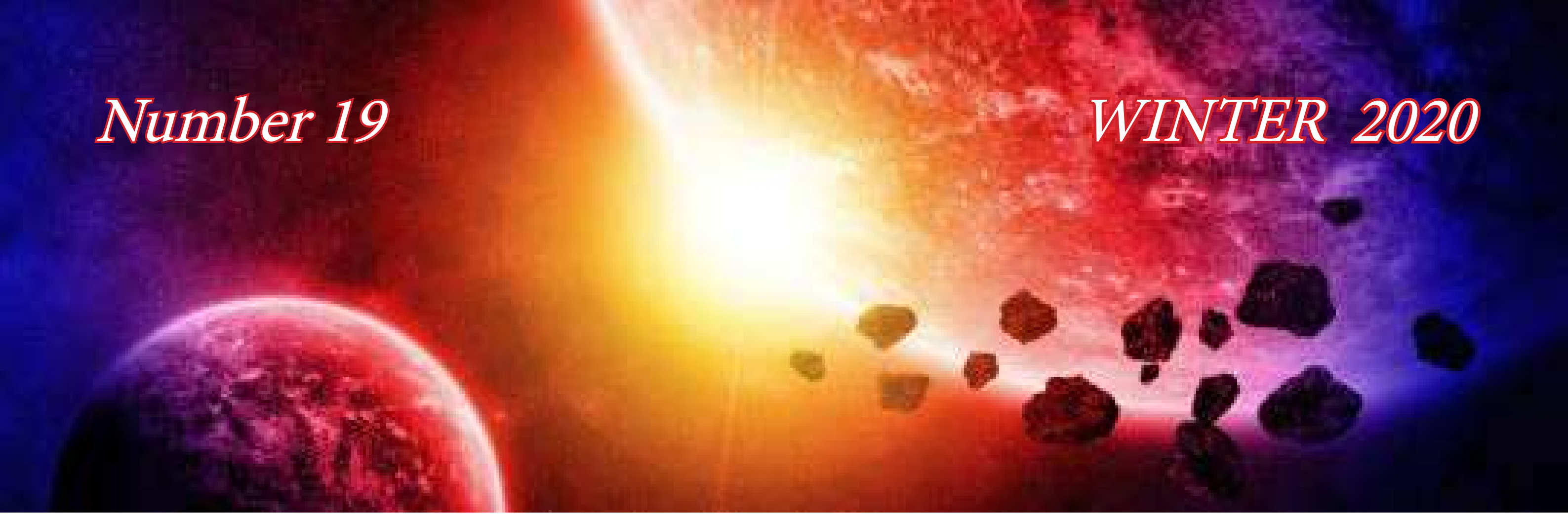

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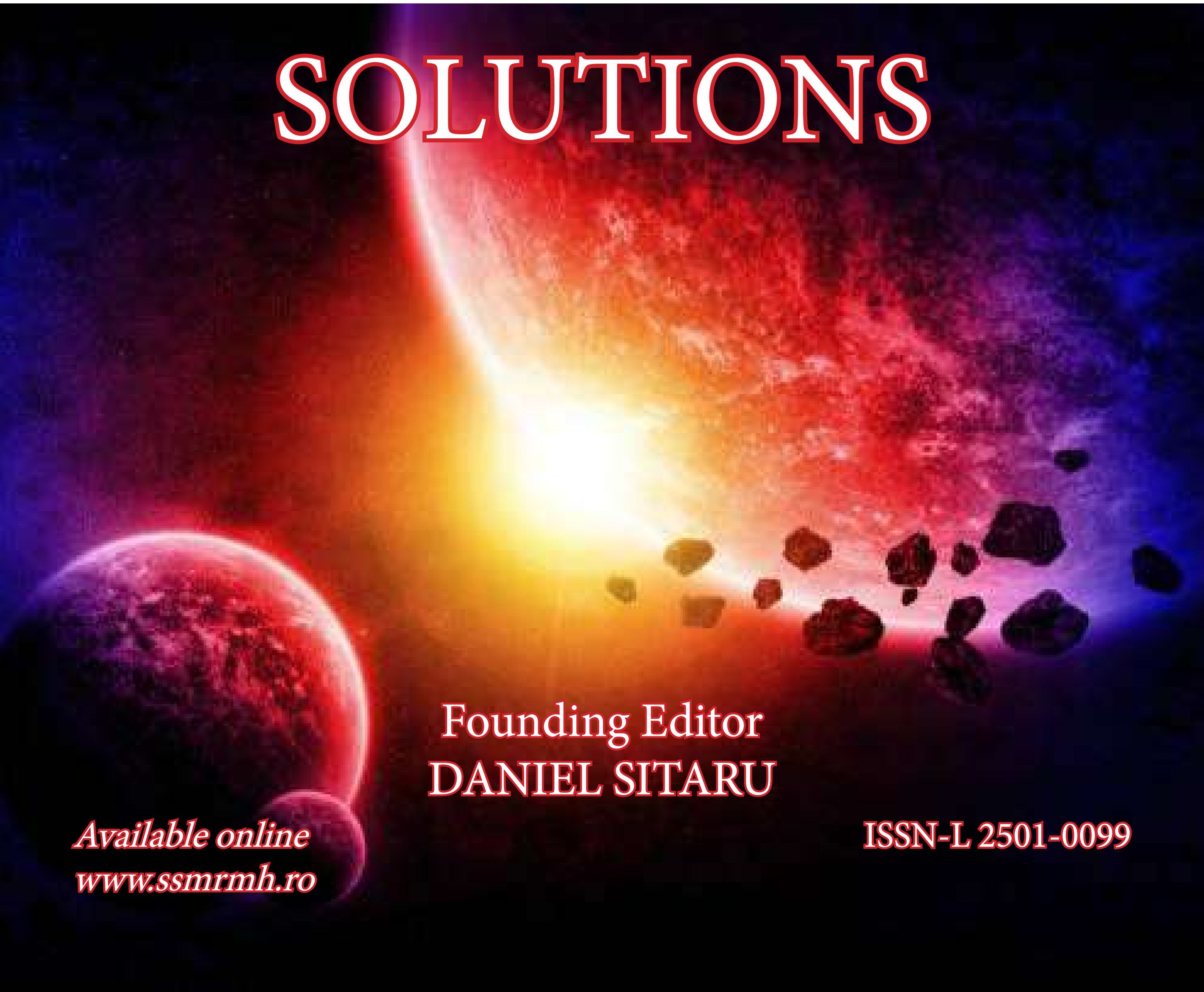

SOLUTIONS

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JP.271. If $a, b, c > 0; abc = a + b + c + 2$ then:

$$(2a + 1)^2 + (2b + 1)^2 + (2c + 1)^2 \geq 25$$

Proposed by Marin Chirciu-Romania

Solution by proposer

$$a, b, c > 0; abc = a + b + c + 2 \Rightarrow \frac{1}{1 + a} + \frac{1}{1 + b} + \frac{1}{1 + c} = 1$$

$$1 = \frac{1}{1 + a} + \frac{1}{1 + b} + \frac{1}{1 + c} \overset{Am-Hm}{\geq} \frac{9}{3 + a + b + c} \Rightarrow a + b + c \geq 6; \ (1)$$

Equality if and only if $a = b = c = 2$.

$$abc = a + b + c + 2 \Rightarrow \frac{a}{1 + a} + \frac{b}{1 + b} + \frac{c}{1 + c} = 2$$

$$2 = \sum \frac{a}{1 + a} = \sum \frac{a^2}{a + a^2} \overset{Bergstrom}{\geq} \frac{(\sum a)^2}{\sum(a + a^2)} \overset{(1)}{\geq} \frac{36}{\sum(a + a^2)}$$

$$2 \geq \frac{36}{\sum(a + a^2)} \Rightarrow \sum(a + a^2) \geq 18 \Rightarrow \sum(2a + 1)^2 \geq 25$$

Equality if only if $a = b = c = 2$

JP.272. If $a, b, c, \lambda > 0, a^2 + b^2 + c^2 = 1$ then:

$$1 \leq a\sqrt{1 + \lambda bc} + b\sqrt{1 + \lambda ca} + c\sqrt{1 + \lambda ab} \leq \sqrt{3 + \lambda}$$

Proposed by Hung Nguyen Viet-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$1 \overset{(*)}{\leq} a\sqrt{1 + \lambda bc} + b\sqrt{1 + \lambda ca} + c\sqrt{1 + \lambda ab} \overset{(**)}{\leq} \sqrt{3 + \lambda}$$

Because: $a^2 + b^2 + c^2 = 1$***, then:***

$$(*) \Leftrightarrow 1 \leq \sum_{cyc} a(1 + \lambda bc) + 2\sum_{cyc}\left(ab\sqrt{(1 + \lambda ca)(1 + \lambda bc)}\right) \Leftrightarrow$$

$$1 \leq \left[\sum_{cyc} a(1 + \lambda bc) + 2\sum_{cyc}\left(ab\sqrt{(1 + \lambda ca)(1 + \lambda bc)}\right)\right]^2 \Leftrightarrow$$

$$1\le\left(\sum_{cyc}a(1+\lambda bc)\right)^2+4\left(\sum_{cyc}a(1+\lambda bc)\right)\left(\sum_{cyc}\left(ab\sqrt{(1+\lambda ca)(1+\lambda bc)}\right)\right)+$$

$$+\left(\sum_{cyc}\left(ab\sqrt{(1+\lambda ca)(1+\lambda bc)}\right)\right)^2$$

Let: $\Omega=4\left(\sum_{cyc}a(1+\lambda bc)\right)\left(\sum_{cyc}\left(ab\sqrt{(1+\lambda ca)(1+\lambda bc)}\right)\right)+$

$\left(\sum_{cyc}\left(ab\sqrt{(1+\lambda ca)(1+\lambda bc)}\right)\right)^2,\ \forall a,b,c,\lambda>0$

$$1\le(a+b+c+3\lambda abc)^2+\Omega\Leftrightarrow$$

$$1\le a^2+b^2+c^2+9(\lambda abc)^2+2\big(ab+bc+ca+3\lambda abc(a+b+c)\big)+\Omega$$

$$\overset{a^2+b^2+c^2=1}{\Longleftrightarrow}0\le 9(\lambda abc)^2+2\big(ab+bc+ca+3\lambda abc(a+b+c)\big)+\Omega$$

Which is clearly true because: $a,b,c,\lambda,\Omega>0$***. Now,***

$$a\sqrt{1+\lambda bc}+b\sqrt{1+\lambda ca}+c\sqrt{1+\lambda ab}\overset{BCS}{\le}\sqrt{a^2+b^2+c^2}\cdot\sqrt{(1+\lambda bc)+(1+\lambda ac)+(1+\lambda ab)}$$

$$\overset{a^2+b^2+c^2=1}{=}\sqrt{3+\lambda(ab+bc+ca)}\overset{\sum ab\le\sum a^2=1}{\le}\sqrt{3+\lambda\cdot 1}=\sqrt{3+\lambda}\Rightarrow(**)$$ ***Is true. Proved.***

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$a\sqrt{1+\lambda bc}+b\sqrt{1+\lambda ca}+c\sqrt{1+\lambda ab}=\sqrt{a^2+\lambda a^2bc}+\sqrt{b^2+\lambda b^2ca}+\sqrt{c^2+\lambda c^2ab}$$

$$\le\sqrt{3(a^2+b^2+c^2)+\lambda(a^2bc+b^2ca+c^2ab)}\le\sqrt{3+\lambda}$$

$$\therefore a^2+b^2+c^2=1\Rightarrow(a^2+b^2+c^2)^2\ge 3(a^2bc+b^2ca+c^2ab)$$

Because: $a^2+b^2+c^2=1$, ***consider***

$1+\lambda bc\ge a^2;\ 1+\lambda ca\ge b^2;1+\lambda ab\ge c^2$ ***hence***

$\sqrt{1+\lambda bc}\ge a,\sqrt{1+\lambda ca}\ge b,\sqrt{1+\lambda ab}\ge c$ ***hence***

$$a\sqrt{1+\lambda bc}\ge a^2,b\sqrt{1+\lambda ca}\ge b^2,c\sqrt{1+\lambda ab}\ge c^2\Rightarrow$$

$$a\sqrt{1+\lambda bc}+b\sqrt{1+\lambda ca}+c\sqrt{1+\lambda ab}\ge a^2+b^2+c^2=1$$

Therefore: $1\le a\sqrt{1+\lambda bc}+b\sqrt{1+\lambda ca}+c\sqrt{1+\lambda ab}\le\sqrt{3+\lambda}$***. Proved.***

JP.273 If $a, b, c > 0$ then:

$$\frac{a^3+b^3+c^3}{3abc}+\frac{ab+bc+ca}{a^2+b^2+c^2}\ge\frac{2(a^2+b^2+c^2)}{ab+bc+ca}$$

Proposed by Nguyen Viet Hung-Hanoi – Vietnam

Solution by Marin Chirciu-Romania

Adding $\frac{a^2+b^2+c^2}{ab+bc+ca}$ to both members, the inequality can be written:

$$\frac{a^2+b^3+c^3}{3abc}+\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge\frac{3(a^2+b^2+c^2)}{ab+bc+ca}\quad (1)$$

Using the means inequality, we obtain:

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge 2\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}}=2\quad (2)$$

From (1) and (2) it suffices to prove that: $\frac{a^3+b^3+c^3}{3abc}+2\ge\frac{3(a^2+b^2+c^2)}{ab+bc+ca}$ *(3)*

Subtracting 3 *from both members of inequality (3), we obtain:*

$$\frac{a^3+b^3+c^3}{3abc}-1\ge\frac{3(a^2+b^2+c^2)}{ab+bc+ca}-3\Leftrightarrow\frac{a^3+b^3+c^3-3abc}{3abc}\ge$$

$$\ge\frac{3(a^2+b^2+c^2-ab-bc-ca)}{ab+bc+ca}\Leftrightarrow$$

$$\Leftrightarrow\frac{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}{3abc}\ge\frac{3(a^2+b^2+c^2-ab-bc-ca)}{ab+bc+ca}\Leftrightarrow$$

$\Leftrightarrow (a^2+b^2+c^2-ab-bc-ca)[(a+b+c)(ab+bc+ca)-9abc]\ge 0$, *which follows from:*

$$(a^2+b^2+c^2-ab-bc-ca)\ge 0\Leftrightarrow(a-b)^2+(b-c)^2+(c-a)^2\ge 0$$

Obviously, with equality for $a=b=c$ *and*

$[(a+b+c)(ab+bc+ca)-9abc]\ge 0$, *true from means inequalities:*

$a+b+c\ge 3\sqrt[3]{abc}$ *and* $ab+bc+ca\ge 3\sqrt[3]{(abc)^2}$, *wherefrom*

$$(a+b+c)(ab+bc+ca)\ge 9abc$$

Above we've used the identity:

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

We deduce that the inequality from enunciation, with equality holds if and only if

$$a = b = c.$$

JP.274. If $x, y, z \geq 0, x + y + z = 1; n \geq 2$ then:

$$(n + 1)(xy + yz + zx) \leq n(x^2 + y^2 + z^2) + 9xyz$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

By Schur's inequality:

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x) \Leftrightarrow$$

$$x^2 + y^2 + z^2 + \frac{9xyz}{x + y + z} \geq 2(xy + yz + zx) \quad (*)$$

Now, because: $x + y + z = 1$

Inequality becomes as: $(n + 1)(xy + yz + zx) \leq n(x^2 + y^2 + z^2) + \frac{9xyz}{x+y+z} \Leftrightarrow$

$$2(xy + yz + zx) + (n - 1)(xy + yz + zx) \leq$$

$$(n - 1)(x^2 + y^2 + z^2) + x^2 + y^2 + z^2 + \frac{9xyz}{x + y + z} \Leftrightarrow$$

$$(n - 1)(x^2 + y^2 + z^2 - xy - yz - zx) +$$

$$+\left(x^2 + y^2 + z^2 + \frac{9xyz}{x+y+z} - 2(xy + yz + zx)\right) \geq 0$$

Which is true because: $n \geq 2 \Rightarrow n - 1 > 0, x^2 + y^2 + z^2 \geq xy + yz + zx$ ***and by (*)***

Proved.

Solution 2 by Marian Dincă-Romania

Let: $x = \frac{a}{a+b+c}, y = \frac{b}{a+b+c}, z = \frac{c}{a+b+c}$

$$(n + 1)(ab + bc + ca)(a + b + c) \leq n(a^2 + b^2 + c^2)(a + b + c) + 9abc$$

Let: $a + b + c = p, ab + bc + ca = q, abc = r$

$$(n + 1)qp \leq n(p^2 - 2q)p + 9r$$

$$f(r) = n(p^2 - 2q)p + 9r - (n + 1)qp, 0 \leq r \leq \frac{pq}{9}$$

Because it is of te first degree in the variable r, it will be necessary and sufficient to:

$$f\left(\frac{pq}{9}\right) \geq 0 \Leftrightarrow n(p^2-2q)p+9r-(n+1)qp \geq \Leftrightarrow np(p^2-2q)-npq \geq 0 \Leftrightarrow$$

$$np(p^2-2q-q) \geq 0 \Leftrightarrow np(p^2-3q) \geq 0, \textit{ true.}$$

$$f(0) \geq 0 \text{ for } r=0 \Rightarrow abc=0$$

$$\text{Let: } c=0 \Rightarrow p=a+b, q=ab \Rightarrow q \leq \tfrac{p^2}{4}$$

$$f(0)=n(p^2-2q)p-(n+1)qp=n\left(p^2-\frac{p^2}{2}\right)p-\frac{(n+1)p^3}{4}=\frac{np^3}{2}-\frac{(n+1)p^3}{4}$$

$$=p^3\left(\frac{n-1}{4}\right)>0$$

JP.275 If in ΔABC, $b^2+c^2=3a^2$ then:

$$\frac{2}{h_a}\sqrt{\frac{bc}{5}}+\frac{w_b}{h_b}+\frac{w_c}{h_c}<1+\frac{r}{R}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$a^2=b^2+c^2-2b\cos A=3a^2-2bc\cos A$$

$$2bc\cos A=2a^2 \Rightarrow bc\cos A=a^2$$

$$\cos A=\tfrac{a^2}{bc}<1 \Rightarrow a^2<bc \quad \textbf{(1)}$$

$$w_a^2=\frac{4bcs(s-a)}{(b+c)^2}=\frac{4bc}{(b+c)^2}\cdot\frac{a+b+c}{2}\cdot\frac{b+c-a}{2}=$$

$$=\frac{bc}{(b+c)^2}\left((b+c)^2-a^2\right)=bc-\frac{bca^2}{(b+c)^2}=$$

$$=bc-\frac{bca^2}{b^2+c^2+2bc}=bc-\frac{bca^2}{3a^2+2bc}=bc-\frac{bc}{3+\frac{2bc}{a^2}}\overset{(1)}{>}bc-\frac{bc}{3+\frac{2bc}{bc}}=$$

$$=bc-\frac{bc}{5}=\frac{4bc}{5} \Rightarrow w_a>2\sqrt{\frac{bc}{5}}$$

$$\frac{2}{h_a}\sqrt{\frac{bc}{5}}+\frac{w_b}{h_b}+\frac{w_c}{h_c}<\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\le 1+\frac{r}{R}$$

JP.276. In ΔABC the following relationship holds:

$$\frac{3-n}{2}+\frac{nr}{2}\le\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\le\frac{3R}{4r};n\ge -1$$

Proposed by Marin Chirciu-Romania

Solution by proposer

$$\textit{LHS:}\ \frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\ge\frac{3-n}{2}+\frac{nr}{2},n\ge -1$$

$$\sum\frac{a^2}{b^2+c^2}=\sum\frac{a^4}{a^2(b^2+c^2)}\overset{Bergstrom}{\ge}\frac{\left(\sum a^2\right)^2}{2\sum b^2c^2}\overset{(1)}{\ge}\frac{R(3-n)+2nr}{2R},\ \textit{where}$$

$(1)\Leftrightarrow R\left(\sum a^2\right)^2\ge 2(2R-r)\sum b^2c^2$, ***true from relationship holds***

$$\sum a^2=2(s^2-r^2-4Rr),\sum b^2c^2=s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2$$

We must show that:

$$4R[s^4-2s^2(r^2+4Rr)+r^2(4R+r)^2]\ge[R(3-n)+2nr][s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2]$$

$$s^2[s^2(R(n+1)-2nr)-r(R^2(8+8n)+Rr(14-18n)+4nr^2)]+$$
$$+r^2(4R+r)^2(R(n+1)-2nr)\ge 0$$

We distinguish the cases:

Case 1) $[s^2(R(n+1)-2nr)-r(R^2(8+8n)+Rr(14-18n)+4nr^2)]\ge 0$ ***the inequality is obvious.***

Case 2) $[s^2(R(n+1)-2nr)-r(R^2(8+8n)+Rr(14-18n)+4nr^2)]<0$

the inequality becomes:

$r^2(4R+r)^2\ge s^2[r(R^2(8+8n)+Rr(14-18n)+4nr^2)-s^2(R(n+1)-2nr)]$ ***true from Blundon-Gerretsen inequality***

$$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}$$

It's suffices to prove:

$$r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}\left[r\left(R^2(8+8n)+Rr(14-18n)+4nr^2\right)-(16Rr-5r^2)(R(n+1)—2nr)\right]\Leftrightarrow$$

$$(8n+8)R^3-(15n+15)R^2r-(4n+2)Rr^2+4nr^3\geq 0\Leftrightarrow$$

$(R-2r)[(8n+8)R^2+(n+1)Rr-2nr^2]\geq 0$ ***true from*** $R\geq 2r$ ***–Euler and with***

$n\geq -1$ ***we obtain:*** $[(8n+8)R^2+(n+1)Rr-2nr^2]\geq 0$

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\leq\frac{3R}{4r}$$

$$\sum\frac{a^2}{b^2+c^2}=\frac{2[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}$$

From $2(b^2+c^2)\geq(b+c)^2\Leftrightarrow(b-c)^2\geq 0$ ***we get:***

$$\sum\frac{a^2}{b^2+c^2}\leq 2\sum\frac{a^2}{(b+c)^2}=\frac{4[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}$$

It's suffices to prove that:

$$\frac{4[s^4-s^2(4Rr+6r^2)+r^2(6R^2+4Rr+r^2)]}{(s^2+r^2+2Rr)^2}\leq\frac{3R}{4r}\Leftrightarrow$$

$$s^2[s^2(3R-16r)+r(12R^2+70Rr+96r^2)]+$$

$$+r^2(12R^3-84R^2r-61Rr^2-16r^3)\geq 0$$

We distinguish the cases:

Case 1) If $3R-16r\geq 0$, ***usig Gerretsen inequality*** $s^2\geq 16Rr-5r^2$

We must show that:

$$(16Rr-5r^2)[(16Rr-5r^2)(3R-16r)+r(12R^2+70Rr+96r^2)]+$$

$$r^2(12R^3-84R^2r-61Rr^2-16r^3)\geq 0\Leftrightarrow$$

$$243R^3-900R^2r+940Rr^2-224r^3\geq 0\Leftrightarrow$$

$(R-2r)(243R^2-414Rr+112r^2)\geq 0$ ***true from*** $R\geq 2r$ ***–Euler.***

Case 2) If $3R-16r<0$, ***the inequality can be rewritten:***

$$r^2(12R^3-84R^2r-61Rr^2-16r^3)\geq s^2[s^2(16r-3R)-r(12R^2+70Rr+96r^2)$$

true from Gerretsen: $s^2\leq 4R^2+4Rr+3r^2$.

We must show that: $r^2(12R^3-84R^2r-61Rr^2-16r^3)\geq$

$$\geq(4R^2+4Rr+3r^2)[(4R^2+4Rr+3r^2)(16r-3R)-r(12R^2+70Rr+96r^2)$$

$$\Leftrightarrow 12R^5-28R^4r-22R^3r^2+21R^2r^3+44Rr^4+32r^5\geq 0$$

$$\Leftrightarrow(R-2r)^2(12R^3+20R^2r+19Rr^2+8r^3)\geq 0.$$

Equality if and only if $R=2r$.

Equality if and only if the triangle is equilateral.

JP.277. In ΔABC the following relationship holds:

$$1 \le \left(\frac{a}{m_b + m_c}\right)^2 + \left(\frac{b}{m_c + m_a}\right)^2 + \left(\frac{c}{m_a + m_b}\right)^2 \le \frac{R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

The $\Delta m_a m_b m_c$ it has the medians $\frac{3a}{4}, \frac{3b}{4}, \frac{3c}{4}$. We show that:

$$1 \le \left(\frac{m_a}{\frac{3b}{4} + \frac{3c}{4}}\right)^2 + \left(\frac{m_b}{\frac{3c}{4} + \frac{3a}{4}}\right)^2 + \left(\frac{m_c}{\frac{3a}{4} + \frac{3b}{4}}\right)^2 \le \frac{R}{2r} \leftrightarrow \frac{9}{16} \le \sum\left(\frac{m_a}{b+c}\right)^2 \le \frac{9R}{32r}$$

Lemma: In any ΔABC

$$\sum\left(\frac{m_a}{b+c}\right)^2 == \frac{15s^6 - s^4(52Rr + 15r^2) + s^2r^2(224R^2 + 432Rr + 85r^2) - 13r^3(4R+r)^3}{16s^2(s^2 + r^2 + 2Rr)^2}$$

Demonstration:

$$\sum\left(\frac{m_a}{b+c}\right)^2 = \frac{\sum m_a^2(a+b)^2(a+c)^2}{(a+b)^2(b+c)^2(c+a)^2}$$

$$\sum m_a^2(a+b)^2(a+c)^2 =$$

$$= \frac{15s^6 - s^4(52Rr + 15r^2) + s^2r^2(224R^2 + 432Rr + 85r^2) - 13r^3(4R+r)^3}{4}$$

$$\prod(b+c) = 2s(s^2 + r^2 + 2Rr)$$

$$\frac{15s^6 - s^4(52Rr + 15r^2) + s^2r^2(224R^2 + 432Rr + 85r^2) - 13r^3(4R+r)^3}{16s^2(s^2 + r^2 + 2Rr)^2} \ge \frac{9}{16}$$

$$\Leftrightarrow 6s^6 - s^4(88Rr + 33r^2) + s^2r^2(188R^2 + 396Rr + 76r^2) \ge 13r^3(4R+r)^3$$

$$\Leftrightarrow s^2[s^2(6s^2 - 88Rr - 33r^2) + r^2(188R^2 + 396Rr + 76r^2)] \ge 13r^3(4R+r)^3$$

from Gerretsen inequality: $s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$. We must show that:

$$\frac{r(4R+r)^2}{R+r} \cdot [(16Rr - 5r^2)(96Rr - 30r^2 - 88Rr - 33r^2) + r^2(188R^2 + 396Rr + 76r^2)] \ge 13r^3(4R+r)^3 \Leftrightarrow$$

$$264R^2-717Rr+378r^2\geq 0\leftrightarrow (R-2r)(264R-189r)\geq 0$$ ***true from Euler*** $R\geq 2r$

Using Lemma the inequality can be rewrite

$$\frac{15s^6-s^4(52Rr+15r^2)+s^2r^2(224R^2+432Rr+85r^2)-13r^3(4R+r)^3}{16s^2(s^2+r^2+2Rr)^2}\leq\frac{9R}{32r}\Leftrightarrow$$

$$s^6(9R-30r)+s^4(36R^2r+122Rr^2+30r^3)+s^2r^2(36R^3-412R^2r-855Rr^2-170r^3)+26r^4(4R+r)^3\geq 0\Leftrightarrow$$

$$s^2[s^4(9R-30r)+s^2(36R^2r+122Rr^2+30r^3)+$$

$$+r^2(36R^3-412R^2r-855Rr^2-170r^3)]+26r^4(4R+r)^3\geq 0$$

We have the cases:

Case 1) If $[s^4(9R-30r)+s^2(36R^2r+122Rr^2+30r^3)+r^2(36R^3-412R^2r-$

855Rr2−170r3]≥0 the inequality is obvious.

Case 2) If $\begin{bmatrix}s^4(9R-30r)+s^2(36R^2r+122Rr^2+30r^3)+\\+r^2(36R^3-412R^2r-855Rr^2-170r^3)\end{bmatrix}<0$ ***we can write:***

$$26r^4(4R+r)^3\geq s^2[s^4(30r-9R)-s^2(36R^2r+122Rr^2+30r^3)-r^2(36R^3-412R^2r-855Rr^2-170r^3)]$$

true from Blundon-Gerretsen: $16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$

We show that: $26r^4(4R+r)^3\geq\frac{R(4R+r)^2}{2(2R-r)}\cdot[(4R^2+4Rr+3r^2)^2(30r-9R)$

$$-(16Rr-5r^2)(36R^2r+122Rr^2+30r^3)-r^2(36R^3-412R^2r-855Rr^2-170r^3]$$

$$72R^6-96R^5r+6R^4r^2+188R^3r^3-60R^2r^4-347Rr^5-26r^6\geq 0$$

$$(R-2r)(72R^5+48R^4r+102R^3r^2+392R^2r^3+180Rr^4+13r^5)\geq 0$$

true from $R\geq 2r$ ***–Euler.***

Equality if and only if the triangle is equilateral.

JP.278. Solve for real numbers ($a\geq 0$;fixed):

$$\sqrt[3]{3x^2-3x+1}+4\sqrt[4]{4x^3-3x^4}=ax^5+(1-5a)x+4a+4$$

Proposed by Marin Chirciu-Romania

Solution by Michael Stergiou-Greece

$$\sqrt[3]{3x^2-3x+1}+4\sqrt[4]{4x^3-3x^4}=ax^5+(1-5a)x+4a+4\quad(1)$$

$$ax^5+a+a+a+a\geq 5\cdot\sqrt[5]{a^5x^5}=5ax$$

So, RHS of (1)$\geq x+4\quad(2)$

$$\sqrt[3]{(3x^2-3x+1)\cdot 1\cdot 1} \overset{Am-Gm}{\leq} \frac{3x^2-3x+1+1+1}{3} = x^2-x+1 \quad (3)$$

$$4\sqrt[4]{(4x^3-3x^4)\cdot 1\cdot 1\cdot 1} \overset{Am-Gm}{\leq} 4\cdot\frac{4x^3-3x^4+1+1+1}{4} = 4x^3-3x^4+3 \quad (4)$$

***Therefore LHS of (1)** $\leq x^2-x+1+4x^3-3x^4+3$ **while RHS of (1))** $\geq x+4$*

***But:** $x^2-x+1+4x^3-3x^4+3-(x+4) = -x(x-1)^2(3x+2) \leq 0$*

Hence we can have only equalities for x=1.

JP.279. RMM NUMBER 19 WINTER 2020

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{1}{r_a(r_a+2r_b)}+\frac{1}{r_b(r_b+2r_c)}+\frac{1}{r_c(r_c+2r_a)} \leq \frac{1}{9r^2}$$

Proposed by Nguyen Viet Hung –Hanoi- Vietnam

Solution:

Using means inequality, we obtain:

$$r_a+2r_b = r_a+r_b+r_b \geq 3\sqrt[3]{\frac{S}{s-a}\cdot\frac{S}{s-b}\cdot\frac{S}{s-b}} = \frac{3S}{\sqrt[3]{(s-a)(s-b)(s-b)}}$$

We obtain: $\frac{1}{r_a(r_a+2r_b)} \leq \frac{1}{r_a\cdot\frac{1}{\sqrt[3]{(s-a)(s-b)(s-b)}}} = \frac{1}{\frac{S}{s-a}\cdot\frac{3S}{\sqrt[3]{(s-a)(s-b)(s-b)}}} = \frac{(s-a)\sqrt[3]{(s-a)(s-b)^2}}{3S^2}$

It follows: $M_s = \sum\frac{1}{r_a(r_a+2r_b)} \leq \sum\frac{(s-a)\sqrt[3]{(s-a)(s-b)^2}}{3S^2} =$

$$= \frac{1}{3S^2}\sum(s-a)\sqrt[3]{(s-a)(s-b)^2} \overset{AM-GM}{\leq}$$

$$\leq \frac{1}{3S^2}\sum(s-a)\cdot\frac{(s-a)+(s-b)+(s-c)}{3} = \frac{1}{9S^2}\sum(s-a)(3s-a-2b) =$$

$= \frac{s^2}{9S^2} = \frac{s^2}{9s^2r^2} = \frac{1}{9r^2} = M_d$, ***which follows from*** $\sum(s-a)(3s-a-2b) = s^2$

Equality holds if and only if ΔABC is equilateral.

Remark.Let's find an inequality having an opposite sense:

2) In ΔABC the following relationship holds:

$$\frac{1}{r_a(r_a+2r_b)}+\frac{1}{r_b(r_b+2r_c)}+\frac{1}{r_c(r_c+2r_a)}\ge\frac{9}{(4R+r)^2}$$

Marin Chirciu – Romania

Solution

Using Bergström's inequality we obtain:

$$M_s=\sum\frac{1}{r_a(r_a+2r_b)}\ge\frac{9}{\sum r_a(r_a+2r_b)}=\frac{9}{\sum(r_a^2+2r_ar_b)}=\frac{9}{(\sum r_a)^2}=\frac{9}{(4R+r)^2}$$
$$=M_d$$

Equality holds if and only if ΔABC is equilateral.

Remark.

We can write the double inequality:

3) In ΔABC the following relationship holds:

$$\frac{9}{(4R+r)^2}\le\frac{1}{r_a(r_a+2r_b)}+\frac{1}{r_b(r_b+2r_c)}+\frac{1}{r_c(r_c+2r_a)}\le\frac{1}{9r^2}$$

Solution

See inequalities 1) and 2).

Equality holds if and only if ΔABC is equilateral.

Remark.

If we replace r_a with h_a we propose:

4) In ΔABC the following relationship holds:

$$\frac{9R^2}{4(R+r)^4}\le\frac{1}{h_a(h_a+2h_b)}+\frac{1}{h_b(h_b+2h_c)}+\frac{1}{h_c(h_c+2h_a)}\le\frac{1}{9r^2}$$

Marin Chirciu – Romania

Solution

Left hand inequality: Using Bergström's inequality, we obtain:

$$\sum\frac{1}{h_a(h_a+2h_b)}\ge\frac{9}{\sum h_a(h_a+h_b)}=\frac{9}{\sum(h_a^2+2h_ah_b)}=\frac{9}{(\sum h_a)^2}=\frac{9}{\left(\frac{s^2+r^2+4Rr}{2R}\right)^2}=$$

$$=\frac{9\cdot 4R^2}{(s^2+r^2+4Rr)^2}\overset{Bergstrom}{\geq}\frac{9\cdot 4R^2}{(4R^2+4Rr+3r^2+r^2+4Rr)^2}=\frac{36R^2}{(4R^2+8Rr+4r^2)^2}=$$

$$=\frac{36R^2}{[4(R+r)^2]^2}=\frac{36R^2}{16(R+r)^4}=\frac{9R^2}{4(R+r)^4}$$

Equality holds if and only if ΔABC is equilateral.

Right hand inequality: Using means inequality, we obtain:

$$h_a+2h_b=h_a+h_b+h_c\geq 3\sqrt[3]{\frac{2S}{a}\cdot\frac{2S}{b}\cdot\frac{2S}{b}}=\frac{3\cdot 2S}{\sqrt[3]{abb}}=\frac{6S}{\sqrt[3]{ab^2}}$$

We obtain: $\frac{1}{h_a(h_a+2h_b)}\leq\frac{1}{h_a\cdot\frac{6S}{\sqrt[3]{ab^2}}}=\frac{1}{\frac{2S}{a}\cdot\frac{6S}{\sqrt[3]{ab^2}}}=\frac{a\sqrt[3]{ab^2}}{12S^2}$. ***It follows:***

$$\sum\frac{1}{h_a(h_a+2h_b)}\leq\sum\frac{a\sqrt[3]{ab^2}}{12S^2}=\frac{1}{12S^2}\sum a\sqrt[3]{ab^2}\overset{AM-GM}{\leq}\frac{1}{12S^2}\sum a\cdot\frac{a+b+b}{3}=$$

$$=\frac{1}{36S^2}\sum a\,(a+2b)=\frac{1}{36S^2}\sum(a^2+2ab)=\frac{1}{36S^2}\left(\sum a\right)^2=\frac{1}{36S^2}\cdot 4s^2=\frac{4s^2}{36s^2r^2}=\frac{1}{9r^2}.$$

Equality holds if and only if ΔABC is equilateral.

JP.280. RMM 19 WINTER EDITION 2020

By Marin Chirciu – Romania

1) In ΔABC:

$$\sqrt[3]{r_a^4r_b^2}+\sqrt[3]{r_b^4r_c^2}+\sqrt[3]{r_c^4r_a^2}\leq\frac{(4R+r)^2}{3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution:

Using means inequality we obtain:

$$\sqrt[3]{r_a^4r_b^2}=\sqrt[3]{r_a^2\cdot r_a^2\cdot r_b^2}\leq\frac{r_a^2+r_a^2+r_b^2}{3}=\frac{2r_a^2+r_b^2}{3}$$ ***and the analogs.***

It follows $\sqrt[3]{r_a^4r_b^2}+\sqrt[3]{r_b^4r_c^2}+\sqrt[3]{r_c^4r_a^2}\leq\frac{2r_a^2+r_b^2}{3}+\frac{2r_b^2+r_c^2}{3}+\frac{2r_c^2+r_a^2}{3}=r_a^2+r_b^2+r_c^2$

We have:

$$r_a^2+r_b^2+r_c^2=\left(\sum r_a\right)^2-2\sum r_br_c=(4R+r)^2-2s^2\overset{(G)}{\le}(4R+r)^2-2\frac{r(4R+r)^2}{R+r}=$$

$$=(4R+r)^2\left(1-\frac{2r}{R+r}\right)=(4R+r)^2\left(\frac{R-r}{R+r}\right)$$, *where (G) is Gerretsen's inequality*

$$s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$$

Equality holds if and only if the triangle is equilateral.

Remark.Let's find and inequality having an opposite sense.

2) In ΔABC

$$\sqrt[3]{r_a^4r_b^2}+\sqrt[3]{r_b^4r_c^2}+\sqrt[3]{r_c^4r_a^2}\ge 27r^2$$

Solution

Using the means inequality we obtain:

$$\sqrt[3]{r_a^4r_b^2}+\sqrt[3]{r_b^4r_c^2}+\sqrt[3]{r_c^4r_a^2}\ge 3\sqrt[3]{\sqrt[3]{r_a^4r_b^2}\cdot\sqrt[3]{r_b^4r_c^2}\cdot\sqrt[3]{r_c^4r_a^2}}=3\sqrt[3]{\sqrt[3]{r_a^6r_b^6r_c^6}}=3\sqrt[3]{r_a^2r_b^2r_c^2}=$$

$$=3\sqrt[3]{r^2s^4}\overset{(M)}{\ge}3\sqrt[3]{r^2(27r^2)^2}=3\sqrt[3]{9^3r^6}=27r^2$$, *where (M) is Mitrinovic's inequality*

$$s\ge 3r\sqrt{3}$$

Equality holds if and only if the triangle is equilateral.

Remark.

If we replace r_a with h_a we propose:

3) In ΔABC:

$$27r^2\left(\frac{2r}{R}\right)^{\frac{2}{3}}\le\sqrt[3]{h_a^4h_b^2}+\sqrt[3]{h_b^4h_c^2}+\sqrt[3]{h_c^4h_a^2}\le\frac{(4R+r)^2}{3}$$

Marin Chirciu-Romania

Solution

The right inequality: Using the means inequality we obtain:

$$\sqrt[3]{h_a^4h_b^2}=\sqrt[3]{h_a^2\cdot h_a^2\cdot h_b^2}\le\frac{h_a^2+h_a^2+h_b^2}{3}=\frac{2h_a^2+h_b^2}{3}$$ ***and the analogs.***

It follows $\sqrt[3]{h_a^4 h_b^2}+\sqrt[3]{h_b^4 h_c^2}+\sqrt[3]{h_c^3 h_a^2}\le\frac{2h_a^2+h_b^2}{3}+\frac{2h_b^2+h_c^2}{3}+\frac{2h_c^2+h_a^2}{3}=h_a^2+h_b^2+h_c^2$

We have: $h_a^2+h_b^2+h_c^2=\left(\sum h_a\right)^2-2\sum h_b h_c=\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-2\cdot\frac{2rs^2}{R}=$

$$=\frac{s^2(s^2+2r^2-8Rr)+r^2(4Rr+r)^2}{4R^2}$$

We prove: $\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4R^2}\le\frac{(4R+r)^2}{3}\Leftrightarrow$

$\Leftrightarrow 3s^2(s^2+2r^2-8Rr)+3(4R+r)^2\le 4R^2(4R+r)^2$, *which follows from Blundon-Gerretsen's inequality* $s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$3\cdot\frac{R(4R+r)^2}{2(2R-r)}(4R^2+4Rr+3r^2+2r^2-8Rr)+3r^2(4R+r)^2\le 4R^2(4R+r)^2\Leftrightarrow$$

$\Leftrightarrow 4R^3+4R^2r-27Rr^2+6r^3\ge 0\Leftrightarrow(R-2r)(4R^2+12Rr-3r^2)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

The left hand inequality: Using the means inequality we obtain:

$$\sqrt[3]{h_a^4 h_b^2}+\sqrt[3]{h_b^4 h_c^2}+\sqrt[3]{h_c^4 h_a^2}\ge 3\sqrt[3]{\sqrt[3]{h_a^4 h_b^2}\cdot\sqrt[3]{h_b^4 h_c^2}\cdot\sqrt[3]{h_c^4 h_a^2}}=3\sqrt[3]{\sqrt[3]{h_a^6 h_b^6 h_c^6}}=$$

$$=3\sqrt[3]{h_a^2 h_b^2 h_c^2}==3\sqrt[3]{\left(\frac{2r^2s^2}{R}\right)^2}\overset{(M)}{\ge}3\sqrt[3]{\left(\frac{2r^2\cdot 27r^2}{R}\right)^2}=3\sqrt[3]{9^3r^6\cdot\frac{4r^2}{R^2}}=27r^2\sqrt[3]{\frac{4r^2}{R^2}}=27r^2\left(\frac{2r}{R}\right)^{\frac{2}{3}}$$

Equality holds if and only if the triangle is equilateral.

Remark.

If we interchange r_a^2 *with* r_a, *we propose:*

4) In ΔABC:

$$9r\le\sqrt[3]{r_a^2 r_b}+\sqrt[3]{r_b^2 r_c}+\sqrt[3]{r_c^2 r_a}\le 4R+r$$

Marin Chirciu – Romania

Solution

Right hand inequality: Using means inequality:

$\sqrt[3]{r_a^2 r_b}=\sqrt[3]{r_a\cdot r_a\cdot r_b}\le\frac{r_a+r_a+r_b}{3}=\frac{2r_a+r_b}{3}$ *and the analogs.*

We obtain $\sqrt[3]{r_a^2 r_b}+\sqrt[3]{r_b^2 r_c}+\sqrt[3]{r_c^2 r_a}\le\frac{2r_a+r_b}{3}+\frac{2r_b+r_c}{3}+\frac{2r_c+r_a}{3}=r_a+r_b+r_c=4R+r$

Equality holds if and only if the triangle is equilateral.

The left hand inequality: Using the means inequality we obtain:

$$\sqrt[3]{r_a^2 r_b}+\sqrt[3]{r_b^2 r_c}+\sqrt[3]{r_c^2 r_a}\ge 3\sqrt[3]{\sqrt[3]{r_a^2 r_b}\cdot\sqrt[3]{r_b^2 r_c}\cdot\sqrt[3]{r_c^2 r_a}}=3\sqrt[3]{\sqrt[3]{r_a^3 r_b^3 r_c^3}}=3\sqrt[3]{r_a r_b r_c}=$$

$$=3\sqrt[3]{rs^2}\overset{(M)}{\ge}3\sqrt[3]{r\cdot 27r^2}=3\sqrt[3]{27r^3}=9r,$$ *where (M) is Mitrinovic's inequality* $s\ge 3r\sqrt{3}$. *Equality holds if and only if the triangle is equilateral.*

Remark.

If we interchange h_a^2 *in* h_a *we propose:*

5) In ΔABC:

$$9r\left(\frac{2r}{R}\right)^{\frac{1}{3}}\le\sqrt[3]{h_a^2 h_b}+\sqrt[3]{h_b^2 h_c}+\sqrt[3]{h_c^2 h_a}\le\frac{2(R+r)^2}{R}$$

Marin Chirciu – Romania

Solution

The right hand inequality: Using means inequality we obtain:

$\sqrt[3]{h_a^2 h_b}=\sqrt[3]{h_a\cdot h_a\cdot h_b}\le\frac{h_a+h_a+h_b}{3}=\frac{2h_a+h_b}{3}$ *and the analogs.*

We obtain $\sqrt[3]{h_a^2 h_b}+\sqrt[3]{h_b^2 h_c}+\sqrt[3]{h_c^2 h_a}\le\frac{2h_a+h_b}{3}+\frac{2h_b+h_c}{3}+\frac{2h_c+h_a}{3}=h_a+h_b+h_c=$

$$=\frac{s^2+r^2+4Rr}{2R}$$

It remains to prove that: $\frac{s^2+r^2+4Rr}{2R}\le\frac{2(R+r)^2}{R}\Leftrightarrow s^2\le 4R^2+4Rr+3r^2$, *(Gerretsen's inequality). Equality holds if and only if the triangle is equilateral.*

The left hand inequality: Using means inequality we obtain:

$$\sqrt[3]{h_a^2 h_b}+\sqrt[3]{h_b h_c}+\sqrt[3]{h_c^2 h_a}\ge 3\sqrt[3]{\sqrt[3]{h_a^2 h_b}\cdot\sqrt[3]{h_b^2 h_c}\cdot\sqrt[3]{h_c^2 h_a}}=3\sqrt[3]{\sqrt[3]{h_a^3 h_b^3 h_c^3}}=$$

$$3\sqrt[3]{h_a h_b h_c}=3\sqrt[3]{\frac{2r^2 s^2}{R}}\overset{(M)}{\ge}3\sqrt[3]{\frac{2r^2\cdot 27r^2}{R}}=9r\sqrt[3]{\frac{2r}{R}}=9r\left(\frac{2r}{R}\right)^{\frac{1}{3}},$$

where (M) is Mitrinovic's inequality $p \geq 3r\sqrt{3}$

Equality holds if and only if the triangle is equilateral.

JP.281. ABOUT PROBLEM JP.281-RMM NUMBER 19 WINTER 2020

By Marin Chirciu – Romania

1) If $a, b, c > 0; abc = 1$ then:

$$\frac{(a+b)^2}{\sqrt{a^2+b^2}} + \frac{(b+c)^2}{\sqrt{b^2+c^2}} + \frac{(c+a)^2}{\sqrt{c^2+a^2}} \geq 6\sqrt{2}$$

Proposed by Nguyen Viet Hung-Hanoi – Vietnam

Solution

Inequality can be written: $\sum \frac{(a+b)^2}{\sqrt{2(a^2+b^2)}} \geq 6$, *which follows from* $\frac{(a+b)^2}{\sqrt{2(a^2+b^2)}} \geq 2\sqrt{ab} \Leftrightarrow$

$\Leftrightarrow \frac{(a+b)^4}{2(a^2+b^2)} \geq 4ab \Leftrightarrow a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0 \Leftrightarrow (a-b)^4 \geq 0$, *obviously*

with equality for $a = b$. We obtain:

$$\sum \frac{(a+b)^2}{\sqrt{2(a^2+b^2)}} \geq 2\sum \sqrt{ab} \overset{AM-GM}{\geq} 2 \cdot 3\sqrt[3]{\sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{ca}} = 6\sqrt[6]{abc} = 6$$

We deduce that the inequality from enunciation holds, with equality if and only if

$a = b = c$.

Remark: The inequality can be developed:

***2)* If $a, b, c > 0; abc = 1$ then:**

$$\frac{(a+b)^3}{\sqrt{a^2+b^2}} + \frac{(b+c)^3}{\sqrt{b^2+c^2}} + \frac{(c+a)^3}{\sqrt{c^2+a^2}} \geq 12\sqrt{2}$$

Marin Chirciu – Romania

Solution

The inequality can be written: $\sum \frac{(a+b)^3}{\sqrt{2(a^2+b^2)}} \geq 12$, *which follows from*

$$\frac{(a+b)^3}{\sqrt{2(a^2+b^2)}} \geq 4ab \Leftrightarrow$$

$$\Leftrightarrow \frac{(a+b)^6}{2(a^2+b^2)} \geq 16a^2b^2 \Leftrightarrow a^6+6a^5b-17a^4b^2+20a^3b^3-17a^2b^4+6ab^5+b^6 \geq 0$$

Dividing with a^3b^3 and grouping based on symmetry, wherefrom we obtain:

$$\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}\right)+6\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)-17\left(\frac{a}{b}+\frac{b}{a}\right)+20 \geq 0$$

We denote $\frac{a}{b}+\frac{b}{a}=t \geq 2$, wherefrom follows: $\frac{a^2}{b^2}+\frac{b^2}{a^2}t^2-2$ and $\frac{a^3}{b^3}+\frac{b^3}{a^3}=t^3-3t$

The last inequality can be written: $t^3-3t+6(t^2-2)-17t+20 \geq 0 \Leftrightarrow$

$\Leftrightarrow t^3+6t^2-20t+8 \geq 0 \Leftrightarrow (t-2)(t^2+8t-4) \geq 0$, *true because $t \geq 2$.*

We obtain:

$$\sum \frac{(a+b)^3}{\sqrt{2(a^2+b^2)}} \geq 4\sum ab \overset{AM-GM}{\geq} 4 \cdot 3\sqrt[3]{ab \cdot bc \cdot ca} = 12\sqrt[6]{(abc)^2} = 12$$

We deduce that the inequality from enunciation holds, with equality if and only if

$$a=b=c.$$

Remark.

The inequality can be generalized:

3) If $a,b,c>0$; $abc=1$ then:

$$\frac{(a+b)^n}{\sqrt{a^2+b^2}}+\frac{(b+c)^n}{\sqrt{b^2+c^2}}+\frac{(c+a)^n}{\sqrt{c^2+a^2}} \geq 3 \cdot 2^{n-1}\sqrt{2}, \text{ where } n \geq 2, n \in \mathbb{N}$$

Marin Chirciu – Romania

Solution

The inequality can be written: $\sum \frac{(a+b)^n}{\sqrt{2(a^2+b^2)}} \geq 3 \cdot 2^{n-1}$, which follows from

$$\frac{(a+b)^n}{\sqrt{2(a^2+b^2)}} \geq \left(2\sqrt{ab}\right)^{n-1} \Leftrightarrow$$

$$\Leftrightarrow \frac{(a+b)^{2n}}{2(a^2+b^2)} \geq (4ab)^{n-1} \Leftrightarrow (a+b)^{2n} \geq 2(a^2+b^2)(4ab)^{n-1} \Leftrightarrow$$

$[(a+b)^2]^n \geq 2(a^2+b^2)(4ab)^n \cdot \frac{1}{4ab} \Leftrightarrow \left[\frac{(a+b)^2}{4ab}\right]^n \geq \frac{a^2+b^2}{2ab}$, *where we prove through mathematical induction after* $n \geq 2, n \in \mathbb{N}$.

Let be $P(n): \left[\frac{(a+b)^2}{4ab}\right]^n \geq \frac{a^2+b^2}{2ab}, n \geq 2, n \in \mathbb{N}$.

$$P(2): \left[\frac{(a+b)^2}{4ab}\right]^2 \geq \frac{a^2+b^2}{2ab} \Leftrightarrow \frac{(a+b)^4}{2(a^2+b^2)} \geq 4ab \Leftrightarrow$$

$\Leftrightarrow a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0 \Leftrightarrow (a-b)^4 \geq 0$, *obviously with equality for* $a = b$.

$P(k) \Rightarrow P(k+1)$, *where* $k \geq 2, k \in \mathbb{N}$

We propose that $P(k): \left[\frac{(a+b)^2}{4ab}\right]^k \geq \frac{a^2+b^2}{2ab}; k \geq 2, k \in \mathbb{N}$, *true and we prove that*

$P(k+1): \left[\frac{(a+b)^2}{4ab}\right]^{k+1} \geq \frac{a^2+b^2}{2ab}$ *true.*

Indeed: $\left[\frac{(a+b)^2}{4ab}\right]^{k+1} = \left[\frac{(a+b)^2}{4ab}\right]^k \cdot \frac{(a+b)^2}{4ab} \overset{P(k)}{\geq} \frac{a^2+b^2}{2ab} \cdot \frac{(a+b)^2}{4ab} \overset{(1)}{\geq} \frac{a^2+b^2}{2ab}$, *where (1)* $\frac{(a+b)^2}{4ab} \geq 1 \Leftrightarrow$

$\Leftrightarrow (a-b)^4 \geq 0$, *obviously with equality for* $a = b$.

We obtain:

$$\sum \frac{(a+b)^n}{\sqrt{2(a^2+b^2)}} \geq \sum \left(2\sqrt{ab}\right)^{n-1} \overset{AM-GM}{\geq} 3\sqrt[3]{\left(2\sqrt{ab}\right)^{n-1} \cdot \left(2\sqrt{bc}\right)^{n-1} \cdot \left(2\sqrt{ca}\right)^{n-1}} =$$

$$= 3\sqrt[3]{(8abc)^{n-1}} = 3\sqrt[3]{8^{n-1}} = 3 \cdot 2^{n-1}$$

We deduce that the inequality from enunciation holds if and only if $a = b = c$.

Note.

For $n = 2$ *we obtain Problem JP.281, RMM Number 19, Winter 2020, proposed by Nguyen Viet Hung, Vietnam*

4) If $a, b, c > 0; abc = 1$ then:

$$\frac{(a+b)^n}{\sqrt{a^2+b^2}} + \frac{(b+c)^n}{\sqrt{b^2+c^2}} + \frac{(c+a)^n}{\sqrt{c^2+a^2}} \geq 3 \cdot 2^{n-\frac{1}{2}}, \text{ where } n \geq 2, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

Solution

We reformulate the enunciation from 3)

JP.282. If $a, b, c > 1$ then:

$$\log a \cdot \log b \cdot \log c \cdot (\log_a e + \log_b e + \log_c e)^2 \geq 3\log(abc)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Florentin Vișescu – Romania

We denote $\ln a = x > 0; \ln b = y > 0; \ln c = z > 0$

$$xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2 \geq 3(x+y+z)$$

$$xyz\left(\frac{yz+xz+xy}{xyz}\right)^2 \geq 3(x+y+z)$$

$$\frac{(xy+xz+yz)^2}{xyz} \geq 3(x+y+z)$$

$$x^2y^2+x^2z^2+y^2z^2+2x^2yz+2xy^2z+2xyz^2 \geq 3x^2yz+3xy^2z+3xyz^2$$

$$x^2y^2+x^2z^2+y^2z^2-x^2yz-xy^2z-xyz^2 \geq 0|2$$

$$(xy-xz)^2+(xy-yz)^2+(xz-yz)^2 \geq 0 \text{ True}$$

Solution 2 by Henry Ricardo-New York-USA

Setting $\log a = A, \log b = B, \log c = C$*, and noting that* $\log_r s = \frac{1}{\log_s r}$*, we have:*

$$\prod_{cyclic}\log a \cdot \left(\sum_{cyclic}\log_a e\right)^2 = ABC\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)$$

$$= (AB+BC+CA)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right) = 2(A+B+C)+\sum_{cyclic}\frac{AB}{C} \geq 3(A+B+C)$$

$$\Leftrightarrow \sum_{cyclic}\frac{AB}{C} \geq A+B+C \Leftrightarrow \sum_{cyclic}(AB)^2 \geq \sum_{cyclic}(AB)(BC),$$

which is true by the AGM inequality. Equality holds if and only if $a = b = c$*.*

JP.283 If $a, b, c \in \mathbb{R}$ then:

$$2\sum_{cyc} sin^2 a + \sum_{cyc} sin^2(a+b) \leq \frac{27}{4}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Marian Ursărescu-Romania

We must show:

$$2\sum_{cyc}(1-cos^2a)+\sum_{cyc}\left(1-cos^2(a+b)\right)\le\frac{27}{4}$$

$$6-\sum_{cyc}cos^2a+3-\sum_{cyc}cos^2(a+b)\le\frac{27}{4}$$

$$2\sum_{cyc}cos^2a+\sum_{cyc}cos^2(a+b)\ge\frac{9}{4}$$

$$cos^2a+cos^2b+cos^2(a+b)+cos^2b+cos^2c+cos^2(b+c)+cos^2c+cos^2a+cos^2(c+a)\ge\frac{9}{4}\dots(1)$$

$$cos^2a+cos^2b+cos^2(a+b)=\frac{1+cos2a}{2}+cos^2b+\frac{1+cos2(a+b)}{2}$$

$$=1+cos^2b+\frac{cos2a+cos2(a+b)}{2}=1+cos^2b+cos(2a+b)cosb$$

$$=\left[cosb+\frac{cos(2a+b)}{2}\right]^2+\frac{1}{4}-\frac{cos^2(2a+b)}{4}+\frac{3}{4}$$

$$=\underbrace{\left[cosb+\frac{cos(2a+b)}{2}\right]^2}_{>0}+\underbrace{\frac{sin^2(2a+b)}{4}}_{>0}+\frac{3}{4}\ge\frac{3}{4}$$

$$cos^2a+cos^2b+cos^2(a+b)\ge\frac{3}{4}$$

and two similary relationship

$$2\sum_{cyc}sin^2a+\sum_{cyc}sin^2(a+b)\le\frac{27}{4}$$

Solution 2 by Adrian Popa-Romania

Firstly we show that: $sin^2a+sin^2b+sin^2(a+b)\le\frac{9}{4}$

$$\therefore\left[\frac{1-cos2a}{2}+\frac{1-cos2b}{2}=1-cos(a+b)cos(a-b)\right]$$

$$1-cos(a+b)cos(a-b)+1-cos^2(a+b)\overset{?}{\le}\frac{9}{4}$$

$$2-cos(a+b)cos(a-b)-cos^2(a+b)\le 2+|cos(a+b)cos(a-b)|-cos^2(a+b)\le$$

$$\le 2+|cos(a+b)|-cos^2(a+b)$$

Denote: $|cos(a+b)| = x$ ***and let:*** $f(x) = 2 + x - x^2$

$f'(x) = 1 - 2x; f'(x) = 0 \Leftrightarrow x = \frac{1}{2}; f_{max} = f\left(\frac{1}{2}\right) = \frac{9}{4}$. ***Then:***

$$sin^2a + sin^2b + sin^2(a+b) \le \frac{9}{4}$$

$$sin^2a + sin^2c + sin^2(a+c) \le \frac{9}{4}$$

$$sin^2b + sin^2c + sin^2(b+c) \le \frac{9}{4}$$

So,

$$2\sum_{cyc} sin^2a + \sum_{cyc} sin^2(a+b) \le \frac{27}{4}$$

JP.284. In acute ΔABC the following relationship holds:

$$\frac{\sqrt{sin2A} + \sqrt{sin2B} + \sqrt{sin2C}}{\sqrt{tanA} + \sqrt{tanB} + \sqrt{tanC}} \ge \sqrt{2\left(\frac{r}{R} + 1\right)^2 - 4}$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$2(x+y) \ge \left(\sqrt{x} + \sqrt{y}\right)^2, \forall x, y > 0$$

Let: $x = sin2A + sin2B - sin2C; y = sin2A - sin2B + sin2C$

$$4sin2A \ge \left(\sqrt{sin2A + sin2B - sin2C} + \sqrt{sin2A - sin2B + sin2C}\right)^2$$

$$\sqrt{sin2A + sin2B - sin2C} + \sqrt{sin2A - sin2B + sin2C} \le 2\sqrt{sin2A}$$

Analogous:

$$\sqrt{sin2A + sin2B - sin2C} + \sqrt{-sin2A + sin2B + sin2C} \le 2\sqrt{sin2B}$$

$$\sqrt{sin2A - sin2B + sin2C} + \sqrt{-sin2A + sin2B + sin2C} \le 2\sqrt{sin2C}$$

$$\sum \sqrt{sin2A + sin2B - sin2C} \le \sqrt{sin2A} + \sqrt{sin2B} + \sqrt{sin2C}; \quad (1)$$

But: $sin2A + sin2B - sin2C = 2sin(A+B)cos(A-B) - 2sinCcosC$

$$= 2sinC[cos(A-B) - cosC] = -4sinCsin\left(\frac{A-B+C}{2}\right)sin\left(\frac{A-B-C}{2}\right)$$

$$= 4\cos A\cos B\sin C = 4\cos A\cos B\cos C\tan C = \frac{s^2-(2R+r)^2}{R^2}\tan C; (2)$$

From (1),(2) we have: $\sqrt{\frac{s^2-(2R+r)^2}{R^2}}\sum\sqrt{\tan A} \le \sum\sqrt{\sin 2A}$

$$\sqrt{\frac{s^2-(2R+r)^2}{R^2}} \le \frac{\sum\sqrt{\sin 2A}}{\sum\sqrt{\tan A}}; (3)$$

$$\sqrt{\frac{s^2-(2R+r)^2}{R^2}} \overset{(*)}{\ge} \sqrt{\frac{2R^2+8Rr+3r^2-4R^2-4Rr-r^2}{R^2}} =$$

$$= \sqrt{\frac{2r^2+4Rr-2R^2}{R^2}} = \sqrt{2\left(\frac{r}{R}+1\right)^2-4}$$

$$(*): s^2 \ge 2R^2+8Rr+3r^2;\ (4)$$

From (3) and (4) we have:

$$\frac{\sqrt{\sin 2A}+\sqrt{\sin 2B}+\sqrt{\sin 2C}}{\sqrt{\tan A}+\sqrt{\tan B}+\sqrt{\tan C}} \ge \sqrt{2\left(\frac{r}{R}+1\right)^2-4}$$

JP.285 In ΔABC the following relationship holds:

$$\frac{m_a^2}{m_b}+\frac{m_b^2}{m_c}+\frac{m_c^2}{m_a} \ge s\sqrt{3}$$

Proposed by Marian Ursărescu-Romania

Solution by Rahim Shahbazov-Baku-Azerbaijan

$$\frac{m_a^2}{m_b}+\frac{m_b^2}{m_c}+\frac{m_c^2}{m_a} \ge s\sqrt{3} \dots (1)$$

Lemma: $x,y,z>0$ *then*: $\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x} \ge 3\cdot\sqrt{\frac{x^2+y^2+y^2}{3}} \overset{(1)}{\Rightarrow}$

$$LHS \ge 3\cdot\sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}} \ge s\sqrt{3} \Rightarrow 3(a^2+b^2+c^2) \ge (a+b+c)^2$$

Prove lemma: $\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x} \ge \frac{(x^2+y^2+z^2)^2}{x^2y+y^2z+z^2x}$

$$\geq \frac{(x^2+y^2+z^2)^2}{\sqrt{(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2y^2)}} \geq 3\cdot\sqrt{\frac{x^2+y^2+y^2}{3}} \Rightarrow$$

$$(x^2+y^2+z^2)^2 \geq x^2y^2+y^2z^2+z^2y^2$$

SP.271. If $a_1, a_2, \dots, a_n > 0; a_1a_2\cdot\dots\cdot a_n = 1;\ \lambda \geq \frac{1}{2}$ then:

$$\frac{1}{\lambda+a_1}+\frac{1}{\lambda+a_2}+\dots+\frac{1}{\lambda+a_n} \leq \frac{a_1+a_2+\dots+a_n}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Michael Sterghiou-Greece

$$\frac{1}{\lambda+a_1}+\frac{1}{\lambda+a_2}+\dots+\frac{1}{\lambda+a_n} \leq \frac{a_1+a_2+\dots+a_n}{\lambda+1} \quad (1)$$

Denote $S_n = \sum_{i=1}^n a_i$. For $n = 1$ we have equality.

For $n = 2 \overset{(1)}{\Rightarrow} \frac{1}{\lambda+a_1}+\frac{1}{\lambda+\frac{1}{a_1}} \leq \frac{a_1+\frac{1}{a_1}}{\lambda+1}$ which reduces to $-\frac{\lambda(a_1-1)^2(a_1^2+\lambda a_1+a_1+1)}{(1+\lambda)(\lambda+a_1)(\lambda a_1+1)} \leq 0$ so, for $n = 2$, (1) holds. Assume that (1) holds for any n numbers (satisfying the conditions of the problem) such $a_1, a_2, \dots, a_{n-1}, \vartheta$ where $\vartheta = a_n a_{n+1}$.

Then $a_1\cdot a_2\cdot\dots\cdot a_{n-1}\cdot\vartheta = 1$ and by induction

$$\sum_{i=1}^{n}\frac{1}{a_i+\lambda} \leq \frac{S_{n-1}+\vartheta}{1+\lambda} \quad (2)$$

Now, we have to prove that: $\sum_{i=1}^{n+1}\frac{1}{a_i+\lambda} \leq \frac{S_{n+1}}{1+\lambda} = \frac{S_{n-1}}{1+\lambda}+\frac{a_n+a_{n+1}}{1+\lambda}$ (3).

From (2) $\left(\sum_{i=1}^{n-1}\frac{1}{a_i+\lambda}\right)+\frac{1}{\lambda+\vartheta} \leq \frac{S_{n-1}+\vartheta}{1+\lambda}$ or $\sum_{i=1}^{n-1}\frac{1}{a_i+\lambda} \leq \frac{S_{n-1}+\vartheta}{1+\lambda}-\frac{1}{\lambda+\vartheta}$

Because of this it suffices to show that: $\frac{S_{n-1}+\vartheta}{1+\lambda}-\frac{1}{\lambda+\vartheta}+\frac{1}{a_n+\lambda}+\frac{1}{a_{n+1}+\lambda} \leq \frac{S_{n-1}}{1+\lambda}+\frac{a_n+a_{n+1}}{1+\lambda}$

Putting: $a_n = x, a_{n+1} = y, \vartheta = xy$ the last inequality becomes:

$$\frac{xy}{1+\lambda}-\frac{1}{\lambda+xy}+\frac{1}{x+\lambda}+\frac{1}{y+\lambda}-\frac{x+y}{1+\lambda} \leq 0 \text{ which reduces to:}$$

$$\frac{(x-1)(y-1)[\lambda^3+\lambda^2xy+\lambda^2x+\lambda^2y+\lambda^2+\lambda x^2y+\lambda xy^2+\lambda xy+x^2y^2-xy]}{(\lambda+1)(\lambda+x)(\lambda+y)(\lambda+xy)} \overset{(4)}{\lesseqgtr} 0$$

The expression in [.] in the nominator of (4) is increasing function of λ and for $\lambda \geq \frac{1}{2}$

it is positive (obvious).

(4) holds if one of x, y is ≥ 1 and the other ≤ 1.

We can clearly assume this as the inequality is cyclic and we cannot have all a_i greater then 1 or smaller then 1.This completes the proof.Done.

SP.272. RMM NUMBER 19 WINTER 2020

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{3}{R} \leq \frac{r_b+r_c}{a^2}+\frac{r_c+r_a}{b^2}+\frac{r_a+r_b}{c^2} \leq \frac{3}{4r}\left(\frac{R^2}{r^2}-2\right)$$

Proposed by George Apostolopoulos-Messolonghi– Greece

Solution

We prove the following lemma:

Lemma:

***2)** In ΔABC the following relationship holds:*

$$\frac{r_b+r_c}{a^2}+\frac{r_c+r_a}{b^2}+\frac{r_a+r_b}{c^2}=\frac{s^2+r^2-8Rr}{4Rr^2}$$

Proof.

Using the formula $r_a=\frac{S}{s-a}$ we obtain:

$$\sum\frac{r_b+r_c}{a^2}=\sum\frac{\frac{S}{s-b}+\frac{S}{s-c}}{a^2}=S\sum\frac{1}{a(s-b)(s-c)}=rs\cdot\frac{s^2+r^2-8Rr}{4Rr^3s}=$$

$$=\frac{s^2+r^2-8Rr}{4Rr^2}$$

Which follows from the known identity in triangle:

$$\sum\frac{1}{a(s-b)(s-c)}=\frac{s^2+r^2-8Rr}{4Rr^3s}$$

Let's get back to the main problem.

The left hand – inequality: Using the Lemma the inequality can be written:

$\frac{s^2+r^2-8Rr}{4Rr^2}\geq\frac{3}{R}\Leftrightarrow s^2\geq 8Rr+11r^2$*, which follows from Gerretsen's inequality*

$s^2\geq 16Rr-5r^2$*. It remains to prove that:*

$16Rr-5r^2\geq 8Rr+11r^2\Leftrightarrow R\geq 2r$*, (Euler's inequality).*

Equality holds if and only if ΔABC is equilateral.

The right hand inequality: Using the Lemma the inequality can be written:

$$\frac{s^2+r^2-8Rr}{4Rr^2}\leq\frac{3}{4r}\left(\frac{R^2}{r^2}-2\right)\Leftrightarrow r(s^2+r^2-8Rr)\leq 3R(R^2-2r^2)$$

which follows from Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$r(4R^2+4Rr+3r^2+r^2-8Rr)\leq 3R(R^2-2r^2)\Leftrightarrow 3R^3-4R^2r-2Rr^2-4r^3\geq 0\Leftrightarrow$$

Equality holds if and only if ΔABC is equilateral.

Remark.

The right-hand inequality can be strengthened:

3) In ΔABC the following relationship holds:

$$\frac{r_b+r_c}{a^2}+\frac{r_c+r_a}{b}+\frac{r_a+r_b}{c^2}\leq\frac{1}{R}\left(\frac{R^2}{r^2}-\frac{R}{r}+1\right)$$

Marin Chirciu – Romania

Solution

Using the Lemma the inequality can be written:

$\frac{s^2+r^2-8Rr}{4Rr^2}\leq\frac{1}{R}\left(\frac{R^2}{r^2}-\frac{R}{r}+1\right)$*, which follows from Gerretsen's inequality:*

$$s^2\leq 4R^2+4Rr+3r^2$$

We obtain:

$$\frac{s^2+r^2-8Rr}{4Rr^2}\leq\frac{4R^2+4Rr+3r^2+r^2-8Rr}{4Rr^2}=\frac{4R^2-4Rr+4r^2}{4Rr^2}=\frac{R^2-Rr+r^2}{Rr^2}=$$

$$=\frac{1}{R}\left(\frac{R^2}{r^2}-\frac{R}{r}+1\right)$$

Equality holds if and only if ΔABC is equilateral.

Remark.

Inequality 3) is stronger than inequality 1) from the right.

4) In ΔABC the following relationship holds:

$$\frac{r_b+r_c}{a^2}+\frac{r_c+r_a}{b^2}+\frac{r_a+r_b}{c^2}\le\frac{1}{R}\left(\frac{R^2}{r^2}-\frac{R}{r}+1\right)\le\frac{3}{4r}\left(\frac{R^2}{r^2}-2\right)$$

Solution

See 3) and $\frac{1}{R}\left(\frac{R^2}{r^2}-\frac{R}{r}+1\right)\le\frac{3}{4r}\left(\frac{R^2}{r^2}-2\right)\Leftrightarrow 3R^3-4R^2r-2Rr^2-4r^3\ge 0\Leftrightarrow$

$\Leftrightarrow(R-2r)(3R^2+2Rr+2r^2)\ge 0$, *obviously from Euler's inequality $R\ge 2r$.*

Equality holds if and only if ΔABC is equilateral.

Remark.

The inequalities can be written:

5) In ΔABC the following relationship holds:

$$\frac{3}{R}\le\frac{r_b+r_c}{a^2}+\frac{r_c+r_a}{b^2}+\frac{r_a+r_b}{c^2}\le\frac{1}{R}\left(\frac{R^2}{r^2}-\frac{R}{r}+1\right)\le\frac{3}{4r}\left(\frac{R^2}{r^2}-2\right)$$

Solution

See inequalities 1) and 4).

Equality holds if and only if ΔABC is equilateral.

Remark

If we replace r_a with h_a we propose:

6) In ΔABC the following relationship holds:

$$\frac{3}{R}\le\frac{h_b+h_c}{a^2}+\frac{h_c+h_a}{b^2}+\frac{h_a+h_b}{c^2}\le\frac{1}{r}\left(\frac{r^2}{R^2}+\frac{r}{2R}+1\right)$$

Marin Chirciu – Romania

Solution

We prove the following lemma:

Lemma

7) In ΔABC the following relationship holds:

$$\frac{h_b+h_c}{a^2}+\frac{h_c+h_a}{b^2}+\frac{h_a+h_b}{c^2}=\frac{s^2+r^2-2Rr}{4R^2r}$$

Proof.

Using the formula $h_a=\frac{2S}{a}$ we obtain:

$$\sum\frac{h_b+h_c}{a^2}=\sum\frac{\frac{2S}{b}+\frac{2S}{c}}{a^2}=\frac{2S}{abc}\sum\frac{b+c}{a}=\frac{2S}{4RS}\cdot\frac{s^2+r^2-2Rr}{2Rr}=$$
$$=\frac{s^2+r^2-2Rr}{4R^2r}$$

which follows from the known identity in triangle: $\sum\frac{b+c}{a}=\frac{s^2+r^2-2Rr}{2Rr}$

Let's get back to the main problem.

The left hand inequality.

Using the Lemma the inequality can be written:

$\frac{s^2+r^2-2Rr}{4R^2r}\ge\frac{3}{R}\Leftrightarrow s^2\ge 14Rr-r^2$, *which follows from Gerretsen's inequality*

$s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow R\ge 2r$, *(Euler's inequality)*

Equality holds if and only if ΔABC is equilateral.

The right hand inequality.

Using the Lemma the inequality can be written:

$$\frac{s^2+r^2-2Rr}{4R^2r}\le\frac{1}{r}\left(\frac{r^2}{R^2}+\frac{r}{2R}+1\right)$$

which follows from Gerretsen's inequality $s^2\le 4R^2+4Rr+3r^2$.

We obtain: $\frac{s^2+r^2-2Rr}{4R^2r}\le\frac{4R^2+4Rr+3r^2+r^2-2Rr}{4R^2r}=\frac{4R^2+2Rr+4r^2}{4R^2r}=\frac{2R^2+Rr+2r^2}{2R^2r}=$

$$=\frac{1}{r}\left(\frac{r^2}{R^2}+\frac{r}{2R}+1\right)$$

Equality holds if and only if ΔABC is equilateral.

Remark.

Between the sums** $\sum\frac{h_b+h_c}{a^2}$ **and** $\sum\frac{r_b+r_c}{a^2}$ **we obtain the relationship:

8) In ΔABC the following relationship holds:

$$\sum\frac{h_b+h_c}{a^2}\le\sum\frac{r_b+r_c}{a^2}$$

Marin Chirciu – Romania

Solution

Using the above Lemmas the inequality can be written:

$\frac{s^2+r^2-2Rr}{4R^2r}\le\frac{s^2+r^2-8Rr}{4Rr^2}\Leftrightarrow s^2(R-r)\ge r(8R^2-3Rr+r^2)$***, which follows from***

Gerretsen's inequality** $s^2\ge 16Rr-5r^2$**. It remains to prove that:

$$(16Rr-5r^2)(R-r)\ge r(8R^2-3Rr+r^2)\Leftrightarrow 4R^2-9Rr+2r^2\ge 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)(4R-r)\ge 0$***, obviously from Euler's inequality** $R\ge 2r$**.***

Equality holds if and only if** ΔABC **is equilateral.

Remark.

The following inequalities can be written:

***9) In** ΔABC* **the following relationship holds:**

$$\frac{3}{R}\le\sum\frac{h_b+h_c}{a^2}\le\sum\frac{r_b+r_c}{a^2}\le\frac{1}{R}\left(\frac{R^2}{r^2}-\frac{R}{r}+1\right)$$

Solution

See inequalities 1), 8) and 3).

Equality holds if and only if** ΔABC **is equilateral.

SP.273. If $x,y\in\mathbb{R}$ then:

$$\sin^4 x+\cos^4 x\sin^4 y+\cos^4 x\cos^4 y\ge\frac{1}{3}$$

When does the equality holds?

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\frac{\sin^4 x+\cos^4 x\sin^4 y+\cos^4 x\cos^4 y}{3}\ge\left(\frac{\sin^2 x+\cos^2 x\sin^2 y+\cos^2 x\cos^2 y}{3}\right)^2$$

But $\sin^2 x+\cos^2 x\sin^2 y+\cos^2 y\cos^z y=\sin^2 x+\cos^2 x(\sin^2 y+\cos^2 y)$

$$=\sin^2 x+\cos^2 x=1$$

Thus, $\sin^4 x+\cos^4 x\sin^4 y+\cos^4 x\cos^2 y\ge\frac{1}{3}$

Equality holds if $\sin^4 x=\cos^4 x\sin^4 y=\cos^4 x\cos^4 y$

x *is an odd multiple of* $\frac{\pi}{2}$ *it is not possible.*

$$\therefore\cos^4 x\ne 0\Rightarrow\sin^4 y=\cos^4 y\Rightarrow y=n\pi\pm\frac{\pi}{4};n\in\mathbb{Z}\Rightarrow\tan^4 x=\frac{1}{4}$$

$$\Rightarrow x=m\pi\pm\tan^{-1}\left(\frac{1}{\sqrt{2}}\right),m\in\mathbb{Z}$$

Thus, equality holds when $x=m\pi\pm\tan^{-1}\left(\frac{1}{\sqrt{2}}\right),y=n\pi\pm\frac{\pi}{4},m,n\in\mathbb{Z}$

Solution 2 by Marin Chirciu-Romania

Using Bergström's inequality we obtain $\sin^4 y+\cos^4 y\ge\frac{(\sin^2 y+\cos^2 y)^2}{2}=\frac{1}{2}$,

with equality if and only if $\sin^2 y=\cos^2 y$. *It follows:*

$$M_s=\sin^4 x+\cos^4 x(\sin^4 y+\cos^4 y)\ge\sin^4 x+\cos^4 x\cdot\frac{1}{2}\overset{(1)}{\ge}\frac{1}{3}=M_d,\text{ where (1)}\Leftrightarrow$$

$$\Leftrightarrow 6\sin^4 x+3\cos^4 x\ge 2\Leftrightarrow 6\sin^4 x+3(1-\sin^2 x)^2\ge 2\Leftrightarrow 9\sin^4 x-6\sin^2 x+1\ge 0\Leftrightarrow$$

$\Leftrightarrow(3\sin^2 x-1)^2\ge 0$, *obviously with equality if and only if* $3\sin^2 x=1$.

We deduce that the inequality from enunciation holds, with equality if and only if

$\sin^2 y=\cos^2 y$ *and* $3\sin^2 x=1$

SP.274 If in $\Delta ABC; s=\frac{1}{2}$ then:

$$a\cdot e^{\frac{m_a}{a}}+b\cdot e^{\frac{m_b}{b}}+c\cdot e^{\frac{m_c}{c}}\ge e^{m_a+m_b+m_c}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let be $f_1, f_2, f_3: (0, \infty) \to \mathbb{R}$

$$f_1(x) = ax\ln x - (a + m_a)x; f_2(x) = bx\ln x - (b + m_b)x$$

$$f_3(x) = cx\ln x - (c + m_c)x$$

$$f'_1(x) = a(\ln x + 1) - (a + m_a) = a\ln x - m_a$$

$$f'_1(x) = 0 \Rightarrow a\ln x = m_a \Rightarrow \ln x = \frac{m_a}{a}$$

$$\ln x = \ln e^{\frac{m_a}{a}} \Rightarrow x = e^{\frac{m_a}{a}}$$

$$\min f_1(x) = f_1\left(e^{\frac{m_a}{a}}\right) = a \cdot e^{\frac{m_a}{a}} \cdot \ln e^{\frac{m_a}{a}} - (a + m_a) \cdot e^{\frac{m_a}{a}} =$$

$$= m_a \cdot e^{\frac{m_a}{a}} - ae^{\frac{m_a}{a}} - m_a \cdot e^{\frac{m_a}{a}} = -ae^{\frac{m_a}{a}}$$

Analogous: $\min f_2\,(x) = -be^{\frac{m_b}{b}}; \min f_3(x) = -ce^{\frac{m_c}{c}}$

$$f_1 + f_2 + f_3: (0, \infty) \to \mathbb{R}$$

$$(f_1 + f_2 + f_3)(x) = f_1(x) + f_2(x) + f_3(x)$$

$$\min(f_1 + f_2 + f_3)\,(x) = -(a + b + c)e^{\frac{m_a+m_b+m_c}{a+b+c}}$$

$$\min f_1(x) + \min f_2(x) + \min f_3(x) \le \min(f_1 + f_2 + f_3)\,(x)$$

$$-ae^{\frac{m_a}{a}} - be^{\frac{m_b}{b}} - ce^{\frac{m_c}{c}} \le -(a + b + c)e^{\frac{m_a+m_b+m_c}{a+b+c}}$$

$$ae^{\frac{m_a}{a}} + be^{\frac{m_b}{b}} + ce^{\frac{m_c}{c}} \ge (a + b + c)e^{\frac{m_a+m_b+m_c}{a+b+c}} =$$

$$= (2s) \cdot e^{\frac{m_a+m_b+m_c}{2s}} = \left(2 \cdot \frac{1}{2}\right) \cdot e^{\frac{m_a+m_b+m_c}{2 \cdot \frac{1}{2}}} = e^{m_a+m_b+m_c}$$

Equality holds for $a = b = c = \frac{1}{6}$.

SP.275. In ΔABC the following relationship holds:

$$\left(\frac{a + b}{m_a + m_b}\right)^2 + \left(\frac{b + c}{m_b + m_c}\right)^2 + \left(\frac{c + a}{m_c + m_a}\right)^2 \ge 4$$

Proposed by Hung Nguyen Viet-Vietnam

Solution by proposer

Lemma 1. In any ΔABC***, then*** $(m_b + m_c)^2 \le 2a^2 + \frac{(b+c)^2}{4}$

Proof. The desired inequality is equivalent to $4m_bm_c \le 2a^2 + bc$

$$[2(c^2 + a^2) - b^2][2(a^2 + b^2) - c^2] \le (2a^2 + bc)^2$$

$$4(a^2 + b^2)(a^2 + c^2) - 2b^2(a^2 + b^2) - 2c^2(c^2 + a^2) + b^2c^2 \le 4a^4 + 2a^2bc + b^2c^2$$

$$2a^2b^2 + 2a^2c^2 + 4b^2c^2 - 2b^4 - 2c^4 \le 2a^2bc$$

$$(ab - ac)^2 \le (b^2 - c^2)^2$$

$$(a + b + c)(b + c - a)(b - c)^2 \ge 0$$

The last inequality is clearly true.

$$\sum_{cyc}\left(\frac{b + c}{m_b + m_c}\right)^2 \ge \sum_{cyc}\frac{4(b + c)^2}{8a^2 + (b + c)^2} \ge \frac{4[(b + c)^2 + (c + a)^2 + (a + b)^2]^2}{\sum_{cyc}[8a^2(b + c)^2 + (b + c)^4]}$$

$$= \frac{16(a^2 + b^2 + c^2 + ab + bc + ca)^2}{2(a^4 + b^4 + c^4) + 22(a^2b^2 + b^2c^2 + c^2a^2) + 16abc(a + b + c) + \sum_{cyc} 4bc(b^2 + c^2)}$$

Hence, it suffices to show that:

$$2(a^2 + b^2 + c^2 + ab + bc + ca)^2 \ge$$

$$(a^4 + b^4 + c^4) + 11(a^2b^2 + b^2c^2 + c^2a^2) + 8abc(a + b + c) + \sum_{cyc} 2bc(b^2 + c^2)$$

Or equivalent to

$$a^4 + b^4 + c^4 + \sum_{cyc} 2bc(b^2 + c^2) \ge 5(a^2b^2 + b^2c^2 + c^2a^2)$$

$$a^4 + b^4 + c^4 + \sum_{cyc} 2bc(b - c)^2 \ge 5(a^2b^2 + b^2c^2 + c^2a^2)$$

The last inequality is true because

$$a^4 + b^4 + c^4 \ge a^2b^2 + b^2c^2 + c^2a^2$$

SP.276. If $x, y, z > 0; n \ge 1$ then:

$$\sum_{cyc}\frac{(nx + y)(nx + z)}{yz} \ge \frac{(n + 1)^2}{2}\sum_{cyc}\frac{y + z}{x}$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Şerban-Romania

$$\sum_{cyc}\frac{(nx+y)(nx+z)}{yz}\geq\frac{(n+1)^2}{2}\sum_{cyc}\frac{y+z}{x}$$

$$\sum_{cyc}\left(\frac{n^2x^2}{yz}+\frac{nxz}{yz}+\frac{nxy}{yz}+\frac{yz}{yz}\right)\geq\frac{n^2+2n+1}{2}\sum_{cyc}\frac{y+z}{x}$$

$$n^2\left(\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}\right)+3\geq\frac{n^2+1}{2}\sum_{cyc}\frac{y+z}{x},\ \frac{n^2(x^3+y^3+z^3)}{xyz}+3\geq\frac{n^2+1}{2}\sum_{cyc}\frac{y+z}{x}$$

$$n^2\left(\frac{x^3+y^3+z^3}{xyz}-\frac{1}{2}\sum_{cyc}\frac{y+z}{x}\right)\geq\frac{1}{2}\sum_{cyc}\frac{y+z}{x}-3$$

$$n^2\left(\frac{x^3+y^3+z^3}{xyz}-\frac{1}{2}\sum_{cyc}\frac{y+z}{x}\right)\geq\frac{x^3+y^3+z^3}{xyz}-\frac{1}{2}\sum_{cyc}\frac{y+z}{x}\overset{?}{\geq}\frac{1}{2}\sum_{cyc}\frac{y+z}{x}-3$$

Because: $x^3+y^3\geq xy(x+y)\Rightarrow\frac{x^3+y^3}{xyz}\geq\frac{xy(x+y)}{xyz}\Rightarrow\frac{x^3+y^3+z^3}{xyz}\geq\frac{1}{2}\sum_{cyc}\frac{y+z}{x}\Rightarrow$

$$\frac{x^3+y^3+z^3}{xyz}\geq\sum_{cyc}\frac{y+z}{x}-3\Rightarrow x^3+y^3+z^3\geq\sum_{cyc}yz(y+z)-3xyz\Rightarrow$$

$x^3+y^3+z^3+3xyz\geq x^2y+xy^2+y^2z+yz^2+z^2x+zx^2$ *true by Schur's inequality.*

SP.277. In ΔABC the following relationship holds:

$$27\left(\frac{R}{2r}\right)^2-\sum_{cyc}\left(\sqrt{\frac{sinA}{sinB}}+\sqrt{\frac{sinA}{sinC}}\right)^3\geq 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

Let a,b,c –be the lengths of sides of the ΔABC.

We know that: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\leq\frac{1}{4r^2}$ *and* $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\leq 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$ *so*

$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\leq\frac{3}{4r^2}$. *We have* $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{2}{ab}+\frac{2}{bc}+\frac{2}{ca}\leq\frac{3}{4r^2}$

$$\frac{3}{4r^2}-\frac{1}{a^2}\ge\left(\frac{1}{b}+\frac{1}{c}\right)^2+\frac{2}{a}\left(\frac{1}{b}+\frac{1}{c}\right)\overset{Am-Gm}{\Longrightarrow}\frac{3}{4r^2}-\frac{1}{a^2}\ge 2\sqrt{\left(\frac{1}{b}+\frac{1}{c}\right)^2\cdot\frac{2}{a}\left(\frac{1}{b}+\frac{1}{c}\right)}$$

$$\frac{3}{4r^2}-\frac{1}{a^2}\ge 2\left(\frac{1}{b}+\frac{1}{c}\right)\sqrt{\frac{2}{a}\left(\frac{1}{b}+\frac{1}{c}\right)}\xRightarrow[x,y>0]{2(x+y)\ge(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{3}{4r^2}-\frac{1}{a^2}\ge\frac{1}{\sqrt{a}}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^2\sqrt{\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^2}=\frac{1}{\sqrt{a}}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^3$$

So: $\frac{1}{a^2}+\frac{1}{\sqrt{a}}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^3\le\frac{3}{4r^2}$ *or*

$$1+\sqrt{a^3}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^3\le\frac{3}{4r^2}\cdot a^2\Leftrightarrow 1+\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{a}{c}}\right)^3\le\frac{3}{4r^2}\cdot a^2$$

Similarly: $1+\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{b}{c}}\right)^3\le\frac{3}{4r^2}\cdot b^2$ *and* $1+\left(\sqrt{\frac{c}{a}}+\sqrt{\frac{c}{b}}\right)^3\le\frac{3}{4r^2}\cdot c^2$

Addind up these inequalities, we have

$$3+\sum_{cyc}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{a}{c}}\right)^3\le\frac{3}{4r^2}\cdot(a^2+b^2+c^2)$$

We know that $a^2+b^2+c^2\le 9R^2$ *and using the law of the sines, we get*

$$3+\sum_{cyc}\left(\sqrt{\frac{sinA}{sinB}}+\sqrt{\frac{sinA}{sinC}}\right)^3\le\frac{27R^2}{4r^2}$$

$$27\left(\frac{R}{2r}\right)^2-\sum_{cyc}\left(\sqrt{\frac{sinA}{sinB}}+\sqrt{\frac{sinA}{sinC}}\right)^3\ge 3$$

SP.278. Let be $f:\left[\frac{\pi}{4},\frac{3\pi}{4}\right]\to\mathbb{R}, f(x)=\frac{cot^2x-2cotx+n-1}{cot^2x+2cotx+n+1}; n\ge 2$. Find Imf.

Proposed by Marin Chirciu-Romania

Solution by proposer

Denote $1+cotx=t$ *we have* $0\le t\le 2, f(t)=\frac{t^2-4t+n+2}{t^2+n}$

We calculate: $f'(t)=\frac{4(t^2-t-n)}{(t^2+n)^2}\le 0, \forall t\in[0,2]$ *and* $f(0)=\frac{n+2}{n}, f(2)=\frac{n-2}{n+4}$ *then*

the function f *–is strictly decreasing on* $[0,2]$*, so*

$$Imf=[f(2),f(0)]=\left[\frac{n-2}{n+4},\frac{n+2}{n}\right]$$

We deduce that $Imf=\left[\frac{n-2}{n+4},\frac{n+2}{n}\right]$ *it is the function image*

$$f:\left[\frac{\pi}{4},\frac{3\pi}{4}\right]\to\mathbb{R}, f(x)=\frac{cot^2x-2cotx+n-1}{cot^2x+2cotx+n+1}$$

SP.279. If in $\Delta ABC;\ \omega$ –Brocard angle then the following relationship holds:

$$\frac{1}{2sin\omega}\ge\sqrt{\frac{w_aw_bw_c}{h_ah_bh_c}}\ge\frac{2cos\omega}{\sqrt{3}}$$

Proposed by Vasile Jiglău-Romania

Solution by proposer

It is a known (and elementary) fact that in triangle $cos\frac{B-C}{2}=\frac{h_a}{l_a};(1)$

Suppose that the sides of the triangle ABC *verify:* $c\ge b\ge a;\ \ (2)$

Clearly the meansures of the angles of the triangle verify $A\ge B\ge C$*, which imply*

$sin\frac{C-B}{2}sin\frac{B-A}{2}\ge 0$*.We have:* $\frac{h_b}{w_b}=cos\frac{C-A}{2}=cos\left(\frac{C-B}{2}+\frac{B-A}{2}\right)$

$$=cos\frac{C-B}{2}cos\frac{B-A}{2}-sin\frac{C-B}{2}sin\frac{B-A}{2}\le cos\frac{C-B}{2}cos\frac{B-A}{2}=\frac{h_a}{w_a}\cdot\frac{h_c}{w_c}$$

Hence, under the hypothesis (1): $\frac{h_b}{w_b}\ge\frac{h_a}{w_a}\cdot\frac{h_c}{w_c};\ \ (3)$

Let's now prove that: $\frac{1}{2sin\omega}\ge\sqrt{\frac{R}{2r}}$

We'll use the formula $sin\omega=\frac{S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$*,where* S *–is area of the given triangle.*

This is equivalent to: $\frac{1}{sin^2\omega}\geq\frac{2R}{r}\Leftrightarrow\frac{\sum a^2b^2}{4S^2}\geq\frac{2R}{r}\Leftrightarrow\sum a^2b^2\geq 8Rrs^2$

$$\Leftrightarrow\sum a^2b^2\geq abc(a+b+c)\Leftrightarrow\sum(ab-bc)^2\geq 0;(4)$$

On the other hand, it is a known fact that: $\frac{R}{2r}\geq\frac{1}{cos^2\frac{B-C}{2}}$ *(see the problem 2382 from "Crux mathematicorum")*

$$\overset{by\ (1)}{\Longrightarrow}\sqrt{\frac{R}{2r}}\geq\frac{w_a}{h_a};\ \ (5)$$

From (3),(4) and (5), we obtain that: $\frac{1}{2sin\omega}\geq\sqrt{\frac{R}{2r}}\geq\frac{w_b}{h_b}\geq\sqrt{\frac{w_aw_bw_c}{h_ah_bh_c}}$*, and the first inequality of the enunciation is proved.*

The proof of the second inequality of the enunciation: We'll use

$$cos\omega=\frac{a^2+b^2+c^2}{2\sqrt{a^2b^2+b^2c^2+c^2a^2}}$$

With the formulas: $w_a=\frac{2\sqrt{bcs(s-a)}}{b+c},h_a=\frac{2S}{a}$ *we can easily see that:*

$$\frac{w_aw_bw_c}{h_ah_bh_c}=\frac{a^2b^2c^2(a+b+c)}{2S^2(a+b)(b+c)(c+a)}$$

The inequality becomes equivalent to:

$$\frac{a^2b^2c^2(a+b+c)}{2S^2(a+b)(b+c)(c+a)}\geq\frac{(a^2+b^2+c^2)^2}{3(a^2b^2+b^2c^2+c^2a^2)}$$

Putting $x=s-a,y=s-b,z=s-c$, *the inequality becomes equivalent to:*

$$3\left(\sum(x+y)^2(y+z)^2\right)\prod(x+y)^2\geq xyz\left(\sum(x+y)^2\right)^2\prod(2x+y+z)\Leftrightarrow$$

$$3\sum x^8y^2+3\sum x^8z^2+12\sum x^7y^3+12\sum x^7z^3+24\sum x^6y^4+24\sum x^6z^4+$$

$$+30\sum x^5y^5+22\sum x^5y^4z+22\sum x^5yz^4\geq 2\sum x^8yz+8\sum x^7y^2z+$$

$$+8\sum x^7yz^2+40\sum x^6y^2z^2+32\sum x^5y^3z^2+32\sum x^5y^2z^3+$$

$$+14\sum x^4y^4z^2+16\sum x^4y^3z^3,(x,y,z\geq 0),$$

Which immediately result by adding the following inequalities, that are simple applications of the Am-Gm inequality and of the Murihead's lemma:

$$\sum x^8y^2 + \sum x^8z^2 \geq 2\sum x^8yz$$

$$8\sum x^7y^3 + 8\sum x^7z^3 \geq 8\sum x^7y^2z + 8\sum x^7yz^2$$

$$2\sum x^8y^2 \geq 2\sum x^6y^2z^2$$

$$2\sum x^8z^2 \geq 2\sum x^6y^2z^2$$

$$4\sum x^7y^3 \geq 4\sum x^6y^2z^2$$

$$4\sum x^7z^3 \geq 4\sum x^6y^2z^2$$

$$14\sum x^6y^4 + 14\sum x^6z^4 \geq 28\sum x^6y^2z^2$$

$$10\sum x^6y^4 + 10\sum x^6z^4 \geq 10\sum x^5y^3z^2 + 10\sum x^5y^2z^3$$

$$22\sum x^5y^4z + 22\sum x^5yz^4 \geq 22\sum x^5y^3z^2 + 22\sum x^5y^2z^3$$

$$14\sum x^5y^5 \geq 14\sum x^4y^4z^2,$$

$$16\sum x^5y^5 \geq 16\sum x^4y^3z^3$$

SP.280 If $x,y,z \geq 0; \{x\}^9 + \{y\}^9 + \{z\}^9 = \frac{1}{64}$ then:

$$x^7\cdot[x]\cdot\{x\} + y^7\cdot[y]\cdot\{y\} + z^7\cdot[z]\cdot\{z\} < 64([x]^9 + [y]^9 + [z]^9) + 1$$

$\{x\} = x - [x]; [*]$ - great integer function

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{[x]^9 + \{x\}^9}{[x]\cdot\{x\}} = \frac{[x]^8}{\{x\}} + \frac{\{x\}^8}{[x]} \overset{BERGSTROM}{\geq} \frac{([x]^4 + \{x\}^4)^2}{\{x\} + [x]} \geq$$

$$\geq \frac{1}{x}\cdot\left(\frac{([x]^2 + \{x\}^2)^2}{2}\right)^2 = \frac{1}{4x}([x]^2 + \{x\}^2)^4 \geq$$

$$\geq \frac{1}{4x}\left(\left(\frac{[x]+\{x\}}{2}\right)^2\right)^4 = \frac{x^8}{16\cdot 4x} = \frac{x^7}{64}$$

$$64([x]^9+\{x\}^9) \geq x^7\cdot[x]\cdot\{x\}; x \geq 0$$

$$x^7\cdot[x]\cdot\{x\} \leq 64([x]^9+\{x\}^9\} \quad \textbf{\textit{(1)}}$$

Analogous:

$$y^7\cdot[y]\cdot\{y\} \leq 64([y]^9+\{y\}^9\} \quad \textbf{\textit{(2)}}$$

$$z^7\cdot[z]\cdot\{z\} \leq 64([z]^9+\{z\}^9\} \quad \textbf{\textit{(3)}}$$

By adding (1); (2); (3):

$$x^7\cdot[x]\cdot\{x\}+y^7\cdot[y]\cdot\{y\}+z^7\cdot[z]\cdot\{z\} \leq$$

$$\leq 64([x]^9+[y]^9+[z]^9\}+64(\{x\}^9+\{y\}^9+\{z\}^9) = 64([x]^9+[y]^9+[z]^9\}+1$$

Inequality is strict because (1); (2); (3) are equalities only for $x=y=z=0$ ***and in our case*** $\{x\}^9+\{y\}^9+\{z^9\}=\frac{1}{64}\neq 0$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Since $x,y,z\geq 0 \Rightarrow [x],[y],[z]\geq 0$ ***and*** $0\leq\{x\},\{y\},\{z\}<1$

Because: $\{x\}^6+\{y\}^6+\{z\}^6=\frac{1}{64}\to 1=64(\{x\}^6+\{y\}^6+\{z\}^6)$

$$RHS = 64([x]^9+\{x\}^9+[y]^9+\{y\}^9+[z]^9+\{z\}^9)$$

Now: $x^7\cdot[x]\cdot\{x\} \overset{Am-Gm}{\leq} x^7\cdot\frac{([x]+\{x\})^2}{4} = x^7\cdot\frac{x^2}{4}=\frac{x^9}{4}$

Analogous: $y^7\cdot[y]\cdot\{y\}\leq\frac{y^9}{4}$ ***and*** $z^7\cdot[z]\cdot\{z\}\leq\frac{z^9}{4}$

$$LHS = x^9\cdot[x]\cdot\{x\}+y^9\cdot[y]\cdot\{y\}+z^9\cdot[z]\cdot\{z\} \leq \frac{x^9+y^9+z^9}{4}$$

$$=\frac{([x]+\{x\})^9+([y]+\{y\})^9+([z]+\{z\})^9}{4}$$

$$=\frac{2^8}{4}([x]^9+\{x\}^9+[y]^9+\{y\}^9+[z]^9+\{z\}^9)$$

$$=64([x]^9+\{x\}^9+[y]^9+\{y\}^9+[z]^9+\{z\}^9)$$

Proved.

Note: For all $\alpha,\beta>0$ ***we have:*** $\alpha^9+\beta^9\geq\frac{(\alpha+\beta)^9}{2^8}$

Equality$\Leftrightarrow [x]=\{x\}=[y]=\{y\}=[z]=\{z\}=0$

But: $\{x\}^9+\{y\}^9+\{z^9\}=\frac{1}{64}\neq 0$. *So, inequality is strict.*

SP.281. If $x\in\left(0,\frac{\pi}{2}\right);a,b>0$ then:

$$\left(\left(\sqrt{\frac{a}{b}}\right)^{\frac{\sin x}{x}}+\left(\sqrt{\frac{b}{a}}\right)^{\frac{\sin x}{x}}\right)\cdot\left(\left(\sqrt{\frac{a}{b}}\right)^{\frac{x}{\tan x}}+\left(\sqrt{\frac{b}{a}}\right)^{\frac{x}{\tan x}}\right)\le\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Florentin Vișescu – Romania

Let be $\sqrt{\frac{a}{b}}=t>0$

$$\left(t^{\frac{\sin x}{x}}+\left(\frac{1}{t}\right)^{\frac{\sin x}{x}}\right)\cdot\left(t^{\frac{x}{\tan x}}+\left(\frac{1}{t}\right)^{\frac{x}{\tan x}}\right)\le\left(t+\frac{1}{t}\right)^2$$

For $x\in\left(0;\frac{\pi}{2}\right),0<\sin x<x<\tan x\,|:x$

$$0<\frac{\sin x}{x}<1<\frac{\tan x}{x}\Rightarrow\frac{\sin x}{x}\in(0,1),\frac{x}{\tan x}\in(0,1)$$

We denote $m=\frac{\sin x}{x}\in(0,1);n=\frac{x}{\tan x}\in(0,1)$

We prove that $\left(t^m+\frac{1}{t^m}\right)\left(t^n+\frac{1}{t^n}\right)\le\left(t+\frac{1}{t}\right)^2$

We prove that $t^k+\frac{1}{t^k}\le t+\frac{1}{t};t>0$ *fixed;* $k\in(0,1)$

We consider $f(k)=t^k+t^{-k};\ f:(0,1)\to\mathbb{R}$

$$f'(k)=t^k\ln t-t^k\ln t=\ln t\left(t^k-t^{-k}\right)$$

$$t^k-t^{-k}=0;\ t^k=t^{-k}\Rightarrow k=-k;\ 2k=0$$

k	0 $\qquad\qquad\qquad\qquad$ 1
$t'(k)$	+ + + + + + + + + + + + + + + + + +
$t(k)$	2 $\nearrow$ $t+\frac{1}{t}$

$t\in(0,1)$

k	0 1
$t'(k)$	+ + + + + + + + + + + + + + + + + +
$t(k)$	2 ↗ $t+\frac{1}{t}$

So, $f(k)\le t+\frac{1}{t}\Rightarrow t^k+\frac{1}{t^k}\le t+\frac{1}{t}$

SP.282. If in ΔABC; H –orthocentre; HD, HE, HF bisectors of angles BHC, CHA respectively AHB; $D\in(BC)$; $E\in(CA)$; $F\in(AB)$ then the following relationship holds:

$$\frac{[DEF]}{[ABC]}\ge 13\left(\frac{r}{R}\right)^2-3$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$\Delta BGC\Rightarrow\frac{BD}{DC}=\frac{HB}{HC}\text{ and analogs}$$

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{HB}{HC}\cdot\frac{HC}{HA}\cdot\frac{HA}{HB}=1;(1)$$

$$\text{Let: }\frac{HA}{HB}=m,\frac{HB}{HC}=k,\frac{HC}{HA}=p$$

$$S_{AEF}=\frac{AF\cdot AE\cdot sinA}{2}=\frac{m}{(m+1)(p+1)}\cdot\frac{bc\cdot sinA}{2}=\frac{m}{(m+1)(p+1)}\cdot S_{ABC}\text{ and analogs}$$

$$S_{DEF}=\frac{1+kmp}{(1+k)(1+m)(1+p)}\overset{(1)}{=}\frac{2}{(1+k)(1+m)(1+p)}=$$

$$=\frac{2AH\cdot BH\cdot CH}{(AH+BH)(AH+CH)(CH+BH)}\cdot S_{ABC}$$

$$\text{But: }AH=2RsinA\Rightarrow$$

$$S_{DEF}=\frac{2cosAcosBcosC}{(cosA+cosB)(cosB+cosC)(cosC+cosA)}\cdot S_{ABC};(2)$$

$$\text{But: }cosAcosBcosC=\frac{s^2-(2R+r)^2}{4R^2};(3)$$

$$(cosA+cosB)(cosB+cosC)(cosC+cosA)=\frac{r(s^2+r^2+2Rr)}{4R^3};(4)$$

$$s^2 \le 27R^2 \cdot s^2 \ge \frac{27}{4} \cdot r^2, R \ge 2r; (5)$$

From (1)+(2)+(3)+(4)+(5) proved.

SP.283. Find $x, y > 0$ such that:

$$\sqrt{\frac{x}{y}} + \sqrt[3]{\frac{3}{x}} + \sqrt[5]{\frac{y}{3}} = \frac{10}{\sqrt[10]{337500}}$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sqrt{\frac{x}{y}} + \sqrt[3]{\frac{3}{x}} + \sqrt[5]{\frac{y}{3}} = \sqrt{\frac{x}{y}} + \frac{3}{\sqrt[3]{9} \cdot \sqrt[3]{x}} + \frac{5\sqrt[5]{3^4} \cdot \sqrt[5]{y}}{15}$$

$$= \frac{1}{2}\sqrt{\frac{x}{y}} + \frac{1}{2}\sqrt{\frac{x}{y}} + \frac{1}{\sqrt[3]{9} \cdot \sqrt[3]{x}} + \frac{1}{\sqrt[3]{9} \cdot \sqrt[3]{x}} + \frac{1}{\sqrt[3]{9} \cdot \sqrt[3]{x}} + \frac{5\sqrt[5]{3^4}}{15} \cdot \sqrt[5]{y} + \frac{5\sqrt[5]{3^4}}{15} \cdot \sqrt[5]{y} + \frac{5\sqrt[5]{3^4}}{15} \cdot \sqrt[5]{y} + \frac{5\sqrt[5]{3^4}}{15} \cdot \sqrt[5]{y} + \frac{5\sqrt[5]{3^4}}{15} \cdot \sqrt[5]{y}$$

$$\overset{Am-Gm}{\ge} 10 \cdot \sqrt[10]{\left(\frac{1}{2}\right)^2 \left(\frac{1}{\sqrt[3]{9}}\right)^3 \left(\frac{\sqrt[5]{3^4}}{15}\right)^5} = \frac{10}{\sqrt[10]{337500}}$$

Equality for: $\frac{1}{2}\sqrt{\frac{x}{y}} = \frac{1}{\sqrt[3]{9x}} = \frac{\sqrt[5]{81y}}{15}$

$$\frac{\sqrt{x}}{2\sqrt{y}} = \frac{1}{\sqrt[3]{9x}} \Leftrightarrow \sqrt{x} \cdot \sqrt[3]{9x} = 2\sqrt{y} \Leftrightarrow x^5 = \frac{64}{81}y^3 \ldots (1)$$

$$\frac{1}{\sqrt[3]{9x}} = \frac{\sqrt[5]{81y}}{15} \Leftrightarrow (9x)^5 \cdot (81y)^3 = 45^{15} \Rightarrow y = \sqrt[6]{\frac{9^4 \cdot 5^{15}}{64}} \Rightarrow x = \sqrt[10]{64 \cdot 5^{15}}$$

Answer: $(x; y) = \left(\sqrt[10]{64 \cdot 5^{15}}; \sqrt[6]{\frac{9^4 \cdot 5^{15}}{64}}\right)$

SP.284. RMM WINTER EDITION 2020

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{1}{3R^2} \le \frac{1}{(r_a+r_b)^2} + \frac{1}{(r_b+r_c)^2} + \frac{1}{(r_c+r_a)^2} \le \frac{16R^2-3r^2}{12r^4}$$

Proposed by George Apostolopoulos-Messolonghi– Greece

Solution.

The left hand inequality.

Using means inequality and $\prod(r_b+r_c) = 4Rs^2$ ***we obtain:***

$$\sum \frac{1}{(r_b+r_c)^2} \ge 3\sqrt[3]{\prod \frac{1}{(r_b+r_c)^2}} = 3\sqrt[3]{\prod \frac{1}{(4Rrs^2)^2}} \overset{(1)}{\ge} \frac{1}{3R^2}$$

where (1) $\Leftrightarrow 9R^2 \ge \sqrt[3]{(4Rs^2)^2} \Leftrightarrow (3R)^3 \ge 4Rs^2 \Leftrightarrow 27R^2 \ge 4s^2$***, obviously from***

Mitrinovic's inequality $s \le \frac{3R\sqrt{3}}{2}$***. Equality holds if and only if*** ΔABC ***is equilateral.***

The right-hand inequality. We prove the strongest inequality:

2) In ΔABC **the following relationship holds:**

$$\sum \frac{1}{(r_b+r_c)^2} \le \frac{1}{12r^2}$$

We have $\sum \frac{1}{(r_b+r_c)^2} \le \sum \frac{1}{4r_br_c} = \frac{\sum r_a}{4\prod r_a} = \frac{4R+r}{4rs^2} \overset{(2)}{\le} \frac{1}{12r^2}$

where (2) $\Leftrightarrow s^2 \ge 3r(4R+r)$***, which follows from Gerretsen's inequality***

$s^2 \ge 16Rr - 5r^2$***. It remains to prove that:***

$16Rr - 5r^2 \ge 3r(4R+r) \Leftrightarrow R \ge 2r$***, (Euler's inequality).***

Equality holds if and only if ΔABC ***is equilateral.***

Let's get back to solving the right-hand inequality:

Using 2) it suffices to prove that:

$\frac{1}{12r^2} \le \frac{16R^2-3r^2}{12r^4} \Leftrightarrow R^2 \ge 4r^2 \Leftrightarrow R \ge 2r$***, (Euler's inequality)***

Equality hold if and only if ΔABC ***is equilateral.***

Remark.

The double inequality can be written:

3) In ΔABC **the following relationship holds:**

$$\frac{1}{3R^2} \le \sum \frac{1}{(r_b + r_c)^2} \le \frac{1}{12r^2}$$

Solution.

See 1) the left hand and 2).

Equality holds if and only if ΔABC is equilateral.

Remark.

If we replace r_a with h_a we propose:

4) In ΔABC the following relationship holds:

$$\frac{1}{3R^2} \le \sum \frac{1}{(h_b + h_c)^2} \le \frac{1}{12r^2}$$

Marin Chirciu – Romania

Solution

The left-hand inequality.

Using the means inequality and $\prod(h_b + h_c) = \frac{rs^2(s^2+r^2+2Rr)}{R}$ we obtain:

$$\sum \frac{1}{(h_b + h_c)^2} \ge 3\sqrt[3]{\prod \frac{1}{(h_b + h_c)^2}} = 3\sqrt[3]{\frac{1}{\left(\frac{rs^2(s^2 + r^2 + 2Rr)}{R^2}\right)^2}} =$$

$$= 3\sqrt[3]{\frac{R^4}{r^2 s^4 (s^2 + r^2 + 2Rr)^2}} \overset{(1)}{\ge} \frac{1}{3R^2}$$

where (1) $\Leftrightarrow 9R^3 \cdot \sqrt[3]{R} \ge \sqrt[3]{r^2 s^4 (s^2 + r^2 + 2Rr)^2} \Leftrightarrow (3R)^3 \cdot R^2 \ge rs^2(s^2 + r^2 + 2Rr) \Leftrightarrow$

$\Leftrightarrow 27R^5 \ge rs^2(s^2 + r^2 + 2Rr)$, ***which follows from Mitrinovic's inequality $s^2 \le \frac{27R^2}{4}$***

and Gerretsen's inequality $s^2 \le 4R^2 + 4Rr + 3r^2$. It remains to prove that:

$$27R^5 \ge r \cdot \frac{27R^2}{4}(4R^2 + 4Rr + 3r^2 + r^2 + 2Rr) \Leftrightarrow 2R^3 - 2R^2 r - 3Rr^2 - 2r^3 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)(2R^2 + 2Rr + r^2) \ge 0$, ***true from Euler's inequality $R \ge 2r$.***

Equality holds if and only if ΔABC is equilateral.

The right-hand inequality.

We have $\sum\frac{1}{(h_b+h_c)^2}\le\sum\frac{1}{4h_bh_c}=\frac{\sum h_a}{4\prod h_a}=\frac{\frac{s^2+r^2+4Rr}{2R}}{4\cdot\frac{2r^2s^2}{R}}=\frac{s^2+r^2+4Rr}{16r^2s^2}\overset{(2)}{\le}\frac{1}{12r^2}$

where (2) $\Leftrightarrow s^2\ge 3r(4R+r)$*, which follows from Gerretsen's inequality*

$s^2\ge 16Rr-5r^2$*. It remains to prove that:*

$16Rr-5r^2\ge 3r(4R+r)\Leftrightarrow R\ge 2r$*, (Euler's inequality).*

Equality holds if and only if ΔABC *is equilateral.*

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By Marin Chirciu – Romania

1) In ΔABC:

$$\frac{3r}{R}\le\frac{h_a}{r_b+r_c}+\frac{h_b}{r_c+r_a}+\frac{h_c}{r_a+r_b}\le\frac{3}{2}$$

Proposed by George Apostolopoulos-Messolonghi– Greece

Proof.

We prove the following lemma:

Lemma.

2) In ΔABC:

$$\frac{h_a}{r_b+r_c}+\frac{h_b}{r_c+r_a}+\frac{h_c}{r_a+r_b}=\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{8R^2s^2}$$

Proof.

Using the following formulas $h_a=\frac{2S}{a}$ *and* $r_a=\frac{S}{s-a}$ *we obtain:*

$$\sum\frac{h_a}{r_a+r_c}=\sum\frac{\frac{2S}{a}}{\frac{S}{s-b}+\frac{S}{s-c}}=2\sum\frac{(s-b)(s-c)}{a^2}=$$

$$=2\cdot\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{16R^2s^2}=$$

$=\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{8R^2s^2}$*, which follows from the known identity in triangle:*

$$\frac{(s-b)(s-c)}{a^2}=\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{16R^2s^2}$$

Let's get back to the main problem:

Left hand inequality: Using the lemma the inequality can be written:

$$\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{8R^2s^2}\geq\frac{3r}{R}\Leftrightarrow s^2(s^2+2r^2-28Rr)+r(4R+r)^3\geq 0$$

We distinguish the following cases:

Case 1). If** $(s^2+2r^2-28Rr)\geq 0$**, the inequality is obvious.

Case 2). If** $(s^2+2r^2-28Rr)<0$**, the inequality can be rewritten:

$r(4R+r)^3\geq s^2(28Rr-2r^2-s^2)$***, which follows from Blundon's – Gerretsen's inequality:***

$16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$***. It remains to prove that:***

$$r(4R+r)^3\geq\frac{R(4R+r)^2}{2(2R-r)}(28Rr-2r^2-16Rr+5r^2)\Leftrightarrow 4R^2-7Rr-2r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)(4R+r)\geq 0$***, obviously from Euler's inequality** $R\geq 2r$.*

Equality holds if and only if** ΔABC **is equilateral.

The right-hand inequality can be written:

$$\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{8R^2s^2}\leq\frac{3}{2}\Leftrightarrow s^2(12R^2+4Rr-2r^2-s^2)\geq r(4R+r)^3,$$

Which follows from Gerretsen's inequality:

$$4R^2+4Rr+3r^2\geq s^2\geq 16Rr-5r^2\geq\frac{r(4R+r)^2}{R+r}$$

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}(12R^2+4Rr-2r^2-4R^2-4Rr-3r^2)\geq r(4R+r)^3\Leftrightarrow$$

$4R^2-5Rr-6r^2\geq 0\Leftrightarrow(R-2r)(4R+3r)\geq 0$***, obviously from Euler's inequality** $R\geq 2r$**. Equality holds if and only if the triangle** ABC **is equilateral.***

Remark.

The double inequality 1) can be strengthened:

3) In** ΔABC**:

$$\frac{7}{2}\cdot\frac{r}{R}-\frac{7}{8}\left(\frac{r}{R}\right)^2+\frac{1}{4}\left(\frac{r}{R}\right)^3\le\frac{h_a}{r_b+r_c}+\frac{h_a}{r_c+r_a}+\frac{h_b}{r_a+r_b}\le 1+\frac{5}{8}\cdot\frac{r}{R}+\frac{3}{4}\left(\frac{r}{R}\right)^2$$

Marin Chirciu – Romania

Solution.

The left hand inequality.

Using Lemma and Blundon-Gerretsen's inequality:

$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$ ***we obtain:***

$$\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{8R^2s^2}=\frac{1}{8R^2}\left[s^2+2r^2-4Rr+\frac{r(4R+r)^3}{s^2}\right]\ge$$

$$\ge\frac{1}{8R^2}\left[16Rr-5r^2+2r^2-4Rr+\frac{r(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=\frac{r(28R^2-7Rr-2r^2)}{8R^3}=$$

$$=\frac{7}{2}\cdot\frac{r}{R}-\frac{7}{8}\left(\frac{r}{R}\right)^2+\frac{1}{4}\left(\frac{r}{R}\right)^3$$

Equality holds if and only if ΔABC is equilateral.

The right hand inequality:

Using Lemma and Gerretsen's inequality: $4R^2+4Rr+3r^2\ge s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$ ***we obtain:***

$$\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)}{8R^2s^2}=\frac{1}{8R^2}\left[s^2+2r^2-4Rr+\frac{r(4R+r)}{s^2}\right]\le$$

$$\le\frac{1}{8R^2}\left[4R^2+4Rr+3r^2+2r^2-4Rr+\frac{r(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{8R^2+5Rr+6r^2}{8R^2}=1+\frac{5}{8}\cdot\frac{r}{R}+\frac{3}{4}\left(\frac{r}{R}\right)^2$$

Equality holds if and only if ΔABC is equilateral.

Remark.

The double inequality 3) is stronger than the double inequality 1)

4) In ΔABC:

$$\frac{3r}{R} \le \frac{7}{2}\cdot\frac{r}{R} - \frac{7}{8}\left(\frac{r}{R}\right)^2 + \frac{1}{4}\left(\frac{r}{R}\right)^3 \le \frac{h_a}{r_b+r_c} + \frac{h_b}{r_c+r_a} + \frac{h_c}{r_a+r_b} \le 1 + \frac{5}{8}\cdot\frac{r}{R} + \frac{3}{4}\left(\frac{r}{R}\right)^2 \le \frac{3}{2}$$

Proof.

See 3) and Euler's inequality $R \ge 2r$.

Equality holds if and only if ΔABC is equilateral.

Remark.

If we interchange r_a with h_a we propose:

5) In ΔABC:

$$\frac{3r}{R} \le \frac{r_a}{h_b+h_c} + \frac{r_b}{h_c+h_a} + \frac{r_c}{h_a+h_b} \le \frac{3R}{4r}$$

Marin Chirciu – Romania

Proof.

We prove the following lemma:

Lemma.

6) In ΔABC:

$$\frac{r_a}{h_b+h_c} + \frac{r_b}{h_c+h_a} + \frac{r_c}{h_a+h_b} = \frac{s^4 + s^2(32R^2+4Rr+2r^2) + r(4R+r)^3}{4s^2(s^2+r^2+2Rr)}$$

Proof.

Using the following formulas $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$$\sum \frac{r_a}{h_b+h_c} = \sum \frac{\frac{S}{s-a}}{\frac{2S}{b}+\frac{2S}{c}} = \frac{1}{2}\cdot\frac{s^4 + s^2(32R^2+4Rr+2r^2) + r(4R+r)^3}{2s^2(s^2+r^2+2Rr)} =$$

$$= \frac{s^4+s^2(32R^2+4Rr+2r^2)+r(4R+r)^3}{4s^2(s^2+r^2+2Rr)}$$, *which follows from the following identity:*

$$\sum \frac{bc}{(s-a)(b+c)} = \frac{s^4+s^2(32R^2+4Rr+2r^2)+r(4R+r)^3}{2s^2(s^2+r^2+2Rr)}$$, *true from the following identities known in triangle:* $\prod(s-a) = r^2 s \cdot \prod(b+c) = 2s(s^2+r^2+2Rr)$ *and*

$$\sum bc\,(a+b)(a+c)(s-b)(s-c) = r^2[s^4 + s^2(32R^2+4Rr+2r^2) + r(4R+r)^3]$$

Let's get back to the main problem:

The left hand identity: Using Lemma the inequality can be written:

$$\frac{s^4+s^2(32R^2+4Rr+2r^2)+r(4R+r)^3}{4s^2(s^2+r^2+2Rr)}\geq\frac{3r}{R}\Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(R-12r)+2(16R^3+2R^2r-11Rr^2-6r^3)]+Rr(4R+r)^3\geq 0$$

We distinguish the following cases:

Case 1). If $[s^2(R-12r)+2(16R^3+2R^2r-11Rr^2-6r^3)]\geq 0$***, the inequality is obvious.***

Case 2). If $[s^2(R-12r)+2(16R^3+2R^2r-11Rr^2-6r^3)]<0$***, the inequality can be written:***

$$Rr(4R+r)^3\geq s^2[s^2(12r-R)-2(16R^3+2R^2r-11Rr^3-6r^3)]$$

which follows from Blundon-Gerretsen's inequality:

$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^3$***. It remains to prove that:***

$$Rr(4R+r)^3\geq\frac{R(4R+r)^2}{2(2Rr-r)}[(4R^2+4Rr+3r^2)(12r-R)-2(16R^3+2R^2r-11Rr^2-6r^3)]\Leftrightarrow$$

$$\Leftrightarrow 36R^3-24R^2r-71Rr^2-50r^3\geq 0\Leftrightarrow(R-2r)(36R^2+48Rr+25r^2)\geq 0$$

obviously from Euler's inequality $R\geq 2r$***.***

Equality holds if and only if ΔABC ***is equilateral.***

The right hand inequality: Using Lemma the inequality can be written:

$$\frac{s^4+s^2(32R^2+4Rr+2r^2)+r(4R+r^3)}{4s^2(s^2+r^2+2Rr)}\leq\frac{3R}{4r}\Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(3R-r)-r(26R^2+Rr+2r^2)]\geq Rr(4R+r)^3$$

which follows from Gerretsen's inequality $s^2\geq 16Rr-5r^2\geq\frac{r(4R+r)^2}{R+r}$

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(3R-r)-r(26R^2+Rr+2r^2)]\geq Rr(4R+r)^3\Leftrightarrow$$

$\Leftrightarrow 18R^2-37Rr+2r^2\geq 0\Leftrightarrow(R-2r)(18R-r)\geq 0$***, obviously from Euler's inequality*** $R\geq 2r$***. Equality holds if and only if*** ΔABC ***is equilateral.***

Remark.

Between the sums $\sum\frac{h_a}{r_b+r_c}$ ***and*** $\sum\frac{r_a}{h_b+h_c}$ ***the following relationship exists:***

7) In ΔABC:

$$\sum \frac{h_a}{r_b + r_c} \le \sum \frac{r_a}{h_b + h_c}$$

Solution.

Using the above Lemmas the inequality can be written:

$$\frac{s^4 + s^2(2r^2 - 4Rr) + r(4R+r)^3}{8R^2s^2} \le \frac{s^4 + s^2(32R^2 + 4Rr + 2r^2) + r(4R+r)^3}{4s^2(s^2 + r^2 + 2Rr)} \Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(2R^2 + 2Rr - 3r^2 - s^2) + (64R^4 - 56R^3r - 36R^2r^2 - 12Rr^3 - 3r^4)] +$$
$$+r(4R+r)^3(2R^2 - 2Rr - r^2) \ge 0$$

We distinguish the following cases:

Case 1). If $s^2\left[s^2\left(2R^2 + 2Rr - 3r^2 - s^2\right) + \left(64R^4 - 56R^3r - 36R^2r^2 - 12Rr^3 - 3r^4\right)\right] \ge 0$

the inequality is obvious.

Case 2). If $s^2\left[s^2\left(2R^2 + 2Rr - 3r^2 - s^2\right) + \left(64R^4 - 56R^3r - 36R^2r^2 - 12Rr^3 - 3r^4\right)\right] < 0$

the inequality can be written:

$$r(4R+r)^3(2R^2 - 2Rr - r^2) \ge$$
$$\ge s^2[s^2(s^2 + 3r^2 - 2Rr - 2R^2) - (64R^4 - 56R^3r - 36R^2r^2 - 12Rr^3 - 3r^4)]$$

which follows from Blundon-Gerretsen's inequality:

$s^2 \le \frac{R(4R+r)^2}{2(2R-r)} \le 4R^2 + 4Rr + 3r^2$***. It remains to prove that:***

$$r(4R+r)^3(2R^2 - 2Rr - r^2) \ge \frac{R(4R+r)^2}{2(2R-r)}$$
$$[(4R^2 + 4Rr + 3r^2)(4R^2 + 4Rr + 3r^2 + 3r^2 - 2Rr - 2R^2) - (64R^4 - 56R^3r - 36R^2r^2 - 12Rr^3 - 3r^4)]$$
$$\Leftrightarrow 56R^5 - 40R^4r - 114R^3r^2 - 54R^2r^3 - 13Rr^4 + 2r^5 \ge 0 \Leftrightarrow$$
$$\Leftrightarrow (R - 2r)(56R^4 + 72R^3r + 30R^2r^2 + 6Rr^3 - r^4) \ge 0$$

Obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if ΔABC ***is equilateral.***

Remark.

We can write the following inequalities:

8) In ΔABC:

$$\frac{3r}{R} \le \sum \frac{h_a}{r_b + r_c} \le \sum \frac{r_a}{h_b + h_c} \le \frac{3R}{4r}$$

Solution.

See inequalities 1), 7) and 5).

Equality holds if and only if ΔABC is equilateral.

UP.271 If $0 < a \le b < \frac{\pi}{4}$ then:

$$\int_a^b \int_a^b \int_a^b \left(\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right)\cos\left(\frac{\pi}{4} - z\right)\right) dxdydz \ge \sin^3(b+a)\cdot \sin^3(b-a)$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\int_a^b \int_a^b \int_a^b \left(\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right)\cos\left(\frac{\pi}{4} - z\right)\right) dxdydz$$

$$= \left(\int_a^b \cos\left(\frac{\pi}{4} - x\right) dx\right)\left(\int_a^b \cos\left(\frac{\pi}{4} - y\right) dy\right)\left(\int_a^b \cos\left(\frac{\pi}{4} - z\right) dz\right)$$

$$= \left(\int_a^b \cos\left(\frac{\pi}{4} - x\right) dx\right)^3 = \left(\sin\left(b - \frac{\pi}{4}\right) - \sin\left(a - \frac{\pi}{4}\right)\right)^3$$

$$= \left(2\cos\left(\frac{b+a}{2} - \frac{\pi}{4}\right)\sin\left(\frac{b-a}{2}\right)\right)^3$$

$$= \left(2\sqrt{2}\left(\cos\left(\frac{b+a}{2}\right) + \sin\left(\frac{b+a}{2}\right)\right)\sin\left(\frac{b-a}{2}\right)\right)^3$$

$$\overset{AM-GM}{\ge} \left(4\sqrt{2}\sqrt{\cos\left(\frac{b+a}{2}\right)\sin\left(\frac{b+a}{2}\right)}\cdot \sin\left(\frac{b-a}{2}\right)\right)^3$$

$$\geq\left(4\sqrt{\sin(a+b)}\cdot\sin\left(\frac{b-a}{2}\right)\right)^3\overset{(1)}{\geqq}\sin^3(a+b)\sin^3(b-a)$$

$$=\sin^3(a+b)\left(2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{b+a}{2}\right)\right)^3$$

$$(1)\Leftrightarrow 4^3\sin(a+b)\cdot\sqrt{\sin(a+b)}\cdot\sin^3\left(\frac{b-a}{2}\right)$$

$$\geq 8\cdot\sin^3(a+b)\cdot\sin^3\left(\frac{b-a}{2}\right)\cos^3\left(\frac{b+a}{2}\right)$$

$$\Leftrightarrow 8\cdot\sin(a+b)\cdot\sqrt{\sin(a+b)}\cdot\sin^3\left(\frac{b-a}{2}\right)\cdot\left(8-\sin(a+b)\cdot\sqrt{\sin(a+b)}\cdot\cos^3\left(\frac{b+a}{2}\right)\right)\geq 0\quad(2)$$

Because: $0<a\leq b<\frac{\pi}{4}\Rightarrow 0<a+b\leq\frac{\pi}{2};0\leq\frac{b-a}{4}<\frac{\pi}{4};0<\frac{a+b}{2}<\frac{\pi}{2}$

$$\Rightarrow 0<\sin(a+b);\cos(a+b)<1;\sin\left(\frac{b-a}{2}\right)\geq 0$$

$$\Rightarrow 8\sin(a+b)\cdot\sqrt{\sin(a+b)}\cdot\cos^3\left(\frac{b+a}{2}\right)<1<8$$

Hence (2) is true then (1) is true.

UP.272. Prove without softs:

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left(\tan(\sqrt[3]{xyz})\right)^3 dxdydz<\frac{\log^3 2}{8}$$

Proposed by Florentin Vişescu-Romania

Solution by Adrian Popa-Romania

$f(x)=\tan x\Rightarrow f'(x)=1+\tan^2 x>0;\ \forall x\in(0,\frac{\pi}{4})\Rightarrow f$ *–increasing*

$f''(x)=2\tan x(1+\tan^2 x)>0,\forall x\in(0,\frac{\pi}{4})\Rightarrow f$ *–concave.*

$$\tan(\sqrt[3]{xyz})\overset{Am-Gm}{\leqq}\tan\left(\frac{x+y+z}{3}\right)\overset{Jensen}{\leqq}\frac{\tan x+\tan y+\tan z}{3}$$

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left(tan(\sqrt[3]{xyz})\right)^3 dxdydz < \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left(\frac{tanx + tany + tanz}{3}\right)^3 dxdydz$$

$$< \left(3\int_0^{\frac{\pi}{4}}\frac{tanx}{3}dx\right)^3 = \left(\int_0^{\frac{\pi}{4}}tanxdx\right)^3 = \left(-log(cosx)|_0^{\frac{\pi}{2}}\right)^3 = \frac{log^3 2}{8}$$

UP.273. In acute ΔABC the following relationship holds:

$$tan(\sqrt{AB}) + tan(\sqrt{BC}) + tan(\sqrt{CA}) \le tanA + tanB + tanC$$

Proposed by Florentin Vişescu-Romania

Solution 1 by George Florin Şerban-Romania

$$f:\left(0,\frac{\pi}{2}\right) \to \mathbb{R}, f(x) = tanx, f'(x) = \frac{1}{cos^2x} > 0 \Rightarrow f \text{ –increasing.}$$

$$\sum_{cyc} tan(\sqrt{AB}) \le \sum_{cyc} tan\left(\frac{A+B}{2}\right) = \sum_{cyc} tan\left(\frac{\pi}{2}-\frac{C}{2}\right) = \sum_{cyc} cot\frac{C}{2} = \frac{s}{r} \overset{?}{\le} \sum_{cyc} tanA =$$

$$\prod_{cyc} tanA = \frac{2rs}{s^2-(2R+r)^2} \Rightarrow s^2-(2R+r)^2 \le 2r^2 \Leftrightarrow s^2 \le (2R+r)^2 + 2r^2 \Rightarrow s^2 \le$$

$$4R^2 + 4Rr + 3r^2 \text{ –true from Gerretsen inequality.}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

In acute ΔABC and $x \to tanx$ is increasing function, hence

$$\sqrt{AB} + \sqrt{BC} + \sqrt{CA} \le \frac{A+B}{2} + \frac{B+C}{2} + \frac{C+A}{2} \text{ then}$$

$$tan(\sqrt{AB}) + tan(\sqrt{BC}) + tan(\sqrt{CA}) \le tan\left(\frac{A+B}{2}\right) + tan\left(\frac{B+C}{2}\right) + tan\left(\frac{C+A}{2}\right) \le tanA + tanB + tanC$$

Remark: for $0 < x, y < \frac{\pi}{2}$, we have

$$tan\left(\frac{x}{2}+\frac{y}{2}\right) = \frac{tan\frac{x}{2} + tan\frac{y}{2}}{1 - tan\frac{x}{2}\cdot tan\frac{y}{2}} = \frac{\frac{sinx}{1+cosx} + \frac{siny}{1+cosy}}{1 - \frac{sinx}{(1+cosx)}\cdot\frac{siny}{(1+cosy)}} =$$

$$= \frac{sinx + siny + sinxcosy + sinycosx}{1 + cosx + cosy + cosxcosy - sinxsiny} \le \frac{1}{2}\left(\frac{sinx}{cosx} + \frac{siny}{cosy}\right); (*)$$

$$2(sinxcosxcosy + sinycosxcosy + sinxcosxcos^2y + sinycosycos^2x \le$$

$$\le sinxcosy + sinycosx + sinxcos^2y + sinycos^2x + sinxcosxcosy +$$

$$+sinycosxcosy + sinxcosxcos^2y + sinycosycos^2x -$$

$$-sin^2xsinycosy - sin^2ysinxcosx$$

Hence

$$sinxcosxcosy + sinycosxcosy + sinxcosxcos^2y + sinycosycos^2x +$$

$$+sin^2xsinycosy + sin^2ysinxcosx \le sinxcosy + sinycosx + sinxcos^2y + sinycos^2x$$

It's true, because $sinxcosxcos^2y + sin^2xsinycosy \le sinxcosx$

$$sinycosycos^2x + sin^2ysinxcosx \le sinycosx$$

And $0 < x, y < \frac{\pi}{2}, (sinxcosy - sinycosx)(cosy - cosx) \ge 0$

Hence

$$sinxcos^2y - sinxcosxcosy - sinycosxcosy + cos^2xsiny \ge 0$$

$$sinxcos^2y + sinycos^2x \ge sinxcosxcosy + sinycosxcosy$$

Therefore it's true.

Solution 3 by Marian Voinea-Romania

$$tan(\sqrt{AB}) + tan(\sqrt{BC}) + tan(\sqrt{CA}) \overset{Am-Gm}{\leq} tan\left(\frac{A+B}{2}\right) + tan\left(\frac{B+C}{2}\right) + tan\left(\frac{C+A}{2}\right)$$

$$\overset{tan-concave}{\leq} \frac{tanA + tanB}{2} + \frac{tanB + tanC}{2} + \frac{tanC + tanA}{2} = tanA + tanB + tanC$$

Equality for $A = B = C = \frac{\pi}{3}$

UP.274. $\omega_n = 1 - \frac{\binom{n}{1}}{3} + \frac{\binom{n}{2}}{5} - \cdots + \frac{(-1)^n\binom{n}{n}}{2n+1}, n \in \mathbb{N}$. **Find:**

$$\Omega = \lim_{n\to\infty}\left(1 + \frac{\sqrt[n]{\omega_n}}{n!}\right)^{\frac{n!}{e^n}}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$(1 - x^2)^n = \binom{n}{0} - \binom{n}{1}x^2 + \binom{n}{2}x^4 - \cdots + (-1)^n\binom{n}{n}x^{2n}$$

$$I_n = \int_0^1 (1-x^2)^n \cdot x' dx = (1-x^2)^n \cdot x \Big|_0^1 + 2n\int_0^1 (1-x^2)^{n-1} \cdot x^2 dx =$$

$$= -2n\int_0^1 (1-x^2-1)(1-x^2)^{n-1}dx = -2n\int_0^1 (1-x^2)^n dx + 2n\int_0^1 (1-x^2)^{n-1}dx =$$

$$= -2nI_n + 2nI_{n-1} \Rightarrow I_n = \frac{2^{2n}\cdot(n!)^2}{(2n+1)!}$$

$$\lim_{n\to\infty} \sqrt[n]{\omega_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^{2n}\cdot(n!)^2}{(2n+1)!}} \overset{C-D'Alembert}{=} \lim_{n\to\infty} \frac{2^{2(n+1)}((n+1)!)^2}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{2n}(n!)^2} = 1 \Rightarrow$$

$$\Omega = \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{\omega_n}}{n!}\right)^{\frac{n!}{e^n}} = e^{\lim_{n\to\infty}\frac{\sqrt[n]{\omega_n}}{n!}\cdot\frac{n!}{e^n}} = e^0 = 1$$

UP.275. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^8}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}(ijkl)\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

We first note that if $k \in \mathbb{N}$,

$$\sum_{r=1}^{n} r^k = \frac{1}{k+1}n^{k+1} + 0(n^k)$$

For $k = 1, \sum_{r=1}^{n} r = \frac{1}{2}n^2 + \frac{1}{2}n$

Assume

$$\sum_{r=1}^{n} r^k = \frac{1}{k+1}n^{k+1} + 0(n^k)$$

For all $k \in \mathbb{N}$ *with* $1 \le k \le m$ *where* $m \in \mathbb{N}, m \ge 1$

For $m + 1$*, we note*

$$(x+1)^{m+2}-x^{m+2}=(m+2)x^{m+1}+0(x^m)\Rightarrow\sum_{x=1}^{n}[(x+1)^{m+2}-x^{m+2}]$$

$\Rightarrow (n+1)^{m+2}-1=(m+2)\sum_{x=1}^{n}x^{m+1}+a$ ***polynomial of degree*** $(m+1)$ ***with rational coefficients.***

$$\Rightarrow\sum_{x=1}^{n}x^{m+1}=\frac{1}{m+2}(n+1)^{m+2}+a$$

polynomial of degree $(m+1)$ ***with rational coefficients. Now,***

$$\sum_{l=1}^{k}(l)=\frac{1}{2}k(k+1)\Rightarrow\sum_{k=1}^{j}k\sum_{l=1}^{k}l=\frac{1}{2}\sum_{k=1}^{j}(k^3+k^2)$$

$=\frac{1}{8}j^4+a$ ***polynomial of degree*** 3 ***in j***

$$\Rightarrow\sum_{j=1}^{i}j\sum_{k=1}^{j}k\sum_{l=1}^{k}l=\sum_{j=1}^{i}j\left(\frac{1}{8}j^4+a\ polynomial\ of\ degree\ 3\ in\ j\right)$$

$=\frac{1}{48}i^6+a$ ***polynomial of degree*** 5 ***in i***

$$\Rightarrow\sum_{i=1}^{n}i\sum_{j=1}^{i}j\sum_{k=1}^{i}k\sum_{l=1}^{k}l=\sum_{i=1}^{n}\left(\frac{1}{48}i^7+a\ polynomial\ of\ degree\ 6\ in\ i\right)$$

$=\frac{1}{384}n^8+$ ***a polynomial of degree*** 7 ***in*** n.

$$\therefore\lim_{n\to\infty}\frac{1}{n^8}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}(ijkl)$$

$$=\lim_{n\to\infty}\left(\frac{1}{384}+a\ polynomial\ of\ degree\ 7\ in\ \frac{1}{n}\right)=\frac{1}{384}$$

UP.276. Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}\left(\frac{1}{2^{i+j+k+l}}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Naren Bhandari-Bajura-Nepal

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}\frac{1}{2^{i+j+k+l}} = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{1}{2^{i+j}}\sum_{k=1}^{j}\sum_{l=1}^{k}\frac{1}{2^{l+k}}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{i=1}^{\infty}\left(\frac{1}{2^{i+j}}\sum_{l=1}^{j}\frac{1}{2^{l}}\left(\sum_{k=1}^{j}\frac{1}{2^{k}}-\sum_{k=1}^{l-1}\frac{1}{2^{k}}\right)\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{1}{2^{i+j}}\sum_{l=1}^{i}\frac{1}{2^{l}}\left(\frac{2}{2^{l}}-\frac{1}{2^{j}}\right)\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2^{i}}\left(\sum_{j=1}^{i}\sum_{l=1}^{j}\frac{2}{2^{j}4^{l}}-\sum_{j=1}^{i}\sum_{l=1}^{j}\frac{1}{2^{l}4^{j}}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2^{i}}\left(\sum_{l=1}^{i}\frac{2}{4^{l}}\left(\sum_{j=1}^{i}\frac{1}{2^{j}}-\sum_{j=1}^{l-1}\frac{1}{2^{j}}\right)-\left(\sum_{l=1}^{i}\frac{1}{2^{l}}\left(\sum_{l=1}^{i}\frac{1}{4^{j}}-\sum_{j=1}^{l-1}\frac{1}{4^{j}}\right)\right)\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2^{i}}\left(\sum_{l=1}^{i}\frac{2}{4^{l}}\left(1-\frac{1}{2^{i}}-1+\frac{1}{2^{l-1}}\right)-\left(\sum_{l=1}^{i}\frac{1}{2^{l}}\left(\frac{1}{3}-\frac{1}{3\cdot 4^{i}}-\frac{1}{3}+\frac{4}{3\cdot 4^{l-1}}\right)\right)\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2^{i}}\left(\sum_{l=1}^{i}\frac{4}{8^{l}}-\sum_{l=1}^{i}\frac{2}{2^{i}\cdot 4^{l}}-\sum_{l=1}^{i}\frac{16}{3\cdot 8^{l}}+\sum_{l=1}^{i}\frac{1}{3\cdot 2^{l}\cdot 4^{i}}\right)$$

$$= \lim_{n\to\infty}\left(\begin{array}{c}\displaystyle\sum_{l=1}^{n}\frac{4}{8^{l}}\left(\sum_{i=1}^{n}\frac{1}{2^{i}}-\sum_{i=1}^{l-1}\frac{1}{2^{i}}\right)-\sum_{l=1}^{n}\frac{2}{4^{l}}\left(\sum_{i=1}^{n}\frac{1}{4^{i}}-\sum_{i=1}^{l-1}\frac{1}{4^{i}}\right)- \\ \displaystyle -\sum_{l=1}^{n}\frac{16}{3\cdot 8^{l}}\left(\sum_{i=1}^{n}\frac{1}{2^{i}}-\sum_{i=1}^{l-1}\frac{1}{2^{i}}\right)+\sum_{l=1}^{n}\frac{1}{3\cdot 2^{l}}\left(\sum_{i=1}^{n}\frac{1}{8^{i}}-\sum_{i=1}^{l-1}\frac{1}{8^{i}}\right)\end{array}\right)$$

$$= \lim_{n\to\infty}\sum_{l=1}^{n}\left(\frac{16}{16^l}-\frac{8}{3\cdot 16^l}-\frac{32}{3\cdot 16^l}+\frac{8}{21\cdot 16^l}\right)=\frac{64}{21\times 15}=\frac{64}{315}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\sum_{l=1}^{k}\frac{1}{2^l}=1-\frac{1}{2^k}$$

$$\Rightarrow \sum_{k=1}^{j}\frac{1}{2^k}\sum_{l=1}^{k}\frac{1}{2^l}=\sum_{k=1}^{j}\left(\frac{1}{2^k}-\frac{1}{2^{2k}}\right)=1-\frac{1}{2^j}-\frac{1}{3}\left(1-\frac{1}{2^{2j}}\right)$$

$$=\frac{2}{3}-\frac{1}{2^j}+\frac{1}{3}\cdot\frac{1}{2^{2j}}\Rightarrow \sum_{j=1}^{i}\frac{1}{2^j}\sum_{k=1}^{j}\frac{1}{2^k}\sum_{l=1}^{k}\frac{1}{2^l}=\sum_{j=1}^{i}\frac{1}{2^j}\left(\frac{2}{3}-\frac{1}{2^j}+\frac{1}{3}\cdot\frac{1}{2^{2j}}\right)$$

$$=\frac{2}{3}\left(1-\frac{1}{2^i}\right)-\frac{1}{3}\left(1-\frac{1}{2^{2i}}\right)+\frac{1}{3}\left(\frac{1}{7}\right)\left(1-\frac{1}{2^{3i}}\right)$$

$$=\frac{8}{21}-\frac{2}{3}\cdot\frac{1}{2^i}+\frac{1}{3}\cdot\frac{1}{2^{2i}}-\frac{1}{21}\cdot\frac{1}{2^{3i}}\Rightarrow \sum_{i=1}^{n}\frac{1}{2^i}\sum_{j=1}^{i}\frac{1}{2^i}\sum_{k=1}^{j}\frac{1}{2^k}\sum_{l=1}^{k}\frac{1}{2^l}$$

$$=\frac{8}{21}\left(1-\frac{1}{2^n}\right)-\frac{2}{3}\cdot\frac{1}{3}\left(1-\frac{1}{2^{2n}}\right)+\frac{1}{3}\left(\frac{1}{7}\right)\left(1-\frac{1}{2^{3n}}\right)-\frac{1}{21}\cdot\frac{1}{15}\left(1-\frac{1}{2^{4n}}\right)$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{i=1}^{k}\frac{1}{2^i 2^j 2^k 2^l}=\frac{8}{21}-\frac{2}{9}+\frac{1}{21}-\frac{1}{315}=\frac{64}{315}$$

Solution 3 by Kartick Chandra Betal-India

$$\Omega=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}\frac{1}{2^{i+j+k+l}}=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\frac{1}{2^{i+j+k}}\cdot\frac{1}{2}\left(\frac{1-\frac{1}{2^k}}{\frac{1}{2}}\right)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{2^{i+j}}\sum_{k=1}^{j}\left\{\frac{1}{2^k}-\frac{1}{2^{2k}}\right\}=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{2^{i+j}}\left\{\left(1-\frac{1}{2^j}\right)-\frac{\frac{1}{4}}{\frac{3}{4}}\left(1-\frac{1}{4^j}\right)\right\}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2^i}\sum_{j=1}^{i}\left\{\frac{2}{3}\cdot\frac{1}{2^j}-\frac{1}{2^{2j}}+\frac{1}{3\cdot 2^{3j}}\right\}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2^i}\left\{\frac{2}{3}\left(1-\frac{1}{2^i}\right)-\frac{1}{3}\left(1-\frac{1}{4^i}\right)+\frac{1}{3}\cdot\frac{1}{7}\left(1-\frac{1}{8^i}\right)\right\}$$

$$=\sum_{i=1}^{\infty}\left\{\left(\frac{2}{3}-\frac{1}{3}+\frac{1}{21}\right)\frac{1}{2^i}-\frac{2}{3}\cdot\frac{1}{2^{2i}}+\frac{1}{3}\cdot\frac{1}{2^{3i}}-\frac{1}{21}\cdot\frac{1}{2^{4i}}\right\}$$

$$=\frac{8}{21}\cdot 1-\frac{2}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{7}-\frac{1}{21}\cdot\frac{1}{15}$$

$$=\frac{8}{21}-\frac{2}{9}+\frac{1}{21}-\frac{1}{315}=\frac{120-70+15-1}{3\cdot 7\cdot 5\cdot 3}=\frac{135-71}{315}=\frac{64}{315}$$

UP.281. If $(a_n)_{n\geq 1}\subset(0,\infty);\ \lim_{n\to\infty}\left(\frac{a_{n+1}}{n^2\cdot a_n}\right)=a>0;\ x_1=a_1;$

$$x_2=a_1\cdot\sqrt{a_2};\ x_3=a_1\cdot\sqrt{a_2}\cdot\sqrt[3]{a_3};\ x_n=a_1\cdot\sqrt{a_2}\cdot\sqrt[3]{a_3}\cdot\ldots\cdot\sqrt[n]{a_n}$$

then find:

$$\Omega=\lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[n+1]{x_{n+1}}}-\frac{n^3}{\sqrt[n]{x_n}}\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

If $(y_n)_{n\geq 1}$ ***–is sequence of real numbers such that:***

i) $\lim_{n\to\infty}\frac{y_{n+1}}{y_n}=1$

ii) $\lim_{n\to\infty}\frac{y_n}{n}=\alpha\in(0,\infty)$

iii) $\lim_{n\to\infty}\left(\frac{y_{n+1}}{y_n}\right)^n=\beta\in(0,\infty)$ ***then***

$$\lim_{n\to\infty}(y_{n+1}-y_n)=\alpha\log\beta$$

Let $y_n=\frac{n^3}{\sqrt[n]{x_n}}$

$$\lim_{n\to\infty}\frac{y_n}{n}=\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{x_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{x_n}}=\lim_{n\to\infty}\frac{(n+1)^{2n+2}}{x_{n+1}}\cdot\frac{x_n}{n^{2n}}$$

$$=\lim_{n\to\infty}\frac{(n+1)^{2n}}{n^{2n}}\cdot\frac{(n+1)^2x_n}{x_{n+1}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}\cdot\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}$$

$$= e^2 \cdot \lim_{n\to\infty} \sqrt[n+1]{\frac{(n+1)^{2n+2}}{a_{n+1}}} = e^2 \cdot \lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{a_n}} = e^2 \cdot \lim_{n\to\infty} \frac{(n+1)^{2n+2}}{a_{n+1}} \cdot \frac{a_n}{n^{2n}}$$

$$= e^2 \cdot \lim_{n\to\infty} \frac{(n+1)^{2n+2}}{n^{2n}} \cdot \underbrace{\frac{(n+1)^2}{n^2}}_{\to 1} \cdot \underbrace{\frac{n^2 \cdot a_n}{a_{n+1}}}_{\to \frac{1}{a}} = e^2 \cdot e^2 \cdot \frac{1}{a} = \frac{e^4}{a} > 0$$

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{(n+1)^3}{\sqrt[n+1]{x_{n+1}}} \cdot \frac{\sqrt[n]{x_n}}{n^3} = \lim_{n\to\infty} \frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} \cdot \frac{\sqrt[n]{x_n}}{n^2} \cdot \frac{n+1}{n} = 1$$

$$\lim_{n\to\infty} \left(\frac{y_{n+1}}{y_n}\right)^n = \lim_{n\to\infty} \left(\frac{\frac{(n+1)^3}{\sqrt[n+1]{x_{n+1}}}}{\frac{n^3}{\sqrt[n]{x_n}}}\right)^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{3n} \cdot \frac{x_n}{x_{n+1}^{\frac{n}{n+1}}} = e^3 \cdot \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \cdot x_{n+1}^{\frac{1}{n+1}}$$

$$= e^3 \cdot \lim_{n\to\infty} \sqrt[n+1]{\frac{x_{n+1}}{a_{n+1}}} = e^3 \cdot \lim_{n\to\infty} \sqrt[n]{\frac{x_n}{a_n}} = e^3 \cdot \lim_{n\to\infty} \frac{\sqrt[n]{x_n}}{n^2} \cdot \frac{n^2}{\sqrt[n]{a_n}} = e > 0$$

So, $\Omega = \frac{e^4}{a} \cdot \log e = \frac{e^4}{a}$

Solution 2 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty} \frac{n^3}{\sqrt[n]{x_n}} \left(\frac{(n+1)^3}{\sqrt[n+1]{x_{n+1}}} \cdot \frac{\sqrt[n]{x_n}}{n^3} - 1\right) = \lim_{n\to\infty} \frac{n^2}{\sqrt[n]{x_n}} \cdot n \left(\left(\frac{n+1}{n}\right)^3 \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}} - 1\right); \quad (1)$$

$$\lim_{n\to\infty} \frac{n^2}{\sqrt[n]{x_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{x_n}} \overset{C.D'A}{=} \lim_{n\to\infty} \frac{(n+1)^{2n+2}}{x_{n+1}} \cdot \frac{x_n}{n^{2n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2n} \cdot \frac{x_n}{x_{n+1}} (n+1)^2$$

$$= e^2 \cdot \lim_{n\to\infty} \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} = e^2 \cdot \lim_{n\to\infty} \frac{n^2}{\sqrt[n]{a_n}} = e^2 \cdot \lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{a_n}} \overset{C.D'A}{=} e^2 \cdot \lim_{n\to\infty} \frac{(n+1)^{2n+2}}{a_{n+1}} \cdot \frac{a_n}{n^{2n}}$$

$$= e^2 \cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2n} \cdot \frac{(n+1)^2 a_n}{a_{n+1}} = e^2 \cdot e^2 \cdot \frac{1}{a} = \frac{e^4}{a}; \quad (2)$$

$$\lim_{n\to\infty} n \left(\left(\frac{n+1}{n}\right)^3 \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}} - 1\right) = \lim_{n\to\infty} n \cdot \left(e^{\log\left(\left(\frac{n+1}{n}\right)^3 \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)} - 1\right)$$

$$= \lim_{n\to\infty} \frac{n \cdot \left(e^{log\left(\left(\frac{n+1}{n}\right)^3 \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)} - 1\right)}{e^{log\left(\left(\frac{n+1}{n}\right)^3 \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)}} \cdot e^{log\left(\left(\frac{n+1}{n}\right)^3 \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{3n} \frac{x_n}{\sqrt[n+1]{x_{n+1}^n}} = log\left(\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{3n} \cdot \frac{x_n}{x_{n+1}} \cdot \sqrt[n+1]{x_{n+1}}\right)\right)$$

$$= log\left(\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{3n} \cdot \frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n+1]{a_{n+1}}}\right)\right) = log\left(\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{3n} \cdot \frac{\sqrt[n]{x_n}}{\sqrt[n]{a_n}}\right)\right)$$

$$= log\left(\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{3n} \cdot \frac{\sqrt[n]{x_n}}{n^2} \cdot \frac{n^2}{\sqrt[n]{a_n}}\right)\right) = log\left(e^3 \cdot \frac{e^2}{a} \cdot \frac{a}{e^4}\right) = loge = 1; \quad (3)$$

From (1),(2),(3) we have: $\Omega = \frac{e^4}{a} \cdot loge = \frac{e^4}{a}$

UP.279. Let $a \in \mathbb{R}_+^*, f, \Gamma: \mathbb{R}_+^* \to \mathbb{R}_+^*, \lim_{x\to\infty} \frac{f(x+1)x^a}{f(x)} = b \in \mathbb{R}_+^*$ then exists

$\lim_{x\to\infty}(f(x))^{\frac{1}{x}} \cdot x^a$ and find

$$\lim_{x\to\infty}\left(\left((\Gamma(x+2))^{\frac{a}{x+1}} - (\Gamma(x+1))^{\frac{a}{x}}\right) \cdot x(f(x))^{\frac{1}{x}}\right)$$

Proposed by D.M.Bătineţu Giurgiu, Neculai Stanciu-Romania

Solution by proposers

$$\lim_{x\to\infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{n\to\infty} \frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

$$\lim_{x\to\infty}(f(x))^{\frac{1}{x}} \cdot x^a = \lim_{n\to\infty}\left(\sqrt[n]{f(n)} \cdot n^a\right)$$

$$= \lim_{n\to\infty} \sqrt[n]{f(n) \cdot n^{na}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{f(n+1)(n+1)^{(n+1)a}}{f(n)n^{na}} =$$

$$\lim_{n\to\infty}\frac{f(n+1)n^a}{f(n)}\cdot\left(\frac{n+1}{n}\right)^{a(n+1)}=b\cdot e^a$$

Let $u(x)=\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^a$ *then* $\lim_{x\to\infty}u(x)=1$ *and* $\lim_{x\to\infty}\frac{u(x)-1}{logu(x)}=1$

$$\lim_{x\to\infty}(u(x))^x=\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^a=\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^a=e^a$$

$$\beta(x)=\left((\Gamma(x+2))^{\frac{a}{x+1}}-(\Gamma(x+1))^{\frac{a}{x}}\right)\cdot x(f(x))^{\frac{1}{x}}$$

$$=(\Gamma(x+1))^{\frac{a}{x}}\cdot(u(x)-1)\cdot x^{1-a}\cdot(f(x))^{\frac{1}{x}}\cdot x^a$$

$$=\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\right)^a\cdot\frac{u(x)-1}{logu(x)}\cdot(f(x))^{\frac{1}{x}}\cdot x^a\cdot log(u(x))^x$$

$$\lim_{x\to\infty}\beta(x)=\left(\frac{1}{e}\right)^a\cdot 1\cdot b\cdot e^a\cdot loge^a=ab$$

UP.284 Let a,b,c be the lengths of the sides of a triangle ABC with iradius r, circumradius R and area F. Prove that:

$$\frac{F}{12R^2(R-r)}\le\sum_{cyc}\frac{ab}{(2a^2+b^2+c^2)(b+c)}\le\frac{\sqrt{3}}{16r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

For the right inequality, we have:

$$2a^2+b^2+c^2=(a^2+b^2)+(a^2+c^2)\ge\frac{(a+b)^2}{2}+\frac{(a+c)^2}{2}\ge(a+b)(b+c)$$

So, $(2a^2+b^2+c^2)(b+c)\ge(a+b)(b+c)(c+a)\ge 8abc$.

Now $\sum_{cyc}\frac{ab}{(2a^2+b^2+c^2)(b+c)}\le\frac{1}{8}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

We know that: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r}$ ***and*** $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\le 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$, **so**

$$\sum_{cyc}\frac{ab}{(2a^2+b^2+c^2)(b+c)}\le\frac{1}{8}\cdot\sqrt{3}\cdot\sqrt{\frac{1}{4r^2}}=\frac{\sqrt{3}}{16r}$$

For the left inequality, we have

$$\sum_{cyc}\frac{ab}{(2a^2+b^2+c^2)(b+c)}=\sum_{cyc}abc\cdot\frac{1}{(2a^2+b^2+c^2)(bc+c^2)}$$

Now, using the Cauchy-Schwartz inequality, we get

$$\sum_{cyc}\frac{ab}{(2a^2+b^2+c^2)(b+c)}=\sum_{cyc}abc\cdot\frac{1}{(2a^2+b^2+c^2)(bc+c^2)}$$

$$\ge abc\cdot\frac{(1+1+1)^2}{2abc(a+b+c)+3(a^2b^2+b^2c^2+c^2a^2)+ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)+a^4+b^4+c^4}$$

$$\ge\frac{9abc}{2abc(a+b+c)+3(a^4+b^4+c^4)+\left(ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\right)+a^4+b^4+c^4}$$

Now, we will prove that

$$2abc\le ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\le 2(a^4+b^4+c^4)$$

We have:

$$2(a^4+b^4+c^4)-ab(a^2+b^2)-bc(b^2+c^2)-ca(c^2+a^2)=$$

$$=(a^2+b^2)^2-ab(a^2+b^2)-2a^2b^2+(b^2+c^2)^2+bc(b^2+c^2)-2b^2c^2+$$

$$+(c^2+a^2)^2+ca(c^2+a^2)-2c^2a^2=$$

$$=(a^2+b^2-2ab)(a^2+b^2+ab)+(b^2+c^2-2bc)(b^2+c^2+bc)+$$

$$+(c^2+a^2-2ca)(c^2+a^2+ca)=(a-b)^2(a^2+b^2+ab)+$$

$$+(b-c)^2(b^2+c^2+bc)+(c-a)^2(c^2+a^2+ca)\ge 0$$

Also, we have: $a^2+b^2\ge 2ab;\ b^2+c^2\ge 2bc;\ \ c^2+a^2\ge 2ca$***, so***

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\ge 2a^2b^2+2b^2c^2+2c^2a^2.$$

Now, we have $a^2b^2+b^2c^2\ge 2ab^2c;\ b^2c^2+c^2a^2\ge 2abc^2;\ c^2a^2+a^2b^2\ge 2a^2bc$

So, $2a^2b^2+2b^2c^2+2c^2a^2\ge 2a^2bc+2ab^2c+2abc^2=2abc(a+b+c)$

We have:

$$\sum_{cyc}\frac{ab}{(2a^2+b^2+c^2)(b+c)}\ge\frac{9abc}{2(a^4+b^4+c^4)+3(a^4+b^4+c^4)+2(a^4+b^4+c^4)+a^4+b^4+c^4}=\frac{9abc}{8(a^4+b^4+c^4)}$$

Now, we will prove that: $a^4+b^4+c^4\le 54R^3(R-r)$

It is well known that: $a^4+b^4+c^4=2(a^2b^2+2b^2c^2+2c^2a^2)-16F^2$

So, $a^4+b^4+c^4=2\left((ab+bc+ca)^2-2abc(a+b+c)\right)-16F^2$.

Now, $ab+bc+ca=s^2+r^2+4Rr$ so that, with a short calculation,

$$a^4+b^4+c^4=2(s^4-2rs^2(4R+3r)+r^2(4R+r)^2)$$

and the inequality becomes:

$$s^4-2r(4R+3r)s^2+r^2(4R+r)^2-27R^3(R-r)\le 0$$

The left hand side is a quadratic in s^2 which writes as $(s^2-\alpha)(s^2-\beta)$ with

$\alpha=r(4R+3r)-\sqrt{\delta}, \beta=r(4R+3r)+\sqrt{\delta}$, the number δ being

$$8r^3(2R+r)+27R^3(R-r)$$

The inequality $\alpha\le s^2$ follows from Gerretsen's inequality $16Rr-5r^2\le s^2$ since

$$\alpha\le 3r^2+4Rr\le 16Rr-5r^2\le s^2.$$

As for the inequality $s^2\le\beta$, using Gerretsen's second inequality

$s^2\le 4R^2+3r^2+4Rr$, we see that it is sufficient to prove $4R^2\le\sqrt{\delta}$ or

$$8r^4+16Rr^3-27R^3r+11R^4\ge 0$$

But, setting $x=\frac{R}{2r}\ge 1$ (Euler) this rewrites as $22x^4-27x^3+4x+1\ge 0$, that is

$$(x-1)\left(11x^3+(x-1)(11x^2+6x+1)\right)\ge 0$$

So, the later inequality holds and we are done.

Equality holds when the triangle is equilateral.

So, $\sum_{cyc}\frac{ab}{(2a^2+b^2+c^2)(b+c)}\ge\frac{9abc}{8\cdot 54R^3(R-r)}$ and we know that $abc=4RF$ then

$$\sum_{cyc}\frac{ab}{(2a^2+b^2+c^2)(b+c)}\ge\frac{F}{12R^2(R-r)}$$

UP.277. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\sum_{1\le i<j<k\le n}\left(\frac{1}{\sqrt[3]{(ijk)^2}}\right)}{e^{H_n}}\right)$$

Proposed by Marian Ursărescu-Romania

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\lim_{n\to\infty}\left(\frac{\sum_{1\le i<j<k\le n}\left(\frac{1}{\sqrt[3]{(ijk)^2}}\right)}{e^{H_n}}\right)=\lim_{n\to\infty}\frac{\frac{1}{n^2}\cdot\sum_{1\le i<j<k\le n}\left(\frac{1}{\sqrt[3]{\left(\frac{ijk}{n^3}\right)^2}}\right)}{ne^{H_n-\log n}}$$

$$=e^{-\gamma}\lim_{n\to\infty}\frac{1}{n^2}\cdot\sum_{1\le i<j<k\le n}\left(\frac{1}{\sqrt[3]{\left(\frac{ijk}{n^3}\right)^2}}\right)=e^{-\gamma}\int_0^1\int_x^1\int_y^1\frac{1}{\sqrt[3]{(xyz)^2}}dxdydz$$

$$=e^{-\gamma}\int_0^1\int_x^1\frac{3(1-y)}{\sqrt[3]{(xy)^2}}dydx=e^{-\gamma}\int_0^1\int_x^1\frac{\frac{9}{2}\left(\sqrt[3]{x^2}-18\sqrt[3]{x}+1\right)}{\sqrt[3]{x^2}}dydx=\frac{9}{2}e^{-\gamma}$$

UP.282. If $m,p,r,s,t\ge 0$; $(a_n)_{n\ge1}$; $(b_n)_{n\ge1}$; $(c_n)_{n\ge1}\subset(0,\infty)$;

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n\cdot n^r}\right)=a>0;\ \lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n\cdot n^s}\right)=b>0;\ \lim_{n\to\infty}\left(\frac{c_{n+1}-c_n}{n^t}\right)=c>0$$

then:

$$\lim_{n\to\infty}\left(\frac{c_{n+1}\cdot\sqrt[n+1]{a_{n+1}^m\cdot b_{n+1}^p}}{(n+1)^{mr+ps+t}}-\frac{c_n\cdot\sqrt[n]{a_n^m\cdot b_n^p}}{n^{mr+ps+t}}\right)=\frac{a^m\cdot b^p\cdot c}{(t+1)\cdot e^{mr+ps}}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\frac{c_n}{n^{t+1}}\overset{L.C-S}{=}\lim_{n\to\infty}\frac{c_{n+1}-c_n}{(n+1)^{t+1}-n^{t+1}}=\lim_{n\to\infty}\left(\frac{c_{n+1}-c_n}{n^t}\cdot\frac{1}{\frac{\left(1+\frac{1}{n}\right)^{t+1}-1}{\frac{1}{n}}}\right)=\frac{c}{t+1}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^r}\overset{C.D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{nr}}=\frac{a}{e^r}$$

$$\lim_{n\to\infty}\left(\frac{n(c_{n+1}-c_n)}{c_n}\right)=\lim_{n\to\infty}\left(\frac{c_{n+1}-c_n}{n^t}\cdot\frac{n^{t+1}}{c_n}\right)\overset{L.C-S}{=}c\lim_{n\to\infty}\frac{n^t}{c_{n+1}-c_n}\cdot\frac{\left(1+\frac{1}{n}\right)^{t+1}-1}{\frac{1}{n}}=t+1$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^s}\overset{C.D'A}{=}\lim_{n\to\infty}\frac{b_{n+1}}{n^s\cdot b_n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{ns}}=\frac{b}{e^s}$$

Let: $u_n=\dfrac{\frac{c_{n+1}\cdot\sqrt[n+1]{a_{n+1}^m\cdot b_{n+1}^p}}{(n+1)^{mr+ps+t}}}{\frac{c_n\cdot\sqrt[n]{a_n^m\cdot b_n^p}}{n^{mr+ps+t}}}$ ***for all*** $n\in\mathbb{N}$

$\therefore \lim_{n\to\infty}u_n=1$ ***hence for all*** $n\to\infty,\frac{u_n-1}{log(u_n)}\to 1$

$$\therefore\lim_{n\to\infty}u_n^n=$$

$$=\lim_{n\to\infty}\left[\left(\frac{c_{n+1}-c_n}{c_n}+1\right)^{\frac{c_n}{c_{n+1}-c_n}\cdot\frac{n(c_{n+1}-c_n)}{c_n}}\cdot\left(\frac{a_{n+1}}{a_n\cdot n^r}\right)^m\cdot\left(\frac{b_{n+1}}{b_n\cdot n^s}\right)^p\cdot\frac{n^{mr+sp}}{\sqrt[n+1]{a_{n+1}^m b_{n+1}^p}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{nt}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n(mr+sp)}}\right]=$$

$$=e^{t+1}\cdot a^m\cdot b^p\cdot\left(\frac{e^r}{a}\right)^m\cdot\left(\frac{e^s}{b}\right)^p\cdot\frac{1}{e^{t+mr+sp}}=e$$

$$\lim_{n\to\infty}\left(\frac{c_{n+1}\cdot\sqrt[n+1]{a_{n+1}^m\cdot b_{n+1}^p}}{(n+1)^{mr+ps+t}}-\frac{c_n\cdot\sqrt[n]{a_n^m\cdot b_n^p}}{n^{mr+ps+t}}\right)=$$

$$=\lim_{n\to\infty}\frac{c_n\cdot\sqrt[n]{a_n^m\cdot b_n^p}}{n^{mr+ps+t}}\cdot\frac{u_n-1}{log(u_n)}\cdot log(u_n^n)=\frac{a^m\cdot b^p\cdot c}{(t+1)\cdot e^{mr+ps}}$$

UP.280. Let a, b, c be sides in ΔABC, $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$, $(z_n)_{n\geq 1}$ sequences of positive numbers such that:

$$\lim_{n\to\infty}(x_{n+1}-x_n)=c,\ \lim_{n\to\infty}\frac{y_{n+1}}{ny_n}=b,\ \lim_{n\to\infty}\frac{z_{n+1}}{nz_n}=c.$$

Prove that:

$$\lim_{n\to\infty}\left(\frac{x_n\cdot\sqrt[n]{y_n}+e\cdot\sqrt[n]{y_nz_n}+x_n\cdot\sqrt[n]{z_n}}{n^2}\right)\geq\frac{4\sqrt{3}F}{e}$$

Proposed by D.M.Bătinețu-Giurgiu and Neculai Stanciu-Romania

Solution by Marian Ursărescu-Romania

$$\lim_{n\to\infty}\left(\frac{x_n\cdot\sqrt[n]{y_n}}{n^2}\right)=\lim_{n\to\infty}\left(\frac{x_n}{n}\cdot\frac{\sqrt[n]{y_n}}{n}\right)\quad(1)$$

$$\lim_{n\to\infty}\frac{x_n}{n}\overset{LC-S}{=}\lim_{n\to\infty}\frac{x_{n+1}-x_n}{n+1-n}=a\quad(2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{y_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{y_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{y_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{y_n}=\lim_{n\to\infty}\frac{y_{n+1}}{ny_n}\cdot\left(\frac{n}{n+1}\right)^n\cdot\frac{n}{n+1}=\frac{b}{e}\quad(3)$$

From (1)+(2)+(3) we have: $\lim_{n\to\infty}\left(\frac{x_n\cdot\sqrt[n]{y_n}}{n^2}\right)=\frac{ab}{e}\quad(4)$

Similarly: $\lim_{n\to\infty}\left(\frac{x_n\cdot\sqrt[n]{z_n}}{n^2}\right)=\frac{ac}{e}\quad(5)$

$$\lim_{n\to\infty}\left(\frac{e\cdot\sqrt[n]{y_nz_n}}{n^2}\right)=\lim_{n\to\infty}\sqrt[n]{\frac{y_n}{n^n}}\cdot\sqrt[n]{\frac{z_n}{n^n}}=\frac{bc}{e}\quad(6)$$

From (4)+(5)+(6) we must show: $\frac{ab+bc+ca}{e}\geq\frac{4\sqrt{3}F}{e}\Leftrightarrow ab+bc+ca\geq 4\sqrt{3}F$,

true it's Gordon inequality.

UP.278 If $a, b\in\mathbb{R}$ then:

$$\int_a^b\int_a^b(\cos x\cos y\cos(x+y))dx\,dy+\frac{1}{8}(b-a)^2\geq 0$$

Proposed by Daniel Sitaru – Romania

Solution by proposer:

$$\textit{We start from: }\left(\cos x+\frac{1}{2}(\cos(x+2y))\right)^2+\frac{1}{4}\sin^2(x+2y)\geq 0$$

$$\cos^2 x+\cos x\cos(x+2y)+\frac{1}{4}\cos^2(x+2y)+\frac{1}{4}\sin^2(x+2y)\geq 0$$

$$\cos^2 x+\cos x\cos(x+2y)\geq -\frac{1}{4}$$

$$2\cos^2 x+2\cos x\cos(x+2y)\geq -\frac{1}{2}$$

$$2\cos^2 x+\cos 2y+\cos(2x+2y)\geq -\frac{1}{2}$$

$$2\cos^2 x-1+\cos^2 y+\cos(2x+2y)\geq -\frac{3}{2}$$

$$\cos 2x+\cos 2y+\cos(2x+2y)\geq -\frac{3}{2}$$

$$2\cos(x+y)\cos(x-y)+2\cos^2(x+y)-1\geq -\frac{3}{2}$$

$$2\cos(x+y)\left(\cos(x-y)+\cos(x+y)\right)\geq -\frac{1}{2}$$

$$\cos(x+y)\cdot 2\cos x\cos y\geq -\frac{1}{4}$$

$$8\cos x\cos y\cos(x+y)+1\geq 0$$

$$\cos x\cos y\cos(x+y)+\frac{1}{8}\geq 0$$

$$\int_a^b\int_a^b \cos x\cos y\cos(x+y)\,dx\,dy+\frac{1}{8}(b-a)^2\geq 0$$

UP.283.RMM WINTER EDITION 2020

By Marin Chirciu – Romania

1) In ΔABC:

$$\frac{r}{4R^4}\leq\frac{h_a}{a^2(b+c)^2}+\frac{h_b}{b^2(c+a)^2}+\frac{h_c}{c^2(a+b)^2}\leq\frac{1}{64r^3}$$

Proposed by George Apostolopoulos-Messolonghi– Greece

Solution

The left hand inequality: Using the means inequality we obtain:

$$\sum\frac{h_a}{a^2(b+c)^2}\ge 3\sqrt[3]{\prod\frac{h_a}{a^2(b+c)^2}}=3\sqrt[3]{\frac{\frac{2r^2s^2}{R}}{16R^2r^2s^2\cdot 4s^2(s^2+r^2+2Rr)^2}}=$$

$$=\frac{3}{\sqrt[3]{32R^3\cdot s^2(s^2+r^2+2Rr)^2}}\overset{(1)}{\ge}\frac{r}{4R^4},\ \textit{where (1)}\Leftrightarrow\frac{27}{32R^3\cdot s^2(s^2+r^2+2Rr)^2}\ge\frac{r^3}{64R^{12}}\Leftrightarrow$$

$\Leftrightarrow 54R^9\ge r^3s^2(s^2+r^2+2Rr)^2$, ***which follows from Mitrinovic's inequality*** $s^2\le\frac{27R^2}{4}$ ***and Gerretsen*** $s^2\le 4R^2+4Rr+3r^2$. ***It remains to prove that:***

$$54R^9\ge r^3\frac{27R^2}{4}(4R^2+4Rr+3r^2+r^2+2Rr)^2\Leftrightarrow 2R^7\ge r^3(2R^2+3Rr+2r^2)^2\Leftrightarrow$$

$$\Leftrightarrow 2R^7-4R^4r^3-12R^3r^4-17R^2r^5-12Rr^6-4r^7\ge 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)\left(2R^6+4R^5r+8R^4r^2+12R^3r^3+12R^2r^4+7Rr^5+2r^6\right)\ge 0$$

obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if ΔABC ***is equilateral.***

The right hand inequality:

We have: $\sum\frac{h_a}{a^2(b+c)^2}\le\sum\frac{h_a}{a^2\cdot 4bc}\le\frac{1}{4abc}\sum\frac{h_a}{a}=\frac{1}{4\cdot 4Rrs}\cdot\frac{s^4+s^2(2r^2-4Rr)+r^2(4Rr+r)^2}{8R^2rs}=$

$$=\frac{s^4+s^2(2r^2-4Rr)+r^2(4Rr+r)^2}{128R^3r^2s^2}\overset{(2)}{\le}\frac{1}{64r^3}$$

where (2) $\Leftrightarrow s^4+s^2(2r^2-4Rr)+r^2(4R+r)^2\le 2R^3s^2$, ***which follows from Gerretsen's inequality*** $s^2\ge 16Rr-5r^2$. ***It remains to prove that:***

$$s^4+s^2(2r^2-4Rr)+r^2(4R+r)^2\le 2R^3(16Rr-5r^2)$$

$\Leftrightarrow s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2\le 2R^3(16-5r)$, ***true from Gerretsen's inequality*** $s^2\le 4R^4+4Rr+3r^2$. ***It suffices to prove that:***

$$\left(4R^4+4Rr+3r^2\right)\left(4R^4+4Rr+3r^2+2r^2-8Rr\right)+r^2(4R+r)^2\le 2R^3(16R-5r)\Leftrightarrow$$

$$\Leftrightarrow 8R^4-5R^3r-16R^2r^2-8Rr^3-8r^4\ge 0\Leftrightarrow (R-2r)\left(8R^3+11R^2r+6Rr^2+4r^4\right)\ge 0$$

obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if ΔABC ***is equilateral.***

Remark.

If we change h_a with r_a we propose:

2) In ΔABC:

$$\frac{r}{4R^4}\le\frac{r_a}{a^2(b+c)^2}+\frac{r_b}{b^2(c+a)^2}+\frac{r_c}{c^2(a+b)^2}\le\frac{1}{64r^3}$$

Marin Chirciu – Romania

Solution.

The left hand inequality:

Using means inequality we obtain:

$$\sum\frac{r_a}{a^2(b+c)^2}\ge 3\sqrt[3]{\prod\frac{r_a}{a^2(b+c)^2}}=3\sqrt[3]{\frac{rs^2}{16R^2r^2s^2\cdot 4s^2(s^2+r^2+2Rr)^2}}=$$

$$=\frac{3}{\sqrt[3]{64R^2r\cdot s^2\left(s^2+r^2+2Rr\right)^2}}\overset{(1)}{\ge}\frac{r}{4R^4},\text{ where (1)}\Leftrightarrow\frac{27}{64R^2r\cdot s^2\left(s^2+r^2+2Rr\right)^2}\ge\frac{r^3}{64R^{12}}\Leftrightarrow$$

$\Leftrightarrow 27R^{10}\ge r^4s^2(s^2+r^2+2Rr)^2$, *which follows from Mitrinovic's inequality*

$s^2\le\frac{27R^2}{4}$ *and Gerretsen* $s^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$27R^{10}\ge r^4\cdot\frac{27R^2}{4}(s^2+r^2+2Rr)^2\Leftrightarrow R^8\ge r^4(2R^2+3Rr+2r^2)^2\Leftrightarrow$$

$$\Leftrightarrow R^8-4R^4r^4-12R^3r^5-17R^2r^6-12Rr^7-4r^8\ge 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)\left(R^7+2R^6r+4R^5r^2+8R^4r^3+12R^3r^4+12R^2r^5+7Rr^6+2r^7\right)\ge 0$$

obviuosly from Euler's inequality $R\ge 2r$.

Equality holds if and only if ΔABC is equilateral.

The right hand inequality

We have $\sum\frac{r_a}{a^2(b+c)^2}\le\sum\frac{r_a}{a^2\cdot 4bc}\le\frac{1}{4abc}\sum\frac{r_a}{a}=\frac{1}{4\cdot 4Rrs}\cdot\frac{s^2+(4R+r)^2}{4Rs}=$

$$=\frac{s^2+(4R+r)^2}{64R^2rs^2}\overset{(2)}{\le}\frac{1}{64r^3}$$

where (2) $\Leftrightarrow r^2(4R+r)^2\le s^2(R^2-r^2)$, *which follows from Gerretsen's inequality*

$s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$$r^2(4R+r)^2\le\frac{r(4R+r)^2}{R+r}(R^2-r^2)\Leftrightarrow R^2-Rr-2r^2\ge 0\Leftrightarrow(R-2r)(R+r)\ge 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if ΔABC is equilateral.

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By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\left(\frac{15}{2}-\frac{3R^2}{4r^2}\right)R \leq \frac{w_a^2}{h_a}+\frac{w_b^2}{h_b}+\frac{w_c^2}{h_c} \leq \frac{9R}{2}$$

By George Apostolopoulos-Messolonghi– Greece

Solution

We prove the strongest inequality:

2) In ΔABC the following relationship holds:

$$\left(10-\frac{2r}{R}\right)r \leq \frac{w_a^2}{h_a}+\frac{w_b^2}{h_b}+\frac{w_c^2}{h_c} \leq 4R+r$$

Marin Chirciu – Romania

Solution

The left hand inequality: Using inequality $w_a \geq h_a$ we obtain:

$$\sum \frac{w_a^2}{h_a} \geq \sum \frac{h_a^2}{h_a} = \sum h_a = \frac{s^2+r^2+4Rr}{2R} \overset{Gerretsen}{\geq} \frac{16Rr-5r^2+r^2+4Rr}{2R} \geq r\left(10-\frac{2r}{R}\right)$$

We prove that $r\left(10-\frac{2r}{R}\right) \geq \left(\frac{15}{2}-\frac{3R^2}{4r^2}\right)R \Leftrightarrow 3R^4-30R^2r^2+40Rr^3-8r^4 \geq 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)(3R^3+6R^2r-18Rr^2+4r^3) \geq 0$, *obvious from Euler's inequality*

$R \geq 2r$.*Equality holds if and only if ΔABC is equilateral.*

The right hand inequality.

Using the formula $h_a=\frac{2S}{a}$ and the inequality $w_a^2 \leq s(s-a)$ we obtain:

$$\sum \frac{w_a^2}{h_a} \leq \sum \frac{s(s-a)}{\frac{2S}{a}} = \frac{s}{2S}\sum a(s-a) = \frac{1}{2r}\cdot 2r(4R+r) = 4R+r$$

Above we've used the known identity in triangle $\sum a(s-a) = 2r(4R+r)$

Remark.

We prove that inequality 2) is stronger than inequality 1).

3) In ΔABC the following relationship holds:

$$\left(\frac{15}{2}-\frac{3R^2}{4r^2}\right)R \le \left(10-\frac{2r}{R}\right)r \le \frac{w_a^2}{h_a}+\frac{w_b^2}{h_b}+\frac{w_c^2}{h_c} \le 4R+r \le \frac{9R}{2}$$

Solution

See inequality 2) and for the left hand inequality we prove that:

$$r\left(10-\frac{2r}{R}\right) \ge \left(\frac{15}{2}-\frac{3R^2}{4r^2}\right)R \Leftrightarrow 3R^4-30R^2r^2+40Rr^3-8r^4 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(3R^3+6R^2r-18Rr^2+4r^3) \ge 0$, ***obviously from Euler's inequality*** $R \ge 2r$.

For the right hand inequality we prove that $4R+r \le \frac{9R}{2} \Leftrightarrow R \ge 2r$, ***(Euler's inequality)***

Remark.

If we change h_a ***with*** r_a ***we propose:***

4) In ΔABC the following relationship holds:

$$\frac{18r^2}{R} \le \frac{w_a^2}{r_a}+\frac{w_b^2}{r_b}+\frac{w_c^2}{r_c} \le \frac{(2R-r)^2}{r}$$

Marin Chirciu – Romania

Solution

Using inequality $w_a \ge h_a$ ***we obtain:***

$$\sum\frac{w_a^2}{r_a} \ge \sum\frac{h_a^2}{r_a} \overset{Bergstrom}{\ge} \frac{(\sum h_a)^2}{\sum r_a} \overset{(1)}{\ge} \frac{3\sum h_bh_c}{\sum r_a} \overset{(3)}{=} \frac{3\cdot\frac{2s^2r}{R}}{4R+r} =$$

$$= \frac{6s^2r}{R(4R+r)} \overset{(4)}{\ge} \frac{6r\cdot\frac{r(4R+r)^2}{R+r}}{R(4R+r)} =$$

$\frac{6r^2(4R+r)}{r(R+r)} = \frac{6r^2}{R}\cdot\frac{4R+r}{R+r} \overset{(5)}{\ge} \frac{6r^2}{R}\cdot 3 = \frac{18r^2}{R}$, ***where (1) it follows from inequality*** $(x+y+z)^2 \ge 3(xy+yz+zx)$, ***(2) it follows from the known identities in triangle***

$\sum h_b h_c = \frac{2s^2r}{R}$ ***and*** $\sum r_a = 4R + r$, ***(4) it follows from Gerretsen***

$s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$, ***and (5)*** $\frac{4R+r}{R+r} \geq 3 \Leftrightarrow R \geq 2r$, ***(Euler's inequality)***

Equality holds if and only if ΔABC is equilateral.

The right hand inequality

Using the formula $h_a = \frac{S}{s-a}$ ***and inequality*** $w_a^2 \leq s(s-a)$ ***we obtain:***

$$\sum \frac{w_a^2}{r_a} \leq \sum \frac{s(s-a)}{\frac{S}{s-a}} = \frac{s}{S}\sum (s-a)^2 = \frac{1}{r}(s^2 - 2r^2 - 8Rr)$$

$$\leq \frac{4R^2 + 4Rr + 3r^2 - 2r^2 - 8Rr}{r} =$$

$$= \frac{4R^2 - 4Rr + r^2}{r} = \frac{(2R-r)^2}{r}$$

Above we've used the known identity in triangle $\sum (s-a)^2 = s^2 - 2r^2 - 8Rr$.

Equality holds if and only if ΔABC is equilateral.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru