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JP.286. If $a,b,c>0, ab+bc+ca=3$ then:

$$\frac{(a^2+b^2)(ab+1)}{a+b}+\frac{(b^2+c^2)(bc+1)}{b+c}+\frac{(c^2+a^2)(ca+1)}{c+a}\geq 6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania

We have:

$$\frac{(a^2+b^2)(ab+1)}{a+b}+\frac{(b^2+c^2)(bc+1)}{b+c}+\frac{(c^2+a^2)(ca+1)}{c+a}\geq 6$$

$$\Leftrightarrow \sum_{cyc}\frac{a^2+b^2}{a+b}+\sum_{cyc}\frac{(a^2+b^2)ab}{a+b}\geq 6$$

We have:

$$\frac{a^2+b^2}{a+b}\geq\frac{a+b}{2}\Rightarrow\sum_{cyc}\frac{a^2+b^2}{a+b}\geq a+b+c$$

On the other hand

$$\frac{(a^2+b^2)ab}{a+b}\geq\frac{2a^2b^2}{a+b}\Rightarrow\sum_{cyc}\frac{(a^2+b^2)ab}{a+b}\geq 2\sum_{cyc}\frac{a^2b^2}{a+b}\geq 2\frac{(\sum ab)^2}{\sum(a+b)}=\frac{9}{a+b+c}$$

It follows

$$\frac{(a^2+b^2)(ab+1)}{a+b}+\frac{(b^2+c^2)(bc+1)}{b+c}+\frac{(c^2+a^2)(ca+1)}{c+a}\geq a+b+c+\frac{9}{a+b+c}$$

Solution 2, generalizations and extensions by Marin Chirciu-Romania

<u>Use</u> inequality $x^2+y^2\geq\frac{(x+y)^2}{2}\Leftrightarrow(x-y)^2\geq 0$***,equality for*** $x=y$***, we get:***

$$M_s=\sum\frac{(b^2+c^2)(bc+1)}{b+c}\geq\sum\frac{\frac{(b+c)^2}{2}(bc+1)}{b+c}=\sum\frac{(b+c)(bc+1)}{2}\overset{(1)}{\geq}6=M_d,$$

where (1) $\Leftrightarrow\sum(b+c)(bc+1)\geq 12$.

From $ab+bc+ca=3$***, and*** $a,b,c>0$***, we can let trigonometric substitutions:***

$$a=\sqrt{3}tg\frac{X}{2}, b=\sqrt{3}tg\frac{Y}{2}, c=\sqrt{3}tg\frac{Z}{2}$$ ***, because:***

$$ab+bc+ca=3 \Leftrightarrow \frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}+\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}+\frac{c}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}}=1 \Leftrightarrow$$

$$\Leftrightarrow tg\frac{X}{2}tg\frac{Y}{2}+tg\frac{Y}{2}tg\frac{Z}{2}+tg\frac{Z}{2}tg\frac{X}{2}=1,$$ ***(true in any triangle*** ΔXYZ ***).***

With substitutions $a=\sqrt{3}tg\frac{X}{2}, b=\sqrt{3}tg\frac{Y}{2}, c=\sqrt{3}tg\frac{Z}{2}$***,***

Inequality $\sum(b+c)(bc+1)\geq 12$ ***becomes:***

$$\sum\left(\sqrt{3}tg\frac{Y}{2}+\sqrt{3}tg\frac{Z}{2}\right)\left(3tg\frac{Y}{2}tg\frac{Z}{2}+1\right)\geq 12 \Leftrightarrow \sum\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)\left(3tg\frac{Y}{2}tg\frac{Z}{2}+1\right)\geq 4\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow 3\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)+2\sum tg\frac{X}{2}\geq 4\sqrt{3}, \quad (2).$$

Following relationship holds in any triangle:

$$\sum tg\frac{X}{2}=\frac{4R+r}{p} \text{ and } \prod tg\frac{X}{2}=\frac{r}{p}, \text{ we get: } \sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)=\frac{2(2R-r)}{p},$$

and ineqality (2) becomes:

$$3\cdot\frac{2(2R-r)}{p}+2\cdot\frac{4R+r}{p}\geq 4\sqrt{3} \Leftrightarrow 5R-r\geq p\sqrt{3},$$

True, from Doucet inequality $4R+r\geq p\sqrt{3}$***.***

We must show that:

$5R-r\geq 4R+r \Leftrightarrow R\geq 2r$***, (Euler inequality).***

Equality if only if triangle is equilateral.

Remark:

Inequality can be extended:

1) If $a,b,c>0$ ***such that*** $ab+bc+ca=3$ ***și*** $n\geq 0$***.Prove:***

$$\frac{(a^2+b^2)(ab+n)}{a+b}+\frac{(b^2+c^2)(bc+n)}{b+c}+\frac{(c^2+a^2)(ca+n)}{c+a}\geq 3(n+1).$$

Proposed by Marin Chirciu-Romania

Solution:

Applying inequality $x^2+y^2\geq\frac{(x+y)^2}{2} \Leftrightarrow (x-y)^2\geq 0$***, equality for*** $x=y$***, we get:***

$$M_s=\sum\frac{(b^2+c^2)(bc+n)}{b+c}\geq\sum\frac{\frac{(b+c)^2}{2}(bc+n)}{b+c}=\sum\frac{(b+c)(bc+n)}{2}\overset{(1)}{\geq}3(n+1)=M_d,$$

where (1) $\Leftrightarrow \sum(b+c)(bc+n)\geq 6(n+1)$***.***

From $ab+bc+ca=3$, ***where*** $a,b,c>0$, ***we can let trigonometric substitutions:***

$$a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}\text{,because:}$$

$$ab+bc+ca=3\Leftrightarrow\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}+\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}+\frac{c}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}}=1\Leftrightarrow$$

$$\Leftrightarrow tg\frac{X}{2}tg\frac{Y}{2}+tg\frac{Y}{2}tg\frac{Z}{2}+tg\frac{Z}{2}tg\frac{X}{2}=1,\text{(true in any }\Delta XYZ\text{).}$$

With substitutions: $a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}$, ***inequality***

$$\sum(b+c)(bc+n)\geq 6(n+1)$$

becomes:

$$\sum\left(\sqrt{3}tg\frac{Y}{2}+\sqrt{3}tg\frac{Z}{2}\right)\left(3tg\frac{Y}{2}tg\frac{Z}{2}+n\right)\geq 6(n+1)\Leftrightarrow$$

$$\Leftrightarrow\sum\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)\left(3tg\frac{Y}{2}tg\frac{Z}{2}+n\right)\geq 2(n+1)\sqrt{3}\Leftrightarrow$$

$$\Leftrightarrow 3\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)+2n\sum tg\frac{X}{2}\geq 2(n+1)\sqrt{3},\ (2).$$

In ΔXYZ ***following relationship holds:***

$$\sum tg\frac{X}{2}=\frac{4R+r}{p}\text{ și }\prod tg\frac{X}{2}=\frac{r}{p}\text{, we get }\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)=\frac{2(2R-r)}{p}\text{, and}$$

inequality (2) becomes:

$$3\cdot\frac{2(2R-r)}{p}+2n\cdot\frac{4R+r}{p}\geq 2(n+1)\sqrt{3}\Leftrightarrow 3(2R-r)+n(4R+r)\geq(n+1)\cdot p\sqrt{3},$$

True from Doucet inequality $4R+r\geq p\sqrt{3}$.

We must show that:

$$3(2R-r)+n(4R+r)\geq(n+1)\cdot(4R+r)\Leftrightarrow R\geq 2r\text{, (Euler inequality).}$$

Equality if only if triangle is equilateral.

Note:

1). For $n=1$ ***we get Problem JP.286 proposed by Daniel Sitaru in RMM, Number 20, Spring Edition 2021.***

2). For $n=0$ ***we get inequality:***

2) If $a,b,c>0$ ***such that*** $ab+bc+ca=3$. ***Prove:***

$$\frac{ab(a^2+b^2)}{a+b}+\frac{bc(b^2+c^2)}{b+c}+\frac{ca(c^2+a^2)}{c+a}\geq 3.$$

Proposed by Marin Chirciu-Romania

Solution:

Applying inequality $x^2+y^2\geq\frac{(x+y)^2}{2}\Leftrightarrow(x-y)^2\geq 0$***, equality for*** $x=y$***, we get:***

$$M_s=\sum\frac{bc(b^2+c^2)}{b+c}\geq\sum\frac{bc\cdot\frac{(b+c)^2}{2}}{b+c}=\sum\frac{bc(b+c)}{2}\overset{(1)}{\geq}3=M_d,$$

where (1) $\Leftrightarrow\sum bc(b+c)\geq 6$.

From $ab+bc+ca=3$***, where*** $a,b,c>0$***, we can let trigonometric substitution:***

$$a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}\text{, because:}$$

$$ab+bc+ca=3\Leftrightarrow\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}+\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}+\frac{c}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}}=1\Leftrightarrow$$

$$\Leftrightarrow tg\frac{X}{2}tg\frac{Y}{2}+tg\frac{Y}{2}tg\frac{Z}{2}+tg\frac{Z}{2}tg\frac{X}{2}=1\text{,(true in any }\Delta XYZ\text{).}$$

With $a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}$***, inequality*** $\sum bc(b+c)\geq 6$ ***becomes:***

$$\sum 3tg\frac{Y}{2}tg\frac{Z}{2}\left(\sqrt{3}tg\frac{Y}{2}+\sqrt{3}tg\frac{Z}{2}\right)\geq 6\Leftrightarrow\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)\geq\frac{2\sqrt{3}}{3}\text{, (2) .}$$

In ΔXYZ ***following relationship holds:***

$$\sum tg\frac{X}{2}=\frac{4R+r}{p}\text{ și }\prod tg\frac{X}{2}=\frac{r}{p}\text{, we get }\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)=\frac{2(2R-r)}{p},$$

and inequality (2) becomes:

$$\frac{2(2R-r)}{p}\geq\frac{2\sqrt{3}}{3}\Leftrightarrow 3(2R-r)\geq p\sqrt{3}\text{, true from Doucet inequality }4R+r\geq p\sqrt{3}.$$

We must show that:

$$3(2R-r)\geq 4R+r\Leftrightarrow R\geq 2r\text{, (Euler inequality).}$$

Equality if only if triangle is equilateral.

JP.287 If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\left(\prod_{cyc} \log(1+2\sin^2 x)\right)\left(\prod_{cyc} \log(1+2\cos^2 x)\right) \le \log^6 2$$

Proposed by Daniel Sitaru-Romania

Solution by Daniel Văcaru-Romania

With Am-Gm, we have:

$$\sqrt{\log(1+2\sin^2 x)\log(1+2\cos^2 x)} \le \frac{\log(1+2\sin^2 x)+\log(1+2\cos^2 x)}{2}$$

$$= \frac{\log(3+\sin^2 2x)}{2} \le \log 2$$

Squaring this, we obtain $\log(1+2\sin^2 x)\log(1+2\cos^2 x) \le \log^2 2, \forall x \in \left(0, \frac{\pi}{2}\right)$

Multiplying relations for $x, y, z \in \left(0, \frac{\pi}{2}\right)$ ***we obtain:***

$$\left(\prod_{cyc} \log(1+2\sin^2 x)\right)\left(\prod_{cyc} \log(1+2\cos^2 x)\right) \le 1$$

JP.288. Solve for real numbers:

$$2\sqrt{\frac{x-2}{x+3}} + \frac{x+4}{\left(\sqrt{x-2}+\sqrt{x+4}\right)^2} = 1$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Miguel Velasquez Culque-Cajamarca-Peru

$$x \in [2, \infty)$$

$$2\sqrt{\frac{x-2}{x+3}} + \frac{x+4}{\left(\sqrt{x-2}+\sqrt{x+4}\right)^2} = 1$$

$$2\sqrt{\frac{x-2}{x+3}}=1-\frac{x+4}{\left(\sqrt{x-2}+\sqrt{x+4}\right)^2}$$

$$2\sqrt{\frac{x-2}{x+3}}=1^2-\left(\frac{\sqrt{x+4}}{\sqrt{x-2}+\sqrt{x+4}}\right)^2$$

$$2\sqrt{\frac{x-2}{x+3}}=\left(1+\frac{\sqrt{x+4}}{\sqrt{x-2}+\sqrt{x+4}}\right)\left(1-\frac{\sqrt{x+4}}{\sqrt{x-2}+\sqrt{x+4}}\right)$$

$$2\sqrt{\frac{x-2}{x+3}}-\left(\frac{\sqrt{x-2}+2\sqrt{x+4}}{\sqrt{x-2}+\sqrt{x+4}}\right)\left(\frac{\sqrt{x-2}}{\sqrt{x-2}+\sqrt{x+4}}\right)=0$$

$$\sqrt{x-2}\left(\frac{2}{\sqrt{x+3}}-\frac{\sqrt{x-2}+2\sqrt{x+4}}{\left(\sqrt{x-2}+\sqrt{x+4}\right)^2}\right)=0$$

$$\sqrt{x-2}=0\to x=2$$

$$f(x)=\frac{2}{\sqrt{x+3}}-\frac{\sqrt{x-2}+2\sqrt{x+4}}{\left(\sqrt{x-2}+\sqrt{x+4}\right)^2}>0;x\in[2,\infty)$$

$$f(2)=\frac{2}{\sqrt{2+3}}-\frac{\sqrt{2-2}+2\sqrt{2+4}}{\left(\sqrt{2-2}+\sqrt{2+4}\right)^2}\approx 0,07$$

$$f(3)=\frac{2}{\sqrt{3+3}}-\frac{\sqrt{3-2}+2\sqrt{3+4}}{\left(\sqrt{3-2}+\sqrt{3+4}\right)^2}\approx 0,34$$

Answer: $x=2$.

Solution 2 by Orlando Irahola Ortega-La Paz-Bolivia

$$2\sqrt{\frac{x-2}{x+3}}+\frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2}=1$$

$$\textit{Let: } t^2=\frac{x-4}{x+2}\to x=\frac{4t^2+2}{1-t^2},\forall t\neq\pm1\to\exists x\in\mathbb{R}$$

$$2\sqrt{\frac{6t^2}{t^2+5}}=\frac{t^2+2t}{(t+1)^2}\to 2\sqrt{6}t(t+1)^2=t(t+2)\sqrt{t^2+5}\to t=0$$

$$\textit{or }\ 2\sqrt{6}(t+1)^2=(t+2)\sqrt{t^2+5}$$

$$\textit{If } t = 0 \to \frac{x-2}{x+4} = 0 \to x = 2.$$

$$\textit{If } 2\sqrt{6}(t+1)^2 = (t+2)\sqrt{t^2+5} \to 24(t+1)^4 = (t^2+5)(t+2)^2 \xrightarrow{t+1=y\geq 1}$$

$$23y^4 - 3y^2 - 10y - 6 = 0$$

$$\textit{How } y \geq 1 \to \nexists t \geq 0 \to \nexists x \in \mathbb{R}$$

$$S = \{2\}$$

Solution 3 by proposer

$$\textit{We have: } x \geq 2$$

$$\textit{Let (1): } 2\sqrt{\frac{x-2}{x+3}} + \frac{x+4}{\left(\sqrt{x-2}+\sqrt{x+4}\right)^2} = 1 \Leftrightarrow 2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\frac{\left(\sqrt{x-2}+\sqrt{x+4}\right)^2}{x+4}} = 1 \Leftrightarrow$$

$$2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}} + \sqrt{\frac{x+4}{x+4}}\right)^2} = 1 \Leftrightarrow$$

$$2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}} + 1\right)^2} = 1; (2)$$

$$\textit{Because } x \geq 2 > 0 \textit{ then } x - 2 \geq 0. \textit{ Hence } 0 < x + 3 < x + 4 \Rightarrow$$

$$\frac{1}{x+3} > \frac{1}{x+4} > 0 \Rightarrow \frac{x-2}{x+3} \geq \frac{x-2}{x+4} \Rightarrow \sqrt{\frac{x-2}{x+3}} \geq \sqrt{\frac{x-2}{x+4}}$$

$$\Leftrightarrow 2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2} \geq 2\sqrt{\frac{x-2}{x+4}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2} \Leftrightarrow$$

$$2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2} \geq 2t + \frac{1}{(t-1)^2}; \text{let } t = \sqrt{\frac{x-2}{x+4}} \geq 0; (3)$$

By AM-GM inequality for three positive real numbers, we have:

$$2t+\frac{1}{(t+1)^2}=(t+1)+(t+1)+\frac{1}{(t+1)^2}-2\ge 3\sqrt[3]{(t+1)(t+1)\cdot\frac{1}{(t+1)^2}}-2$$

$$=3\sqrt[3]{1}-2=1\Rightarrow$$

$2t+\frac{1}{(t-1)^2}\ge 1$.*Let with (3)*$\Rightarrow 2\sqrt{\frac{x-2}{x+3}}+\frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2}\ge 1;(4)$

Let (2),(4)$\Rightarrow 2\sqrt{\frac{x-2}{x+3}}+\frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2}=1$ *and equality occurs if* $\begin{cases} x-2=0\\ t+1=\frac{1}{(t+1)^2}\end{cases}\Leftrightarrow$

$$\begin{cases} x=2\\ \sqrt{\frac{x-2}{x+4}}=0\end{cases}\Leftrightarrow x=2.$$

The solution of equation is $S=\{2\}$

JP.289. If $a,b,c>0; abc(a+b+c)=3$ then:

$$(a^5-2a+4)(b^5-2b+4)(c^5-2c+4)\ge 9\sqrt{3(a^2+b^2+c^2)}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by Rahim Shahbazov-Baku-Azerbaijan

Let's show that: $a^5-2a+4\ge a^3+2\Rightarrow (a-1)^2(a^3+2a^2+2a+2)\ge 0$ *true, then*

$$(a^3+2)(b^3+2)(c^3+2)=$$

$$=(a^3+1^3+1^3)(b^3+1^3+1^3)(c^3+1^3+1^3)\overset{Holder}{\ge}(a+b+c)^3$$

We have: $(a+b+c)^3\ge 9\sqrt{3(a^2+b^2+c^2)}$ *then*

$$(a+b+c)^5\ge 81abc(a^2+b^2+c^2)\therefore (x+y+z)^3\ge 27xyz$$

We take $x=y=ab+bc+ca$ *and* $z=a^2+b^2+c^2$. *We have:*

$$(a+b+c)^6\ge 27(ab+bc+ca)^2(a^2+b^2+c^2)\ge 81abc(a+b+c)(a^2+b^2+c^2)$$

Then $(a+b+c)^5\ge 81abc(a^2+b^2+c^2)$

JP.290- RMM 20 NUMBER, SPRING EDITION 2021

1) **JP.290. In ΔABC the following relationship holds:**

$$\frac{w_b w_c}{w_a}+\frac{w_c w_a}{w_b}+\frac{w_a w_b}{w_c}\ge\frac{h_b h_c}{h_a}+\frac{h_c h_a}{h_b}+\frac{h_a h_b}{h_c}$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by Marin Chirciu-Romania

Use inequality $\sum\frac{w_b w_c}{w_a}=2\cdot\frac{p^2(2R+r)+r^2(4R+r)}{p^2+r^2+2Rr}$ ***and*** $\sum\frac{h_b h_c}{h_a}=\frac{p^2-r^2-4Rr}{R}$***, the inequality can be write:***

$$2\cdot\frac{p^2(2R+r)+r^2(4R+r)}{p^2+r^2+2Rr}\ge\frac{p^2-r^2-4Rr}{R}\Leftrightarrow p^2\left(4R^2+4Rr-p^2\right)+r^2(4R+r)^2\ge 0.$$

We distinct the case:

Case1). If $\left(4R^2+4Rr-p^2\right)\ge 0$***, inequality is obvious.***

Case 2). If $\left(4R^2+4Rr-p^2\right)<0$***,inequality can be rewrite:***

$r^2(4R+r)^2\ge p^2\left(p^2-4R^2-4Rr\right)$***, true from Blundon-Gerretsen inequalitiy:***

$$p^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2.$$

We must show that:

$r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}\left(4R^2+4Rr+3r^2-4R^2-4Rr\right)\Leftrightarrow R\ge 2r$***, (Euler inequality).***

Equality if only if the triangle ie equilateral.

Remark:

Let's determine an inequality of the opposite direction for the sum: $\sum\frac{w_b w_c}{w_a}$***.***

2) In ΔABC ***the following relationship holds:***

$$\frac{w_b w_c}{w_a}+\frac{w_c w_a}{w_b}+\frac{w_a w_b}{w_c}\le\frac{r_b r_c}{r_a}+\frac{r_c r_a}{r_b}+\frac{r_a r_b}{r_c}.$$

Proposed by Marin Chirciu- Romania

Solution by Marin Chirciu-Romania

Use inequality $\sum\frac{w_b w_c}{w_a}=2\cdot\frac{p^2(2R+r)+r^2(4R+r)}{p^2+r^2+2Rr}$ ***and*** $\sum\frac{r_b r_c}{r_a}=\frac{p^2-2r^2-8Rr}{r}$***,the inequality can be write:***

$$2\cdot\frac{p^2(2R+r)+r^2(4R+r)}{p^2+r^2+2Rr}\le\frac{p^2-2r^2-8Rr}{r}\Leftrightarrow p^2\left(p^2-3r^2-10Rr\right)\ge 4r^2\left(4R^2+5Rr+r^2\right)$$

,from Gerretsen inequality: $p^2\ge 16Rr-5r^2$

We must show that:

$$\left(16Rr-5r^2\right)\left(16Rr-5r^2-3r^2-10Rr\right)\ge 4r^2\left(4R^2+5Rr+r^2\right)\Leftrightarrow$$

$40R^2-89Rr+18r^2\ge 0 \Leftrightarrow (R-2r)(40R-9r)\ge 0$, ***true from Euler inequality*** $R\ge 2r$.

Equality if only if the triangle is equilateral.

Remark: It can be doubled:

3) In ΔABC ***the following relationship holds:***

$$\frac{r_br_c}{r_a}+\frac{r_cr_a}{r_b}+\frac{r_ar_b}{r_c}\ge\frac{w_bw_c}{w_a}+\frac{w_cw_a}{w_b}+\frac{w_aw_b}{w_c}\ge\frac{h_bh_c}{h_a}+\frac{h_ch_a}{h_b}+\frac{h_ah_b}{h_c}.$$

Proposed by Marin Chirciu- Romania

Solution by Marin Chirciu-Romania

See relation 1) and 2).

Equality if only if the triangle is equilateral.

Remark:

We can be write the sequences of the inequalities:

4) In ΔABC ***the following relationship holds:***

$$\frac{(2R-r)^2}{r}\ge\frac{r_br_c}{r_a}+\frac{r_cr_a}{r_b}+\frac{r_ar_b}{r_c}\ge\frac{w_bw_c}{w_a}+\frac{w_cw_a}{w_b}+\frac{w_aw_b}{w_c}\ge\frac{h_bh_c}{h_a}+\frac{h_ch_a}{h_b}+\frac{h_ah_b}{h_c}\ge\frac{6r(2R-r)}{R}.$$

Proposed by Marin Chirciu- Romania

Solution by Marin Chirciu-Romania

See the double inequality 3), $\frac{(2R-r)^2}{r}\ge\sum\frac{r_br_c}{r_a}$ ***și*** $\sum\frac{h_bh_c}{h_a}\ge\frac{6r(2R-r)}{R}$, ***which rezult from the identities*** $\sum\frac{r_br_c}{r_a}=\frac{p^2-2r^2-8Rr}{r}$, $\sum\frac{h_bh_c}{h_a}=\frac{p^2-r^2-4Rr}{R}$ ***and the Gerretsen inequality:*** $16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$.

Equality if only if the triangle is equilateral.

JP.291. Solve for real numbers:

$$\begin{cases}\dfrac{1}{\sqrt{a}}+\dfrac{1}{\sqrt{b}}+\dfrac{1}{\sqrt{c}}=3\\ \dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}=3\sqrt[3]{\dfrac{a^3+b^3+c^3}{3}}\end{cases}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by Rahim Shahbazov-Baku-Azerbaijan

Lemma: If $a,b,c>0$ ***then***

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3\sqrt[4]{\frac{a^4+b^4+c^4}{3}}\ge 3\sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

Proof:

$$\sum\frac{a^2}{b}=\sum\frac{a^4}{a^2b}\overset{Bergstrom}{\ge}\frac{(a^2+b^2+c^2)^2}{a\cdot ab+b\cdot bc+c\cdot ca}$$

$$\overset{BCS}{\ge}\frac{(a^2+b^2+c^2)^2}{\sqrt{(a^2+b^2+c^2)((a^2b^2+b^2c^2+c^2a^2)}}\ge 3\sqrt[4]{\frac{a^4+b^4+c^4}{3}}$$

Then: $(a^2+b^2+c^2)^6\ge 27(a^2b^2+b^2c^2+c^2a^2)^2(a^4+b^4+c^4)$

Let: $a^2=x;\ b^2=y;\ c^2=z$ *then*

$(x+y+z)^6\ge 27(xy+yz+zx)^2(x^2+y^2+z^2)$ *true by*

$$(m+n+p)^3\overset{Am-Gm}{\ge}27mnp$$

We take: $m=x^2+y^2+z^2$ *and* $n=xy+yz+zx$ *then:*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=3\sqrt[3]{\frac{a^3+b^3+c^3}{3}}\Rightarrow a=b=c=1.$$

$$S=\{(a,b,c)=(1,1,1)\}$$

JP.292. If $x,y,z>0;a\ge 3;xy+yz+zx=3$ then:

$$\frac{1}{a+x^2}+\frac{1}{a+y^2}+\frac{1}{a+z^2}\le\frac{3}{a+1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Daniel Văcaru-Romania

"Reversing" this inequality, we obtain:

$$-\sum_{cyc}\left(\frac{1}{x^2+a}\right)\ge\frac{-3}{a+1}\Rightarrow\sum_{cyc}\left(\frac{x^2}{x^2+a}\right)\ge\frac{3}{a+1}$$

By Bergstrom,

$$\sum_{cyc}\left(\frac{x^2}{x^2+a}\right)\ge\frac{(x+y+z)^2}{x^2+y^2+z^2+3a}=\frac{x^2+y^2+z^2+6}{x^2+y^2+z^2+a}\overset{(1)}{\ge}\frac{3}{a+1}$$

But (1) is equivalent to

$$a\sum x^2+6a+\sum x^2+6\geq 3\sum x^2\geq +3a\Rightarrow (a-2)\sum x^2\geq 3(a-2).$$

But $\sum x^2\geq \sum xy=3$, and together to this end our proof.

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$a\geq 3; xy+yz+zx=3;$$

$$\frac{1}{a+x^2}+\frac{1}{a+y^2}+\frac{1}{a+z^2}\leq \frac{3}{a+1}$$

$$\left(\frac{1}{a+x^2}-\frac{1}{a}\right)+\left(\frac{1}{a+y^2}-\frac{1}{a}\right)+\left(\frac{1}{a+z^2}-\frac{1}{a}\right)\leq \frac{3}{a+1}-\frac{3}{a}$$

$$\frac{x^2}{a+x^2}+\frac{y^2}{a+y^2}+\frac{z^2}{a+z^2}\overset{(*)}{\geq}\frac{3}{a+1}$$

$$LHS_{(*)}=\sum_{cyc}\frac{x^2}{a+x^2}\overset{BCS}{\geq}\frac{(x+y+z)^2}{3a+x^2+y^2+z^2}=\frac{x^2+y^2+z^2+6}{3a+x^2+y^2+z^2}\overset{(**)}{\geq}\frac{3}{a+1}$$

$$(**)\overset{\substack{t=x^2+y^2+z^2\\ \geq xy+yz+zx=3}}{\Longleftrightarrow}(a+1)(t+6)\geq 3(3a+t)\Leftrightarrow$$

$$at+6a+t+6-9a-3t\geq 0\Leftrightarrow at-3a-2t+6\geq 0\Leftrightarrow$$

$$(a-2)(t-3)\geq 0$$

Which is clearly true, because: $a\geq 3>2\Rightarrow (a-2)>0; t\geq 3\Rightarrow t-3\geq 0$

Then (**) is true, then (*) is true.

JP.293. If $a,b,c\geq 0; a+b+c=3; k\geq \frac{3}{2}$ then:

$$(k+a^2)(k+b^2)(k+c^2)\geq (k+1)^3$$

Proposed by Marin Chirciu-Romania

Solution by Michael Sterghiou-Greece

$$(k+a^2)(k+b^2)(k+c^2)\geq (k+1)^3 \quad (1)$$

Let: $(a+b+c, ab+bc+ca, abc)=(p,q,r); p=3; q\leq 3; r\leq 1$

Let: $f(k)=(k+a^2)(k+b^2)(k+c^2)-(k+1)^3$

$\frac{df}{dk} = (a^2+b^2+c^2)\cdot 2k + a^2b^2 + b^2c^2 + c^2a^2 - 6k - 3$ *which reduces to*

$$q^2 - 6r + \underbrace{4(3-q)}_{\geq 0}k - 3 \overset{k\geq\frac{3}{2}}{\geq} \underbrace{q^2 - 6q - 6r + 15}_{f(q)} \geq 0$$

But: $f'(q) = 2(q-3) \leq 0$ *so,* $f(q)$ *–decreasing and*

$f(q) \geq f(3) = 6(1-r)$ *as* $r \leq 1$.

Hence $f(k)$ *–increasing and* $f(k) \geq f\left(\frac{3}{2}\right)$ *which yields after some computation*

$$3q^2 - 6q + 2r^2 - 18r + 16 \geq 0 \quad (2).$$

We will use V.Cîrtoaje theorem that with p, q *fixed the product* $abc = r$ *get's maximal when* $a = b$ *assuming WLOG* $a \leq b \leq c$ *and minimal when* $b = c$.

As (2) is a decreasing function of r; $[2r^2 - 18r = 2r(r-9) \leq 0]$

It suffices to prove (2) for maximal or when $a = b \leq 1 \leq c$.

Now: $q = a^2 + 2ac = a^2 + 2a(3-2a), r = a^2(3-2a)$ *and (2) becomes after computation and factoring:*

$(a-1)^2(8a^4 - 8a^3 + 21a^2 - 22a + 16) \geq 0$ *with* $0 \leq a \leq 1$.

It is easy to see that: $8a^4 - 8a^3 \geq -\frac{27}{32}$ *and* $21a^2 - 22a \geq -\frac{121}{21}$ *hence*

$8a^4 - 8a^3 + 21a^2 - 22a + 16 > 0$.*Done!*

JP.294. If $a, b, c > 0; a + b + c = 3$ then:

$$\frac{a}{b(b+2c)^2} + \frac{b}{c(c+2a)^2} + \frac{c}{a(a+2b)^2} \geq \frac{9}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^3}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c \geq 0, a + b + c = 3$, *we give:* $a = x^2, b = y^2, c = z^2$ *hence*

$x^2 + y^2 + z^2 = a + b + c = 3$ *and*

$$\frac{a}{b(b+2c)^2}+\frac{b}{c(c+2a)^2}+\frac{c}{a(a+2b)^2}\ge\frac{9}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^3}\Leftrightarrow$$

$$\frac{x^2}{y^2(y^2+2z^2)^2}+\frac{y^2}{z^2(z^2+2x^2)^2}+\frac{z^2}{x^2(x^2+2y^2)^2}\ge\frac{9}{(x+y+z)^3}\Leftrightarrow$$

$$\frac{\frac{x^4}{(y^2+2z^2)^2}}{x^2y^2}+\frac{\frac{y^4}{(y^2+2z^2)^2}}{y^2z^2}+\frac{\frac{z^4}{(y^2+2z^2)^2}}{x^2z^2}\ge\frac{9}{(x+y+z)^3}\Leftrightarrow$$

$$\frac{\left(\frac{x^2}{y^2+2z^2}+\frac{y^2}{y^2+2z^2}+\frac{z^2}{y^2+2z^2}\right)^2}{x^2y^2+y^2z^2+x^2z^2}\ge\frac{9}{(x+y+z)^3}\Leftrightarrow$$

$$\left(\frac{(x^2+y^2+z^2)^2}{3(x^2y^2+y^2z^2+x^2z^2)}\right)^2\ge\frac{9(x^2y^2+y^2z^2+x^2z^2)}{(x+y+z)^3}\overset{x^2+y^2+z^2=3}{\Longleftrightarrow}$$

$$(x^2+y^2+z^2)^2(x+y+z)^3\ge 9(x^2y^2+y^2z^2+x^2z^2)^3\overset{x^2+y^2+z^2=3}{\Longleftrightarrow}$$

$$(x+y+z)^3\ge(x^2y^2+y^2z^2+x^2z^2)^3$$

$$x+y+z\ge x^2y^2+y^2z^2+x^2z^2 \text{ true for } x^2+y^2+z^2=3$$

Solution 2 by proposer

By Cauchy-Schwarz inequality we have:

$$\frac{a}{b(b+2c)^2}+\frac{b}{c(c+2a)^2}+\frac{c}{a(a+2b)^2}=\frac{\left(\frac{a}{b+2c}\right)^2}{ab}+\frac{\left(\frac{b}{c+2a}\right)^2}{bc}+\frac{\left(\frac{c}{a+2b}\right)^2}{ca}\ge$$

$$\ge\frac{\left(\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\right)^2}{ab+bc+ca}\Rightarrow$$

$$\frac{a}{b(b+2c)^2}+\frac{b}{c(c+2a)^2}+\frac{c}{a(a+2b)^2}\ge\frac{\left(\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\right)^2}{ab+bc+ca};(1)$$

$$\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}=\frac{a^2}{a(b+2c)}+\frac{b^2}{b(c+2a)}+\frac{c^2}{c(a+2b)}\ge$$

$$\ge\frac{(a+b+c)^2}{a(b+2c)+b(c+2a)+c(a+2b)}=\frac{(a+b+c)^2}{3(ab+bc+ca)}\ge$$

$$\ge\frac{3^2}{3(ab+bc+ca)}=\frac{3}{ab+bc+ca};(2)$$

From (1),(2) we have:

$$\frac{a}{b(b+2c)^2}+\frac{b}{c(c+2a)^2}+\frac{c}{a(a+2b)^2}\ge\frac{\left(\frac{3}{ab+bc+ca}\right)^2}{ab+bc+ca}=\frac{9}{(ab+bc+ca)^2};(3)$$

By Am-Gm inequality for three positive real numbers, we have:

$$\left(\sqrt{a}+\sqrt{a}+a^2\right)+\left(\sqrt{b}+\sqrt{b}+b^2\right)+\left(\sqrt{c}+\sqrt{c}+c^2\right)\ge$$

$$\ge 3\sqrt[3]{\sqrt{a}\cdot\sqrt{a}\cdot a^2}++3\sqrt[3]{\sqrt{b}\cdot\sqrt{b}\cdot b^2}+3\sqrt[3]{\sqrt{c}\cdot\sqrt{c}\cdot c^2}\Rightarrow$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+a^2+b^2+c^2\ge 3\sqrt[3]{a^3}+3\sqrt[3]{b^3}+3\sqrt[3]{c^3}=3(a+b+c)=$$

$$\overset{a+b+c=3}{=}(a+b+c)^2\Leftrightarrow$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+a^2+b^2+c^2\ge a^2+b^2+c^2+2(ab+bc+ca)\Leftrightarrow$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\ge 2(ab+bc+ca)\Leftrightarrow$$

$$ab+bc+ca\le\sqrt{a}+\sqrt{b}+\sqrt{c};(4)$$

From (3),(4) we get:

$$\frac{a}{b(b+2c)^2}+\frac{b}{c(c+2a)^2}+\frac{c}{a(a+2b)^2}\ge\frac{9}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^3}$$

Equality occurs if $\begin{cases} a,b,c>0; a+b+c=3\\ a=b=c\\ \sqrt{a}=a^2;\sqrt{b}=b^2;\sqrt{c}=c^2\end{cases}\Leftrightarrow a=b=c=1.$

JP.295 Let a,b,c positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^2}{b(a+5c)^3}+\frac{b^2}{c(b+5a)^3}+\frac{c^2}{a(c+5b)^3}\ge\frac{1}{24\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by proposer

By Cauchy-Schwartz inequality we have:

$$\frac{a^2}{b(a+5c)^3}+\frac{b^2}{c(b+5a)^3}+\frac{c^2}{a(c+5b)^3}=\frac{\left(\frac{a}{a+5c}\right)^2}{b(a+5c)}+\frac{\left(\frac{b}{b+5a}\right)^2}{c(b+5a)}+\frac{\left(\frac{c}{c+5b}\right)^2}{a(c+5b)}\ge$$

$$\ge \frac{\left(\frac{a}{a+5c}+\frac{b}{b+5a}+\frac{c}{c+5b}\right)^2}{b(a+5c)+c(b+5a)+a(c+5b)} \Rightarrow$$

$$\frac{a^2}{b(a+5c)^3}+\frac{b^2}{c(b+5a)^3}+\frac{c^2}{a(c+5b)^3} \ge \frac{\left(\frac{a}{a+5c}+\frac{b}{b+5a}+\frac{c}{c+5b}\right)^2}{6(ab+bc+ca)}; (1)$$

$$\frac{a}{a+5c}+\frac{b}{b+5a}+\frac{c}{c+5b} = \frac{a^2}{a^2+5ac}+\frac{b^2}{b^2+5ab}+\frac{c^2}{c^2+5bc} \ge$$

$$\ge \frac{(a+b+c)^2}{a^2+b^2+c^2+5(ab+bc+ca)} = \frac{(a+b+c)^2}{(a+b+c)^2+3(ab+bc+ca)} \ge$$

$$\ge \frac{(a+b+c)^2}{(a+b+c)^2+(a+b+c)^2} = \frac{1}{2}; (2)$$

Because $(x+y+z)^2 \ge 3(xy+yz+zx);\ \forall x,y,z \in \mathbb{R}$

From (1),(2) we have:

$$\frac{a^2}{b(a+5c)^3}+\frac{b^2}{c(b+5a)^3}+\frac{c^2}{a(c+5b)^3} \ge \frac{\left(\frac{1}{2}\right)^2}{6(ab+bc+ca)} = \frac{1}{24(ab+bc+ca)}; (3)$$

By Am-Gm inequality for three positive real numbers, we have:

$$\left(\sqrt{a}+\sqrt{a}+a^2\right)+\left(\sqrt{b}+\sqrt{b}+b^2\right)+\left(\sqrt{c}+\sqrt{c}+c^2\right) \ge$$

$$\ge 3\sqrt[3]{\sqrt{a}\cdot\sqrt{a}\cdot a^2} + +3\sqrt[3]{\sqrt{b}\cdot\sqrt{b}\cdot b^2} + 3\sqrt[3]{\sqrt{c}\cdot\sqrt{c}\cdot c^2} \Rightarrow$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+a^2+b^2+c^2 \ge 3\sqrt[3]{a^3}+3\sqrt[3]{b^3}+3\sqrt[3]{c^3} = 3(a+b+c) =$$

$$\overset{a+b+c=3}{=} (a+b+c)^2 \Leftrightarrow$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+a^2+b^2+c^2 \ge a^2+b^2+c^2+2(ab+bc+ca) \Leftrightarrow$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge 2(ab+bc+ca) \Leftrightarrow$$

$$ab+bc+ca \le \sqrt{a}+\sqrt{b}+\sqrt{c}; (4)$$

From (3),(4) we get:

$$\frac{a^2}{b(a+5c)^3}+\frac{b^2}{c(b+5a)^3}+\frac{c^2}{a(c+5b)^3} \ge \frac{1}{24\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}$$

Equality occurs if $\begin{cases} a,b,c>0; a+b+c=3 \\ a=b=c \\ \sqrt{a}=a^2; \sqrt{b}=b^2; \sqrt{c}=c^2 \end{cases} \Leftrightarrow a=b=c=1.$

JP.296. If $a, b, c > 0; abc = 1; n \geq 2$ then:

$$a+b+c+\frac{1}{n+a}+\frac{1}{n+b}+\frac{1}{n+c}\geq\frac{3(n+2)}{n+1}$$

Proposed by Marin Chirciu-Romania

Solution by Daniel Văcaru-Romania

Consider $f:(0,\infty)\to\mathbb{R}, f(t)=t+\frac{1}{n+t}, n\geq 2, f$ *−concave.*

It follows that $a+b+c+\frac{1}{n+a}+\frac{1}{n+b}+\frac{1}{n+c}\geq 3\left(\frac{a+b+c}{3}+\frac{1}{n+\frac{a+b+c}{3}}\right)$

Let's prove that

$$\frac{a+b+c}{3}+\frac{1}{n+\frac{a+b+c}{3}}\geq\frac{3(n+2)}{n+1}\Leftrightarrow$$

$$n^2\left(\frac{a+b+c}{3}\right)+n\left(\frac{a+b+c}{3}\right)^2+n+n\left(\frac{a+b+c}{3}\right)+\left(\frac{a+b+c}{3}\right)^2+1$$

$$\geq n^2+2n+n\left(\frac{a+b+c}{3}\right)+2\left(\frac{a+b+c}{3}\right)\Leftrightarrow$$

$$(n^2-2)\left(\frac{a+b+c}{3}\right)+(n+1)\left(\frac{a+b+c}{3}\right)^2+n+1\geq n^2+2$$

By Am-Gm, we have $\frac{a+b+c}{3}\geq 1;\ (1)$. *It follows:*

$$(n^2-2)\left(\frac{a+b+c}{3}\right)+(n+1)\left(\frac{a+b+c}{3}\right)^2+n+1\overset{(1)}{\geq}(n^2-2)+(n+1)+n+1=n^2+2n,$$

As desired.

RMM-NUMBER 20 SPRING 2021-JP.297

By Marin Chirciu,Octavian Stroe-Romania

1) *If* $a,b,c\geq 0$, $ab+bc+ca=3$ *then find min* $\Omega(a,b,c)$,

$$\Omega(a,b,c)=\frac{1}{(a+b)^5}+\frac{1}{(b+c)^5}+\frac{1}{(c+a)^5}.$$

Proposed by Hoang Le Nhat Tung- Hanoi-Vietnam

Solution by Marin Chirciu and Octavian Stroe-Romania

Lema:

2) If $a,b,c\geq 0$***,*** $ab+bc+ca=3$ ***then find min*** $f(a,b,c)$***,***

$$f(a,b,c)=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Demonstration:

We have $f(a,b,c)=\sum\frac{1}{b+c}=\frac{\sum(a+b)(a+c)}{\prod(b+c)}=\frac{(a+b+c)^2+ab+bc+ca}{(a+b+c)(ab+bc+ca)-abc}=$

$$=\frac{(a+b+c)^2+3}{3(a+b+c)-abc}=\frac{x^2+3}{3x-abc}\overset{abc\geq 0}{\geq}\frac{x^2+3}{3x}=\frac{x}{3}+\frac{1}{x}\text{, (1), then } x=a+b+c.$$

Also $\sum\frac{a}{b+c}=\sum a\sum\frac{1}{b+c}-3$ ***we get:***

$$f(a,b,c)=\frac{1}{a+b+c}\left(3+\sum\frac{a}{b+c}\right)\overset{Bergstrom}{\geq}\frac{1}{a+b+c}\left(3+\frac{(a+b+c)^2}{2\sum bc}\right)=\frac{1}{x}\left(3+\frac{x^2}{2\cdot 3}\right)=$$

$$=\frac{x}{6}+\frac{3}{x}\text{, (2).}$$

From: (1) +3· (2) we have: $4f(a,b,c)\geq\frac{5x}{6}+\frac{10}{x}\Leftrightarrow f(a,b,c)\geq\frac{5x}{24}+\frac{5}{2x}$.

We get $f(a,b,c)\geq\frac{5x}{24}+\frac{5}{2x}=\frac{5}{2}\left(\frac{x}{12}+\frac{1}{x}\right)\overset{AM-GM}{\geq}\frac{5}{2}\cdot 2\sqrt{\frac{x}{12}\cdot\frac{1}{x}}=5\sqrt{\frac{1}{12}}=\frac{5}{2\sqrt{3}}$,

Equality for $\frac{x}{12}=\frac{1}{x}\Leftrightarrow x^2=12\Leftrightarrow x=2\sqrt{3}$.

We deduce min $f(a,b,c)$ ***is*** $\frac{5}{2\sqrt{3}}$ ***for*** $a+b+c=2\sqrt{3}$***, (for example*** $a=b=\sqrt{3},c=0$***).***

Using Lema and Hőlder inequality we get:

$$\Omega(a,b,c)=\frac{1}{(a+b)^5}+\frac{1}{(b+c)^5}+\frac{1}{(c+a)^5}\geq\frac{\left(\sum\frac{1}{b+c}\right)^5}{3^4}\geq\frac{\left(\frac{5}{2\sqrt{3}}\right)^5}{3^4}=3\cdot\left(\frac{5}{6\sqrt{3}}\right)^5.$$

We deduce min $\Omega(a,b,c)$ ***is*** $3\cdot\left(\frac{5}{6\sqrt{3}}\right)^5$ ***for*** $a+b+c=2\sqrt{3}$***, (for example*** $a=b=\sqrt{3},c=0$***).***

Remark.

The problem can be developed:

3) If $a,b,c\geq 0$***,*** $ab+bc+ca=3$ ***then find min*** $\Omega(a,b,c)$***,***

$$\Omega(a,b,c)=\frac{1}{(a+b)^n}+\frac{1}{(b+c)^n}+\frac{1}{(c+a)^n},\ n\in\mathbf{N}^*.$$

Marin Chirciu and Octavian Stroe-Romania

Solution

Using Lema and Hőlder inequality we get:

$$\Omega(a,b,c)=\frac{1}{(a+b)^n}+\frac{1}{(b+c)^n}+\frac{1}{(c+a)^n}\geq\frac{\left(\sum\frac{1}{b+c}\right)^n}{3^{n-1}}\geq\frac{\left(\frac{5}{2\sqrt{3}}\right)^n}{3^{n-1}}=3\cdot\left(\frac{5}{6\sqrt{3}}\right)^n.$$

We deduce min $\Omega(a,b,c)$ ***is*** $3\cdot\left(\frac{5}{6\sqrt{3}}\right)^n$ ***for*** $a+b+c=2\sqrt{3}$***, (for example*** $a=b=\sqrt{3},c=0$***).***

Note:

For $n=5$ ***we find the Problem JP.297 from RMM number 20, Spring 2021, proposed by Hoang Le Nhat Tung, Hanoi, Vietnam.***

For $n=1$ ***we get Lema.***

JP.298 If $x,y,z\in\mathbb{R}$ such that:

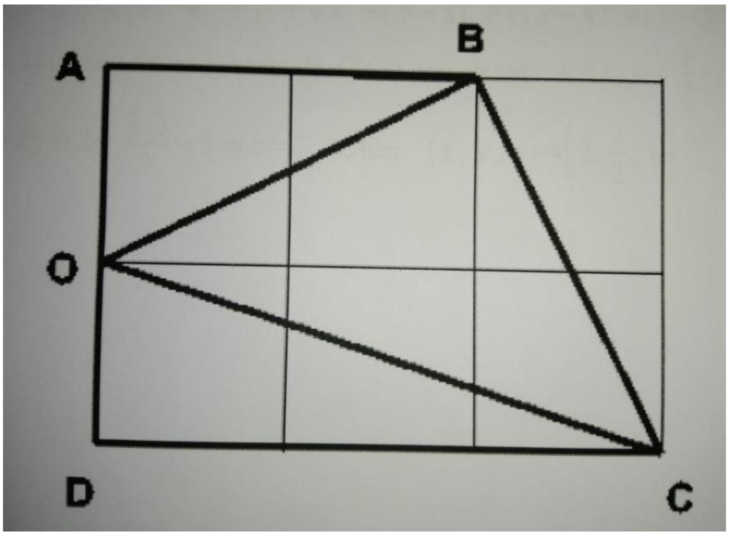
$$\begin{cases}\tan x(\tan y+\tan z)=5\\ \tan z(\tan x+\tan y)=9\\ \tan y(\tan z+\tan x)=8\end{cases}$$

then find $\Omega=x+y+z$.

Proposed by Daniel Sitaru-Romania

Solution by Daniel Văcaru-Romania

Solving system, we obtain $\tan x=1;\tan y=2;\tan z=3$



This figure shows that $\Omega = tan^{-1}1 + tan^{-1}2 + tan^{-1}3 = \pi$, *because*

$OB^2 = 5; BC^2 = 5; OC^2 = 10$, *showing that* $\widehat{BOC} = tan^{-1}1$

JP.299 If $a, b, c > 0; abc = 4$ **then:**

$$\frac{2(a^5+b^5)+c^5}{(a+b)^2} + \frac{2(b^5+c^5)+a^5}{(b+c)^2} + \frac{2(c^5+a^5)+b^5}{(c+a)^2} \geq 15$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$3a^5 + b^5 + c^5 \overset{AM-GM}{\geq} 5 \cdot \sqrt[5]{(a^5)^3 b^5 c^5} = 5a^3bc \quad (1)$$

$$a^5 + 3b^5 + c^5 \overset{AM-GM}{\geq} 5 \cdot \sqrt[5]{a^5 (b^5)^3 c^5} = 5ab^3c \quad (2)$$

$$2a^5 + 2b^5 + c^5 \overset{AM-GM}{\geq} 5 \cdot \sqrt[5]{(a^5)^2 \cdot (b^5)^2 \cdot c^5} = 5a^2b^2c \quad (3)$$

$$2a^5 + 2b^5 + c^5 \overset{AM-GM}{\geq} 5 \cdot \sqrt[5]{(a^5)^2 \cdot (b^5) \cdot c^5} = 5a^2b^2c \quad (4)$$

By adding (1); (2); (3); (4):

$$8a^5 + 8b^5 + 4c^5 \geq 5abc(a^2 + 2ab + b^2)$$

$$2a^5 + 2b^5 + c^5 \geq \frac{5}{4}abc(a+b)^2, \qquad \frac{2(a^5+b^5)+c^5}{(a+b)^2} \geq \frac{5}{4}abc$$

$$\sum_{cyc}\frac{2(a^5+b^5)+c^5}{(a+b)^2}\geq\frac{5}{4}\sum_{cyc}abc=\frac{5}{4}\cdot 3abc=\frac{15}{4}\cdot 4=15$$

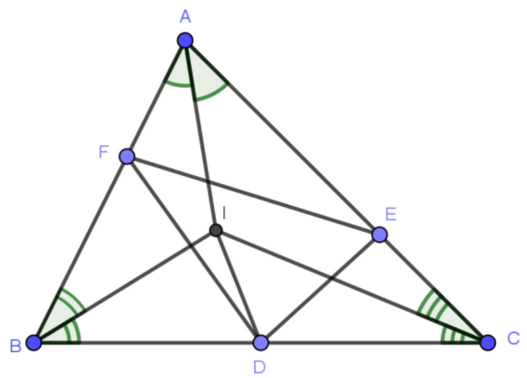
Equality holds for $a=b=c=\sqrt[3]{4}$

JP.300. In $\Delta ABC, I$ - incenter, ID, IE, IF – symedians in $\Delta BIC, \Delta CIA, \Delta AIB, D\in(BC), E\in(CA), F\in(AB)$. Prove that:

$$\frac{[DEF]}{[ABC]}\geq\frac{r^2}{2R^2-Rr-2r^2}$$

Proposed by Marian Ursărescu – Romania

Solution by Tran Hong-Dong Thap-Vietnam



In ΔIBC:

$$\frac{DB}{DC}=\frac{IB^2}{IC^2}\Rightarrow\frac{DB}{BC}=\frac{IB^2}{IB^2+IC^2}$$

$$\Rightarrow DB=\frac{IB^2}{IB^2+IC^2}\cdot a; DC=\frac{IC^2}{IB^2+IC^2}\cdot a$$

Similarly:

$$FA=\frac{IA^2}{IA^2+IB^2}\cdot c; FB=\frac{IB^2}{IA^2+IB^2}$$

$$EA=\frac{IA^2}{IA^2+IC^2}\cdot b; EC=\frac{IC^2}{IA^2+IC^2}\cdot b$$

$$\Rightarrow\Omega=[AFE]+[BFD]+[DCE]$$

$$=\frac{1}{2}\cdot AF\cdot AE\cdot \sin A+\frac{1}{2}BF\cdot BD\cdot \sin B+\frac{1}{2}\cdot CD\cdot CE\cdot \sin C$$

$$=\frac{1}{2}\cdot\frac{IA^2\cdot IA^2\cdot bc\cdot \sin A}{(IA^2+IB^2)(IA^2+IC^2)}+\frac{1}{2}\cdot\frac{IB^2\cdot IB^2\cdot c\cdot a\cdot \sin B}{(IA^2+IB^2)(IB^2+IC^2)}+\frac{1}{2}\cdot\frac{IC^2\cdot IC^2\cdot ab\cdot \sin C}{(IB^2+IC^2)(IA^2+IC^2)}$$

$$=\left[\frac{IA^4}{(IA^2+IB^2)(IA^2+IC^2)}+\frac{IB^4}{(IB^2+IA^2)(IB^2+IC^2)}+\frac{IC^4}{(IC^2+IB^2)(IC^2+IA^2)}\right]\cdot[ABC]$$

$$\Rightarrow [EFD]=[ABC]-\Omega=\frac{(IA^2+IB^2)(IB^2+IC^2)(IC^2+IA^2)-[IA^4(IB^2+IC^2)+IB^4(IA^2+IC^2)+IC^4(IA^2+IB^2)]}{(IA^2+IB^2)(IB^2+IC^2)(IC^2+IA^2)}\cdot[ABC]$$

$$=\frac{2(IA\cdot IB\cdot IC)^2}{(IA^2+IB^2)(IB^2+IC^2)(IC^2+IA^2)}[ABC]=\frac{2\left(\prod_{cyc}\sin\frac{A}{2}\right)^2}{\prod_{cyc}\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}\right)}\cdot[ABC]$$

$$\prod\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}\right)=\left(\sum_{cyc}\sin^2\frac{A}{2}\right)\cdot\left(\sum_{cyc}\left(\sin\frac{A}{2}\sin\frac{B}{2}\right)^2\right)-\prod_{cyc}\left(\sin\frac{A}{2}\right)^2$$

$$=\left(\frac{2R-r}{2R}\right)\cdot\left(\frac{s^2+r^2-8Rr}{16R^2}\right)-\left(\frac{r}{4R}\right)^2$$

$$=\frac{(2R-r)(s^2+r^2-8Rr)-2Rr^2}{32R^3}$$

$$\Rightarrow\frac{[EFD]}{[ABC]}=\frac{2\left(\frac{r}{4R}\right)^2}{\frac{(2R-r)(s^2+r^2-8Rr)-2Rr^2}{32R^3}}=\frac{4Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}\overset{(*)}{\geq}\frac{r^2}{2R^2-Rr-2r^2}$$

$$(*)\Leftrightarrow 4R(2R^2-Rr-2r^2)\geq(2R-r)(s^2+r^2-8Rr)-2Rr^2$$

But:

$$s^2\leq 4R^2+4Rr+3r^2\Rightarrow(2R-r)\left(s^2+r^2-8Rr\right)\leq(2R-r)\left(4R^2+4r^2-4Rr\right)-2Rr^2$$

We need to prove:

$$4R(2R^2-Rr-2r^2)\geq 4(2R-r)(R^2-Rr+r^2)-2Rr^2$$

$$\Leftrightarrow 4R^3-2rR^2-4Rr^2\geq 4R^3-4rR^2+4r^2R-2rR^2+2Rr^2-2r^2-Rr^2$$

$$\Leftrightarrow 4rR^2\geq 9Rr^2-2r^3\Leftrightarrow 4R^2-9Rr+2r^2\geq 0\Leftrightarrow(R-2r)(4R-r)\geq 0$$

True because: $R\geq 2r\Rightarrow R-2r\geq 0; 4R-r\geq 8r-r=7r>0$

$\Rightarrow$ ***(*) is true*** $\Rightarrow$ ***proved.***

SP.286. In ΔABC the following relationship holds:

$$\frac{\sqrt[4]{27}}{\sqrt{(n+1)R}} \le \frac{1}{\sqrt{na+b}} + \frac{1}{\sqrt{nb+c}} + \frac{1}{\sqrt{nc+a}} \le \frac{\sqrt[4]{27}}{\sqrt{(n+1)2r}}; n \in \mathbb{N}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Tran Hong-Dong Thap-Vietnam

$$\text{Let: } \varphi(t) = \frac{1}{\sqrt{t}}; (t > 0); \varphi'(t) = -\frac{1}{2}t^{-\frac{3}{2}}; \varphi''(t) = \frac{3}{4}t^{-\frac{5}{2}} > 0, \forall t > 0$$

$$\text{Let: } \Omega = \frac{1}{\sqrt{na+b}} + \frac{1}{\sqrt{nb+c}} + \frac{1}{\sqrt{nc+a}}; (n \in \mathbb{N})$$

$$\Omega = \varphi(na+b) + \varphi(nb+c) + \varphi(nc+a) \overset{Jensen}{\ge} 3\varphi\left(\frac{(n+1)(a+b+c)}{3}\right)$$

$$= 3\varphi\left(\frac{2(n+1)s}{3}\right) = \frac{3}{\sqrt{\frac{2(n+1)s}{3}}} = \frac{3\sqrt{3}}{\sqrt{2(n+1)s}} \overset{(*)}{\ge} \frac{\sqrt[4]{27}}{\sqrt{(n+1)R}}$$

$$(*) \Leftrightarrow 3\sqrt{3R} \ge \sqrt{2s} \cdot \sqrt[4]{27} \Leftrightarrow 729R^2 \ge 4 \cdot 27s^2 \Leftrightarrow 27R^2 \ge 4s^2 \Leftrightarrow s \le \frac{3\sqrt{3}}{2}R \Rightarrow (*) \text{ is true.}$$

$$\Omega = \frac{1}{\sqrt{na+b}} + \frac{1}{\sqrt{nb+c}} + \frac{1}{\sqrt{nc+a}} \overset{BCS}{\le} \sqrt{3} \cdot \sqrt{\frac{1}{na+b} + \frac{1}{nb+c} + \frac{1}{nc+a}}$$

$$= \sqrt{3} \cdot \sqrt{\frac{1}{\underbrace{a+a+\dots+a}_{n-timmes}+b} + \frac{1}{\underbrace{b+b+\dots+b}_{n-timmes}+c} + \frac{1}{\underbrace{c+c+\dots+c}_{n-timmes}+a}}$$

$$\overset{BCS}{\le} \sqrt{3} \cdot \sqrt{\frac{1}{(n+1)^2}(n+1)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} = \sqrt{3} \cdot \sqrt{\frac{(n+1)}{(n+1)^2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}$$

$$= \sqrt{\frac{3}{n+1}} \cdot \sqrt{\frac{ab+bc+ca}{abc}} = \sqrt{\frac{3}{n+1}} \cdot \sqrt{\frac{s^2+4Rr+r^2}{4Rrs}} \overset{(**)}{\le} \frac{\sqrt[4]{27}}{\sqrt{(n+1)2r}}$$

$$(**) \Leftrightarrow \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{s^2+4Rr+r^2}{Rs}} \le \sqrt[4]{27} \Leftrightarrow \frac{3}{2} \cdot \frac{s^2+4Rr+r^2}{Rs} \le 3\sqrt{3}$$

$$s^2 + 4Rr + r^2 \le 2Rs\sqrt{3}$$

But: $s \le \frac{3\sqrt{3}}{2}R \Rightarrow s^2 \le \frac{3\sqrt{3}}{2}sR \Rightarrow s^2 + 4Rr + r^2 \le \frac{3\sqrt{3}}{2}sR + 4Rr + r^2 \overset{(1)}{\le} 2Rs\sqrt{3}$

$$(1) \Leftrightarrow \frac{3\sqrt{3}}{2}sR + 4Rr + r^2 \le 2Rs\sqrt{3} \Leftrightarrow \frac{\sqrt{3}}{2}sR \ge 4Rr + r^2$$

But: $s \ge 3\sqrt{3}r \Rightarrow \frac{\sqrt{3}}{2}sR \ge \frac{9Rr}{2}$

We need to prove: $\frac{9Rr}{2} \ge 4Rr + r^2 \Leftrightarrow 9Rr \ge 8Rr + 2r^2 \Leftrightarrow Rr \ge 2r^2 \Leftrightarrow R \ge 2r$ *(Euler)*

$\Rightarrow (1)$ *is true.Proved.*

SP.287. In ΔABC the following relationship holds:

$$\frac{2}{(n+1)R} \le \frac{1}{nm_a + m_b} + \frac{1}{nm_b + m_c} + \frac{1}{nm_c + m_a} \le \frac{1}{(n+1)r}, n \in \mathbb{N}$$

Proposed by George Apostolopoulos-Greece

Solution 1 by Marian Ursărescu-Romania

From Bergstrom inequality, we have:

$$\frac{1}{nm_a + m_b} + \frac{1}{nm_b + m_c} + \frac{1}{nm_c + m_a} \ge \frac{(1+1+1)^2}{(n+1)(m_a + m_b + m_c)}$$

We must show:

$$\frac{9}{(n+1)(m_a + m_b + m_c)} \ge \frac{2}{(n+1)R} \Leftrightarrow m_a + m_b + m_c \le \frac{9R}{2}; (1)$$

From Cauchy inequality, we have:

$$(m_a + m_b + m_c)^2 \le 3(m_a^2 + m_b^2 + m_c^2) \text{ and}$$

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2) \Rightarrow$$

$$m_a^2 + m_b^2 + m_c^2 \le \frac{9}{4}(a^2 + b^2 + c^2) \le \frac{9}{4} \cdot 9R^2 = \frac{81R^2}{4} \Rightarrow$$

$$m_a + m_b + m_c \le \frac{9R}{2} \Rightarrow (1) \text{it's true.}$$

Again from Cauchy inequality,we have:

$$(m_a + m_a + \cdots + m_a + m_b)\left(\frac{1}{m_a} + \frac{1}{m_a} + \cdots + \frac{1}{m_a} + \frac{1}{m_b}\right) \ge (n+1)^2 \Rightarrow$$

$$(nm_a+m_b)\left(\frac{n}{m_a}+\frac{1}{m_b}\right)\geq(n+1)^2\Rightarrow$$

$$nm_a+m_b\geq\frac{(n+1)^2}{\frac{n}{m_a}+\frac{1}{m_b}}\Rightarrow\frac{1}{nm_a+m_b}\leq\frac{\frac{n}{m_a}+\frac{1}{m_b}}{(n+1)^2}\ \textit{and similary}$$

$$\frac{1}{nm_a+m_b}+\frac{1}{nm_b+m_c}+\frac{1}{nm_c+m_a}\leq$$

$$\leq\frac{1}{(n+1)^2}\left(n\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)+\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)$$

$$\frac{1}{nm_a+m_b}+\frac{1}{nm_b+m_c}+\frac{1}{nm_c+m_a}\leq\frac{1}{n+1}\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)$$

We must show that: $\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\leq\frac{1}{r};(2)$

But $m_a\geq\sqrt{s(s-a)}$ *and* $r=\frac{S}{s};(3)$

From $(2),(3)$ *we must show:*

$$\frac{1}{\sqrt{s(s-a)}}+\frac{1}{\sqrt{s(s-b)}}+\frac{1}{\sqrt{s(s-c)}}\leq\frac{s}{S}=\frac{s}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$\frac{1}{\sqrt{s-a}}+\frac{1}{\sqrt{s-b}}+\frac{1}{\sqrt{s-c}}\leq\frac{s}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$\sqrt{(s-a)(s-b)}+\sqrt{(s-b)(s-c)}+\sqrt{(s-c)(s-a)}\leq s;(4)$$

But $\sqrt{(s-a)(s-b)}\leq\frac{s-a+s-b}{2}=\frac{c}{2}$ *and similary,then*

$$\sqrt{(s-a)(s-b)}+\sqrt{(s-b)(s-c)}+\sqrt{(s-c)(s-a)}\leq\frac{a+b+c}{2}=s$$

$\Rightarrow(4)$ **it's true.**

Solution 2 by Marin Chirciu-Romania

Using Bergstrőm Inequality, we get:

$$\sum_{cyc}\frac{1}{n\cdot m_a+m_b}\geq\frac{9}{\sum(n\cdot m_a+m_b)}=\frac{9}{(n+1)\sum m_a}\overset{(1)}{\geq}\frac{2}{(n+1)R}$$

where $(1)\Leftrightarrow\sum m_a\leq\frac{9R}{2}$*, which result from*

$$\sum m_a\leq 4R+r\leq\frac{9R}{2}.$$

$\sum_{cyc}\frac{1}{n\cdot m_a+m_b}\leq\frac{1}{(n+1)r}$ *which result from*

$$(x_1+x_2+\dots+x_k)\left(\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_k}\right)\geq k^2$$

We obtain

$$n\cdot m_a+m_b=(m_a+m_a+\dots+m_a+m_b)\left(\frac{1}{m_a}+\frac{1}{m_a}+\dots+\frac{1}{m_a}+\frac{1}{m_b}\right)\geq(n+1)^2 \text{ where}$$

$$n\cdot m_a+m_b\geq\frac{(n+1)^2}{\frac{n}{m_a}+\frac{1}{m_b}}\Leftrightarrow\frac{1}{n\cdot m_a+m_b}\leq\frac{\frac{n}{m_a}+\frac{1}{m_b}}{(n+1)^2} \text{ and then result}$$

$$\sum_{cyc}\frac{1}{n\cdot m_a+m_b}\leq\sum_{cyc}\frac{\frac{n}{m_a}+\frac{1}{m_b}}{(n+1)^2}=\frac{1}{(n+1)^2}\sum_{cyc}\left(\frac{n}{m_a}+\frac{1}{m_b}\right)=\frac{1}{(n+1)^2}\cdot(n+1)\sum_{cyc}\frac{1}{m_a}$$

$$=\frac{1}{n+1}\sum_{cyc}\frac{1}{m_a}\overset{(2)}{\leq}\frac{1}{(n+1)}\cdot\frac{1}{r}$$

Where $(2)\Leftrightarrow\sum_{cyc}\frac{1}{m_a}\leq\frac{1}{r}$ *which result from* $\sum_{cyc}\frac{1}{m_a}\leq\sum_{cyc}\frac{1}{h_a}=\frac{1}{r}$.

Equality holds if and only if the triangle is equilateral.

Remark.

In same class of the problems:

1) In ΔABC the following relationship holds:

$$\frac{2}{(n+1)R}\leq\sum_{cyc}\frac{1}{n\cdot h_a+h_b}\leq\frac{1}{(n+1)r}, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Bergstrőm Inequality, we get:

$$\sum_{cyc}\frac{1}{n\cdot h_a+h_b}\geq\frac{9}{\sum n\cdot h_a+h_b}=\frac{9}{(n+1)\sum h_a}\overset{(1)}{\geq}\frac{2}{(n+1)R}$$

where $(1)\Leftrightarrow\sum h_a\leq\frac{9R}{2}$ *which result from*

$$\sum h_a\leq\sum m_a\leq 4R+r\leq\frac{9R}{2}$$

$\sum_{cyc}\frac{1}{n\cdot h_a+h_b}\le\frac{1}{(n+1)r}$ *which result from*

$$(x_1+x_2+\cdots+x_k)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}\right)\ge k^2$$

We obtain

$$n\cdot h_a+h_b=(h_a+h_a+\cdots+h_a+h_b)\left(\frac{1}{h_a}+\frac{1}{h_a}+\cdots+\frac{1}{h_a}+\frac{1}{h_b}\right)\ge(n+1)^2 \text{ where}$$

$$n\cdot h_a+h_b\ge\frac{(n+1)^2}{\frac{n}{h_a}+\frac{1}{h_b}}\Leftrightarrow\frac{1}{n\cdot h_a+h_b}\le\frac{\frac{n}{h_a}+\frac{1}{h_b}}{(n+1)^2} \text{ and then result}$$

$$\sum_{cyc}\frac{1}{n\cdot h_a+h_b}\le\sum_{cyc}\frac{\frac{n}{h_a}+\frac{1}{h_b}}{(n+1)^2}=\frac{1}{(n+1)^2}\sum_{cyc}\left(\frac{n}{h_a}+\frac{1}{h_b}\right)=\frac{1}{(n+1)^2}\cdot(n+1)\sum_{cyc}\frac{1}{h_a}$$

$$=\frac{1}{n+1}\sum_{cyc}\frac{1}{h_a}\overset{(2)}{\le}\frac{1}{(n+1)}\cdot\frac{1}{r}$$

Where $(2)\Leftrightarrow\sum_{cyc}\frac{1}{h_a}=\frac{1}{r}$.

Equality holds of and only if the triagle is equilateral.

2) In ΔABC the following relationship holds:

$$\frac{2}{(n+1)R}\le\sum_{cyc}\frac{1}{n\cdot w_a+w_b}\le\frac{1}{(n+1)r},n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Bergstrőm Inequality,we get:

$$\sum_{cyc}\frac{1}{n\cdot w_a+w_b}\ge\frac{9}{\sum(n\cdot w_a+w_b)}=\frac{9}{(n+1)\sum w_a}\overset{(1)}{\ge}\frac{2}{(n+1)R}$$

where $(1)\Leftrightarrow\sum w_a\le\frac{9R}{2}$ *which result from*

$$\sum w_a\le\sum m_a\le 4R+r\le\frac{9R}{2}$$

$\sum_{cyc}\frac{1}{n\cdot w_a+w_b}\le\frac{1}{(n+1)r}$ *which result from*

$$(x_1+x_2+\cdots+x_k)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}\right)\geq k^2$$

We obtain

$$n\cdot w_a+w_b=(w_a+w_a+\cdots+w_a+w_b)\left(\frac{1}{w_a}+\frac{1}{w_a}+\cdots+\frac{1}{w_a}+\frac{1}{w_b}\right)\geq(n+1)^2 \text{ where}$$

$$n\cdot w_a+w_b\geq\frac{(n+1)^2}{\frac{n}{w_a}+\frac{1}{w_b}}\Leftrightarrow\frac{1}{n\cdot w_a+w_b}\leq\frac{\frac{n}{w_a}+\frac{1}{w_b}}{(n+1)^2} \text{ and then result}$$

$$\sum_{cyc}\frac{1}{n\cdot w_a+w_b}\leq\sum_{cyc}\frac{\frac{n}{w_a}+\frac{1}{w_b}}{(n+1)^2}=\frac{1}{(n+1)^2}\sum_{cyc}\left(\frac{n}{w_a}+\frac{1}{w_b}\right)=\frac{1}{(n+1)^2}\cdot(n+1)\sum_{cyc}\frac{1}{w_a}$$

$$=\frac{1}{n+1}\sum_{cyc}\frac{1}{w_a}\overset{(2)}{\leq}\frac{1}{(n+1)}\cdot\frac{1}{r}$$

Where $(2)\Leftrightarrow\sum_{cyc}\frac{1}{w_a}\leq\frac{1}{r}$ *which result from* $\sum_{cyc}\frac{1}{w_a}\leq\sum_{cyc}\frac{1}{h_a}=\frac{1}{r}$.

Equality holds if and only if the triangle is equilateral.

In ΔABC the following relationship holds:

$$\frac{2}{(n+1)R}\leq\sum_{cyc}\frac{1}{n\cdot r_a+r_b}\leq\frac{1}{(n+1)r}, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Bergstrőm Inequality,we get:

$$\sum_{cyc}\frac{1}{n\cdot r_a+r_b}\geq\frac{9}{\sum(n\cdot r_a+r_b)}=\frac{9}{(n+1)\sum r_a}\overset{(1)}{\geq}\frac{2}{(n+1)R}$$

where $(1)\Leftrightarrow\sum r_a\leq\frac{9R}{2}$ *which result from*

$$\sum r_a\leq\sum m_a\leq 4R+r\leq\frac{9R}{2}$$

$\sum_{cyc}\frac{1}{n\cdot r_a+r_b}\leq\frac{1}{(n+1)r}$ *which result from*

$$(x_1+x_2+\cdots+x_k)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}\right)\ge k^2$$

We obtain

$$n\cdot r_a+r_b=(r_a+r_a+\cdots+r_a+r_b)\left(\frac{1}{r_a}+\frac{1}{r_a}+\cdots+\frac{1}{r_a}+\frac{1}{r_b}\right)\ge(n+1)^2 \text{ where}$$

$$n\cdot r_a+r_b\ge\frac{(n+1)^2}{\frac{n}{r_a}+\frac{1}{r_b}}\Leftrightarrow\frac{1}{n\cdot r_a+r_b}\le\frac{\frac{n}{r_a}+\frac{1}{r_b}}{(n+1)^2} \text{ and then result}$$

$$\sum_{cyc}\frac{1}{n\cdot r_a+r_b}\le\sum_{cyc}\frac{\frac{n}{r_a}+\frac{1}{r_b}}{(n+1)^2}=\frac{1}{(n+1)^2}\sum_{cyc}\left(\frac{n}{r_a}+\frac{1}{r_b}\right)=\frac{1}{(n+1)^2}\cdot(n+1)\sum_{cyc}\frac{1}{r_a}$$

$$=\frac{1}{n+1}\sum_{cyc}\frac{1}{r_a}\overset{(2)}{=}\frac{1}{(n+1)}\cdot\frac{1}{r}$$

Where $(2)\Leftrightarrow\sum_{cyc}\frac{1}{r_a}=\frac{1}{r}$

Equality holds if and only if the triangle is equilateral.

SP.288. Let $a,b,c>0, a+b+c=3$. Find $min(\Omega)$

$$\Omega=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sqrt[3]{\frac{ab}{a^2-ab+b^2}}+\sqrt[3]{\frac{bc}{b^2-bc+c^2}}+\sqrt[3]{\frac{ca}{c^2-ca+a^2}}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

For $a,b>0$ we have:

$$\sqrt[3]{\frac{ab}{a^2-ab+b^2}}=\sqrt[3]{1\cdot1\cdot\frac{ab}{a^2-ab+b^2}}\overset{Gm-Hm}{\ge}\frac{3}{\frac{1}{1}+\frac{1}{1}+\frac{1}{\frac{ab}{a^2-ab+b^2}}}=$$

$$=\frac{3}{1+1+\frac{1}{\frac{ab}{a^2-ab+b^2}}}=\frac{3ab}{2ab+a^2-ab+b^2}=\frac{3ab}{a^2+ab+b^2};(1)$$

Similary: $\sqrt[3]{\frac{bc}{b^2-bc+c^2}}\ge\frac{3bc}{b^2+bc+c^2};(2)$ *and* $\sqrt[3]{\frac{ca}{c^2-ca+a^2}}\ge\frac{3ca}{c^2+ca+a^2};(3)$

From (1),(2),(3) we have:

$$\sum_{cyc}\sqrt[3]{\frac{ab}{a^2-ab+b^2}}\geq 3\left(\frac{ab}{a^2+ab+b^2}+\frac{bc}{b^2+bc+c^2}+\frac{ca}{c^2+ca+a^2}\right)=$$

$$=3abc\left(\frac{1}{a^2c+abc+b^2c}+\frac{1}{b^2a+abc+c^2a}+\frac{1}{bc^2+abc+ba^2}\right)\overset{Bergström}{\geq}$$

$$\geq 3abc\cdot\frac{(1+1+1)^2}{ab(a+b)+bc(b+c)+ca(c+a)+3abc}=\frac{9\cdot 3abc}{(a+b+c)(ab+bc+ca)}=$$

$$\overset{a+b+c=3}{=}\frac{9abc}{ab+bc+ca}=\frac{9}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$ ***Hence,***

$$\Omega=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum_{cyc}\sqrt[3]{\frac{ab}{a^2-ab+b^2}}\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{9}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\overset{Am-Gm}{\geq}$$

$$\geq 2\sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\cdot\frac{9}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}}=2\sqrt{9}=6\Rightarrow\Omega\geq 6$$

$$min(\Omega)=6\Leftrightarrow a=b=c=1.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x\in\mathbb{R},x\geq 2$ ***we have:*** $x+1+\sqrt[3]{\frac{27}{x-1}}\geq 6\Leftrightarrow(x+1)\sqrt[3]{x-1}+3\geq 6\sqrt[3]{x-1}\overset{a=\sqrt[3]{x-1}}{\Longleftrightarrow}$

$$(a^3+2)a+3\geq 6a\Leftrightarrow a^4+2a+3\geq 6a\Leftrightarrow a^4+3\geq 4a\Leftrightarrow a^3\cdot a+1+2\geq 4a$$

$$a^3+a+2\geq 4a\Leftrightarrow a^3+2\geq 3a\Leftrightarrow a^2\cdot a+1+1\geq 3a\Leftrightarrow a^2+a+1\geq 3a\Leftrightarrow$$

$$a^2+1\geq 2a\Leftrightarrow a\cdot a+1\geq 2a \text{ true.}$$

Hence for $a,b,c>0,a+b+c=3$ ***we give*** $a=\frac{3x}{x+y+z};b=\frac{3y}{x+y+z};c=\frac{3z}{x+y+z}$***. Hence,***

$$\Omega=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sqrt[3]{\frac{ab}{a^2-ab+b^2}}+\sqrt[3]{\frac{bc}{b^2-bc+c^2}}+\sqrt[3]{\frac{ca}{c^2-ca+a^2}}=$$

$$\frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}+1\right)+\sqrt[3]{\frac{27}{\frac{x}{y}+\frac{y}{x}-1}}+\frac{1}{3}\left(\frac{y}{z}+\frac{z}{y}+1\right)+\sqrt[3]{\frac{27}{\frac{y}{z}+\frac{z}{y}-1}}+\frac{1}{3}\left(\frac{x}{z}+\frac{z}{x}+1\right)$$

$$+\sqrt[3]{\frac{27}{\frac{x}{z}+\frac{z}{x}-1}}\geq 6$$

$$\frac{x}{y}+\frac{y}{x}+1+\sqrt[3]{\frac{27}{\frac{x}{y}+\frac{y}{x}-1}}+\frac{y}{z}+\frac{z}{y}+1+\sqrt[3]{\frac{27}{\frac{y}{z}+\frac{z}{y}-1}}+\frac{x}{z}+\frac{z}{x}+1+\sqrt[3]{\frac{27}{\frac{x}{z}+\frac{z}{x}-1}}\geq 18$$

Because: $\frac{x}{y}+\frac{y}{x}\geq 2;\ \frac{y}{z}+\frac{z}{y}\geq 2;\ \frac{x}{z}+\frac{z}{x}\geq 2$

Hence, $\Omega\geq 6\Rightarrow min(\Omega)=6\Leftrightarrow a=b=c=1.$

Solution 3 by proposer

Using Am-Gm inequality for three positive real numbers, we have:

$$a^2+ab+b^2=(a^2-ab+b^2)+ab+ab\geq 3\sqrt[3]{(a^2-ab+b^2)\cdot ab\cdot ab}$$

$$=3\sqrt[3]{(ab)^2(a^2-ab+b^2)}\Rightarrow$$

$$\sqrt[3]{ab}(a^2+ab+b^2)\geq 3ab\sqrt[3]{(a^2-ab+b^2)}\Leftrightarrow$$

$$\sqrt[3]{\frac{ab}{a^2-ab+b^2}}\geq\frac{3ab}{a^2+ab+b^2}$$

Similary: $\sqrt[3]{\frac{bc}{b^2-bc+c^2}}\geq\frac{3bc}{b^2+bc+c^2}$ *and* $\sqrt[3]{\frac{ca}{c^2-ca+a^2}}\geq\frac{3ca}{c^2+ca+a^2}$

$$\sqrt[3]{\frac{ab}{a^2-ab+b^2}}+\sqrt[3]{\frac{bc}{b^2-bc+c^2}}+\sqrt[3]{\frac{ca}{c^2-ca+a^2}}\geq$$

$$\geq\frac{3ab}{a^2+ab+b^2}+\frac{3bc}{b^2+bc+c^2}+\frac{3ca}{c^2+ca+a^2};(1)$$

Other, because $a+b+c=3$ *then*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{3}\cdot 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{3}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\Leftrightarrow$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{3}\left(\frac{a}{b}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{b}+1+\frac{a}{c}+\frac{c}{a}+1\right)=$$

$$=\frac{1}{3}\left(\frac{a^2+ab+b^2}{ab}+\frac{b^2+bc+c^2}{bc}+\frac{c^2+ca+a^2}{ca}\right)\Leftrightarrow$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{a^2+ab+b^2}{3ab}+\frac{b^2+bc+c^2}{3bc}+\frac{c^2+ca+a^2}{3ca};(2)$$

Let $(1),(2)$ with Am-GM inequality, we get:

$$\Omega=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sqrt[3]{\frac{ab}{a^2-ab+b^2}}+\sqrt[3]{\frac{bc}{b^2-bc+c^2}}+\sqrt[3]{\frac{ca}{c^2-ca+a^2}}\geq$$

$$\geq\frac{a^2+ab+b^2}{3ab}+\frac{3ab}{a^2+ab+b^2}+\frac{b^2+bc+c^2}{3bc}+\frac{3bc}{b^2+bc+c^2}+\frac{c^2+ca+a^2}{3ca}$$

$$+\frac{3ca}{c^2+ca+a^2}\geq$$

$$\geq 2\sqrt{\frac{a^2+ab+b^2}{3ab}\cdot\frac{3ab}{a^2+ab+b^2}}+2\sqrt{\frac{b^2+bc+c^2}{3bc}\cdot\frac{3bc}{b^2+bc+c^2}}$$

$$+2\sqrt{\frac{c^2+ca+a^2}{3ca}\cdot\frac{3ca}{c^2+ca+a^2}}=2\sqrt{1}+2\sqrt{1}+2\sqrt{1}=6$$

$$\Omega\geq 6\Rightarrow min(\Omega)=6$$

Equality holds if $\begin{cases}a,b,c>0;a+b+c=3\\ a=b=c\end{cases}\Leftrightarrow a=b=c=1.$

SP.289. If $x,y,z>0;x+y+z=1$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{xsinA+ysinB+zsinC}\leq\sum_{cyc}\frac{1}{sinA+sinB-sinC}$$

Proposed by George Apostolopoulos-Greece

Solution by Marian Ursărescu-Romania

From sine law: $\frac{a}{sinA}=2R\Rightarrow sinA=\frac{a}{2R}$

We must show that:

$$\sum_{cyc}\frac{1}{xa+yb+zc}\leq\sum_{cyc}\frac{1}{a+b-c};(1)$$

From Cauchy inequality,we have:

$$(xa+yb+zc)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\geq(x+y+z)^2=1\Rightarrow$$

$xa+yb+zc\geq\frac{1}{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}}\Rightarrow\frac{1}{xa+yb+zc}\leq\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ *and simillary (2)*

From (1),(2) we must show:

$$\sum_{cyc}\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\le\sum_{cyc}\frac{1}{a+b-c}\Leftrightarrow$$

$$(x+y+z)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le\frac{1}{a+b-c}+\frac{1}{a-b+c}+\frac{1}{-a+b+c}\Leftrightarrow$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{1}{a+b-c}+\frac{1}{a+b-c}+\frac{1}{-a+b+c};(3)$$

Let: $\begin{cases}-a+b+c=x\\ a-b+c=y\\ a+b-c=z\end{cases}\Rightarrow\begin{cases}a=\frac{y+z}{2}\\ b=\frac{x+z}{2};(4)\\ c=\frac{x+y}{2}\end{cases}$

From (3),(4) we must show:

$$\frac{2}{x+y}+\frac{2}{x+z}+\frac{2}{y+z}\le\frac{1}{x}+\frac{1}{y}+\frac{1}{z};(5)$$

But: $\frac{1}{x+y}\le\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right);\frac{1}{y+z}\le\frac{1}{4}\left(\frac{1}{y}+\frac{1}{z}\right);\frac{1}{x+z}\le\frac{1}{4}\left(\frac{1}{x}+\frac{1}{z}\right)\Rightarrow$

$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\le\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\Rightarrow(5)$ *it's true.Proved.*

SP.290. If $x,y,z>0;x+y+z=3$ then prove:

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+12\ge 5(xy+yz+zx)$$

When the equality occurs if and if?

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

For all $x>0$ we have:

$$\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+x^3+x^3+x^3+1\overset{AGM}{\ge}10\sqrt[10]{x^{10}}=10x\Rightarrow$$

$$6\sqrt[6]{x}+3x^3+1\ge 10x\Rightarrow 6\sqrt[6]{x}\ge 10x-3x^3-1 \textit{ simillary}:$$

$$6\sqrt[6]{y}\ge 10y-3y^3-1 \textit{ and } 6\sqrt[6]{z}\ge 10z-3z^3-1.$$

$$6\left(\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}\right)\ge 10(x+y+z)-3(x^3+y^3+z^3)-3\overset{x+y+z=3}{=}$$

$$=10\cdot 3-3(x^3+y^3+z^3)-3=27-3(x^3+y^3+z^3);(1)$$

More, from $x,y,z>0;x+y+z=3\Rightarrow 0<x,y,z<3\Rightarrow x-3<0,y-3<0,$

$z-3<0$.*We have:*

$$(x-3)(x-1)^2+(y-3)(y-1)^2+(z-3)(z-1)^2\le 0\Leftrightarrow$$

$$\sum_{cyc}[(x-3)(x^2-2x+1)]\le 0\Leftrightarrow\sum_{cyc}(x^3-5x^2+7x-3)\le 0\Leftrightarrow$$

$$\sum_{cyc}x^3-5\sum_{cyc}x^2+7\sum_{cyc}x-9\le 0\Leftrightarrow$$

$$x^3+y^3+z^3\le 5(x^2+y^2+z^2)-7(x+y+z)+9\Leftrightarrow$$

$$x^3+y^3+z^3\le 5[(x+y+z)^2-2(xy+yz+zx)]-7(x+y+z)+9\Leftrightarrow$$

$$x^3+y^3+z^3\le 5(x+y+z)^2-10(xy+yz+zx)-7\left(\underbrace{x+y+z}_{3}\right)+9\Leftrightarrow$$

$$x^3+y^3+z^3\le 5\cdot 3^2-10(xy+yz+zx)-7\cdot 3+9\Leftrightarrow$$

$$x^3+y^3+z^3\le 33-10(xy+yz+zx)\Leftrightarrow$$

$$27-3(x^3+y^3+z^3)\ge 27[33-10(xy+yz+zx)]=30(xy+yz+zx)-72;(2)$$

From (1),(2) we have:

$$6(\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z})\ge 30(xy+yz+zx)-72$$

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+12\ge 5(xy+yz+zx)$$

Equality holds if $x=y=z=1$.*Proved.*

Solution 2 by Dimitris Skouteris-Greece

Writing $xy+yz+zx=\frac{1}{2}(9-x^2-y^2-z^2)$ *the inequality becomes:*

$$\sum_{cyc}\left(\sqrt[6]{x}+\frac{5}{2}x^2\right)\ge\frac{21}{2}$$

Now, from weighted Am-Gm, we have: $\sqrt[6]{x}+\frac{5}{2}x^2\ge\frac{7}{2}x^{\frac{31}{21}}$ *and, from the power inequality,*

$$\frac{1}{3}\sum_{cyc}x^{\frac{31}{21}}\ge\left(\frac{1}{3}\sum_{cyc}x\right)^{\frac{31}{21}}=1$$

Whence the result, with equality for $x=y=z=1$.*Proved.*

Solution3 by Marin Chirciu-Romania

From $x+y+z=3$ we have: $xy+yz+zx=\frac{1}{2}(9-x^2-y^2-z^2)$.

The inequality becomes:

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}\ge 5\cdot\frac{1}{2}(9-x^2-y^2-z^2)\Leftrightarrow$$

$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+\frac{5}{2}x^2+\frac{5}{2}y^2+\frac{5}{2}z^2\ge\frac{21}{2}$; $(*)$ *which result from weighted Am-Gm:*

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+\frac{5}{2}x^2+\frac{5}{2}y^2+\frac{5}{2}z^2=x^{\frac{1}{6}}+y^{\frac{1}{6}}+z^{\frac{1}{6}}+\frac{5}{2}x^2+\frac{5}{2}y^2+\frac{5}{2}z^2=$$

$$=\sum_{cyc}\left(x^{\frac{1}{6}}+\frac{5}{2}x^2\right)\ge\sum_{cyc}\left(1+\frac{5}{2}\right)x^{\frac{\frac{1}{6}+2\cdot\frac{5}{2}}{1+\frac{5}{2}}}=\sum_{cyc}\frac{7}{2}x^{\frac{13}{21}}=\frac{7}{2}\sum_{cyc}x^{\frac{13}{21}};(1)$$

From Jensen inequality, we have $\frac{x^r+y^r+z^r}{3}\ge\left(\frac{x+y+z}{3}\right)^{\frac{1}{r}}$ with $r=\frac{13}{21}$, therefore

$$\frac{\sum x^{\frac{13}{21}}}{3}\ge\left(\frac{\sum x}{3}\right)^{\frac{21}{13}}=\left(\frac{3}{3}\right)^{\frac{21}{13}}=1\Rightarrow\sum x^{\frac{13}{21}}\ge 3;(2)$$

From (1),(2) we have:

$$Lhs_{(*)}=\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+\frac{5}{2}x^2+\frac{5}{2}y^2+\frac{5}{2}z^2\overset{(1)}{\ge}\frac{7}{2}\sum x^{\frac{13}{21}}\overset{(2)}{\ge}\frac{7}{2}\cdot 3=\frac{21}{2}=Rhs_{(*)}$$

Equality holds if and only if $x=y=z=1$.

Remark.

The problem it can be developed.

1) If $x,y,z>0$ such that $x+y+z=3$ and $n\ge 0$ then

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+3n\ge(n+1)(xy+yz+zx)$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using $x+y+z=3$ we have $xy+yz+zx=\frac{1}{2}(9-x^2+y^2+z^2)$.

The inequality becomes:

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+3n\ge(n+1)\cdot\frac{1}{2}(9-x^2-y^2-z^2)\Leftrightarrow$$

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+\frac{n+1}{2}\cdot x^2+\frac{n+1}{2}\cdot y^2+\frac{n+1}{2}\cdot z^2\geq\frac{3n+9}{2};(*)$$

Which result from weighted Am-Gm:

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+\frac{n+1}{2}\cdot x^2+\frac{n+1}{2}\cdot y^2+\frac{n+1}{2}\cdot z^2=$$

$$=x^{\frac{1}{6}}+y^{\frac{1}{6}}+z^{\frac{1}{6}}+\frac{n+1}{2}\cdot x^2+\frac{n+1}{2}\cdot y^2+\frac{n+1}{2}\cdot z^2=\sum_{cyc}\left(x^{\frac{1}{6}}+\frac{n+1}{2}\cdot x^2\right)\geq$$

$$\geq\sum_{cyc}\left(1+\frac{n+1}{2}\right)x^{\frac{\frac{1}{6}+1}{1+\frac{n+1}{2}}}=\sum_{cyc}\frac{n+3}{2}\cdot x^{\frac{13}{3n+9}}=\frac{n+3}{2}\sum_{cyc}x^{\frac{13}{3n+9}};(1)$$

From Jensen Inequaity, we have: $\frac{x^r+y^r+z^r}{3}\geq\left(\frac{x+y+z}{3}\right)^{\frac{1}{r}}$ *with* $r=\frac{13}{3n+9}$*, therefore:*

$$\frac{\sum x^{\frac{13}{3n+9}}}{3}\geq\left(\frac{\sum x}{3}\right)^{\frac{3n+9}{13}}=\left(\frac{3}{3}\right)^{\frac{3n+9}{13}}=1\Rightarrow\sum x^{\frac{13}{3n+9}}\geq 3;(2)$$

From (1),(2) we have:

$$Lhs_{(*)}=\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+\frac{n+1}{2}x^2+\frac{n+1}{2}y^2+\frac{n+1}{2}z^2\overset{(1)}{\geq}\frac{n+3}{2}\sum x^{\frac{13}{3n+9}}\overset{(2)}{\geq}$$

$$\geq\frac{n+3}{2}\cdot 3=\frac{3n+9}{2}=Rhs_{(*)}$$

Equality holds if and only if $x=y=z=1$.

Note.

For $n=4$ *we get the Problem SP.290 from 20-RMM Spring Edition 2021, Proposed by Hoang Le Nhat Tung- Hanoi- Vietnam.*

2) If $x,y,z>0$ such that $x+y+z=3, n\in\mathbb{N}, n\geq 2$ and $\lambda\geq 0$ then

$$\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+3\lambda\geq(\lambda+1)(xy+yz+zx)$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using $x+y+z=3$ *we have* $xy+yz+zx=\frac{1}{2}(9-x^2-y^2-z^2)$.

The inequality becomes:

$$\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+3\lambda \geq (\lambda+1)\cdot\frac{1}{2}(9-x^2-y^2-z^2)$$

$$\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+\frac{\lambda+1}{2}x^2+\frac{\lambda+1}{2}y^2+\frac{\lambda+1}{2}z^2 \geq \frac{3\lambda+9}{2};(*)$$

Which result from weighted Am-Gm:

$$\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+\frac{\lambda+1}{2}x^2+\frac{\lambda+1}{2}y^2+\frac{\lambda+1}{2}z^2 =$$

$$= x^{\frac{1}{n}}+y^{\frac{1}{n}}+z^{\frac{1}{n}}+\frac{\lambda+1}{2}x^2+\frac{\lambda+1}{2}y^2+\frac{\lambda+1}{2}z^2 = \sum_{cyc}\left(x^{\frac{1}{n}}+\frac{\lambda+1}{2}x^2\right) \geq$$

$$\geq \sum_{cyc}\left(1+\frac{\lambda+1}{2}\right)x^{\frac{\frac{1}{n}+2}{1+\frac{\lambda+1}{2}}} = \sum_{cyc}\frac{\lambda+3}{2}x^{\frac{2n+1}{3\lambda+9}} = \frac{\lambda+3}{2}\sum_{cyc}x^{\frac{2n+1}{3\lambda+9}};(1).$$

From Jensen Inequality,we have:

$$\frac{x^r+y^r+z^r}{3} \geq \left(\frac{x+y+z}{3}\right)^{\frac{1}{r}} \text{ with } r=\frac{2n+1}{3\lambda+9}, \text{ therefore:}$$

$$\frac{\sum x^{\frac{2n+1}{3\lambda+9}}}{3} \geq \left(\frac{\sum x}{3}\right)^{\frac{3\lambda+9}{2n+1}} = \left(\frac{3}{3}\right)^{\frac{3\lambda+9}{2n+1}} = 1 \Rightarrow \sum x^{\frac{2n+1}{3\lambda+9}} \geq 3;(2)$$

From (1) , (2) we get:

$$Lhs_{(*)} = \sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+\frac{\lambda+1}{2}x^2+\frac{\lambda+1}{2}y^2+\frac{\lambda+1}{2}z^2 \overset{(1)}{\geq} \frac{\lambda+3}{2}\sum x^{\frac{2n+1}{3\lambda+9}} \overset{(2)}{\geq}$$

$$\geq \frac{3\lambda+9}{2} = Rhd_{(*)}$$

Equality holds if and only if $x=y=z=1$.

Note.

For $n=6, \lambda=4$ we get the Problem SP.290 from 20-RMM Spring Edition 2021, Proposed by Hoang Le Nhat Tung- Hanoi-Vietnam.

Solution 4 by proposer

For all $x>0$ we have:

$$\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+\sqrt[6]{x}+x^3+x^3+x^3+1 \overset{AGM}{\geq} 10\sqrt[10]{x^{10}} = 10x \Rightarrow$$

$$6\sqrt[6]{x}+3x^3+1 \geq 10x \Rightarrow 6\sqrt[6]{x} \geq 10x-3x^3-1 \text{ simillary:}$$

$6\sqrt[6]{y} \geq 10y - 3y^3 - 1$ and $6\sqrt[6]{z} \geq 10z - 3z^3 - 1$. *Hence,*

$$6\left(\sqrt[6]{x} + \sqrt[6]{y} + \sqrt[6]{z}\right) \geq 10(x+y+z) - 3(x^3+y^3+z^3) - 3 \overset{x+y+z=3}{=}$$

$$= 10 \cdot 3 - 3(x^3+y^3+z^3) - 3 = 27 - 3(x^3+y^3+z^3)$$

$$6\left(\sqrt[6]{x} + \sqrt[6]{y} + \sqrt[6]{z}\right) \geq 27 - 3(x^3+y^3+z^3); (1)$$

Other, because $\begin{cases} x,y,z > 0 \\ x+y+z = 3 \end{cases} \Rightarrow 0 < x,y,x < 3 \Rightarrow x-3 < 0; y-3 < 0; z-3 < 0$

$$(x-3)(x-1)^2 + (y-3)(y-1)^2 + (z-3)(z-1)^2 \leq 0 \Leftrightarrow$$

$$\sum_{cyc}[(x-3)(x^2-2x+1)] \leq 0 \Leftrightarrow \sum_{cyc}(x^3 - 5x^2 + 7x - 3) \leq 0 \Leftrightarrow$$

$$\sum_{cyc} x^3 - 5\sum_{cyc} x^2 + 7\sum_{cyc} x - 9 \leq 0 \Leftrightarrow$$

$$x^3+y^3+z^3 \leq 5(x^2+y^2+z^2) - 7(x+y+z) + 9 \Leftrightarrow$$

$$x^3+y^3+z^3 \leq 5[(x+y+z)^2 - 2(xy+yz+zx)] - 7(x+y+z) + 9 \Leftrightarrow$$

$$x^3+y^3+z^3 \leq 5(x+y+z)^2 - 10(xy+yz+zx) - 7\left(\underbrace{x+y+z}_{3}\right) + 9 \Leftrightarrow$$

$$x^3+y^3+z^3 \leq 5 \cdot 3^2 - 10(xy+yz+zx) - 7 \cdot 3 + 9 \Leftrightarrow$$

$$x^3+y^3+z^3 \leq 33 - 10(xy+yz+zx) \Leftrightarrow$$

$$27 - 3(x^3+y^3+z^3) \geq 27[33 - 10(xy+yz+zx)] = 30(xy+yz+zx) - 72; (2)$$

From (1),(2) we have:

$$6\left(\sqrt[6]{x} + \sqrt[6]{y} + \sqrt[6]{z}\right) \geq 30(xy+yz+zx) - 72$$

$$\sqrt[6]{x} + \sqrt[6]{y} + \sqrt[6]{z} + 12 \geq 5(xy+yz+zx)$$

Equality holds if $x = y = z = 1$.Proved.

SP.291. In acute ΔABC, let h_1, h_2, h_3 be the altitudes of orthic triangle.

Prove that:

$$\frac{F^4}{sR^2} \leq \frac{\pi^3}{486\sqrt{3}}\left(\sum_{cyc} h_1 \cdot r_a\right)\left(\sum_{cyc} \frac{h_a^3}{\mu^3(A)}\right)$$

Proposed by Radu Diaconu-Romania

Solution by Adrian Popa-Romania

$$\text{Let be } h_a < h_b < h_c \Rightarrow \widehat{A} > \widehat{B} > \widehat{C} \Rightarrow \frac{1}{\widehat{A}} < \frac{1}{\widehat{B}} < \frac{1}{\widehat{C}} \Rightarrow \begin{cases} h_a^3 < h_b^3 < h_c^3 \\ \left(\frac{1}{\widehat{A}}\right)^3 < \left(\frac{1}{\widehat{B}}\right)^3 < \left(\frac{1}{\widehat{C}}\right)^3 \end{cases}$$

From Chebyshev's inequality, we have:

$$3\sum_{cyc}\frac{h_a^3}{\mu^3(A)} \geq \sum_{cyc} h_a^3 \sum_{cyc}\frac{1}{\mu^3(A)} \overset{Radon}{\geq} \left(\left(\frac{2F}{a}\right)^3 + \left(\frac{2F}{b}\right)^3 + \left(\frac{2F}{c}\right)^3\right)\left(\frac{(1+1+1)^4}{(A+B+C)^3}\right) =$$

$$= 8F^3\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right)\cdot\frac{81}{\pi^3} \overset{Radon}{\geq} 8F^3 \cdot \frac{(1+1+1)^4}{(a+b+c)^3}\cdot\frac{81}{\pi^3} = \frac{8F^3\cdot 81^2}{8s^3\cdot\pi^3} = \frac{81^2\cdot F^3}{s^3\cdot\pi^3}$$

$$\sum_{cyc} h_a r_a = \sum_{cyc}\frac{2F}{a}\cdot\frac{F}{s-a} = 2F^2\sum_{cyc}\frac{1}{as-a^2} \overset{Radon}{\geq} 2F^2\cdot\frac{(1+1+1)^2}{2s^2-a^2-b^2-c^2} =$$

$$= \frac{18F^2}{8Rr+2r^2} = \frac{9F^2}{4Rr+r^2}. \text{ Now,}$$

$$\frac{\pi^3}{486\sqrt{3}}\cdot\frac{81\cdot 27\cdot F^3}{s^3\cdot\pi^3}\cdot\frac{9F^2}{4Rr+r^2} \overset{(1)}{\geq} \frac{F^4}{sR^2}$$

$$(1) \Leftrightarrow 815\cdot sR^2 \geq 3\sqrt{3}s^3(4Rr+r^2) \Leftrightarrow 81s^2rR^2 \geq 3\sqrt{3}s^3(4Rr+r^2) \Leftrightarrow$$

$$\begin{cases} 27rR^2 \geq \sqrt{3}s(4Rr+r^2) \\ s < \frac{3\sqrt{3}}{2}R \end{cases} \Rightarrow 27rR^2 \geq \sqrt{3}\cdot\frac{3\sqrt{3}}{2}R(4Rr+r^2)$$

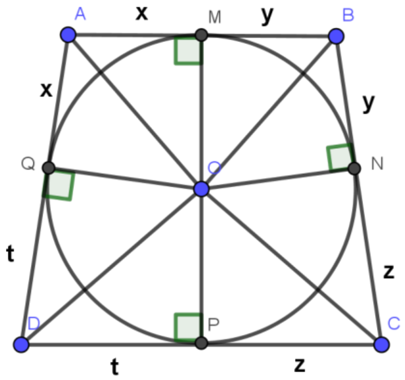
$$\Leftrightarrow 54rR^2 \geq 9R(4Rr+r^2) \Leftrightarrow 6rR^2 \geq 4rR + Rr^2 \Leftrightarrow R \geq 2r(Euler).$$

SP.292. Let $ABCD$ be a tangential quadrilateral circumscribed to a circle with radii $r = 1$. Prove that:

$$[ABCD]\cdot\sum_{cyc}\mu(A)\sqrt{\mu(A)+\mu(B)+\mu(C)} \leq 2\pi\sqrt{\frac{3\pi}{2}\left(\sum_{cyc}\cot\frac{A}{2}\right)}$$

Proposed by Radu Diaconu – Romania

Solution by Șerban George Florin – Romania



$$\mu(A) \le \mu(B) \le \mu(C) \le \mu(D)$$

$$\Rightarrow 360^\circ - \mu(\hat{A}) \ge 360^\circ - \mu(\hat{B}) \ge 360^\circ \mu(\hat{C}) \ge 360^\circ - \mu(\hat{D})$$

$$\mu(\hat{B}) + \mu(\hat{C}) + \mu(\hat{D}) \ge \mu(\hat{A}) + \mu(\hat{C}) + \mu\,(\hat{D}) \ge \mu(\hat{A}) + \mu(\hat{B}) + \mu(\hat{D}) \ge$$

$$\ge \mu(\hat{A}) + \mu(\hat{B}) + \mu(\hat{C})$$

Applying Cebyshev's inequality:

$$\sum_{cyc} \mu(A) \cdot \sqrt{\mu(A) + \mu(B) + \mu(C)} \le \frac{1}{4}\left(\sum_{cyc} \mu(A)\right) \cdot \left(\sum_{cyc} \sqrt{\mu(A) + \mu(B) + \mu(C)}\right)$$

$$= \frac{2\pi}{3} \sum_{cyc} \sqrt{2\pi - \mu(A)} \overset{Jensen}{\le} \frac{2\pi}{4} \cdot 4\sqrt{\frac{3\pi}{2}} = 2\pi\sqrt{\frac{3\pi}{2}}$$

$$f: (0, 2\pi) \to \mathbb{R}, f(x) = \sqrt{2\pi - x} = (2\pi - x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(2\pi - x)^{-\frac{1}{2}}(-1) = -\frac{1}{2}(2\pi - x)^{-\frac{1}{2}}$$

$$f''(x) = -\tfrac{1}{4}(2\pi - x)^{-\frac{3}{2}} < 0 \Rightarrow f \textit{ concave.}$$

$$\Rightarrow f\left(\frac{A+B+C+D}{4}\right) \ge \frac{\sum f(A)}{4} \Rightarrow \sum_{cyc} f(A) \le 4f\left(\frac{A+B+C+D}{4}\right) = 4f\left(\frac{2A}{4}\right) =$$

$$= 4f\left(\frac{D}{2}\right) = 4\sqrt{2\pi - \frac{D}{2}} = 4\sqrt{\frac{3D}{2}}$$

$$AM = AQ = x, MB = BM = y, CN = CP = z, DQ = DP = t$$

$$\Rightarrow \Delta AMO, \textit{right-angled}, \cot\tfrac{A}{2} = \frac{AM}{OM} = AM = x$$

$$\Rightarrow \sum_{cyc} \cot\frac{A}{2} = x + y + z + t = \frac{P_{ABCD}}{2}, OM = ON = OP = OQ = r = 1$$

$$S_{ABCD} = [ABCD] = S_{AOB} + S_{BOC} + S_{DOC} + S_{AOD} = \frac{AB \cdot OM}{2} + \frac{BC \cdot OM}{2} +$$

$$+\frac{OP + DC}{2} + \frac{OQ \cdot AD}{2} = \frac{P_{ABCD}}{2}$$

$$\Rightarrow [ABCD] \cdot \sum_{cyc} \mu(A)\sqrt{\mu(D) + \mu(B) + \mu(C)} \le \frac{P_{ABCD}}{2} \cdot 2\pi\sqrt{\frac{3\pi}{2}} =$$

$$= 2\pi\sqrt{\frac{3\pi}{2}}\left(\sum_{cyc} \cot\frac{A}{2}\right), \textit{true}$$

SP.293. If $0 < a \le b < e; 0 \le x \le y \le z; x + y + z = 3$ then:

$$(x-1)\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}} + (y-1)\left(\sqrt{ab}\right)^{\frac{1}{ab}} + (z-1)\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}} \ge 0$$

Proposed by Daniel Sitaru-Romania

Solution by George Florin Şerban-Romania

$$a \le \frac{2ab}{a+b} \le b \Rightarrow a + b \le 2b \Rightarrow a \le b \textit{ true.}$$

$$a \le \sqrt{ab} \le b \Rightarrow a^2 \le ab \le b^2 \Rightarrow a \le b \textit{ true.}$$

$$a \le \frac{a+b}{2} \le b \Rightarrow 2a \le a + b \le 2b \textit{ true.}$$

Let be the function $f: (0, e] \to \mathbb{R}, f(x) = \frac{logx}{x}, \ f'(x) = \frac{1-logx}{x^2}$

$$f'(x) = 0 \Leftrightarrow logx = 1 \Leftrightarrow x = e$$

If $x \le e, g(x) = logx$ *–increasing, then* $1 - logx \ge 0 \Rightarrow$

f *–increasing on* $(0, e]$

$$\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}} \le \left(\sqrt{ab}\right)^{\frac{1}{ab}} \le \left(\frac{a+b}{2}\right)^{\frac{2}{a+b}} \Leftrightarrow$$

$$log\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}} \le log\left(\sqrt{ab}\right)^{\frac{1}{ab}} \le log\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}} \Leftrightarrow$$

$$\frac{log\left(\frac{2ab}{a+b}\right)}{\frac{2ab}{a+b}} \le \frac{log\left(\sqrt{ab}\right)}{\sqrt{ab}} \le \frac{log\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \Leftrightarrow$$

$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2}\ (Hm - Gm - Am)$$

$$x \le y \le z \Rightarrow x - 1 \le y - 1 \le z - 1$$

Applying Cebyshev's inequality, we have:

$$(x-1)\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}} + (y-1)\left(\sqrt{ab}\right)^{\frac{1}{ab}} + (z-1)\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}} \ge$$

$$\ge \frac{1}{3}\underbrace{(x-1+y-1+z-1)}_{=0}\left[\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}} + \left(\sqrt{ab}\right)^{\frac{1}{ab}} + \left(\frac{a+b}{2}\right)^{\frac{2}{a+b}}\right] \ge 0.\ \textit{Proved.}$$

SP.294.If $a, b, c > 0, a + b + c = 3$. Then prove:

$$\frac{a^2}{\sqrt[3]{4(b^6+1)}+4b} + \frac{b^2}{\sqrt[3]{4(c^6+1)}+4c} + \frac{c^2}{\sqrt[3]{4(a^6+1)}+4a} \ge \frac{1}{2}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

For $x, y > 0$ we have: $4(x^6 + y^6) \le (3x^2 - 4xy + 3y^2)^3 \Leftrightarrow$

$$(x-y)^4(23x^2 - 16xy + 23y^2) \ge 0 \Leftrightarrow (x-y)^4[23(x-y)^2 + 3xy] \ge 0(true).$$

Now, choose: $x = a; y = 1 \Rightarrow 4(a^6 + 1) \le (3a^2 - 4a + 3)^3$;

Similarly: $4(b^6 + 1) \le (3b^2 - 4b + 3)^3$ ***and*** $4(c^6 + 1) \le (3c^2 - 4c + 3)^3 \Rightarrow$

$$\Omega = \sum_{cyc}\frac{a^2}{\sqrt[3]{4(b^6+1)}+4b} \ge \sum_{cyc}\frac{a^2}{3b^2 - 4b + 3 + 4b} = \sum_{cyc}\frac{a^2}{3b^2+3} = \frac{1}{3}\sum_{cyc}\frac{(a^2)^2}{a^2b^2 + a^2} \ge$$

$$\overset{Bergström}{\ge} \frac{1}{3}\cdot\frac{(\sum a^2)^2}{\sum a^2 + \sum a^2b^2} \overset{(*)}{\ge} \frac{1}{2}$$

$$(*)\Leftrightarrow 2\left(\sum a^4+2\sum a^2b^2\right)\geq 3\left(\sum a^2+\sum a^2b^2\right)\Leftrightarrow$$

$$2\sum a^4+\sum a^2b^2\geq 3\sum a^2=\frac{1}{3}\cdot 9\left(\sum a^2\right)\overset{a+b+c=3}{=}\frac{(\sum a)^2\sum a^2}{3};(1)$$

But: $a^2+b^2+c^2\overset{Bergström}{\geq}\frac{(a+b+c)^2}{3}\Rightarrow\frac{(\sum a)^2\sum a^2}{3}\leq(\sum a^2)^2;(2)$

From (1),(2) we need to prove:

$$2\sum a^4+\sum a^2b^2\geq\sum a^4+2\sum a^2b^2\Leftrightarrow\sum a^4\geq\sum a^2b^2$$

Which is true, because: $X^2+Y^2+Z^2\geq XY+YZ+ZX$

Chose: $X=a^2;Y=b^2;Z=c^2\Rightarrow(*)$ *is true.Proved.*

Solution 2 by proposer

By Am-Gm inequality, we have: $\sqrt[3]{4(b^6+1)}=\sqrt[3]{4(b^2+1)(b^4-b^2+1)}=$

$$=2\sqrt[3]{\frac{b^2+1}{2}(2-\sqrt{3})(b^2+b\sqrt{3}+1)(2+\sqrt{3})(b^2-b\sqrt{3}+1)}\leq$$

$$\leq 2\cdot\frac{\frac{b^2+1}{2}+(2-\sqrt{3})(b^2+b\sqrt{3}+1)+(2+\sqrt{3})(b^2-b\sqrt{3}+1)}{3}=3b^2-4b+3$$

$$\Rightarrow\sqrt[3]{4(b^6+1)}+4b\leq 3(b^2+1)\Leftrightarrow\frac{1}{\sqrt[3]{4(b^6+1)}+4b}\geq\frac{1}{3(b^2+1)}\Leftrightarrow$$

$\frac{a^2}{\sqrt[3]{4(b^6+1)}+4b}\geq\frac{a^2}{3(b^2+1)}$. *Similarly:*

$$\frac{b^2}{\sqrt[3]{4(c^6+1)}+4c}\geq\frac{b^2}{3(c^2+1)};\frac{c^2}{\sqrt[3]{4(a^6+1)}+4a}\geq\frac{c^2}{3(a^2+1)}$$

Therefore: $\frac{a^2}{\sqrt[3]{4(b^6+1)}+4b}+\frac{b^2}{\sqrt[3]{4(c^6+1)}+4c}+\frac{c^2}{\sqrt[3]{4(a^6+1)}+4a}\geq$

$$\geq\frac{1}{3}\left(\frac{a^2}{b^2+1}+\frac{b^2}{c^2+1}+\frac{c^2}{a^2+1}\right);(1)$$

Other $\frac{a^2}{b^2+1}+\frac{b^2}{c^2+1}+\frac{c^2}{a^2+1}=$

$$=\frac{a^2(1+b^2)-a^2b^2}{b^2+1}+\frac{b^2(1+c^2)-b^2c^2}{c^2+1}+\frac{c^2(1+a^2)-c^2a^2}{a^2+1}\geq$$

$$\geq(a^2+b^2+c^2)-\frac{a^2b^2}{2b}-\frac{b^2c^2}{2c}-\frac{c^2a^2}{2a}=$$

$$=(a^2+b^2+c^2)-\frac{1}{2}(a^2b+b^2c+c^2a)\overset{Chebyshev's}{\geq}$$

$$\geq a^2+b^2+c^2-\frac{1}{2}\cdot\frac{(a+b+c)(a^2+b^2+c^2)}{3}$$

$$\geq a^2+b^2+c^2-\frac{1}{2}(a^2+b^2+c^2)=\frac{a^2+b^2+c^2}{2}$$

$$\Rightarrow\frac{a^2}{b^2+1}+\frac{b^2}{c^2+1}+\frac{c^2}{a^2+1}\geq\frac{a^2+b^2+c^2}{2}\overset{QAM}{\geq}\frac{\frac{(a+b+c)^2}{3}}{2}=\frac{(a+b+c)^2}{6}=\frac{3}{2};(2)$$

For (1),(2) we have:

$$\frac{a^2}{\sqrt[3]{4(b^6+1)}+4b}+\frac{b^2}{\sqrt[3]{4(c^6+1)}+4c}+\frac{c^2}{\sqrt[3]{4(a^6+1)}+4a}\geq\frac{3}{2}\cdot\frac{1}{3}\geq\frac{1}{2}$$

Equality occurs if $a=b=c=1$.

SP.295. In ΔABC the following relationship holds:

$$(a+2r)(b+2r)(c+2r)\geq 2R^3(3\sqrt{3}+5)$$

Proposed by Florentin Vişescu-Romania

Solution 1 by Daniel Văcaru-Romania

We have: $(a+2r)(b+2r)(c+2r)=8R^3\prod_{cyc}\left(sinA+\frac{r}{R}\right)\quad(1)$

Consider the function $f:(0,\pi)\to\mathbb{R}, f(x)=log\left(sinx+\frac{r}{R}\right)$.

We obtain second derivate as $f''(x)=\frac{-1+\frac{r}{R}sinx}{\left(sinx+\frac{r}{R}\right)^2}\leq 0,\forall x\in(0,\pi)$

Then $\sum_{cyc}log\left(sinx+\frac{r}{R}\right)\leq 3log\left(sin\left(\frac{\sum A}{3}\right)+\frac{r}{R}\right)\Rightarrow$

$$\left(\prod_{cyc}\left(sinA+\frac{r}{R}\right)\right)\leq\left(\frac{\sqrt{3}}{2}+\frac{r}{R}\right)^3\leq\frac{\left(\sqrt{3}+1\right)^3}{8}=\frac{2(3\sqrt{3}+5)}{8}$$

From (1) we obtain

$$(a+2r)(b+2r)(c+2r)=8R^3\prod_{cyc}\left(\sin A+\frac{r}{R}\right)\le\frac{8R^3\cdot 2(3\sqrt{3}+5)}{8}=2R^3(3\sqrt{3}+5)$$

Solution 2 by Marin Chirciu-Romania

Lema:

1) *In* ΔABC *we have* $(a+2r)(b+2r)(c+2r)=2r\left[p^2+2p(R+2r)+r(4R+5r)\right]$.

Demonstration:

We have:

$$\prod(a+2r)=abc+4r^2\sum a+2r\sum bc+8r^3=4Rrp+4r^2\cdot 2p+2r\left(p^2+r^2+4Rr\right)+8r^3=$$
$$=2r\left[p^2+2p(R+2r)+r(4R+5r)\right],$$

Then $abc=4Rrp$, $\sum a=2p$, $\sum bc=p^2+r^2+4Rr$.

Using lema Lema the inequality can be rewrite

$2r\left[p^2+2p(R+2r)+r(4R+5r)\right]\le 2R^3\left(5+3\sqrt{3}\right)$, ***from Gerretsen inequality***

$p^2\le 4R^2+4Rr+3r^2$ ***and Doucet inequality*** $p\sqrt{3}\le 4R+r$.

We must show that:

$$2r\left[4R^2+4Rr+3r^2+2\cdot\frac{4R+r}{\sqrt{3}}(R+2r)+r(4R+5r)\right]\le 2R^3\left(5+3\sqrt{3}\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{2r}{\sqrt{3}}\left[\left(8+4\sqrt{3}\right)R^2+\left(18+8\sqrt{3}\right)Rr+\left(4+8\sqrt{3}\right)r^2\right]\le 2R^3\left(5+3\sqrt{3}\right)\Leftrightarrow$$

$$\Leftrightarrow R^3\left(9+5\sqrt{3}\right)-\left(8+4\sqrt{3}\right)R^2r-\left(18+8\sqrt{3}\right)Rr^2-\left(4+8\sqrt{3}\right)r^3\ge 0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)\left[\left(9+5\sqrt{3}\right)R^2+\left(10+6\sqrt{3}\right)Rr+\left(2+4\sqrt{3}\right)r^2\right]\ge 0,$$

True from Euler $R\ge 2r$.

Equality if only if the triangle is equilateral.

Remark

Let's determine an opposite inequality.

2) In ΔABC the following relationship holds

3)

$$(a+2r)(b+2r)(c+2r)\geq 16r^3\left(5+3\sqrt{3}\right).$$

Proposed by Marin Chirciu-Romania

Using lema we can rewrite

$2r\left[p^2+2p(R+2r)+r(4R+5r)\right]\geq 16r^3\left(5+3\sqrt{3}\right)$, ***true from Gerretsen*** $p^2\geq 16Rr-5r^2$ ***and Mitrinović*** $p\geq 3r\sqrt{3}$.

We must show:

$$2r\left[16Rr-5r^2+2\cdot 3r\sqrt{3}(R+2r)+r(4R+5r)\right]\geq 16r^3\left(5+3\sqrt{3}\right)\Leftrightarrow$$

$\Leftrightarrow R\left(5+3\sqrt{3}\right)\geq 2r\left(5+3\sqrt{3}\right)$, ***true from Euler*** $R\geq 2r$.

Equality if only if the triangle is equilateral.

Remark

4) In ΔABC the following relationship holds

$$16r^3\left(5+3\sqrt{3}\right)\leq(a+2r)(b+2r)(c+2r)\leq 2R^3\left(5+3\sqrt{3}\right).$$

Proposed by Marin Chirciu-Romania

Solution:

See the relation 1) and 2).

Equality if only if the triangle is equilateral.

SP.296. In acute ΔABC, H −orthocenter, O −circumcenter, I −incenter the following relationship holds:

$$\sum_{cyc}\left(\frac{(r_a+AH)^2}{\mu(A)+r_a cosA}\right)\geq\frac{36(4OI^2+12Rr+r^2)}{4\pi+9R}$$

Proposed by Radu Diaconu-Romania

Solution by Daniel Văcaru-Romania

We known that in acute triangle $r_a + AH = 2R + r$. On the other hand,

$4OI^2 + 12Rr + r^2 = 4R^2 - 8Rr + 12Rr + r^2 = (2R + r)^2$. We obtain:

$$\sum_{cyc}\left(\frac{(r_a + AH)^2}{\mu(A) + r_a cosA}\right) = (2R + r)^2\sum_{cyc}\left(\frac{1}{\mu(A) + r_a cosA}\right)$$

$$\overset{Bergstrom}{\geq} (2R + r)^2\frac{9}{\sum\mu(A) + \sum r_a cosA} = \frac{9(2R + r)^2}{\pi + \sum r_a cosA}$$

We have $\sum r_a cosA = \sum r_a\left(2cos^2\frac{A}{2} - 1\right) = 2\sum r_a cos^2\frac{A}{2} - \sum r_a = \frac{s^2}{R} - 4R - r$.

We intend to prove that: $\frac{s^2}{R} - 4R - r \leq \frac{9R}{4} \Leftrightarrow \frac{s^2}{R} - r \leq \frac{25R}{4}$. With Gerretsen,

$$s^2 \leq 4R^2 + 4Rr + 3r^2 \Rightarrow 4R^2 + 3Rr + 3r^2 \leq \frac{25R^2}{4}$$

$$\Rightarrow 3R^2 \geq 4Rr + 4r^2 \Rightarrow 3\left(\frac{R}{r}\right)^2 - 4\left(\frac{R}{r}\right) - 4 \geq 0$$

But we obtain $\Delta = 64$ and $\frac{R}{r} = 2$ or $\frac{R}{r} = -\frac{2}{3}$.

It's clearly that: $3\left(\frac{R}{r}\right)^2 - 4\left(\frac{R}{r}\right) - 4 \geq 0$. We have

$$\sum_{cyc}\left(\frac{(r_a + AH)^2}{\mu(A) + r_a cosA}\right) \geq \frac{(2R + r)^2}{\pi + \sum r_a cosA} \geq \frac{9(2R + r)^2}{\pi + \frac{9R}{4}} = \frac{36(2R + r)^2}{4\pi + 9R}$$

$$= \frac{36(4OI^2 + 12Rr + r^2)}{4\pi + 9R}$$

SP.297 If $a, b, c > 0$ such that $a + b + c = 3$. Prove that:

$$\frac{a}{7bc + \sqrt[3]{4(b^6 + c^6)}} + \frac{b}{7ca + \sqrt[3]{4(c^6 + a^6)}} + \frac{c}{7ab + \sqrt[3]{4(a^6 + b^6)}} + \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} \geq \frac{7}{12}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

For $a, b > 0$ we have $4(a^6 + b^6) \leq (3a^2 - 4ab + 3b^2)^3 \Leftrightarrow$

$(a - b)^4(23a^2 - 16ab + 23b^2) \geq 0 (true) \forall a, b > 0$. Equality $\Leftrightarrow a = b$.

Similary: $4(a^6+c^6)\le(3a^2-4ac+3c^2)^3$ *and* $4(c^6+b^6)\le(3c^2-4cb+3b^2)^3$

$$\Rightarrow\Omega_1=\sum_{cyc}\frac{a}{7bc+\sqrt[3]{4(b^6+c^6)}}\ge\sum_{cyc}\frac{a}{3b^2-4bc+3c^2+7bc}=\sum_{cyc}\frac{a}{3(b^2+bc+c^2)}=\Omega_2$$

$$\Phi=\sum_{cyc}\frac{a}{(b^2+bc+c^2)}=\sum_{cyc}\frac{a^2}{a(b^2+bc+c^2)}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(a+b+c)^2}{ab(a+b)+bc(b+c)+ca(c+a)+3abc}=\frac{(a+b+c)^2}{(a+b+c)(ab+bc+ca)}=$$

$$=\frac{a+b+c}{ab+bc+ca}=\frac{3}{ab+bc+ca}\Rightarrow\Omega_2\ge\frac{1}{3}\cdot\Phi=\frac{1}{ab+bc+ca}\Rightarrow$$

$$\Omega_1=\sum_{cyc}\frac{a}{7bc+\sqrt[3]{4(b^6+c^6)}}\ge\Omega_2\ge\frac{1}{ab+bc+ca};(1)$$

$$\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\overset{AGM}{\ge}5\sqrt[5]{\left(\sqrt[3]{a}\right)^3\cdot a^2\cdot a^2}=5\sqrt[5]{a^5}=5a$$

$$\Rightarrow 3\sqrt[3]{a}+2a^2\ge 5a\Rightarrow 3\sqrt[3]{a}\ge 5a-2a^2$$

Similary: $3\sqrt[3]{b}\ge 5b-2b^2$ and $3\sqrt[3]{c}\ge 5c-2c^2\Rightarrow$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 5(a+b+c)-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 15-2(a^2+b^2+c^2);(a+b+c=3)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 18-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 2(a+b+c)^2-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 2(a^2+b^2+c^2+2ab+2bc+2ca)-2(a^2+b^2+c^2)$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 4(ab+bc+ca)-3\Leftrightarrow$$

$$\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{4(ab+bc+ca)-3}{36}\Leftrightarrow\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{ab+bc+ca}{9}-\frac{1}{12};(2)$$

From (1),(2) we have:

$$\sum_{cyc}\frac{a}{7bc+\sqrt[3]{4(b^6+c^6)}}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\overset{AGM}{\ge}$$

$$\ge 2\sqrt{\frac{1}{ab+bc+ca}\cdot\frac{ab+bc+ca}{9}}-\frac{1}{12}=\frac{2}{3}-\frac{1}{12}=\frac{7}{12}.$$

Proved.Equality if $a=b=c=1$.

Solution 2 by proposer

We have: $b^6+c^6=(b^2+c^2)(b^4-b^2c^2+c^4)=(b^2+c^2)\left((b^2+c^2)^2-\left(bc\sqrt{3}\right)^2\right)=$

$$=(b^2+c^2)\left(b^2-bc\sqrt{3}+c^2\right)\left(b^2+bc\sqrt{3}+c^2\right)$$

By inequality Am-Gm for three positive real numbers:

$$\sqrt[3]{4(b^6+c^6)}=\sqrt[3]{(b^2+c^2)2\left(2+\sqrt[3]{3}\right)\left(b^2-bc\sqrt{3}+c^2\right)2\left(2-\sqrt[3]{3}\right)\left(b^2+bc\sqrt{3}+c^2\right)}\le$$

$$\le\frac{(b^2+c^2)+2\left(2+\sqrt[3]{3}\right)\left(b^2-bc\sqrt{3}+c^2\right)+2\left(2-\sqrt[3]{3}\right)\left(b^2+bc\sqrt{3}+c^2\right)}{3}=$$

$$=\frac{9b^2-12bc+9c^2}{3}\Leftrightarrow\frac{1}{7bc+\sqrt[3]{4(b^6+c^6)}}\ge\frac{1}{3(b^2+bc+c^2)}$$

$$\Leftrightarrow\frac{a}{7bc+\sqrt[3]{4(b^6+c^6)}}\ge\frac{a}{3(b^2+bc+c^2)};(2)$$

Similary: $\frac{b}{7ac+\sqrt[3]{4(a^6+c^6)}}\ge\frac{b}{3(a^2+ac+c^2)};(3)$

$$\frac{c}{7ab+\sqrt[3]{4(a^6+b^6)}}\ge\frac{c}{3(a^2+ab+b^2)};(4)$$

For (2),(3),(4) we have:

$$\sum_{cyc}\frac{a}{7bc+\sqrt[3]{4(b^6+c^6)}}\ge\sum_{cyc}\frac{a}{3(b^2+bc+c^2)}=\frac{1}{3}\sum_{cyc}\frac{a}{b^2+bc+c^2}=$$

$$=\frac{1}{3}\sum_{cyc}\frac{a^2}{a(b^2+bc+c^2)}\overset{Bergstrom}{\ge}\frac{1}{3}\cdot\frac{(a+b+c)^2}{\sum a(b^2+bc+c^2)}=$$

$$=\frac{1}{3}\cdot\frac{(a+b+c)^2}{ab(a+b)+bc(b+c)+ca(c+a)+3abc}=\frac{1}{3}\cdot\frac{(a+b+c)^2}{(a+b+c)(ab+bc+ca)}=$$

$$=\frac{1}{3}\cdot\frac{a+b+c}{ab+bc+ca}\overset{a+b+c=3}{=}\frac{1}{ab+bc+ca};(5)$$

$$\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\overset{AGM}{\ge}5\sqrt[5]{\left(\sqrt[3]{a}\right)^3\cdot a^2\cdot a^2}=5\sqrt[5]{a^5}=5a\Rightarrow$$

$3\sqrt[3]{a}+2a^2\ge 5a\Rightarrow 3\sqrt[3]{a}\ge 5a-2a^2$. *Similary:*

$3\sqrt[3]{b}\ge 5b-2b^2$ *and* $3\sqrt[3]{c}\ge 5c-2c^2\Rightarrow$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 5(a+b+c)-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 15-2(a^2+b^2+c^2);(a+b+c=3)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 18-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 2(a+b+c)^2-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 2(a^2+b^2+c^2+2ab+2bc+2ca)-2(a^2+b^2+c^2)$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 4(ab+bc+ca)-3\Leftrightarrow$$

$$\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{4(ab+bc+ca)-3}{36}\Leftrightarrow\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{ab+bc+ca}{9}-\frac{1}{12};(6)$$

From (1),(2) we have:

$$\sum_{cyc}\frac{a}{7bc+\sqrt[3]{4(b^6+c^6)}}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\overset{AGM}{\ge}$$

$$\ge 2\sqrt{\frac{1}{ab+bc+ca}\cdot\frac{ab+bc+ca}{9}}-\frac{1}{12}=\frac{2}{3}-\frac{1}{12}=\frac{7}{12}.$$

Proved.Equality if $\begin{cases}a,b,c>0,a+b+c=3\\ a=b=c\\ \frac{1}{a^2+ab+b^2}=\frac{1}{b^2+bc+c^2}=\frac{1}{c^2+ca+a^2}\\ \frac{1}{ab+bc+ca}=\frac{ab+bc+ca}{9}\end{cases}\Leftrightarrow a=b=c=1.$

SP.298. Find without softs:

$$\Omega=\int_0^{\frac{\pi}{4}}\left(\log(\cos 2x)+\left(\frac{1-\tan x}{1+\tan x}\right)\log(1+\sin 2x)\right)dx$$

Proposed by Pedro Pantoja-Natal-Brazil

Solution 1 by Tobi Joshua-Nigeria

$$I=\int_0^{\frac{\pi}{4}}\left(\log(\cos 2x)+\left(\frac{1-\tan x}{1+\tan x}\right)\log(1+\sin 2x)\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\log(\cos 2x)dx+\int_0^{\frac{\pi}{4}}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\log(\sin x+\cos x)dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log(\cos 2x)dx+2\int_0^{\frac{\pi}{4}}\left(\frac{\log(\cos x+\sin x)}{\cos x+\sin x}\right)d(\cos x+\sin x)$$

$$=\frac{1}{2}\left[\frac{\partial}{\partial s}\right]_{s=0}\int_0^{\frac{\pi}{2}}\cos^s x\,dx+2\left[\frac{\log^2(\cos x+\sin x)}{2}\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\left[\frac{\partial}{\partial s}\right]_{s=0}\int_0^{\frac{\pi}{2}}\cos^s x\sin^0 x\,dx+2\left(\frac{\log^2\left(\frac{2}{\sqrt{2}}\right)}{2}\right)$$

$$=\frac{1}{2}\left[\frac{\partial}{\partial s}\right]_{s=0}\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{s+2}{2}\right)}+2\left(\frac{\log^2\left(\frac{2}{\sqrt{2}}\right)}{2}\right)$$

$$=\frac{\sqrt{\pi}}{4}\left[\frac{\partial}{\partial s}\right]_{s=0}\left[\frac{\frac{1}{2}\Gamma'\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+2}{2}\right)}-\frac{\frac{1}{2}\Gamma'\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma^2\left(\frac{s+2}{2}\right)}\right]_{s=0}+\log^2\left(\frac{2}{\sqrt{2}}\right)$$

$$=\frac{\sqrt{\pi}}{4}\left[\frac{\frac{1}{2}\Gamma'\left(\frac{1}{2}\right)}{\Gamma(1)}-\frac{\frac{1}{2}\Gamma'(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma^2(1)}\right]+\log^2\left(\frac{2}{\sqrt{2}}\right)$$

$$=\frac{\sqrt{\pi}}{4}\left[\frac{\frac{1}{2}\psi\left(\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(1)}-\frac{\frac{1}{2}\psi(1)\sqrt{\pi}}{\Gamma^2(1)}\right]+\log^2\left(\frac{2}{\sqrt{2}}\right)$$

$$=\frac{\pi}{4}\left[\frac{\psi\left(\frac{1}{2}\right)}{2}-\frac{\psi(1)}{2}\right]+\log^2\left(\frac{2}{\sqrt{2}}\right)$$

$$=\frac{\pi}{4}\left[\frac{(-\gamma-2\log 2)}{2}+\frac{\gamma}{2}\right]++\log^2\left(\frac{2}{\sqrt{2}}\right)=\frac{-\pi\log 2}{4}+\log^2\left(\frac{2}{\sqrt{2}}\right)$$

Solution 2 by Avishek Mitra-West Bengal-India

$$I=\int_0^{\frac{\pi}{4}}\left(\left(\frac{1-\tan x}{1+\tan x}\right)\log(1+\sin 2x)\right)dx$$

$$= 2\int_0^{\frac{\pi}{4}} \left(\frac{cosx - sinx}{cosx + sinx}\right) log(cosx + sinx)\, dx \overset{\substack{sinx+cosx=z \\ (cosx-sinx)dx=dz}}{=}$$

$$= 2\int_1^{\sqrt{2}} \frac{logz}{z} dz = 2\int_1^{\sqrt{2}} logz\, d(logz) = 2\left[\frac{log^2 z}{2}\right]_1^{\sqrt{2}} = log^2\sqrt{2} - log^2 1 = \frac{1}{4}log^2 2$$

$$I = \int_0^{\frac{\pi}{4}} log(cos2x)\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} log(cosx)\, dx = I_1$$

$$I_1 = \int_0^{\frac{\pi}{2}} log(cosx)\, dx = \int_0^{\frac{\pi}{2}} log(sinx)\, dx = I_1$$

$$\Rightarrow 2I_1 = \int_0^{\frac{\pi}{2}} log(sinx \cdot cosx)\, dx = \int_0^{\frac{\pi}{2}} log\left(\frac{sin2x}{2}\right) dx$$

$$\Rightarrow 2I_1 = \frac{1}{2}\int_0^{\pi} log(sinx)\, dx - \frac{\pi}{2}log2$$

$$\therefore \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx; \text{if } f(2a-x) = f(x)$$

$$\Rightarrow 2I_1 = \frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} log(sinx)\, dx - \frac{\pi}{2}log2$$

$$\Rightarrow 2I_1 - I_1 = -\frac{\pi}{2}log2 \Rightarrow I_1 = -\frac{\pi}{4}log2$$

$$\Omega = \frac{1}{4}log^2 2 - \frac{\pi}{4}log2$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\int_0^{\frac{\pi}{4}} \left(log(cos2x) + \left(\frac{1 - tan\, x}{1 + tan\, x}\right) log(1 + sin2x)\right) dx$$

$$= I = \int_0^{\frac{\pi}{4}} \log(\cos 2x)\,dx + \int_0^{\frac{\pi}{4}} \left(\frac{1-\tan x}{1+\tan x}\right)\log(1+\sin 2x)dx$$

$$= \frac{1}{8}\lim_{y\to\frac{1}{2}}\frac{d}{dy}B\left(\frac{1}{2},y\right) + \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\log(1+\sin 2x)\right)'\log(1+\sin 2x)dx$$

$$= -\frac{\pi}{4}\log 2 + \frac{1}{4}\left[\log^2(1+\sin 2x)\right]_0^{\frac{\pi}{4}} = \frac{\log 2(\log 2-\pi)}{4}$$

Solution 4 by Zaharia Burghelea-Romania

$$\Omega = \int_0^{\frac{\pi}{4}} \left(\log(\cos 2x) + \left(\frac{1-\tan x}{1+\tan x}\right)\log(1+\sin 2x)\right)dx$$

$$= I + J = \frac{1}{4}\log^2 2 - \frac{\pi}{4}\log 2$$

$$J := \int_0^{\frac{\pi}{4}} \left(\left(\frac{1-\tan x}{1+\tan x}\right)\log(1+\sin 2x)\right)dx$$

$$\overset{x\to\frac{\pi}{4}-x}{=} \int_0^{\frac{\pi}{4}} \tan x\cdot\log(1+\cos 2x)dx$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\log(1+\cos 2x)\right)'\log(1+\cos 2x)\,dx = -\frac{1}{4}\left[\log^2(1+\cos 2x)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\log^2 2}{4}$$

$$I := \int_0^{\frac{\pi}{4}} \log(\cos 2x)\,dx \overset{x\to\frac{\pi}{4}-x}{=} \int_0^{\frac{\pi}{4}} \log(\sin 2x)\,dx$$

$$2I = \int_0^{\frac{\pi}{4}} \log(\sin 2x\cdot\cos 2x)\,dx = \int_0^{\frac{\pi}{4}} \left(\log\left(\frac{1}{2}\right) + \log(\sin 4x)\right)dx$$

$$\stackrel{2x\to x}{=}-\frac{\pi}{4}log2+\frac{1}{2}\int_0^{\frac{\pi}{2}}log(sin2x)\,dx$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}log(sin2x)\,dx\stackrel{x-\frac{\pi}{4}\to x}{=}\int_0^{\frac{\pi}{4}}log(sin2x)\,dx=I$$

$$2I=-\frac{\pi}{4}log2+I\Rightarrow I=-\frac{\pi}{4}log2$$

SP.299. Solve for real numbers:

$$\begin{cases}\sqrt{x^2+y^2}+\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}+\sqrt{(x-4)^2+(y-3)^2}=10\\ x+2y=5z\end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania

Consider the rectangle $ABCD$ with $A(0,0)$; $B(4,0)$; $C(4,3)$; $D(0,3)$ and $M(x,y)$.

With triangle inequalities, we have $MA+MC\geq AC=5$ and $MB+MD\geq BD=5$

We obtain

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}+\sqrt{(x-4)^2+(y-3)^2}\geq 10.$$

Equality is attained when M is the center of rectangle $ABCD$.

We obtain $x=2, y=\frac{3}{2}$, and $x+2y=2+\frac{2\cdot 3}{2}=5\Rightarrow z=1$.

Then $(x,y,z)=\left(2,\frac{3}{2},1\right)$

Solution 2 by Abner Chinga Bazo-Lima-Peru

By Minkowsky inequality we have:

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}\geq\sqrt{3^2+4^2}=5$$

Equality occurs when: $\frac{x-4}{x}=\frac{y-3}{y}=k\Rightarrow\begin{cases}x=4k\\ y=3k\end{cases}$

If $\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}=5$

$$\Rightarrow\sqrt{(3k)^2+(4k)^2}+\sqrt{(3k-4)^2+(4k-3)^2}=5$$

$$\Rightarrow \sqrt{25k^2} + \sqrt{25(k^2-2k+1)} = 5 \Rightarrow 5|k| + 5|k-1| = 5$$

$$\Rightarrow |k| + |k-1| = 1$$

Case 1: $k \in (-\infty, 0) \Rightarrow -k-(k-1) = 1 \Rightarrow k = 0 \Rightarrow k \in \emptyset$

Case 2: $k \in [0,1) \Rightarrow k-(k-1) = 1 \Rightarrow k \in \mathbb{R} \Rightarrow k \in [0,1)$

Case 3: $k \in [1,\infty) \Rightarrow k+k-1 = 1 \Rightarrow k = 1$

$$\sqrt{x^2+y^2} + \sqrt{(x-4)^2+(y-3)^2} = 5 \Rightarrow k \in [0,1) \quad (i)$$

$$\sqrt{(x-4)^2+y^2} + \sqrt{x^2+(y-3)^2} \ge \sqrt{3^2+4^2} = 5$$

Equality occurs when: $\frac{x-4}{x} = \frac{y-4}{y} = k \Rightarrow \begin{cases} x = 4k \\ y = 3k \end{cases}$

If $\sqrt{(x-4)^2+y^2} + \sqrt{x^2+(y-3)^2} = 5$

$$\Rightarrow \sqrt{(4k-4)^2+(3k)^2} + \sqrt{(4k)^2+(3k-3)^2} = 5$$

$\Rightarrow \sqrt{25k^2-32k+16} + \sqrt{25k^2-18k+9} = 5$ *solving the equation:* $k = \frac{1}{2}$

$$\sqrt{(x-4)^2+y^2} + \sqrt{x^2+(y-3)^2} = 5 \Rightarrow k = \frac{1}{2} \quad (ii)$$

From (i),(ii) we get:

$$\sqrt{x^2+y^2} + \sqrt{(x-4)^2+y^2} + \sqrt{(x-4)^2+(y-3)^2} = 10,$$

$$x = 4k, y = 3k, k = \frac{1}{2}$$

If $\sqrt{x^2+y^2} + \sqrt{(x-4)^2+y^2} + \sqrt{(x-4)^2+(y-3)^2} = 10 \Rightarrow x = 2; y = \frac{3}{2}$

But: $x + 2y = 5z \Rightarrow z = 1$

So, $(x,y,z) = \left(2, \frac{3}{2}, 1\right)$

SP.300. If $A \in M_n(\mathbb{R})$; $A^3 = 2A^2 + 7A + 4I_n$ then find:

$$\Omega = det(A^2 - 3A + 3I_n)$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$A^3 - 2A^2 - 7A - 4I_n = O_n$$

Let: $f(x) = x^3 - 2x^2 - 7x - 4 = (x+1)^2(x-4) \Rightarrow f(A) = O_n$

Let be m_A –the minimal polynomial $m_A/f \Rightarrow m_A(x) = (x+1)^{k_1}(x-4)^{k_2}$, from

Frobenius theorem, result $p_A(x) = (x+1)^{p_1}(x-4)^{p_2}, p_1 + p_2 = n; (1)$

$$p_A(x) = det(xI_n - A) = (-1)^n det(A - xI_n); (2)$$

Let be the equation: $x^2 - 3x + 3 = 0$ with roots $x_{1,2} = \frac{-3\pm i\sqrt{3}}{2} = 1 \pm \frac{1\pm i\sqrt{3}}{2}$

Where the equation: $x^2 - x + 1 = 0$ have the roots $\frac{1\pm i\sqrt{3}}{2}$.

$$\alpha = 1 + \varepsilon; \varepsilon^2 - \varepsilon + 1 = 0, \varepsilon^3 = -1, \varepsilon^6 = 1, \varepsilon + \bar{\varepsilon} = 1, \varepsilon \cdot \bar{\varepsilon} = 1.$$

$$det(A^2 - 3A + 3I_n) = det(A - \alpha I_n) \cdot det(A - \bar{\alpha} I_n); (3)$$

From (2) result: $p_A(\alpha) = (-1)^n det(A - \alpha I_n)$; $p_A(\bar{\alpha}) = (-1)^n det(A - \bar{\alpha} I_n)$ then

$$p_A(\alpha) \cdot p_A(\bar{\alpha}) = det(A - \alpha I_n) \cdot det(A - \bar{\alpha} I_n) = det(A^2 - 3A + 3I_n); (4)$$

$$p_A(\alpha) \cdot p_A(\bar{\alpha}) = (\alpha+1)^{p_1}(\alpha-4)^{p_2}(\bar{\alpha}+1)^{p_1}(\bar{\alpha}-4)^{p_2} =$$

$$= \left((\varepsilon+2)(\bar{\varepsilon}+2)\right)^{p_1}\left((\varepsilon-3)(\bar{\varepsilon}-3)\right)^{p_2} = (1 + 2(\varepsilon+\bar{\varepsilon}) + 4)^{p_1}(1 - 3(\varepsilon+\bar{\varepsilon}) + 9)^{p_2} =$$

$$= 7^{p_1} \cdot 7^{p_2} = 7^{p_1+p_2} = 7^n \Rightarrow \Omega = det(A^2 - 3A + 3I_n) = 7^n.$$

UP.286. Prove:

$$\int_0^n (n+x)^{\frac{1}{\sqrt{n^2+x^2}}} dx \geq n\sqrt[n]{(1+\sqrt{2})^{\log(n\sqrt{2})}}, n \in \mathbb{N}, n \geq 2$$

Proposed by Florică Anastase-Romania

Solution by proposer:

Inequality of means about integral forms:

$$\therefore \frac{b-a}{\int_a^b \frac{dx}{\varphi(x)}} \leq e^{\frac{1}{b-a}\int_a^b \log\varphi(x)dx} \leq \frac{1}{b-a}\int_a^b \varphi(x)dx$$

$$Let\ \varphi: [0, n] \to (0, \infty), \varphi(x) = (n+x)^{\frac{1}{\sqrt{n^2+x^2}}} \to \frac{1}{n}\int_0^n (n+x)^{\frac{1}{\sqrt{n^2+x^2}}} dx$$

$$\geq e^{\frac{1}{n}\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx} \quad (i)$$

$$\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx = \int_0^n \sqrt{n^2+x^2}\cdot\frac{1}{n^2+x^2}\cdot\log(n+x)dx$$

Applying Chebyshev inequality for function: $p(x)=\frac{1}{n^2+x^2}$ *integrable and*

$$f(x)=\sqrt{n^2+x^2}, h(x)=log(n+x)\ increasing\ , we\ get:$$

$$\left(\int_0^n \frac{1}{n^2+x^2}\right)\cdot\left(\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx\right) \ge \left(\int_0^n \frac{\sqrt{n^2+x^2}}{n^2+x^2}dx\right)\cdot\left(\int_0^n \frac{\log(n+x)}{n^2+x^2}dx\right) \leftrightarrow$$

$$\left(\int_0^n \frac{1}{n^2+x^2}dx\right)\cdot\left(\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx\right) \ge \left(\int_0^n \frac{1}{\sqrt{n^2+x^2}}dx\right)\cdot\left(\int_0^n \frac{\log(n+x)}{n^2+x^2}dx\right) \quad (ii)$$

$$\int_0^n \frac{1}{n^2+x^2}dx = \frac{\pi}{4n}\ and\ \int_0^n \frac{1}{\sqrt{n^2+x^2}}dx = \log(1+\sqrt{2})$$

$$\int_0^n \frac{\log(n+x)}{n^2+x^2}dx \overset{x=ny,\ dx=ndy}{\cong} n\int_0^1 \frac{log(n(1+y))}{n^2(1+y^2)}dy = \int_0^1 \frac{log(n(1+y))}{n(1+y^2)}dy =$$

$$= \frac{logn}{n}\int_0^1 \frac{dy}{1+y^2} + \frac{1}{n}\int_0^1 \frac{\log(1+y)}{1+y^2}dy = \frac{\pi logn}{4n} + \frac{1}{n}\int_0^1 \frac{\log(1+y)}{1+y^2}dy\ , but:$$

$$\int_0^1 \frac{\log(1+y)}{1+y^2}dy \overset{y=tanz, dy=\frac{dz}{cos^2z}}{\cong} \int_0^{\frac{\pi}{4}} \frac{\log(1+tanz)}{\frac{1}{cos^2z}}\cdot\frac{dz}{cos^2z}$$

$$= \int_0^{\frac{\pi}{4}} \log(1+tanz)\,dz = \frac{\pi}{8}log2 \quad \to$$

$$\int_0^n \frac{\log(n+x)}{n^2+x^2}dx = \frac{\pi logn}{4n} + \frac{\pi}{8n}log2 = \frac{\pi log(n\sqrt{2})}{4n} \quad (iii)$$

From (ii)+(iii) we get:

$$\int_0^n \frac{\log(n+x)}{\sqrt{n^2+x^2}}dx \ge \log(n\sqrt{2})\cdot\log(1+\sqrt{2}) = \log(1+\sqrt{2})^{\log(n\sqrt{2})} \quad (iv)$$

From (i)+(iv), we have:

$$\int_0^n (n+x)^{\frac{1}{\sqrt{n^2+x^2}}}dx \ge n\sqrt[n]{(1+\sqrt{2})^{\log(n\sqrt{2})}}, n\in\mathbb{N}, n\ge 2$$

UP.287. If $0 < a \le b$ then

$$\tan^{-1}\left(\frac{2ab}{a+b}\right) + \int_{\frac{2ab}{a+b}}^{\sqrt{ab}} e^{-t^2}dt \le \tan^{-1}\sqrt{ab}$$

Proposed by Daniel Sitaru-Romania

Solution by Daniel Văcaru-Romania

We know that $e^x \ge x+1, \forall t \in \mathbb{R}$.

Let's integrate this on $\left[\frac{2ab}{a+b};\sqrt{ab}\right]$. It follows that

$$\int_{\frac{2ab}{a+b}}^{\sqrt{ab}} e^{-t^2}dt \le \tan^{-1}\sqrt{ab} - \tan^{-1}\left(\frac{2ab}{a+b}\right) \Rightarrow$$

$\tan^{-1}\left(\frac{2ab}{a+b}\right) + \int_{\frac{2ab}{a+b}}^{\sqrt{ab}} e^{-t^2}dt \le \tan^{-1}\sqrt{ab}$*, as desired.*

UP.288 Solve for positive real numbers:

$$\begin{cases} \frac{x^2}{y}+\frac{y^2}{x} = \sqrt[8]{128(x^8+y^8)} \\ 4x^3-3y = \sqrt{\frac{1+\sqrt{1-xy}}{2}} \end{cases}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by proposer

$$\begin{cases} \frac{x^2}{y}+\frac{y^2}{x}=\sqrt[8]{128(x^8+y^8)} \\ 4x^3-3y=\sqrt{\frac{1+\sqrt{1-xy}}{2}} \end{cases};(1)$$

By CBS inequality, we have:

$$\left(\sqrt{2(x^8+y^8)}+2x^2y^2\right)^2\le 2(2(x^8+y^8)+4x^4y^4)=4(x^8+2x^4y^4+y^8)$$

$$=4(x^4+y^4)^2\Rightarrow\sqrt{2(x^8+y^8)}+2x^2y^2\le 2(x^4+y^4)$$

$$\Rightarrow\sqrt[4]{\frac{x^8+y^8}{2}}\le\sqrt{x^4-x^2y^2+y^4};(2)$$

Other, $\sqrt{x^4-x^2y^2+y^4}=\sqrt{(2+\sqrt{3})(x^2-xy\sqrt{3}+y^2)(2-\sqrt{3})(x^2+xy\sqrt{3}+y^2)}\le$

$$\le\frac{(2+\sqrt{3})(x^2-xy\sqrt{3}+y^2)+(2-\sqrt{3})(x^2+xy\sqrt{3}+y^2)}{2}=2x^2-3xy+2y^2;(3)$$

From (2),(3) we have

$$\sqrt[4]{\frac{x^8+y^8}{2}}\le\ 2x^2-3xy+2y^2;(4)$$

By AM-GM we have: $\frac{x^2}{y}+\frac{y^2}{x}=\frac{x^3+y^3}{xy}=\frac{(x+y)[(2x^2-3xy+2y^2)+xy]}{2xy}\ge\frac{2\sqrt{xy}\cdot 2\sqrt{xy(2x^2-3xy+2y^2)}}{2xy}\Rightarrow$

$$\frac{x^2}{y}+\frac{y^2}{x}\ge 2\sqrt{xy(2x^2-3xy+2y^2)}\overset{(4)}{\Rightarrow}\frac{x^2}{y}+\frac{y^2}{x}\ge\sqrt[8]{128(x^8+y^8)};(5)$$

From (1),(5) we get:

$$\frac{x^2}{y}+\frac{y^2}{x}=\sqrt[8]{128(x^8+y^8)}\Leftrightarrow\begin{cases}\sqrt{2(x^8+y^8)}=2x^2y^2\\ x=y>0\end{cases}\Leftrightarrow x=y>0.$$

From (1) result: $4x^3-3x=\sqrt{\frac{1+\sqrt{1-x^2}}{2}}.$

Because: $0<x\le 1$***, put:*** $x=sin\alpha>0;\forall\alpha\in(0,\pi)$ ***then***

$$4\sin^3\alpha - 3\sin\alpha = \sqrt{\frac{1+\sqrt{1-\sin^2\alpha}}{2}} \Leftrightarrow \sin(-3\alpha) = \sqrt{\frac{1+|\cos\alpha|}{2}};(6)$$

Case 1. $\cos\alpha \geq 0;\ \alpha \in (0,\pi) \Rightarrow \alpha \in [0,\frac{\pi}{2})$

$$(6) \Leftrightarrow \sin(-3\alpha) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+2\cos^2\frac{\alpha}{2}-1}{2}} = \sqrt{\cos^2\frac{\alpha}{2}} = \cos\frac{\alpha}{2} \Leftrightarrow$$

$$\sin(-3\alpha) = \cos\frac{\alpha}{2} \Leftrightarrow \sin(-3\alpha) = \sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) \Rightarrow \begin{cases} -3\alpha = \frac{\pi}{2} + 2k\pi \\ -3\alpha = \pi - \left(\frac{\pi}{2}-\frac{\alpha}{2}\right) + 2k\pi \end{cases} \Leftrightarrow$$

$$\begin{cases} \alpha = -\frac{\pi}{5} - \frac{4k\pi}{5} \\ \alpha = -\frac{\pi}{7} - \frac{4k\pi}{7} \end{cases}$$

If $\alpha = -\frac{\pi}{5} - \frac{4k\pi}{5}$, *because* $\alpha \in [0,\frac{\pi}{2})$ *we get:*

$$0 \leq -\frac{\pi}{5} - \frac{4k\pi}{5} < \frac{\pi}{2} \Leftrightarrow \frac{1}{4} \leq -k < \frac{7}{8} \Leftrightarrow -\frac{1}{4} \geq k > -\frac{7}{8};\ k \in \mathbb{Z} \text{ absurd!}$$

If $\alpha = -\frac{\pi}{7} - \frac{4k\pi}{7}$, *because* $\alpha \in [0,\frac{\pi}{2})$ *we get:*

$$0 \leq -\frac{\pi}{7} - \frac{4k\pi}{7} < \frac{\pi}{2} \Leftrightarrow \frac{1}{4} \leq -k < \frac{9}{8} \Leftrightarrow -\frac{1}{4} \geq k > -\frac{9}{8};\ k \in \mathbb{Z} \Rightarrow k = -1 \Rightarrow \alpha = \frac{3\pi}{7} \Rightarrow x =$$

$$y = \sin\frac{3\pi}{7} = \sin\frac{4\pi}{7}.$$

Case 2. $\cos\alpha < 0;\ \alpha \in (0,\pi) \Rightarrow \alpha \in \left(\frac{\pi}{2},\pi\right)$

$$(6) \Leftrightarrow \sin(-3\alpha) = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(1-2\sin^2\frac{\alpha}{2}\right)}{2}} = \sqrt{\sin^2\frac{\alpha}{2}} = \sin\frac{\alpha}{2}$$

$$\Leftrightarrow \sin(-3\alpha) = \sin\frac{\alpha}{2} \Leftrightarrow \begin{cases} -3\alpha = \frac{\alpha}{2} + 2k\pi \\ -3\alpha = \pi - \frac{\alpha}{2} + 2k\pi \end{cases} \Leftrightarrow \begin{cases} \alpha = -\frac{4k\pi}{7} \\ \alpha = -\frac{2k\pi}{5} - \frac{4k\pi}{5} \end{cases}$$

If $\alpha = -\frac{4k\pi}{7}$, *because* $\alpha \in \left(\frac{\pi}{2},\pi\right) \Rightarrow \frac{\pi}{2} < -\frac{4k\pi}{7} < \pi \Leftrightarrow \frac{7}{8} \leq -k < \frac{7}{4} \Leftrightarrow$

$$-\frac{7}{8} > k > -\frac{7}{4};\ k \in \mathbb{Z} \text{ then } k = -1 \Rightarrow \alpha = \frac{4\pi}{7} \Rightarrow x = y = \sin\frac{4\pi}{7}.$$

If $\alpha = -\frac{2k\pi}{5} - \frac{4k\pi}{5}$*, because* $\alpha \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{\pi}{2} < -\frac{2\pi}{5} - \frac{4k\pi}{5} < \pi \Leftrightarrow \frac{9}{8} < -k < \frac{7}{4} \Leftrightarrow$

$-\frac{9}{8} > k > -\frac{7}{4}; k \in \mathbb{Z}$ *absurd!*

Therefore: $(x, y) = \left(\sin\frac{4\pi}{7}; \sin\frac{4\pi}{7}\right)$

UP.289. If $0 \le k \le m; 0 \le l \le n$ then:

$$\left(\frac{4}{\pi}\right)^{k+l} \int_0^1 \left(\tan^{-1}(1+x^2)\right)^{m+n} dx \ge \int_0^1 \frac{\left(\tan^{-1}(1+x^2)\right)^{m+n}}{\left(\tan^{-1}(1+x^2)\right)^{k+l}} dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania

We have $\tan^{-1}(1+x^2) \ge \frac{\pi}{4}, \forall x \in [0,1]$*. It follows*

$$\frac{1}{\tan^{-1}(1+x^2)^{k+l}} \le \left(\frac{4}{\pi}\right)^{k+l} \Rightarrow \left(\frac{4}{\pi}\right)^{k+l} \cdot \tan^{-1}(1+x^2)^{m+n} \le \frac{\tan^{-1}(1+x^2)^{m+n}}{\tan^{-1}(1+x^2)^{k+l}}, \forall x \in [0,1]$$

Integrating this on $[0,1]$*, we obtain:*

$$\left(\frac{4}{\pi}\right)^{k+l} \int_0^1 \left(\tan^{-1}(1+x^2)^{m+n}\right) dx \ge \int_0^1 \frac{\tan^{-1}(1+x^2)^{m+n}}{\tan^{-1}(1+x^2)^{k+l}} dx$$

Solution 2 by Adrian Popa-Romania

Let be the function $f(x) = \tan^{-1}x; f'(x) = \frac{1}{1+x^2} > 0, \forall x \in \mathbb{R} \Rightarrow f$ *–increasing.*

$$\tan^{-1}(1+x^2) \ge \tan^{-1}1 = \frac{\pi}{4} \to \left(\tan^{-1}(1+x^2)\right)^{k+l} \ge \left(\frac{\pi}{4}\right)^{k+l}$$

$$\frac{1}{\left(\tan^{-1}(1+x^2)\right)^{k+l}} \le \left(\frac{4}{\pi}\right)^{k+l} \leftrightarrow \frac{\left(\tan^{-1}(1+x^2)\right)^{m+n}}{\left(\tan^{-1}(1+x^2)\right)^{k+l}} \le \left(\frac{4}{\pi}\right)^{k+l} \cdot \left(\tan^{-1}(1+x^2)\right)^{m+n}$$

$$\int_0^1 \frac{\left(\tan^{-1}(1+x^2)\right)^{m+n}}{\left(\tan^{-1}(1+x^2)\right)^{k+l}} dx \le \left(\frac{4}{\pi}\right)^{k+l} \int_0^1 \left(\tan^{-1}(1+x^2)\right)^{m+n} dx$$

UP.290 If $a, b \geq 1$; $f \in C^2([0,1])$, $f(0) \leq 0$, f convexe then:

$$2\int_0^1 f(x)\,dx \geq a^2\int_0^{\frac{1}{a}} f(x)\,dx + b^2\int_0^{\frac{1}{b}} f(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$A = a^2\int_0^{\frac{1}{a}} f(x)\,dx = a\int_0^1 f\left(\frac{x}{a}\right)dx$$

$$B = b^2\int_0^{\frac{1}{b}} f(x)\,dx = b\int_0^1 f\left(\frac{x}{b}\right)dx$$

$$A + B = \int_0^1 \left(af\left(\frac{x}{a}\right) + bf\left(\frac{x}{b}\right)\right)dx$$

***f*-convexe and $a, b \geq 1$, then: $f(tx + (1-t)y) \leq tf(x) + (1-t)f)y)$**

$$f\left(\frac{x}{a}\right) \leq \frac{1}{a}f(x) + \left(1 - \frac{1}{a}\right)f(0)$$

$$f\left(\frac{x}{b}\right) \leq \frac{1}{b}f(x) + \left(1 - \frac{1}{b}\right)f(0)$$

$$A + B \leq 2\int_0^{\frac{1}{a}} f(x)\,dx + (a-1)f(0) + (b-1)f(0)$$

$$a^2\int_0^{\frac{1}{a}} f(x)\,dx + b^2\int_0^{\frac{1}{b}} f(x)\,dx \leq 2\int_0^1 f(x)dx + (a+b-2)f(0);$$

$$a, b \geq 1 \Rightarrow a + b - 2 \geq 0$$

So, if $f(0) \leq 0$ then:

$$2\int_0^1 f(x)\,dx \ge a^2\int_0^{\frac{1}{a}} f(x)\,dx + b^2\int_0^{\frac{1}{b}} f(x)\,dx$$

Solution 2 by proposer

$f \in C^2([0,1]); f$ *convexe* $\Rightarrow f''(x) \ge 0; (\forall)x \in [0,1]$. *Let be* $g:(0,1] \to \mathbb{R}; g(x) = \frac{f(x)}{x}$

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{h(x)}{x^2}; h(x) = xf'(x) - f(x)$$

$$h'(x) = f'(x) + xf''(x) - f'(x) = xf''(x) > 0$$

$$h \text{ increasing } h(x) \ge h(0) = 0 \Rightarrow g'(x) > 0 \Rightarrow g \text{ increasing}$$

$$\text{If } x \in \left(0,\frac{1}{a}\right]; g(x) \le g\left(\frac{1}{a}\right) \Rightarrow \frac{f(x)}{x} \le \frac{f\left(\frac{1}{a}\right)}{\frac{1}{a}}$$

$$f(x) \le axf\left(\frac{1}{a}\right) \Rightarrow \int_0^{\frac{1}{a}} f(x)\,dx \le \int_0^{\frac{1}{a}} ax\,f\left(\frac{1}{a}\right)dx = af\left(\frac{1}{a}\right)\int_0^{\frac{1}{a}} x\,dx = a\cdot\frac{1}{2a^2}f\left(\frac{1}{a}\right) = \frac{1}{2a}f\left(\frac{1}{a}\right)$$

$f\left(\frac{1}{a}\right) \ge 2a\int_0^{\frac{1}{a}} f(x)\,dx$. *If* $x \in \left[\frac{1}{a},1\right]; g(x) \ge g\left(\frac{1}{a}\right) \Rightarrow \frac{f(x)}{x} \ge \frac{f\left(\frac{1}{a}\right)}{\frac{1}{a}}$

$$f(x) \ge ax\,f\left(\frac{1}{a}\right)$$

$$\int_{\frac{1}{a}}^1 f(x)\,dx \ge \int_{\frac{1}{a}}^a ax\,f\left(\frac{1}{a}\right) = af\left(\frac{1}{a}\right)\int_{\frac{1}{a}}^a x\,dx = af\left(\frac{1}{a}\right)\cdot\frac{1}{2}\left(1-\frac{1}{a^2}\right) = \frac{a}{2}\cdot\frac{a^2-1}{a^2}f\left(\frac{1}{a}\right) \ge$$

$$\ge \frac{a^2-1}{2a}\cdot 2a\int_0^{\frac{1}{a}} f(x)\,dx = (a^2-1)\int_0^{\frac{1}{a}} f(x)\,dx$$

$$\int_0^1 f(x)\,dx = \int_0^{\frac{1}{a}} f(x)\,dx + \int_{\frac{1}{a}}^1 f(x)\,dx \ge \int_0^{\frac{1}{a}} f(x)\,dx + (a^2-1)\int_0^{\frac{1}{a}} f(x)\,dx == a^2\int_0^{\frac{1}{a}} f(x)\,dx$$

$\int_0^1 f(x)\,dx \ge a^2\int_0^{\frac{1}{a}} f(x)\,dx$ *(1). Analogous:* $\int_0^1 f(x)\,dx \ge b^2\int_0^{\frac{1}{b}} f(x)\,dx$ *(2)*

By adding (1); (2):

$$2\int_0^1 f(x)\,dx \geq a^2\int_0^{\frac{1}{a}} f(x)\,dx + b^2\int_0^{\frac{1}{b}} f(x)\,dx$$

UP.291 Prove that:

$$\int_0^{\frac{1}{2}}\left(\log(1+x)\cdot\log\left(\frac{3}{2}+x\right)\right)dx < \frac{1}{2}\left(\int_0^1 \log(1+x)\,dx\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by proposer:

$$\textit{Let be } f:[0,1]\to[0,\infty);\, f(x)=\log(1+x)$$

$$f'(x)=\frac{1}{1+x}>0;\, f \textit{ increasing}$$

$$\textit{If } x\in\left[0,\frac{1}{2}\right]\Rightarrow f(x)\le f\left(\frac{1}{2}\right)\Rightarrow f(x)-f\left(\frac{1}{2}\right)\le 0$$

$$f\left(x+\frac{1}{2}\right)\ge f\left(\frac{1}{2}\right)\Rightarrow f\left(x+\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\le 0$$

$$\left(f(x)-f\left(\frac{1}{2}\right)\right)\left(f\left(x+\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\right)\le 0$$

$$\left(\log(1+x)-\log\frac{3}{2}\right)\left(\log\left(\frac{3}{2}+x\right)-\log\frac{3}{2}\right)\le 0$$

$$\log(1+x)\log\left(\frac{3}{2}+x\right)-\log\frac{3}{2}\cdot\log(1+x)-\log\frac{3}{2}\cdot\log\left(\frac{3}{2}+x\right)+\log^2\frac{3}{2}\le 0$$

$$\log(1+x)\log\left(\frac{3}{2}+x\right)\le\log\frac{3}{2}\cdot\log(1+x)+\log\frac{3}{2}\cdot\log\left(\frac{3}{2}+x\right)-\log^2\frac{3}{2}$$

$$\int_0^{\frac{1}{2}}\left(\log(1+x)\log\left(\frac{3}{2}+x\right)\right)dx<\log\frac{3}{2}\int_0^{\frac{1}{2}}\log(1+x)\,dx+$$

$$+\log\frac{3}{2}\int_0^{\frac{1}{2}}\log\left(\frac{3}{2}+x\right)dx-\int_0^{\frac{1}{2}}\log^2\frac{3}{2}\,dx=$$

$$= \log\frac{3}{2}\int_0^{\frac{1}{2}} \log(1+x)\,dx + \log\frac{3}{2}\int_{\frac{1}{2}}^{1} \log(1+x)\,dx +$$

$$-\frac{1}{2}\log^2\frac{3}{2} = \log\frac{3}{2}\int_0^1 \log(1+x)\,dx - \frac{1}{2}\log^2\frac{3}{2}$$

$$2\int_0^{\frac{1}{2}} \left(\log(1+x)\cdot\log\left(\frac{3}{2}+x\right)\right)dx < 2\log\frac{3}{2}\int_0^1 \log(1+x)\,dx - \log^2\left(\frac{3}{2}\right)$$

Remains to prove:

$$2\log\frac{3}{2}\int_0^1 \log(1+x)\,dx - \log^2\left(\frac{3}{2}\right) < \left(\int_0^1 \log(1+x)\,dx\right)^2$$

$$\left(\int_0^1 \log(1+x)\,dx\right)^2 - 2\log\frac{3}{2}\int_0^1 \log(1+x)\,dx + \log^2\left(\frac{3}{2}\right) > 0$$

$$\left(\int_0^1 \log(1+x)\,dx - \log\frac{3}{2}\right)^2 > 0$$

UP.292. Prove that:

$$\int_0^1 \left(tan^{-1}x + \frac{x}{1+x^2}\right)^2 dx + 4\int_0^1 \frac{1}{(1+x^2)^4}\,dx > \frac{(\pi+2)^2}{16}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Şerban-Romania

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right) - C.B.S$$

$$\int_0^1 \left(tan^{-1}x + \frac{x}{1+x^2}\right)^2 dx \geq \left(\int_0^1 \left(tan^{-1}x + \frac{x}{1+x^2}\right) dx\right)^2$$

$$= \left(\int_0^1 x' tan^{-1}x\, dx + \int_0^1 \frac{xdx}{1+x^2}\right)^2 = \left(xtan^{-1}x\Big|_0^1 - \int_0^1 \frac{xdx}{1+x^2} + \int_0^1 \frac{xdx}{1+x^2}\right)^2 = \frac{\pi^2}{16}$$

$$4\int_0^1 \frac{dx}{(1+x^2)^4} = 4\int_0^1 \left(\frac{1}{(1+x^2)^2}\right)^2 dx \geq 4\left(\int_0^1 \frac{dx}{(1+x^2)^2}\right)^2$$

$$\overset{x=tan(t)}{=} \int_0^{\frac{\pi}{4}} cos^2 t dt = \int_0^{\frac{\pi}{4}} \frac{1+cos(2t)}{2} dt = \frac{1}{2}\int_0^{\frac{\pi}{4}} dt + \frac{1}{2}\int_0^{\frac{\pi}{4}} cos(2t)dt$$

$$= \frac{1}{2}\cdot t\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\cdot \frac{sin(2t)}{2}\Big|_0^{\frac{\pi}{4}} = \frac{\pi+2}{8} \Rightarrow 4\int_0^1 \frac{dx}{(1+x^2)^2} > \frac{(\pi+2)^2}{16}$$

$$\int_0^1 \left(tan^{-1}x + \frac{x}{1+x^2}\right)^2 dx + 4\int_0^1 \frac{1}{(1+x^2)^4} dx > \frac{\pi^2}{16} + \frac{(\pi+2)^2}{16} > \frac{(\pi+2)^2}{16}$$

Solution 2 by Adrian Popa-Romania

Let: $f(x) = tan^{-1}x + \frac{x}{1+x^2} \Rightarrow f'(x) = \frac{2}{(1+x^2)^2} > 0$

We must show:

$$\int_0^1 f^2(x)dx + \int_0^1 (f'(x))^2 dx > \frac{(\pi+2)^2}{16}$$

$$\int_0^1 f^2(x)dx + \int_0^1 \left(f'(x)\right)^2 dx \overset{Am-Gm}{\geq} 2\sqrt{\int_0^1 f^2(x)dx \cdot \int_0^1 \left(f'(x)\right)^2 dx} \overset{CBS}{\geq}$$

$$\geq 2\sqrt{\left(\int_0^1 f(x)\cdot f'(x)dx\right)^2} = 2\cdot \frac{f^2(x)}{2}\Big|_0^1 = \left(tan^{-1}x + \frac{x}{1+x^2}\right)\Big|_0^1 = \left(tan^{-1}1 + \frac{1}{2}\right)^2$$

$$= \left(\frac{\pi+2}{4}\right)^2 = \frac{(\pi+2)^2}{16}$$

Solution 3 by Avishek Mitra-West Bengal

$$\left(\tan^{-1}x+\frac{x}{1+x^2}\right)^2+\frac{4}{(1+x^2)^4}\overset{Am-Gm}{\geq}2\left(\tan^{-1}x+\frac{x}{1+x^2}\right)\cdot\frac{2}{(1+x^2)^2}$$

$$=\frac{4\tan^{-1}x}{(1+x^2)^2}+\frac{4x}{(1+x^2)^3}$$

$$I_1=\int_0^1\frac{4x}{(1+x^2)^3}dx\overset{\substack{x^2=z\\2xdx=dz}}{=}2\int_0^1\frac{dz}{(1+z)^3}=2\left.\frac{(1+z)^{-2}}{-2}\right|_0^1=\frac{3}{4}$$

$$I_2=\int_0^1\frac{\tan^{-1}x}{(1+x^2)^2}dx\overset{\substack{x=\tan\theta\\dx=\sec^2\theta d\theta}}{=}\int_0^{\frac{\pi}{4}}\theta\cos^2\theta d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\theta(1+\cos(2\theta))d\theta=\frac{1}{2}\left[\frac{\theta^2}{2}+\frac{\theta\sin(2\theta)}{2}+\frac{\cos(2\theta)}{4}\right]_0^{\frac{\pi}{4}}=\frac{\pi^2}{64}+\frac{\pi}{16}-\frac{1}{8}$$

$$4I_2=\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{1}{2}$$

$$\Omega=\int_0^1\left(\tan^{-1}x+\frac{x}{1+x^2}\right)^2dx+4\int_0^1\frac{1}{(1+x^2)^4}dx\geq 4I_2+I_1=\frac{(\pi+2)^2}{16}$$

Solution 4 by Tobi Joshua-Nigeria

$$\int_0^1\left(\tan^{-1}x+\frac{x}{1+x^2}\right)^2dx+4\int_0^1\frac{1}{(1+x^2)^4}dx$$

$$=\int_0^1\left((\tan^{-1}x)^2dx+\frac{2x\tan^{-1}x}{1+x^2}+\frac{x^2}{(1+x^2)^2}\right)dx+$$

$$+4\int_0^1\frac{1}{(1+x^2)^4}dx>\frac{(\pi+2)^2}{16}\overset{x=\tan(y)}{\Longleftrightarrow}$$

$$\int_0^{\frac{\pi}{4}}\left(y^2+\frac{2y\tan(y)}{\sec^2y}+\frac{\tan^2y}{\sec^4y}\right)\sec^2ydy+4\int_0^{\frac{\pi}{4}}\frac{\sec^2ydy}{\sec^8y}>\frac{(\pi+2)^2}{16}$$

$$\int_0^{\frac{\pi}{4}}\left(y^2+2ytan(y)+\frac{tan^2y}{sec^2y}\right)dy+4\int_0^{\frac{\pi}{4}}\frac{dy}{sec^6y}>\frac{(\pi+2)^2}{16}$$

$$\int_0^{\frac{\pi}{4}}\left(d\left(y^2tan(y)\right)+\frac{(sec^2y-1)dy}{sec^2y}\right)+4\int_0^{\frac{\pi}{4}}\frac{dy}{sec^6y}>\frac{(\pi+2)^2}{16}$$

$$\int_0^{\frac{\pi}{4}}\left(d\left(y^2tan(y)\right)+\frac{(sec^2y-1)dy}{sec^2y}\right)+4\int_0^{\frac{\pi}{4}}cos^6ydy>\frac{(\pi+2)^2}{16}$$

Recall that: $cos^6x=\frac{1}{32}(cos(6x)+6cos(4x)+16cos(2x)+10)$

$$\int_0^{\frac{\pi}{4}}d\left(y^2tan(y)\right)+\int_0^{\frac{\pi}{4}}\frac{(sec^2y-1)dy}{sec^2y}$$

$$+\frac{4}{32}\int_0^{\frac{\pi}{4}}(cos(6y)+6cos(4y)+16cos(2y)+10)dy>\frac{(\pi+2)^2}{16}$$

$$\left(y^2tan(y)\right)_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}(1-cos^2y)dy+$$

$$+\frac{4}{32}\left(\frac{sin(6y)}{6}+\frac{6sin(4y)}{4}+8sin(2y)+10y\right)_0^{\frac{\pi}{4}}>\frac{(\pi+2)^2}{16}$$

$$\frac{\pi^2}{16}+\frac{1}{2}\int_0^{\frac{\pi}{4}}(1-cos(2y))dy+\frac{4}{32}\left(-\frac{1}{6}+8+\frac{5\pi}{2}\right)>\frac{(\pi+2)^2}{16}$$

$$\frac{\pi^2}{16}+\frac{7\pi}{4}+\frac{35}{48}>\frac{(\pi+2)^2}{16}$$

$$\frac{1}{16}\left(\frac{\pi^2}{16}+28\pi+\frac{35}{3}\right)>\frac{(\pi+2)^2}{16}$$

$$\frac{\pi^2}{16}+28\pi+\frac{35}{3}>(\pi+2)^2$$

UP.293. Solve for real numbers:

$$\begin{cases} \dfrac{x^2}{2y}+\dfrac{y^2}{2x}=\sqrt[4]{\dfrac{x^4+y^4}{2}} \\ x^2y^2-y^3+1=\sqrt{2x^2-2y+1} \end{cases}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Orlando Irahola Ortega-La Paz-Bolivia

$$\begin{cases} \dfrac{x^2}{2y}+\dfrac{y^2}{2x}=\sqrt[4]{\dfrac{x^4+y^4}{2}};(1) \\ x^2y^2-y^3+1=\sqrt{2x^2-2y+1};(2) \end{cases}$$

From (1) we have $x^3+y^3=2xy\sqrt[4]{\frac{x^4+y^4}{2}}\Rightarrow x^6+y^6+2x^3y^3=4x^2y^2\sqrt{\frac{x^4+y^4}{2}}\Rightarrow$

$$\frac{x^3}{y^3}+\frac{y^3}{x^3}+2=4\sqrt{\frac{1}{2}\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)}$$

Let: $t=\frac{x}{y}+\frac{y}{x}\Rightarrow t^3-3t+2=2\sqrt{2t^2-4}\Rightarrow t(t^2-1)=2\left(\sqrt{2t^2-4}-1\right)\Rightarrow$

$$t^2(t^2-3)^2=4\left(\sqrt{2t^2-4}-1\right)^2\Rightarrow\left(\sqrt{2t^2-4}^2+4\right)\left(\sqrt{2t^2-4}^2-2\right)^2=$$

$$32\left(\sqrt{2t^2-4}-1\right)^2.$$

Let: $m=\sqrt{2t^2-4}\Rightarrow(m^2+4)(m^2-2)^2=32(m-1)^2\Rightarrow$

$$m^6-44m^2+64m-16=0\Leftrightarrow(m-2)\left(m^5+2m^4+4m^3+8m^2-28m+8\right)=0$$

$m-2=0\Rightarrow m_1=2\Rightarrow\sqrt{2t^2-4}=2\Rightarrow t=\pm2$, but $t=\frac{x}{y}+\frac{y}{x}\Rightarrow\frac{t}{2}\geq\sqrt{\frac{x}{y}\cdot\frac{y}{x}}$

$\Rightarrow t\geq2; x,y\in\mathbb{R}$. For $t=2\Rightarrow\frac{x}{y}+\frac{y}{x}=2\Rightarrow(x-y)^2=0\Rightarrow x=y;(3)$

$$m^5+2m^4+4m^3+8m^2-28m+8=0=f(m)$$

Condition: $m\geq0\Rightarrow\sqrt{2t^2-4}=|m|$

If $m=0\Rightarrow f(0)=8>0, m=1\Rightarrow f(1)=-5<0\Rightarrow\exists x_0\in[0,1]$ *solution.*

If $m>1\Rightarrow f(m>1)>0\Rightarrow$ *don't exist solution.*

Fom (2) we have $y^2(2x^2-2y)+2=2\sqrt{2x^2-2y+1}\Rightarrow$

$$y^2\sqrt{2x^2-2y+1}^2+2-y^2=2\sqrt{2x^2-2y+1}$$

Let: $t=\sqrt{2x^2-2y+1}\Rightarrow t^2y^2+2-y^2=2t\Rightarrow(t-1)(ty^2+y^2-2)=0\Rightarrow$

$$t=1;(*);ty^2=2-y^2;(**)$$

From (*)$\Rightarrow t=1\Rightarrow\sqrt{2x^2-2y+1}=1\Rightarrow y=x^2;(4)$

From (**)$\Rightarrow ty^2=2-y^2\Rightarrow t^2y^4=y^4-4y^2+4\Rightarrow y^4(2x^2-2y+1)=y^4-4y^2+4$

$$x^2y^4+2y^2=y^5+2$$

We have: $\begin{cases}y=x\\y=x^2\end{cases}\Rightarrow x^2-x=0\Rightarrow\begin{cases}x=y=0\\x=y=1\end{cases}$ *and* $x,y\neq0\Rightarrow\begin{cases}x=y\\x^2y^4+2y^2=y^5+2\end{cases}\Rightarrow$

$$\Rightarrow(x-1)(x^5+2x+2)=0$$

If $x=1$ and $x^5+2x+2=0,\forall x\geq0\nexists$ *solution.*

$\forall x\in\mathbb{R},-1\leq x<0\Rightarrow\exists x<0,y<0$ *but from (2) result* $2x^2-2y+1\geq0\Rightarrow$

$2x^2+1\geq2y\Rightarrow y\geq0$ *and how* $y<0,\nexists x,y$ *for satisfying the system.*

So, $x=y=1$ *unique solution.*

Solution 2 by Abner Chinga Bazo-Lima-Peru

$$\frac{1}{2}\left(\frac{x^2}{y}+\frac{y^2}{x}\right)\geq\frac{1}{2}\frac{(x+y)^2}{x+y}=\frac{1}{2}(x+y);(i)$$

Equality occurs when $x=y$.

The problem: $\frac{x^2}{2y}+\frac{y^2}{2x}=\sqrt[4]{\frac{x^4+y^4}{2}};(ii)$

From (ii) in (i) result: $\sqrt[4]{\frac{x^4+y^4}{2}}\geq\frac{x+y}{2};x,y>0$

Equality occurs when $x=y=k$.

$$x^2y^2-y^3+1=\sqrt{2x^2-2y+1}\Leftrightarrow k^4-k^3+1=\sqrt{2k^2-2k+1}\Leftrightarrow$$

$$(k^4-k^3+1)^2=2k^2-2k+1\Leftrightarrow$$

$$k^8-2k^7+k^6+k^5+2k^3-2k^2-2k+2=0\Leftrightarrow k(k-1)^2(k^5+2k+2)=0;k>0$$

Since $k>0\Rightarrow k^5+2k+2>2\Rightarrow k=x=y=1.$

So, $(x,y)=(1,1)$ *unique solution.*

Solution 3 by proposer

$$\text{Let (1): } \frac{x^2}{y}+\frac{y^2}{x}=2\sqrt[4]{\frac{x^4+y^4}{2}}\Leftrightarrow\frac{x^3+y^3}{xy}=2\sqrt[4]{\frac{x^4+y^4}{2}}\Leftrightarrow\left(\frac{x^3+y^3}{xy}\right)^4=2^4\cdot\frac{x^4+y^4}{2}\Leftrightarrow$$

$$\frac{(x^3+y^3)^4}{(xy)^4}=8(x^4+y^4)\Leftrightarrow(x^3+y^3)^4=8(xy)^4(x^4+y^4);(2)$$

We have:

$$2(x^2-xy+y^2)^2-(x^4+y^4)=2(x^4-2x^3y+3x^2y^2-2xy^3+y^4)-(x^4+y^4)=$$

$$=x^4-4x^3y+6x^2y^2-4xy^3+y^4=(x-y)^4\geq 0\Rightarrow$$

$$2(x^2-xy+y^2)^2-(x^4+y^4)\geq 0\Rightarrow(x^2-xy+y^2)^2=\frac{x^4+y^4}{2};(3)$$

By AM-GM inequality for two positive real numbers, we have:

$$(x+y)^4(x^2-xy+y^2)^2\geq\left(2\sqrt{xy}\right)^4(2xy-xy)^2=16(xy)^4$$

From (3) result:

$$(x+y)^4(x^2-xy+y^2)^2(x^2-xy+y^2)^2\geq 16(xy)^4\cdot\frac{x^4+y^4}{2}\Rightarrow$$

$$[(x+y)(x^2-xy+y^2)]^4\geq 8x^4y^4(x^4+y^4)\Rightarrow$$

$$(x^3+y^3)^4\geq 8x^4y^4(x^4+y^4);(4)$$

From (2),(4) result the equality $(x^3+y^3)^4=8x^4y^4(x^4+y^4)$ ***occurs if***

$$\begin{cases}x-y=0\\x=y>0\end{cases}\Leftrightarrow x=y>0$$

Let $x=y$ ***in (1):*** $x^4-x^3+1=\sqrt{2x^2-2x+1};(5)$

Because $2x^2-2x+1=2\left(x^2-x+\frac{1}{4}\right)+\frac{1}{2}=2\left(x-\frac{1}{2}\right)^2+\frac{1}{2}\geq\frac{1}{2}>0,\forall x>0$ ***hence by AM-GM inequality, we have***

$$\sqrt{(2x^2-2x+1)\cdot 1}\leq\frac{2x^2-2x+1+1}{2}=\frac{2x^2-2x+2}{2}=x^2-x+1;(6)$$

From (5),(6) we get: $x^4-x^3+1\leq x^2-x+1\Leftrightarrow x^4-x^3-x^2+x\leq 0\Leftrightarrow$

$$x(x^3-x^2-x+1)\leq 0\Leftrightarrow x[x^2(x-1)-(x-1)]\leq 0\Leftrightarrow x(x+1)(x-1)^2\leq 0;(7)$$

Because $x>0$ ***then*** $x(x+1)(x-1)^2\geq 0$ ***from (7) we get:*** $x(x+1)(x-1)^2=0$ ***and***

equality occurs if $\begin{cases} 2x^2 - 2x + 1 = 1 \\ x - 1 = 0 \\ x = y > 0 \end{cases} \Rightarrow \begin{cases} 2x(x-1) = 0 \\ x = 1 \\ x = y > 0 \end{cases} \Leftrightarrow x = y = 1.$

Hence $(x, y) = (1, 1)$ *is the solution for the problem.*

Solution 4 by Miguel Velasquez Culque-Peru

Condition $x > 0, y > 0$ *and* $x \neq 0, y \neq 0$. *By AM-GM we have:*

$$\frac{x^2}{2y} + \frac{y^2}{2x} \geq 2\sqrt{\frac{x^2}{2y} \cdot \frac{y^2}{2x}} = \sqrt{xy} \Leftrightarrow \sqrt[4]{\frac{x^4 + y^4}{2}} \geq \sqrt{xy} \Leftrightarrow \frac{x^4 + y^4}{2} \geq x^2y^2 \Leftrightarrow$$

$$x^4 + y^4 \geq 2x^2y^2 \Leftrightarrow (x^2 - y^2)^2 \geq 0. \text{ Equality holds if } x = y.$$

Replace in the second equation:

$$x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1} \Rightarrow x^4 - x^3 + 1 = \sqrt{2x^2 - 2x + 1} \Leftrightarrow$$

$$(x^4 - x^3 + 1)^2 = 2x^2 - 2x + 1 \Leftrightarrow x^8 - 2x^7 + x^5 + 2x^3 - 2x^2 + 2x = 0 \Leftrightarrow$$

$$x(x^7 - 2x^6 + x^5 + 2x^3 - 2x^2 - 2x + 2) = 0 \Leftrightarrow x(x-1)^2(x^5 + 2x + 2) = 0 \Leftrightarrow$$

$$x = 1, \text{ because } x \neq 0 \text{ and } x^5 + 2x + 2 > 2 > 0$$

UP.294. In $\Delta ABC, AD, BE, CF$ –medians, G –centroid; $AM = MG, M \in (AG)$;

$$2cotA = cotB + cotC$$

Prove that: $DEMF$ is a cyclic quadrilateral.

Proposed by Marian Ursărescu-Romania

Solution by Daniel Văcaru-Romania

We have: $2cotA = cotB + cotC$

$$\Rightarrow \frac{2cosA}{sinA} = \frac{cosB}{sinB} + \frac{cosC}{sinC} \Rightarrow \frac{2cosA}{sinA} = \frac{\sin(B+C)}{sinBsinC} \Rightarrow 2sinA \cdot sinB \cdot sinC = sin^2A$$

$$\Rightarrow 2bccosA = a^2 \Rightarrow b^2 + c^2 = 2a^2$$

For $DEMF$ is a cyclic quadrilateral it suffices to prove that:

$$MF \cdot DE + ME \cdot FD = FE \cdot MD \Rightarrow 2m_a = m_b \cdot b + m_c \cdot c$$

But $4m_b^2 = 2(a^2 + c^2) - b^2 = 2a^2 + 2c^2 - b^2 = 3c^2 \Rightarrow m_b = \frac{c\sqrt{3}}{2}, m_c = \frac{b\sqrt{3}}{2},$

$$4m_a^2 = 2(b^2+c^2) - a^2 \Rightarrow m_a = \frac{a\sqrt{3}}{2}$$

It follows $m_b \cdot b + m_c \cdot c = b \cdot \left(\frac{c\sqrt{3}}{2}\right)^2 + c \cdot \left(\frac{b\sqrt{3}}{2}\right)^2 = \frac{(b^2+c^2)\sqrt{3}}{2} = \frac{2a^2\sqrt{3}}{2} = a^2\sqrt{3} = 2m_a \cdot a$

UP.295 Find $a, b, c, d \in \mathbb{R}$ such that:

$$\begin{cases} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^3 + \begin{pmatrix} c & d \\ -d & c \end{pmatrix}^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Daniel Văcaru-Romania

We note: $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = X$ *and* $\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = Y$.

Our system could be written as $\begin{cases} X + Y = I_2 \\ X^3 + Y^3 = 2I_2 \end{cases}$

Let's observe that $XY = YX$, *then* $b = 0; d = 0$.

It follows $a_{1,2} = \frac{3\pm\sqrt{21}}{6} \Rightarrow c_{1,2} = 1 - \frac{3\pm\sqrt{21}}{6} = \frac{3\mp\sqrt{21}}{6}$

We obtain:

$$X = \begin{pmatrix} \frac{3+\sqrt{21}}{6} & 0 \\ 0 & \frac{3+\sqrt{21}}{6} \end{pmatrix}, Y = \begin{pmatrix} \frac{3-\sqrt{21}}{6} & 0 \\ 0 & \frac{3-\sqrt{21}}{6} \end{pmatrix} \text{ and}$$

$$X = \begin{pmatrix} \frac{3-\sqrt{21}}{6} & 0 \\ 0 & \frac{3-\sqrt{21}}{6} \end{pmatrix}, Y = \begin{pmatrix} \frac{3+\sqrt{21}}{6} & 0 \\ 0 & \frac{3+\sqrt{21}}{6} \end{pmatrix}$$

UP.296. If $x, y, z > 0; xy + yz + zx = 3$ then:

$$\sqrt[3]{\frac{2x}{(y+z)^7}}+\sqrt[3]{\frac{2y}{(z+x)^7}}+\sqrt[3]{\frac{2z}{(x+y)^7}}\ge\frac{3}{4}$$

Proposed by Pedro Pantoja-Brazil

Solution by Tran Hong-Dong Thap-Vietnam

$$\textit{With } a,b,c>0,\textit{ let } t=a+b+c\ge\sqrt{3(ab+bc+ca)}=3$$

$$\sqrt[3]{\frac{2a}{(b+c)^7}}=\frac{1}{(b+c)^2}\cdot\sqrt[3]{\frac{2a}{b+c}}=\frac{1}{(b+c)^2}\cdot\sqrt[3]{\frac{2a}{b+c}\cdot 1\cdot 1}\overset{Am-Hm}{\ge}$$

$$\ge\frac{1}{(b+c)^2}\cdot\frac{3}{1+1+\frac{1}{\frac{2a}{b+c}}}=\frac{1}{(b+c)^2}\cdot\frac{3}{1+1+\frac{b+c}{2a}}$$

$$=\frac{6a}{(b+c)^2(4a+b+c)}=\frac{6\left(\frac{a}{b+c}\right)^2}{4a^2+ab+ac}$$

$$\sqrt[3]{\frac{2b}{(c+a)^7}}\ge\frac{6\left(\frac{b}{c+a}\right)^2}{4b^2+ba+bc}$$

$$\sqrt[3]{\frac{2c}{(a+b)^7}}\ge\frac{6\left(\frac{c}{a+b}\right)^2}{4c^2+cb+ca}\Rightarrow LHS=\sum_{cyc}\sqrt[3]{\frac{2a}{(b+c)^7}}$$

$$\ge 6\left[\frac{\left(\frac{a}{b+c}\right)^2}{4a^2+ab+ac}+\frac{\left(\frac{b}{c+a}\right)^2}{4b^2+ba+bc}+\frac{\left(\frac{c}{a+b}\right)^2}{4c^2+cb+ca}\right]$$

$$\overset{C-B-S}{\ge}6\cdot\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2}{4(a^2+b^2+c^2)+2(ab+bc+ca)}$$

$$=\frac{3\cdot\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2}{2(a^2+b^2+c^2)+ab+bc+ca}=\frac{3\cdot\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2}{2(a^2+b^2+c^2)-9}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a^2}{ab+ac}+\frac{b^2}{bc+ba}+\frac{c^2}{ac+bc}\overset{B-C-S}{\ge}$$

$$\geq \frac{(a+b+c)^2}{2(ab+bc+ca)} = \frac{(a+b+c)^2}{6}$$

So, we need to prove: $\frac{3\cdot\frac{t^4}{6^2}}{2t^2-9} \geq \frac{3}{4} \Leftrightarrow 4t^4 \geq 36(2t^2-9)$

$\Leftrightarrow t^4 - 18t^2 + 81 \geq 0 \Leftrightarrow (t^2-1)^2 \geq 0$ *true for* $t \geq 3$. *Proved.*

Equality $\Leftrightarrow a = b = c = 1$

UP.297. Let $(a_n)_{n\geq 1}$ be a sequence real numbers such that $a_n \in \mathbb{R}_+^* = (0,\infty), \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*$. Find:

$$\lim_{n\to\infty} \sqrt[n^2]{a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdot \ldots \cdot \sqrt[n]{a_n}}$$

Proposed By D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers:

Let be $b_n = a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdot \ldots \cdot \sqrt[n]{a_n}, \forall n \geq 1.$

From Cauchy-D'Alembert Criterion (C-D'A), we have:

$$\lim_{n\to\infty} \sqrt[n^2]{b_n} = \lim_{n\to\infty} \sqrt[n]{\sqrt[n]{b_n}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}; (1)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{b_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{\sqrt[n+1]{a_{n+1}}}{n+1} = \frac{1}{e} \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n}$$

$$= \frac{1}{e} \cdot \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} \overset{C-D'A}{=} \frac{1}{e} \cdot \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} = \frac{1}{e} \lim_{n\to\infty} \frac{a_{n+1}}{na_n} \left(\frac{n}{n+1}\right)^n = \frac{a}{e^2} \overset{(1)}{\Rightarrow}$$

$$\lim_{n\to\infty} \sqrt[n^2]{b_n} = \lim_{n\to\infty} \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} = \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{b_n}} \cdot \frac{n+1}{n}\right) = \frac{a}{e^2} \cdot \frac{e^2}{a} \cdot 1 = 1.$$

Solution 2 by proposers

$$\lim_{n\to\infty} \frac{1}{n^2} \log b_{n+1} = \lim_{n\to\infty} \frac{\log b_{n+1} - \log b_n}{(n+1)^2 - n^2} = \lim_{n\to\infty} \frac{n}{2n+1} \cdot \frac{\log b_{n+1} - \log b_n}{n} \overset{C-S}{=}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{logb_{n+2}-2logb_{n+1}+logb_n}{(n+1)-n}=\frac{1}{2}log\left(\lim_{n\to\infty}\frac{b_{n+2}\cdot b_n}{b_{n+1}^2}\right)=$$

$$=\frac{1}{2}log\left(\lim_{n\to\infty}\frac{\sqrt[n+2]{a_{n+2}}\cdot\sqrt[n+1]{a_{n+1}}}{\sqrt[n+1]{a_{n+1}^2}}\right)=\frac{1}{2}log\left(\lim_{n\to\infty}\frac{\sqrt[n+2]{a_{n+2}}}{n+2}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n+2}{n+1}\right)=$$

$$=\frac{1}{2}log\left(\frac{a}{e^2}\cdot\frac{e^2}{a}\cdot 1\right)=\frac{1}{2}log1=0\Rightarrow\lim_{n\to\infty}\sqrt[n^2]{b_n}=e^{\lim_{n\to\infty}\frac{1}{n^2}logb_{n+1}}=e^0=1.$$

UP.298. Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[2n+2]{(2n+1)!!\cdot(n+1)!}-\sqrt[2n]{(2n)!!\cdot n!}\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Adrian Popa-Romania

$$\therefore L=\lim_{n\to\infty}(y_{n+1}-y_n)=\alpha log\beta$$

$$i)\ \lim_{n\to\infty}\frac{y_{n+1}}{y_n}=1$$

$$ii)\ \lim_{n\to\infty}\frac{y_n}{n}=\alpha$$

$$iii)\ \lim_{n\to\infty}\left(\frac{y_{n+1}}{y_n}\right)^n=\beta$$

$$(2n+1)!!=\frac{(2n+1)!}{2\cdot 4\cdot\ldots\cdot 2n}=\frac{(2n+1)!}{2^n\cdot n!}$$

$$(2n-1)!!=\frac{(2n-1)!}{2\cdot 4\cdot\ldots\cdot(2n-2)}=\frac{(2n-1)!}{2^{n-1}\cdot(n-1)!}$$

$$\Omega=\lim_{n\to\infty}\left(\sqrt[2n+2]{\frac{(2n+1)!\cdot(n+1)!}{2^n\cdot n!}}-\sqrt[2n]{\frac{(2n-1)!\cdot n!}{2^{n-1}\cdot(n-1)!}}\right)$$

$$=\lim_{n\to\infty}\left(\underbrace{\sqrt[2n+2]{\frac{(2n+1)!\cdot(n+1)}{2^n}}}_{y_{n+1}}-\underbrace{\sqrt[2n]{\frac{(2n-1)!\cdot n}{2^{n-1}}}}_{y_n}\right)$$

$$\lim_{n\to\infty}\frac{y_n}{n}=\lim_{n\to\infty}\sqrt[2n]{\frac{(2n-1)!\cdot n}{2^{n-1}\cdot n^{2n}}}=\lim_{n\to\infty}\sqrt{\sqrt[n]{\frac{(2n-1)!\cdot n}{2^{n-1}\cdot n^{2n}}}}$$

$$\overset{C-D'A}{=}\lim_{n\to\infty}\sqrt{\frac{(2n+1)!\cdot(n+1)}{2^n\cdot(n+1)^{2n}\cdot(n+1)^2}\cdot\frac{2^{n-1}\cdot n^{2n}}{(2n-1)!\cdot n}}$$

$$=\lim_{n\to\infty}\sqrt{\frac{2n(2n+1)}{2n(n+1)}\cdot\left(\frac{n}{n+1}\right)^{2n}}=\frac{\sqrt{2}}{e}=\alpha$$

$$\lim_{n\to\infty}\frac{y_{n+1}}{y_n}=\lim_{n\to\infty}\frac{\sqrt[2n+2]{\frac{(2n+1)!\cdot(n+1)}{2^n}}}{n+1}\cdot\frac{n}{\sqrt[2n]{\frac{(2n-1)!\cdot n}{2^{n-1}}}}\cdot\frac{n+1}{n}=1$$

$$\lim_{n\to\infty}\left(\frac{y_{n+1}}{y_n}\right)^n=\lim_{n\to\infty}\left(\frac{\sqrt[2n+2]{\frac{(2n+1)!\cdot(n+1)}{2^n}}}{\sqrt[2n]{\frac{(2n-1)!\cdot n}{2^{n-1}}}}\right)^n$$

$$=\lim_{n\to\infty}\frac{\left(\frac{(2n+1)!\cdot(n+1)}{2^n}\right)^{\frac{n}{2n+2}}}{\left(\frac{(2n-1)!\cdot n}{2^{n-1}}\right)^{\frac{n}{2n}}}=\lim_{n\to\infty}\sqrt{\frac{\left(\frac{(2n+1)!\cdot(n+1)}{2^n}\right)^{\frac{n+1-1}{n+1}}}{\frac{(2n-1)!\cdot n}{2^{n-1}}}}$$

$$=\lim_{n\to\infty}\sqrt{\frac{\frac{(2n+1)!\cdot(n+1)}{2^n}}{\frac{(2n-1)!\cdot n}{2^{n-1}}}\cdot\frac{1}{\sqrt[n+1]{\frac{(2n+1)!\cdot(n+1)}{2^n}}}}$$

$$=\lim_{n\to\infty}\sqrt{\frac{2n(2n+1)(n+1)}{2n}\cdot\frac{1}{\sqrt[n+1]{\frac{(2n+1)!\cdot(n+1)}{2^n}}}}$$

$$=\lim_{n\to\infty}\sqrt{\frac{(2n+1)(n+1)}{(n+1)^2}\cdot\frac{n+1}{y_{n+1}}}=\sqrt{2}\cdot\frac{e}{\sqrt{2}}=e=\beta$$

$$\Omega=\alpha\log\beta=\frac{\sqrt{2}}{e}\cdot\log e=\frac{\sqrt{2}}{e}$$

UP.299. If $m, p \in \mathbb{N}^*$ −fixed values; $a_n \in (0,\infty)$; $b_n \in (0,\infty), n \in \mathbb{N}^*$

$$\lim_{n\to\infty}\frac{a_{n+1}}{(3n+m)a_n} = a > 0;\ \lim_{n\to\infty}\frac{b_{n+1}}{(4n+p)b_n} = b > 0$$

Then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}} - \frac{n^3}{\sqrt[n]{a_n\cdot b_n}}\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}} - \frac{n^3}{\sqrt[n]{a_n\cdot b_n}}\right)$$

$$= \lim_{n\to\infty}\frac{n^3}{\sqrt[n]{a_n\cdot b_n}}\left(\frac{(n+1)^3}{n^3}\cdot\frac{\sqrt[n]{a_n\cdot b_n}}{\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}} - 1\right)$$

$$= \lim_{n\to\infty}\frac{3n+m}{\sqrt[n]{a_n}}\cdot\frac{4n+p}{\sqrt[n]{b_n}}\cdot\frac{n^3}{(3n+m)(4n+p)}\cdot\left(\frac{(n+1)^3}{n^3}\cdot\frac{\sqrt[n]{a_n\cdot b_n}}{\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}} - 1\right)$$

$$= \frac{1}{12}\lim_{n\to\infty}\left(\frac{(3n+3+m)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{(3n+m)^n}\right)\cdot\lim_{n\to\infty}\left(\frac{(4n+4+p)^{n+1}}{b_{n+1}}\cdot\frac{b_n}{(4n+p)^n}\right)$$

$$\cdot\lim_{n\to\infty} n\left(\frac{(n+1)^3}{n^3}\cdot\frac{\sqrt[n]{a_n\cdot b_n}}{\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}} - 1\right)$$

$$= \frac{1}{12}\cdot\frac{e^2}{ab}log\left(e^3\lim_{n\to\infty}\frac{a_nb_n}{a_{n+1}b_{n+1}}\cdot\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}\right)$$

$$= \frac{e^2}{12ab}\cdot log\left(e^3\cdot\frac{1}{ab}\cdot\frac{ab}{e^2}\right) = \frac{e^2}{12ab} \Rightarrow \Omega = \frac{e^2}{12ab}$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}} - \frac{n^3}{\sqrt[n]{a_n\cdot b_n}}\right) \overset{C-S}{=} \lim_{n\to\infty}\frac{n^2}{\sqrt[n]{a_n\cdot b_n}}$$

$$= \lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{a_nb_n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{\frac{(n+1)^{2n+2}}{a_{n+1}b_{n+1}}}{\frac{n^{2n}}{a_nb_n}}$$

$$= e^2 \lim_{n\to\infty} \frac{(3n+m)a_n}{a_{n+1}} \cdot \frac{(4n+p)b_n}{b_{n+1}} \cdot \frac{(n+1)^2}{(4n+p)(3n+m)} = \frac{e^2}{12ab}$$

UP.300. Let be $f, \Gamma: \mathbb{R}_+^* \to \mathbb{R}_+^* = (0, \infty)$ such that $\lim_{x\to\infty} \frac{f(x+1)}{x^2 f(x)} = a \in \mathbb{R}_+^*$ and exist $\lim_{x\to\infty} \frac{(f(x))^{\frac{1}{x}}}{x^2}$ and Γ —Euler's function by class 2.Find:

$$\lim_{x\to\infty} \left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x^2} - \frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(x+1)^2} \right) \cdot (f(x))^{\frac{1}{x}}$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution by proposers

$$\lim_{x\to\infty} \frac{(f(x))^{\frac{1}{x}}}{x^2} = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}^*}} \frac{(f(n))^{\frac{1}{n}}}{n^2} = \lim_{n\to\infty} \sqrt[n]{\frac{f(n)}{n^{2n}}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{f(n+1)}{(n+1)^{2(n+1)}} \cdot \frac{n^{2n}}{f(n)} =$$

$$= \lim_{n\to\infty} \frac{f(n+1)}{n^2 f(n)} \cdot \left(\frac{n}{n+1}\right)^{2(n+1)} = \frac{a}{e^2}$$

Similarly:

$$\lim_{x\to\infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x^2} = \lim_{n\to\infty} \frac{(\Gamma(n+1))^{\frac{1}{n}}}{n^2} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \overset{C-D'A}{=}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{1}{e}.$$

Let: $u(x) = \frac{(\Gamma(x+1))^{\frac{1}{x}}}{(\Gamma(x+2))^{\frac{1}{x+1}}} \cdot \left(\frac{x+1}{x}\right)^2$ *then* $\lim_{x\to\infty} u(x) = 1$ *and* $\lim_{x\to\infty} \frac{u(x)-1}{\log u(x)} = 1$ *so, we have:*

$$\lim_{x\to\infty} (u(x))^x = \lim_{x\to\infty} \frac{\Gamma(x+1)}{\Gamma(x+2)} \cdot \left(\frac{x+1}{x}\right)^{2x} \cdot (\Gamma(x+2))^{\frac{1}{x+1}} = e^2 \lim_{x\to\infty} \frac{(\Gamma(x+1))^{\frac{1}{x+1}}}{x+1} = e^2 \cdot \frac{1}{e} = e.$$

Therefore:

$$B(x) = \left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x^2} - \frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(x+1)^2} \right) \cdot (f(x))^{\frac{1}{x}} = \frac{(f(x))^{\frac{1}{x}} (\Gamma(x+2))^{\frac{1}{x+1}}}{(x+1)^2} \cdot (u(x) - 1)$$

$$=\frac{\left(f(x)\right)^{\frac{1}{x}}\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}{(x+1)^2}\cdot\left(\frac{u(x)-1}{log u(x)}\right)log u(x)=$$

$$\frac{\left(f(x)\right)^{\frac{1}{x}}}{x^2}\cdot\frac{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}{(x+1)^2}\cdot\frac{x}{x+1}\cdot\left(\frac{u(x)-1}{log u(x)}\right)log\left(u(x)\right)^x.$$

$$\lim_{x\to\infty}B(x)=\frac{a}{e^2}\cdot\frac{1}{e}\cdot 1\cdot log e=\frac{a}{e^3}.$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

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