frac{(a^3+b^3)(b^3+c^3)(c^3+b^2)}{8}\ge (abc)^3$

and it is true because $\frac{(a^3+b^3)(b^3+c^3)(c^3+a^3)}{8}\ge\frac{(abc+abc)^3}{8}=\frac{(2abc)^3}{8}=(abc)^3$ ***ok***

Therefore, it is true.

JP.219. Let be $a,b,c>0$ such that: $a^2b^2+b^2c^2+c^2a^2=3a^2b^2c^2$. Find the maximum value of:

$$P=\frac{ab}{2a^6-a^5+b^4+a^2+1}+\frac{bc}{2b^6-b^5+c^4+b^2+1}+\frac{ca}{2c^6-c^5+a^4+c^2+1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Amit Dutta-Jamshedpur-India

First of all, we need to minimize the denominators.

i.e., $2a^6-a^5+b^4+a^2+1$ ***so,*** $2a^6-a^5+a^2+1=(a^6+1)+(a^6-a^5+a^2)$

Now, $a^6+1\overset{AM-GM}{\ge}2a^3$. ***Equality holds when*** $a=1$.

$$a^6-a^5+a^2=a^6-a^2(a^3-1)=a^6-a^3+a^3-a^2(a^3-1)=$$

$$= a^3(a^3-1) - a^2(a^3-1) + a^3 = (a^3-a^2)(a^3-1) + a^3$$

$$a^6 - a^5 + a^2 = a^2(a-1)^2(a^2+a+1) + a^3 \geq a^3$$

Equality holds when $a = 1.$ $\therefore a^6 - a^5 + a^2 \geq a^3$

so, $2a^6 - a^5 + a^2 + 1 = (a^6+1) + \left(a^6 - a^5 + a^2\right) \geq 2a^3 + a^3 \geq 3a^3$

$$\therefore 2a^6 - a^5 + b^4 + a^2 + 1 \geq 3a^3 + b^4 \geq a^3 + a^3 + a^3 + b^4 \overset{AM-GM}{\geq} 4ba^{\frac{9}{4}}$$

$$\therefore P = \sum_{cyc(a,b,c)} \frac{ab}{2a^6 - a^5 + b^5 + a^2 + 1}$$

$$P \leq \sum_{cyc} \frac{ab}{4ba^{\frac{9}{4}}} = \sum_{cyc} \frac{1}{4a^{\frac{5}{4}}}$$

$$P \leq \sum_{cyc} \frac{1}{4aa^{\frac{1}{4}}} = \sum_{cyc} \frac{1 \times 1}{4a \cdot 4}\left\{\frac{1}{a} + 1 + 1 + 1\right\}$$

$$P \leq \sum_{cyc} \frac{1 \times 1}{4a \times 4}\left(\frac{1}{a} + 3\right) \leq \sum_{cyc} \frac{1}{16a}\left(\frac{1}{a} + 3\right)$$

$$P \leq \frac{1}{16}\left\{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right\} + \frac{3}{16}\left\{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right\}$$

$$\because a^2b^2 + b^2c^2 + c^2a^2 = 3a^2b^2c^2$$

$$\therefore \tfrac{1}{a^2} + \tfrac{1}{b^2} + \tfrac{1}{c^2} = 3 \quad \textit{(1)}$$

Using Cauchy's Schwarz inequality: $\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)(1^2 + 1^2 + 1^2) \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2$

$$9 \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2$$

$$\therefore \tfrac{1}{a} + \tfrac{1}{b} + \tfrac{1}{c} \leq 3 \quad \textit{(2)}$$

$$\therefore P \leq \frac{1}{16}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{3}{16}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Using (1) and (2): $P \leq \frac{1}{16} \times 3 + \frac{3}{16} \times 3$

$$P \leq \frac{3}{16} + \frac{9}{16} \leq \frac{12}{16} \leq \frac{3}{4};\ P \leq \frac{3}{4}$$

Equality holds when $a = b = c = 1.$

$$\therefore P_{\max} = \frac{3}{4}$$

JP.220. Let a,b,c be positive real numbers. Prove that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\ge\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Bogdan Fustei-Romania

$$\Rightarrow\frac{1}{2}\sum\frac{b+c}{a}=\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)=\frac{1}{2}\sum\frac{a^2+b^2}{ab}\overset{Bergstrom}{\ge}\frac{\left(\sum\sqrt{a^2+b^2}\right)^2}{2(ab+bc+ac)}=$$

$$=\frac{a^2+b^2+c^2+\sum\sqrt{(a^2+b^2)(a^2+c^2)}}{ab+bc+ac}\ge\frac{(a^2+b^2+c^2)+\sum(a^2+bc)}{ab+bc+ac}=$$

$$=\frac{2(a^2+b^2+c^2)}{ab+bc+ac}+1\ge\frac{2(a^2+b^2+c^2)}{ab+bc+ac}+\frac{ab+bc+ac}{a^2+b^2+c^2};\ a^2+b^2+c^2\ge ab+bc+ac\text{ – true}$$

$$\Rightarrow\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\ge\frac{4(a^2+b^2+c^2)}{ab+bc+ac}+\frac{2(ab+bc+ac)}{a^2+b^2+c^2}.\ \text{Q.E.D.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $b+c=x,c+a=y,a+b=z$. Then, x,y,z are sides of a triangle with semi-perimeter, circumradius, inradius $=s,R,r$ (say)

$$\because 2\sum a=\sum x=2s,\therefore a=s-x,b=s-y,c=s-z$$

Now, $\sum a^2=\sum(s^2-2sx+x^2)=3s^2-2s(2s)+2(s^2-4Rr-r^2)\overset{(1)}{=}s^2-8Rr-2r^2$

and $\sum ab=\sum(s-x)(s-y)=\sum(s^2-s(x+y)+xy)$

$$=3s^2-4s^2+s^2+4Rr+r^2\overset{(2)}{=}4Rr+r^2$$

Also, $\sum\frac{b+c}{a}=\sum\frac{x}{s-x}=\sum\frac{x-s+s}{s-x}=-3+\frac{s}{r^2s}\sum(s^2-s(y+z)+yz)$

$$=-3+\frac{3s^2-4s^2+s^2+4Rr+r^2}{r^2}=-3+\frac{4R+r}{r}\overset{(3)}{=}\frac{4R-2r}{r}$$

(1), (2), (3) ⇒ given inequality ⇔

$$\frac{2R-r}{r}\ge\frac{2(s^2-8Rr-2r^2)}{4Rr+r^2}+\frac{4Rr+r^2}{s^2-8Rr-2r^2}=\frac{2(s^2-8Rr-2r^2)^2+(4Rr+r^2)^2}{(4Rr+r^2)(s^2-8Rr-2r^2)}$$

$$\Leftrightarrow(2R-r)(4R+r)(s^2-8Rr-2r^2)\ge2(s^2-8Rr-2r^2)^2+(4Rr+r^2)^2$$

$$\Leftrightarrow2s^4-s^2(8R^2+30Rr+7r^2)+64R^3r+144R^2r^2+60Rr^3+7r^4\overset{(4)}{\le}0$$

Now, Rouche $\Rightarrow s^2 \ge m-n \Rightarrow s^2-m+n \overset{(a)}{\ge} 0$ *and* $s^2 \le m+n \Rightarrow s^2-m-n \overset{(b)}{\le} 0$,

where $m = 2R^2+10Rr-r^2$ *and* $n = 2(R-2r)\sqrt{R^2-2Rr}$

(a). (b)$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0$

$$\Rightarrow 2s^4 - s^2(8R^2+40Rr-4r^2) + 128R^3r + 96R^2r^2 + 24Rr^3 + 2r^4 \overset{(i)}{\le} 0$$

(4), (i) $\Rightarrow$ *it suffices to prove:*

$$2s^4 - s^2(8R^2+30Rr+7r^2) + 64R^3r + 144R^2r^2 + 60Rr^3 + 7r^4 \le$$

$$\le 2s^4 - s^2(8R^2+40Rr-4r^2) + 128R^3r + 96R^2r^2 + 24Rr^3 + 2r^4 \Leftrightarrow$$

$$\Leftrightarrow s^2(10Rr-11r^2) \overset{(5)}{\le} r(64R^3-48R^2r-36Rr^2-5r^3)$$

Now, LHS of (5) $\overset{Gerretsen}{\le} (4R^2+4Rr+3r^2)(10Rr-11r^2)$

$$\overset{?}{\le} r(64R^3-48R^2r-36Rr^2-5r^3) \Leftrightarrow 12t^3-22t^2-11t+14 \overset{?}{\ge} 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{(t-2)(12t+26)+45\} \overset{?}{\ge} 0$$

$\rightarrow$ *true* $\because t \overset{Euler}{\ge} 2 \Rightarrow$ *(5)* $\Rightarrow$ *(4)* $\Rightarrow$ *given inequality is true (Proved)*

Solution 3 by Tran Hong-Dong Thap-Vietnam

Let $p = a+b+c; q = ab+bc+ca; r = abc \quad (p,q,r>0)$. *Inequality:*

$$\leftrightarrow [bc(b+c)+ca(c+a)+ab(a+b)](a^2+b^2+c^2)(ab+bc+ca) \ge$$

$$\ge abc[4(a^2+b^2+c^2)^2 + 2(ab+bc+ca)^2]$$

$$\leftrightarrow [pq-3r][p^2-2q]q \ge r[4(p^2-2q)^2+2q^2]$$

$$\leftrightarrow p^3q^2 - 3qrp^2 - 2pq^3 + 6q^2r \ge r(4p^4 - 16p^2q + 18q^2)$$

$$\leftrightarrow p^3q^2 + 13qrp^2 - 2pq^3 - 12q^2r - 4p^2r \ge 0$$

$$\leftrightarrow (p^3q^2 - 3p^4r) + (13qrp^2 - 2pq^3 - 12q^2r - p^4r) \ge 0$$

$$\leftrightarrow p^3(q^2-3pr) + 4qr(p^2-3q) + p(9pqr - 2q^3 - p^3r) \ge 0$$

It is true because: $q^2 - 3qr \ge 0;\ p^2 - 3q \ge 0$

$9pqr - 2q^3 - p^3r \ge 0$ *(1)*

By Schur's inequality: $9r \ge 4pq - p^3 \rightarrow 9pqr \ge 4(pq)^2 - pq\cdot p^3$

(1) is true because: $(pq)^2 \ge 3q^3 \leftrightarrow p^2 \ge 3q$

$3(pq)^2 - pq\cdot p^3 + q^2 - p^3r \ge 0$ *(*)*

(*) is true, because: $\frac{q^2}{3}\ge pr \to \frac{(pq)^2}{3}\ge p^3r$

$$\frac{2(pq)^2}{3}-pq\cdot p^3+q^2\ge 0 \leftrightarrow 2(pq)^2-3pq\cdot p^3+3q^2\ge 0$$

Solution 4 by Anant Bansal-India

$$\sum_{cyc}\frac{a+b}{c}=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3\overset{AM\ge HM}{\ge}9-3=6\quad (i)$$

Let x,y ***be two positive real numbers: By QM≥*** AM: $\sqrt{32x^4+8y^4}\ge 4x^2+2y^2=k$

Maximum k ***stands for*** $4x^2=2y^2;y=x\sqrt{2}$

Maximum value of $k=8$.

Maximum value of $\frac{4x^2+2y^2}{xy}=\frac{8}{\sqrt{2}}<6$

Putting $x=a^2+b^2+c^2$ ***and*** $y=ab+bc+ca$

we get $6\ge\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{(a^2+b^2+c^2)}$

From (i): $\Rightarrow\sum_{cyc}\frac{a+b}{c}\ge\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{(a^2+b^2+c^2)}$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$***, we have:*** $\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)(ab+bc+ca)(a^2+b^2+c^2)$

$$=\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)(ab+bc+ca)(a^2+b^2+c^2)$$

$$=2(a^2+b^2+c^2)(a^2+b^2+c^2)+$$

$$+\left(\frac{a^2b}{c}+\frac{a^2c}{b}+\frac{b^2c}{a}+\frac{b^2a}{c}+\frac{c^2a}{b}+\frac{c^2b}{a}\right)(a^2+b^2+c^2)+2(ab+bc+ca)(a^2+b^2+c^2)$$

$$\ge 2(a^2+b^2+c^2)^2+2(a^2+b^2+c^2)(a^2+b^2+c^2)+2(ab+bc+ca)(ab+bc+ca)$$

$$=4(a^2+b^2+c^2)^2+2(ab+bc+ca)^2$$

Hence $\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\ge\frac{4(a^2+b^2+c^2)^2}{(ab+bc+ca)(a^2+b^2+c^2)}+\frac{2(ab+bc+ca)^2}{(ab+bc+ca)(a^2+b^2+c^2)}$

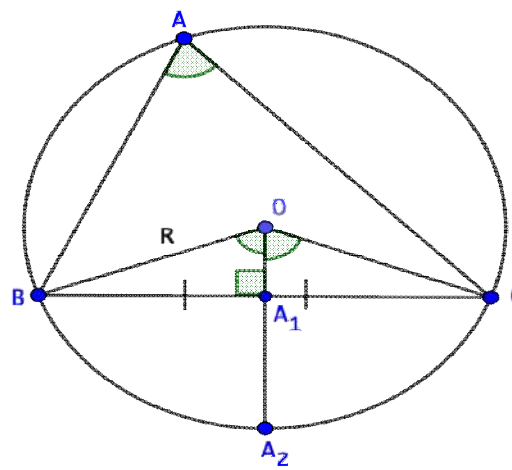
$=\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$***. Therefore, it is true.***

JP.221. Let ABC be an acute-angled triangle. The perpendiculars from O on ΔABC sides, intersect BC, AC and AB sides in A_1, A_2, A_3 and the circumcircle of ΔABC in the points A_2, B_2, C_2. Prove that:

$$A_1A_2^n + B_1B_2^n + C_1C_2^n \geq 3r^n, \forall n \in \mathbb{N}^*$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



$\Delta BOA_1 \cong \Delta COA_1 \therefore \angle BOA_1 = \angle COA_1$ *and* $\because \angle BOC = 2A$ *($\because$ angle at center is true angle at circumference)* $\therefore \angle BOA_1 = A$. *Using* $\Delta BOA_1, OA_1 = R\cos A$

$$\therefore A_1A_2 = OA_2 - OA_1 = R - R\cos a \overset{(1)}{=} R(1-\cos A)$$

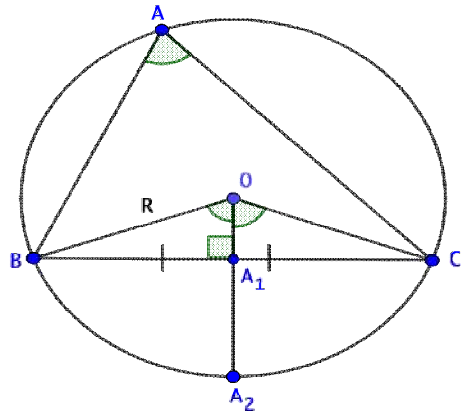
Similarly, $B_1B_2 \overset{(2)}{=} R(1-\cos B)$ *and* $C_1C_2 \overset{(3)}{=} R(1-\cos C)$

Applying Chebysev successively, and $\because n \in \mathbb{N}^*$

$$A_1A_2^n + B_1B_2^n + C_1C_2^n \geq \frac{1}{3^{n-1}}(A_1A_2 + B_1B_2 + C_1C_2)^n$$

$$= \frac{1}{3^{n-1}}\left(\sum R(1-\cos A)\right)^n \quad \text{(by (1)+(2)+(3))}$$

$$= \frac{1}{3^{n-1}}\left(3R - R\left(\frac{R+r}{R}\right)\right)^n = \frac{(2R-r)^n}{3^{n-1}} \overset{Euler}{\geq} \frac{(2(2r)-r)^n}{3^{n-1}} = \frac{3^n r^n}{3^{n-1}} = 3r^n \quad \text{(Proved)}$$



$$\Delta BOA_1 \cong \Delta COA_1 \therefore \angle BOA_1 = \angle COA_1 \text{ and} \therefore \angle BOC = 2A \therefore \angle BOA_1 = A$$

$$\text{From } \Delta BOA_1, OA_1 = R\cos A \therefore A_1A_2 = OA_2 - OA_1 = R - R\cos A \stackrel{(1)}{=} R(1-\cos A)$$

$$\text{Similarly, } B_1B_2 \stackrel{(2)}{=} R(1-\cos B) \text{ and } C_1C_2 \stackrel{(3)}{=} R(1-\cos C)$$

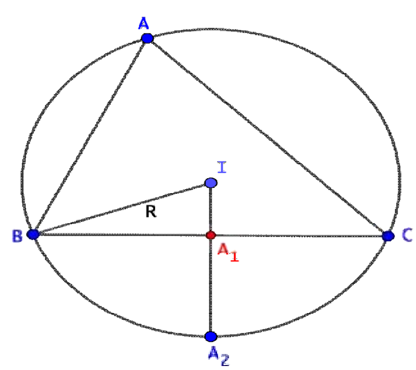
$$\text{Let } f(x) = x^n \therefore f''(x) = n(n-1)x^{n-2} \geq 0 \forall n \geq 1 \text{ and } \forall x > 0$$

$$\therefore (A_1A_2)^n + (B_1B_2)^n + (C_1C_2)^n \stackrel{Jensen}{\geq} 3\left(\frac{A_1A_2 + B_1B_2 + C_1C_2}{3}\right)^n$$

$$= 3\left(\frac{R(3-\sum\cos A)}{3}\right)^2 \text{ (by (1)+(2)+(3))}$$

$$= 3\left(\frac{3R - \frac{R(R+r)}{R}}{3}\right)^n = 3\left(\frac{2R-r}{3}\right)^n \stackrel{Euler}{\geq} 3\left(\frac{3r}{3}\right)^n = 3r^n \text{ (proved)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam



$$IA_1 = \sqrt{R^2 - \left(\frac{BC}{2}\right)^2} = \sqrt{R^2 - \frac{a^2}{4}} = \sqrt{R^2 - \frac{(2R\sin A)^2}{4}}$$

$$= R\sqrt{1-\sin^2 A} = R\sqrt{\cos^2 A} \stackrel{acute}{=} R\cos A$$

$$\rightarrow A_1A_2 = IA_2 - IA_1 = R - R\cos A = R(1-\cos A) = 2R\sin^2\frac{A}{2} \text{ (etc)}$$

$$\rightarrow LHS = \left(2R\sin^2\frac{A}{2}\right)^n + \left(2R\sin^2\frac{B}{2}\right)^n + \left(2R\sin^2\frac{C}{2}\right)^n =$$

$$= (2R)^n\left(\sin^{2n}\frac{A}{2} + \sin^{2n}\frac{B}{2} + \sin^{2n}\frac{C}{2}\right) = \omega$$

Let $f(x) = \sin^{2n}\frac{x}{2}; \left(0 < x < \frac{\pi}{2}, n \ge 1\right) \rightarrow f''(x)$

$$= \frac{n}{2}\sin^{2n-2}\frac{x}{2}\left[(2n-1)\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right] = \frac{n}{2}\sin^{2n-2}\frac{x}{2}\left[2n\cos^2\frac{x}{2} - 1\right]$$

$$\ge \frac{n}{2}\sin^{2n-2}\frac{x}{2}\left[2\cos^2\frac{x}{2} - 1\right] = n\cos x\sin^{2n-2}\frac{x}{2} > 0; \left(0 < x < \frac{\pi}{2}, n \ge 1\right)$$

$$\rightarrow \omega \overset{Jensen}{\ge} (2R)^n \cdot 3 \cdot \sin^{2n}\frac{A+B+C}{6}$$

$$= (2R)^n \cdot 3 \cdot \sin^{2n}\frac{\pi}{6} = (2R)^n \cdot 3 \cdot \frac{1}{2^{2n}} \overset{Euler}{\ge} (2\cdot 2r)^n \cdot 3 \cdot \frac{1}{2^{2n}} = 3 \cdot r^n \text{ (Proved)}$$

JP.222. In ABC triangle the following relationship holds:

$$a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} + b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}} + b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}} + c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}} + c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}} + a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}} \le 4s$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Bogdan Fustei-Romania

$$\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} = \left(1 + \frac{b}{a} - 1\right)^{\frac{h_c}{w_c}} = \left(1 + \frac{b-a}{a}\right)^{\frac{h_c}{w_c}}$$

$\frac{h_c}{w_c} \le 1$ *(and the analogs) because* $h_c \le w_c$ *(and the analogs)*

$\frac{b-a}{a} = \frac{b}{a} - 1 > -1$; *We will apply Bernoulli's inequality:*

$$\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} = \left(1 + \frac{b}{a} - 1\right)^{\frac{h_c}{w_c}} \le 1 + \frac{h_c}{w_c}\left(\frac{b-a}{a}\right)\Bigg| \cdot a \Rightarrow a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} \le a + \frac{h_c}{w_c}(b-a) \text{ (and the}$$

analogs). Summing we will obtain:

$$a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} + b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}} + b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}} + c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}} + c\left(\frac{a}{c}\right)^{\frac{h_a}{w_a}} + a\left(\frac{c}{a}\right)^{\frac{h_a}{w_b}} \le$$

$$\le 4s+\sum\frac{h_a}{w_a}(b-c+c-b)=4s$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $\frac{a}{b}=t$ ***and let*** $0<\theta\le 1$

$$a\left(\frac{b}{a}\right)^{\theta}+b\left(\frac{a}{b}\right)^{\theta}\overset{(1)}{\le}a+b\Leftrightarrow bt\frac{1}{t^{\theta}}+bt^{\theta}\le bt+b$$

$$\Leftrightarrow\frac{t}{t^{\theta}}+t^{\theta}\le t+1\Leftrightarrow t^{\theta}-1\le t\left(1-\frac{1}{t^{\theta}}\right)$$

$$\Leftrightarrow\left(t^{\theta}-1\right)\left(1-\frac{t}{t^{\theta}}\right)\le 0\Leftrightarrow\left(t^{\theta}-1\right)\left(t^{\theta-1}-1\right)\overset{(2)}{\le}0$$

Case 1) $t\ge 1$***. Then,*** $\theta\ln t\ge 0\Rightarrow\ln t^{\theta}\ge\ln 1\Rightarrow t^{\theta}-1\overset{(a)}{\ge}0$

Also, $(\theta-1)\ln t\le 0\Rightarrow\ln t^{\theta-1}\le\ln 1\Rightarrow t^{\theta-1}\le 1\Rightarrow t^{\theta-1}-1\overset{(b)}{\le}0$

(a).(b)⇒(2)⇒ (1) is true.

Case 2) $t<1$***. Then,*** $\theta\ln t<0\Rightarrow\ln t^{\theta}<1\Rightarrow t^{\theta}-1\overset{(c)}{<}0$

Also, $(\theta-1)\ln t\ge 0\Rightarrow\ln t^{\theta-1}\ge\ln 1\Rightarrow t^{\theta-1}-1\overset{(d)}{\ge}0$

(c).(d)⇒(2)⇒ (1) is true. $\therefore\ \forall\theta\in(0,1],\ a\left(\frac{b}{a}\right)^{\theta}+b\left(\frac{a}{b}\right)^{\theta}\le a+b$

$$\Rightarrow a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\overset{(i)}{\le}a+b\ \ \textbf{(choosing } \theta=\frac{h_c}{w_c}\textbf{)}$$

Similarly, $b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}+c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}\overset{(ii)}{\le}b+c$ ***and*** $c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}+a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}\overset{(iii)}{\le}c+a$

(i)+(ii)+(iii)⇒ $LHS\le\sum(a+b)=4s$ ***(proved)***

Solution 3 by Tran Hong-Dong Thap-Vietnam

Let $f(x)=x^{\alpha},(x>0,0<\alpha\le 1)\to f''(x)=\alpha(\alpha-1)x^{\alpha-2}\le 0,(x>0,0<\alpha\le 1)$

We have: $0<\frac{h_a}{w_a},\frac{h_b}{w_b},\frac{h_c}{w_c}\le 1$***. Now, using Jensen's inequality:***

$$\Omega_1=\frac{a}{2s}\cdot\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+\frac{b}{2s}\cdot\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}+\frac{c}{2s}\cdot\left(\frac{c}{a}\right)^{\frac{h_c}{w_c}}\le\left(\frac{b+a+c}{2s}\right)^{\frac{h_c}{w_c}}=1$$

$$\Omega_2=\frac{a}{2s}\cdot\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}+\frac{c}{2s}\cdot\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}+\frac{b}{2s}\cdot\left(\frac{b}{b}\right)^{\frac{h_b}{w_b}}\le\left(\frac{a+c+b}{2s}\right)^{\frac{h_b}{w_b}}=1$$

$$\Omega_2 = \frac{b}{2s}\cdot\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}} + \frac{c}{2s}\cdot\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}} + \frac{a}{2s}\cdot\left(\frac{a}{a}\right)^{\frac{h_a}{w_a}} \le \left(\frac{b+c+a}{2s}\right)^{\frac{h_a}{w_a}} = 1$$

$$\to \Omega_1 + \Omega_2 + \Omega_2 = \frac{a}{2s}\cdot\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} + \frac{b}{2s}\cdot\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}} + \frac{c}{2s}\cdot\left(\frac{c}{c}\right)^{\frac{h_c}{w_c}} + \frac{a}{2s}\cdot\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}} +$$

$$+\frac{c}{2s}\cdot\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}} + \frac{b}{2s}\cdot\left(\frac{b}{b}\right)^{\frac{h_b}{w_b}} + \frac{b}{2s}\cdot\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}} + \frac{c}{2s}\cdot\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}} + \frac{a}{2s}\cdot\left(\frac{a}{a}\right)^{\frac{h_a}{w_a}} \le 3$$

$$\leftrightarrow \frac{a}{2s}\cdot\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} + \frac{b}{2s}\cdot\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}} + \frac{a}{2s}\cdot\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}} + \frac{c}{2s}\cdot\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}} + \frac{b}{2s}\cdot\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}} + \frac{c}{2s}\cdot\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}} \le$$

$$\le 3 - \frac{a+b+c}{2s} = 2$$

$$\leftrightarrow a\left[\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} + \left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}\right] + b\left[\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}} + \left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}\right] + c\left[\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}} + \left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}\right] \le 4s.$$ ***Proved.***

JP.223. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that:

$$a(a^3 + (b+c)^3) + b(b^3 + (c+a)^3) + c(c^3 + (a+b)^3) \le 243R^4$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Marian Ursărescu-Romania

In any ΔABC ***we have:*** $a^2 + b^2 + c^2 \le 9R^2 \Rightarrow$

$$\Rightarrow 81R^4 \ge (a^2+b^2+c^2)^2 \Rightarrow 243R^4 \ge 3(a^2+b^2+c^2)^2 \Rightarrow$$

We must show:

$$3(a^2+b^2+c^2)^2 \ge a^4 + b^4 + c^4 + a(b+c)^3 + b(c+a)^3 + c(a+b)^3 \Leftrightarrow$$

$$\Leftrightarrow 2(a^4+b^4+c^4) + 6(a^2b^2+b^2c^2+c^2a^2) - a(b+c)^3 - b(c+a)^3 - c(a+b)^3 \ge 0 \quad \textbf{(1)}$$

Let $f_4(a,b,c) = 2(a^4+b^4+c^4) + 6(a^2b^2+b^2c^2+c^2a^2) - a(b+c)^3 -$

$$-b(c+a)^3 - c(a+b)^3$$

We use Cârtoaje's theorem: If $f_4(a,b,c)$ ***is a homogeneous and symmetric polygon of degree*** 4 ***then*** $f_4(a,b,c) \ge 0 \ \ \forall a,b,c \in \mathbb{R} \Leftrightarrow f_4(a,1,1) \ge 0, \forall a \in \mathbb{R}$

$$\textbf{(1)} \Leftrightarrow 2(a^4+2) + 6(2a^2+1) - 8a - 2(a+1)^3 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow a^4+2+6a^2+3-4a-a^3-3a^2-3a-1\ge 0\Leftrightarrow$$

$$\Leftrightarrow a^4-a^3+3a^2-7a+4\ge 0\Leftrightarrow (a-1)^2(a^2+a+4)\ge 0,$$ ***which is true.***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum(a^4+b^4)\ge\sum\frac{1}{2}(a^2+b^2)^2\overset{A-G}{\ge}\sum\left[\frac{1}{2}\cdot 2ab(a^2+b^2)\right]$$

$$=\sum(a^3b+ab^3)\Rightarrow\sum(a^3b+ab^3)\overset{(1)}{\le}2\sum a^4$$

Now, LHS $=\sum a^4+\sum a\left(b^3+c^3+3bc(b+c)\right)$

$$=\sum a^4+\sum(ab^3+a^3b)+3abc\cdot 4s\overset{by\ (1)}{\le}3\sum a^4+3\cdot 16Rrs^2\overset{?}{\le}243R^4$$

$$\Leftrightarrow\left(\sum a^2\right)^2-2\left[\left(\sum ab\right)^2-2abc(2s)\right]+16Rrs^2\overset{?}{\le}81R^4$$

$$\Leftrightarrow 4(s^2-4Rr-r^2)^2-2(s^2+4Rr+r^2)^2+48Rrs^2\overset{?}{\le}81R^4$$

$$\Leftrightarrow 2s^4-12r^2s^2+2r^2(4R+r)^2\underset{(2)}{\overset{?}{\le}}81R^4$$

Now, LHS of (2) $\overset{Gerretsen}{\le}(2(4R^2+4Rr+3r^2)-12r^2)s^2+2r^2(4R+r)^2$

$$=(8R^2+8Rr-6r^2)s^2+2r^2(4R+r)^2$$

$$\overset{Gerretsen}{\le}(4R^2+4Rr+3r^2)(8R^2+8Rr-6r^2)+2r^2(4R+r)^2\overset{?}{\le}81R^4$$

$$\Leftrightarrow 49t^4-64t^3-64t^2-16t+16\overset{?}{\ge}0$$

$$\Leftrightarrow(t-2)\left(49t^3+34t^2+4(t-2)\right)\overset{?}{\ge}0\to$$ ***true*** $\because t\overset{Euler}{\ge}2$

$\Rightarrow$ ***(2)*** $\Rightarrow$ ***given inequality is true (proved)***

JP.224. Let a,b,c be the lengths of the sides of a triangle with circumradius R. Prove that:

$$\frac{\left(\frac{a+b}{c}\right)^3+\left(\frac{b+c}{a}\right)^3+\left(\frac{c+a}{b}\right)^3+3}{\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}}\le(3R)^4$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\textit{Given inequality} \Leftrightarrow (3R)^4\frac{\sum a^4b^4}{(abc)^4} \geq \frac{\sum a^3b^3(a+b)^3+3a^3b^3c^3}{(abc)^3}$$

$$\Leftrightarrow \frac{81(abc)^4\cdot\sum a^4b^4}{256s^2(\prod(s-a))^2(abc)^4} \geq \frac{\sum a^3b^3\,(a+b)^3+3a^3b^3c^3}{(abc)^3}$$

$$\Leftrightarrow \left[81\sum(y+z)^4(z+x)^4\right]\prod(y+z)^3 \geq$$

$$256x^2y^2z^2\left(\sum x\right)^2\left[\sum(y+z)^3(z+x)^3(x+y+2z)^3+3\prod(y+z)^3\right]$$

$$\begin{pmatrix}a=y+z\\b=z+x\\c=x+y\end{pmatrix} \Leftrightarrow 81\left(\sum x^{14}y^3+\sum x^3y^{14}\right)+567\left(\sum x^{13}y^4+\sum x^4y^{13}\right)+$$

$$+243xyz\left(\sum x^{13}y+\sum xy^{13}\right)+2268xyz\left(\sum x^{12}y^2+\sum x^2y^{12}\right)+$$

$$+1354x^2y^2z^2\left(\sum x^{11}\right)+1701\left(\sum x^{12}y^5+\sum x^5y^{12}\right)+$$

$$+9072xyz\left(\sum x^{11}y^3+\sum x^3y^{11}\right)+2835\left(\sum x^{11}y^6+\sum x^6y^{11}\right)+$$

$$+5399(xyz)^2\left(\sum x^{10}y+\sum xy^{10}\right)+18757x^2y^2z^2\left(\sum x^9y^2+\sum x^2y^9\right)+$$

$$+79112x^3y^3z^3\left(\sum x^5y^3+\sum x^3y^5\right)+123720(xyz)^3\left(\sum x^4y^4\right)+$$

$$+20412xyz\left(\sum x^{10}y^4+\sum x^4y^{10}\right)+2997\left(\sum x^{10}y^7+\sum x^7y^{10}\right)+$$

$$+29484xyz\left(\sum x^9y^5+\sum x^5y^9\right)+46747x^2y^2z^2\left(\sum x^8y^3+\sum x^3y^8\right)+$$

$$+2511\left(\sum x^9y^8+\sum x^8y^9\right)+31428xyz\left(\sum x^8y^6+\sum x^6y^8\right)+$$

$$+81317x^2y^2z^2\left(\sum x^7y^4+\sum x^4y^7\right)+4256x^3y^3z^3\left(\sum x^6y^2+\sum x^2y^6\right)+$$

$$+30618xyz\left(\sum x^7y^7\right)+104659x^2y^2z^2\left(\sum x^6y^5+\sum x^5y^6\right)\overset{(1)}{\geq}$$

$$\geq 3784x^3y^3z^3\left(\sum x^8\right)+19992x^3y^3z^3\left(\sum x^7y+\sum xy^7\right)+$$

$$+102432x^4y^4z^4\left(\sum x^5\right)+121880x^4y^4z^4\left(\sum x^4y+\sum xy^4\right)+$$

$$+69816x^4y^4z^4\left(\sum x^3y^2+\sum x^2y^3\right)+236664x^5y^5z^5\left(\sum x^2\right)+$$

$$+277128x^5y^5z^5\left(\sum xy\right)$$

$$\sum x^4y^4 \geq x^2y^2z^2\left(\sum x^2\right) \Rightarrow 123720x^3y^3z^3 \overset{(2)}{\geq} 123720x^5y^5z^5\left(\sum x^2\right)$$

$$\sum x^5y^3 + \sum x^3y^5 \geq 2\sum x^4y^4 \geq 2x^2y^2z^2\left(\sum x^2\right) \Rightarrow$$

$$\Rightarrow 56472x^3y^3z^3\left(\sum x^5y^3 + \sum x^3y^5\right) \overset{(3)}{\geq} 112944x^5y^5z^5\left(\sum x^2\right)$$

$$\sum x^5y^3 + \sum x^3y^5 \geq 2x^2y^2z^2\left(\sum x^2\right) \geq 2x^2y^2z^2\left(\sum xy\right)$$

$$\Rightarrow 22640x^3y^3z^3\left(\sum x^5y^3 + \sum x^3y^5\right) \overset{(4)}{\geq} 45280x^5y^5z^5\left(\sum xy\right)$$

$$\sum x^6y^5 + \sum x^5y^6 = \sum x^5\left(y^6 + z^6\right) \overset{A-G}{\geq} 2\sum x^5y^3z^3 = 2x^3y^3z^3\left(\sum x^2\right) \geq 2(xyz)^3\sum xy$$

$$\Rightarrow 104659x^2y^2z^2\left(\sum x^6y^5 + \sum x^5y^6\right) \overset{(5)}{\geq} 209318x^5y^5z^5\left(\sum xy\right)$$

$$2\sum x^7y^7 = \sum x^7\left(y^7 + z^7\right) \overset{CBC}{\geq} \frac{1}{2}\sum x^7\left(y^4 + z^4\right)(y^3 + z^3) \overset{A-G}{\geq} \sum x^7y^2z^2\left(y^3 + z^3\right)$$

$$= x^2y^2z^2\left(\sum x^5y^3 + \sum x^3y^5\right) \overset{A-G}{\geq} 2x^2y^2z^2\sum x^4y^4 \overset{earlier}{\geq} 2x^4y^4z^4\left(\sum xy\right)$$

$$\Rightarrow 14018xyz\left(\sum x^7y^7\right) \overset{(6)}{\geq} 14018x^5y^5z^5\left(\sum xy\right)$$

$$2\sum x^7y^7 \overset{earlier}{\geq} x^2y^2z^2\sum x^5\left(y^3 + z^3\right) \geq x^2y^2z^2\sum x^5\,yz(y + z)$$

$$= x^3y^3z^3\left(\sum xy(x^3 + y^3)\right) \geq x^3y^3z^3\sum x^2y^2(x + y) = x^3y^3z^3\left(\sum x^3y^2 + \sum x^2y^3\right)$$

$$\Rightarrow 16600xyz\left(\sum x^7y^7\right) \overset{(7)}{\geq} 8300x^4y^4z^4\left(\sum x^3y^2 + \sum x^2y^3\right)$$

$$\sum x^7y^4 + \sum x^4y^7 = \sum x^7\left(y^4 + z^4\right) \overset{A-G}{\geq} 2\sum x^7y^2z^2 = x^2y^2z^2\sum\left(x^5 + y^5\right)$$

$$\overset{CBC}{\geq} \frac{1}{2}x^2y^2z^2\sum\left(x^2 + y^2\right)\left(x^3 + y^3\right) \overset{A-G}{\geq} x^2y^2z^2\sum xy \cdot xy(x + y) =$$

$$= x^2y^2z^2\left(\sum x^3y^2 + \sum x^2y^3\right)$$

$$\Rightarrow 61516x^2y^2z^2\left(\sum x^7y^4 + \sum x^4y^7\right) \overset{(8)}{\geq} 61516x^4y^4z^4\left(\sum x^3y^2 + \sum x^2y^3\right)$$

$$\sum x^7y^4+\sum x^4y^7\overset{earlier}{\geq}\frac{1}{2}x^2y^2z^2\sum(x^2+y^2)(x^3+y^3)\overset{A-G}{\geq}x^2y^2z^2\sum xy(x^3+y^3)$$

$$\Rightarrow 19801x^2y^2z^2\left(\sum x^7y^4+\sum x^4y^7\right)\overset{(9)}{\geq}19801x^4y^4z^4\left(\sum x^4y+\sum xy^4\right)$$

$$\sum x^9y^5+\sum x^5y^9=\sum x^9(y^5+z^5)\overset{(CBC)}{\geq}\frac{1}{2}\sum x^9\,(y^2+z^2)(y^3+z^3)\overset{A-G}{\geq}$$

$$\geq\sum x^9yz(y^3+z^3)$$

$$=xyz\sum x^8(y^3+z^3)\geq x^2y^2z^2\sum x^7\,(y+z)\overset{CBC}{\geq}\frac{1}{2}x^2y^2z^2\sum x\,(y^2+z^2)(y^5+z^5)$$

$$\overset{A-G}{\geq}x^2y^2z^2\sum xyz\,(y^5+z^5)=2x^3y^3z^3\left(\sum x^5\right)\Rightarrow 5580xyz\left(\sum x^9y^5+\sum x^5y^9\right)$$

$$\overset{(10)}{\geq}11160x^4y^4z^4\left(\sum x^5\right)$$

$$\sum x^9y^8+\sum x^8y^9=\sum x^9(y^8+z^8)\overset{A-G}{\geq}2x^9y^4z^4=2x^4y^4z^4\left(\sum x^5\right)$$

$$\Rightarrow 2511\left(\sum x^9y^8+\sum x^8y^9\right)\overset{(11)}{\geq}5022x^4y^4z^4\left(\sum x^5\right)$$

$$\sum x^8y^6+\sum x^6y^8\overset{A-G}{\geq}2\sum x^7y^7\overset{earlier}{\geq}x^2y^2z^2\cdot\sum x^5\,(y^3+z^3)\geq$$

$$\geq x^3y^3z^3\sum x^4(y+z)$$

$$\Rightarrow 31428xyz\left(\sum x^8y^6+\sum x^6y^8\right)\overset{(12)}{\geq}31428x^4y^4z^4\left(\sum x^4y+\sum xy^4\right)$$

$$\sum x^8y^3+\sum x^3y^8=\sum x^3\,(y^8+z^8)\overset{CBC}{\geq}\frac{1}{2}\sum x^3(y^2+z^2)\,(y^6+z^6)\overset{A-G}{\geq}$$

$$\geq\sum x^3yz\,(y^6+z^6)$$

$$=xyz\sum x^2(y^6+z^6)\overset{CBC}{\geq}\frac{1}{2}xyz\sum x^2(y^2+z^2)\,(y^4+z^4)\overset{A-G}{\geq}xyz\sum x^2yz\,(y^4+z^4)$$

$$=x^2y^2z^2\left(\sum x^4y+\sum xy^4\right)\Rightarrow 46747x^2y^2z^2\left(\sum x^8y^3+\sum x^3y^8\right)\overset{(13)}{\geq}$$

$$\geq 46747x^4y^4z^4\left(\sum x^4y+\sum xy^4\right)$$

$$\sum x^9y^5+\sum x^5y^9=\sum x^5\,(y^9+z^9)\overset{CBS}{\geq}\frac{1}{2}\sum x^5\,(y^2+z^2)(y^7+z^7)\overset{A-G}{\geq}$$

$$\geq \sum x^5yz\left(y^7+z^7\right)$$

$$= xyz\sum x^4(y^7+z^7) \overset{earlier}{\geq} x^3y^3z^3\sum xy(x^3+y^3) = x^3y^3z^3\left(\sum x^4y+\sum xy^4\right)$$

$$\Rightarrow 23904xyz\left(\sum x^9y^5+\sum x^5y^9\right) \overset{(14)}{\geq} 23904x^4y^4z^4\left(\sum x^4y+\sum xy^4\right)$$

$$\sum x^{10}y^7+\sum x^7y^{10} = \sum x^{10}\left(y^7+z^7\right) \overset{CBC}{\geq} \frac{1}{2}\sum x^{10}\left(y^6+z^6\right)(y+z)$$

$$\overset{A-G}{\geq} \sum x^{10}y^3z^3\,(y+z) = x^3y^3z^3\sum x^7\,(y+z) = x^3y^3z^3\sum x(y^7+z^7)$$

$$\overset{CBC}{\geq} \frac{1}{2}x^3y^3z^3\sum x\left(y^2+z^2\right)\left(y^5+z^5\right) \overset{A-G}{\geq} x^3y^3z^3\sum xyz\left(y^5+z^5\right)$$

$$= 2x^4y^4z^4\left(\sum x^5\right) \Rightarrow 2997\left(\sum x^{10}y^7+\sum x^7y^{10}\right) \overset{(15)}{\geq} 5994x^4y^4z^4\left(\sum x^5\right)$$

$$\sum x^{10}y^4+\sum x^4y^{10} = \sum x^{10}\left(y^4+z^4\right) \overset{CBC}{\geq} \frac{1}{2}\sum x^{10}\,(y+z)(y^3+z^3)$$

$$\geq \frac{1}{2}\sum x^{10}\,(y+z)^2yz \overset{A-G}{\geq} 2xyz\sum x^9\,yz = x^2y^2z^2\sum(x^8+y^8)$$

$$\overset{(CBC)}{\geq} \frac{1}{2}x^2y^2z^2\sum(x^2+y^2)(x^6+y^6) \overset{A-G}{\geq} x^2y^2z^2\sum xy(x^6+y^6)$$

$$= x^2y^2z^2\sum x(y^7+z^7) \overset{earlier}{\geq} 2x^3y^3z^3\left(\sum x^5\right)$$

$$\Rightarrow 20412xyz\left(\sum x^{10}y^4+\sum x^4y^{10}\right) \overset{(16)}{\geq} 40824x^4y^4z^4\left(\sum x^5\right)$$

$$\sum x^9y^2+\sum x^2y^9 = \sum x^9(y^2+z^2) \overset{A-G}{\geq} 2xyz\sum x^8 = xyz\sum(x^8+y^8)$$

$$\overset{earlier}{\geq} 2x^2y^2z^2\left(\sum x^5\right) \Rightarrow 18757x^2y^2z^2\left(\sum x^9y^2+\sum x^2y^9\right) \overset{(17)}{\geq}$$

$$\geq 37514x^4y^4z^4\left(\sum x^5\right)$$

$$\sum x^{10}+\sum xy^{10} = \sum xy(x^9+y^9) \overset{CBC}{\geq} \frac{1}{2}\sum xy\,(x^2+y^2)(x^7+y^7)$$

$$\overset{A-G}{\geq} \sum x^2y^2\,(x^7+y^7) = \sum x^9y^2+\sum x^2y^9 \overset{earlier}{\geq} 2x^2y^2z^2\left(\sum x^5\right)$$

$$\Rightarrow 959x^2y^2z^2\left(\sum x^{10}y+\sum xy^{10}\right) \overset{(18)}{\geq} 1918x^4y^4z^4\left(\sum x^5\right)$$

$$\sum x^{10}y + \sum xy^{10} \overset{earlier}{\geq} \sum x^9y^2 + \sum x^2y^9 = \sum x^9(y^2+z^2) \overset{A-G}{\geq} 2xyz\left(\sum x^8\right)$$

$$= xyz\sum(x^8+y^8) \overset{CBC}{\geq} \frac{1}{2}xyz\sum(x^2+y^2)(x^6+y^6) \overset{A-G}{\geq} xyz\sum xy\,(x^6+y^6)$$

$$\Rightarrow 4440x^2y^2z^2 \overset{(19)}{\geq} 4440x^3y^3z^3\left(\sum x^7y + \sum xy^7\right)$$

$$\sum x^{11}y^6 + \sum x^6y^{11} = \sum x^{11}\,(y^6+z^6) \overset{A-G}{\geq} 2\sum x^{11}y^3z^3 = 2x^3y^3z^3\left(\sum x^8\right)$$

$$\overset{earlier}{\geq} x^3y^3z^3\left(\sum x^7y + \sum xy^7\right) \Rightarrow$$

$$\Rightarrow 2835\left(\sum x^{10}y + \sum xy^{10}\right) \overset{(20)}{\geq} 2835x^3y^3z^3\left(\sum x^7y + \sum xy^7\right)$$

$$\sum x^{11}y^3 + \sum x^3y^{11} = \sum x^{11}(y^3+z^3) \geq \sum x^{11}yz\,(y+z) = xyz\sum x^{10}\,(y+z)$$

$$= xyz\sum x\,(y^{10}+z^{10}) \overset{CBC}{\geq} \frac{1}{2}xyz\sum x(y^2+z^2)\,(y^8+z^8) \overset{A-G}{\geq} 2x^2y^2z^2\sum x^8$$

$$\overset{earlier}{\geq} x^2y^2z^2\left(\sum x^7y + \sum xy^7\right) \Rightarrow 9072xyz\left(\sum x^{11}y^3 + \sum x^3y^{11}\right) \overset{(21)}{\geq}$$

$$\geq 9072x^3y^3z^3\left(\sum x^7y + \sum xy^7\right)$$

$$\sum x^{12}y^5 + \sum x^5y^{12} = \sum x^5y^5(x^7+y^7) \overset{CBC}{\geq} \frac{1}{2}\sum x^5y^5(x^2+y^2)\left(x^5+y^5\right)$$

$$\overset{A-G}{\geq} \sum x^6y^6\left(x^5+y^5\right) = \sum x^{11}y^6 + \sum x^6y^{11} \overset{earlier}{\geq} x^3y^3z^3\left(\sum x^7y + \sum xy^7\right)$$

$$\Rightarrow 1701\left(\sum x^{12}y^5 + \sum x^5y^{12}\right) \overset{(22)}{\geq} 1701x^3y^3z^3\left(\sum x^7y + \sum xy^7\right)$$

$$2\sum x^{11} = \sum(x^{11}+y^{11}) \overset{CBC}{\geq} \frac{1}{2}\sum(x^2+y^2)(x^9+y^9) \overset{A-G}{\geq} \sum xy(x^9+y^9)$$

$$= \sum x^{10}y + \sum xy^{10} \overset{earlier}{\geq} xyz\left(\sum x^7y + \sum xy^7\right) \Rightarrow 1354x^2y^2z^2\left(\sum x^{11}\right) \overset{(23)}{\geq}$$

$$\geq 677x^3y^3z^3\left(\sum x^7y + \sum xy^7\right)$$

$$\sum x^{12}y^2 + \sum x^2y^{12} = \sum x^{12}\,(y^2+z^2) \overset{A-G}{\geq} 2\sum x^{12}yz = 2xyz\left(\sum x^{11}\right)$$

$$\overset{earlier}{\geq} x^2y^2z^2\left(\sum x^7\,y + \sum xy^7\right) \Rightarrow 1267xyz\left(\sum x^{12}y^2 + \sum x^2y^{12}\right) \overset{(24)}{\geq}$$

$$\geq 1267x^3y^3z^3\left(\sum x^7y+\sum xy^7\right)$$

$$\sum x^{12}y^2+\sum x^2y^{12}\overset{earlier}{\geq} xyz\sum(x^{11}+y^{11})\overset{CBC}{\geq}\frac{1}{2}xyz\sum(x^2+y^2)(x^9+y^9)$$

$$\overset{A-G}{\geq} xyz\sum xy\,(x^9+y^9)=xyz\left(\sum x^{10}y+\sum xy^{10}\right)\overset{earlier}{\geq} 2x^2y^2z^2\left(\sum x^8\right)$$

$$\Rightarrow 1001xyz\left(\sum x^{12}y^2+\sum x^2y^{12}\right)\overset{(25)}{\geq} 2002x^3y^3z^3\left(\sum x^8\right)$$

$$\sum x^{13}y+\sum xy^{13}=\sum xy\,(z^{12}+y^{12})\overset{CBC}{\geq}\frac{1}{2}\sum xy\,(x^2+y^2)(x^{10}+y^{10})$$

$$\overset{A-G}{\geq}\sum x^2y^2\,(x^{10}+y^{10})=\sum x^{12}y^2+\sum x^2y^{12}\overset{earlier}{\geq} 2x^2y^2z^2\left(\sum x^8\right)$$

$$\Rightarrow 243xyz\left(\sum x^{13}y+\sum xy^{13}\right)\overset{(26)}{\geq} 486x^3y^3z^3\left(\sum x^8\right)$$

$$\sum x^{13}y^4+\sum x^4y^{13}=\sum x^4y^4\,(x^9+y^9)\overset{CBC}{\geq}\frac{1}{2}\sum x^4y^4(x^2+y^2)\,(x^7+y^7)$$

$$\overset{A-G}{\geq}\sum x^5y^5\,(x^7+y^7)\overset{CBC}{\geq}\frac{1}{2}\sum x^5y^5\,(x^2+y^2)(x^5+y^5)$$

$$\overset{A-G}{\geq}\sum x^6y^6\,(x^5+y^5)=\sum x^{11}y^6+\sum x^6y^{11}\overset{earlier}{\geq} 2x^3y^3z^3\left(\sum x^8\right)$$

$$\Rightarrow 567\left(\sum x^{13}y^4+\sum x^4y^{13}\right)\overset{(27)}{\geq} 1134x^3y^3z^3\left(\sum x^8\right)$$

$$\sum x^{14}y^3+\sum x^3y^{14}=\sum x^3y^3\,(x^{11}+y^{11})\overset{CBC}{\geq}\frac{1}{2}\sum x^3y^3(x^2+y^2)\,(x^9+y^9)$$

$$\overset{A-G}{\geq}\sum x^4y^4(x^9+y^9)=\sum x^{13}y^4+\sum x^4y^{13}\overset{earlier}{\geq} 2x^3y^3z^3\left(\sum x^8\right)$$

$$\Rightarrow 81\left(\sum x^{14}y^3+\sum x^3y^{14}\right)\overset{(28)}{\geq} 162x^3y^3z^3\left(\sum x^8\right)$$

(2)+(3)+(4)+(5)+(6)+(7)+(8) +(9) +(10) +(11) +(12) +(13) +(14) +(15) +(16) + +(17) +(18) +(19) +(20) +(21) +(22) +(23) +(24) +(25) +(26) +(27) +(28)⇒ (1) is true (Proved)

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\textit{Let } \Omega=\frac{\left(\frac{a+b}{c}\right)^3+\left(\frac{b+c}{a}\right)^3+\left(\frac{c+a}{b}\right)^3+3}{\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}}=\frac{abc\left[(ab(a+b)^3+\left(bc(b+c)\right)^3+\left(ca(c+a)\right)^3+3(abc)^3\right]}{a^4b^4+b^4c^4+c^4a^4}$$

$$9R^2\geq a^2+b^2+c^2\to(3R)^4=(9R^2)^2\geq(a^2+b^2+c^2)^2$$

We must show that:

$$\frac{abc\left[\left(ab(a+b)\right)^3+\left(bc(b+c)\right)^3+\left(ca(c+a)\right)^3+3(abc)^3\right]}{a^4b^4+b^4c^4+c^4a^4}\le(a^2+b^2+c^2)^2$$

$$\leftrightarrow abc\left[\left(ab(a+b)\right)^3+\left(bc(b+c)\right)^3+\left(ca(c+a)\right)^3+3(abc)^3\right]$$

$$\le(a^4b^4+b^4c^4+c^4a^4)(a^2+b^2+c^2)^2$$

$$\leftrightarrow(a^4b^8+a^8b^4+a^4c^8+a^8c^4+b^4c^8+b^8c^4)+2(a^6b^6+b^6c^6+c^6a^6)+$$

$$+2a^2b^2c^2(a^4b^2+a^4c^2+b^4a^2+b^4c^2+c^4a^2+c^4b^2)\ge$$

$$\ge abc(a^6b^3+a^6c^3+b^6a^3+b^6c^3+c^6a^3+c^6b^3)+$$

$$+3abc\left(a^5b^4+a^5c^4+b^5c^4+b^5a^4+c^5a^4+c^5b^4\right)\quad \textbf{(*)}$$

$$3abc\le a^3+b^3+c^3\to 3abc\left(a^5b^4+a^5c^4+b^5c^4+b^5a^4+c^5a^4+c^5b^4\right)\le$$

$$\le(a^3+b^3+c^3)\left(a^5b^4+a^5c^4+b^5c^4+b^5a^4+c^5a^4+c^5b^4\right)=$$

$$=a^8(b^4+c^4)+b^8(c^4+a^4)+c^8(b^4+a^4)+$$

$$+a^3\left(b^5c^4+b^5a^4+c^5a^4+c^5b^4\right)+b^3\left(a^5b^4+a^5c^4+c^5a^4+c^5b^4\right)+$$

$$+c^3\left(a^5b^4+a^5c^4+b^5c^4+b^5a^4\right)$$

We must show that:

$$2(a^6b^6+b^6c^6+c^6a^6)+2a^2b^2c^2(a^4b^2+a^4c^2+b^4a^2+b^4c^2+c^4a^2+c^4b^2)\ge$$

$$\ge abc(a^6b^3+a^6c^3+b^6a^3+b^6c^3+c^6a^3+c^6b^3)$$

$$+a^3\left(b^5c^4+b^5a^4+c^5a^4+c^5b^4\right)+b^3\left(a^5b^4+a^5c^4+c^5a^4+c^5b^4\right)+$$

$$+c^3\left(a^5b^4+a^5c^4+b^5c^4+b^5a^4\right)$$

$$=a^3b^3c^3(a^2b+b^2a+b^2c+c^2b+c^2a+ca^2)+a^7\left(b^5+c^5\right)+b^7\left(a^5+c^5\right)+$$

$+c^7\left(a^5+b^5\right)$***. It is true because:***

$$2(a^6b^6+b^6c^6+c^6a^6)\ge a^3b^3c^3(a^2b+b^2a+b^2c+c^2b+c^2a+ca^2)\quad \textbf{(1)}$$

$$+a^6b^6+b^6c^6+c^6a^6\ge a^3b^3c^3(a^3+b^3+c^3)\ge a^3b^3c^3(a^2b+b^2c+c^2a)$$

$+a^6b^6+b^6c^6+c^6a^6\ge a^3b^3c^3(a^3+b^3+c^3)\ge a^3b^3c^3(b^2a+ca^2+c^2b)\to$ ***(1) true.***

$$2a^2b^2c^2(a^4b^2+a^4c^2+b^4a^2+b^4c^2+c^4a^2+c^4b^2)\ge$$

$$\ge abc(a^6b^3+a^6c^3+b^6a^3+b^6c^3+c^6a^3+c^6b^3)+$$

$$+a^7\left(b^5+c^5\right)+b^7\left(a^5+c^5\right)+c^7\left(a^5+b^5\right)\quad \textbf{(2)}$$

(2) true because: By ABC theorem: $2(a^4b^2+a^4c^2+b^4a^2+b^4c^2+c^4a^2+c^4b^2)>0$

With $f(a^2b^2c^2) = 2a^2b^2c^2(a^4b^2 + a^4c^2 + b^4a^2 + b^4c^2 + c^4a^2 + c^4b^2) -$

$$-\begin{bmatrix} abc(a^6b^3 + a^6c^3 + b^6a^3 + b^6c^3 + c^6a^3 + c^6b^3) + a^7(b^5 + c^5) + \\ +b^7(a^5 + c^5) + c^7(a^5 + b^5) \end{bmatrix}$$

So, (*) true. Proved

JP.225. Solve the following system of equations:

$$\begin{cases} x^3 + 2x + 3 = 8y^3 - 6xy + 4y \\ \sqrt{x^2 - 2y + 2} + \sqrt{x^2 - 4y + 4} = x^2 - 3y + 4 \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Amit Dutta-Jamshedpur-India

Domain $(x^2 - 2y + 2) \geq 0, (x^2 - 4y + 4) \geq 0$

$$GM \leq AM$$

$$\sqrt{x^2 - 2y + 2} = \sqrt{(x^2 - 2y + 2) \cdot 1} \leq \frac{(x^2 - 2y + 2) + 1}{2}$$

$$\sqrt{x^2 - 2y + 2} \leq \left(\frac{x^2 - 2y + 3}{2}\right)$$

$$\sqrt{x^2 - 4y + 4} = \sqrt{(x^2 - 4y + 4) \cdot 1} \leq \left(\frac{x^2 - 4y + 5}{2}\right)$$

Adding these: $\sqrt{x^2 - 2y + 2} + \sqrt{x^2 - 4y + 4} \leq (x^2 - 3y + 4)$

But we have: $\sqrt{x^2 - 2y + 2} + \sqrt{x^2 - 4y + 4} = (x^2 - 3y + 4)$

So, for equality, we must have: $\begin{cases} x^2 - 2y + 2 = 1 \Rightarrow x^2 = 2y - 1 \\ x^2 - 4y + 4 = 1 \Rightarrow x^2 = 4y - 3 \end{cases}$

Solve these two equations, we get: $\begin{Bmatrix} x = \pm 1 \\ y = 1 \end{Bmatrix}$

But for the system of equation, we must check these solutions for the other equation also: i.e., $x^3 + 2x + 3 = 8y^3 - 6xy + 4y$

For $(x, y) = (1, 1)$*;* $LHS = 6; RHS = 6$

Equality holds, so $(1, 1)$ *is a solution for other possible solution:* $(x, y) = (-1, 1)$

$$LHS = 0; RHS = 18$$

Equality do not hold. So, $(-1,1)$ ***is not the solution for this system of equation.***

Hence, $(1,1)$ ***is the only solution.***

Solution 2 by Orlando Irahola Ortega-Bolivia

$$\begin{cases} x^3+2x+3=8y^3-6xy+4y \quad (1) \\ \sqrt{x^2-2y+2}+\sqrt{x^2-4y+4}=x^2-3y+4 \quad (2) \end{cases}$$

$$2\times(2) \quad 2\left(\sqrt{x^2-2y+2}+\sqrt{x^2-4y+4}\right)=\sqrt{x^2-2y+2}^2+\sqrt{x^2-4y+4}^2+2$$

Sea: $a=\sqrt{x^2-2y+2};\ b=\sqrt{x^2-4y+4}$

$$\Rightarrow 2a+2b=a^2+b^2+2\Rightarrow (a-1)^2+(b-1)^2=0\Rightarrow a-1=0\wedge b-1=0$$

$$a=1\wedge b=1$$

Si: $a=1\Rightarrow\sqrt{x^2-2y+2}=1\Rightarrow 2y=x^2+1$ ***(2.1)***

Si: $b=1\Rightarrow\sqrt{x^2-4y+4}=1\Rightarrow 4y=x^2+3$ ***(2.2)***

(A) (1)$\rightarrow x^3+2x+3=(2y)^3-3x(2y)+2(2y)$***; (2.1) en (1)***$\Rightarrow$

$$\Rightarrow x^3+2x+3=(x^2+1)^3-3x(x^2+1)+2(x^2+1)$$

$$\Rightarrow x^6+3x^4-4x^3+5x^2-5x=0\Rightarrow x(x-1)(x^4+x^3+4x^2+5)=0\Rightarrow$$

$$x_1=0\Rightarrow y_1=y_2\ ;x_2=1\Rightarrow y_2=1$$

$$x^4+x^3+4x^2+5=0 \quad (**)$$

$$4^4(**)\Rightarrow (4x)^4+4(4x)^3+64\,(4x^2+1280)=0$$

$$\Rightarrow [(4x)^2+2(4x)]^2+60(4x)^2+1280=0$$

$$\Rightarrow \underbrace{(16x^2+8x)^2}_{\geq 0}+\underbrace{960x^2+1280}_{>0}=0\rightarrow (16x^2+8x)^2+960x^2+1280>0\Rightarrow x\in\mathbb{C}$$

(B) $8\times$ ***(1)***$\Rightarrow 8x^3+16x+24=(4y)^3-12x(4y)+8(4y)$***; (2.2) en (1)***$\Rightarrow$

$$\Rightarrow 8x^3+16x+24=(x^2+3)^3-12x(x^2+3)+8(x^3+3)$$

$$\Rightarrow x^6+9x^4-20x^3+35x^2-52x+27=0\Rightarrow (x-1)^2(x^4+2x^3+12x^2+2x+27)=0$$

$$\Rightarrow x=1\wedge x^4+2x^3+12x^2+2x+27=0$$

$$\underbrace{(x^2+x)^2}_{\geq 0}+\underbrace{11x^2+2x+27}_{>0}=0$$

$$(x^2+x)^2+11x^2+2x+27>0\Rightarrow x\in\mathbb{C}$$

$$(x,y)\in\mathbb{R}^2|(x,y)=(1;1)$$

SP.211. Find all real roots of the following equation:

$$\sqrt[2n]{2-x^2}+\sqrt[2n]{2|x|-1}=(x^2-1)^{2m}+2$$

where m, n are positive integers.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Michael Sterghiou-Greece

$$\sqrt[2n]{2-x^2}+\sqrt[2n]{2|x|-1}=(x^2-1)^{2m}+2 \quad (1)$$

Let $y=|x|\ge 0$ *(1)* $\rightarrow \sqrt[2n]{2-y^2}+\sqrt[2n]{2y-1}=(y^2-1)^{2m}+2$ *(1)'*

Consider the function $f(t)=\sqrt[2n]{t}, t\ge 0$ *with* $f''(t)=\frac{(1-2n)t^{\frac{1}{2n}-2}}{4n^2}<0$

for $n\in\mathbb{N}, n\ge 1$*, hence* $f(t)$ *concave and from Jensen:*

LHS of (1) $\le 2\cdot\sqrt[2n]{\frac{2-y^2+2y-1}{2}}=2\sqrt[2n]{\frac{1}{2}(-y^2+2y+1)}$ *(2)*

From (1)' we have $2-y^2\ge 0\rightarrow y\le\sqrt{2}$ *and* $2y-1\ge 0\rightarrow y\ge\frac{1}{2}$ *or*

$\frac{1}{2}\le y\le\sqrt{2}$*. Now,* $-y^2+2y+1>0$ *and equality in (2) when*

$2-y^2=2y-1\leftrightarrow y=1$*. From (1) and (2)*

$\rightarrow 2\cdot\sqrt[2n]{\frac{1}{2}(-y^2+2y+1)}\ge(y^2-1)^{2m}+2$ *(3). Consider the function*

$f(y)=\frac{1}{2}(-y^2+2y+1)$ *with* $f'(y)=-y+1$ *with root* $y=1$ *and* $f''(y)<0$ *with*

$\max f=1$ *at* $y=1$*. As* $\frac{1}{2}(-y^2+2y+1)\le 1\rightarrow\sqrt[2n]{\frac{1}{2}(-y^2+2y+1)}\le 1$ *and*

$2\cdot\sqrt[2n]{\frac{1}{2}(-y^2+2y+1)}\le 2$*. The last inequality and (3) give*

$2\ge$ *LHS of (3)* $\ge 2+(y^2-1)^{2m}$ *which can happen only if* $y^2-1=0$ *or* $y=1$ *or*

$|x|=1$ *or* $x=\pm 1$ *which are the only real solution of (1).*

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\sqrt[2n]{2-x^2}+\sqrt[2n]{2|x|-1}=(x^2-1)^{2m}+2$$

$$\begin{array}{c} 2-x^2\geq 0 \\ x\in[-\sqrt{2},\sqrt{2}] \end{array}\left|\begin{array}{c} 2|x|\geq 1 \\ |x|\geq\frac{1}{2} \\ \left]-\infty,-\frac{1}{2}\right]\cup\left[\frac{1}{2},+\infty\right[\end{array}\right| D=\left[-\sqrt{2},-\frac{1}{2}\right]\cup\left[\frac{1}{2},\sqrt{2}\right]$$

if we have substituted $-x$ instead of x then the relation is True. So, we will solve the equation and the interval $\left[\frac{1}{2},\sqrt{2}\right]$

Note in the interval $\left[\frac{1}{2},\sqrt{2}\right]$: $|x|=x$

$$\sqrt[2n]{2-x^2}+\sqrt[2n]{2x-1}=(x^2-1)^{2m}+2$$

$$(x^2-1)^{2m}+2\leq 2\sqrt[2n]{\frac{2-x^2+2x-1}{2}}$$

$$\frac{(x^2-1)^{2m}+2}{2}\leq\sqrt[2n]{\frac{-x^2+2x+1}{2}}$$

$$\frac{(x^2-1)^{2m}+2}{2}\leq\sqrt[2n]{1-\frac{(x-1)^2}{2}}\leq 1$$

$$\frac{(x^2-1)^{2m}}{2}+1\leq 1\Rightarrow\frac{(x^2-1)^{2m}}{2}\leq 0$$

This holds when: $x^2-1=0\Rightarrow x^2=1;\ x=1$. *So:* $S'=\{-1,+1\}$

SP.212. Evaluate:

$$\lim_{n\to\infty}\frac{\left\lfloor e^{\frac{1}{n}}\right\rfloor+\left\lfloor e^{\frac{2}{n}}\right\rfloor+\cdots+\left\lfloor e^{\frac{n}{n}}\right\rfloor}{n}$$

where $\lfloor x\rfloor$ denotes the integer part of x.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Samir HajAli-Damascus-Syria

$$\lim_{n\to\infty}\frac{\left[e^{\frac{1}{n}}\right]+\left[e^{\frac{2}{n}}\right]+\cdots+\left[e^{\frac{n}{n}}\right]}{n}=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left[e^{\frac{k}{n}}\right]}{n}=\int_0^1[e^x]\,dx$$

Let put $e^x = t \Rightarrow dx = \frac{dt}{t}$ *then* $\int_0^1 [e^x]\,dx = \int_0^e [t]\frac{dt}{t} = \int_1^2 [t]\frac{dt}{t} + \int_2^e [t]\frac{dt}{t}$

$$= \int_1^2 \frac{dt}{t} + \int_2^e 2\frac{dt}{t} = \ln 2 + 2(1 - \ln 2) = 2 - \ln 2$$

SP.213. Prove that in any ABC triangle the following inequality holds:

$$\frac{9r^2}{4R^2}(2R^2 - 5r^2) \le \sum m_a^2 \sin^2\frac{B}{2}\sin^2\frac{C}{2} \le \frac{1}{4R^2}(4R^4 - 37r^4)$$

Proposed by Marin Chirciu – Romania

Solution by proposer

We prove the following lemma:

Lemma: In ΔABC:

$$\sum m_a^2 \sin^2\frac{B}{2}\sin^2\frac{C}{2} = \frac{s^2(s^2 - 12Rr) + r(4R + r)(2R - r)}{16R^2}$$

Proof: Using $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$ ***and*** $\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$ ***we obtain:***

$$\sum m_a^2 \sin^2\frac{B}{2}\sin^2\frac{C}{2} = \sum \frac{2b^2 + 2c^2 - a^2}{4}\cdot\frac{(s-a)(s-c)}{ac}\cdot\frac{(s-a)(s-b)}{ab} =$$

$$= \frac{(s-a)(s-b)(s-c)}{4abc}\sum \frac{(2b^2 + 2c^2 - a^2)(s-a)}{a}$$

$$= \frac{r}{4R}\cdot\frac{s^2(s^2 - 12Rr) + r(4R + r)(2R - r)}{4Rr} = \frac{s^2(s^2 - 12Rr) + r(4R + r)(2R - r)}{16R^2}$$

Let's get back to the main problem: Using the Lemma, the inequality can be written:

$$\frac{9r^2}{4R^2}(2R^2 - 5r^2) \le \frac{s^2(s^2 - 12Rr) + r(4R + r)(2R - r)}{16R^2} \le \frac{1}{4R^2}(4R^4 - 37r^4)$$

which follows from Gerretsen's inequality: $16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2$ ***and Euler's inequality*** $R \ge 2r$***. Equality holds if and only if the triangle is equilateral.***

SP.214. Prove that in any ABC triangle the following inequality holds:

$$\frac{3r^2}{4R^2}(4R + r)^2 \le \sum m_a^2 \cos^2\frac{B}{2}\cos^2\frac{C}{2} < \frac{3}{16}(4R + r)^2$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3r^2}{4R^2}(4R+r)^2 \overset{(1)}{\leq} \sum m_a^2 \cos^2\frac{B}{2}\cos^2\frac{C}{2} \overset{(2)}{\leq} \frac{3}{16}(4R+r)^2$$

$$\sum m_a^2 \cos^2\frac{B}{2}\cos^2\frac{C}{2} = \left(\frac{s}{4R}\right)^2\left(\sum m_a^2 \sec^2\frac{A}{2}\right)$$

$$= \frac{s^2}{16R^2}\sum\left\{\left(\frac{2b^2+2c^2+2a^2-3a^2}{4}\right)\left(\frac{bc}{s(s-a)}\right)\right\}$$

$$= \left(\frac{s}{64R^2}\right)\left[2\left(\sum a^2\right)\sum\frac{bc}{s-a} - 3abc\sum\frac{a}{s-a}\right]$$

$$= \left(\frac{s}{64R^2}\right)\left[\frac{4(s^2-4Rr-r^2)}{r^2s}\sum bc\,(s-b)(s-c) - 12Rrs\left(\sum\frac{a-s+s}{s-a}\right)\right]$$

$$= \left(\frac{s}{64R^2}\right)\left[\frac{4(s^2-4Rr-r^2)}{r^2s}\sum bc\,(s^2-s(2s-a)+bc)\right]$$

$$-\left(\frac{s}{64R^2}\right)12Rrs\left(-3+\frac{s\sum(s-b)(s-c)}{r^2s}\right)$$

$$= \left(\frac{s}{64R^2}\right)\left[\frac{4(s^2-4Rr-r^2)}{r^2s}\left\{s^2\sum ab - 2s^2\sum ab + 3sabc + \left(\sum ab\right)^2 - 2abc(2s)\right\}\right]$$

$$-\left(\frac{s}{64R^2}\right)12Rrs\left(-3+\frac{\sum(s^2-s(b+c)+bc)}{r^2}\right)$$

$$= \left(\frac{s}{64R^2}\right)\left[\frac{4(s^2-4Rr-r^2)}{r^2s}\left\{\left(\sum ab\right)(4Rr+r^2) - 4Rrs^2\right\}\right]$$

$$-\left(\frac{s}{64R^2}\right)12Rrs\left(-3+\frac{3s^2-4s^2+s^2+4Rr+r^2}{r^2}\right)$$

$$= \left(\frac{s}{64R^2}\right)\left[\frac{4(s^2-4Rr-r^2)}{s}\{s^2+(4R+r)^2\} - 12Rrs\left(\frac{4R-2r}{r}\right)\right]$$

$$= \frac{(s^2-4Rr-r^2)(s^2+(4R+r)^2) - 6Rs^2(2R-r)}{16R^2}$$

$$\overset{(a)}{=} \frac{s^4+s^2(4R^2+10Rr)-(64R^3r+48R^2r^2+12Rr^3+r^4)}{16R^2}$$

(a) $\Rightarrow$ ***(1)*** $\Leftrightarrow \frac{s^4+s^2(4R^2+10Rr)-(64R^3r+48R^2r^2+12Rr^3+r^4)}{16R^2} - \frac{3r^2}{4R^2}(4R+r)^2 \geq 0$

$$\Leftrightarrow s^4+s^2(4R^2+10Rr)-(64R^3r+48R^2r^2+12Rr^3+r^4)-12r^2(4R+r)^2 \overset{(1a)}{\geq} 0$$

Now, LHS of (1a) $\overset{Gerretsen}{\geq} s^2(4R^2+26Rr-5r^2)-(64R^3r+48R^2r^2+12Rr^3+r^4)-$

$$-12r^2(4R+r)^2$$

$\overset{Gerretsen}{\geq} (16Rr-5r^2)(4R^2+26Rr-5r^2)-(64R^3r+48R^2r^2+12Rr^3+r^4)-$

$$-12r^2(4R+r)^2\overset{?}{\geq}0\Leftrightarrow 26R^2-53Rr+2r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(26R-r)\overset{?}{\geq}0$$

$\rightarrow$ ***true*** $\because R\overset{Euler}{\geq}2r\Rightarrow$ ***(1a)⇒ (1) is true. Again, (a)⇒ (2)***

$$\Leftrightarrow\frac{s^4+s^2(4R^2+10Rr)-(64R^3r+48R^2r^2+12Rr^3+r^4)}{16R^2}\leq\frac{3}{16}(4R+r)^2$$

$$\Leftrightarrow s^4+s^2(4R^2+10Rr)-(64R^3r+48R^2r^2+12Rr^3+r^4)-3R^2(4R+r)^2\overset{(2a)}{\leq}0$$

Now, LHS of (2a) $\overset{Gerretsen}{\leq} s^2(8R^2+14Rr+3r^2)-(64R^3r+48R^2r^2+12Rr^3+r^4)-$

$$-3R^2(4R+r)^2$$

$\overset{Gerretsen}{\leq}(4R^2+4Rr+3r^2)(8R^2+14Rr+3r^2)-(64R^3r+48R^2r^2+12Rr^3+r^4)-$

$$-3R^2(4R+r)^2\overset{?}{\leq}0\Leftrightarrow 16t^4-41t^2-42t-8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$\Leftrightarrow(t-2)(16t^3+32t^2+23t+4)\overset{?}{\geq}0\rightarrow$ ***true*** $\because t\overset{Euler}{\geq}2\Rightarrow$ ***(2a)⇒ (2) is true***

(proved)

SP.215. Let a,b,c be positive real numbers such that $a+b+c+1=4abc$. Prove that:

$$\frac{a^2b}{b+5c}+\frac{b^2c}{c+5a}+\frac{c^2a}{a+5b}\geq\frac{1}{2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Michael Sterghiou-Greece

$$a+b+c+1=4abc\ \ (c)$$

$$\sum_{cyc}\frac{a^2b}{b+5c}\geq\frac{1}{2}\ \ (1)$$

$$(1)\rightarrow\sum_{cyc}\frac{a^2b^2}{b^2+5bc}\overset{BCS}{\geq}\frac{\left(\sum_{cyc}ab\right)^2}{\left(\sum_{cyc}a^2\right)+5\sum_{cyc}ab}\overset{?}{\geq}\frac{1}{2}\ \ (2)$$

Let $(p,q,r) = \left(\sum_{cyc} a, \sum_{cyc} ab, abc\right)$

(2)$\rightarrow \frac{q^2}{p^2-2q+5q} \ge \frac{1}{2} \rightarrow 2q^2 \ge p^2 + 3q \rightarrow 2q^2 - p^2 - 3q \ge 0$ *and as* $-3q \ge -p^2$ *it suffices that:* $2(q^2 - p^2) \ge 0 \rightarrow q \ge p$ *(3)*

(3)$\rightarrow ab + bc + ca - a - b - c \ge 0$. *From (c):* $p + 1 = 4r$ *so we have to show that:* $ab + bc + ca - 4abc + 1 \ge 0$ *or* $q - 4r + 1 \ge 0$ *(4). This is a decreasing function of* r *so we need to show (4) where* r *becomes maximal. This according to* V*. Cîrtoaje theorem with fixed happens when* $a = b$ *assuming WLOG that* $a \le b \le c$*. Assuming* $a = b$ *we have:*

From (c) $2a + c + 1 = 4a^2c$ *or* $c = \frac{2a+1}{4a^2-1} = \frac{1}{2a-1}$*. As* $c > 0$ *we have* $a > \frac{1}{2}$*. Now, (4) becomes:* $a^2 + 2ac - 4a^2c + 1 \ge 0$ *or* $a^2 + \frac{2a}{2a-1} - \frac{4a^2}{2a-1} + 1 \ge 0$ *which reduces to:* $2a^3 - 5a^2 + 4a - 1 \ge 0$ *or* $(a-1)^2(2a-1) \ge 0$ *which holds. Done!*

Solution 2 by Marian Ursărescu-Romania

One of my student, asked my if it is possible to decondition relationship $a + b + c + 1 = 4abc$*. The answer is yes: first using Bergström inequality:*

$$\frac{a^2b^2}{b^2+5cb} + \frac{b^2c^2}{c^2+5ac} + \frac{c^2a^2}{a^2+5ab} \ge \frac{(ab+ac+bc)^2}{a^2+b^2+c^2+5(ab+ac+bc)} \Rightarrow$$

We must show: $\frac{(ab+bc+ac)^2}{a^2+b^2+c^2+5(ab+ac+bc)} \ge \frac{1}{2}$ *(1)*

Now let $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z} \Rightarrow a + b + c + 1 = 4abc \Leftrightarrow$

$$\Leftrightarrow xy + xz + yz + xyt = 4(2) \Rightarrow$$

(1)$\Leftrightarrow \frac{(x+y+z)^2}{x^2y^2+x^2z^2+y^2z^2+5xyz(x+y+z)} \ge \frac{1}{2}$ *(3)*

Because $xy + xz + yz + xyz = 4 \Rightarrow \exists m,n,p > 0$ *such that:* $x = \frac{2m}{n+p}, y = \frac{2n}{m+p},$

$z = \frac{2p}{m+n} \Rightarrow$*(3)*$\Leftrightarrow \frac{\left(\sum m(m+n)(m+p)\right)^2}{2\sum m^2n^2(m+n)+10mnp\sum m(m+n)(m+p)} \ge 1$ *(4)*

Relation (4) it is true because using Cîrtoaje's theorem: If $f_6(m,n,p)$ *it's a symmetric polygon of degree* 6 *then*

$$f_6(a,b,c) \ge 0, \forall a,b,c \in \mathbb{R} \Leftrightarrow f_6(a,1,1) \ge 0, \forall a \in \mathbb{R}$$

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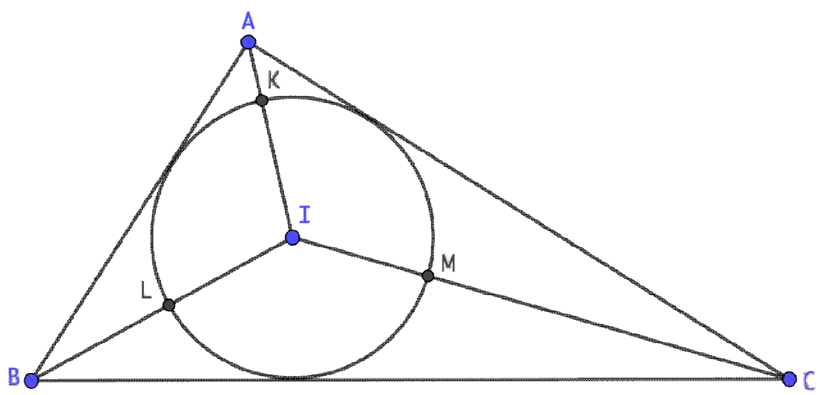
SP.216. Let I be the incentre of a triangle ABC with inradius r, and let K, L, M be the intersection points of the segments AI, BI, CI with the inscribed of the triangle ABC, respectively. Prove that:

$$AK^n + BL^n + CM^n \geq 3 \cdot r^n$$

for each positive integer n.

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Marian Ursărescu – Romania



From Hölder's inequality we have: $AK^n + BL^n + CM^n \geq \frac{(AK+BL+CM)^n}{3^{n-1}} \Rightarrow$

We must show: $\frac{(AK+BL+CM)^n}{3^{n-1}} \geq 3r^n \Leftrightarrow AK + BL + CM \geq 3r$ ***(1)***

But $AK = AI - r = \frac{r}{\sin\frac{A}{2}} - r = r\left(\frac{1}{\sin\frac{A}{2}} - 1\right)$ ***(2)***

From (1)+(2) we must show that: $\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}} \geq 6$ ***(3)***

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \leq \frac{a}{2\sqrt{bc}} \Rightarrow \frac{1}{\sin\frac{A}{2}} \geq \frac{2\sqrt{bc}}{a} \Rightarrow$$

$\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}} \geq 2\left(\frac{\sqrt{bc}}{a} + \frac{\sqrt{ac}}{b} + \frac{\sqrt{ab}}{c}\right) \geq 2 \cdot 3\sqrt[3]{\frac{abc}{abc}} = 6 \Rightarrow$ ***(3) it is true.***

SP.217. Let a, b, c be positive real numbers such that:

$(a^3+b^3)(b^3+c^3)(c^3+a^3)=8$. Find the minimum value of:

$$T=\frac{a}{(b^2+bc+c^2)(a+2b)^2}+\frac{b}{(c^2+ca+a^2)(b+2c)^2}+\frac{c}{(a^2+ab+b^2)(c+2a)^2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Firstly, $\frac{a}{a+2b}+\frac{b}{b+2c}+\frac{c}{c+2a}=\frac{a^2}{a^2+2ab}+\frac{b^2}{b^2+2bc}+\frac{c^2}{c^2+2ca}$

$$\overset{Bergstrom}{\geq}\frac{(\sum a)^2}{\sum a^2+2\sum ab}=\frac{(\sum a)^2}{(\sum a)^2}=1\Rightarrow\sum\frac{a}{a+2b}\overset{(1)}{\geq}1$$

Now, $a^3+b^3\overset{Chebyshev}{\geq}\frac{1}{2}(a+b)(a^2+b^2)\geq\frac{1}{2}(a+b)\cdot\frac{1}{2}(a+b)^2\Rightarrow a^3+b^3\overset{(a)}{\geq}\frac{(a+b)^3}{4}$

Similarly, $b^3+c^3\overset{(b)}{\geq}\frac{(b+c)^3}{4}$ ***and*** $c^3+a^3\overset{(c)}{\geq}\frac{(c+a)^3}{4}$

(a).(b).(c)$\Rightarrow\frac{\{\prod(a+b)\}^3}{64}\leq\prod(a^3+b^3)=8\Rightarrow\prod(a+b)\overset{(2)}{\leq}8$

Now, $T=\sum\frac{\left(\frac{a}{a+2b}\right)^2}{a(b^2+bc+c^2)}\overset{Bergstrom}{\geq}\frac{\left(\sum\frac{a}{a+2b}\right)^2}{\sum a^2b+\sum ab^2+3abc}$

$$\overset{by\ (1)}{\geq}\frac{1}{\prod(a+b)+abc}\overset{by\ (2)}{\geq}\frac{1}{8+abc}\overset{Cesaro}{\geq}\frac{1}{8+\frac{\prod(a+b)}{2}}$$

$\overset{by\ (2)}{\geq}\frac{1}{8+\frac{8}{8}}=\frac{1}{9}\therefore T_{\min}=\frac{1}{9}$, ***equality when*** $a=b=c=1$. ***(Answer)***

Solution 2 by Tran Hong-Dong Thap-Vietnam

For $a, b>0$ ***we have:*** $a^2+ab+b^2\leq 3(a^2-ab+b^2)\leftrightarrow 2(a^2-2ab+b^2)\geq 0$

$\leftrightarrow 2(a-b)^2\geq 0$ ***(true)***

$$(b^3+a^3)(a^3+c^3)(1^3+1^3)\overset{Holder}{\geq}(b\cdot a\cdot 1+a\cdot c\cdot 1)^3=(ba+ac)^3$$

$\rightarrow(b^3+a^3)(a^3+c^3)\geq\frac{(ba+ac)^3}{2}$. ***Now,*** $\frac{a}{(b^2+bc+c^2)(a+2b)^2}\geq\frac{a}{3(b^2-bc+c^2)(a+2b)^2}$

$$=\frac{a(b+c)}{3(b^3+c^3)(a+2b)^2}=\frac{a(b+c)(b^3+c^3)(a^3+c^3)}{24(a+2b)^2}$$

$$\geq \frac{(ab+ac)}{24(a+2b)^2}\cdot\frac{(ba+ac)^3}{2}=\frac{(ab+ac)^4}{48(a+2b)^2}$$

Similarly: $\frac{b}{(c^2+ca+a^2)(b+2c)^2}\geq\frac{b(c+a)(a^3+b^3)(b^3+c^3)}{24(b+2c)^2}\geq\frac{(bc+ba)^4}{48(b+2c)^2}$

And: $\frac{c}{(a^2+ab+b^2)(c+2a)^2}\geq\frac{c(a+b)(a^3+c^3)(b^3+c^3)}{24(c+2a)^2}\geq\frac{(ca+cb)^4}{48(c+2a)^2}$

$$\to\frac{a}{(b^2+bc+c^2)(a+2b)^2}+\frac{b}{(c^2+ca+a^2)(b+2c)^2}+\frac{c}{(a^2+ab+b^2)(c+2a)^2}\geq$$

$$\geq\frac{(ab+ac)^4}{48(a+2b)^2}+\frac{(bc+ba)^4}{48(b+2c)^2}+\frac{(ca+cb)^4}{48(c+2a)^2}=$$

$$=\frac{1}{48}\left(\frac{(ab+ac)^4}{(a+2b)^2}+\frac{(bc+ba)^4}{(b+2c)^2}+\frac{(ca+cb)^4}{(c+2a)^2}\right)=\omega$$

We have: $\frac{(ab+ac)^4}{(a+2b)^2}+\frac{(bc+ba)^4}{(b+2c)^2}+\frac{(ca+cb)^4}{(c+2a)^2}\geq\frac{16}{3}$

It is true because: $\frac{(ab+ac)^4}{(a+2b)^2}+\frac{(bc+ba)^4}{(b+2c)^2}+\frac{(ca+cb)^4}{(c+2a)^2}\overset{Schwarz}{\geq}\frac{[(ab+ac)^2+(bc+ba)^2+(ca+cb)^2]^2}{(a+2b)^2+(b+2c)^2+(c+2a)^2}$**. And:**

$$3[(ab+ac)^2+(bc+ba)^2+(ca+cb)^2]^2\geq 16((a+2b)^2+(b+2c)^2+(c+2a)^2)$$

(By ABC theorem) $\to\omega\geq\frac{1}{48}\cdot\frac{16}{3}=\frac{1}{9}$**. Equality** $\leftrightarrow a=b=c=1$.

SP.218. Let x,y,z be positive real numbers such that:

$x^2+y^2+z^2=3$. Find the minimum of the expression:

$$P=\frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}+\frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}}+3zx}+\frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}}+3xy}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a^4+b^4\leq 2(a^2-ab+b^2)^2\Leftrightarrow a^4+b^4\leq 2(a^2+b^2)^2-4ab(a^2+b^2)+2a^2b^2$$

$$\Leftrightarrow a^4+b^4\leq 2(a^4+b^4)+4a^2b^2-4ab(a^2+b^2)+2a^2b^2$$

$$\Leftrightarrow (a^2+b^2)^2-4ab(a^2+b^2)+4a^2b^2\geq 0\Leftrightarrow (a^2+b^2-2ab)^2\geq 0$$

$$\to \textit{true}\therefore a^4+b^4\overset{(1)}{\leq}2(a^2-ab+b^2)^2$$

Choosing $a = y^2$ *and* $b = z^2$ *in (1),* $y^8 + z^8 \le 2(y^4 - y^2z^2 + z^4)^2$

$$\Rightarrow \sqrt[4]{\frac{y^8 + z^8}{2}} + 3yz \le \sqrt{y^4 - y^2z^2 + z^4} + \sqrt{3}(\sqrt{3}yz)$$

$$\overset{CBS}{\le} \sqrt{1+3}\sqrt{y^4 - y^2z^2 + z^4 + 3y^2z^2} = 2\sqrt{(y^2+z^2)^2} = 2(y^2+z^2)$$

$$\therefore \sqrt[4]{\frac{y^8 + z^8}{2}} + 3yz \overset{(a)}{\le} 2(y^2+z^2)$$

Similarly, $\sqrt[4]{\frac{z^8+x^8}{2}} + 3zx \overset{(b)}{\le} 2(z^2+x^2)$ *and,* $\sqrt[4]{\frac{x^8+y^8}{2}} + 3xy \overset{(c)}{\le} 2(x^2+y^2)$

(a), (b), (c)$\Rightarrow P \ge \frac{1}{2}\sum \frac{x}{y^2+z^2} = \frac{1}{2}\sum \frac{x}{3-x^2} \therefore P \overset{(i)}{\ge} \frac{1}{2}\sum \frac{x}{3-x^2}$

Now, $\frac{x}{3-x^2} \ge \frac{x^2}{2} \Leftrightarrow 2 \ge x(3-x^2) \Leftrightarrow x^3 - 3x + 2 \ge 0$

$\Leftrightarrow (x+2)(x-1)^2 \ge 0 \to$ *true* $\therefore \frac{x}{3-x^2} \overset{(d)}{\ge} \frac{x^2}{2}$*. Similarly,* $\frac{y}{3-y^2} \overset{(e)}{\ge} \frac{y^2}{2}$ *and,* $\frac{z}{3-z^2} \overset{(f)}{\ge} \frac{z^2}{2}$

(d), (e), (f), (i) $\Rightarrow P \ge \frac{1}{4}\sum x^2 = \frac{3}{4}$

$\therefore P_{\min} = \frac{3}{4}$ *and it occurs when* $x = y = z = 1$*.*

Solution 2 by Tran Hong-Dong Thap-Vietnam

For all $a, b > 0$ *we have:* $\sqrt[4]{\frac{a^8+b^8}{2}} + 3ab \le 2(a^2+b^2)$ *(*)*

$$\leftrightarrow \sqrt[4]{\frac{a^8 + b^8}{2}} \le 2(a^2+b^2) - 3ab \leftrightarrow \frac{a^8+b^8}{2} \le (2(a^2+b^2) - 3ab)^4$$

$$\leftrightarrow \frac{1}{2}(a-b)^4[31(a^4+b^4) + 102a^2b^2 - 68(ab^3 + ba^3)] \ge 0$$

Which is true because: $\frac{1}{2}(a-b)^4 \ge 0$

$$\begin{cases} 31a^4 + 51a^2b^2 \overset{AM-GM}{\ge} 2\sqrt{31\cdot 51\cdot a^6b^2} = 2\sqrt{1581}\cdot ba^3 > 68\cdot ba^3 \\ 31b^4 + 51a^2b^2 \overset{AM-GM}{\ge} 2\sqrt{31\cdot 51\cdot a^6b^2} = 2\sqrt{1581}\cdot ab^3 > 68\cdot ab^3 \end{cases}$$

$$\to 31(a^4+b^4) + 102a^2b^2 - 68(ab^3 + ba^3) > 0$$

So, (*) is true. Equality if and only if $a = b$*. Now, using (*) inequality:*

$$P=\frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}+\frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}}+3zx}+\frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}}+3xy}$$

$$\geq\frac{x}{2(y^2+z^2)}+\frac{y}{2(z^2+x^2)}+\frac{z}{2(x^2+y^2)}=\frac{x}{2(3-x^2)}+\frac{y}{2(3-y^2)}+\frac{z}{2(3-z^2)}$$

(Because:** $x^2+y^2+z^2=3\to 0<x^2,y^2,z^2<3$**)

Lastly, we must show that: $\frac{x}{2(3-x^2)}+\frac{y}{2(3-y^2)}+\frac{z}{2(3-z^2)}\geq\frac{3}{4}$

$\leftrightarrow\frac{x}{(3-x^2)}+\frac{y}{(3-y^2)}+\frac{z}{(3-z^2)}\geq\frac{3}{2}$ ***(**)***

We have: $\frac{x}{(3-x^2)}\geq\frac{x^2}{2}\overset{0<x<\sqrt{3}}{\leftrightarrow}2\geq x(3-x^2)\leftrightarrow(x-1)^2(x+2)\geq 0$ ***(true)***

Similarly: $\frac{y}{(3-y^2)}\geq\frac{y^2}{2}$ ***and*** $\frac{z}{(3-z^2)}\geq\frac{z^2}{2}$

$$\to\frac{x}{(3-x^2)}+\frac{y}{(3-y^2)}+\frac{z}{(3-z^2)}\geq\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{3}{2}$$

So, (**) true. $\to P_{\min}=\frac{3}{2}\leftrightarrow x=y=z=1.$

SP.219. Prove the following inequality:

$$\sum_{k=1}^{n}\frac{a_k^2}{a_k+(n+1)(S-a_k)}\geq\frac{1}{n^2}\sum_{k}^{n}a_k$$

where $a_1,a_2,\dots,a_n$ are any strictly positive real numbers and we make the notation: $S=a_1+a_2+\dots+a_n$

Proposed by Vasile Mircea Popa – Romania

Solution by Marian Ursărescu – Romania

From Bergström inequality, we have:

$$\sum_{k=1}^{n}\frac{a_k^2}{a_k+(n+1)(S-a_k)}\geq\frac{(a_1+a_2+\dots+a_n)^2}{(a_1+a_2+\dots+a_n)+(n+1)(nS-a_1-a_2-\dots-a_n)}$$

$$=\frac{(a_1+a_2+\dots+a_n)^2}{S+(n+1)(n-1)S}=\frac{S^2}{S+(n^2-1)S}=\frac{S^2}{n^2S}=\frac{S}{n^2}=\frac{1}{n^2}\sum_{k=1}^{n}a_k$$

SP.220. Let $a, b, c > 0$ such that: $a + b + c = 3$. Find the minimum of the expression:

$$P = \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc} + \frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca} + \frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab} + \frac{(a+b)(b+c)(c+a)}{24}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

With $a, b > 0$ ***we have:*** $4(a^6+b^6) \le (3a^2-4ab+3b^2)^3$

$$\leftrightarrow (a-b)^4(23a^2-16ab+23b^2) \ge 0 \leftrightarrow (a-b)^4[23(a-b)^2+3ab] \ge 0 \text{ (true)}$$

Equality if and only if $a = b$. ***Similarly:*** $4(a^6+c^6) \le (3a^2-4ac+3c^2)^3$

$$4(b^6+c^6) \le (3b^2-4bc+3c^2)^3 \to \sqrt[3]{4(b^6+c^6)} \le 3b^2-4bc+3c^2$$

$$\to \sqrt[3]{4(b^6+c^6)} \le 3b^2-4bc+3c^2$$

$$\to \sqrt[3]{4(b^6+c^6)} + 7bc \le 3b^2-4bc+3c^2+7bc = 3(b^2+bc+c^2)$$

Similarly: $\sqrt[3]{4(a^6+c^6)} + 7ac \le 3(a^2+ac+c^2)$

$$\sqrt[3]{4(a^6+b^6)} + 7ab \le 3(a^2+ab+b^2)$$

$$\to \Omega = \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc} + \frac{b}{\sqrt[3]{4(a^6+c^6)}+7ac} + \frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}$$

$$\ge \frac{a}{3(b^2+bc+c^2)} + \frac{b}{3(a^2+ac+c^2)} + \frac{c}{3(a^2+ab+b^2)}$$

$$= \frac{1}{3}\left(\frac{a}{b^2+bc+c^2} + \frac{b}{a^2+ac+c^2} + \frac{c}{a^2+ab+b^2}\right)$$

We have: $\frac{a}{b^2+bc+c^2} + \frac{b}{a^2+ac+c^2} + \frac{c}{a^2+ab+b^2}$

$$= \frac{a^2}{a(b^2+bc+c^2)} + \frac{b^2}{b(a^2+ac+c^2)} + \frac{c^2}{c(a^2+ab+b^2)} \overset{Schwarz}{\ge}$$

$$\ge \frac{(a+b+c)^2}{ab^2+ba^2+ac^2+ca^2+bc^2+cb^2+3abc}$$

$$= \frac{9}{ab^2+ba^2+ac^2+ca^2+bc^2+cb^2+3abc}$$

$$= \frac{9}{(ab^2+ba^2+ac^2+ca^2+bc^2+cb^2+2abc)+abc}$$

$$=\frac{9}{(a+b)(b+c)(c+a)+abc}\overset{(a+b)(b+c)(c+a)\geq 8abc}{\geq}$$

$$\geq\frac{9}{(a+b)(b+c)(c+a)+\frac{(a+b)(b+c)(c+a)}{8}}=\frac{8}{(a+b)(b+c)(c+a)}$$

Then: $P\geq\frac{8}{3(a+b)(b+c)(c+a)}+\frac{(a+b)(b+c)(c+a)}{24}\overset{AM-GM}{\geq}2\sqrt{\frac{8}{3(a+b)(b+c)(c+a)}\cdot\frac{(a+b)(b+c)(c+a)}{24}}=\frac{2}{3}$

$$\rightarrow P_{\min}=\frac{2}{3}\leftrightarrow a=b=c=1.$$

SP.221. Prove that in any ΔABC the following inequality holds:

$$\sqrt[3]{(\pi-A)^{m_a^2}\cdot(\pi-B)^{m_b^2}\cdot(\pi-C)^{m_c^2}}\geq\sqrt[4]{(\pi-A)^{a^2}(\pi-B)^{b^2}(\pi-C)^{c^2}}$$

A, B, C the measures in radians of the angles.

Proposed by Marian Ursărescu – Romania

Solution by Adrian Popa – Romania

$$\sqrt[3]{(\pi-A)^{m_a^2}\cdot(\pi-B)^{m_b^2}\cdot(\pi-C)^{m_c^2}}\geq\sqrt[4]{(\pi-A)^{a^2}(\pi-B)^{b^2}(\pi-C)^{c^2}}\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{3}\left[\ln(\pi-A)^{m_a^2}+\ln(\pi-B)^{m_b^2}+\ln(\pi-C)^{m_c^2}\right]\geq\frac{1}{4}\left[\ln(\pi-A)^{a^2}+\ln(\pi-B)^{b^2}+\ln(\pi-C)^{c^2}\right]$$

We suppose: $A\geq B\geq C\Rightarrow\begin{cases}a\geq b\geq c\Rightarrow a^2\geq b^2\geq c^2\\ m_a\leq m_b\leq m_c\Rightarrow m_a^2\leq m_b^2\leq m_c^2\end{cases}$

$\Rightarrow B+C\leq A+C\leq A+B\Rightarrow\pi-A\leq\pi-B\leq\pi-C$. ***From Cebyshev we have:***

$$\frac{1}{3}\left[m_a^2\ln(\pi-A)+m_b^2\ln(\pi-B)+m_c^2\ln(\pi-C)\right]\geq$$

$$\geq\frac{m_a^2+m_b^2+m_c^2}{3\cdot 3}(\ln(\pi-A)+\ln(\pi-B)+\ln(\pi-C))=$$

$$=\frac{\frac{3}{4}(a^2+b^2+c^2)}{3\cdot 3}(\ln(\pi-A)+\ln(\pi-B)+\ln(\pi-C))=$$

$$=\frac{a^2+b^2+c^2}{4\cdot 3}(\ln(\pi-A)+\ln(\pi-B)+\ln(\pi-C))\geq$$

$$\overset{Cebyshev}{\geq}\frac{1}{4}(a^2\ln(\pi-A)+b^2\ln(\pi-B)+c^2\ln(\pi-C))=$$

$$=\frac{1}{4}\left(\ln(\pi-A)^{a^2}(\pi-B)^{b^2}(\pi-C)^{c^2}\right)=\ln\left((\pi-A)^{a^2}(\pi-B)^{b^2}(\pi-C)^{c^2}\right)^{\frac{1}{4}}\Rightarrow$$

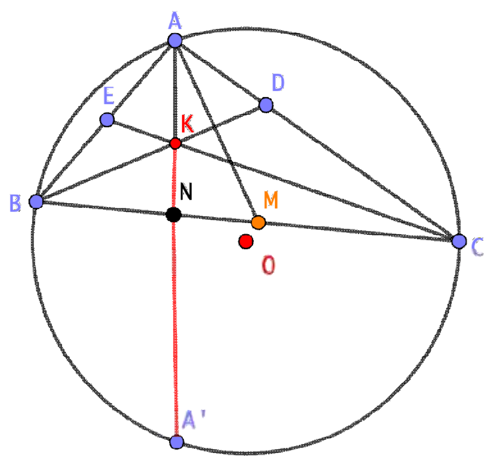
$$\Rightarrow\sqrt[3]{(\pi-A)^{m_a^2}(\pi-B)^{m_b^2}(\pi-C)^{m_c^2}}\geq\sqrt[4]{(\pi-A)^{a^2}(\pi-B)^{b^2}(\pi-C)^{c^2}}$$

SP.222. Let ABC be a triangle and A', B', C' the intersection points of the simedians with circumcircle. Prove that:

$$\frac{6r}{R^2}\leq\frac{1}{KA'}+\frac{1}{KB'}+\frac{1}{KC'}\leq\frac{3R}{4r^2}$$

Proposed by Marian Ursărescu – Romania

Solution by Tran Hong-Dong Thap-Vietnam



We have: $AN=\frac{bc\sqrt{2(b^2+c^2)-a^2}}{(b^2+c^2)}$

By Van Aubel's theorem we have: $\frac{KA}{KN}=\frac{EA}{EB}+\frac{DA}{DC}=\frac{b^2}{a^2}+\frac{c^2}{a^2}=\frac{b^2+c^2}{a^2}$

$$\rightarrow KA=\frac{b^2+c^2}{a^2+b^2+c^2}\cdot AN=\frac{bc\sqrt{2(b^2+c^2)-a^2}}{a^2+b^2+c^2}$$

Now, we compute AA': $\begin{cases}\frac{NB}{NC}=\frac{c^2}{b^2}\\ NB+NC=a\end{cases}\rightarrow\begin{cases}NB=\frac{ac^2}{b^2+c^2}\\ NC=\frac{ab^2}{b^2+c^2}\end{cases}$

If AN, AM are symedian and median of triangle, respectively, then $\widehat{BAN}=\widehat{CAM}$

More, $\widehat{BA'A}=\widehat{BAC}\to\Delta ABA'\sim\Delta AMC$

$$\to\frac{c}{AM}=\frac{AA'}{b}\to AA'=\frac{bc}{AM}=\frac{2bc}{\sqrt{2(b^2+c^2)-a^2}}$$

$$\to KA'=AA'-AK=\frac{2bc}{\sqrt{2(b^2+c^2)-a^2}}-\frac{bc\sqrt{2(b^2+c^2)-a^2}}{a^2+b^2+c^2}$$

$=\frac{3bca^2}{(a^2+b^2+c^2)\left(\sqrt{2(b^2+c^2)-a^2}\right)}$. *Similarly:* $KB'=\frac{3acb^2}{(a^2+b^2+c^2)\left(\sqrt{2(a^2+c^2)-b^2}\right)}$

And: $KC'=\frac{3abc^2}{(a^2+b^2+c^2)\left(\sqrt{2(a^2+b^2)-c^2}\right)}\to\Omega=\frac{1}{KA'}+\frac{1}{KB'}+\frac{1}{KC'}$

$$=\frac{(a^2+b^2+c^2)}{3abc}\cdot\left(\frac{\sqrt{2(b^2+c^2)-a^2}}{a}+\frac{\sqrt{2(a^2+c^2)-b^2}}{b}+\frac{\sqrt{2(a^2+b^2)-c^2}}{c}\right)=$$

$=\frac{2(a^2+b^2+c^2)}{3abc}\cdot\left(\frac{AM}{a}+\frac{BM}{b}+\frac{CM}{c}\right)$. *We have:*

$$\frac{AM}{a}+\frac{BM}{b}+\frac{CM}{c}\ge\frac{3\sqrt{3}}{2};a^2+b^2+c^2\ge4\sqrt{3}S=4\sqrt{3}\cdot\frac{abc}{4R}=\frac{abc\sqrt{3}}{R}$$

$$\to\frac{2(a^2+b^2+c^2)}{3abc}\cdot\left(\frac{AM}{a}+\frac{BM}{b}+\frac{CM}{c}\right)\ge2\cdot\frac{3\sqrt{3}}{2}\cdot\frac{abc\sqrt{3}}{3R\cdot abc}=\frac{3}{R}$$

We must show that: $\frac{3}{R}\ge\frac{6r}{R^2}\leftrightarrow R\ge2R$ *(Euler)*$\to\Omega=\frac{1}{KA'}+\frac{1}{KB'}+\frac{1}{KC'}=\frac{6r}{R^2}$

Lastly, we show that: $\Omega\le\frac{3R}{4r^2}\leftrightarrow\frac{2(a^2+b^2+c^2)}{3abc}\cdot\left(\frac{AM}{a}+\frac{BM}{b}+\frac{CM}{c}\right)\le\frac{3R}{4r^2}$

$$(a^2+b^2+c^2)\le9R^2;abc=4Rrs\to\frac{2(a^2+b^2+c^2)}{3abc}\le\frac{2\cdot3R}{4rs}$$

We must show that: $\frac{AM}{a}+\frac{BM}{b}+\frac{CM}{c}\le\frac{s}{2r}\leftrightarrow bcAM+acBM+abCM\le\frac{4Rrs}{2r}=2Rs^2$

$$bcAM+acBM+abCM\le\sqrt{(bc)^2+(ac)^2+(ab)^2}\cdot\sqrt{AM^2+BM^2+CM^2}$$

Because: $AM^2+BM^2+CM^2=\frac{3}{4}(a^2+b^2+c^2)\le\frac{3}{4}\cdot9R^2$

$(bc)^2+(ac)^2+(ab)^2\le\frac{16}{27}s^4$. *Hence:*

$$bcAM+acBM+abCM\le\sqrt{(bc)^2+(ac)^2+(ab)^2}\cdot\sqrt{AM^2+BM^2+CM^2}\le\frac{4Rrs}{2r}=2Rs^2$$

Proved.

SP.223. In ΔABC the following relationship holds:

$$\left(a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\right)\left(b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}+c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}\right)\left(c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}+a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}\right)\ge 8abc$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

Because: $0<\frac{h_a}{w_a};\frac{h_b}{w_b};\frac{h_c}{w_c}\le 1$. *Using AM-GM we have:*

$$a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\ge 2\sqrt{a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}\cdot b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}}=2\sqrt{ab\left(\frac{b}{a}\cdot\frac{a}{b}\right)^{\frac{h_c}{w_c}}}=2\sqrt{ab}$$

$$b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}+c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}\ge 2\sqrt{b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}\cdot c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}}=2\sqrt{bc\left(\frac{c}{b}\cdot\frac{b}{c}\right)^{\frac{h_a}{w_a}}}=2\sqrt{bc}$$

$$c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}+a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}\ge 2\sqrt{c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}\cdot a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}}=2\sqrt{ca\left(\frac{a}{c}\cdot\frac{c}{a}\right)^{\frac{h_b}{w_b}}}=2\sqrt{ca}$$

$$\to LHS\ge 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}=8abc$$

SP.224. In ΔABC the following relationship holds:

$$\frac{(s^2+r_ar_b)(s^2+r_br_c)(s^2+r_cr_a)}{(s^2-r_ar_b)(s^2-r_br_c)(s^2-r_cr_a)}\ge 8$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We have: $r_ar_br_c=s^2r\Rightarrow r_br_c=\frac{s^2r}{r_a}\Rightarrow\frac{s^2+r_br_c}{s^2-r_br_c}=\frac{s^2+\frac{s^2r}{r_a}}{s^2-\frac{s^2r}{r_a}}=\frac{1+\frac{r}{r_a}}{1-\frac{r}{r_a}}\Rightarrow$ *we must show:*

$$\frac{\left(1+\frac{r}{r_a}\right)\left(1+\frac{r}{r_b}\right)\left(1+\frac{r}{r_c}\right)}{\left(1-\frac{r}{r_a}\right)\left(1-\frac{r}{r_b}\right)\left(1-\frac{r}{r_c}\right)}\ge 8\ (1)$$

Let $\frac{r}{r_a}=x,\frac{r}{r_b}=y,\frac{r}{r_c}=z$, *because* $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Rightarrow x+y+z=1\ \ (2)$

From (1)+(2) we must show: $\frac{(1+x)(1+y)(1+z)}{(1-x)(1-y)(1-z)} \geq 8$, with $x + y + z = 1 \Leftrightarrow$

$$\frac{(x+y+x+z)(x+y+y+z)(y+z+x+z)}{(x+y)(y+z)(x+z)} \geq 8 \quad (3)$$

Let $x + y = m, y + z = n$ and $z + x = p$ (4)

From (3)+(4) we must show:

$$\frac{(m+n)(n+p)(p+m)}{mnp} \geq 8 \Leftrightarrow (m+n)(n+p)(p+m) \geq 8mnp \quad (5)$$

$$\text{But } \left.\begin{matrix} m+n \geq 2\sqrt{mn} \\ n+p \geq 2\sqrt{np} \\ p+m \geq 8\sqrt{pm} \end{matrix}\right\} \Rightarrow (m+n)(n+p)(p+m) \geq 8mnp$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$s^2 + r_a r_b = s^2 + \frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)} = s(s+s-c) \overset{(1)}{=} s(a+b)$$

Similarly, $s^2 + r_b r_c \overset{(2)}{=} s(b+c)$ and $s^2 + r_c r_a \overset{(3)}{=} s(c+a)$

Also, $s^2 - r_a r_b = s^2 - s(s-c) \overset{(4)}{=} sc$,

$s^2 - r_b r_c \overset{(5)}{=} sa$ and $s^2 - r_c r_a \overset{(6)}{=} sb$

(1), (2), (3), (4), (5), (6) $\Rightarrow$ given inequality $\Leftrightarrow \frac{s^3\prod(a+b)}{s^3abc} \geq 8 \Leftrightarrow \prod(a+b) \geq 8abc$

$\rightarrow$ true (Cesaro) (Proved)

SP.225. Let a, b, c, d be positive real numbers with $abcd = 1$. Prove that:

$$\sum_{cyc} \frac{1}{a(b+c+d)} \leq \frac{1}{9}\left(\sum_{cyc} \frac{1}{a^2} + 2\sum_{cyc} a^2\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Ivan Mastev-Maribor-Slovenia

$a, b, c, d > 0$ and $abcd = 1$

$$\sum_{cyc} \frac{1}{a(b+c+d)} \leq \frac{1}{9}\left(\sum_{cyc} \frac{1}{a^2} + 2\sum_{cyc} a^2\right)$$

$$\sum_{cyc}\frac{9}{a(b+c+d)}\overset{HM-AM}{\le}\sum_{cyc}\frac{1}{a}\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=2\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}+\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}\right)=$$

$$=\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{ad}\right)+\left(\frac{1}{ab}+\frac{2}{ac}+\frac{1}{ad}+\frac{1}{bc}+\frac{2}{bd}+\frac{1}{cd}\right)\le$$

$$\le\left(\sum_{cyc}\frac{1}{a^2}\right)+2\left(\frac{1}{ac}+\frac{1}{bd}\right)+\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{ad}\right)=$$

$$=\left(\sum_{cyc}\frac{1}{a^2}\right)+2(bd+ac)+(cd+ad+ab+bc)\le$$

$$\le\left(\sum_{cyc}\frac{1}{a^2}\right)+(a^2+b^2+c^2+d^2)+\frac{(c^2+d^2)+(a^2+d^2)+(a^2+b^2)+(b^2+c^2)}{2}=$$

$$=\sum_{cyc}\frac{1}{a^2}+2\sum_{cyc}a^2$$

Solution 2 by Marian Ursărescu-Romania

$$\frac{3}{b+c+d}=\frac{3}{\frac{1}{acd}+\frac{1}{abd}+\frac{1}{abc}}\le\frac{acd+abd+abc}{3}\Rightarrow$$

$$\Rightarrow\frac{3}{b+c+d}\le\frac{a(bc+bd+cd)}{3}\Rightarrow\frac{1}{a(b+c+d)}\le\frac{bc+bd+cd}{9}\Rightarrow$$

We must show: $2(ab+ac+ad+bc+bd+cd)\le\sum\frac{1}{a^2}+2\sum a^2$ ***(1)***

Now, using the inequality:

$$x_1^2+x_2^2+x_3^2+x_4^2\ge\frac{2}{3}(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)\quad\textbf{(2)}$$

From (2) $\Rightarrow 2\sum a^2\ge\frac{4}{3}(ab+ac+ad+bc+bd+cd)$ ***(3)***

$$\sum\frac{1}{a^2}\ge\frac{2}{3}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}+\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}\right)=\frac{2}{3}(cd+bd+bc+ad+ac+ab)\quad\textbf{(4)}$$

From (3)+(4)$\Rightarrow 2\sum a^2+\sum\frac{1}{a^2}\ge\frac{4}{3}\sum ab+\frac{2}{3}\sum ab=2\sum ab\Rightarrow$ ***(1) it is true.***

UP.211. Calculate the integral:

$$\int_0^1 \frac{\sqrt{x}\ln x}{x^2+1}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\int_0^1 \frac{\sqrt{x}\log x}{1+x^2}dx = \sum_{n=0}^{+\infty}(-1)^n\int_0^1 x^{2n+\frac{1}{2}}\log x\,dx = \sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{\left(2n+\frac{3}{2}\right)^2} =$$

$$= \sum_{n=0}^{+\infty}\left(\frac{1}{\left(4n+\frac{7}{2}\right)^2}-\frac{1}{\left(4n+\frac{3}{2}\right)^2}\right) = \frac{\Psi_1\left(\frac{7}{8}\right)-\Psi_1\left(\frac{3}{8}\right)}{16}$$

Solution 2 by Samir HajAli-Damascus-Syria

$$I = \int_0^1 \sqrt{x}\ln x\sum_{n=0}^{\infty}(-x^2)^n\,dx = \sum_{n=0}^{\infty}(-1)^n\cdot\int_0^1 \ln x\cdot x^{2n+\frac{1}{2}}dx$$

Let put: $J = \int_0^1 x^{2n+\frac{1}{2}}\ln x\,dx$

$$= \int_0^1 \frac{\partial}{\partial a}x^a dx\bigg|_{a=2n+\frac{1}{2}} = \frac{\partial}{\partial a}\int_0^1 x^a\,dx\bigg|_{a=2n+\frac{1}{2}} = \frac{\partial}{\partial a}\left(\frac{1}{a+1}\right)\bigg|_{a=2n+\frac{1}{2}} = \frac{-1}{\left(2n+\frac{3}{2}\right)^2}$$

So: $I = \sum_{n=0}^{\infty}(-1)^{n+1}\cdot\frac{1}{\left(2n+\frac{3}{2}\right)^2}$

$$= -\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{3}{4}\right)^2} = -\frac{1}{4}\left[\sum_{n=0}^{\infty}\frac{1}{\left(2n+\frac{3}{4}\right)^2}-\sum_{n=0}^{\infty}\frac{1}{\left(2n+\frac{7}{4}\right)^2}\right]$$

$$= -\frac{1}{16}\left[\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{3}{8}\right)^2}-\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{7}{8}\right)^2}\right] = -\frac{1}{16}\left(\zeta\left(2,\frac{3}{8}\right)-\zeta\left(2,\frac{7}{8}\right)\right) \approx -0,3847$$

Solution 3 by Abdul Hafeez Ayinde-Nigeria

$$\Omega = \int_0^1 \frac{\sqrt{x}\ln x}{x^2+1}dx$$

$$\Omega = \sum_{k=0}^{\infty}(-1)^k \int_0^1 x^{2k+\frac{1}{2}}\ln x\, dx$$

$$\Omega = \sum_{k=0}^{\infty}(-1)^k \cdot \frac{\partial}{\partial b}\bigg|_{b=2k+\frac{1}{2}} \int_0^1 x^b\, dx$$

$$\Omega = \sum_{k=0}^{\infty}(-1)^k \cdot \frac{\partial}{\partial b}\bigg|_{b=2k+\frac{1}{2}} \left(\frac{1}{b+1}\right)$$

$$\Omega = -\sum_{k=0}^{\infty}(-1)^k \bigg|_{b=2k+\frac{1}{2}} \frac{1}{(b+1)^2}$$

$$\Omega = -\sum_{k=0}^{\infty}(-1)^k \frac{1}{\left(2k+\frac{3}{2}\right)^2}$$

$$\Omega = -\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(2k+\frac{3}{2}\right)^2}$$

$$\Omega = -\frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(k+\frac{3}{4}\right)^2}$$

$$\Omega = -\frac{1}{4}\left(\frac{1}{4}\left(\psi_1\left(\frac{3}{8}\right) - \psi_1\left(\frac{7}{8}\right)\right)\right)$$

$$\Omega = \frac{1}{16}\left(\psi_1\left(\frac{7}{8}\right) - \psi_1\left(\frac{3}{8}\right)\right)$$

Solution 4 by Nelson Javier Villaherrera Lopez-El Salvador

$$\int_0^1 \frac{\sqrt{x}\ln(x)}{x^2+1}dx = -\int_0^1 \frac{-\ln(x)\sqrt{x}}{1+x^2}dx = \int_{\infty}^{0} \frac{ye^{-\frac{y}{2}}}{1+e^{-2y}}e^{-y}dy = -\int_0^{\infty} ye^{-\frac{y}{2}}\frac{e^{-y}}{1+e^{-2y}}dy =$$

$$= -\int_0^{\infty} ye^{-\frac{y}{2}}\sum_{k=1}^{\infty}(-1)^{k-1}e^{-(2k-1)y}\,dy = -\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\infty} ye^{-\left(2k-\frac{1}{2}\right)y}\,dy$$

$$= -\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\left(2k-\frac{1}{2}\right)^2}\int_0^{\infty}\left(2k-\frac{1}{2}\right)ye^{-\left(2k-\frac{1}{2}\right)y}\left(2k-\frac{1}{2}\right)dy = -4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(4k-1)^2}\int_0^{\infty} ze^{-z}\,dz =$$

$$= -4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Gamma(1+1)}{(4k-1)^2} = -4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(4k-1)^2}$$

$$= -\frac{1}{16}\left[\psi_1\left(\frac{3}{8}\right)-\psi_1\left(\frac{7}{8}\right)\right], \psi_n(x) = \{\ln[\Gamma[x]]\}^{n+1}$$

UP.212. Calculate the limit of the sequence $(a_n)_{n\ge 1}$ defined by the following relationship:

$$a = \frac{1}{n}\int_1^2 \ln(1+e^{n\cdot\arctan x})\,dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by Remus Florin Stanca – Romania

$$\lim_{n\to\infty} a_n =$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\int_1^2\ln(1+e^{n\arctan x})\,dx - \frac{1}{n}\int_1^2\ln(e^{n\arctan x})\,dx\right) + \int_1^2\arctan x\,dx \quad (1)$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\int_1^2\ln(1+e^{n\arctan x})\,dx - \frac{1}{n}\int_1^2\ln(e^{n\arctan x})\,dx\right) = \lim_{n\to\infty}\frac{\int_1^2\ln(e^{-n\arctan x}+1)\,dx}{n}$$

$$= \lim_{n\to\infty}\int_1^2\frac{\ln(e^{-n\arctan x}+1)}{n}\cdot x'\,dx =$$

$$= \lim_{n\to\infty}\left(\frac{2\ln(e^{-n\arctan 2}+1)}{n} - \frac{\ln\left(e^{-n\frac{\pi}{4}+1}\right)}{n} + \int_1^2\frac{1}{n}\cdot\frac{x}{e^{-n\arctan x}+1}\cdot ne^{-n\arctan x}\cdot\frac{1}{x^2+1}dx\right) =$$

$$= \lim_{n\to\infty}\int_1^2\frac{x}{x^2+1}\cdot\frac{e^{-n\arctan x}}{e^{-n\arctan x}+1}dx \quad (2)$$

$\frac{1}{x^2+1}\le\frac{1}{2}$ *and* $e^{-n\arctan x}+1>1 \Rightarrow \frac{x}{(x^2+1)(e^{-n\arctan x}+1)} < \frac{x}{2} \Rightarrow \frac{xe^{-n\arctan x}}{(x^2+1)(e^{-n\arctan x}+1)} < \frac{\pi}{2}\cdot e^{-n\frac{\pi}{4}} \Rightarrow$

$$\Rightarrow \int_1^2 \frac{xe^{-n\arctan x}}{(x^2+1)(e^{-n\arctan x}+1)}dx < \int_1^2 \frac{x}{2}e^{-\frac{n\pi}{4}}dx = e^{-\frac{n\pi}{4}} - e^{-\frac{n\pi}{4}} \cdot \frac{1}{4} \Rightarrow$$

$\Rightarrow \lim_{n\to\infty} \int_1^2 \frac{xe^{-n\arctan x}}{(x^2+1)(e^{-n\arctan x}+1)}dx = 0$ ***because*** $\lim_{n\to\infty} e^{-\frac{n\pi}{4}} - \frac{e^{-\frac{n\pi}{4}}}{4} - 0$ ***and***

$$\int_1^2 \frac{xe^{-n\arctan x}}{(x^2+1)(e^{-n\arctan x}+1)} > 0 \overset{(1);(2)}{\Rightarrow} \lim_{n\to\infty} a_n = \int_1^2 \arctan x\, dx = \int_1^2 \arctan x \cdot x' =$$

$$= 2\arctan 2 - \frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{2x}{x^2+1}dx$$

$$= 2\arctan 2 - \frac{\pi}{4} - \frac{1}{2}(\ln 5 - \ln 2) = 2\arctan 2 - \frac{\pi}{4} + \ln\left(\sqrt{\frac{2}{5}}\right) \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} a_n = 2\arctan 2 - \frac{\pi}{4} + \ln\left(\sqrt{\frac{2}{5}}\right)$$

UP.213. Let $A \in M_3(\mathbb{R})$ invertible such that: $Tr\ A = Tr\ A^{-1} = 1$. Prove that:

$$\det(A^2 + A + I_3) \geq 3\det A$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

As A^{-1}***, exists,*** $\det(A) \neq 0.$ $\det(A^2 + A + I_3) = \det\left((A - \omega I_3)(A - \omega^2 I_3)\right)$

$$= \det\left((A - \omega I_3)\right)\overline{(A - \omega I_3)} = \det(A - \omega I_3)\overline{\det(A - \omega I_3)} = |\det(A - \omega I_3)|^2 \geq 0$$

$\therefore$ ***If*** $\det(A) < 0$***, then there is nothing to show. We assume*** $\det(A) > 0$***. Let*** $\det(A) = \alpha^2$***, where*** $\alpha > 0$***. We have*** $A^* = \det(A)\,A^{-1} = \alpha^2 A^{-1} \Rightarrow Tr\ (A^*) = \alpha^2 Tr\ (A^{-1}) = \alpha^2$ ***. Characteristic polynomial of*** A ***is:*** $P(t) = \det(tI_3 - A)$

$$= t^3 - Tr\ (A)t^2 + Tr\ (A^*)t - \det(A) = (t^3 + \alpha^2 t) - (t^2 + \alpha) = (t^2 + \alpha^2)(t - 1)$$

Now, from (1): $\det(A^2 + A + I_3) = |\det(A - \omega I_3)|^2 = |\det(\omega I_3 - A)|^2$

$$= |(\omega - 1)|\left|\omega^2 + \alpha^2\right|$$

But $|\omega-1| = \left|-\frac{3}{2}+\frac{\sqrt{3}}{2}i\right| = 3$ *and* $|\omega^2+\alpha^2| = \left(-\frac{1}{2}+\alpha\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \alpha^4-\alpha^2+1 \geq \alpha^2$

Thus, $\det(A^2+A+I_3) \geq 3\alpha^2 = 3\det(A)$

UP.214. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}}{e^n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$a(n) = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}$$

$$= 1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+\cdots+\frac{1}{15}\right)+\cdots+\left(\frac{1}{2^{n-1}}+\cdots+\frac{1}{2^n-1}\right)$$

$$< 1+2\left(\frac{1}{2}\right)+4\left(\frac{1}{4}\right)+8\left(\frac{1}{8}\right)+\cdots+2^{n-1}\left(\frac{1}{2^{n-1}}\right) = n+1$$

Now, $0 < a(n) < n+1 \Rightarrow 0 < \frac{a(n)}{e^n} < \frac{n+1}{e^n} < \frac{2e}{n+1}\left[e^{n+1} > \frac{(n+1)^2}{2}\right]$

As $\lim_{n\to\infty}\frac{2e}{n+1} = 0$, *we get* $\lim_{n\to\infty}\frac{a(n)}{e^n} = 0$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}-\ln(2^n-1)+\ln(2^n-1)}{e^n} = \lim_{n\to\infty}\frac{\gamma}{e^n}+\lim_{n\to\infty}\frac{\ln(2^n-1)}{e^n} =$$

$$= \lim_{n\to\infty}\frac{\ln(2^n-1)}{e^n} \overset{Stolz-Cesaro}{=} \lim_{n\to\infty}\frac{\ln\frac{2^{n+1}-1}{2^n-1}}{e^{n+1}-e^n} = \lim_{n\to\infty}\frac{\ln 2}{e^n(e-1)} = 0 \Rightarrow \Omega = 0$$

Solution 3 by Naren Bhandari-Bajura-Nepal

$$\Omega = \lim_{n\to\infty}\frac{1}{e^n}\left(\sum_{k=1}^{n}\frac{1}{2^k-1}\right) = \lim_{n\to\infty}\frac{1}{e^n}\left(\sum_{k=1}^{n}\frac{1}{2^k-1}\right) \leq \lim_{n\to\infty}\frac{1}{e^n}\left(\sum_{k=0}^{n}\frac{1}{e^n}\right)$$

Here $\sum_{k=1}^{n}\frac{1}{2^k}$ *is a convergent series this* $\sum_{k=1}^{n}\frac{1}{2^k-1}$ *is also convergent series*

$$\Omega = \lim_{n\to\infty}\frac{1}{e^n}\left(\sum_{k=1}^{n}\frac{1}{2^k-1}\right) \leq \lim_{n\to\infty}\frac{1}{e^n}\left(\sum_{k=0}^{n}\frac{1}{2^k}\right) = \lim_{n\to\infty}\frac{1}{e^n}(2) = 2\cdot 0 = 0$$

UP.215. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b\int_a^b\left(\frac{\cot x + \cot y + \tan(x+y)}{\cot x \cot y \tan(x+y)}\right)dx\,dy \leq \frac{\pi(b-a)}{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$NUM = \cot x + \cot y + \tan(x+y) == \frac{\cos x}{\sin x} + \frac{\cos y}{\sin y} + \tan(x+y)$$

$$= \frac{\sin(x+y)}{\sin x \sin y} + \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x+y)}{\sin x \sin y \cos(x+y)}[\cos(x+y) + \sin x \sin y]$$

$$= \frac{\sin(x+y)\cos x \cos y}{\sin x \sin y \cos(x+y)} = \tan(x+y)\cot x \cot y = DEN$$

$$\therefore \Omega = \int_a^b\int_a^b 1\,dx\,dy = (b-a)^2 < \frac{\pi}{2}(b-a)$$

$$\left[\because 0 < a \leq b < \frac{\pi}{2} \Rightarrow b - a < \frac{\pi}{2}\right]$$

Solution 2 by Andrew Okukura-Romania

$$\frac{\cot x + \cot y + \tan(x+y)}{\cot x \cot y \tan(x+y)} = \frac{\cot x + \cot y + \frac{\tan x + \tan y}{1 - \tan x \tan y}}{\cot x \cot y \frac{\tan x + \tan y}{1 - \tan x \tan y}} =$$

$$= \frac{\cot x + \cot y + \frac{\tan x + \tan y}{1 - \tan x \tan y}}{\frac{\cot x + \cot y}{1 - \tan x \tan y}} = 1 - \tan x \tan y + \frac{\tan x + \tan y}{\cot x + \cot y} =$$

$= 1 - \tan x \tan y + \tan x \tan y = 1$. *By noting the left side, I, we have:*

$I = \int_a^b\left(\int_a^b dx\right)dy = \int_a^b (b-a)\,dy = (b-a)^2$. *But* $(b-a) \leq \frac{\pi}{2} \Rightarrow y \leq \frac{\pi}{2}(b-a)$

Solution 3 by Remus Florin Stanca-Romania

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{\frac{1}{\cot x}+\frac{1}{\cot y}}{1-\frac{1}{\cot x\cot y}}=\frac{\cot x+\cot y}{\cot x\cot y-1}\Rightarrow$$

$$\Rightarrow \tan(x+y)\cot x\cot y-\tan(x+y)=\cot(x)+\cot(y)\Rightarrow$$

$$\Rightarrow \tan(x+y)\cot x\cot y=\tan(x+y)+\cot x+\cot y\Rightarrow$$

$$\Rightarrow \int_a^b \frac{\cot x+\cot y+\tan(x+y)}{\cot x\cot(y)\tan(x+y)}dx=\int_a^b 1\,dx=b-a\Rightarrow$$

$$\Rightarrow \int_a^b\int_a^b \frac{\cot x+\cot y+\tan(x+y)}{\cot x\cot y\tan(x+y)}dx\,dy=(b-a)^2 \quad (1)$$

$$b<\frac{\pi}{2} \text{ and } -a<0 \overset{by\ adding}{\Rightarrow} b-a<\frac{\pi}{2}|\cdot\left((b-a)\geq 0\right)\Rightarrow (b-a)^2\leq\frac{\pi(b-a)}{2} \quad (2)$$

$$\overset{(1);(2)}{\Rightarrow} \int_a^b\int_a^b \frac{\cot x+\cot y+\tan(x+y)}{\cot x\cot y\tan(x+y)}dx\,dy\leq\frac{\pi(b-a)}{2}$$

Solution 4 by Avishek Mitra-West Bengal-India

$$\Omega=\int_a^b\int_a^b \frac{\cot x+\cot y+\tan(x+y)}{\cot x\cdot\cot y\cdot\tan(x+y)}dx\,dy$$

$$=\int_a^b\int_a^b \frac{\cot x+\cot y}{\cot x\cot y\cdot\frac{(\tan x+\tan y)}{(1-\tan x\tan y)}}+\frac{1}{\cot x\cot y}dxdy$$

$$=\int_a^b\int_a^b\left\{\frac{(\cot x+\cot y)(1-\tan x\tan y)}{(\cot x+\cot y)}+\tan x\cdot\tan y\right\}dx\,dy=$$

$$=\int_a^b\int_a^b(1-\tan x\cdot\tan y+\tan x\cdot\tan y)\,dx\,dy$$

$$=\int_a^b\int_a^b dx\,dy=(b-a)\int_a^b dy=(b-a)^2$$

$$\Leftrightarrow \textbf{Or, }\Rightarrow \int_a^b\int_a^b \frac{\cot x+\cot y+\tan(x+y)}{\cot x\cot y\tan(x+y)}dx\,dy=\int_a^b\int_a^b\frac{\frac{1}{\tan x}+\frac{1}{\tan y}+\tan(x+y)}{\cot x\cdot\cot y\cdot\tan(x+y)}dx\,dy$$

$$= \int_a^b \int_a^b \frac{\frac{\tan x + \tan y}{\tan x \tan y} + \tan(x+y)}{\cot x \cot y \tan(x+y)} dx\, dy =$$

$$= \int_a^b \int_a^b \left\{ \frac{\tan x + \tan y}{\tan(x+y)} + \frac{\tan(x+y) \cdot \tan x \tan y}{\tan(x+y)} \right\} dx\, dy$$

$$= \int_a^b \int_a^b \left\{ \frac{(\tan x + \tan y)}{\frac{(\tan x + \tan y)}{(1 - \tan x \tan y)}} + \tan x \tan y \right\} dx\, dy =$$

$$= \int_a^b \int_a^b (1 - \tan x \tan y + \tan x \tan y)\, dx\, dy = \int_a^b \int_a^b dx\, dy = (b-a)^2$$

$\Leftrightarrow$ ***need to show*** $\Rightarrow (b-a)^2 \le \frac{\pi(b-a)}{2} \Rightarrow (b-a) \le \frac{\pi}{2} \Leftrightarrow$ ***(* true as*** $a \le b < \frac{\pi}{2}, b < \frac{\pi}{2}$***)***

$\Leftrightarrow (b-a)^2 \le \frac{\pi(b-a)}{2}$ ***(* true)***

$$\Leftrightarrow \int_a^b \int_a^b \frac{\cot x + \cot y + \tan(x+y)}{\cot x \cot y \tan(x+y)} dx\, dy \le \frac{\pi(b-a)}{2}$$

UP.216. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4} - x - y\right)\right)}{1 + \tan x \tan y \tan\left(\frac{\pi}{4} - x - y\right)} dx\, dy \le \pi(b-a)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jovica Mikic-Sarajevo-Bosnia

Let $x, y, z \ge 0$ ***such that:*** $x + y + z = \frac{\pi}{4}$ ***then:***

$\sum_{cyc} \tan x + \sum_{cyc} \tan x \tan y = 1 + \tan x \tan y + \tan z$ ***(*)***

$$x + y = \frac{\pi}{4} - z;\ \tan(x+y) = \tan\left(\frac{\pi}{4} - z\right)$$

$$\frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{1 - \tan z}{1 + \tan z} \Leftrightarrow$$

$$\Leftrightarrow (\tan x + \tan y)(1 + \tan z) = (1 - \tan x \tan y)(1 - \tan z) \quad \textbf{(**)}$$

$$\tan x + \tan y + \tan x \tan z + \tan y \tan z = 1 - \tan z - \tan x \tan y + \tan x \tan y \tan z$$

$$\sum_{cyc} \tan x + \sum_{cyc} \tan x \tan y = 1 + \tan x \tan y \tan z \quad \textbf{(*)}$$

then: $(1 + \tan x)(1 + \tan y)(1 + \tan z) = 2(1 + \tan x \tan y \tan z)$ ***(**)***

proof: $(1 + \tan x)(1 + \tan y)(1 + \tan z) =$

$$= 1 + \sum_{cyc} \tan x + \sum_{cyc} \tan x \tan y + \tan x \tan y \tan z \overset{*}{=} 2(1 + \tan x \tan y \tan z)$$

Finally,

$$\int_a^b \int_a^b \frac{(1 + \tan x)(1 + \tan y)(1 + \tan z)}{1 + \tan x \tan y \tan z} dx\, dy = \int_a^b \int_a^b 2\, dx\, dy =$$

$= 2(b - a)^2 < 2(b - a) \cdot \frac{\pi}{2} = (b - a)\pi$ ***Q.E.D. Since,*** $0 < a \le b < \frac{\pi}{2} \Rightarrow b - a < \frac{\pi}{2}$

Solution 2 by Amit Dutta-Jamshedpur-India

$$\tan\left(\frac{\pi}{4} - x - y\right) = \tan\left(\frac{\pi}{4} - (x + y)\right) = \frac{1 - \tan(x + y)}{1 + \tan(x + y)} = \frac{1 - \frac{(\tan x + \tan y)}{1 - \tan x \tan y}}{1 + \left(\frac{\tan x + \tan y}{1 - \tan x \tan y}\right)}$$

$$\tan\left(\frac{\pi}{4} - x - y\right) = \frac{1 - \tan x \tan y - \tan x - \tan y}{1 - \tan x \tan y + \tan x + \tan y}$$

$$1 + \tan\left(\frac{\pi}{4} - (x + y)\right) = \frac{2(1 - \tan x \tan y)}{1 - \tan x \tan y + \tan x + \tan y}$$

$\therefore$ ***Numerator of the integrand*** $= \frac{2(1+\tan x)(1+\tan y)(1-\tan x \tan y)}{1-\tan x \tan y+\tan x+\tan y}$

Denominator of the integrand $= 1 + \tan x \tan y \tan\left(\frac{\pi}{4} - x - y\right)$

$$= 1 + \tan x \tan y \left\{\frac{1 - \tan x \tan y - \tan x - \tan y}{1 - \tan x \tan y + \tan x + \tan y}\right\}$$

$$= \frac{1 - \tan x \tan y + \tan x + \tan y + \tan x \tan y - \tan^2 x \tan^2 y - \tan^2 x \tan y - \tan x \tan^2 y}{1 - \tan x \tan y + \tan x \tan y}$$

$$= \frac{(1 + \tan x + \tan y + \tan x \tan y) - \tan x \tan y\,(1 + \tan x + \tan y + \tan x \tan y)}{1 - \tan x \tan y + \tan x + \tan y}$$

$$= \frac{(1 + \tan x)(1 + \tan y) - \tan x \tan y\,(1 + \tan x)(1 + \tan y)}{1 - \tan x \tan y + \tan x + \tan y}$$

$$= \frac{(1+\tan x)(1+\tan y)(1-\tan x\tan y)}{1-\tan x\tan y+\tan x+\tan y}$$

∴ Putting the values of numerator and denominator so obtained in the integration, we are left with

$$I = \int_a^b\int_a^b \frac{2(1+\tan x)(1+\tan y)(1-\tan x\tan y)}{(1+\tan x)(1+\tan y)(1-\tan x\tan y)}dx\,dy$$

$$I = \int_a^b\int_a^b 2\,dx\,dy = 2(b-a)^2$$

$$I = 2(b-a)(b-a) \therefore 0 < a \le b \le \frac{\pi}{2} \therefore (b-a) \le \frac{\pi}{2} \therefore I \le 2(b-a)\cdot\left(\frac{\pi}{2}\right)$$

$I \le \pi(b-a)$. ***Proved.***

Solution 3 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \tan\left(\frac{\pi}{4}-x-y\right) = \frac{1-\tan(x+y)}{1+\tan(x+y)} \Rightarrow 1+\tan\left(\frac{\pi}{4}-x-y\right) = 1+\frac{1-\tan(x+y)}{1+\tan(x+y)} =$$

$$= \frac{2}{1+\tan(x+y)} \Rightarrow (1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right) =$$

$$= \frac{2(1+\tan x)(1+\tan y)}{(1+\tan(x+y))}$$

$$\Leftrightarrow 1+\tan x\tan y\tan\left(\frac{\pi}{4}-x-y\right) = \frac{\tan x\tan y\,(1-\tan(x+y))}{1+\tan(x+y)}+1 =$$

$$= \frac{\tan x\tan y\,(1-(\tan x+y))+1+\tan(x+y)}{1+\tan(x+y)}$$

$$\Leftrightarrow \frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+\tan x\tan y\tan\left(\frac{\pi}{4}-x-y\right)} =$$

$$= \frac{2(1+\tan x)(1+\tan y)}{1+\tan x\tan y+\tan(x+y)-\tan x\tan y\tan(x+y)}$$

$$= \frac{2(1+\tan x)(1+\tan y)}{\frac{\tan x+\tan y}{1-\tan x\tan y}+\tan(x+y)-\tan x\tan y\frac{(\tan x+\tan y)}{(1-\tan x\tan y)}+1}$$

$$= \frac{2(1+\tan x)(1+\tan y)(1-\tan x\tan y)}{\tan x+\tan y+\tan x\tan y-\tan^2 x\tan^2 y-\tan^2 x\tan y-\tan x\tan^2 y+1-\tan x\tan y}$$

$$=\frac{2(1-\tan x\tan y+\tan x-\tan^2 x\tan y+\tan y-\tan x\tan^2 y+\tan x\tan y-\tan^2 x\tan^2 y)}{(\tan x+\tan y-\tan^2 x\tan^2 y-\tan^2 x\tan y-\tan x\tan^2 y+1)}$$

$$=2$$

$$\Leftrightarrow \Omega=\int_a^b\int_a^b\frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+\tan x\tan y\tan\left(\frac{\pi}{4}-x-y\right)}dx\,dy=$$

$$=\int_a^b\int_a^b 2\,dx\,dy=2(b-a)^2$$

$\Leftrightarrow$ ***need to show*** $\Rightarrow 2(b-a)^2\le\pi(b-a)$

$\Rightarrow 2(b-a)^2\le\pi(b-a)\Rightarrow(b-a)\le\frac{\pi}{2}$ ***[* true as*** $0\le a$ ***and*** $b\le\frac{\pi}{2}$***]***

$\Leftrightarrow 2(b-a)^2\le\pi(b-a)\Rightarrow$ ***(* true)***

$$\Leftrightarrow\int_a^b\int_a^b\frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+\tan x\tan y\tan\left(\frac{\pi}{4}-x-y\right)}\le\pi(b-a)$$

UP.217. Find:

$$\Omega=\int\left(\tan\left(\frac{\pi-9x}{3}\right)\tan\left(\frac{\pi-3x}{3}\right)\tan x\tan\left(\frac{\pi+3x}{3}\right)\tan\left(\frac{\pi+9x}{3}\right)\right)dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaijan

It is known that: $\tan\left(\frac{\pi}{3}-x\right)\tan x\tan\left(\frac{\pi}{x}+x\right)=\tan 3x$ ***(1)***

we have:

$$\Omega=\int\underbrace{\tan\left(\frac{\pi}{3}-3x\right)\tan\left(\frac{\pi}{3}+3x\right)\tan 3x}_{(1)=\tan 9x}\cdot\underbrace{\tan\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}+x\right)\tan x}_{(1)=\tan 3x}\cdot\frac{1}{\tan 3x}dx$$

$$=\int\tan 9x\cdot\tan 3x\cdot\frac{1}{\tan 3x}dx=\int\tan 9x\,dx$$

$$=\int\frac{\sin 9x}{\cos 9x}=-\frac{1}{9}\int\frac{d(\cos 9x)}{\cos 9x}=-\frac{1}{9}\ln|\cos 9x|+C$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \tan x \tan\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}+x\right)=\frac{\sin x\sin\left(\frac{\pi}{3}-x\right)\cdot\sin\left(\frac{\pi}{3}+x\right)}{\cos x\cos\left(\frac{\pi}{3}-x\right)\cdot\cos\left(\frac{\pi}{3}+x\right)}$$

$$=\frac{\sin x}{\cos x}\cdot\frac{2\sin(60-x)\cdot\sin(60+x)}{2\cos(60-x)\cos(60+x)}=$$

$$=\frac{\sin x}{\cos x}\cdot\frac{[\cos(60+x-60+x)-\cos(60+x+60-x)]}{[\cos(60+x+60-x)+\cos(60-x-60+x)]}$$

$$=\frac{\sin x(\cos 2x-\cos 120)}{\cos(\cos 120+\cos 2x)}=\frac{\cos 2x\cdot\cos x+\frac{\sin x}{2}}{\cos 2x\cdot\cos x-\frac{\cos x}{2}}$$

$$=\frac{2\cos 2x\cdot\sin x+\sin x}{2\cos 2x\cdot\cos x-\cos x}=\frac{\sin 3x-\sin x+\sin x}{\cos 3x+\cos x-\cos x}=\frac{\sin 3x}{\cos 3x}=\tan 3x$$

$$\Leftrightarrow \tan\left(\frac{\pi}{3}-3x\right)\cdot\tan 3x\tan\left(\frac{\pi}{3}+3x\right)=\frac{\sin 3x}{\cos 3x}\cdot\frac{2\sin(60-3x)\cdot\sin(60+3x)}{2\cos(60-3x)\cdot\cos(60+3x)}$$

$$=\frac{\sin 3x}{\cos 3x}\cdot\frac{[\cos(60+3x-60+3x)-\cos(60+3x+60-3x)]}{[\cos(60+3x+60-3x)+\cos(60+3x-60+3x)]}$$

$$=\frac{\sin 3x}{\cos 3x}\cdot\frac{(\cos 6x-\cos 120)}{(\cos 120+\cos 6x)}=\frac{\cos 6x\cdot\sin 3x+\frac{\sin 3x}{2}}{\cos 3x\cdot\cos 6x-\frac{\cos 3x}{2}}$$

$$=\frac{2\sin 3x\cdot\cos 6x+\sin 3x}{2\cos 6x\cdot\cos 3x-\cos 3x}=\frac{\sin 9x-\sin 3x+\sin 3x}{\cos 9x+\cos 3x-\cos 3x}=\frac{\sin 9x}{\cos 9x}=\tan 9x$$

$$\Leftrightarrow \Omega=\int\tan\left(\frac{\pi-9x}{3}\right)\cdot\tan\left(\frac{\pi-3x}{3}\right)\tan x\cdot\tan\left(\frac{\pi+3x}{3}\right)\cdot\tan\left(\frac{\pi+9x}{3}\right)dx$$

$$=\int\tan\left(\frac{\pi}{3}-3x\right)\cdot\tan\left(\frac{\pi}{3}+3x\right)\tan x\tan\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}+x\right)dx$$

$$=\int\tan 3x\tan\left(\frac{\pi}{3}-3x\right)\tan\left(\frac{\pi}{3}+3x\right)dx=\int\tan 9x\,dx=\frac{1}{9}\log|\sec(9x)|+c$$

$$\Leftrightarrow \tan x\tan\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}+x\right)=\tan x\cdot\frac{\sqrt{3}-\tan x}{1+\sqrt{3}\tan x}\cdot\frac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x}$$

$$=\tan x\cdot\frac{3-\tan^2 x}{1-3\tan^2 x}=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}=\tan 3x$$

$$\Leftrightarrow \tan 3x\cdot\tan\left(\frac{\pi}{3}-3x\right)\tan\left(\frac{\pi}{3}+3x\right)$$

$$= \tan 3x \cdot \frac{\sqrt{3} - \tan 3x}{1 + \sqrt{3}\tan 3x} \cdot \frac{\sqrt{3} + \tan 3x}{1 - \sqrt{3}\tan 3x} = \tan 3x \cdot \frac{3 - \tan^2 3x}{1 - 3\tan^2 3x}$$

$$= \frac{3\tan 3x - \tan^3 3x}{1 - 3\tan^2 3x} = \tan(3 \cdot 3x) = \tan 9x$$

$$\Leftrightarrow \int \tan\left(\frac{\pi - 9x}{3}\right)\tan\left(\frac{\pi - 3x}{3}\right) \cdot \tan x \tan\left(\frac{\pi + 3x}{3}\right)\tan\left(\frac{\pi + 9x}{3}\right) dx$$

$$= \int \tan x \tan\left(\frac{\pi}{3} - x\right)\tan\left(\frac{\pi}{3} + x\right)\tan\left(\frac{\pi}{3} + 3x\right)\tan\left(\frac{\pi}{3} - 3x\right) dx$$

$$= \int \tan 3x \tan\left(\frac{\pi}{3} - 3x\right)\tan\left(\frac{\pi}{3} + 3x\right) dx = \int \tan 9x \, dx = \frac{1}{9}\log|\sec(9x)| + c$$

UP.218. Let be $G = \{a + b\sqrt[3]{5} + c\sqrt[3]{25} | a, b, c \in \mathbb{Q}\}$. Prove that: $x \in G \Rightarrow x^{2019} \in G$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jovika Mikic-Sarajevo-Bosnia

Let $x \in G, y \in G$. Let us prove that $xy \in G$

$$x = a + b\sqrt[3]{5} + c\sqrt[3]{25}; \{a, b, c, , d, e, f\} \subset \mathbb{Q}$$

$$y = f + e\sqrt[3]{5} + f^3\sqrt{25}$$

$$xy = (a + b\sqrt[3]{5} + c\sqrt{25})(f + e\sqrt[3]{5} + f\sqrt[3]{25})$$

$$= af + ae\sqrt[3]{5} + af\sqrt[3]{25}$$

$$+bf\sqrt[3]{5} + be\sqrt[3]{25} + bf5$$

$$+cf\sqrt[3]{25} + ce5 + cf5\sqrt[3]{5}$$

$$\text{So, } xy = \underbrace{(af + bf5 + ce5)}_{\in \mathbb{Q}} + \underbrace{(ae + bd + 5cf)}_{\in \mathbb{Q}}\sqrt[3]{5} + \underbrace{(af + be + cd)}_{\in \mathbb{Q}}\sqrt[3]{25}$$

Therefore, the set G is closed under multiplication.

It follows, $x^{2019} \in G$, as well as $x^n \in G, n \in \mathbb{N}$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } x = a + b(5)^{\frac{1}{3}} + c\left(5^{\frac{2}{3}}\right) \in G$$

$y = a_1 + b_1\left(5^{\frac{1}{3}}\right) + c_1\left(5^{\frac{2}{3}}\right) \in G$ where $a, a, b, b, c, c \in \mathbb{Q}$

$$xy = aa_1 + (a_1b)\left(5^{\frac{1}{3}}\right) + a_1c\left(5^{\frac{2}{3}}\right) + 5b_1c + ab_1\left(5^{\frac{1}{3}}\right) + bb_1\left(5^{\frac{2}{3}}\right) +$$

$$+5bc_1 + 5cc_1\left(5^{\frac{1}{3}}\right) + ac_1\left(5^{\frac{2}{3}}\right) = a_2 + b_2\left(5^{\frac{1}{3}}\right) + c_2\left(5^{\frac{2}{3}}\right)$$

where $a_2 = aa_1 + 5b_1c + 5bc_1 \in \mathbb{Q}$

$b_2 = a_1b + ab_1 + 5cc_1 \in \mathbb{Q}$

$c_2 = a_1c + bb_1 + ac_1 \in \mathbb{Q}$

Thus, G is closed under multiplication.

If $x \in G, x \cdot x \in G = x^2 \in G \Rightarrow x^2 \cdot x \in G$ or $x^3 \in G$

Continue like this, $x^{2019} \in G$.

Solution 3 by Marian Ursărescu-Romania

First, we prove: if $x, y \in G \Rightarrow xy \in G$ (1)

Let $x \in G \Rightarrow x = a + b\sqrt[3]{5} + c\sqrt[3]{25}$ and $y \in G \Rightarrow$

$y = a' + b'\sqrt[3]{5} + c'\sqrt[3]{25}, a, b, c, a', b', c' \in \mathbb{Q}$

$$xy = (aa' + 5bc' + 5b'c) + \sqrt[3]{5}(ab' + a'b + 5cc') + \sqrt[3]{25}(ac' + a'c + bb') \Rightarrow xy \in G$$

Now, we prove by induction: if $x \in G \Rightarrow x^{3n} \in G, \forall n \geq 1$

$P(1): x \in G \Rightarrow x^3 \in G, x = a + b\sqrt[3]{5} + c\sqrt[3]{25} \Rightarrow$

$$x^3 = a^3 + 5b^3 + 25c^3 + 30abc + \sqrt[3]{5}(3a^2b + 15ac^2 + 15b^2c) +$$

$$+\sqrt[3]{25}(3ab^2 + 3a^2c + 15bc^2) \in G$$

$P(k)$: if $x \in G \Rightarrow x^{3k} \in G$

$P(k+1)$: if $x \in G \Rightarrow x^{3k+3} \in G$

$x^{3k+3} = x^{3k} \cdot x^3 \in G$ from (1)

Let $n = 673 \Rightarrow x^{2019} \in G$

UP.219. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\frac{a^3}{b^4c(a^2 + ac + c^2)} + \frac{b^3}{c^4a(b^2 + ba + a^2)} + \frac{c^3}{a^4b(c^2 + cb + b^2)} \geq 1$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

For $abc = 1, a, b, c > 0$ *we get as follows:*

$$1.\ \frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2} \geq \frac{\left(\frac{a^2}{c}+\frac{c^2}{b}+\frac{b^2}{a}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{3} \geq \frac{a^2}{c}+\frac{c^2}{b}+\frac{b^2}{a}$$

$$2.\ \frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}+\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2} \geq 2\left(\frac{a^3}{b^2c}+\frac{b^3}{c^2a}+\frac{c^3}{a^2b}\right)$$

$$2.1.\ \frac{a^3}{b^2c}+\frac{b^3}{c^2a}+\frac{c^3}{a^2b} \geq \frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)}{3} \geq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

$$2.2.\ \frac{a^3}{b^2c}+\frac{b^3}{c^2a}+\frac{c^2}{a^2b} \geq \frac{\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)}{3} \geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$

and since $\frac{a^3}{b^4c(a^2+ac+a^2)}+\frac{b^3}{c^4a(b^2+ab+a^2)}+\frac{c^3}{a^4b(c^2+bc+b^2)}$

$$= \frac{\frac{a^4}{b^4}}{ac(a^2+ac+c^2)}+\frac{\frac{b^4}{c^4}}{ab(b^2+ab+a^2)}+\frac{\frac{c^4}{a^4}}{bc(c^2+bc+b^2)}$$

$$\geq \frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{a^3c+c^3a+a^2c^2+ab^3+ba^3+a^2b^2+bc^2+b^3c+b^2c^2}$$

$$= \frac{\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}+2\left(\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{b^2}{a^2}\right)}{\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{a^2}{c}+\frac{c^2}{b}+\frac{b^2}{a}+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} \geq 1$$ *ok. Therefore, it is true.*

UP.220. If $e_n = \left(1+\frac{1}{n}\right)^n; n \in \mathbb{N}^*$ **then find:**

$$\Omega = \lim_{n\to\infty}\left((e-e_n)\cdot e^{H_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty}(e-e_n)\,e^{H_n} = \lim_{n\to\infty}(e-e_n)n\cdot\frac{e^{H_n}}{n} \quad (1)$$

$$\lim_{n\to\infty}\frac{e^{H_n}}{n} = \lim_{n\to\infty}\frac{e^{H_n}}{e^{\ln n}} = \lim_{n\to\infty}e^{H_n-\ln n} =$$

$$= \lim_{n\to\infty}e^{1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n} = e^H \quad (2)$$

$$\lim_{n\to\infty}(e-e_n)n = \lim_{n\to\infty}\frac{e-e_n}{\frac{1}{n}} = \lim_{n\to\infty}\frac{e-\left(1+\frac{1}{n}\right)^n}{\frac{1}{n}} \quad (3)$$

$$\textit{Let } \frac{1}{n} = x, n\to\infty \Rightarrow x\to 0 \Rightarrow$$

$$(3) \Leftrightarrow \lim_{n\to 0}\frac{e-(1+x)^{\frac{1}{x}}}{x} \overset{L'H}{=} \lim_{x\to 0}\frac{-(1+x)^{\frac{1}{x}}\left[-\frac{1}{x^2}(1+x)+\frac{1}{x}\cdot\frac{1}{1+x}\right]}{1}$$

$$= \lim_{x\to 0} -(1+x)^{\frac{1}{x}}\frac{(-(1+x)\ln(1+x)+x)}{x^2(1+x)} =$$

$$= -e\lim_{x\to 0}\frac{-(1+x)\ln(1+x)+x}{x^3+x^2} \overset{L'H}{=} -e\lim_{x\to 0}\frac{-\ln(1+x)-1+1}{3x^2+2x} =$$

$$= e\lim_{x\to 0}\frac{\ln(1+x)}{x(2+3x)} = \frac{e}{2} \quad (4)$$

$$\textit{From } (1)+(2)+(3)+(4) \Rightarrow \Omega = \frac{e}{2}\cdot e^H = \frac{e^{H+1}}{2}$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\lim_{n\to+\infty}\left(e-\left(1+\frac{1}{n}\right)^n\right)e^{H_n} = \lim_{n\to+\infty}\left[n\left(e-\left(1+\frac{1}{n}\right)^n\right)e^{H_n-\log n}\right]$$

$$= e^{\gamma}\lim_{n\to 0}\frac{e-(1+n)^{\frac{1}{n}}}{n} = e^{\gamma}\lim_{n\to 0}\frac{1-e^{\frac{\ln(1+n)}{n}-1}}{n} = e^{\gamma+1}\lim_{n\to 0}\frac{1-e^{-\frac{n}{2}+o(n^2)}}{n} =$$

$$= e^{\gamma+1}\lim_{n\to 0}\frac{1-\left(1-\frac{n}{2}+o(n^2)\right)}{n} = \frac{e^{\gamma+1}}{2}$$

Solution 3 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\frac{e-e_n}{e^{-H_n}} = \lim_{n\to\infty}(e-e_n)\,e^{H_n-\ln n+\ln n} = e^{\gamma}\lim_{n\to\infty}(e-e_n)n =$$

$$= e^{\gamma}\lim_{n\to\infty}\left(e-\left(1+\frac{1}{n}\right)^n\right)n = e^{\gamma}\lim_{n\to\infty} n\cdot\frac{e^1-e^{\ln\left(1+\frac{1}{n}\right)^n}}{1-\ln\left(1+\frac{1}{n}\right)^n}\cdot\left(1-\ln\left(1+\frac{1}{n}\right)^n\right) \quad (1)$$

$$\textit{It's known that } \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \overset{(1)}{\Rightarrow} \Omega = e^{\gamma+1}\lim_{n\to\infty} n\left(1-n\ln\left(1+\frac{1}{n}\right)\right) =$$

$$= -e^{\gamma+1}\lim_{n\to\infty} n\ln\left(n\ln\left(1+\frac{1}{n}\right)\right) = -e^{\gamma+1}\lim_{n\to\infty}\frac{\ln n+\ln\left(\ln\left(1+\frac{1}{n}\right)\right)}{\frac{1}{n}}$$

Let $\frac{1}{n} = x \Rightarrow x \to 0 \Rightarrow \Omega = -e^{\gamma+1} \lim_{x\to 0} \frac{-\ln x + \ln(\ln(x+1))}{x} \overset{L'H}{\underset{\frac{0}{0}}{=}} -e^{\gamma+1} \lim_{x\to 0}\left(-\frac{1}{x} + \frac{1}{\ln(x+1)} \cdot\right.$

1x+1=

$$= -e^{\gamma+1} \lim_{x\to 0} \frac{1}{x}\left(\frac{x}{(x+1)\ln(x+1)} - 1\right) = -e^{\gamma+1} \lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x+1} - \frac{\ln(x+1)}{x}\right) \overset{L'H}{\underset{\frac{0}{0}}{=}}$$

$$= -e^{\gamma+1} \lim_{x\to 0}\left(-\frac{1}{(x+1)^2} - \frac{\frac{x}{x+1} - \ln(x+1)}{x^2}\right) =$$

$$= e^{\gamma+1}\left(-1 - \lim_{x\to 0} \frac{\frac{1}{(x+1)^2} - \frac{1}{x+1}}{2x}\right) = e^{\gamma+1}\left(1 + \lim_{x\to 0} \frac{-\frac{2}{(x+1)^3} + \frac{1}{(x+1)^2}}{2}\right) = \frac{e^{\gamma+1}}{2} \Rightarrow$$

$$\Rightarrow \Omega = \frac{e^{\gamma+1}}{2}$$

UP.221. If $(x_n)_{n\geq 1} \subset (0,\infty)$; $\lim_{n\to\infty}\left(\frac{x_n}{\sqrt{n}} \cdot e^{2\sqrt{n}}\right) = b \in (0,\infty)$, $a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ then find:

$$\Omega = \lim_{n\to\infty}\left(\left(e^{a_{n+1}} - e^{a_n}\right) \cdot x_n\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty}\left(e^{a_{n+1}} - e^{a_n}\right) x_n = \lim_{n\to\infty} e^{a_n}\left(e^{a_{n+1}-a_n} - 1\right) x_n$$

$$= \lim_{n\to\infty} \frac{e^{\frac{1}{\sqrt{n+1}}} - 1}{\frac{1}{\sqrt{n+1}}} \cdot \frac{1}{\sqrt{n+1}} e^{a_n} \cdot x_n = \lim_{n\to\infty} \frac{x_n}{\sqrt{n}} e^{2\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot e^{a_n} \cdot e^{-2\sqrt{n}}$$

$$= b \lim_{n\to\infty} e^{1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2\sqrt{n}} = b \cdot e^L, \textit{ where}$$

$$L = \lim_{n\to\infty}\left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}\right), L \in (-2, e)$$

It is Ioachimescu limit.

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega = \lim_{n\to+\infty}\left(x_n\left(e^{e_{n+1}} - e^{a_n}\right)\right) = \lim_{n\to+\infty}\frac{x_n e^{2\sqrt{n}}}{\sqrt{n}}\cdot\frac{\sqrt{n}\left(e\sum_{k=1}^{n+1}\frac{1}{\sqrt{k}} - e\sum_{k=1}^{n+1}\frac{1}{\sqrt{k}}\right)}{e^{2\sqrt{n}}}$$

$$= b\lim_{n\to+\infty}\frac{\sqrt{n}\left(e^{\zeta\left(\frac{1}{2}\right)+2\sqrt{n+1}} + \frac{1}{2\sqrt{n+1}} - \frac{1}{24(n+1)^{\frac{3}{2}}} + \frac{5}{8(n+1)^{\frac{5}{2}}} + o\left(\frac{1}{n^{\frac{5}{2}}}\right) - e^{\zeta\left(\frac{1}{2}\right)+2\sqrt{n}} + \frac{1}{2\sqrt{n}} - \frac{1}{24n^{\frac{3}{2}}} + o\left(\frac{1}{n^{\frac{3}{2}}}\right)\right)}{e^{2\sqrt{n}}}$$

$$= be^{\zeta\left(\frac{1}{2}\right)}\lim_{n\to+\infty}\left(\frac{e^{\frac{3}{2\sqrt{n+1}}+o\left(\left(\frac{1}{n}\right)^{\frac{3}{2}}\right)} - 1}{\frac{1}{\sqrt{n}}} - \frac{e^{\frac{1}{2\sqrt{n+1}}+o\left(\left(\frac{1}{n}\right)^{\frac{3}{2}}\right)} - 1}{\frac{1}{\sqrt{n}}}\right) = be^{\zeta\left(\frac{1}{2}\right)}$$

Note:

$$\sum_{k}^{n}\frac{1}{k^\alpha} = \zeta(\alpha) + \frac{n^{1-\alpha}}{2} + \frac{1}{2n^\alpha} - \frac{\alpha}{24n^{1+\alpha}} + o\left(\frac{1}{n^{\alpha+2}}\right), 0 < \alpha \neq 1$$

Solution 3 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty} e^{a_n}\left(e^{a_{n+1}-a_n} - 1\right)x_n = \lim_{n\to\infty} e^{a_n - 2\sqrt{n} + 2\sqrt{n}}\cdot\left(\frac{1}{e^{\sqrt{n+1}}} - 1\right)x_n =$$

$$= e^S\cdot\lim_{n\to\infty} e^{2\sqrt{n}}x_n\left(e^{\frac{1}{\sqrt{n+1}}} - 1\right) =$$

UP.222. If $a > 0$; $(x_n)_{n\geq1} \subset (0,\infty)$ such that:

$\log(n + ax_n) = H_n - \gamma$ then find $\Omega = \lim_{n\to\infty} x_n$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\ln(n + ax_n) = H_n - \gamma \Rightarrow n + ax_n = e^{H_n-\gamma} \Rightarrow$$

$$x_n = \frac{1}{a}\left(e^{H_n-\gamma} - n\right) \quad (1)$$

$$\lim_{n\to\infty}\left(e^{H_n-\gamma} - n\right) = \lim_{n\to\infty} e^{H_n-\gamma} - e^{\ln n} = \lim_{n\to\infty} e^{\ln n}\left(e^{H_n-\gamma-\ln n} - 1\right) =$$

$$= \lim_{n\to\infty}\frac{n\left(e^{H_n-\ln n-\gamma}-1\right)}{H_n-\ln n-\gamma}\left(H_n - \ln n - \gamma\right) \quad (2)$$

$$\lim_{n\to\infty}\frac{e^{H_n-\ln n-\gamma}-1}{H_n-\ln n-\gamma} = \ln e = 1 \quad (3)$$

$$\lim_{n\to\infty} n(H_n - \ln n - \gamma) = \lim_{n\to\infty} \frac{H_n - \ln n - \gamma}{\frac{1}{n}} \underset{for\ \frac{0}{0}}{\overset{C.S.}{=}}$$

$$= \lim_{n\to\infty} \frac{H_{n+1} - \ln(n+1) - H_n + \ln n}{\frac{1}{n+1} - \frac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right)}{\frac{n-n-1}{n(n+1)}} = \lim_{n\to\infty} \frac{1-(n+1)\ln\left(1+\frac{1}{n}\right)}{-\frac{1}{n}} =$$

$$= \lim_{n\to 0} \frac{1-\left(\frac{1}{x}+1\right)\ln(1+x)}{-x} = \lim_{n\to 0} \frac{x-(1+x)\ln(1+x)}{-x^2} \overset{L'H}{=}$$

$$= \lim_{n\to 0} \frac{1-\ln(1+x)-1}{-2x} = \lim_{n\to 0} \frac{\ln(1+x)}{2x} = \frac{1}{2} \quad (4)$$

From (1)+(2)+(3)+(4) $\Rightarrow \Omega = \frac{1}{a} \cdot \frac{1}{2} = \frac{1}{2a}$

Solution 2 by Michael Sterghiou-Greece

$H_n = \ln n + \gamma + e_n$ ***where*** $e_n \sim \frac{1}{2n}$ ***therefore:***

$$\log(n + ax_n) = \log n + \frac{1}{2n} \to x_n = \frac{1}{a}\left[e^{\ln n + \frac{1}{2n}} - n\right] =$$

$\frac{1}{a}\left[n \cdot e^{\frac{1}{2n}} - n\right] = \frac{1}{a} n\left[e^{\frac{1}{2n}} - 1\right] = \frac{1}{a} \cdot \frac{\frac{1}{2n} - 1}{\frac{1}{n}}$. ***This limit is of the form*** $\frac{0}{0}$ ***as*** $e^{\frac{1}{2n}} \to 1, n \to \infty$.

Taking the respective function $\frac{e^{\frac{1}{2x}}-1}{\frac{1}{x}}$ ***and using DLH we have:***

$\lim_{x\to\infty} \frac{e^{\frac{1}{2x}}-1}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{1}{2x}-e^{\frac{1}{2x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\left(\frac{1}{2} \cdot e^{\frac{1}{2x}}\right) = \frac{1}{2}$ ***as*** $\lim_{x\to\infty} e^{\frac{1}{2x}} = 1$. ***Therefore***

$$\lim_{n\to\infty} x_n = \frac{1}{2a}$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \frac{e^{H_n-\gamma} - n}{a} = \frac{1}{a}\lim_{n\to+\infty}\left(e^{\left(\gamma+\log\left(n+\frac{1}{2}\right)+o\left(\frac{1}{n^2}\right)\right)-\gamma} - n\right) =$$

$$= \frac{1}{a}\lim_{n\to+\infty}\left(n + \frac{1}{2} - n\right) = \frac{1}{2a}$$

Solution 4 by Khaled Abd Imouti-Damascus-Syria

As you know: $0 \leq \gamma - H_n + \frac{1}{2^n} + L_n(x) \leq \frac{1}{2n(n-1)}, \forall n \geq 2$

So: $\frac{1}{2n} \leq \gamma - H_n + \frac{1}{n} + L_n(n) = \frac{1}{2(n-1)}$

$$-\frac{1}{2n} \geq H_n - \gamma - L_n(n) \geq \frac{1}{n} - \frac{1}{2(n-1)}$$

$$L_n(n) + \frac{n-2}{2n(n-1)} \leq H_n - \gamma \leq \frac{1}{n} + L_n(n), \forall n \geq 2$$

$$n \cdot e^{\frac{n-2}{2n(n-1)}} \leq e^{H_n - \gamma} \leq e^{\frac{1}{2n}} \cdot n$$

$$n \cdot e^{\frac{n-2}{2n(n-1)}} \leq n + a \cdot x_n \leq e^{\frac{1}{2n}} \cdot n - n$$

$$\frac{1}{a}\left(n \cdot e^{\frac{n-2}{2n(n-1)}} - n\right) \leq x_n \leq \frac{1}{a}\left(n \cdot e^{\frac{1}{2n}} - n\right)$$

Suppose: $f(x) = x \cdot e^{\frac{1}{2x}} - x, \lim_{x\to+\infty} f(f(x)) = ?$

$$y = \frac{1}{2x}, x \to +\infty \Rightarrow y \to 0$$

$$\lim_{x\to+\infty}(f(x)) = \lim_{y\to 0}\left[\frac{e^y - 1}{2y}\right] = \frac{1}{2}$$

Suppose: $g(x) = x \cdot e^{\frac{x-2}{2x(x-1)}} - x, \lim_{x\to+\infty}(g(x)) = ?$

$$y = \frac{1}{2x}, \lim_{x\to+\infty}(g(x)) = \lim_{y\to 0}\left[\frac{e^{\frac{y-4y^2}{1-2y}} - 1}{2y}\right] = \lim_{y\to 0}\left[\frac{\left(\frac{y-4y^2}{1-2y}\right)}{2y} \cdot \frac{e^{\frac{y-4y^2}{1-2y}}}{\left(\frac{y-4y^2}{1-2y}\right)}\right]$$

$= \lim_{y\to 0}\left[\frac{1-4y}{2(1-2y)} \cdot \frac{e^{\frac{y-4y^2}{1-2y}} - 1}{\left(\frac{y-4y^2}{1-2y}\right)}\right] = \frac{1}{2}$. ***By using Sandwich Theorem*** $\lim_{n\to+\infty}(x_n) = \frac{1}{2a}$

UP.223. If $(a_n)_{n\geq 1}; (b_n)_{n\geq 1} \subset (0, \infty)$ such that:

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n} \cdot \frac{1}{n\sqrt{n}}\right) = a > 0; \lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n} \cdot \sqrt{n}\right) = b > 0$$

then find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n]{a_nb_n}\cdot\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right)\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty}\frac{\sqrt[n]{a_nb_n}}{n}\cdot n\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right) \quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_nb_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{a_nb_n}{n^n}} \overset{C.D.}{=} \lim_{n\to\infty}\frac{a_{n+1}b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_nb_n} =$$

$$= \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{n\sqrt{n}}\cdot\frac{b_{n+1}}{b_n}\cdot\sqrt{n}\cdot\frac{n}{n+1}\cdot\left(\frac{n}{n+1}\right)^n = \frac{ab}{e} \quad (2)$$

$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right) = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n+1}-e}{\frac{1}{n}} = \lim_{t\to\infty}\frac{\left(1+\frac{1}{t}\right)^{t+1}-e}{\frac{1}{t}}$$

$$= \lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}+1}-e}{x} \overset{L'H}{=} \lim_{x\to 0}(1+x)^{\frac{1}{x}+1}\left[-\frac{1}{x^2}\ln(1+x)+\frac{\frac{1}{x}+1}{1+x}\right]$$

$$= \lim_{x\to 0} e\left(-\frac{\ln(1+x)}{x^2}+\frac{1}{x}\right) = \lim_{x\to 0} e\left(-\frac{\ln(1+x)+x}{x^2}\right) =$$

$$\overset{L'H}{=} e\lim_{x\to 0}\frac{-\frac{1}{1+x}+1}{2x} = e\lim_{x\to 0}\frac{-1+1+x}{2x(1+x)} = \frac{e}{2}$$ ***(3). From (1)+(2)+(3)*** $\Rightarrow \Omega = \frac{ab}{2}$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\cdot\frac{1}{n\sqrt{n}}\right) = a > 0 \text{ and } \lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\cdot\sqrt{n}\right) = b > 0$$

Let $u_n = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+1}$ ***for all*** $n\in\mathbb{N}$***, then*** $\lim_{n\to\infty}u_n = 1$***. Hence*** $\frac{u_n-1}{\ln u_n}\to 1$ ***for all*** $n\to\infty$

Let $\Omega = n\ln\frac{\left(1+\frac{1}{n}\right)^{n+1}}{e}$ ***for all*** $n\in\mathbb{N}$***, then*** $\lim_{n\to\infty}\Omega = \lim_{n\to\infty}\frac{(n+1)\ln\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}}$

$$\overset{Cesaro-Stolz}{=} \lim_{n\to\infty}\frac{(n+1)\ln\left(1+\frac{1}{n}\right)-(n+2)\ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n}-\frac{1}{n+1}}$$

$$= \lim_{n\to\infty}\left\{n(n+1)^2\ln\left(1+\frac{1}{n}\right)-n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right)\right\}$$

$$= \lim_{n\to 0}\frac{(1+x)^2\ln(1+x)-(1+x)(2x+1)\ln\left(1+\frac{x}{1+x}\right)}{x^3}$$

$$= \lim_{n\to 0}\frac{(3x+2)(1+x)\ln(1+x)-(1+x)(2x+1)\ln(2x+1)}{x^3}\left[\frac{0}{0}format\right]$$

$$\overset{L'Hospital's\ Rule}{=}\lim_{x\to 0}\frac{x+(6x+5)\ln(1+x)-(4x+3)\ln(2x+1)}{3x^2}\left[\frac{0}{0}format\right]$$

$$\overset{L'Hospital's\ Rule}{=}\lim_{x\to 0}\frac{1+\frac{6x+5}{1+x}+6\ln(1+x)-\frac{2(4x+3)}{2x+1}-4\ln(2x+1)}{6x}\left[\frac{0}{0}format\right]$$

$$\overset{L'Hospital's\ Rule}{=}\lim_{x\to 0}\frac{\frac{6}{1+x}-\frac{6x+5}{(1+x)^2}+\frac{6}{1+x}-\frac{8}{2x+1}+\frac{4(4x+3)}{(2x+1)^2}-\frac{8}{2x+1}}{6}=\frac{1}{2}$$

$$\therefore \lim_{n\to\infty}\left(\sqrt[n]{a_nb_n}\cdot\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right)\right)=e\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_nb_n}}{n}\cdot\left(\frac{1}{e}\left(1+\frac{1}{n}\right)^{n+1}-1\right)\right)$$

$$\overset{\substack{Cauch\\ D'Alembert}}{=} e\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\cdot\frac{1}{n\sqrt{n}}\right)\cdot\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\sqrt{n}\right)\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\lim_{n\to\infty}\frac{n}{1+n}\cdot$$

$$\cdot\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}\cdot\lim_{n\to\infty}\ln u_n^n=\frac{ab}{2}\quad \textit{(Answer)}$$

Solution 3 by Remus Florin Stanca-Romania

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n\sqrt{n}}\cdot\sqrt[n]{b_n}\cdot\sqrt{n}\cdot n\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right)\quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n\sqrt{n}}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n\left(\sqrt{n}\right)^n}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}\left(\sqrt{n+1}\right)^{n+1}}\cdot\frac{n^n\left(\sqrt{n}\right)^n}{a_n}=$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\left(\sqrt{\frac{n}{n+1}}\right)^n\cdot\frac{1}{\sqrt{n+1}(n+1)}=\frac{1}{e\sqrt{e}}\cdot\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{n\sqrt{n}}=\frac{1}{e\sqrt{e}}a\quad (2)$$

$$\lim_{n\to\infty}\sqrt{n}\cdot\sqrt[n]{b_n}=\lim_{n\to\infty}\sqrt[n]{\left(\sqrt{n}\right)^n\cdot b_n}=\lim_{n\to\infty}\frac{\left(\sqrt{n+1}\right)^{n+1}b_{n+1}}{\left(\sqrt{n}\right)^n b_n}=$$

$$\sqrt{e}\cdot\lim_{n\to\infty}\frac{b_{n+1}}{b_n}\sqrt{n}=\sqrt{e}b\ (3)$$

$$\overset{(1);(2);(3)}{\Rightarrow}\ \Omega=\frac{1}{e\sqrt{e}}a\sqrt{e}\lim_{n\to\infty}n\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right)=\frac{ab}{e}\lim_{n\to\infty}n\left(e^{\ln\left(1+\frac{1}{n}\right)^{n+1}}-e^{1}\right)=$$

$$=\frac{ab}{e}\lim_{n\to\infty}n\frac{e^{\ln\left(1+\frac{1}{n}\right)^{n+1}}-e^{1}}{\ln\left(1+\frac{1}{n}\right)^{n+1}-1}\left(\ln\left(1+\frac{1}{n}\right)^{n+1}-1\right)\quad \textbf{(4)}$$

It's known that $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f(x_0)\overset{(4)}{\Rightarrow}\Omega=ab\cdot\lim_{n\to\infty}n\left((n+1)\ln\left(1+\frac{1}{n}\right)-\right.$

1=

$$=ab\lim_{n\to\infty}n\ln\left((n+1)\ln\left(1+\frac{1}{n}\right)\right)=ab\lim_{n\to\infty}\frac{\ln(n+1)+\ln\left(\ln\left(1+\frac{1}{n}\right)\right)}{\frac{1}{n}}=$$

Let $\frac{1}{n}=x\Rightarrow\Omega=ab\cdot\lim_{x\to0}\frac{\ln\left(\frac{1}{x}+1\right)+\ln(\ln(x+1))}{x}=$

$$=ab\lim_{x\to0}\frac{\ln(x+1)-\ln x+\ln(\ln(x+1))}{x}\underset{\frac{0}{0}}{\overset{L'H}{=}}ab\lim_{x\to0}\left(\frac{1}{x+1}-\frac{1}{x}+\frac{1}{(x+1)\ln(x+1)}\right)=$$

$$=ab\left(1+\lim_{x\to0}\left(\frac{1}{(x+1)\ln(x+1)}-\frac{1}{x}\right)\right)=ab+ab\lim_{x\to0}\frac{1}{x}\left(\frac{x}{(x+1)\ln(x+1)}-1\right)=$$

$$=ab+ab\lim_{x\to0}\frac{\ln(x+1)}{x}\cdot\frac{1}{x}\left(\frac{x}{(x+1)\ln(x+1)}-1\right)=ab+ab\lim_{x\to0}\frac{1}{x}\left(\frac{1}{x+1}-\frac{\ln(x+1)}{x}\right)=$$

$$\underset{\frac{0}{0}}{\overset{L'H}{=}}ab+ab\lim_{x\to0}\left(-\frac{1}{(x+1)^2}-\frac{\frac{x}{x+1}-\ln(x+1)}{x^2}\right)=$$

$$=ab-ab\left(1+\lim_{x\to0}\frac{-\frac{2}{(x+1)^3}+\frac{1}{(x+1)^2}}{2}\right)=ab-ab\left(1-\frac{1}{2}\right)=\frac{ab}{2}\Rightarrow\Omega=\frac{ab}{2}$$

Solution 4 by Mokhtar Khassani-Mostaganem-Algerie

$$\lim_{n\to+\infty}\sqrt[n]{a_nb_n}\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right)=e\lim_{n\to+\infty}\sqrt[n]{\frac{a_n}{n^{\frac{3n}{2}}}b_nn^{\frac{3}{2}}}\,n\left(e^{(1+n)\log\left(1+\frac{1}{n}\right)-1}-1\right)$$

$$=\lim_{n\to+\infty}\frac{\frac{a_{n+1}}{(n+1)^{\frac{3(n+1)}{2}}}b_{n+1}(n+1)^{\frac{n+1}{2}}}{\frac{a_n}{n^{\frac{3n}{2}}}b_nn^{\frac{n}{2}}}\cdot\frac{e^{(1+n)\log\left(1+\frac{1}{n}\right)-1}-1}{(1+n)\log\left(1+\frac{1}{n}\right)-1}\cdot\frac{(1+n)\log\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}}$$

$$= abe\frac{1}{2}\lim_{n\to+\infty}\frac{n^{n+1}}{(n+1)^{n+1}} = \frac{abe}{2}\lim_{n\to+\infty}\left(1-\frac{1}{n+1}\right)^{n+1} = \frac{ab}{2}$$

UP.224. If $(a_n)_{n\geq 1}; (b_n)_{n\geq 1} \subset (0,\infty)$ such that:

$$\lim_{n\to\infty}\left(\frac{a_n}{n}\right) = a > 0; \lim_{n\to\infty}\left(\frac{b_{n+1}}{a_n b_n}\right) = b > 0 \text{ then find:}$$

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\frac{a_n}{n} = a > 0 \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{a_n b_n} = b > 0$$

Now, $\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}} \overset{CAUCHY-D'ALEMBERT}{=} \lim_{n\to\infty}\left(\frac{b_{n+1}}{a_n b_n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{n}{n+1}\cdot\frac{a_n}{n}\right) = \frac{ab}{e}$

Let $u_n = \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}$ *for all* $n\in\mathbb{N}$ *then* $\lim_{n\to\infty}u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\left(1+\frac{1}{n}\right)\right) = 1$

Hence $\frac{u_n-1}{\ln u_n}\to 1$ *for* $n\to\infty$. $\text{Lim}_{n\to\infty}u_n^n = \lim_{n\to\infty}\left(\frac{b_{n+1}}{a_n b_n}\cdot\frac{a_n}{n}\cdot\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\right) = e$

$\therefore \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{b_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) = \frac{ab}{e}\cdot 1\cdot\ln e = \frac{ab}{e}$ ***(Answer)***

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega = \lim_{n\to+\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \lim_{n\to+\infty}\left((n+1)\sqrt[n]{\frac{b_{n+1}}{(n+1)^{n+1}}} - n\sqrt[n]{\frac{b_n}{n^n}}\right) =$$

$$= \lim_{n\to+\infty}\frac{\frac{b_{n+1}}{(n+1)^{n+1}}}{\frac{b_n}{n^n}} = \lim_{n\to+\infty}\frac{b_{n+1}}{b_n a_n}\cdot\frac{a_n}{n}\left(1-\frac{1}{n+1}\right)^{n+1} = \frac{ab}{e}$$

UP.225. If $m \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$3^m\left(\left(\frac{a}{h_a}\cot A\right)^{m+1}+\left(\frac{b}{h_b}\cot B\right)^{m+1}+\left(\frac{c}{h_c}\cot C\right)^{m+1}\right)\ge m+2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

From Hölder's inequality we have:

$$\left(\frac{a}{h_a}\cot A\right)^{m+1}+\left(\frac{b}{h_b}\cot B\right)^{m+1}+\left(\frac{c}{h_c}\cot C\right)^{m+1}\ge\frac{\left(\frac{a}{h_a}\cot A+\frac{b}{h_b}\cot B+\frac{c}{h_c}\cot C\right)^{m+1}}{3^m}$$

$\Rightarrow$ *we must show:* $\left(\frac{a}{h_a}\cot A+\frac{b}{h_b}\cot B+\frac{c}{h_c}\cot C\right)^{m+1}\ge m+2$ *(1)*

But $\frac{a}{h_a}=\frac{a}{\frac{2S}{a}}=\frac{a^2}{2S}$ *(2)*

From (1)+(2)⇒ we must show: $\left(\frac{a^2\cot A+b^2\cot B+c^2\cot C}{2S}\right)^{m+1}\ge m+2$ *(3)*

But in any ΔABC *we have:* $a^2\cot A+b^2\cot B+c^2\cot C=4S$ *(4)*

From (3)+(4) we must show:

$2^{m+1}\ge m+2, \forall m\in\mathbb{N}$, *which it's true, with equality for* $m=0$.

Solution 2 by Tran Hong-Dong Thap-Vietnam

Using AM-GM inequality we have:

$$\left(\frac{a}{h_a}\cot A\right)^{m+1}+\left(\frac{b}{h_b}\cot B\right)^{m+1}+\left(\frac{c}{h_c}\cot C\right)^{m+1}\ge 3\sqrt[3]{\left(\frac{a}{h_a}\cot A\frac{b}{h_b}\cot B\frac{c}{h_c}\cot C\right)^{m+1}}$$

$$=3\left(\frac{abc\cot A\cot B\cot C}{h_ah_bh_c}\right)^{m+1}=3\left(\frac{4rRs}{\frac{2s^2r^2}{R}}\cdot\frac{s^2-(2R+r)^2}{2sr}\right)^{\frac{m+1}{3}}$$

$$=3\left(\frac{R^2(s^2-(2R+r)^2)}{s^2r^2}\right)^{\frac{m+1}{3}}$$

$$\to 3^m\left(\left(\frac{a}{h_a}\cot A\right)^{m+1}+\left(\frac{b}{h_b}\cot B\right)^{m+1}+\left(\frac{c}{h_c}\cot C\right)^{m+1}\right)$$

$$\geq 3^{m+1}\left(\frac{R^2(s^2-(2R+r)^2)}{s^2r^2}\right)^{\frac{m+1}{3}}=\Omega.\text{ We have: }\frac{R^2(s^2-(2R+r)^2)}{s^2r^2}>\left(\frac{2}{3}\right)^3$$

In ΔABC *(acute) we have:* $s^2-(2R+r)^2>0\leftrightarrow s^2>(2R+r)^2\leftrightarrow s>2R+r$

$$R\geq 2r\to\frac{R^2}{r^2}\geq 4;\ \frac{s^2-(2R+r)^2}{s^2}>\frac{2}{27}\leftrightarrow 5s>2R+r$$

(true because: $s>2R+r\to 5s>2R+r$*)*

$$\text{So, }\ \Omega\geq 3^{m+1}\cdot\left(\frac{2}{3}\right)^{3\cdot\frac{m+1}{3}}\geq 2^{m+1}\geq m+2\ \ (\text{true with } m\in\mathbb{N})$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$m\in\mathbb{N},\text{ repeated Chebyshev}\Rightarrow\sum\left(\frac{a}{h_a}\cot A\right)^{m+1}\geq\frac{1}{3^m}\left(\sum\frac{a}{h_a}\cot A\right)^{m+1}$$

$$=\frac{1}{3^m}\left(\sum\frac{2R\sin A\cot A}{\frac{2rs}{a}}\right)^{m+1}=\frac{1}{3^m}\left(\sum\frac{2R\cdot 2R\sin A\cos A}{2rs}\right)^{m+1}$$

$$=\frac{1}{3^m}\left(\left(\frac{2R^2}{2rs}\right)\sum\sin 2A\right)^{m+1}=\frac{1}{3^m}\left(\left(\frac{R^2}{rs}\right)4\sin A\sin B\sin C\right)^{m+1}$$

$$=\frac{1}{3^m}\left(\frac{4R^2}{rs}\cdot\frac{4Rrs}{8R^3}\right)^{m+1}=\frac{1}{3^m}(2)^{m+1}=\frac{1}{3^m}(1+1)^{m+1}$$

$$\overset{Bernoulli}{\geq}\frac{1}{3^m}(1+m+1)(\because m+1\geq 1\because m\in\mathbb{N})$$

$$=\frac{m+2}{3^m}\Rightarrow 3^m\sum\left(\frac{a}{h_a}\cot A\right)^{m+1}\geq m+2\quad(\text{Proved})$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru