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SOLUTIONS

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JP.301. Prove that in any ABC triangle the following relationship holds:

$$\tan^2\frac{A}{2}\cot\frac{B}{2}+\tan^2\frac{B}{2}\cot\frac{C}{2}+\tan^2\frac{C}{2}\cot\frac{A}{2}\geq\sqrt{3}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Daniel Văcaru- Romania

$$\tan^2\frac{A}{2}\cot\frac{B}{2}+\tan^2\frac{B}{2}\cot\frac{C}{2}+\tan^2\frac{C}{2}\cot\frac{A}{2}=\frac{\tan^2\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan^2\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan^2\frac{C}{2}}{\tan\frac{A}{2}}$$

$$\overset{Bergstrom}{\geq}\frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\overset{\substack{t\to\tan t\\ convex}}{\geq}3\tan\frac{A+B+C}{3}=\sqrt{3}$$

Solution 2 by Henry Ricardo-New York-USA

Engel's form of the Cauchy-Schwarz inequality gives us

$$\sum_{cyc}\tan^2\frac{A}{2}\cot\frac{B}{2}=\sum_{cyc}\frac{\tan^2\frac{A}{2}}{\tan\frac{B}{2}}\geq\frac{\left(\sum_{cyc}\tan\frac{A}{2}\right)^2}{\sum_{cyc}\tan\frac{B}{2}}\geq\sum_{cyc}\tan\frac{A}{2}\geq\sqrt{3}$$

Where the last inequality is known (see 2.33 *Geometric Inequalities* **by Bottema)**

JP.302. In any ΔABC the following relationship holds:

$$a^2r_a+b^2r_b+c^2r_c\geq 54Rr^2$$

Proposed by Marian Ursărescu-Romania

Solution by Daniel Văcaru-Romania

We can write:

$$a^2r_a+b^2r_b+c^2r_c=\frac{a^2}{\frac{1}{r_a}}+\frac{b^2}{\frac{1}{r_b}}+\frac{c^2}{\frac{1}{r_c}}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}}=$$

$$=\frac{(a+b+c)^2}{\frac{(s-a)+(s-b)+(s-c)}{S}}=\frac{(a+b+c)^2}{\frac{1}{r}}=(a+b+c)^2r$$

$$=\frac{(a+b+c)^3\cdot r}{2s}$$

But: $(a+b+c)^3\ge 27abc\Rightarrow$

$$a^2r_a+b^2r_b+c^2r_c\ge\frac{27abc\cdot r}{2s}=\frac{27\cdot 4RS\cdot r}{2s}=54Rr\left(\frac{S}{s}\right)=54Rr^2$$

JP.303. If $x,y,z\ge 1, x^2+y^2+z^2-2xyz=1, n\ge 0$ prove:

$$2(n+x)(n+y)(n+z)\le(n+1)^3(1+xyz)$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Denote $t=xyz$ and from $x^2+y^2+z^2-2xyz=1$ we get

$$x^2+y^2+z^2=1+2xyz=1+2t$$

From C.B.S. Inequality we have

$$(x+y+z)^2\le 3(x^2+y^2+z^2)=3(1+2xyz)=3(1+2t),\text{ where}$$

$$x+y+z\le\sqrt{3(1+2t)}\text{ and }xy+yz+zx\le x^2+y^2+z^2=1+2t$$

Therefore,

$$(n+x)(n+y)(n+z)-\frac{(n+1)^3}{2}\cdot xyz=$$

$$=n^3+n^2(x+y+z)+n(xy+yz+zx)+xyz-\frac{(n+1)^3}{2}\cdot xyz\le$$

$$\le n^3+n^2\sqrt{3(1+2t)}+n(1+2t)+t-\frac{(n+1)^3}{2}\cdot t=$$

$$=n^3+n+n^2\sqrt{3(1+2t)}-t\cdot\frac{n^3+3n^2-n-1}{2}$$

Remains to prove that:

$$n^3+n+n^2\sqrt{3(1+2t)}-t\cdot\frac{n^3+3n^2-n-1}{2}\le\frac{(n+1)^3}{2}\Leftrightarrow$$

$$n^2\sqrt{3(1+2t)}\le\frac{-n^3+3n^2+n+1}{2}+t\cdot\frac{n^3+3n^2-n-1}{2}\Leftrightarrow$$

$$3n^4(1+2t)\le\left(\frac{-n^3+3n^2+n+1}{2}+t\cdot\frac{n^3+3n^2-n-1}{2}\right)^2\Leftrightarrow$$

$$3n^4(1+2t)\le\left(\frac{-n^3+3n^2+n+1}{2}\right)^2+2\cdot\frac{-n^3+3n^2+n+1}{2}\cdot t\cdot\frac{n^3+3n^2-n-1}{2}+\left(t\cdot\frac{n^3+3n^2-n-1}{2}\right)^2\Leftrightarrow$$

$$3n^4(1+2t)\le\frac{n^6-6n^5+7n^4+4n^3+7n^2+2n+1}{4}+$$

$$+t\cdot\frac{-n^6+11n^5+2n^3-n^2-2n-1}{2}+t^2\cdot\frac{n^6+6n^5+7n^4-8n^3-5n^2+2n+1}{4}\Leftrightarrow$$

$$t^2\left(n^6+6n^5+7n^4-8n^3-5n^2+2n+1\right)+2t(-n^6+n^4+2n^3-n^2-2n-1)+$$

$$+n^6-6n^5-5n^4+4n^3+7n^2+2n+1\ge 0\Leftrightarrow$$

$$(t-1)\left[t\left(n^6+6n^5+7n^4-8n^3-5n^2+2n+1\right)-(n^6-6n^5-5n^4+4n^3+7n^2+\right.$$

$$\left.2n+1)\right]\ge 0 \text{ which result from } t-1\ge 0 \text{ and}$$

$$\left[t\left(n^6+6n^5+7n^4-8n^3-5n^2+2n+1\right)-(n^6-6n^5-5n^4+4n^3+7n^2+2n+1)\right]\ge 0$$

true from

$$\left[t\left(n^6+6n^5+7n^4-8n^3-5n^2+2n+1\right)-(n^6-6n^5-5n^4+4n^3+7n^2+2n+1)\right]\overset{t\ge1}{\ge}$$

$$\overset{t\ge1}{\ge}\left[\left(n^6+6n^5+7n^4-8n^3-5n^2+2n+1\right)-(n^6-6n^5-5n^4+4n^3+7n^2+2n+1)\right]=$$

$$=12n^5+12n^4-12n^3-12n^2=12(n^2-1)(n^3-1)=$$

$$=12(n-1)^2(n+1)(n^2+n+1)\ge 0 \text{ true.}$$

Equality holds if and only if $x=y=z=1$

JP.304. Solve the equation in real numbers:

$$3\cdot\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}}=2(x^4-3x+4)$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by proposer

$$3\cdot\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}}=2(x^4-3x+4);(1)$$

By C-B-S Inequality, we have

$$\left(\sqrt{2(x^8+1)}+2x^2\right)^2 \le 2[2(x^8+1)+4x^4] = 4(x^8+2x^4+1) = 4(x^4+1)^2 \Rightarrow$$

$$\sqrt{2(x^8+1)}+2x^2 \le 2(x^4+1) \Leftrightarrow \sqrt{2(x^8+1)} \le 2(x^4-x^2+1) \Leftrightarrow$$

$$\sqrt[4]{\frac{x^8+1}{2}} \le \sqrt{x^4-x^2+1};\ \ (2)$$

Other

$$\sqrt{x^4-x^2+1} = \sqrt{(x^2+1)^2-\left(x\sqrt{3}\right)^2} =$$

$$= \sqrt{(2+\sqrt{3})(x^2-x\sqrt{3}+1)(2-\sqrt{3})(x^2+x\sqrt{3}+1)} \le$$

$$\le \frac{(2+\sqrt{3})(x^2-x\sqrt{3}+1)+(2-\sqrt{3})(x^2+x\sqrt{3}+1)}{2} =$$

$$= \frac{4x^2-6x+4}{2} = 2x^2-3x+2;\ \ (3)$$

From (2),(3) we get: $\sqrt[4]{\frac{x^8+1}{2}} \le \sqrt{x^4-x^2+1} \le 2x^2-3x+2; (4)$

By AM-GM inequality, we have:

$$3\cdot\sqrt[3]{(x^2-x+1)\cdot 1\cdot 1} \le (x^2-x+1)+1+1 = 2x^2-3x+2;\ \ (5)$$

From (4),(5) result

$$3\cdot\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}} \le x^2-x+3+2x^2-3x+2 = 3x^2-4x+5; (6)$$

From (1),(6) result

$$2(x^4-3x+4) \le 3x^2-4x+5 \Leftrightarrow 2x^4-3x^2-2x+3 \le 0 \Leftrightarrow$$

$$2x^3(x-1)+2x^2(x-1)-x(x-1)-3(x-1) \le 0 \Leftrightarrow$$

$$(x-1)(2x^3+2x^2-x-3) \le 0 \Leftrightarrow (x-1)\left(2x^2(x-1)+4x(x-1)+3(x-1)\right) \le 0$$

$$(x-1)^2(2x^2+4x+3) \le 0;\ \ (7)$$

$(x-1)^2 \ge 0, \forall x \in \mathbb{R}; 2x^2+4x+3 = 2(x+1)^2+1 \ge 1 > 0$.**Hence**

$(x-1)^2(2x^2+4x+3) \ge 0$ **and from (7) we get**

$(x-1)^2(2x^2+4x+3)=0$. Equality occurs if:

$$\begin{cases} x-1=0 \\ \sqrt{2(x^8+1)}=2x^2 \\ (2+\sqrt{3})(x^2-x\sqrt{3}+1)=(2-\sqrt{3})(x^2+x\sqrt{3}+1) \\ x^2-x+1=1 \end{cases} \Leftrightarrow x=1.$$

Hence, $x=1$ is the solution of the equation.

JP.305. Solve the equation:

$$\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}=7-3x$$

Proposed by Hoang Le Nhat-Hanoi-Vietnam

Solution 1 by Agayev Sedreddin-Baku-Azerbaijan

$$\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}=7-3x$$

$$\begin{cases} 3x-2x^4\geq 0 \\ 7-3x\geq 0 \end{cases} \Rightarrow x\in\left[0;\sqrt[3]{\frac{3}{2}}\right]$$

$$7-3x=\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}\leq\frac{2+x^4+1}{2}+2\cdot\frac{1+3x-2x^4}{2}=$$

$$=\frac{x^4}{2}+\frac{3}{2}+1+3x-2x^4=-\frac{3}{2}x^4+3x+\frac{5}{2}\Rightarrow\frac{3}{2}x^4-6x+\frac{9}{2}\leq 0$$

$$x^4-4x+3\leq 0\Leftrightarrow x^4+3\leq 4x;\ (*)$$

$$x\geq 0;\ x^4+3=x^4+1+1+1\geq 4\sqrt[4]{x^4}=4x;\ (**)$$

By (*),(**)$\Rightarrow x^4+3=4x\Rightarrow x=1$.

Solution 2 by Khaled Abd Imouti-Damascus-Siria

$$\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}=7-3x;\ (1)$$

$$\begin{cases} 3x-2x^4\geq 0 \\ 7-3x\geq 0 \end{cases} \Rightarrow x\in\left[0;\sqrt[3]{\frac{3}{2}}\right]$$

Denote: $\begin{cases} u=2x^4+1 \\ v=3x-2x^4 \end{cases}\Rightarrow u+v=3x+2$

So, the equation (1) can be written as:

$$\sqrt{u}+2\sqrt{v}=7-(u+v-2)\Leftrightarrow\sqrt{u}+2\sqrt{v}=9-(u+v);(2)$$

Now, let be: $\alpha=u-4;\beta=v-1$ then: $u=\alpha+4;v=\beta+1\overset{(2)}{\Rightarrow}$

$$\sqrt{\alpha+4}+2\sqrt{\beta+1}=4-\alpha-\beta\Leftrightarrow 2\sqrt{\beta+1}+\beta=4-\alpha-\sqrt{\alpha+4};(3)$$

Let be the function: $f_1(\beta)=2\sqrt{\beta+1}+\beta;f_1(0)=2;f'_1(\beta)=1+\frac{1}{\sqrt{\beta+1}}>0$

Let be the function: $f_2(\alpha)=4-\left(\alpha+\sqrt{\alpha+4}\right);f_2(0)=2;f'_2(\alpha)=-\left(1+\frac{1}{2\sqrt{\alpha+4}}\right)<0$

So, the equation (3)is satisfying when $\alpha=\beta=0\Rightarrow\begin{cases}3x-2x^4=0\\7-3x=0\end{cases}\Rightarrow x=1$

Solution 3 by proposer

$$\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}=7-3x;(1)$$

$$\begin{cases}3x-2x^4\geq 0\\7-3x\geq 0\end{cases}\Rightarrow x\in\left[0;\sqrt[3]{\frac{3}{2}}\right]$$

Using AM-GM for two positive real numbers, we have:

$$\sqrt{2(x^4+1)}=\sqrt{2\left[(x^2+1)^2-\left(x\sqrt{2}\right)^2\right]}=$$

$$=\sqrt{\left(2+\sqrt{2}\right)\left(x^2-x\sqrt{2}+1\right)\left(2-\sqrt{2}\right)\left(x^2+x\sqrt{2}+1\right)}\leq$$

$$\leq\frac{\left(2+\sqrt{2}\right)\left(x^2-x\sqrt{2}+1\right)+\left(2-\sqrt{2}\right)\left(x^2+x\sqrt{2}+1\right)}{2}=2x^2-2x+2\Rightarrow$$

$$\sqrt{2(x^4+1)}\leq 2x^2-2x+2;(2)$$

Other: $2\sqrt{3x-2x^4}=2\sqrt{x(3-2x^3)}\leq x+3-2x^3=-2x^3+x+3;(3)$

From (2),(3) result:

$$\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}\leq 2x^2-2x+2-2x^3+x+3=-2x^3+2x^2-x+5;(4)$$

From (1),(4) result: $7-3x\leq -2x^3+2x^2-x+5\Leftrightarrow$

$$2x^3-2x^2-2x+2\leq 0\Leftrightarrow (x-1)^2(x+1)\leq 0;(5)$$

Because: $x\in\left[0;\sqrt[3]{\frac{3}{2}}\right]\Rightarrow x+1>0$ and how $(x-1)^2\geq 0\Rightarrow (x-1)^2(x+1)\geq 0;(6)$

From (5),(6) equality occurs if (2),(3) simultaneous occurrence. Hence:

$$\begin{cases} x \in \left[0; \sqrt[3]{\frac{3}{2}}\right] \\ (2+\sqrt{2})(x^2 - x\sqrt{2}+1) = (2-\sqrt{2})(x^2+x\sqrt{2}+1) \\ x = 3-2x^3 \\ x-1=0 \end{cases} \Leftrightarrow$$

$$\begin{cases} x \in \left[0; \sqrt[3]{\frac{3}{2}}\right] \\ 2x^2-4x+2=0 \\ 2x^3+x-3=0 \\ x=1 \end{cases} \Leftrightarrow x=1$$

Solution of equation is: $S = \{1\}$

JP.306. If $a,b,c > 0$ then:

$$(a+2c)\sqrt{a}+(b+2a)\sqrt{b}+(c+2b)\sqrt{c} \le (a+b+c)\sqrt{3(a+b+c)}$$

Proposed by Daniel Sitaru-Romania

Solution by Daniel Văcaru-Romania

Function $t \to \sqrt{t}$ is concave.That implies:

$$\frac{(a+2c)\sqrt{a}+(b+2a)\sqrt{b}+(c+2b)\sqrt{c}}{\big((a+2c)+(b+2a)+(c+2b)\big)} \le \sqrt{\frac{(a+2c)a+(b+2a)b+(c+2b)c}{3(a+b+c)}}$$

$$(a+2c)\sqrt{a}+(b+2a)\sqrt{b}+(c+2b)\sqrt{c} \le 3(a+b+c)\sqrt{\frac{(a+b+c)^2}{3(a+b+c)}} \to$$

$$(a+2c)\sqrt{a}+(b+2a)\sqrt{b}+(c+2b)\sqrt{c} \le (a+b+c)\sqrt{3(a+b+c)}$$

JP.307. Solve the equation in real numbers:

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4} = \frac{x^4-3x^3}{2}+7$$

Proposed by Hoang Le Nhat-Hanoi-Vietnam

Solution by proposer

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7;(1)$$

$$\text{Let: }\begin{cases}x^3-2x^2+2x\geq 0\\4x-3x^4\geq 0\end{cases}\Rightarrow\begin{cases}x(x^2-2x+2)\geq 0\\x(3x^3-4)\leq 0\end{cases}\Rightarrow\begin{cases}x((x-1)^2+1)\geq 0\\0\leq x\leq\sqrt[3]{\frac{4}{3}}\end{cases}\Leftrightarrow$$

$$0\leq x\leq\sqrt[3]{\frac{4}{3}}$$

$$x^2-x+1=\left(x^2-x+\frac{1}{4}\right)+\frac{3}{4}=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\geq\frac{3}{4}>0$$

Hence, by AM-GM inequality for positive real numbers:

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=$$

$$=\sqrt{x(x^2-2x+2)}+3\sqrt[3]{(x^2-x+1)\cdot 1\cdot 1}+2\sqrt[4]{x(4-3x^3)\cdot 1\cdot 1}\leq$$

$$\leq\frac{x+x^2-2x+2}{2}+(x^2-x+1)+1+1+2\cdot\frac{x+4-3x^3+1+1}{4}\Leftrightarrow$$

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}\leq\frac{-3x^3+3x^2-2x+14}{2};(2)$$

From (1),(2) we get:

$$\frac{x^4-3x^3}{2}+7\leq\frac{-3x^3+3x^2-2x+14}{2}\Leftrightarrow\frac{x^4-3x^3+14}{2}\leq\frac{-3x^3+3x^2-2x+14}{2}\Leftrightarrow$$

$$x^4-3x^3+14\leq -3x^3+3x^2-2x+14\Leftrightarrow x(x^3-3x+2)\leq 0\Leftrightarrow$$

$$x\left(x^2(x-1)+x(x-1)-2(x-1)\right)\leq 0\Leftrightarrow x(x-1)(x^2+x-2)\leq 0$$

$$x(x+2)(x-1)^2\leq 0;(3)$$

Because: $x\geq 0\Rightarrow x(x+2)\geq 0\Rightarrow x(x+2)(x-1)^2\geq 0;(4)$

From (3),(4) equality $x(x+2)(x-1)^2=0$ **occurs if and only if**

$$\begin{cases}x=x^2-2x+2\\x^2-x+1=1\\x=4-3x^3=1\\x(x+2)(x-1)^2=0\end{cases}\Leftrightarrow\begin{cases}(x-1)(x-2)=0\\x(x-1)=0\\3x^3+x-4=0\\x(x+2)(x-1)^2=0\end{cases}\Leftrightarrow x=1$$

Solution of equation is $S=\{1\}$.

JP.308. Let $a,b,c\in[1,3]$ such that $a+b+c=6$. Find the maximum value of the expression:

$$P=a^4+b^4+c^4$$

Proposed by Hoang Le Nhat Tung-Vietnam

Solution 1 by Daoudi Abdessattar-Sbiba-Tunisia

Suppose $c\ge b\ge a$ and $3>2>1$ we have:

$3\ge c\Rightarrow 3+2\ge c+b$ because $a+b+c=6$ and $a>1$; $3+2+1=a+b+c$

$f(x)=x^4$ −convex function⇒ $f(a)+f(b)+f(c)\le f(1)+f(2)+f(3)$

Equality holds if: $c=3,b=2,a=1$ or permutation.

Solution 2 by proposer

Let $a-2=x,b-2=y,c-2=z;x,y,z\in\mathbb{R}\Rightarrow a=x+2,b=y+2,c=z+2$

Because $a,b,c\in[1,3]\Rightarrow x+2,y+2,z+2\in[1,3]\Leftrightarrow x,y,z\in[-1,1]$

We have: $P=a^4+b^4+c^4=(x+2)^4+(y+2)^4+(z+2)^4=$

$$=(x^4+8x^3+24x^2+32x+16)+(y^4+8y^3+24y^2+32y+16)+$$
$$+(z^4+8z^3+24z^2+32z+16)=$$
$$=(x^4+y^4+z^4)+8(x^3+y^3+z^3)+24(x^2+y^2+z^2)+32(x+y+z)+48;(1)$$

Because $a+b+c=6\Rightarrow x+2+y+2+z+2=6\Rightarrow x+y+z=0;(2)$

Other, $x+y+z=0\Leftrightarrow y+z=-x\Leftrightarrow (y+z)^3=-x^3\Leftrightarrow$

$$x^3+y^3+z^3=-3yz(y+z)=3xyz$$

From (1),(2) we get: $P=(x^4+y^4+z^4)+24xyz+32(x+y+z)+48$

Because $x+y+z=0\Rightarrow$

$$P=(x^4+y^4+z^4)+24xyz+24[(x+y+z)^2-2(xy+yz+zx)]+48\ \Leftrightarrow$$
$$P=(x^4+y^4+z^4)-48(xy+yz+zx)+24xyz+48;(3)$$

Because $x,y,z\in[-1,1]\Rightarrow 0\le x^2,y^2,z^2\le 1\Rightarrow$

$$x^2(x^2-1)+y^2(y^2-1)+z^2(z^2-1)\le 0\Leftrightarrow$$
$$x^4+y^4+z^4\le x^2+y^2+z^2=-2(xy+yz+zx);(\therefore x+y+z=0);(4)$$

From $(3),(4)\Rightarrow P\le -50(xy+yz+zx)+24xyz+48;(5)$

How $x,y,z\in[-1,1]\Rightarrow(x+1)(y+1)(z+1)\geq 0;(1-x)(1-y)(1-z)\geq 0$ hence

$$13(x+1)(y+1)(z+1)+37(1-x)(1-y)(1-z(\geq 0\Leftrightarrow$$

$$13(xyz+xy+yz+zx+x+y+z+1)+37(1+xy+yz+zx-x-y-z-xyz)\geq 0\Leftrightarrow$$

$$-24xyz+50(xy+yz+zx)-24(x+y+z)+50\geq 0\Leftrightarrow$$

$$-50(xy+yz+zx)+24xyz\leq 50;(6)$$

From $(5),(6)\Rightarrow P\leq 50+48=98\Rightarrow P_{Max}=98$

Hence $P_{Max}=98\Leftrightarrow\begin{cases}x+y+z=0;x,y,z\in[-1,1]\\(x+1)(y+1)(z+1)=(1-x)(1-y)(1-z)\end{cases}$

$$\Leftrightarrow x=1,y=0,z=-1\Leftrightarrow a=3;b=2;c=1$$

JP.309. If $m\in\mathbb{N},h_A,h_B,h_C,h_D$ –be the lengths of altitudes of tetrahedron $[ABCD]$ and r –the radius of the insphere, then:

$$m+\frac{1}{4}\left(\left(\frac{h_A-3r}{h_A+3r}\right)^{m+1}+\left(\frac{h_B-3r}{h_B+3r}\right)^{m+1}+\left(\frac{h_C-3r}{h_C+3r}\right)^{m+1}+\left(\frac{h_D-3r}{h_D+3r}\right)^{m+1}\right)\geq\frac{m+1}{7}$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution by proposers

Applying J.Radon Inequality, we have

$$W_m=m+\frac{1}{4}\sum_{cyc}\left(\frac{h_A-3r}{h_A+3r}\right)^{m+1}\geq m+\frac{1}{4^{m+1}}\left(\sum_{cyc}\frac{h_A-3r}{h_A+3r}\right)^{m+1}\overset{AM-GM}{\geq}$$

$$\geq(m+1)\sqrt[m+1]{\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{m-times}\cdot\frac{1}{4^{m+1}}\left(\sum_{cyc}\frac{h_A-3r}{h_A+3r}\right)^{m+1}}=$$

$$=(m+1)\cdot\frac{1}{4}\cdot\sum_{cyc}\frac{h_A-3r}{h_A+3r}=(m+1)W_0=\frac{m+1}{4}\cdot U_0;\ \ (1),where$$

$$U_0=\sum_{cyc}\frac{h_A-3r}{h_A+3r}=\sum_{cyc}\frac{h_AS_A-3rS_A}{h_AS_A+3rS_A}=\sum_{cyc}\frac{V-rS_A}{V+rS_A}\Leftrightarrow$$

$$U_0+4=\sum_{cyc}\left(\frac{V-rS_A}{V+rS_A}+1\right)=2V\cdot\sum_{cyc}\frac{1}{V+rS_A}\overset{Bergstrom}{\geq}2V\cdot\frac{(1+1+1+1)^2}{4V+r\sum_{cyc}S_A}=$$

$$= 32V \cdot \frac{1}{4V + rS} = \frac{32V}{4V + rS} = \frac{32}{7} \Leftrightarrow U_0 \geq \frac{32}{7} - 4 = \frac{4}{7}; \ (2)$$

From $(1), (2)$ we deduce that

$$W_m \geq \frac{m+1}{4} \cdot \frac{4}{7} = \frac{m+1}{7}$$

Note:

Equality occurs if and only if the tetrahedron is regular.

So, V –volume of tetrahedron; S_A –lateral area of the face and analogs; S – area of tetrahedron and $3V = rS$.

JP.310. If $a, b, c > 0; abc = 1$ then:

$$\sum_{cyc} \frac{c(a^2 + b^2) + 1}{a + b} \geq \frac{3}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania

$$\sum_{cyc} \frac{c(a^2 + b^2) + 1}{a + b} = \sum_{cyc} \frac{c(a + b)^2 - 1}{a + b} = \sum_{cyc}\left(c(a + b) - \frac{abc}{a + b}\right) \overset{Am-Gm}{\geq}$$

$$\geq \sum_{cyc}\left(c(a + b) - \frac{c\sqrt{ab}}{2}\right) \overset{Am-Gm}{\geq} \sum_{cyc}\left(c(a + b) - \frac{(a + b)c}{4}\right) =$$

$$= \frac{3}{4}\sum_{cyc} c(a + b) = \frac{3}{2}\sum_{cyc} ab = \frac{3}{2}\sum_{cyc} \frac{ab}{abc} = \frac{3}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Solution 2 by Henry Ricardo-New York-USA

First, we denote that the AGM inequality gives us:

$c(a^2 + b^2) + 1 \geq 2abc + 1 = 3$, and then we see that:

$\frac{2}{a+b} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$ by the Harmonic Mean-Arithmetic Mean inequality.

Therefore,

$$\sum_{cyc}\frac{c(a^2+b^2)+1}{a+b}\geq 3\sum_{cyc}\frac{1}{a+b}\geq\frac{3}{4}\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{3}{4}\cdot 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=$$

$$=\frac{3}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

JP.311. If $x,y,z>0$; $\sqrt{x}+\sqrt{y}+\sqrt{z}=3$ then:

$$x+y+z\geq\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania

We have that: $(\sqrt{x},\sqrt{y},\sqrt{z})$ and $(\sqrt{x},\sqrt{y},\sqrt{z})$ has the same orientation, then

$$\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2\overset{Chebyshev's}{\leq}3(x+y+z)\to x+y+z\geq 3;(1)$$

We can write:

$$\sqrt[3]{x}=\sqrt[3]{\sqrt{x}\cdot\sqrt{x}\cdot 1}\overset{Am-Gm}{\leq}\frac{2\sqrt{x}+1}{3}\to\sum_{cyc}\sqrt[3]{x}\leq\frac{2}{3}\sum_{cyc}\sqrt{x}+1=2+1=2;(2)$$

We obtain: $x+y+z\overset{(1)}{\geq}3\overset{(2)}{\geq}\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}$

Solution 2 by Daniel Văcaru-Romania

We have: $3=\sqrt{x}+\sqrt{y}+\sqrt{z}\overset{CBS}{\leq}\sqrt{(1+1+1)\left(\sqrt{x}^2+\sqrt{y}^2+\sqrt{z}^2\right)}$

$$\to x+y+z\geq 3;(1)$$

We can write:

$$\sqrt[3]{x}=\sqrt[3]{\sqrt{x}\cdot\sqrt{x}\cdot 1}\overset{Am-Gm}{\leq}\frac{2\sqrt{x}+1}{3}\to\sum_{cyc}\sqrt[3]{x}\leq\frac{2}{3}\sum_{cyc}\sqrt{x}+1=2+1=2;(2)$$

We obtain: $x+y+z\overset{(1)}{\geq}3\overset{(2)}{\geq}\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}$

Solution 3 by Daniel Văcaru-Romania

We have:

$$x+1\geq 2\sqrt{x}\to x+y+z+3\geq 2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\to x+y+z\geq 3;(1)$$

We can write:

$$\sqrt[3]{x}=\sqrt[3]{\sqrt{x}\cdot\sqrt{x}\cdot 1}\overset{Am-Gm}{\leq}\frac{2\sqrt{x}+1}{3}\to\sum_{cyc}\sqrt[3]{x}\leq\frac{2}{3}\sum_{cyc}\sqrt{x}+1=2+1=2;(2)$$

We obtain: $x+y+z\overset{(1)}{\geq}3\overset{(2)}{\geq}\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}$

Solution 4 by Henry Ricardo-New York-USA

The power mean inequality gives us:

$$\frac{x+y+z}{3}\geq\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}\right)^2\geq\left(\frac{\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}}{3}\right)^3$$

Or

$$\frac{x+y+z}{3}\geq 1\geq\left(\frac{\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}}{3}\right)^3\Rightarrow\frac{x+y+z}{3}\geq 1\geq\frac{\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}}{3}$$

Multiplying through by 3 gives us the desired result.

Equality holds if and only if $x=y=z=1$.

JP.312. If $a,b,c>0$

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}\right)\geq\frac{9}{\lambda+1},\lambda\geq 0$$

Proposed by Marin Chirciu-Romania

Solution by Daniel Văcaru-Romania

We have:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}\right)\geq\frac{9}{\lambda+1}\leftrightarrow$$

$$(\lambda+1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}\right)\geq 9$$

But $(\lambda+1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}\right)=$

$$=\left[\lambda\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\right]\left(\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}\right)\geq$$

$$\left[3\lambda+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\right]\left(\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}\right)=$$

$$=\left(\frac{a+\lambda b}{b}+\frac{b+\lambda c}{c}+\frac{c+\lambda a}{a}\right)\left(\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}\right)\overset{Am-Gm}{\geq}$$

$$\geq 3\sqrt[3]{\frac{a+\lambda b}{b}\cdot\frac{b+\lambda c}{c}\cdot\frac{c+\lambda a}{a}}\cdot 3\sqrt[3]{\frac{a}{a+\lambda b}\cdot\frac{b}{b+\lambda c}\cdot\frac{c}{c+\lambda a}}=9$$

JP.313. Solve in $\mathbb{R}$ the system of equations:

$$\begin{cases}4(x+y)=\sqrt[4]{8(x^4+y^4)}+6\sqrt{xy}\\ 16x^5-20x^3=\sqrt{1-y^2}-5y\end{cases}$$

Proposed by Hoang Le Nhat-Hanoi-Vietnam

Solution by proposer

$$\begin{cases}4(x+y)=\sqrt[4]{8(x^4+y^4)}+6\sqrt{xy}\\ 16x^5-20x^3=\sqrt{1-y^2}-5y\end{cases};(1)$$

$$\begin{cases}xy\geq 0\\ 1-y^2\geq 0\end{cases}\overset{(1)}{\Rightarrow}4(x+y)=\sqrt[4]{8(x^4+y^4)}+6\sqrt{xy}\geq 0\Rightarrow x+y\geq 0\Rightarrow x\geq 0;y\geq 0$$

By CBS Inequality, we have:

$$\left(\sqrt{2(x^4+y^4)}+2xy\right)^2\leq 2(2(x^4+y^4)+4x^2y^2)=4(x^4+2x^2y^2+y^4)=4(x^2+y^2)^2$$

$$\sqrt{2(x^4+y^4)}+2xy\leq 2(x^2+y^2)\Leftrightarrow\sqrt{2(x^4+y^4)}\leq 2(x^2-xy+y^2)\Leftrightarrow$$

$$\sqrt[4]{8(x^4+y^4)}\leq 2\sqrt{x^2-xy+y^2}\Leftrightarrow$$

$$\sqrt[4]{8(x^4+y^4)}+6\sqrt{xy}\leq 2\left(\sqrt{x^2-xy+y^2}+3\sqrt{xy}\right);(2)$$

Other hand: $\sqrt{x^2-xy+y^2}+3\sqrt{xy}=\sqrt{x^2-xy+y^2}+\sqrt{xy}+\sqrt{xy}+\sqrt{xy}\leq$

$$\leq\sqrt{4(x^2-xy+y^2+xy+xy+xy)}=\sqrt{4(x^2+2xy+y^2)}=\sqrt{4(x+y)^2}=2(x+y)$$

$$\Rightarrow\sqrt{x^2-xy+y^2}+3\sqrt{xy}\leq 2(x+y);(3)$$

From (2),(3) result: $\sqrt[4]{8(x^4+y^4)}+6\sqrt{xy}\leq 4(x+y);(4)$

From (1),(4) result: $\sqrt[4]{8(x^4+y^4)}+6\sqrt{xy}=4(x+y)\Leftrightarrow$

$$\begin{cases}\sqrt{2(x^4+y^4)}=2xy\\ x^2-xy+y^2=xy\end{cases}\Leftrightarrow x=y\geq 0$$

Let: $x=y\geq 0$ in (1): $16x^5-20x^3=\sqrt{1-y^2}-5y\Leftrightarrow$

$$16x^5-20x^3=\sqrt{1-x^2}-5x;(5)$$

$$\begin{cases}1-x^2\geq 0\\ x\geq 0\end{cases}\Leftrightarrow 0\leq x\leq 1\Rightarrow x=\cos\alpha$$

Let: $\alpha\in[-\pi,\pi]$. Because $\cos\alpha\geq 0$ then $\alpha\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

From (5)$\Rightarrow 16\cos^5\alpha-20\cos^3\alpha+5\cos\alpha=\sqrt{1-\cos^2\alpha}\Leftrightarrow \cos5\alpha=\sqrt{\sin^2\alpha}=|\sin\alpha|$

Case 1: $\cos5\alpha=\sin\alpha\geq 0\left(\sin\alpha\geq 0;\alpha\in\left[0,\frac{\pi}{2}\right]\right)\Leftrightarrow \cos5\alpha=\cos\left(\frac{\pi}{2}-\alpha\right)$

$$\Leftrightarrow\begin{cases}5\alpha=\dfrac{\pi}{2}-\alpha+2k\pi\\ 5\alpha=\alpha-\dfrac{\pi}{2}+2k\pi\end{cases};k\in\mathbb{Z}\Leftrightarrow$$

$$\begin{cases}\alpha=\dfrac{\pi}{12}+\dfrac{k\pi}{3}\\ \alpha=-\dfrac{\pi}{8}+\dfrac{k\pi}{2}\end{cases};k\in\mathbb{Z}\xRightarrow{\alpha\in\left[0,\frac{\pi}{2}\right]}\begin{cases}0\leq\dfrac{\pi}{12}+\dfrac{k\pi}{3}\leq\dfrac{\pi}{2}\\ 0\leq-\dfrac{\pi}{8}+\dfrac{k\pi}{2}\leq\dfrac{\pi}{2}\end{cases}\Rightarrow\alpha\in\left\{\frac{\pi}{12};\frac{5\pi}{12};\frac{3\pi}{8}\right\}$$

Case 2: $\cos5\alpha=-\sin\alpha\geq 0;\left(\sin\alpha\leq 0;\alpha\in\left[-\frac{\pi}{2},0\right]\right);k\in\mathbb{Z}$

$$\begin{cases}\alpha=-\dfrac{\pi}{12}+\dfrac{k\pi}{3}\\ \alpha=\dfrac{\pi}{8}+\dfrac{k\pi}{2}\end{cases};k\in\mathbb{Z}\xRightarrow{\alpha\in\left[-\frac{\pi}{2},0\right]}\begin{cases}-\dfrac{\pi}{2}\leq-\dfrac{\pi}{12}+\dfrac{k\pi}{3}\leq 0\\ -\dfrac{\pi}{2}\leq\dfrac{\pi}{8}+\dfrac{k\pi}{2}\leq 0\end{cases}\Rightarrow\alpha\in\left\{-\frac{\pi}{12};-\frac{5\pi}{12};-\frac{3\pi}{8}\right\}$$

Other, $\cos\beta=\cos(-\beta);\cos\left(\frac{\pi}{12}\right)=\cos\left(-\frac{\pi}{12}\right);\cos\left(\frac{5\pi}{12}\right)=\cos\left(-\frac{5\pi}{12}\right);$

$$\cos\left(\frac{3\pi}{8}\right)=\cos\left(-\frac{3\pi}{8}\right)$$

$$(x,y)\in\left\{\left(\cos\frac{\pi}{12};\cos\frac{\pi}{12}\right);\left(\cos\frac{5\pi}{12};\cos\frac{5\pi}{12}\right);\left(\cos\frac{3\pi}{8};\cos\frac{3\pi}{8}\right)\right\}$$

JP.314. Solve in $\mathbb{R}$ the system of equations:

$$\begin{cases} a^2+b^2+c^2=a^3+b^3+c^3 \\ a^3b+b^3c+c^3a=3 \end{cases}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by proposer

$$\begin{cases} a^2+b^2+c^2=a^3+b^3+c^3 \\ a^3b+b^3c+c^3a=3 \end{cases};(1)$$

Lemma: If $a,b,c>0$ then: $(a^2+b^2+c^2)^2\ge 3(a^3b+b^3c+c^3a);(2)$

Proof.

$$\left(a^2-ab+2bc-b^2\right)^2+\left(b^2-bc+2ca-c^2-ab\right)^2+\left(c^2-ca+2ab-a^2-bc\right)^2\ge 0\Leftrightarrow$$

$$(a^2-ab)^2+2a(a^2-ab)(2bc-b^2-ca)+(2bc-b^2-ca)^2+(b^2-bc)^2+$$

$$+2(b^2-bc)(2ca-c^2-ab)+(2ca-c^2-ab)^2+(c^2-ca)^2+$$

$$+2(c^2-ca)(2ab-a^2-bc)+(2ab-a^2-bc)^2\ge 0\Leftrightarrow$$

$$2(a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2)-6(a^3b+b^3c+c^3a)\ge 0\Leftrightarrow$$

$$2(a^2+b^2+c^2)^2-6(a^3b+b^3c+c^3a)\ge 0\Leftrightarrow$$

$$2(a^2+b^2+c^2)^2\ge 6(a^3b+b^3c+c^3a)\ \Leftrightarrow (a^2+b^2+c^2)^2\ge 3(a^3b+b^3c+c^3a)$$

Lemma is proved.

By AM-GM inequality, we have:

$$(a^3+a^3+1)+(b^3+b^3+1)+(c^3+c^3+1)\ge 3\sqrt[3]{a^6}+3\sqrt[3]{b^6}+3\sqrt[3]{c^6}\Leftrightarrow$$

$$2(a^3+b^3+c^3)+3\ge 3(a^2+b^2+c^2);(4)$$

From (1),(4) result: $2(a^2+b^2+c^2)+3\ge 3(a^2+b^2+c^2)\Leftrightarrow 3\ge a^2+b^2+c^2;(5)$

From (3),(5) result: $3^2\ge (a^2+b^2+c^2)^2\ge 3(a^3b+b^3c+c^3a)\Leftrightarrow$

$$a^3b+b^3c+c^3a\le 3;(6)$$

From (1),(6) we get: $a^3b+b^3c+c^3a=3$ occurs if: $\begin{cases} a,b,c>0 \\ a=b=c=1 \\ a^2+b^2+c^2=3 \end{cases}\Leftrightarrow a=b=c=1$

The solution of system is $(a,b,c)=(1,1,1)$

JP.315. If s is the semiperimeter of ABC triangle and r_a, r_b, r_c the radii of excircles, then

$$\frac{s^2-r_ar_b}{s^2+r_ar_b}+\frac{s^2-r_br_c}{s^2+r_br_c}+\frac{s^2-r_cr_a}{s^2+r_cr_a}\geq\frac{3}{2}$$

Proposed by D.M.Bătineţu Giurgiu, Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania

We have:

$$\frac{s^2-r_ar_b}{s^2+r_ar_b}=\frac{s^2-\frac{S^2}{(s-a)(s-b)}}{s^2+\frac{S^2}{(s-a)(s-b)}}=\frac{s^2-s(s-c)}{s^2+s(s-c)}=\frac{sc}{s(2s-c)}=\frac{c}{a+b}$$

Then: $\sum_{cyc}\frac{s^2-r_ar_b}{s^2+r_ar_b}=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=$

$$=\frac{a^2}{ab+ab}+\frac{b^2}{bc+ba}+\frac{c^2}{c+cb}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{2(ab+bc+ca)}=$$

$$=1+\frac{a^2+b^2+c^2}{2(ab+bc+ca)}\geq 1+\frac{1}{2}=\frac{3}{2}$$

Solution 2 and generalizations by Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{s^2-r_br_c}{s^2+r_br_c}\geq\frac{3}{2}$$

2) Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{s^2-r_br_c}{s^2+r_br_c}=\sum_{cyc}\frac{a}{b+c}=\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}$$

Proof. Using the identity: $r_a=\frac{S}{s-a}$ we get:

$$\sum_{cyc}\frac{s^2-r_br_c}{s^2+r_br_c}=\sum_{cyc}\frac{s^2-\frac{S}{s-b}\cdot\frac{S}{s-c}}{s^2+\frac{S}{s-b}\cdot\frac{S}{s-c}}=$$

$$=\sum_{cyc}\frac{s^2(s-b)(s-c)-s(s-a)(s-b)(s-c)}{s^2(s-b)(s-c)-s(s-a)(s-b)(s-c)}=\sum_{cyc}\frac{s-(s-a)}{s+(s-a)}=\sum_{cyc}\frac{a}{b+c}$$

Let's solve the proposed problem.

Using lemma and $\sum_{cyc}\frac{a}{b+c}\ge\frac{3}{2}$ $(Nesbit\ I.)$ **we get:**

$$Lhs=\sum_{cyc}\frac{s^2-r_br_c}{s^2+r_br_c}=\sum_{cyc}\frac{a}{b+c}\ge\frac{3}{2}=Rhs$$

Equality holds if and only if triangle is equilateral.

Remark. Let's find reverse inequality.

3) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{s^2-r_br_c}{s^2+r_br_c}\le\frac{3R}{4r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using lemma, inequality it can be written as:

$$\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}\le\frac{3R}{4r}\Leftrightarrow s^2(3R-8r)+r(6R^2+11Rr+8r^2)\ge 0$$

We distinguish the cases:

Case 1) If $3R-8r\ge 0$ the inequality is obviously

Case 2) If $3R-8r\le 0$ the inequality it can be written as:

$$r(6R^2+11Rr+8r^2)\ge s^2(8r-3R)$$

which result from $s^2\le 4R^2+4Rr+3r^2$ $(Gerretsen)$

Remain to prove that:

$$r(6R^2+11Rr+8r^2)\ge(4R^2+4Rr+3r^2)(8r-3R)\Leftrightarrow$$

$$6R^3-7R^2r-6Rr^2-8r^3\ge 0\Leftrightarrow(R-2r)(6R^2+5Rr+4r^2)\ge 0$$

which is true from $R\ge 2r$ $(Euler)$.Equality holds if and only if triangle is equilateral.

Remark. The inequality it can be doubled.

4) In ΔABC the following relationship holds:

$$\frac{3}{2} \le \sum_{cyc} \frac{s^2 - r_b r_c}{s^2 + r_b r_c} \le \frac{3R}{4r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

See inequalities 1) and 3)

5) In ΔABC the following relationship holds:

$$\frac{3r}{R} \le \sum_{cyc} \frac{s^2 - r_a^2}{s^2 + r_a^2} \le \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

6) Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{s^2 - r_a^2}{s^2 + r_a^2} = 1 + \frac{r}{R}$$

Proof. Using the identity: $r_a = \frac{S}{s-a}$ we get: $Lhs = \sum_{cyc} \frac{s^2 - r_a^2}{s^2 + r_a^2} = \sum_{cyc} \frac{s^2 - \left(\frac{S}{s-a}\right)^2}{s^2 + \left(\frac{S}{s-a}\right)^2} =$

$$= \sum_{cyc} \frac{s^2(s-a)^2 - s(s-a)(s-b)(s-c)}{s^2(s-a)^2 + s(s-a)(s-b)(s-c)} = \sum_{cyc} \frac{2s(s-a) - bc}{bc} =$$

$$= 2s \sum_{cyc} \frac{s-a}{bc} - 3 = 2s \cdot \frac{4R + r}{2Rs} - 3 = \frac{R + r}{r} = Rhs$$

Let's solve the proposed problem. Using Lemma and $R \ge 2r$ $(Euler)$ we get:

$$\frac{3r}{R} \le \frac{R+r}{r} \le \frac{3}{2}$$

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$3(2 - \sqrt{3}) \le \sum_{cyc} \frac{s - r_a}{s + r_a} \le 3(2 - \sqrt{3}) \frac{R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

8) Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{s-r_a}{s+r_a}=\frac{s+2R-r}{s+2R+r}$$

Proof. Using the identity: $r_a=\frac{S}{s-a}$ we get:

$$\sum_{cyc}\frac{s-r_a}{s+r_a}=\sum_{cyc}\frac{s-\frac{S}{s-a}}{s+\frac{S}{s-a}}=\sum_{cyc}\frac{s(s-a)-rs}{s(s-a)+rs}=\sum_{cyc}\frac{(s-a)-r}{(s-a)+r}$$

Using Ravi substitution: $a=y+z; b=z+x; c=x+y; s-a=x; s-b=y; s-c=z$

we get:

$$\sum_{cyc}\frac{(s-a)-r}{(s-a)+r}=\sum_{cyc}\frac{x-r}{x+r}=\frac{\sum(x-r)(y+r)(z+r)}{\prod(x+r)}=$$

$$=\frac{2r^2(s+2R-2r)}{2r^2(s+2R+r)}=\frac{s+2R-r}{s+2R+r},$$

which result from

$$\sum(x-r)(y+r)(z+r)=3xyz+r\sum yz-r^2\sum x-3r^3=$$

$$=3\prod(s-a)+r\sum(s-b)(s-c)-r^2\sum(s-a)-3r^3=$$

$$=3r^2s+r\cdot r(4R+r)-r^2s-3r^3=2r^2(s+2R-r)$$

$$\prod_{cyc}(x+r)=xyz+r\sum_{cyc}yz+r^2\sum_{cyc}x+r^3=\prod_{cyc}(s-a)+r\sum_{cyc}(s-b)(s-c)+$$

$$+r^2\sum_{cyc}(s-a)+r^3=r^2s+r\cdot r(4R+r)+r^2s+r^3=2r^2(s+2R+r)$$

Let's solve the proposed problem. Using Lemma the inequality it can be written as:

$$\frac{s+2R-r}{s+2R+r}\geq 3(2-\sqrt{3})\Leftrightarrow s(3\sqrt{3}-5)\geq 2R(5-3\sqrt{3})+r(7-3\sqrt{3})$$

which result from $s\geq 3\sqrt{3}r$ $(Mitrinovic)$

Remain to prove that: $3r\sqrt{3}(3\sqrt{3}-5)\geq 2R(5-3\sqrt{3})+r(7-3\sqrt{3})\Leftrightarrow$

$$2R(3\sqrt{3}-5)\geq 4r(3\sqrt{3}-5)\Leftrightarrow R\geq 2r\ (Euler)$$

Equality holds if and only if triangle is equilateral.

$$\frac{s+2R-r}{s+2R+r}\le 3(2-\sqrt{3})\frac{R}{2r}\Leftrightarrow$$

$$s\left[3(2-\sqrt{3})R-2r\right]\ge 6R^2(\sqrt{3}-2)+Rr(3\sqrt{3}-2)-2r^2$$

which result from $s\ge 3\sqrt{3}r$ $(Mitrinovic)$. Remains to prove that:

$$3\sqrt{3}r\left[3(2-\sqrt{3})R-2r\right]\ge 6R^2(\sqrt{3}-2)+Rr(3\sqrt{3}-2)-2r^2\Leftrightarrow$$

$$6(2-\sqrt{3})R^2+5(3\sqrt{3}-5)Rr+2(1-3\sqrt{3})r^2\ge 0\Leftrightarrow$$

$(R-2r)\left[6(2-\sqrt{3})R+(3\sqrt{3}-1)r\right]\ge 0$ which result from $R\ge 2r$ $(Euler)$

Equality holds if and only if triangle is equilateral.

SP.301. Let $a,b,c>0, a+b+c=3$. Find the minimum of value:

$$T=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Michael Sterghiou-Greece

$$T=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc};\quad (1)$$

Let $(p,q,r)=(\sum a,\sum ab,abc)$ with $p=3, q\le 3, r\le 1$

We will to show that $T\ge 4$. We will use the following lemma.

Lemma. If $a,b,c>0, a+b+c=3$ then $\sum\frac{a}{b}\ge\sum a^2$; $\quad(L)$

Proof. Consider the inequality: $\cdot\sum\frac{a}{b}+3\ge 7\cdot\sum a^2$; (2) which written as homogeneous from becomes $2(\sum a)^2\cdot(\sum ab^2)+abc(\sum a)^2\ge 21abc(\sum a^2)$ which can be written as $2\cdot\sum a(a-b)^2(b-2c)^2\ge 0$ which is true, so (2) is true.

By adding (2) to the obvious $a^2+b^2+c^2\ge 3$ we get (L).

Now (1) using (L) and BCS we get: $T\ge\sum a^2+\frac{9}{2\sum a^3+3r}\ge 4$ or as:

$\sum a^2=p^2-2q=9-2q$; $\sum a^3=p^3-3pq+3r=27-9q+3r$ after simplification reduces to: $4p^2-2qr-22q+5r+31\ge 0;\ (3)\Rightarrow$

$$f(q)=4q^2-(2q-5)r-22q+31\ge 0;\quad (4)$$

$(2q-5)\begin{cases}\geq 0 \text{ or} \\ \leq 0\end{cases}$ so (4) must work either $r = \begin{cases} max \text{ or} \\ min\end{cases}$ with every fixed $q \in (0,3]$.

This according to V.Cîrtoaje theorem happens when any two of a, b, c are equal.

Assume $a \leq b \leq c$ WLOG we have to show (4) with either $a = b$ or $b = c$.

In either case $0 < b < \frac{3}{2}$. Now, $q = 3b(2-b)$ and $r = b^2(3-2b)$; $(a+b+c=3)$ and

(4)$\Rightarrow f(b) = -(b-1)^2(12b^3 - 54b^2 + 70b - 31) \geq 0, 0 < b < \frac{3}{2}$

It is easy to show that $h(b) = 12b^3 - 54b^2 + 70b - 31$ has a max on $\left(0, \frac{3}{2}\right)$ of $\frac{11\sqrt{11}}{9} - 7$

at $b = \frac{3}{2} - \frac{\sqrt{11}}{6}$ as $f'(b)$ is a trinomial.

This means $max(h) < 0$ and $f(b) \geq 0$. Equality for $a = b = c = 1$.

Solution 2 by proposer

By Cauchy-Schwartz inequality, we have:

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} =$$

$$= \frac{c}{c(a^3+b^3+abc)} + \frac{a}{a(b^3+c^3+abc)} + \frac{b}{b(c^3+a^3+abc)} \geq$$

$$\geq \frac{\left(\sqrt{c}+\sqrt{a}+\sqrt{b}\right)^2}{c(a^3+b^3+abc) + a(b^3+c^3+abc) + b(c^3+a^3+abc)} \Rightarrow$$

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \geq \frac{\left(\sqrt{c}+\sqrt{a}+\sqrt{b}\right)^2}{(a^2+b^2+c^2)(ab+bc+ca)}; \ (1)$$

Other, by AM-GM inequality for three positive real numbers, with $a+b+c=3$ we have:

$$\left(a^2+\sqrt{a}+\sqrt{a}\right) + \left(b^2+\sqrt{b}+\sqrt{b}\right) + \left(c^2+\sqrt{c}+\sqrt{c}\right) \geq$$

$$\geq 3\cdot\sqrt[3]{a^2\cdot\sqrt{a}\cdot\sqrt{a}} + 3\cdot\sqrt[3]{b^2\cdot\sqrt{b}\cdot\sqrt{b}} + 3\cdot\sqrt[3]{c^2\cdot\sqrt{c}\cdot\sqrt{c}} \Rightarrow$$

$$a^2+b^2+c^2+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \geq 3\cdot\sqrt[3]{a^3} + 3\cdot\sqrt[3]{b^3} + 3\cdot\sqrt[3]{c^3} =$$

$$= 3(a+b+c) = (a+b+c)(a+b+c) \Leftrightarrow$$

$$a^2+b^2+c^2+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \geq (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \Leftrightarrow$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \geq 2(ab+bc+ca) \Leftrightarrow$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \geq ab+bc+ca; \quad (2)$$

From (1) and (2) we get:

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\ge\frac{(ab+bc+ca)^2}{(a^2+b^2+c^2)(ab+bc+ca)}\Leftrightarrow$$

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\ge\frac{ab+bc+ca}{a^2+b^2+c^2};\ \ (3)$$

Other, by Cauchy-Schwartz inequality, we have:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{a^2}{ab}+\frac{b^2}{bc}+\frac{c^2}{ca}\ge\frac{(a+b+c)^2}{ab+bc+ca}=\frac{a^2+b^2+c^2+2(ab+bc+ca)}{ab+bc+ca}=$$

$$=\frac{a^2+b^2+c^2}{ab+bc+ca}+2;\ \ (4)$$

From (3),(4), using AM-GM inequality, we have

$$T=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\ge$$

$$\ge\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{ab+bc+ca}{a^2+b^2+c^2}+2\overset{AM-GM}{\ge}2\cdot\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}\cdot\frac{ab+bc+ca}{a^2+b^2+c^2}}+2=4$$

$$\Rightarrow T\ge 4\Rightarrow T_{min}=4 \text{ and equality occurs if }\begin{cases}a,b,c>0;a+b+c=3\\ a^2=\sqrt{a};b^2=\sqrt{b};c^2=\sqrt{c}\\ a=b=c\\ a^2+b^2+c^2=ab+bc+ca\end{cases}\Leftrightarrow$$

$$a+b+c=1$$

Hence, $T_{min}=4$ **for** $a=b=c=1$

SP.302 In acute $\Delta ABC, r_1, r_2, r_3$ **−inradii in** $\Delta BOC, \Delta COA, \Delta AOB, O$ **−center of circumcircle** ΔABC **and** H **−orthocenter. Prove that:**

$$\left(\frac{r_1}{AH}+\frac{r_2}{BH}+\frac{r_3}{CH}\right)\left(\sum_{cyc}\frac{A}{a}\right)<\frac{\pi\sqrt{3}s}{12Rr}$$

Proposed by Radu Diaconu-Romania

Solution 1 by George Florin Şerban-Romania

WLOG, suppose: $a \le b \le c \Rightarrow \begin{cases} A \le B \le C \\ \frac{1}{a} \ge \frac{1}{b} \ge \frac{1}{c} \end{cases}$ and from Chebyshev's inequality, we get:

$$\sum_{cyc}\frac{A}{a} \le \frac{1}{3}\cdot\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}A\right) \overset{(Petrivic)}{\le} \frac{\pi}{3}\cdot\frac{s}{3Rr} = \frac{\pi s}{9Rr}$$

$$\left(\sum_{cyc}\frac{r_1}{AH}\right)\left(\sum_{cyc}\frac{A}{a}\right) < \frac{\pi\sqrt{3}\cdot s}{12Rr};\quad (1)$$

$$\left(\sum_{cyc}\frac{r_1}{AH}\right)\left(\sum_{cyc}\frac{A}{a}\right) \le \frac{s\pi}{9Rr}\cdot\left(\sum_{cyc}\frac{r_1}{AH}\right) < \frac{\pi\sqrt{3}\cdot s}{12Rr} \Leftrightarrow \sum_{cyc}\frac{r_1}{AH} < \frac{\pi\sqrt{3}\cdot s}{12Rr}\cdot\frac{9Rr}{s\pi} = \frac{3\sqrt{3}}{4}$$

$$S_{[BOC]} = \frac{OB\cdot OC\cdot \sin(\widehat{BOC})}{2} = \frac{R^2\cdot \sin 2A}{2};\ s_{[BOC]} = \frac{BO+OC+a}{2} = \frac{2R+a}{2}$$

$$\Rightarrow r_1 = \frac{S_{[BOC]}}{s_{[BOC]}} = \frac{R^2\cdot\sin 2A}{2}\cdot\frac{2}{2R+a} = \frac{R^2\cdot \sin 2A}{2R+2R\sin A} = \frac{R\cdot \sin 2A}{2+2\sin A} \Rightarrow$$

$$\sum_{cyc}\frac{r_1}{AH} = \sum_{cyc}\frac{R\cdot\sin 2A}{(2+2\sin A)\cdot 2R\cos A} = \sum_{cyc}\frac{2\sin A\cos A}{2\cos A(2+2\sin A)} = \frac{1}{2}\sum_{cyc}\frac{\sin A}{1+\sin A} \overset{(2)}{\le} \frac{3\sqrt{3}}{4}$$

$$(2) \Leftrightarrow \sum_{cyc}\frac{\sin A}{1+\sin A} \le \frac{3\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\left(1-\frac{1}{1+\sin A}\right) \le \frac{3\sqrt{3}}{2} \Leftrightarrow$$

$$\sum_{cyc}\frac{1}{1+\sin A} > 3-\frac{3\sqrt{3}}{2};\quad (3)$$

From Bergstrom inequality, we have:

$$\sum_{cyc}\frac{1}{1+\sin A} \overset{Bergstrom}{\ge} \frac{(1+1+1)^2}{\sum_{cyc}(1+\sin A)} = \frac{9}{3+\sum_{cyc}\sin A} = \frac{9}{3+\frac{s}{R}} \overset{Mitrinovic}{\ge}$$

$$\ge \frac{9}{3+\frac{3\sqrt{3}}{2}} = \frac{18}{6+3\sqrt{3}} = \frac{6}{2+\sqrt{3}} = 6\left(2-\sqrt{3}\right) = 12-6\sqrt{3} > 3-\frac{3\sqrt{3}}{2} \text{ true from}$$

$$12-3 \ge 6\sqrt{3}-\frac{3\sqrt{3}}{2} \Leftrightarrow 9 > \frac{9\sqrt{3}}{2} \Leftrightarrow 2 > \sqrt{3} \text{ true} \Rightarrow (3)\text{true} \Rightarrow (1)\text{true}.$$

Solution 2 by proposer

We have: $AH = 2R\cos A; BH = 2R\cos B; CH = 2R\cos C$

$$S_{[BOC]}=\frac{OB\cdot OC\cdot sin2A}{2}=\frac{R^2\cdot sin2A}{2};\ s_{[BOC]}=\frac{BO+OC+a}{2}=\frac{2R+a}{2}$$

$\Rightarrow r_1=\frac{S_{[BOC]}}{s_{[BOC]}}=\frac{R^2\cdot sin2A}{2R+a}$. Similarly: $r_2=\frac{R^2\cdot sin2B}{2R+b}$; $r_3=\frac{R^2\cdot sin2C}{2R+c}$. Therefore,

$$\frac{r_1}{AH}+\frac{r_2}{BH}+\frac{r_3}{CH}=\frac{RsinA}{2R+a}+\frac{RsinB}{2R+b}+\frac{RsinC}{2R+c}=R\left(\frac{sinA}{2R+a}+\frac{sinB}{2R+b}+\frac{sinC}{2R+c}\right)$$

WLOG, suppose: $a\le b\le c\Rightarrow\begin{cases}sinA\ge sinB\ge sinC\\ \frac{1}{2R+a}\ge\frac{1}{2R+b}\ge\frac{1}{2R+c}\end{cases}$ and from Chebyshev's inequality, we get:

$$R\left(\frac{sinA}{2R+a}+\frac{sinB}{2R+b}+\frac{sinC}{2R+c}\right)\le\frac{R}{3}\cdot\left(\sum_{cyc}sinA\right)\left(\sum_{cyc}\frac{1}{2R+a}\right)\le$$

$$\le\frac{R}{3}\cdot\frac{3\sqrt{3}}{2}\left(\sum_{cyc}\frac{1}{2R+a}\right)=\frac{R\sqrt{3}}{2}\cdot\frac{1}{2R}\left(\sum_{cyc}\frac{1}{1+sinA}\right)=\frac{\sqrt{3}}{4}\left(\sum_{cyc}\frac{1}{1+sinA}\right)<\frac{3\sqrt{3}}{4}$$

How: $\frac{1}{1+sinA}<1;\frac{1}{1+sinB}<1;\frac{1}{1+sinC}<1\Rightarrow\sum_{cyc}\frac{1}{1+sinA}<3$

$$\frac{r_1}{AH}+\frac{r_2}{BH}+\frac{r_3}{CH}<\frac{3\sqrt{3}}{4};\quad (1)$$

WLOG, suppose: $a\le b\le c\Rightarrow\begin{cases}A\le B\le C\\ \frac{1}{a}\ge\frac{1}{b}\ge\frac{1}{c}\end{cases}$ and from Chebyshev's inequality, we get:

$$\sum_{cyc}\frac{A}{a}\le\frac{1}{3}\cdot\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}A\right)=\frac{\pi}{3}\cdot\left(\sum_{cyc}\frac{1}{a}\right)\overset{(*)}{\le}\frac{\pi}{3}\cdot\frac{s}{3Rr}=\frac{\pi s}{9Rr};\quad (2)$$

Where $(*)\Leftrightarrow\sum_{cyc}\frac{1}{a}\le\frac{s}{3Rr}$ which following from: $3(ab+bc+ca)\le(a+b+c)^2\Rightarrow$

$$\sum_{cyc}\frac{1}{a}\le\frac{(a+b+c)^2}{3abc}=\frac{4s^2}{12RS}=\frac{s}{3Rr}$$

$$\left(\frac{r_1}{AH}+\frac{r_2}{BH}+\frac{r_3}{CH}\right)\left(\sum_{cyc}\frac{A}{a}\right)<\frac{3\sqrt{3}}{4}\cdot\frac{\pi s}{9Rr}=\frac{\pi\sqrt{3}\cdot s}{12Rr}$$

SP.303. Let $x, y, z > 0$ such that $x + y + z = 3$. Find the minimum of value:

$$P = \frac{x^3}{y\sqrt{x^3+8}} + \frac{z^3}{z\sqrt{y^3+8}} + \frac{z^3}{x\sqrt{z^3+8}}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$\sqrt{x^3+8} = \sqrt{(x+2)(x^2-2x+4)} \overset{AM-GM}{\leq} \frac{(x+2)+(x^2-2x+4)}{2} = \frac{x^2-x+6}{2}$$

$$\frac{1}{\sqrt{x^3+8}} \geq \frac{2}{x^2-x+6} \Rightarrow \frac{x^3}{\sqrt{x^3+8}} \geq \frac{2x^3}{x^2-x+6}$$

But: $P = \frac{x^3}{y\sqrt{x^3+8}} + \frac{y^3}{z\sqrt{y^3+8}} + \frac{z^3}{x\sqrt{z^3+8}} \geq$

$$\geq \frac{1}{3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{x^3}{\sqrt{x^3+8}} + \frac{y^3}{\sqrt{y^3+8}} + \frac{z^3}{\sqrt{z^3+8}}\right)$$

Let be the function: $f(x) = \frac{x^3}{\sqrt{x^3+8}}; f(x) \geq \frac{2x^3}{x^2-x+6} \Rightarrow$

$$P \geq \frac{1}{3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{2x^3}{x^2-x+6} + \frac{2y^3}{y^2-y+6} + \frac{2z^3}{z^2-z+6}\right)$$

From AM-HM and $x + y + z = 3$, we have: $\frac{x+y+z}{3} \geq \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \Rightarrow 1 \geq \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq \frac{1}{3} \Rightarrow$

$$\frac{1}{3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 1; \quad (*)$$

Let be the function: $g(x) = \frac{2x^3}{x^2-x+6} = 2x + 2 - 2 \cdot \frac{5x+6}{x^2-x+6}$

$$g'(x) = 2 - 2 \cdot \frac{-5x^2-12x+36}{(x^2-x+6)^2}$$

$g''(x) = \frac{-2}{(x^2-x+6)^2} \cdot \frac{-10x^3-7x^2-26x-108}{(x^2-x+6)^2} > 0 \Rightarrow g$ —convex function, so

$$g(x)+g(y)+g(z) \geq \frac{1}{3}g\left(\frac{x+y+z}{3}\right) \Leftrightarrow g(x)+g(y)+g(z) \geq \frac{1}{3}g(1) \Leftrightarrow$$

$g(x)+g(y)+g(z) \geq 1; \quad (**)$ and hence: $P \geq 1$.

$minP(x, y, z) = 1$, for $x = y = z = 1$

Solution 2 by proposer

By AM-GM inequality for positive real numbers, we have:

$$P=\frac{x^3}{y\sqrt{x^3+8}}+\frac{y^3}{z\sqrt{y^3+8}}+\frac{z^3}{x\sqrt{z^3+8}}=$$

$$=\frac{x^3}{y\sqrt{(x+2)(x^2-x+4)}}+\frac{y^3}{z\sqrt{(y+2)(y^2-y+4)}}+\frac{z^3}{x\sqrt{(z+2)(z^2-z+4)}}\ge$$

$$\ge\frac{x^3}{y\cdot\frac{(x+2)+(x^2-x+4)}{2}}+\frac{y^3}{z\cdot\frac{(y+2)+(y^2-y+4)}{2}}+\frac{z^3}{x\cdot\frac{(z+2)+(z^2-z+4)}{2}}=$$

$$=2\left(\frac{x^3}{y(x^2-x+6)}+\frac{y^3}{z(y^2-y+6)}+\frac{z^3}{x(z^2-z+6)}\right)\Rightarrow$$

$$P\ge2\left(\frac{x^3}{y(x^2-x+6)}+\frac{y^3}{z(y^2-y+6)}+\frac{z^3}{x(z^2-z+6)}\right);\quad(1)$$

By Cauchy Schwartz inequality, we have:

$$\frac{x^3}{y(x^2-x+6)}+\frac{y^3}{z(y^2-y+6)}+\frac{z^3}{x(z^2-z+6)}=$$

$$=\frac{x^4}{xy(x^2-x+6)}+\frac{y^4}{yz(y^2-y+6)}+\frac{z^4}{zx(z^2-z+6)}\ge$$

$$\ge\frac{(x^2+y^2+z^2)^2}{xy(x^2-x+6)+yz(y^2-y+6)+zx(z^2-z+6)};\quad(2)$$

From (1),(2) we get:

$$P\ge\frac{(x^2+y^2+z^2)^2}{xy(x^2-x+6)+yz(y^2-y+6)+zx(z^2-z+6)};\quad(3)$$

We will to prove that:

$$\frac{(x^2+y^2+z^2)^2}{xy(x^2-x+6)+yz(y^2-y+6)+zx(z^2-z+6)}\ge\frac{1}{2};\quad(4)$$

$$\Leftrightarrow 2(x^2+y^2+z^2)^2\ge xy(x^2-x+6)+yz(y^2-y+6)+zx(z^2-z+6)\Leftrightarrow$$

$$2(x^2+y^2+z^2)^2+(x^2y+y^2z+z^2x)\ge(x^3y+y^3z+z^3x)+6(xy+yz+zx)\Leftrightarrow$$

$$6(x^2+y^2+z^2)^2+3(x^2y+y^2z+z^2x)\ge3(x^3y+y^3z+z^3x)+18(xy+yz+zx)\Leftrightarrow$$

$$6(x^2+y^2+z^2)^2+(x+y+z)(x^2y+y^2z+z^2x)\ge3(x^3y+y^3z+z^3x)+$$

$$+2(x+y+z)^2(xy+yz+zx)\Leftrightarrow$$

$$6(x^2+y^2+z^2)^2+(x^3y+y^3z+z^3x)+xyz(x+y+z)+(x^2y^2+y^2z^2+z^2x^2)\ge$$

$$\ge 3(x^3y+y^3z+z^3x)+4(xy+yz+zx)^2+2(x^2+y^2+z^2)(xy+yz+zx)\Leftrightarrow$$

$$6(x^4+y^4+z^4)+9(x^2y^2+y^2z^2+z^2x^2)\ge 4(x^3y+y^3z+z^3x)+$$

$$+2(xy^3+yz^3+zx^3)+9xyz(x+y+z);\ \ (5)$$

By AM-GM inequality, we have:

$$(x^4+x^4+x^4+y^4)+(y^4+y^4+y^4+z^4)+(z^4+z^4+z^4+x^4)\ge$$

$$\ge 4\sqrt[4]{x^{12}\cdot y^4}+4\sqrt[4]{y^{12}\cdot z^4}+4\sqrt[4]{z^{12}\cdot x^4}\Rightarrow$$

$$4(x^4+y^4+z^4)\ge 4(x^3y+y^3z+z^3x);(6)$$

$$(x^4+y^4+y^4+y^4)+(y^4+z^4+z^4+z^4)+(z^4+x^4+x^4+x^4)\ge$$

$$\ge 4\sqrt[4]{x^4\cdot y^{12}}+4\sqrt[4]{y^4\cdot z^{12}}+4\sqrt[4]{z^4\cdot x^{12}}\Rightarrow$$

$$2(x^4+y^4+z^4)\ge 2(xy^3+yz^3+zx^3);(7)$$

$$9(x^2y^2+y^2z^2+z^2x^2)=9\left(\frac{x^2(y^2+z^2)}{2}+\frac{y^2(z^2+x^2)}{2}+\frac{z^2(x^2+y^2)}{2}\right)\ge$$

$$\ge 9(x^2yz+xy^2z+xyz^2)=9xyz(x+y+z);\ \ (8)$$

From (6),(7),(8) we have: $6(x^4+y^4+z^4)+9(x^2y^2+y^2z^2+z^2x^2)\ge$

$\ge 4(x^3y+y^3z+z^3x)+2(xy^3+yz^3+zx^3)+9xyz(x+y+z)\Rightarrow(5)$ **true** $\Rightarrow(4)$ **true.**

From (3),(4) we have: $minP(x,y,z)=1$, **for** $x=y=z=1$

SP.304. Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$.

Find the minimum value of the expression:

$$P=\frac{a}{b(b+2c)(a+3c)^2}+\frac{b}{c(c+2a)(b+3a)^2}+\frac{c}{a(a+2b)(c+3b)^2}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by proposer

By Cauchy-Schwartz inequality, we have:

$$P=\frac{a}{b(b+2c)(a+3c)^2}+\frac{b}{c(c+2a)(b+3a)^2}+\frac{c}{a(a+2b)(c+3b)^2}=$$

$$=\frac{\left(\frac{a}{a+3c}\right)^2}{ab(b+2c)}+\frac{\left(\frac{b}{b+3a}\right)^2}{bc(c+2a)}+\frac{\left(\frac{c}{c+3b}\right)^2}{ca(a+2b}\geq\frac{\left(\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}\right)^2}{ab(b+2c)+bc(c+2a)+ca(a+2b)}\Rightarrow$$

$$P\geq\frac{\left(\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}\right)^2}{ab^2+bc^2+ca^2+6abc};\ \ (1)$$

By AM-GM inequality, we have

$$(a+b)(b+c)(c+a)\geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}=8\sqrt{(abc)^2}=8abc\Leftrightarrow$$

$$abc\leq\frac{(a+b)(b+c)(c+a)}{8};(2)$$

From (2) we have:

$$(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc\leq$$

$$\leq(a+b)(b+c)(c+a)+\frac{(a+b)(b+c)(c+a)}{8}=\frac{9(a+b)(b+c)(c+a)}{8}$$

Then

$$(a+b)(b+c)(c+a)\geq\frac{8(a+b+c)(ab+bc+ca)}{9}\Leftrightarrow$$

$$(a+b+c)(ab+bc+ca)\leq\frac{9}{8}\Leftrightarrow$$

$$\frac{9}{8}\geq(ab^2+bc^2+ca^2)+(a^2b+b^2c+c^2a)+3abc;\ \ (3)$$

By AM-GM inequality, we have

$$a^2b+b^2c+c^2a\geq 3\sqrt[3]{(a^2b)\cdot(b^2c)\cdot(c^2a)}=3\sqrt[3]{(abc)^3}=3abc\overset{(3)}{\Rightarrow}$$

$$\frac{9}{8}\geq(ab^2+bc^2+ca^2)+3abc+3abc\Leftrightarrow ab^2+bc^2+ca^2+6abc\leq\frac{9}{8};\ \ (4)$$

On the other hand, we have:

$$\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}=\frac{a^2}{a^2+3ac}+\frac{b^2}{b^2+3ab}+\frac{c^2}{c^2+3bc}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a+b+c)^2}{a^2+3ac+b^2+3ab+c^2+3bc}\geq\frac{(a+b+c)^2}{(a+b+c)^2+\frac{(a+b+c)^2}{3}}=\frac{3}{4};(5)$$

From (1),(4),(5) we have:

$$P \geq \frac{\left(\frac{3}{4}\right)^2}{\frac{9}{8}} = \frac{1}{2} \Rightarrow P_{min} = \frac{1}{2} \Rightarrow \begin{cases} (a+b)(b+c)(c+a) = 1 \\ a = b = c > 0 \end{cases} \Leftrightarrow a = b = c = \frac{1}{2}$$

Hence, the minimum expression value of expression P is $\frac{1}{2}$ then $a = b = c = \frac{1}{2}$

SP.305. Let $a, b, c > 0$ such that $abc = 1$. Find the maximum expression:

$$P = \frac{1}{\sqrt{3a^4 - 4a + 2b^2 + 11}} + \frac{1}{\sqrt{3b^4 - 4b + 2c^2 + 11}} + \frac{1}{\sqrt{3c^4 - 4c + 2a^2 + 11}}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by proposer

We have:

$$3a^4 - 2a^2 - 8a + 7 = 3a^3(a-1) + 3a^2(a-1) + a(a-1) - 7(a-1) =$$
$$= (a-1)(3a^3 + 3a^2 + a - 7) = (a-1)[3a^2(a-1) + 6a(a-1) + 7(a-1)] =$$
$$= (a-1)^2(3a^2 + 6a + 7) \geq 0;\ \forall a > 0$$

Hence

$$3a^4 - 4a + 2b^2 + 11 \geq 4a + 2(a^2 + b^2) + 4 \geq 4a + 2 \cdot 2a + 4 =$$
$$= 4(a + ab + 1) \Leftrightarrow \frac{1}{3a^4 - 4a + 2b^2 + 11} \leq \frac{1}{4(ab + a + 1)}$$

By AM-GM inequality we have:

$$\frac{1}{\sqrt{(3a^4 - 4a + 2b^2 + 11) \cdot 12 \cdot 12}} \leq \frac{1}{3}\left(\frac{1}{3a^4 - 4a + 2b^2 + 11} + \frac{1}{12} + \frac{1}{12}\right)$$

Hence $\frac{1}{\sqrt{(3a^4-4a+2b^2+11) \cdot 12 \cdot 12}} \leq \frac{1}{3}\left(\frac{1}{4(ab+a+1)} + \frac{1}{6}\right) = \frac{1}{12(ab+a+1)} + \frac{1}{18}$

Then $\frac{1}{\sqrt{3a^4-4a+2b^2+11}} \leq \frac{\sqrt[3]{12^2}}{12(ab+a+1)} + \frac{\sqrt[3]{12^2}}{18} = \frac{1}{\sqrt[3]{12}(ab+a+1)} + \frac{\sqrt[3]{12^2}}{18}$

Similarly: $\frac{1}{\sqrt{3b^4-4b+2c^2+11}} \leq \frac{1}{\sqrt[3]{12}(bc+b+1)} + \frac{\sqrt[3]{12^2}}{18}$

And $\frac{1}{\sqrt{3c^4-4c+2a^2+11}} \leq \frac{1}{\sqrt[3]{12}(ca+c+1)} + \frac{\sqrt[3]{12^2}}{18}$

Hence

$$P=\frac{1}{\sqrt{3a^4-4a+2b^2+11}}+\frac{1}{\sqrt{3b^4-4b+2c^2+11}}+\frac{1}{\sqrt{3c^4-4c+2a^2+11}}\le$$

$$\le\frac{1}{\sqrt[3]{12}}\left(\frac{1}{ab+a+1}+\frac{1}{bc+b+1}+\frac{1}{ac+c+1}\right)+\frac{\sqrt[3]{12^2}}{6}$$

Other, because $abc=1$ then

$$\frac{1}{ab+a+1}+\frac{1}{bc+b+1}+\frac{1}{ac+c+1}=$$

$$=\frac{1}{ab+a+1}+\frac{a}{abc+ab+a}+\frac{ab}{a\cdot abc+abc+ab}=$$

$$=\frac{1}{ab+a+1}+\frac{a}{1+ab+a}+\frac{ab}{a+1+ab}=\frac{ab+a+1}{ab+a+1}=1;\quad(2)$$

From (1),(2) we have:

$$P\le\frac{1}{\sqrt[3]{12}}\cdot 1+\frac{\sqrt[3]{12^2}}{6}=\frac{18}{6\sqrt[3]{12}}=\frac{3}{\sqrt[3]{12}}\Rightarrow P\le\frac{3}{\sqrt[3]{12}}\Rightarrow P_{Max}=\frac{3}{\sqrt[3]{12}}$$

Equality occurs if $\begin{cases}abc=1;a,b,c>0\\a-1=b-1=c-1=0\end{cases}\Leftrightarrow a=b=c=1.$

Hence the maximum value of expression P is $\frac{3}{\sqrt[3]{12}}$ then $a=b=c=1$.

SP.306. In ΔABC the following relationship holds:

$$\frac{16}{9}\cdot\frac{4R+r}{3R-2r}\le\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\le 4\left(\frac{R}{r}-1\right)$$

Proposed by Marin Chirciu-Romania

Solution 1 by George Florin Şerban-Romania

First, we prove that:

$$\sum_{cyc}\frac{a}{m_a}\ge 2\sqrt{3}$$

$$\sum_{cyc}\frac{a}{m_a}=2\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}\ge 2\sqrt{3}\Leftrightarrow\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}\ge 2\sqrt{3}$$

Applying Holder Inequality, we get:

$$\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}\cdot\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}\cdot\sum_{cyc}a(2b^2+2c^2-a^2)\geq\left(\sum_{cyc}a\right)^3\Leftrightarrow$$

$$\left(\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}\right)^2\geq\frac{\left(\sum_{cyc}a\right)^3}{\sum_{cyc}a(2b^2+2c^2-a^2)}\Leftrightarrow$$

$$\left(\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}\right)^2\geq\frac{\left(\sum_{cyc}a\right)^3}{2\sum_{cyc}ab(a+b)-\sum_{cyc}a^3}\geq 3\Leftrightarrow$$

$$\left(\sum_{cyc}a\right)^3\geq 6\sum_{cyc}ab(a+b)-3\sum_{cyc}a^3\geq 4\sum_{cyc}a^3+6abc\geq 3\sum_{cyc}ab(a+b)$$

true by Schur's Inequality.

$$\sum_{cyc}\frac{a^2}{m_a^2}\overset{CBS}{\geq}\frac{\left(\sum_{cyc}\frac{a}{m_a}\right)^2}{3}\geq\frac{(2\sqrt{3})^2}{3}=\frac{12}{3}=4\overset{(1)}{\geq}\frac{16}{9}\cdot\frac{4R+r}{3R-2r}$$

$$(1)\Leftrightarrow 27R-18r\geq 16R+4r\Leftrightarrow 11R\geq 22r\Leftrightarrow R\geq 2r\ (Euler)$$

$$\sum_{cyc}\frac{a^2}{m_a^2}\overset{(m_a^2\geq s(s-a))}{\leq}\sum_{cyc}\frac{a^2}{s(s-a)}\leq\frac{1}{s}\sum_{cyc}\frac{a^2}{s-a}=\frac{1}{s}\cdot\frac{4s(R-r)}{r}=\frac{4(R-r)}{r}=4\left(\frac{R}{r}-1\right)$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{a^2}{m_a^2}\overset{(m_a^2\geq s(s-a))}{\leq}\sum_{cyc}\frac{a^2}{s(s-a)}\leq\frac{1}{s}\sum_{cyc}\frac{a^2}{s-a}=\frac{\sum_{cyc}a^2(s-b)(s-c)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{s^2\sum_{cyc}a^2-s\sum_{cyc}(a^2b+ab^2)+\sum_{cyc}a^2bc}{S^2}=$$

$$=\frac{s^2\sum_{cyc}a^2-s\sum_{cyc}ab(a+b)+\sum_{cyc}a^2bc}{S^2}=$$

$$=\frac{s^2\sum_{cyc}a^2-s\sum_{cyc}ab(2s-c)+abc\sum_{cyc}a}{S^2}=$$

$$=\frac{s^2\sum_{cyc}a^2-2s^2\sum_{cyc}ab+s\cdot 3abc+2s\cdot abc}{S^2}=$$

$$=\frac{s^2(2s^2-8Rr-2r^2-2s^2-2r^2-8Rr)+5\cdot s\cdot 4Rrs}{S^2}=$$

$$=\frac{s^2(20Rr-16Rr-4r^2)}{r^2s^2}=\frac{4Rr-4r^2}{r^2}=4\left(\frac{R}{r}-1\right)$$

$$\sum_{cyc}\frac{a^2}{m_a^2}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{\sum_{cyc}m_a^2}=\frac{(2s)^2}{\frac{3}{4}\sum_{cyc}c^2}=\frac{4s^2}{\frac{3}{4}\cdot 2(s^2-4Rr-r^2)}=\frac{16s^2}{6(s^2-4Rr-r^2)}$$

Now, need to prove:

$$\frac{16s^2}{6(s^2-4Rr-r^2)}\geq\frac{16}{9}\cdot\frac{4R+r}{3R-2r}\Leftrightarrow 3s^2(3R-2r)\geq 2(s^2-4Rr-r^2)(4R+r)\Leftrightarrow$$

$$9s^2R-6s^2r\geq 2(4s^2R+s^2r-16R^2r-4Rr^2-4Rr^2-r^3)\Leftrightarrow$$

$$9s^2R-8s^2r+32R^2r+16Rr^2+2r^3\geq 0$$

$$\left(\because s^2\geq 16Rr-5r^2(Gerretsen)\right)$$

We need to prove that:

$$(16Rr-5r^2)(R-8r)+32R^2r+16Rr^2+2r^3\geq 0\Leftrightarrow$$

$48R^2r-117Rr^2+42r^3\geq 0\Leftrightarrow(R-2r)(48R-21r)\geq 0$ **true by** $R\geq 2r(Euler)$

Proved.

Solution 3 by Tran Hong-Dong Thap-Vietnam

WLOG, suppose: $a\leq b\leq c\Rightarrow\begin{cases}a^2\leq b^2\leq c^2\\ m_a^2\geq m_b^2\geq m_c^2\end{cases}\Rightarrow\begin{cases}a^2\leq b^2\leq c^2\\ \frac{1}{m_a^2}\leq\frac{1}{m_b^2}\leq\frac{1}{m_c^2}\end{cases}$

$$\Omega=\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\overset{Chebyshev's}{\geq}\frac{1}{3}(a^2+b^2+c^2)\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)\overset{Bergstrom}{\geq}$$

$$\geq\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{9}{m_a^2+m_b^2+m_c^2}=\frac{3(a^2+b^2+c^2)}{\frac{3}{4}(a^2+b^2+c^2)}=4\overset{(1)}{\geq}\frac{16}{9}\cdot\frac{4R+r}{3R-2r}$$

$$(1)\Leftrightarrow 9(3R-2r)\geq 4(4R+r)\Leftrightarrow 27R-18r\geq 4(4R+r)\Leftrightarrow$$

$11R\geq 22r\Leftrightarrow R\geq 2r(Euler)\Leftrightarrow(1)$ **is true.**

$$m_a^2=\frac{2(b^2+c^2)-a^2}{4}\overset{BCS}{\geq}\frac{(b+c)^2-a^2}{4}\ (and\ analogs)$$

$$\Omega=\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\leq 4\cdot\sum_{cyc}\frac{a^2}{(b+c)^2-a^2}=\frac{4}{a+b+c}\cdot\sum_{cyc}\frac{a^2}{b+c-a}=$$

$$=\frac{4}{2s}\sum_{cyc}\frac{a^2}{2(s-a)}=\frac{1}{s}\sum_{cyc}\frac{a^2}{s-a};\ (2)$$

$$\therefore\sum_{cyc}\frac{a^2}{s-a}=\sum_{cyc}\frac{(s+a-s)^2}{s-a}=\sum_{cyc}\frac{s^2+2s(a-s)+(a-s)^2}{s-a}=$$

$$=s^2\sum_{cyc}\frac{1}{s-a}-\sum_{cyc}2s+\sum_{cyc}(s-a)=s^2\cdot\frac{4R+r}{sr}-6s+s=$$

$$=s\left(\frac{4R+r}{r}-5\right)=4s\cdot\frac{R-r}{r};\ (3)$$

From (2),(3) we get: $\Omega\le\frac{1}{s}\cdot 4s\cdot\frac{R-r}{r}=4\left(\frac{R}{r}-1\right)$

SP.307. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\left(2+\frac{\sqrt{h_bh_c}}{a}-\frac{2(s-a)^2}{bc}\right)\le\sum_{cyc}(1+cscA)^{\frac{1}{1+cotA}}\cdot(1+secA)^{\frac{1}{1+tanA}}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\text{Let: } f:(0,1)\to\mathbb{R}, f(x)=log\left(\frac{1}{x}+1\right)$$

$$f'(x)=\frac{-1}{x(x+1)}<0,\forall x\in(0,1)\Rightarrow f-\text{decreasing.}$$

$$f''(x)=\frac{2x+1}{x^2(x+1)^2}>0,\forall x\in(0,1)\Rightarrow f-\text{convexe.}$$

$$log(1+sinx+cosx)=f\left(\frac{1}{sinx+cosx}\right)=f\left(\frac{sin^2x+cos^2x}{sinx+cosx}\right)$$

$$=f\left(\frac{sinx\cdot sinx+cosx\cdot cosx}{sinx+cosx}\right)\le\frac{sinxf(sinx)+cosxf(cosx)}{sinx+cosx}$$

$$=\frac{1}{1+cotx}log\left(\frac{1}{sinx}+1\right)+\frac{1}{1+tanx}log\left(\frac{1}{cosx}+1\right)$$

$$=log\left(\left(\frac{1}{sinx}+1\right)^{\frac{1}{1+cotx}}\cdot\left(\frac{1}{cosx}+1\right)^{\frac{1}{1+tanx}}\right)\Rightarrow$$

$$1+sinx+cosx \le \left(\frac{1}{sinx}+1\right)^{\frac{1}{1+cotx}} \cdot \left(\frac{1}{cosx}+1\right)^{\frac{1}{1+tanx}}$$

$$1+sinx+cosx \le (cscx+1)^{\frac{1}{1+cotx}} \cdot (secx+1)^{\frac{1}{1+tanx}}$$

$$\sum_{cyc}(1+cscA)^{\frac{1}{1+cotA}} \cdot (1+secA)^{\frac{1}{1+tanA}} \ge 3+\sum_{cyc} sinA+\sum_{cyc} cosA$$

$$= 3+\frac{s}{R}+\left(1+\frac{r}{R}\right) = 4+\left(\frac{1}{2}\sum_{cyc}\frac{h_b+h_c}{a}\right)+\left(2-2\sum_{cyc}\frac{(s-a)^2}{bc}\right)$$

$$\overset{Am-Gm}{\ge} 6+\sum_{cyc}\left(\frac{\sqrt{h_b \cdot h_c}}{a}-2\cdot\frac{(s-a)^2}{bc}\right)$$

SP.308. Let $(x_n)_{\ge 1}, x_1 = 0, x_n = \frac{(1-n)x_{n-1}+1-2n}{nx_{n-1}+2n}$. Find:

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^{n}(2+x_k)$$

Proposed by Florică Anastase-Romania

Solution by Kamel Benaicha-Algiers-Algerie

$$\Omega_n = \prod_{k=1}^{n}(2+x_k)$$

$$x_n = -1+\frac{1}{n}\cdot\frac{x_{n-1}+1}{x_{n-1}+2} \Rightarrow x_n+1 = \frac{1}{n}\cdot\frac{x_{n-1}+1}{x_{n-1}+2} \Rightarrow x_{n-1}+2 = \frac{1}{n}\cdot\frac{x_{n-1}+1}{x_n+1}$$

$$\Omega_n = \prod_{k=1}^{n}(2+x_k) = \prod_{k=1}^{n}\frac{1}{k+1}\cdot\frac{x_k+1}{x_{k+1}+1} =$$

$$= \frac{1}{2}\cdot\frac{x_1+1}{x_2+1}\cdot\frac{1}{3}\cdot\frac{x_2+1}{x_3+1}\cdot\ldots\cdot\frac{1}{n+1}\cdot\frac{x_n+1}{x_{n+1}+1} = \frac{1}{(n+1)!}\cdot\frac{x_n+1}{x_{n+1}+1} =$$

$$= \frac{1}{2(n+1)!}\cdot\frac{1}{x_{n+1}+1} \Rightarrow \Omega_n = \frac{1}{2n!}\cdot\frac{1}{x_n+1}$$

We have:

$$x_n+2=\frac{1}{n+1}\cdot\frac{x_n+1}{x_{n+1}+1}\Rightarrow\frac{1}{x_{n+1}+1}=(n+1)\cdot\frac{x_n+2}{x_n+1}=(n+1)\left(1+\frac{1}{1+x_n}\right)$$

$$\therefore\Omega_n=\frac{1}{2n!}\left(1+\frac{1}{1+x_n}\right)=\frac{1}{2n!}(1+2n!\cdot\Omega_{n-1})$$

$$\Omega_n-\Omega_{n-1}=\frac{1}{2n!}$$

$$\text{So: }\Omega_2-\Omega_1+\Omega_3-\Omega_2+\cdots+\Omega_n-\Omega_{n-1}=\frac{1}{2}\left(\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)\Leftrightarrow$$

$$\Omega_n-\Omega_1=\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k!}\Rightarrow\Omega=\lim_{n\to\infty}\Omega_n=\Omega_1+\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k!}$$

$$\Omega_1=\frac{3}{2}\Rightarrow\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}(2+x_k)=\frac{3}{2}+\frac{1}{2}(e-2)=\frac{e+1}{2}$$

$$\left(\therefore\sum_{k=0}^{\infty}\frac{1}{k!}=e\text{ denote Napier's constant}\right)$$

SP.309. In any ΔABC the following relationship holds:

$$\frac{r}{4R}\le sin\left(\frac{\pi-A}{4}\right)sin\left(\frac{\pi-B}{4}\right)sin\left(\frac{\pi-C}{4}\right)\le\frac{1}{8}$$

Proposed by Marian Ursărescu-Romania

Solution by Daniel Văcaru-Romania

We have: $\frac{r}{4R}=sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$.

Observe that: $\prod_{cyc}sin\left(\frac{\pi-A}{4}\right)=\frac{\sum_{cyc}sin\frac{A}{2}-1}{4}\ge\frac{3\sqrt[3]{\prod_{cyc}sin\frac{A}{2}}-1}{4}\ge\prod_{cyc}sin\frac{A}{2};(1)$

Let's denote $\sqrt[3]{\prod_{cyc}sin\frac{A}{2}}=x$.

Then (1) became: $3x-1\ge4x^3\leftrightarrow(2x-1)^2(1-x)\ge0$, true because $x<1$.

For the right side, we have: $\frac{\sum_{cyc}sin\frac{A}{2}-1}{4}\overset{t\to sin\frac{t}{2}\ concave}{\le}\frac{3sin\left(\frac{A+B+C}{6}\right)}{4}=\frac{1}{8}$

SP.310. Let $\Delta A'B'C'$ the extouch triangle of ΔABC.Prove that:

$B'C'$ is tangent of the incircle in ΔABC if and only if

$$(s-b)^2+(s-c)^2=(s-a)^2$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$B'C'$ –is tangent $\Leftrightarrow$ $BCB'C$ –circumscribe $\Leftrightarrow B'C'+BC=BC'+CB'$ (1)

In $\Delta A'B'C'$ we have: $B'C'=\sqrt{x^2+y^2-2xy\cos A}$ (2)

From (1)+(2) we have:

$$\sqrt{x^2+y^2-2xy\cos A}=b+c-a-(x+y)=2(s-a)-(x+y)\Leftrightarrow$$

$$x^2+y^2-2xy\cos A=4(s-a)^2+4(s-a)(x-y)+x^2+y^2+2xy \quad (3)$$

where $x+y<2(s-a)$

$$4(s-a)^2-4(s-a)(x+y)+2xy\left(1+\frac{b^2+c^2-a^2}{2bc}\right)=0\Leftrightarrow$$

$$4(s-a)^2-4(s-a)(x+y)+\frac{4xy}{bc}\cdot s(s-a)=0$$

$$s-a-x-y+\frac{xy}{bc}\cdot s=0$$

$$b(s-c)\left(\frac{y}{b(b-y)}+\frac{1}{b}\right)+c(s-b)\left(\frac{x}{c(c-x)}+\frac{1}{c}\right)=s$$

$$(s-c)\frac{y}{b-y}+(s-b)\frac{x}{c-x}=s-(s-c)-(s-b)=b+c-s=s-a \quad (4)$$

But B', C' –the contact points of the external circumscription circle, then

$$x=s-b, y=s-c, s-y=s-a, c-x=s-a \quad (5)$$

From (4)+(5) $\Leftrightarrow \frac{(s-c)^2}{s-a}+\frac{(s-b)^2}{s-a}=s-a\Leftrightarrow(s-b)^2+(s-c)^2=(s-a)^2$

SP.311. If $A\in M_2(\mathbb{R})$ such that $\det(A^4+4I_2)=0$. Prove that:

$$(\det A)^2=(TrA)^2$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$det(A^4+4I_2)=$$
$$=det(A-(1+i)I_2)det(A-(1-i)I_2)\,det(A-(-1+i)I_2)det(A-(-1-i)I_2)=0$$
$$P_A(x)=x^2\pm 2x+2=x^2-Tr(A)x+det(A)\Rightarrow (detA)^2=(TrA)^2$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ we have:

$$A^4+4I_2=A^4+4A^2+4I_2-4A^2=(A^2+2I_2)^2-(2A)^2=$$
$$=(A^2+2A+2I_2)(A^2-2A+2I_2)=$$
$$=(A-(1+i)I_2)(A-(1-i)I_2)(A-(-1+i)I_2)(A-(-1-i)I_2)$$

Now, $det(A^4+4I_2)=$

$$=det(A-(1+i)I_2)det(A-(1-i)I_2)\,det(A-(-1+i)I_2)det(A-(-1-i)I_2)=0$$
$$\Rightarrow det(A-(1+i)I_2)=det(A-(1-i)I_2)=det(A-(-1+i)I_2)=$$
$$=det(A+(1+i)I_2)=0$$

Suppose that: $det(A+(1+i)I_2)=0\Rightarrow$

$$(a+1+i)(d+1+i)-bc=0\Leftrightarrow$$
$$ad-bc+(1+i)(a+d)+2i=0\Leftrightarrow$$
$$\begin{cases} ad-bc+a+d=0 \\ a+d+2=0\end{cases}\Rightarrow\begin{cases} a+d=-2 \\ ad-bc=2\end{cases}\Rightarrow (detA)^2=(TrA)^2$$

Similarly, for the other cases.

SP.312. In ΔABC let the point $A'\in(BC)$ such that the incircle of $\Delta AA'B$ and $\Delta AA'C$ have same radius. Prove that:

$$\sqrt[3]{AA'\cdot BB'\cdot CC'}\geq 3r$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

Let: r_A –the radius of circle inscribed $\Delta AA'B$ and $\Delta AA'C$.

$$S = S_{ABA'} + S_{ACA'} = s_{ABA'} \cdot r_A + s_{ACA'} \cdot r_A = r_A \cdot (s_{ABA'} + s_{ACA'}) = r_A \cdot (s + AA'); \ (1)$$

$$\Delta II_1I_2 \sim \Delta IBC \Rightarrow \frac{I_1I_2}{BC} = \frac{r - r_A}{r} = 1 - \frac{r_A}{r} \Rightarrow \frac{r_A}{r} = 1 - \frac{I_1I_2}{a}; \quad (2)$$

Let D, E –the points of intersection with sides BC of inscribed circle I_1I_2ED –rectangle,

then: $I_1I_2 = ED = DA' + A'E = s_{ABA'} - c + s_{ACA'} - b = s - b - c + AA'; \quad (3)$

From (2)+(3) we have: $\frac{r_A}{r} = 1 - \frac{s-b-c-AA'}{a} = \frac{s-AA'}{a} \Rightarrow r_A = \frac{r}{a}(s - AA'); \quad (4)$

From (1)+(4) $\Rightarrow \frac{r}{a}(s - AA')(s + AA') = S \Rightarrow s^2 - AA'^2 = as$

$\Rightarrow AA'^2 = s^2 - sa \Rightarrow AA' = \sqrt{s(s-a)}$ and analogous

$$BB' = \sqrt{s(s-b)}; CC' = \sqrt{s(s-c)}$$

$$AA' \cdot BB' \cdot CC' = s\sqrt{s(s-a)(s-b)(s-c)} = s \cdot S = s^2 r \geq 27r^3$$

$$\sqrt[3]{AA' \cdot BB' \cdot CC'} \geq 3r$$

SP.313. Let $x, y, z > 0$ such that $xyz = 1$.

Find the minimum of the expression:

$$P = 2(x+y+z) + \frac{x}{y^3+z^3+1} + \frac{y}{z^3+x^3+1} + \frac{z}{x^3+y^3+1}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let

$$\Omega = \frac{x}{y^3+z^3+1} + \frac{y}{z^3+x^3+1} + \frac{z}{x^3+y^3+1} \overset{xyz=1}{=} \sum_{cyc} \frac{x}{y^3+z^3+xyz} =$$

$$= \sum_{cyc} \frac{x^2}{xy^3+xz^3+x^2yz} \overset{Bergstrom}{\geq} \frac{(x+y+z)^2}{\sum xy(x^2+y^2) + (x+y+z)xyz} =$$

$$= \frac{(x+y+z)^2}{(x^2+y^2+z^2)(xy+yz+zx)} = \frac{(x+y+z)^2(xy+yz+zx)}{(x^2+y^2+z^2)(xy+yz+zx)^2} \overset{AM-GM}{\geq}$$

$$= \frac{(x+y+z)^2(xy+yz+zx)}{\frac{(x^2+y^2+z^2+2xy+2yz+2zx)^3}{27}} = \frac{27(x+y+z)^2(xy+yz+zx)}{((x+y+z)^2)^3} =$$

$$=\frac{27(xy+yz+zx)}{(x+y+z)^4}\overset{AM-GM}{\geq}\frac{27\cdot 3\sqrt[3]{(xyz)^2}}{(x+y+z)^4}=\frac{27\cdot 3\sqrt[3]{1^2}}{(x+y+z)^4}=\frac{3^4}{(x+y+z)^4}\Rightarrow$$

$$P=2(x+y+z)+\Omega\geq 2(x+y+z)+\left(\frac{3}{x+y+z}\right)^4=$$

$$=2\cdot 3\cdot\frac{x+y+z}{3}+\left(\frac{3}{x+y+z}\right)^4=6t+\frac{1}{t^4};\left(t=\frac{x+y+z}{3}\geq\sqrt[3]{xyz}=1\right)$$

$$\varphi(t)=6t+\frac{1}{t^4}=t+\cdots+t+\frac{1}{t^4}\overset{AM-GM}{\geq}7\sqrt[7]{t^6\cdot\frac{1}{t^4}}=7\cdot\sqrt[7]{t^2}\geq 7\Rightarrow P\geq\varphi(t)\geq 7$$

$$P_{Min}=7\Leftrightarrow\begin{cases}x=y=z\\ xyz=1\end{cases}\Leftrightarrow x=y=z=1.$$

Solution 2 by proposer

By Cauchy-Schwartz inequality, we have:

$$\frac{x}{y^3+z^3+1}+\frac{y}{z^3+x^3+1}+\frac{z}{x^3+y^3+1}=$$

$$=\frac{x}{y^3+z^3+xyz}+\frac{y}{z^3+x^3+xyz}+\frac{z}{x^3+y^3+xyz}=$$

$$=\frac{x^2}{xy^3+xz^3+x^2yz}+\frac{y^2}{yz^3+yx^3+xy^2z}+\frac{z^2}{zx^3+zy^3+xyz^2}\geq$$

$$\geq\frac{(x+y+z)^2}{(xy^3+xz^3+x^2yz)+(yz^3+yx^3+xy^2z)+(zx^3+zy^3+xyz^2)}\Rightarrow$$

$$\frac{x}{y^3+z^3+1}+\frac{y}{z^3+x^3+1}+\frac{z}{x^3+y^3+1}\geq$$

$$\geq\frac{(x+y+z)^2}{xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)+xyz(x+y+z)};\quad (1)$$

Other,

$$\frac{(x+y+z)^2}{xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)+xyz(x+y+z)}=$$

$$=\frac{(x+y+z)^2}{(xy+yz+zx)(x^2+y^2+z^2)}\geq\frac{3}{x^2+y^2+z^2}$$

From (1) we get:

$$\frac{x}{y^3+z^3+1}+\frac{y}{z^3+x^3+1}+\frac{z}{x^3+y^3+1}\geq\frac{3}{x^2+y^2+z^2}\Rightarrow$$

$$P=2(x+y+z)+\frac{x}{y^3+z^3+1}+\frac{y}{z^3+x^3+1}+\frac{z}{x^3+y^3+1}\ge$$

$$\ge 2(x+y+z)+\frac{3}{x^2+y^2+z^2};\ \ (2)$$

By AM-GM inequality, we have:

$$xy+yz+zx\ge 3\sqrt[3]{(xyz)^2}=3\Leftrightarrow 2(xy+yz+zx)\ge 6\Leftrightarrow$$

$$\frac{3}{x^2+y^2+z^2}=\frac{3}{(x+y+z)^2-2(xy+yz+zx)}\ge\frac{3}{(x+y+z)^2-6};\ \ (3)$$

From (2),(3) we get:

$$P\ge 2(x+y+z)+\frac{3}{(x+y+z)^2-6}=2t+\frac{3}{t^2-6};(t=a+b+c>0);(4)$$

Now, we have:

$$2t+\frac{3}{t^2-6}\ge 7\Leftrightarrow\frac{2t^3-12t+3}{t^2-6}\ge 7\Leftrightarrow 2t^3-12t+3\ge 7t^2-42\Leftrightarrow$$

$$2t^3-7t^2-12t+45\ge 0\Leftrightarrow(t-3)(2t^2-t-15)\ge 0\Leftrightarrow(t-3)^2(2t+5)\ge 0(true)$$

From (4),(5) we get:

$P\ge 7\Rightarrow P_{Min}=7$. Equality occurs if

$$\begin{cases} x,y,z>0;xyz=1\\ \dfrac{x}{y^3+z^3+1}=\dfrac{y}{z^3+x^3+1}=\dfrac{z}{x^3+y^3+1}\\ x=y=z\\ t=x+y+z=3\end{cases}\Leftrightarrow\begin{cases}x=y=z>0\\ xyz=1\end{cases}\Leftrightarrow x=y=z=1.$$

SP.314 Let $a,b,c>0$ such that $a+b+c=3$. Prove that:

$$\frac{a^2}{b^4c\cdot\sqrt[3]{4(b^6+1)}}+\frac{b^2}{c^4a\cdot\sqrt[3]{4(c^6+1)}}+\frac{c^2}{a^4b\cdot\sqrt[3]{4(a^6+1)}}\ge\frac{3}{2}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by proposer

$$\frac{a^2}{b^4c\cdot\sqrt[3]{4(b^6+1)}}+\frac{b^2}{c^4a\cdot\sqrt[3]{4(c^6+1)}}+\frac{c^2}{a^4b\cdot\sqrt[3]{4(a^6+1)}}\ge\frac{3}{2};\ \ (1)$$

We have: $b^6+1=(b^2+1)(b^4-b^2+1)=(b^2+1)\left((b^2+1)^2-\left(b\sqrt{3}\right)^2\right)=$

$$=(b^2+1)\left(b^2-b\sqrt{3}+1\right)\left(b^2+b\sqrt{3}+1\right)$$

By AM-GM inequality, we have:

$$\sqrt[3]{4(b^6+1)}=\sqrt[3]{(b^2+1)\left[2\left(2+\sqrt{3}\right)\left(b^2-b\sqrt{3}+1\right)\right]\left[2\left(2-\sqrt{3}\right)\left(b^2+b\sqrt{3}+1\right)\right]}\le$$

$$\le\frac{(b^2+1)+\left[2\left(2+\sqrt{3}\right)\left(b^2-b\sqrt{3}+1\right)\right]+\left[2\left(2-\sqrt{3}\right)\left(b^2+b\sqrt{3}+1\right)\right]}{3}=$$

$$=\frac{9b^2-12b+9}{3}=3b^2-4b+3\Rightarrow$$

$$\sqrt[3]{4(b^6+1)}\le 3b^2-4b+3\Leftrightarrow\frac{1}{\sqrt[3]{4(b^6+1)}}\ge\frac{1}{3b^2-4b+3}\Leftrightarrow$$

$$\frac{a^2}{b^4c\cdot\sqrt[3]{4(b^6+1)}}\ge\frac{a^2}{b^4c(3b^2-4b+3)}$$

Similarly:

$$\frac{b^2}{c^4a\cdot\sqrt[3]{4(c^6+1)}}\ge\frac{b^2}{c^4a(3c^2-4c+3)}$$

$$\frac{c^2}{a^4b\cdot\sqrt[3]{4(a^6+1)}}\ge\frac{c^2}{a^4b(3a^2-4a+3)}$$

Hence,

$$\frac{a^2}{b^4c\cdot\sqrt[3]{4(b^6+1)}}+\frac{b^2}{c^4a\cdot\sqrt[3]{4(c^6+1)}}+\frac{c^2}{a^4b\cdot\sqrt[3]{4(a^6+1)}}\ge$$

$$\ge\frac{a^2}{b^4c(3b^2-4b+3)}+\frac{b^2}{c^4a(3c^2-4c+3)}+\frac{c^2}{a^4b(3a^2-4a+3)};\quad(2)$$

By Cauchy-Schwartz inequality, we have:

$$\frac{a^2}{b^4c(3b^2-4b+3)}+\frac{b^2}{c^4a(3c^2-4c+3)}+\frac{c^2}{a^4b(3a^2-4a+3)}=$$

$$=\frac{\left(\frac{a^2}{b^2}\right)^2}{a^2c(3b^2-4b+3)}+\frac{\left(\frac{b^2}{c^2}\right)^2}{b^2a(3c^2-4c+3)}+\frac{\left(\frac{c^2}{a^2}\right)^2}{c^2b(3a^2-4a+3)}\ge$$

$$\geq \frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{a^2c(3b^2-4b+3)+b^2a(3c^2-4c+3)+c^2b(3a^2-4a+3)}=$$

$$=\frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{3abc(ab+bc+ca)-4abc(a+b+c)+3(ab^2+bc^2+ca^2)}\Rightarrow$$

$$\frac{a^2}{b^4c(3b^2-4b+3)}+\frac{b^2}{c^4a(3c^2-4c+3)}+\frac{c^2}{a^4b(3a^2-4a+3)}\geq$$

$$\geq \frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{3abc(ab+bc+ca)-4abc(a+b+c)+3(ab^2+bc^2+ca^2)};\ (3)$$

By AM-GM inequality, we have:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=\frac{\frac{a^2}{b^2}+\frac{a^2}{b^2}+\frac{b^2}{c^2}}{3}+\frac{\frac{b^2}{c^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}{3}+\frac{\frac{c^2}{a^2}+\frac{c^2}{a^2}+\frac{a^2}{b^2}}{3}\geq$$

$$\geq \frac{3\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{a^2}{b^2}\frac{b^2}{c^2}}}{3}+\frac{3\sqrt[3]{\frac{b^2}{c^2}\cdot\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}}}{3}+\frac{3\sqrt[3]{\frac{c^2}{a^2}\cdot\frac{c^2}{a^2}\cdot\frac{a^2}{b^2}}}{3}=$$

$$=\sqrt[3]{\frac{a^4}{b^2c^2}}+\sqrt[3]{\frac{b^4}{c^2a^2}}+\sqrt[3]{\frac{c^4}{a^2b^2}}\Rightarrow$$

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\sqrt[3]{\frac{a^4}{b^2c^2}}+\sqrt[3]{\frac{b^4}{c^2a^2}}+\sqrt[3]{\frac{c^4}{a^2b^2}};\ (4)$$

Other, $3=a+b+c\geq 3\sqrt[3]{abc}\Leftrightarrow\sqrt[3]{abc}\leq 1\Leftrightarrow\sqrt[3]{(abc)^2}\leq 1;(5)$

From (4),(5) we get:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{a^2+b^2+c^2}{1}\Leftrightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq a^2+b^2+c^2\ (6)$$

From (3),(6) we get:

$$\frac{a^2}{b^4c(3b^2-4b+3)}+\frac{b^2}{c^4a(3c^2-4c+3)}+\frac{c^2}{a^4b(3a^2-4a+3)}\geq$$

$$\geq\frac{(a^2+b^2+c^2)^2}{3abc(ab+bc+ca)-4abc(a+b+c)+3(ab^2+bc^2+ca^2)};\ (7)$$

Now, we have:

$$3abc(ab+bc+ca)\le abc(a+b+c)^2=9abc=3abc(a+b+c)\Rightarrow$$

$$3abc(a+b+c)-4abc(a+b+c)+3(ab^2+bc^2+ca^2)\le$$

$$\le -abc(a+b+c)+3(ab^2+bc^2+ca^2)\Leftrightarrow$$

$$3abc(a+b+c)-4abc(a+b+c)+3(ab^2+bc^2+ca^2)\le$$

$$\le -abc(a+b+c)+(a+b+c)(ab^2+bc^2+ca^2)\Leftrightarrow$$

$$3abc(a+b+c)-4abc(a+b+c)+3(ab^2+bc^2+ca^2)\le$$

$$\le -abc(a+b+c)+(a^2b^2+b^2c^2+c^2a^2)+(ab^3+bc^3+ca^3)+abc(a+b+c)$$

$$3abc(a+b+c)-4abc(a+b+c)+3(ab^2+bc^2+ca^2)\le$$

$$\le (a^2b^2+b^2c^2+c^2a^2)+(ab^3+bc^3+ca^3)\Leftrightarrow$$

$$\frac{(a^2+b^2+c^2)^2}{3abc(ab+bc+ca)-4abc(a+b+c)+3(ab^2+bc^2+ca^2)}\ge$$

$$\ge \frac{(a^2+b^2+c^2)^2}{(a^2b^2+b^2c^2+c^2a^2)+(ab^3+bc^3+ca^3)};\ \ (8)$$

From (7),(8) we get:

$$\frac{a^2}{b^4c(3b^2-4b+3)}+\frac{b^2}{c^4a(3c^2-4c+3)}+\frac{c^2}{a^4b(3a^2-4a+3)}\ge$$

$$\ge \frac{(a^2+b^2+c^2)^2}{(a^2b^2+b^2c^2+c^2a^2)+(ab^3+bc^3+ca^3)};\ \ (9)$$

We will prove:

$$\frac{(a^2+b^2+c^2)^2}{(a^2b^2+b^2c^2+c^2a^2)+(ab^3+bc^3+ca^3)}\ge\frac{3}{2};\ \ (10)$$

$$\Leftrightarrow 2(a^2+b^2+c^2)^2\ge 3(a^2b^2+b^2c^2+c^2a^2)+3(ab^3+bc^3+ca^3)$$

$$\Leftrightarrow 2(a^4+b^4+c^4)+(a^2b^2+b^2c^2+c^2a^2)\ge 3(ab^3+bc^3+ca^3);\ \ (11)$$

By AM-GM inequality, we have:

$$a^2(a^2+c^2)+b^2(b^2+a^2)+c^2(c^2+b^2)\ge a^2\cdot 2ac+b^2\cdot 2bc+c^2\cdot 2cb$$

$$=2(ab^3+bc^3+ca^3)\Leftrightarrow$$

$$\Leftrightarrow (a^4+b^4+c^4)+(a^2b^2+b^2c^2+c^2a^2)\ge 2(ab^3+bc^3+ca^3);\ \ (12)$$

Other,

$$a^4+b^4+c^4=\frac{a^4+b^4+b^4+b^4}{4}+\frac{b^4+c^4+c^4+c^4}{4}+\frac{c^4+a^4+a^4+a^4}{4}\ge$$

$$\geq \frac{4\sqrt[4]{a^4b^{12}}}{4}+\frac{4\sqrt[4]{b^4c^{12}}}{4}+\frac{4\sqrt[4]{c^4a^{12}}}{4}=ab^3+bc^3+ca^3 \Rightarrow$$

$$a^4+b^4+c^4 \geq ab^3+bc^3+ca^3;\ (13)$$

From (12),(13) we get:

$$2(a^4+b^4+c^4)+(a^2b^2+b^2c^2+c^2a^2) \geq 3(ab^3+bc^3+ca^3) \Rightarrow (11) \text{ is true} \Rightarrow (10)$$

is true.

From (9),(10) we get:

$$\frac{a^2}{b^4c(3b^2-4b+3)}+\frac{b^2}{c^4a(3c^2-4c+3)}+\frac{c^2}{a^4b(3a^2-4a+3)} \geq \frac{3}{2} \Rightarrow (1)$$

is true and we get the result.

Equality occurs if $\begin{cases} a,b,c>0; a+b+c=3 \\ a=b=c \end{cases} \Leftrightarrow a=b=c=1.$

SP.315. Find:

$$\Omega=\cos^3\frac{2\pi}{7}\sin^6\frac{2\pi}{7}\sin^6\frac{3\pi}{7}-\cos^3\frac{3\pi}{7}\sin^6\frac{3\pi}{7}\sin^6\frac{\pi}{7}-\cos^3\frac{\pi}{7}\sin^6\frac{\pi}{7}\sin^6\frac{2\pi}{7}$$

Proposed by Pedro Pantoja-Natal-Brazil

Solution by proposer

The polynomial equation $8y^3+4y^2-4y-1=0$ **have roots equal to**

$$\cos\frac{2k\pi}{7}, k=1,2,3,..$$

We will do the transformation $t=-\frac{x}{1-x}=1-\frac{1}{1-x}; x=\cos\frac{2k\pi}{7}, k=1,2,3,\ldots$

Note that $1-\cos\frac{2k\pi}{7}=2\sin^2\frac{k\pi}{7}$**, then the polynomial equation** $8y^3-4y^2-4y+1=0$

have roots equal to $x=-\cos\frac{2k\pi}{7}, =1,2,3,..$

The polynomial equation $8(y-1)^3-4(y-1)^2-4(y-1)+1=0 \Leftrightarrow$

$8y^3-28y^2+28y-7=0$ **have roots equal to** $1-\cos\frac{2k\pi}{7}, k=1,2,3,\ldots$

The polynomial equation $7y^3-28y^2+28y-8=0$ **have roots equal to**

$$\frac{1}{1-\cos\frac{2k\pi}{7}}, k=1,2,3,\dots$$

The polynomial equation $7y^3+28y^2+28y+8=0$ have roots equal to

$$\frac{-1}{1-\cos\frac{2k\pi}{7}}, k=1,2,3,\dots$$

The polynomial equation $7(y-1)^3+28(y-1)^2+28(y-1)+8=0 \Leftrightarrow$

$7y^3+7y^2-7y+1=0$ have roots $b_k=1+\frac{-1}{1-\cos\frac{2k\pi}{7}}, k=1,2,3,\dots$

Therefore $\begin{cases} b_1+b_2+b_3=-1 \\ b_1b_2+b_2b_3+b_3b_1=-1 \\ b_1b_2b_3=-\frac{1}{7} \end{cases}$

$$\Rightarrow b_1^2+b_2^2+b_3^2=(-1)^2-2\cdot(-1)=3 \text{ and}$$

$$b_1^3+b_2^3+b_3^3-3b_1b_2b_3=(b_1+b_2+b_3)\left(b_1^2+b_2^2+b_3^2-b_1b_2-b_2b_3-b_3b_1\right)\Rightarrow$$

$$b_1^3+b_2^3+b_3^3=-\frac{25}{7}$$

Hence, $\frac{\cos^3\frac{2\pi}{7}}{\sin^3\frac{\pi}{7}}+\frac{\cos^3\frac{4\pi}{7}}{\sin^3\frac{2\pi}{7}}+\frac{\cos^3\frac{6\pi}{7}}{\sin^3\frac{3\pi}{7}}=\frac{25}{56}$

Because $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{8}$ we have:

$$\Omega=\cos^3\frac{2\pi}{7}\sin^6\frac{2\pi}{7}\sin^6\frac{3\pi}{7}-\cos^3\frac{3\pi}{7}\sin^6\frac{3\pi}{7}\sin^6\frac{\pi}{7}-\cos^3\frac{\pi}{7}\sin^6\frac{\pi}{7}\sin^6\frac{2\pi}{7}=$$

$$=\frac{1225}{2^{21}}$$

UP.301. If $S_n=\sum_{k=1}^{n}3^{k-1}\cdot\sin^3\frac{\pi}{3^{k+1}}$ and $I=\pi\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx$ then find:

$$\Omega=\lim_{n\to\infty}([I]\cdot S_n)^n;\ [*]-GIF$$

Proposed by Florică Anastase-Romania

Solution 1 by Adrian Popa-Romania

$$\sin3x=3\sin x-4\sin^3x\Rightarrow4\sin^3x=3\sin x-\sin3x\Rightarrow\sin^3x=\frac{3\sin x-\sin3x}{4}$$

$$S_n=\sum_{k=1}^{n}3^{k-1}\cdot\sin^3\frac{\pi}{3^{k+1}}=\frac{1}{4}\cdot\sum_{k=1}^{n}3^{k-1}\left(3\sin\frac{\pi}{3^{k+1}}-\sin\frac{\pi}{3^k}\right)=$$

$$=\frac{1}{4}\cdot\sum_{k=1}^{n}\left(3^k\sin\frac{\pi}{3^{k+1}}-3^{k-1}\sin\frac{\pi}{3^k}\right)=\frac{1}{4}\cdot\left(3^n\sin\frac{\pi}{3^{n+1}}-\sin\frac{\pi}{3}\right)=$$

$$=\frac{1}{4}\cdot\left(\frac{\sin\frac{\pi}{3^{n+1}}}{\frac{1}{3^{n+1}}}\cdot\frac{1}{3}-\frac{\sqrt{3}}{2}\right)=\frac{\pi}{12}\cdot\frac{\sin\frac{\pi}{3^{n+1}}}{\frac{\pi}{3^{n+1}}}-\frac{\sqrt{3}}{8}\underset{n\to\infty}{\longrightarrow}\frac{\pi}{12}-\frac{\sqrt{3}}{8}$$

Now, $\frac{1}{\sqrt{3}}<x<1\Rightarrow\frac{\pi}{6}<\tan^{-1}x<\frac{\pi}{4}\Rightarrow\frac{4}{\pi}<\frac{1}{\tan^{-1}x}<\frac{6}{\pi}\Rightarrow\frac{4x}{\pi}<\frac{x}{\tan^{-1}x}<\frac{6x}{\pi}$

$$\int_{\frac{1}{\sqrt{3}}}^{1}\frac{4x}{\pi}dx<\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx<\int_{\frac{1}{\sqrt{3}}}^{1}\frac{6x}{\pi}dx\Leftrightarrow$$

$$\frac{4}{\pi}\cdot\frac{x^2}{2}\bigg|_{\frac{1}{\sqrt{3}}}^{1}<\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx<\frac{6}{\pi}\cdot\frac{x^2}{2}\bigg|_{\frac{1}{\sqrt{3}}}^{1}\Leftrightarrow$$

$$\frac{2}{\pi}\left(1-\frac{1}{3}\right)<\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx<\frac{3}{\pi}\left(1-\frac{1}{3}\right)\Leftrightarrow$$

$$\frac{4}{3}<I<2\Rightarrow[I]=1$$

$$\Omega=\lim_{n\to\infty}([I]\cdot S_n)^n=\lim_{n\to\infty}\left(\frac{\pi}{12}-\frac{\sqrt{3}}{8}\right)^n=0;\ \frac{\pi}{12}-\frac{\sqrt{3}}{8}<1$$

Solution 2 by Kamel Benaicha-Algiers-Algerie

$$I=\pi\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx$$

Let be the function $f:\left[\frac{1}{\sqrt{3}},1\right]\to\mathbb{R},f(x)=\frac{x}{\tan^{-1}x};f'(x)=\frac{(1+x^2)\tan^{-1}x-x^2}{(1+x^2)(\tan^{-1}x)^2}$

$$g(x)=(1+x^2)\tan^{-1}x-x^2\Rightarrow g'(x)=2x(\tan^{-1}x-1)+1$$

$$\frac{\pi}{6}\leq\tan^{-1}x\Rightarrow\frac{2}{\sqrt{3}}\left(\frac{\pi}{6}-1\right)+1\leq g'(x)\Rightarrow$$

$$g'(x)=1-\frac{2}{\sqrt{3}}+\frac{\pi}{3\sqrt{3}}=\frac{3\sqrt{3}+\pi-6}{3\sqrt{3}}>\frac{3+\frac{9}{2}-6}{3\sqrt{3}}=\frac{1}{2\sqrt{3}}>0$$

$$\therefore g\nearrow\Rightarrow g(x)\geq\frac{\pi}{2}-1>0\Rightarrow f'(x)>0\Rightarrow f\nearrow x\in\left[\frac{1}{\sqrt{3}},1\right]\Rightarrow$$

$$\frac{6}{\pi\sqrt{3}}<f(x)<\frac{4}{\pi}$$

$$\frac{6}{\sqrt{3}}\left(1-\frac{1}{\sqrt{3}}\right)\leq\pi\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\tan^{-1}x}dx\leq 4\left(1-\frac{1}{\sqrt{3}}\right)$$

$$2-\frac{1}{\sqrt{3}}\leq I\leq 4-\frac{4}{\sqrt{3}}\Rightarrow\left[2-\frac{1}{\sqrt{3}}\right]\leq[I]\leq\left[4-\frac{4}{\sqrt{3}}\right]$$

$$-\frac{1}{\sqrt{3}}<-\frac{2}{3}\Rightarrow 4-\frac{4}{\sqrt{3}}<\frac{4}{3}\Rightarrow\left[4-\frac{4}{\sqrt{3}}\right]\leq 1,\left[2-\frac{1}{\sqrt{3}}\right]=1\Rightarrow[I]=1$$

$$\sin^3x=\frac{1}{2}(\sin x-\sin x\cdot\cos 2x)=\frac{1}{2}\left(\sin x-\frac{1}{2}(\sin 3x-\sin x)\right)=$$

$$=\frac{1}{4}(3\sin x-\sin 3x)$$

$$S_n=\sum_{k=1}^{n}3^{k-1}\cdot\sin^3\frac{\pi}{3^{k+1}}=\frac{1}{4}\cdot\sum_{k=1}^{n}3^{k-1}\left(3\sin\frac{\pi}{3^{k+1}}-\sin\frac{\pi}{3^k}\right)=$$

$$=\frac{1}{4}\cdot\sum_{k=1}^{n}\left(3^k\sin\frac{\pi}{3^{k+1}}-3^{k-1}\sin\frac{\pi}{3^k}\right)=\frac{1}{4}\cdot\left(3^n\sin\frac{\pi}{3^{n+1}}-\sin\frac{\pi}{3}\right)=$$

$$=\frac{1}{4}\cdot\left(3^n\sin\frac{\pi}{3^{n+1}}-\frac{\sqrt{3}}{2}\right)$$

$$\Omega=\lim_{n\to\infty}([I]\cdot S_n)^n=\lim_{n\to\infty}(S_n)^n=\lim_{n\to\infty}\left(\frac{1}{4}\cdot\left(3^n\sin\frac{\pi}{3^{n+1}}-\frac{\sqrt{3}}{2}\right)\right)^n=$$

$$=\lim_{n\to\infty}\frac{1}{4^n}\cdot e^{n\cdot\log\left(3^n\sin\frac{\pi}{3^{n+1}}-\frac{\sqrt{3}}{2}\right)}$$

$$\lim_{n\to\infty}\left(3^n\sin\frac{\pi}{3^{n+1}}-\frac{\sqrt{3}}{2}\right)\overset{t=\frac{1}{3^{n+1}}}{=}\frac{1}{3}\cdot\lim_{t\to 0^+}\frac{\sin\pi t}{t}-\frac{\sqrt{3}}{2}=\frac{\pi}{3}-\frac{\sqrt{3}}{2}<1\Rightarrow$$

$$\lim_{n\to\infty} n\cdot log\left(3^n sin\frac{\pi}{3^{n+1}}-\frac{\sqrt{3}}{2}\right)=-\infty$$

So,

$$\Omega=\lim_{n\to\infty}([I]\cdot S_n)^n=\lim_{n\to\infty}\frac{1}{4^n}\cdot e^{n\cdot log\left(3^n sin\frac{\pi}{3^{n+1}}-\frac{\sqrt{3}}{2}\right)}=0$$

UP.302. Let $x,y,z>0$ real numbers such that $x+y+z=\frac{3}{xyz}$

Find the minimum of value expression:

$$Q=(2x^2-xy+2y^2)(2y^2-yz+2z^2)(2z^2-zx+2x^2)$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

For any $x,y,z>0$ we have: $2x^2-xy+2y^2\ge\frac{3(x+y)^2}{4}\Leftrightarrow$

$$4(2x^2-xy+2y^2)\ge 3(x+y)^2\Leftrightarrow 8x^2+8y^2-4xy\ge 3(x^2+2xy+y^2)\Leftrightarrow$$

$$5x^2+5y^2-10xy\ge 0\Leftrightarrow 5(x-y)^2\ge 0\ (true\ \ \forall x,y>0)$$

Similarly: $2y^2-yz+2z^2\ge\frac{3(y+z)^2}{4};\ 2z^2-zx+2x^2\ge\frac{3(z+x)^2}{4}$

$$Q=(2x^2-xy+2y^2)(2y^2-yz+2z^2)(2z^2-zx+2x^2)$$

$$\ge\frac{27}{16}\left((x+y)(y+z)(z+x)\right)^2=P$$

Let $\Omega=(x+y)(y+Z)(z+x)=(x+y+z)(xy+yz+zx)-xyz;$

$$x+y+z=\frac{3}{xyz}\Rightarrow 3=(x+y+z)xyz\overset{AM-GM}{\le}(x+y+z)\cdot\frac{(x+y+z)^3}{27}=$$

$$=\frac{(x+y+z)^4}{27}\Rightarrow 81\le(x+y+z)^4\Rightarrow x+y+z\ge 3$$

$$(xy+yz+zx)^2\ge 3xyz(x+y+z)=3\cdot\frac{3}{x+y+z}\cdot(x+y+z)=9\Rightarrow$$

$xy+yz+zx\ge 3$. So,

$$\Omega=(x+y+z)(xy+yz+zx)-xyz\ge 3(x+y+z)-\frac{3}{x+y+z}\overset{x+y+z\ge 3}{\ge}$$

$$\geq 9-\frac{3}{3}=8\Rightarrow Q\geq P\geq \frac{27}{64}\cdot 8^2=27$$

$$Q_{Min}=27\Leftrightarrow \begin{cases} x=y=z \\ x+y+z=\dfrac{3}{xyz} \end{cases}\Leftrightarrow x=y=z=1$$

Solution 2 by Michael Sterghiou-Greece

$$x+y+z=\frac{3}{xyz};(c)$$

$$Q=(2x^2-xy+2y^2)(2y^2-yz+2z^2)(2z^2-zx+2x^2);\ \ (1)$$

Let $(p,q,r)=(\sum_{cyc}x,\sum_{cyc}xy,\prod_{cyc}x)$ from (c) we can easily to show that

$$p\geq 3,q\geq 3,r\leq 1$$

$$\left[\therefore \frac{3}{r}=p\geq 3\sqrt[3]{r}\Rightarrow r\leq 3,\text{so } p=\frac{3}{r}\geq 3 \text{ and } q^2\geq 3pr=9\Rightarrow q\geq 3\right]$$

As $xy\leq \frac{x^2+y^2}{2}$ and analogs then (1) reduces to the stronger

$$Q\geq \frac{27}{8}\cdot\prod_{cyc}(x^2+y^2)=\frac{27}{8}\cdot\left[\left(\sum_{cyc}x^2\right)\cdot\left(\sum_{cyc}x^2y^2\right)-r^2\right];\ \ (2)$$

But $\sum_{cyc}x^2=p^2-2q,\sum_{cyc}x^2y^2=q^2-2pr\overset{(c)}{=}q^2-6$ and $r^2=\frac{3}{p^2}$

So, (2) reduces to:

$$\frac{27}{8}\cdot\left[(p^2-2q)(q^2-6)-\frac{9}{p^2}\right]=\frac{27}{8}\cdot f(p,q)$$

Now, $f(p,q)\geq 8$ as we will show. This is equivalent to:

$p^2(p^2-2q)(q^2-6)-8p^2\geq 9$ or $p^2[(p^2-2q)(q^2-6)-8]\geq 9$ or as $p^2\geq 9$ to the stronger $(p^2-2q)(q^2-6)-8\geq 1$ which reduces to:

$(q-3)(q^2+3q+3)\geq 0$ which is true.

Equality for $x=y=z=1$ and $Q\geq \frac{27}{8}\cdot f(p,q)\geq \frac{27}{8}\cdot 8=27$. Done!

Solution 3 by proposer

We have:

$$2x^2-xy+2y^2=\frac{5}{4}(x^2-2xy+y^2)+\frac{3}{4}(x+y)^2=\frac{5}{4}(x-y)^2+\frac{3}{4}(x+y)^2$$

$$\geq\frac{3}{4}(x+y)^2\Rightarrow 2x^2-xy+2y^2\geq\frac{3}{4}(x+y)^2$$

Similarly: $2y^2-yz+2z^2\geq\frac{3}{4}(y+z)^2;\ 2z^2-zx+2x^2\geq\frac{3}{4}(z+x)^2$. Hence,

$$Q=(2x^2-xy+2y^2)(2y^2-yz+2z^2)(2z^2-zx+2x^2)\geq$$

$$\geq\frac{3}{4}(x+y)^2\cdot\frac{3}{4}(y+z)^2\cdot\frac{3}{4}(z+x)^2\Rightarrow Q\geq\frac{27}{64}\cdot(x+y)^2(y+z)^2(z+x)^2;\ \ (1)$$

We have $\forall x,y,z>0$

$$x(y-z)^2+y(z-x)^2+z(x-y)^2\geq 0\Leftrightarrow$$

$$x^2y+xy^2+y^2z+yz^2+z^2x+xz^2\geq 6xyz\Leftrightarrow$$

$$9(x^2y+xy^2+y^2z+yz^2+z^2x+xz^2+2xyz)\geq$$

$$\geq 8(x^2y+xy^2+y^2z+yz^2+z^2x+xz^2+3xyz)\Leftrightarrow$$

$$9(x+y)(y+z)(z+x)\geq 8(x+y+z)(xy+yz+zx)\Leftrightarrow$$

$$(x+y)(y+z)(z+x)\geq\frac{8(x+y+z)(xy+yz+zx)}{9};\ \ (2)$$

From (1),(2) we get:

$$Q\geq\frac{27\left(\frac{8(x+y+z)(xy+yz+zx)}{9}\right)^2}{27}=\frac{27\cdot 8^2(x+y+z)^2(xy+yz+zx)^2}{9^2\cdot 64}\Leftrightarrow$$

$$Q\geq\frac{(x+y+z)^2(xy+yz+zx)^2}{3};\ \ (3)$$

Because $x+y+z=\frac{3}{xyz}\Rightarrow 3=(x+y+z)xyz\leq\frac{(xy+yz+zx)^2}{3}\Leftrightarrow$

$$(xy+yz+zx)^2\geq 9\Leftrightarrow xy+yz+zx\geq 3;\ \ (4)$$

$$\left(\text{using } abc(a+b+c)\leq\frac{(ab+bc+ca)^2}{3}\right)$$

$$\Rightarrow(x+y+z)^2\geq 3(xy+yz+zx)\geq 3\cdot 3=9\Rightarrow x+y+z\geq 3;\ \ (5)$$

From (3),(4),(5) we get:

$$Q\geq\frac{(x+y+z)^2(xy+yz+zx)^2}{3}\geq\frac{3^2\cdot 3^2}{3}=27\Rightarrow Q\geq 27\Rightarrow Q_{Min}=27.$$

Equality holds if $\begin{cases} x=y=z>0 \\ x+y+z=\frac{3}{xyz} \end{cases} \Leftrightarrow \begin{cases} x=y=z>0 \\ 3x=\frac{3}{x^3} \end{cases} \Leftrightarrow x=y=z=1.$

Hence, the minimum of value Q is 27 for $x=y=z=1$.

UP. 303. Let be $(I_n)_{n\ge1}, I_n=\int_1^{a^2}\frac{dx}{x(1+\sqrt{x})^n}; a\in\mathbb{R}, a\ge2;$

$\Omega(a)=\lim_{n\to\infty}(1+I_n)\cdot\sum_{k=1}^n\frac{a^k-2^k}{k\cdot(2a)^k}$. Then prove:

$$\frac{a-2}{4a}\le\Omega(a)\le\frac{a-1}{a+1}$$

Proposed by Florică Anastase-Romania

Solution by Adrian Popa-Romania

$$I_n=\int_1^{a^2}\frac{dx}{x(1+\sqrt{x})^n}\overset{t=\sqrt{x}}{=}\int_1^a\frac{2tdt}{t^2(1+t)^n}=2\int_1^a\frac{dt}{t(1+t)^n}\overset{1+t=u}{=}2\int_2^{a+1}\frac{du}{(u-1)u^n}=$$

$$=2\int_2^{a+1}\left(\frac{1}{u-1}-\frac{1}{u}-\frac{1}{u^2}-\cdots-\right)du=2\int_2^{a+1}\left(\frac{1}{u-1}-\frac{1}{u}\cdot\frac{\frac{1}{u^n}-1}{\frac{1}{u}-1}\right)du=$$

$$=2\int_2^{a+1}\left(\frac{1}{u-1}+\frac{1}{u}\cdot\frac{u}{1-u}\right)du=2\int_2^{a+1}\left(\frac{1}{u-1}+\frac{1}{1-u}\right)du=0\Rightarrow I_n\to0$$

$$\sum_{k=1}^n\frac{a^k-2^k}{k\cdot(2a)^k}=\sum_{k=1}^n\left(\frac{1}{k\cdot2^k}-\frac{1}{k\cdot a^k}\right)=\sum_{k=1}^n\left(\frac{\frac{1}{2^k}}{k}-\frac{\frac{1}{a^k}}{k}\right)$$

Let be the sum:

$$1+x+x^2+\cdots+x^{n-1}=\frac{-1}{x-1}=\frac{1}{1-x};(x\in(0,1),x^n\to0)\Big|\int\Leftrightarrow$$

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}=-log(1-x)\Rightarrow\sum_{k=1}^n\frac{x^k}{k}=-log(1-x)$$

Therefore,

$$\sum_{k=1}^{n}\frac{a^k-2^k}{k\cdot(2a)^k}=\sum_{k=1}^{n}\left(\frac{\frac{1}{2^k}}{k}-\frac{\frac{1}{a^k}}{k}\right)=-log\frac{1}{2}+log\left(1-\frac{1}{a}\right)=log\frac{2(a-1)}{a}$$

So, $\Omega(a)=log\frac{2(a-1)}{a}$

We must show: $\frac{a-2}{4a}\le\Omega(a)\le\frac{a-1}{a+1}$

Case 1) Let be the function $f(a)=log\frac{2(a-1)}{a}-\frac{a-2}{4a};a\in[2,\infty)$

$$f'(a)=\frac{a+1}{2a^2(a-1)}>0,\forall a>2\Rightarrow f(a)\nearrow[2,\infty)\Rightarrow f(a)\ge f(2)=0,\forall a\in[2,\infty)$$

$$\Rightarrow log\frac{2(a-1)}{a}\ge\frac{a-2}{4a},\forall a\in[2,\infty);\ \ (1)$$

Case 2) Let be the function $g(a)=log\frac{2(a-1)}{a}-\frac{a-1}{a+1},a\in[2,\infty)$

$$g'(a)=\frac{-a^2+4a+1}{a(a-1)(a+1)^2};g(a)=0\Leftrightarrow a_1=2+\sqrt{5};(a>2);\ g(a_1)<0$$

a	2 $\qquad$ a_1 $\qquad$ ∞
$g'(a)$	$+++++0-----------$
$g(a)$	$-\frac{1}{3}\nearrow\nearrow\nearrow\ g(a_1)\ \searrow\searrow\searrow\quad(log2-1)$

We get, $g(a)<0,,\forall a\in[2,\infty);(2)$

From (1),(2) we get: $\frac{a-2}{4a}\le\Omega(a)\le\frac{a-1}{a+1}$

UP.304. Let $(a_n)_{n\ge1}$ sequence of real numbers such that

$$a_n=\prod_{k=1}^{n}\left(2sin\frac{k\pi}{2n}\right),n\in\mathbb{N},n>0$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2-na_k\cdot a_n}{a_k}\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by Sergio Esteban-Argentina

By complex numbers we can deduce that:

$$\frac{x^{2n}-1}{x^2-1}=\prod_{k=1}^{n}\left(x^2-2\cos\left(\frac{k\pi}{n}x\right)+1\right)\overset{x=1}{\Longrightarrow}$$

$$\lim_{x\to 1}\frac{x^{2n}-1}{x^2-1}=\prod_{k=1}^{n}2\left(1-\cos\left(\frac{k\pi}{n}\right)\right)\to$$

$$n=\prod_{k=1}^{n-1}2^2\sin^2\frac{k\pi}{2n}\to n=2^{2(n-1)}\prod_{k=1}^{n-1}\sin^2\frac{k\pi}{2n}=\left(2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}\right)^2\to$$

$$2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}=\sqrt{n}\leftrightarrow\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}=\frac{\sqrt{n}}{2^{n-1}}$$

Then,

$$(\mathrm{i})x_n=2^n\cdot\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}=2^n\cdot\frac{\sqrt{n}}{2^{n-1}}=2\sqrt{n}$$

$$(\mathrm{ii})\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2-na_k\cdot a_n}{a_k}\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2}{x_k}-\sum_{k=1}^{n}2n\sqrt{n}\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2}{x_k}-2n^2\sqrt{n}\right)$$

(iii) Notice that $\sum_{k=1}^{n}\left(\frac{1}{k}\right)^{\frac{1}{p}}$ represent the lower Darboux sum of $f(x)=x^{-\frac{1}{p}}$ on $[0,n]$, while

$$\frac{p}{p-1}n^{1-\frac{1}{p}}=\int_0^n x^{-\frac{1}{p}}dx\to$$

$\sum_{k=1}^{n}\left(\frac{1}{k}\right)^{\frac{1}{p}}<\frac{p}{p-1}n^{1-\frac{1}{p}}$ **for all integers $1\le n$ and any real number $p>1$.**

Another way to prove is by induction.

So, put $p=2$ and by (ii) $\Omega<\lim_{n\to\infty}\left(2\sqrt{n}-2n^2\sqrt{n}\right)=-\infty$

Solution by proposer

Let: $x_k, k=1,2,..,2n$ the roots of the unity.

$$x_k=\cos\frac{k\pi}{2n}+i\sin\frac{k\pi}{2n}, k=1,2,\dots,2n$$

$$x^{2n}-1=\prod_{k=1}^{2n}(x-x_k)\overset{x_{1,2}=\pm 1-roots}{\triangleq}(x^2-1)\prod_{k=1}^{n-1}(x-x_k)(x-\overline{x_k})$$

$$=(x^2-1)\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right)$$

$$\Rightarrow x^{2n-2}+x^{2n-4}+\dots+x^2+1=\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right)\overset{x=1}{\Longrightarrow}$$

$$n=\prod_{k=1}^{n-1}\left(2-2\cos\frac{k\pi}{n}\right)=\prod_{k=1}^{n-1}\left(4\sin^2\frac{k\pi}{2n}\right)$$

$$n=2^{2(n-1)}\cdot\sin^2\frac{\pi}{2n}\cdot\sin^2\frac{2\pi}{2n}\cdot\ldots\cdot\sin^2\frac{(n-1)\pi}{2n}$$

$$2^{n-1}\cdot\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\ldots\cdot\sin\frac{(n-1)\pi}{2n}=\sqrt{n}\Rightarrow a_n=2\sqrt{n}$$

$$\sum_{k=1}^{n}\frac{2-na_k\cdot a_n}{a_k}=\sum_{k=1}^{n}\frac{2}{a_k}-n^2a_n=\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2n^2\sqrt{n}$$

$$=\left(\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2\sqrt{n}\right)+2\sqrt{n}-2n^2\sqrt{n}$$

$b_n=\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2\sqrt{n}$ –Ioachimescu sequence.

$$b_{n+1}-b_n=-2\sqrt{n+1}+2\sqrt{n}+\frac{1}{2\sqrt{n+1}}<-2(\sqrt{n+1}-\sqrt{n})+2(\sqrt{n+1}-\sqrt{n})=0$$

$\Rightarrow (b_n)_{n\ge 1}$ –decreasing $\Rightarrow b_n<b_1=-1$

From $\sqrt{k+1}-\sqrt{k}=\frac{1}{\sqrt{k+1}+\sqrt{k}}<\frac{1}{2\sqrt{k}}, k>0\Rightarrow$

$$\sqrt{n+1}-1<\frac{1}{2}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$$

$$\Rightarrow -2<-2\sqrt{n+1}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\le -2\sqrt{n}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}}=b_n<b_1=1$$

$$\Rightarrow b_n\in(-2,-1)$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2-na_k\cdot a_n}{a_k}\right)=\lim_{n\to\infty}\left(\left(\underbrace{\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2\sqrt{n}}_{=c}\right)+2\sqrt{n}(1-n^2)\right)=-\infty$$

UP.305. Let $(a_n)_{n\geq 1}, a_1=e, a_n=e^n a_{n-1}^n$ and $(b_n)_{n\geq 1}$ such that:

$$\left(1+\frac{1}{n}\right)^{n+b_n}=\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right)$$

Find: $\Omega=\lim_{n\to\infty} b_n$

Proposed by Florică Anastase-Romania

Solution 1 by Sergio Esteban-Argentina

Noticed that: $log(a_n)=n(log(a_{n-1})+1)$

$$u_2=loga_2=2(1+1)=2!\left(\frac{1}{0!}+\frac{1}{1!}\right)=4$$

$$u_3=loga_3=3(4+1)=3!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}\right)=15$$

$$u_4=loga_4=3(15+1)=3!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\right)=64$$

$$\vdots$$

$$u_n=loga_n=n!\cdot\sum_{k=0}^{n-1}\frac{1}{k!}$$

Now, if $q_n=\sum_{k=0}^{n-1}\frac{1}{k!}\Rightarrow\frac{q_n}{q_{n+1}}=1+\frac{1}{n!q_{n-1}}$ but $n!\,q_{n-1}=u_n\Rightarrow\frac{q_n}{q_{n+1}}=1+\frac{1}{u_n}\Rightarrow$

$$q_n=q_0\prod_{k=1}^{n}\frac{q_k}{q_{k-1}}=\prod_{k=1}^{n}\left(1+\frac{1}{u_k}\right)$$

Finally,

$$\left(1+\frac{1}{n}\right)^{n+b_n}=\sum_{k=0}^{n}\frac{1}{k!}\Rightarrow b_n=\frac{log\left(\sum_{k=0}^{n}\frac{1}{k!}\right)}{log\left(1+\frac{1}{n}\right)}-n$$

$$\Omega=\lim_{n\to\infty}b_n=\frac{1}{2}$$

Solution 2 by Adrian Popa-Romania

$$a_1=e;\ a_n=(ea_{n-1})^n\Rightarrow loga_n=log(ea_{n-1})^n=nlog(ea_{n-1})\Rightarrow$$

$$1+\frac{1}{loga_k}=\frac{1+loga_k}{loga_k}=\frac{loge+loga_k}{loga_k}=\frac{log(ea_k)}{loga_k}\Rightarrow$$

$$\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right)=\prod_{k=1}^{n}\frac{log(ea_k)}{loga_k}=\frac{log(ea_n)}{n!}$$

$$a_1=e;a_2=e^2\cdot e^2=e^{2+2\cdot1};a_3=e^{3+2\cdot3+1\cdot2\cdot3}$$

Applying Mathematical Induction we get:

$$a_n=e^{n+(n-1)n+(n-2)(n-1)n+\cdots+1\cdot2\cdot3\cdot\ldots\cdot n}=e^{n!+\frac{n!}{1!}+\frac{n!}{2!}+\cdots+\frac{n!}{(n-2)!}+\frac{n!}{(n-1)!}}=e^{n!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}\right)}$$

$$\frac{log(ea_n)}{n!}=\frac{log\left(e\cdot e^{n!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}\right)}\right)}{n!}=\frac{loge+log\left(e^{n!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}\right)}\right)}{n!}=$$

$$=\frac{1+n!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}\right)}{n!}=\frac{1}{n!}+\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}\right)=$$

$$=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}+\frac{1}{n!}$$

$$\left(1+\frac{1}{n}\right)^{n+b_n}=\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right)\Rightarrow$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+b_n}=\lim_{n\to\infty}\left(\underbrace{\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}+\frac{1}{n!}}_{e}\right)\overset{\circ log}{\Longrightarrow}$$

$$(n+b_n)log\left(1+\frac{1}{n}\right)=1\Rightarrow b_n=\frac{1}{log\left(1+\frac{1}{n}\right)}-n$$

$$\Omega=\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1-nlog\left(1+\frac{1}{n}\right)}{log\left(1+\frac{1}{n}\right)}=\lim_{x\to\infty}\frac{1-xlog\left(1+\frac{1}{x}\right)}{log\left(1+\frac{1}{x}\right)}=$$

$$\overset{L'H}{=}\lim_{x\to\infty}\frac{-log\left(1+\frac{1}{x}\right)+\frac{1}{x+1}}{-\frac{1}{x(x+1)}}\overset{L'H}{=}\lim_{x\to\infty}\frac{1}{x(x+1)^2}\cdot\frac{x^2(x+1)^2}{2x+1}=\frac{1}{2}$$

Finally,

$$\Omega=\lim_{n\to\infty}b_n=\frac{1}{2}$$

Solution by proposer

$$a_n=e^n a_{n-1}^n \Leftrightarrow loga_n=n+nloga_{n-1}=n(1+loga_{n-1})$$

$$\textit{Let: } x_n=loga_n;\ x_1=1\Rightarrow x_n=n(1+x_{n-1}),x_1=1$$

$$1+x_k=k(1+x_{k-1})+1\Rightarrow\frac{1+x_k}{k!}-\frac{1+x_{k-1}}{(k-1)!}=\frac{1}{k!}\Rightarrow$$

$$\frac{1+x_n}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}=E_n$$

$$\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right)=\prod_{k=1}^{n}\left(1+\frac{1}{x_k}\right)=\prod_{k=1}^{n}\left(\frac{1}{k+1}\cdot\frac{x_{k+1}}{x_k}\right)=\frac{x_{n+1}}{(n+1)!}=\frac{1+x_n}{n!}$$

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{loga_k}\right)=\lim_{n\to\infty}\frac{1+x_n}{n!}=e$$

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)}{log\left(1+\frac{1}{n}\right)}-n\right)$$

$$=\lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)-1}{log\left(1+\frac{1}{n}\right)}+\frac{1}{log\left(1+\frac{1}{n}\right)}-n\right);\quad(1)$$

$$\lim_{n\to\infty}\left(\frac{log\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)-1}{log\left(1+\frac{1}{n}\right)}\right)\overset{L.C-S}{=}\lim_{n\to\infty}\frac{log\left(1+\frac{\frac{1}{(n+1)!}}{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}\right)}{log\left(1-\frac{1}{(n+1)^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\log\left(1+\frac{\frac{1}{(n+1)!}}{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}\right)^{n^2}}{\log\left(1-\frac{1}{(n+1)^2}\right)^{n^2}} = 0;\ \ (2)$$

$$\lim_{n\to\infty}\left(\frac{1}{\log\left(1+\frac{1}{n}\right)} - n\right) = \lim_{x\to 0}\left(\frac{1}{\log(1+x)} - \frac{1}{x}\right) = \frac{1}{2}; (3)$$

From (1)+(2)+(3) we have:

$$\Omega = \lim_{n\to\infty} b_n = \frac{1}{2}$$

UP.306. Let $(x_n)_{n\ge 1}, x_0 = 2, x_n = n(x_{n-1} - (n-1)! - 2) - 2$

Find:

$$\Omega = \lim_{n\to\infty} \frac{\sqrt[n]{1+2n-\sum_{k=1}^{n} x_k}}{n}$$

Proposed by Florică Anastase-Romania

Solution 1 by Samir HajAli-Damascus-Syria

$$x_0 = 2, x_n = n(x_{n-1} - (n-1)! - 2) - 2$$

$$x_1 = x_0 - 1 - 2 - 2 = -3$$

$$x_2 = 2(x_1 - 1 - 2) - 2 = -14$$

$$x_3 = 3(x_2 - 2 - 2) - 2 = -56$$

Thus $x_n \underset{n\to\infty}{\longrightarrow} -\infty$ then $-x_n > 0, \forall n \ge 1$ therefore,

$$1 + 2n - \sum_{k=1}^{n} x_k > 0$$

$$\Omega = \lim_{n\to\infty} \frac{\sqrt[n]{1+2n-\sum_{k=1}^{n} x_k}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{1+2n-\sum_{k=1}^{n} x_k}{n^n}} =$$

$$\overset{C-D'A}{=} \lim_{n\to\infty} \frac{1+2(n+1)-\sum_{k=1}^{n+1} x_k}{(n+1)^{n+1}} \cdot \frac{n^n}{1+2n-\sum_{k=1}^{n} x_k} =$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \cdot \frac{1+2(n+1)-\sum_{k=1}^{n+1} x_k}{(n+1)\left(1+2n-\sum_{k=1}^{n} x_k\right)} =$$

$$= \frac{1}{e}\cdot \lim_{n\to\infty}\frac{1}{n+1}\cdot \underbrace{\frac{1+2(n+1)-\sum_{k=1}^{n+1} x_k}{1+2n-\sum_{k=1}^{n} x_k}}_{\Omega_n}$$

$$\lim_{n\to\infty}\Omega_n = \lim_{n\to\infty}\frac{1+2(n+1)-\sum_{k=1}^{n+1} x_k}{1+2n-\sum_{k=1}^{n} x_k} \overset{LC-S}{=} \lim_{n\to\infty}\frac{2-x_{n+1}}{2-x_n} =$$

$$= \lim_{n\to\infty}\frac{2-(n+1)(x_n-n!-2)+2}{2-x_n} = \lim_{n\to\infty}\frac{2-x_n+n(2-x_n)+4+(n+1)!}{2-x_n} =$$

$$= \lim_{n\to\infty}\left(n+1+\frac{4}{2-x_n}+\frac{(n+1)!}{2-x_n}\right) \text{ then:}$$

$$\Omega = \frac{1}{e}\cdot \lim_{n\to\infty}\frac{\Omega_n}{n+1} = \frac{1}{e}\cdot \lim_{n\to\infty}\left(1+\underbrace{\frac{4}{(n+1)(2-x_n)}}_{\to 0}+\frac{n!}{2-x_n}\right)$$

Let put: $y_n = \frac{n!}{2-x_n}$, clearly $y_n > 0, \forall n \geq 1$ and let prove that:

$$-x_n > (n+1)!$$

For $n = 1$: $-x_1 = 3 > 2! = 2$

Suppose: $-x_n > (n+1)!$ then

$$-x_{n+1} = -(n+1)(x_n-n!-2)+2 = -(n+1)x_n+(n+1)!+2(n+1)+2 >$$

$$> (n+1)(n+1)!+(n+1)!+2(n+1)+2 > (n+2)!$$

Then: $y_n = \frac{n!}{2-x_n} \leq \frac{n!}{-x_n} < \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$$0 \leq \lim_{n\to\infty} y_n \leq \lim_{n\to\infty}\frac{1}{n+1} = 0 \Rightarrow y_n \to 0$$

Hence,

$$\Omega = \frac{1}{e}\cdot \lim_{n\to\infty}\frac{\Omega_n}{n+1} = \frac{1}{e}\cdot \lim_{n\to\infty}\left(1+\underbrace{\frac{4}{(n+1)(2-x_n)}}_{\to 0}+y_n\right) = \frac{1}{e}$$

Solution 2 by Remus Florin Stanca-Romania

$$x_n = n(x_{n-1}-(n-1)!-2)-2 \,|\cdot \frac{1}{n!} \Rightarrow \frac{x_n}{n!} = \frac{x_{n-1}}{(n-1)!} - 1 - \frac{2}{(n-1)!} - \frac{2}{n!}$$

$$\sum_{k=1}^{n}\left(\frac{x_k}{k!}-\frac{x_{k-1}}{(k-1)!}\right)=\sum_{k=1}^{n}\left(-1-\frac{2}{(k-1)!}-\frac{2}{k!}\right)=-n-2\sum_{k=1}^{n}\frac{1}{(k-1)!}-2\sum_{k=1}^{n}\frac{1}{k!}\Rightarrow$$

$$\frac{x_n}{n!}-2=-n-2-2\sum_{k=1}^{n}\frac{1}{k!}-2\sum_{k=1}^{n}\frac{1}{k!}\Rightarrow\frac{x_n}{n!}=-n-4\sum_{k=1}^{n}\frac{1}{k!}\Rightarrow$$

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{1+2n-\sum_{k=1}^{n}x_k}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{1+2n-\sum_{k=1}^{n}x_k}{n^n}}=$$

$$\overset{C-D'A}{=}\lim_{n\to\infty}\frac{1+2(n+1)-\sum_{k=1}^{n+1}x_k}{(n+1)^{n+1}}\cdot\frac{n^n}{1+2n-\sum_{k=1}^{n}x_k}=$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{1}{n}\left(1+\frac{2-x_{n+1}}{2n+1-\sum_{k=1}^{n}x_k}\right)=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{1}{n}\cdot\frac{2-x_{n+1}}{2n+1-\sum_{k=1}^{n}x_k}\overset{L.C-S}{=}$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{2x_{n+2}-x_{n+3}-x_{n+1}}{4n-(n+1)x_{n+2}+nx_{n+1}-x_{n+2}}=$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{2x_{n+2}-x_{n+3}-x_{n+1}}{4n-(n+2)x_{n+2}+x_{n+2}}=$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{2x_{n+2}-(n+3)x_{n+2}+(n+3)!+2n+8-\frac{x_{n+2}+2}{n+2}-(n+1)!-2}{4n-(n+2)x_{n+2}+\frac{n}{n+2}(x_{n+2}+2)+(n+1)!+2n}=$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{(-n^2-3n-3)x_{n+2}+(n+2)(n+3)!+(2n+6)(n+2)-(n+2)!}{6n^2+14n+(n+2)!+(-n^3-3n-4)x_{n+2}}=$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{\frac{(-n^2-3n-3)x_{n+2}}{(n+2)!}+(n+2)(n+3)+\frac{(2n+6)(n+2)}{(n+2)!}-1}{\frac{6n^2+14n}{(n+2)!}+1+\frac{(-n^3-3n-4)x_{n+2}}{(n+2)!}};\quad(1)$$

Let: $\frac{x_{n+2}}{(n+2)!}=-n-2-4\sum_{k=1}^{n}\frac{1}{k!}=x\Rightarrow\lim_{n\to\infty}x=-\infty\overset{(1)}{\Rightarrow}$

$$\Omega=\lim_{n\to\infty}\frac{\frac{(n+2)(n+3)}{x(-n^2-3n-3)}+\frac{(2n+6)(n+2)}{(n+2)!\,x(-n^2-3n-3)}+\frac{1}{x(-n^2-3n-3)}}{\frac{-n^2-3n-4}{-n^2-3n-3}+\frac{1}{x(-n^2-3n-3)}+\frac{6n^2+14n}{x(-n^2-3n-3)(n+2)!}}=\frac{1}{e}$$

UP.307. Let $(a_n)_{n\ge1}, a_n \in (0,\infty)$ be sequence of real numbers such that $a_1 = \sqrt{a}, a > 0, a_{n+1}^2 = n\cdot a_n + 1$ then find:

$$\Omega = \lim_{n\to\infty} \frac{a_n}{n^3}\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}, n\in\mathbb{N}, n\ge 2$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$a_1 = \sqrt{a}, a > 0;\ a_{n+1}^2 = n\cdot a_n + 1$$

Let: $a_{n+1}^2 = x_{n+1} > 0 \Rightarrow x_1 = a > 0, a_{n+1} = \sqrt{x_{n+1}};$

$$x_{n+1} = n\cdot\sqrt{x_n} + 1$$

How $x_{n+1} = n\cdot\sqrt{x_n} + 1 > 1 \Rightarrow x_n > 1, \forall n \ge 2, n\in\mathbb{N}$ then

$$n\cdot\sqrt{x_n} < x_{n+1} < (n+1)\sqrt{x_n}, \forall n \ge 2 \Leftrightarrow$$

$$\log n + \frac{1}{2}\log x_n < \log x_{n+1} < \log(n+1) + \frac{1}{2}\log x_n \Leftrightarrow$$

$$2^{n+1}\log n + 2^n\log x_n < 2^{n+1}\log x_{n+1} < 2^{n+1}\log(n+1) + 2^n\log x_n$$

$$2^{n+1}\log n < 2^{n+1}\log x_{n+1} - 2^n\log x_n < 2^{n+1}\log(n+1)$$

Let: $y_n = 2^n\log x_n \Rightarrow 2^{n+1}\log n < y_{n+1} - y_n < 2^{n+1}\log(n+1)$ and summing, we get:

$$\sum_{k=3}^n 2^k\log(k-1) < y_n - y_2 < \sum_{k=3}^n 2^k\log k$$

$$\frac{y_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^n 2^k\log(k-1) - 2\log n < \log\left(\frac{x_n}{n^2}\right) < \frac{y_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^n 2^k\log k - 2\log n$$

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^n 2^k\log(k-1) - 2\log n\right) = \lim_{n\to\infty}\frac{1}{2^n}\left(\sum_{k=3}^n 2^k\log k - 2^{n+1}\log n\right) \overset{L.C-Stolz}{=}$$

$$= \lim_{n\to\infty}\frac{2^{n+1}\log n - 2^{n+2}\log(n+1) + 2^{n+1}\log n}{2^{n+1} - 2^n} = 4\lim_{n\to\infty}\log\left(\frac{n+1}{n}\right) = 0; (1)$$

Analogously,

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^n 2^k\log k - 2\log n\right) = 0; (2)$$

From (1),(2) we get: $\lim_{n\to\infty} log\left(\frac{x_n}{n^2}\right) = 0 \Rightarrow \lim_{n\to\infty}\left(\frac{x_n}{n^2}\right) = \lim_{n\to\infty}\left(\frac{a_n^2}{n^2}\right) = 1 \Rightarrow \lim_{n\to\infty}\left(\frac{a_n}{n}\right) = 1$

Now,

$$\frac{x^{2n}+1}{x^n+1} > \sqrt{2}-1 \Leftrightarrow x^{2n} - (\sqrt{2}-1)x^n + 2 - \sqrt{2} > 0; t = x^n > 0 \Rightarrow$$

$$t^2 - (\sqrt{2}-1)t + 2 - \sqrt{2} > 0, \Delta_t = -5 + 2\sqrt{2} < 0 \Rightarrow$$

$$\sqrt{2} - 1 < \int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}\,dx; \ \ (3)$$

$$\sqrt[n]{\frac{x^{2n}+1}{x^n+1}} = \sqrt[n]{(x^{2n}+1)\cdot\underbrace{1\cdot 1\ldots.1}_{(n-2)}\cdot\frac{1}{x^n+1}} \overset{AM-GM}{\le} \frac{(x^{2n}+1)+n-2+\frac{1}{x^n+1}}{n} \Leftrightarrow$$

$$\sqrt[n]{\frac{x^{2n}+1}{x^n+1}} \le \frac{1}{n}\left(x^{2n}+n-1+\frac{1}{x^n+1}\right)$$

$$\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}\,dx \le \frac{1}{n}\int_0^1 x^{2n}\,dx + \frac{n-1}{n} + \frac{1}{n}\underbrace{\int_0^1 \frac{1}{x^n+1}}_{\le 1} \le \frac{1}{n(2n+1)} + 1; (4)$$

From (3),(4) we have:

$$\sqrt{2}-1 < \int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}\,dx < \frac{1}{n(2n+1)} + 1 \Leftrightarrow$$

$$(\sqrt{2}-1)\frac{a_n}{n}\cdot\frac{1}{n^2} < \frac{a_n}{n^3}\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}\,dx < \frac{a_n}{n}\cdot\frac{1}{n^2}\left(\frac{1}{n(2n+1)}+1\right)$$

$$\text{So}, \Omega = \lim_{n\to\infty}\frac{a_n}{n^3}\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}} = 0$$

Solution 2 by proposer

$$a_1 = \sqrt{a}, a > 0;\ a_{n+1}^2 = n\cdot a_n + 1$$

Let: $a_{n+1}^2 = x_{n+1} > 0 \Rightarrow x_1 = a > 0, a_{n+1} = \sqrt{x_{n+1}};$

$$x_{n+1} = n\cdot\sqrt{x_n} + 1$$

How $x_{n+1} = n\cdot\sqrt{x_n} + 1 > 1 \Rightarrow x_n > 1, \forall n \ge 2, n \in \mathbb{N}$ then

$$n\cdot\sqrt{x_n} < x_{n+1} < (n+1)\sqrt{x_n}, \forall n \ge 2 \Leftrightarrow$$

$$log\, n+\frac{1}{2}logx_n<logx_{n+1}<log(n+1)+\frac{1}{2}logx_n \Leftrightarrow$$

$$2^{n+1}log\, n+2^n logx_n<2^{n+1}logx_{n+1}<2^{n+1}log(n+1)+2^n logx_n$$

$$2^{n+1}log\, n<2^{n+1}logx_{n+1}-2^n logx_n<2^{n+1}log(n+1)$$

Let: $y_n=\ 2^n logx_n \Rightarrow 2^{n+1}log\, n<y_{n+1}-y_n<2^{n+1}log(n+1)$ and summing, we get:

$$\sum_{k=3}^{n}2^k log(k-1)<y_n-y_2<\sum_{k=3}^{n}2^k log\, k$$

$$\frac{y_2}{2^n}+\frac{1}{2^n}\sum_{k=3}^{n}2^k log(k-1)-2log\, n<log\left(\frac{x_n}{n^2}\right)<\frac{y_2}{2^n}+\frac{1}{2^n}\sum_{k=3}^{n}2^k log\, k-2log\, n$$

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n}2^k log(k-1)-2log\, n\right)=\lim_{n\to\infty}\frac{1}{2^n}\left(\sum_{k=3}^{n}2^k log\, k-2^{n+1}log\, n\right)\overset{L.C-Stolz}{=}$$

$$=\lim_{n\to\infty}\frac{2^{n+1}log\, n-2^{n+2}log(n+1)+2^{n+1}log\, n}{2^{n+1}-2^n}=4\lim_{n\to\infty}log\left(\frac{n+1}{n}\right)=0;(1)$$

Analogously,

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n}2^k log\, k-2log\, n\right)=0;(2)$$

From (1),(2) we get: $\lim\limits_{n\to\infty}log\left(\frac{x_n}{n^2}\right)=0\Rightarrow\lim\limits_{n\to\infty}\left(\frac{x_n}{n^2}\right)=\lim\limits_{n\to\infty}\left(\frac{a_n^2}{n^2}\right)=1\Rightarrow\lim\limits_{n\to\infty}\left(\frac{a_n}{n}\right)=1$

Now, let be the function:

$$f:[0,1]\to\mathbb{R}, f(x)=\frac{x^{2n}+1}{x^n+1}; f'(x)=\frac{nx^{n-1}(x^{2n}+2x^n-1)}{(x^n+1)^2}$$

x	0	$\sqrt[n]{\sqrt{2}-1}$	1
$f'(x)$	$-\ -\ -\ -\ -\ -\ -\ -$	0	$+\ +\ +\ +\ +\ +\ +$
$f(x)$	1 ↘	$2(\sqrt{2}-1)$	↗ 1

We have: $2(\sqrt{2}-1)\le f(x)\le 1;\forall x\in[0,1]\Rightarrow$

$$\sqrt[n]{2(\sqrt{2}-1)}<\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}<1;\forall x\in[0,1]\Rightarrow$$

$$\frac{a_n}{n}\cdot\frac{1}{n^2}\sqrt[n]{2\left(\sqrt{2}-1\right)}<\frac{a_n}{n^3}\cdot\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}<\frac{a_n}{n}\cdot\frac{1}{n^2};\forall x\in[0,1]\Rightarrow$$

$$\frac{a_n}{n}\cdot\frac{1}{n^2}\sqrt[n]{\frac{2}{\sqrt{2}+1}}<\frac{a_n}{n^3}\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}<\frac{a_n}{n}\cdot\frac{1}{n^2};\forall x\in[0,1]$$

$$\text{So},\Omega=\lim_{n\to\infty}\frac{a_n}{n^3}\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}=0$$

UP.308. Find:

$$\Omega=\lim_{n\to\infty}\frac{log\left(\sum_{k=0}^{n}(n+k)\binom{n+k}{k}\right)}{\sqrt[n]{n!}}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Daniel Văcaru-Romania

We have:

$$\sum_{k=0}^{n}(n+k)\binom{n+k}{k}=\frac{n(2n+3)\binom{2n+1}{n}}{n+2}$$

$$\Omega=\lim_{n\to\infty}\frac{log\left(\sum_{k=0}^{n}(n+k)\binom{n+k}{k}\right)}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{log\left(\frac{n(2n+3)\binom{2n+1}{n}}{n+2}\right)}{\sqrt[n]{n!}}\overset{S-Cesaro}{=}$$

$$=\lim_{n\to\infty}\frac{log\left(\frac{(n+1)(2n+5)\binom{2n+3}{n+1}}{n+3}\right)-log\left(\frac{n(2n+3)\binom{2n+1}{n}}{n+2}\right)}{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}=$$

$$=\lim_{n\to\infty}\frac{log\left(\frac{(n+1)(n+2)(2n+5)\binom{2n+3}{n+1}}{n(n+3)(2n+3)\binom{2n+1}{n}}\right)}{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}$$

We have:

$$\lim_{n\to\infty}\frac{(n+1)(n+2)(2n+5)\binom{2n+3}{n+1}}{n(n+3)(2n+3)\binom{2n+1}{n}}=\lim_{n\to\infty}\frac{\binom{2n+3}{n+1}}{\binom{2n+1}{n}}=$$

$$= \lim_{n\to\infty} \frac{(2n+3)!\,n!\,(n+1)!}{(2n+1)!\,(n+1)!\,(n+2)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+3)}{(n+1)(n+2)} = 4$$

On the other hand,

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{(n+1)-n} \overset{S-Cesaro}{=}$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{(n+1)!\,n^n}{n!\,(n+1)^{n+1}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

We obtain: $\Omega = \frac{log4}{\frac{1}{e}} = elog4 = log4^e$

Solution 2 by Sergio Esteban-Argentina

First, we calculate:

$$\omega_n = \sum_{k=0}^{n} (n+k)\binom{n+k}{k} = n\sum_{k=0}^{n}\binom{n+k}{k} + \sum_{k=0}^{n} k\binom{n+k}{k}$$

We will use the following fundamental properties of the binomial coefficients:

i) $k\binom{n}{k} = (n-k+1)\binom{n}{k-1}$

ii) $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$

Then, by (ii) and (i) we get:

$$\omega_n = n\binom{2n+1}{n} + \sum_{k=0}^{n} k\binom{n+k}{k} = n\binom{2n+1}{n} + (n+1)\sum_{k=0}^{n}\binom{n+k}{k-1} =$$

$$= n\binom{2n+1}{n} + (n+1)\binom{2n+1}{n-1} = \frac{(2n+1)!}{(n-1)!\,(n+1)!}\left(1 + \frac{n+1}{n+2}\right)$$

Now,

$$\Omega = \lim_{n\to\infty} \frac{log\left(\sum_{k=0}^{n}(n+k)\binom{n+k}{k}\right)}{\sqrt[n]{n!}} = \lim_{n\to\infty} \frac{log\left(\frac{2(2n+1)!}{(n-1)!\,(n+1)!}\right)}{\sqrt[n]{n!}}$$

By Stirling's approximation:

$$\lim_{n\to\infty} \frac{log\left(\frac{2(2n+1)!}{(n-1)!\,(n+1)!}\right)}{\sqrt[n]{n!}} =$$

$$=\lim_{n\to\infty}\frac{log2+(2n+1)log(2n+1)-(2n+1)-(n-1)log(n-1)+(n-1)-(n+1)log(n+1)+(n+1)}{\frac{n}{e}}$$

$$=\lim_{n\to\infty}\frac{e}{n}log\left(\frac{(2n+1)^{2n+1}}{(n-1)^{n-1}(n+1)^{n+1}}\right)=$$

$$=\lim_{n\to\infty}\frac{e}{n}log\left(\frac{(2n+1)^2}{(n-1)(n+1)}\sqrt[n]{\frac{(2n+1)(n-1)}{n+1}}\right)^n=elog4$$

$$\Omega=elog4=log4^e$$

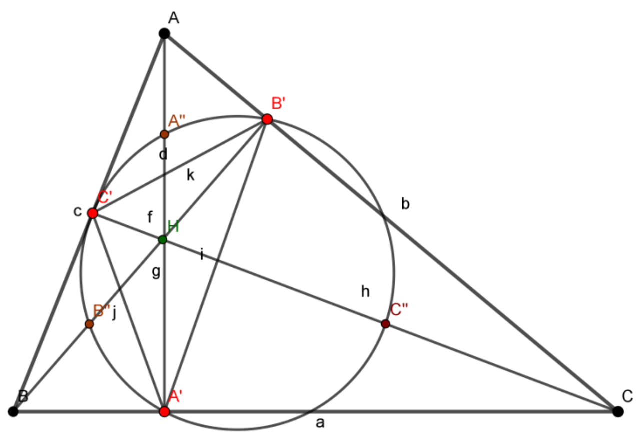
UP.309.In acute ΔABC the altitudes AA', BB', CC' intersect at all second times the determined circle by the points A', B', C' in A",B", C".

Prove that:

$$(2r)^{2s}\geq(A'A")^a\cdot(B'B")^b\cdot(C'C")^c$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Adrian Popa-Romania



$$A"\in(AH), B"\in(BH), C"\in(CH); AA"=A"H; BB"=B"H; CC"=C"H$$

$$(2r)^{2s}\geq(A'A")^a\cdot(B'B")^b\cdot(C'C")^c$$

$$A'A=A'H+HA=A'H+\frac{AH}{2}$$

$$\Delta AC'H\ (\hat{C}=90^\circ): cos\widehat{C'AH}=\frac{AC'}{AH}\Rightarrow AH=\frac{AC'}{cos(90^\circ-B)}=\frac{AC'}{sinB}$$

$$\Delta AC'C\ (\hat{C'}=90^\circ): cosA=\frac{AC'}{b}\Rightarrow AC'=bcosA=2RsinBcosA\Rightarrow$$

$$AH=\frac{2RsinBcosA}{sinB}=2RcosA\Rightarrow HA''=RcosA\ (and\ analogous)$$

$$HB''=RcosB; HC''=RcosC$$

$$S_{BHC}=\frac{BH\cdot HC\cdot sinH}{2}=\frac{BH\cdot CH\cdot sinA}{2}=\frac{2RcosB\cdot 2RcosC\cdot sinA}{2}=$$

$$=2R^2cosBcosCsinA; (1)$$

$$(\therefore \hat{H}=\widehat{BHA'}+\widehat{A'HC}=90^\circ-(90^\circ-\hat{C})+90^\circ-(90^\circ-\hat{B})=\hat{B}+\hat{C}=90^\circ-\hat{A})$$

$$S_{BHC}=\frac{HA'\cdot a}{2}=\frac{2RsinA\cdot HA'}{2}; (2)$$

From $(1),(2)\Rightarrow 2R^2cosBcosCsinA=RsinA\cdot HA'\Rightarrow HA'=2RcosBcosC$

So, $A'A''=R(cosA+cosBcosC)$

$$(A'A'')^a\cdot(B'B'')^b\cdot(C'C'')^c\overset{AM-GM}{\leq}\left(\frac{a\cdot A'A''+b\cdot B'B''+c\cdot C'C''}{a+b+c}\right)^{a+b+c}\overset{(1)}{\leq}(2r)^{2s}$$

We must show that:

$$\frac{a\cdot A'A''+b\cdot B'B''+c\cdot C'C''}{2s}\leq 2r$$

$$a\cdot A'A''=2RsinA\cdot R(cosA+cosBcosC)=2R^2(sinAcosA+sinAcosBcosC)$$

$$\therefore sinAcosBcosC=sinA\cdot\frac{cos(B+C)+cos(B-C)}{2}=$$

$$=\frac{sin(B+C)cos(B+C)+sin(B+C)cos(B-C)}{2}=$$

$$=\frac{\frac{sin2(B+C)}{2}+\frac{sin2B+sin2C}{2}}{2}=\frac{-sinA+sin2B+sin2C}{4}$$

$$a\cdot A'A''+b\cdot B'B''+c\cdot C'C''=R^2\left(\sum_{cyc}sin2A+\frac{-sinA+sin2B+sin2C}{4}\right)=$$

$$=\frac{3}{2}R^2\sum_{cyc}sin2A; (3)$$

$$\sum_{cyc} sin2A = 2sin(A+B)cos(A-B) + 2sinCcosC =$$

$$= 2sinC\big(cos(A-B) - cos(A+B)\big) = 2sinC \cdot 2sinAsinB =$$

$$= 4sinAsinBsinC = \frac{abc}{2R^3} = \frac{4Rrs}{2R^3} = \frac{2rs}{R^2}; (4)$$

From $(3), (4)$ we have:

$$\frac{3}{2}R^2 \sum_{cyc} sin2A = \frac{3}{2}R^2 \cdot \frac{2rs}{R^2} = \frac{3rs}{2s} = \frac{3r}{2} < 2r$$

Solution 2 by proposer

$$\frac{AA"}{AA'} = \frac{AA" \cdot AA'}{AA'^2} \quad (1)$$

$$AA" \cdot AA' = \rho(A) = A\Omega^2 - \frac{R^2}{4} \quad (2)$$

From $(1) + (2) \Rightarrow AA" \cdot AA' = \frac{b^2+c^2-a^2}{4} \Rightarrow AA' = h_a = 2R \cdot sinB \cdot sinC \quad (3)$

From $(1) + (2) + (3) \Rightarrow$

$$\frac{AA"}{AA'} = \frac{b^2+c^2-a^2}{16R^2 \cdot sin^2B \cdot sin^2C} = \frac{2bc \cdot cosA}{16R^2 \cdot sin^2B \cdot sin^2C} = \frac{4R^2 \cdot cosA \cdot sinB \cdot sinC}{8R^2 \cdot sin^2B \cdot sin^2C} =$$

$$= \frac{2bc \cdot cosA}{2sinB \cdot sinC}$$

Therefore

$$\sum \frac{AA"}{AA'} = \frac{1}{2} \underbrace{\sum \frac{cosA}{sinBsinC}}_{1} = 1 \Rightarrow \sum \frac{AA"}{\frac{2S}{a}} = 1 \Rightarrow \sum a \cdot AA" = 2S$$

$$\Rightarrow \sum a(h_a - A'A") = 2S \Rightarrow \sum ah_a - \sum a \cdot A'A" = 2S$$

$$\Rightarrow \sum a \cdot A'A" = 4S \Rightarrow \sum \frac{a \cdot A'A"}{a+b+c} = \frac{4S}{2s} = \frac{4sr}{2s} = 2r \quad (4)$$

Applying weighted AM-GM inequality, we have:

$$2r = \sum \frac{a}{a+b+c} \cdot A'A" \geq (A'A")^{a/2s} \cdot (B'B")^{b/2s} \cdot (C'C")^{c/2s}$$

So: $(2r)^{2s} \geq (A'A")^a \cdot (B'B")^b \cdot (C'C")^c$

UP.310. Let $x,y,z>0$ real numbers such that $x+y+z=3$. Prove that:

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+1\right)\ge 4(xy+yz+zx)$$

Hence, find the minimum value of expression:

$$P=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)}{8}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

By AM-GM inequality, we have:

$$\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\ge 5\sqrt[5]{\left(\sqrt[3]{a}\right)^3\cdot a^2\cdot a^2}=5\sqrt[5]{a^5}=5a\Leftrightarrow$$

$$3\sqrt[3]{a}+2a^2\ge 5a\Leftrightarrow 3\sqrt[3]{a}\ge 5a-2a^2$$

Similarly: $3\sqrt[3]{b}\ge 5b-2b^2;\ 3\sqrt[3]{c}\ge 5c-2c^2$

Therefore,

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 5(a+b+c)-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 15-2(a^2+b^2+c^2);(\because a+b+c=3)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 18-2(a^2+b^2+c^2)\overset{a+b+c=3}{=}$$

$$=2(a+b+c)^2-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 2(a^2+b^2+c^2+2ab+2bc+2ca)-2(a^2+b^2+c^2)\Leftrightarrow$$

$$3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 4(ab+bc+ca)-3$$

Choose: $a=x;b=y;c=z\ (x,y,z>0,x+y+z=3)$ we have:

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\ge 4(xy+yz+zx)-3;\ \ (*)\Leftrightarrow$$

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+1\right)\ge 4(xy+yz+zx);$$

Now, $\Omega\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{x^2}{x(y+z)}+\frac{y^2}{y(z+x)}+\frac{z^2}{z(x+y)}\ge$

$$\overset{Bergstrom}{\ge}\frac{(x+y+z)^2}{2(xy+yz+zx)}=\frac{3^2}{2(xy+yz+zx)}=\frac{9}{2(xy+yz+zx)}$$

$$P=\Omega+\frac{3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)}{8}\ge\frac{9}{2(xy+yz+zx)}+\frac{3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)}{8}\overset{(*)}{\ge}$$

$$\ge\frac{9}{2(xy+yz+zx)}+\frac{4(xy+yz+zx)-3}{8}=\frac{9}{2(xy+yz+zx)}+\frac{xy+yz+zx}{2}-\frac{3}{8}\ge$$

$$\overset{AM-GM}{\ge}2\sqrt{\frac{9}{2(xy+yz+zx)}\cdot\frac{xy+yz+zx}{2}}-\frac{3}{8}=2\sqrt{\frac{9}{4}}-\frac{3}{8}=3-\frac{3}{8}=\frac{21}{8}$$

Hence $P\ge\frac{21}{8}$ and $P_{Min}=\frac{21}{8}\Leftrightarrow\begin{cases}x=y=z\\x+y+z=3\end{cases}\Leftrightarrow x=y=z=1.$

Solution 2 by proposer

By AM-GM inequality, we have:

$$\sqrt[3]{x}+\sqrt[3]{x}+\sqrt[3]{x}+x^2+x^2\ge5\sqrt[5]{\sqrt[3]{x}\cdot\sqrt[3]{x}\cdot\sqrt[3]{x}\cdot x^2\cdot x^2}=5\sqrt[5]{x^5}=5x\Leftrightarrow$$

$$3\sqrt[3]{x}+2x^2\ge5x\Leftrightarrow3\sqrt[3]{x}\ge5x-2x^2$$

Similarly: $3\sqrt[3]{y}\ge5y-2y^2;\ 3\sqrt[3]{z}\ge5z-2z^2$

Therefore, $3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\ge5(x+y+z)-2(x^2+y^2+z^2)\Leftrightarrow$

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\ge5(x+y+z)-2[(x+y+z)^2-2(xy+yz+zx)]\Leftrightarrow$$

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\ge5\cdot3-2[(3^2-2(xy+yz+zx)]\Leftrightarrow$$

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\ge4(xy+yz+zx)-3\Leftrightarrow$$

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+1\right)\ge4(xy+yz+zx);\ \ (1)$$

By Cauchy-Schwartz inequality, we have:

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{x^2}{x(y+z)}+\frac{y^2}{y(z+x)}+\frac{z^2}{z(x+y)}\ge\frac{(x+y+z)^2}{x(y+z)+y(z+x)+z(x+y)}$$

Hence, $P=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)}{8}\ge$

$$\ge\frac{(x+y+z)^2}{2(xy+yz+zx)}+\frac{3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)}{8}=\frac{9}{2(xy+yz+zx)}+\frac{3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)}{8};(2)$$

From (1): $3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+1\right)\ge4(xy+yz+zx)\Rightarrow$

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\ge4(xy+yz+zx)-3;\ \ (3)$$

From (2),(3) and AM-GM inequality, we have:

$$P\geq\frac{9}{2(xy+yz+zx)}+\frac{4(xy+yz+zx)-3}{8}=\frac{1}{2}\left(\frac{9}{xy+yz+zx}+(xy+yz+zx)\right)-\frac{3}{8}$$

$$\geq\frac{1}{2}\cdot 2\sqrt{\frac{9}{xy+yz+zx}\cdot(xy+yz+zx)}-\frac{3}{8}=\frac{1}{2}\cdot 2\cdot 3-\frac{3}{8}=3-\frac{3}{8}=\frac{21}{8}$$

$P\geq\frac{21}{8}\Rightarrow P_{Min}=\frac{21}{8}$. Equality occurs if

$$\begin{cases} x,y,z>0; x+y+z=3\\ \sqrt[3]{x}=x^2;\sqrt[3]{y}=y^2;\sqrt[3]{z}=z^2\\ x=y=z>0\\ xy+yz+zx=3\end{cases}\Leftrightarrow x=y=z=1.$$

UP.311. Find:

$$\lim_{n\to\infty}\sqrt[3]{n^2}\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}-\frac{\sqrt[3]{n^2}}{\sqrt[3n]{n!}}\right)$$

Proposed by D.M.Bătineţu-Giurgiu,Neculai Stanciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$L=\lim_{n\to\infty}\sqrt[3]{n^2}\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}-\frac{\sqrt[3]{n^2}}{\sqrt[3n]{n!}}\right)=\lim_{n\to\infty}\sqrt[3]{n^2}\cdot\frac{\sqrt[3]{n^2}}{\sqrt[3n]{n!}}\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}\cdot\frac{\sqrt[3n]{n!}}{\sqrt[3]{n^2}}-1\right)=$$

$$=\lim_{n\to\infty}n\cdot\frac{\sqrt[3]{n}}{\sqrt[3n]{n!}}\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}\cdot\frac{\sqrt[3n]{n!}}{\sqrt[3]{n^2}}-1\right);\quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n}}{\sqrt[3n]{n!}}=\lim_{n\to\infty}\sqrt[3]{\frac{n}{\sqrt[n]{n!}}}=\sqrt[3]{\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}}\overset{C-D'A}{=}\sqrt[3]{\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}}=$$

$$=\sqrt[3]{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n}=\sqrt[3]{e};\quad (2)$$

$$\lim_{n\to\infty}n\cdot\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}\cdot\frac{\sqrt[3n]{n!}}{\sqrt[3]{n^2}}-1\right)=$$

$$=\lim_{n\to\infty} n\cdot\left(\frac{e^{log\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}\cdot\frac{\sqrt[3n]{n!}}{\sqrt[3]{n^2}}\right)}-1}{log\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}\cdot\frac{\sqrt[3n]{n!}}{\sqrt[3]{n^2}}\right)}\right)log\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}\cdot\frac{\sqrt[3n]{n!}}{\sqrt[3]{n^2}}\right)=$$

$$=\lim_{n\to\infty} n\cdot log\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}\cdot\frac{\sqrt[3n]{n!}}{\sqrt[3]{n^2}}\right)=\lim_{n\to\infty} log\left[\left(\sqrt[3]{\left(\frac{n+1}{n}\right)^2}\right)^n\cdot\sqrt[3]{\frac{n!}{\left(\sqrt[n+1]{(n+1)!}\right)^n}}\right]=$$

$$=\frac{1}{3}\lim_{n\to\infty} log\left[\left(1+\frac{1}{n}\right)^{2n}\cdot\frac{n!}{(n+1)!}\cdot\sqrt[n+1]{(n+1)!}\right]=$$

$$=\frac{1}{3}\lim_{n\to\infty} log\left(e^2\cdot\frac{\sqrt[n+1]{(n+1)!}}{n+1}\right)=\frac{1}{3}\lim_{n\to\infty} log\left(e^2\cdot\frac{\sqrt[n]{n!}}{n}\right)\overset{(2)}{=}$$

$$\frac{1}{3}log\left(e^2\cdot\frac{1}{e}\right)=\frac{1}{3}loge=\frac{1}{3};\ \ (3)$$

From $(1),(2),(3)$ we get

$$L=\lim_{n\to\infty}\sqrt[3]{n^2}\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}-\frac{\sqrt[3]{n^2}}{\sqrt[3n]{n!}}\right)=\frac{\sqrt[3]{e}}{3}$$

Solution 2 by Hemn Hsain-Uzbekistan

$$n!\cong\left(\frac{n}{e}\right)^n\Rightarrow\sqrt[3n]{n!}\cong\sqrt[3]{\frac{n}{e}};\ (n+1)!\cong\left(\frac{n+1}{e}\right)^{n+1}\Rightarrow\sqrt[3n+3]{(n+1)!}\cong\sqrt[3]{\frac{n+1}{e}}$$

$$L=\lim_{n\to\infty}\sqrt[3]{n^2}\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}-\frac{\sqrt[3]{n^2}}{\sqrt[3n]{n!}}\right)=\sqrt[3]{e}\lim_{n\to\infty}\sqrt[3]{n^2}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)=$$

$$=\sqrt[3]{e}\lim_{n\to\infty}\left[\sqrt[3]{n^2}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)\cdot\frac{\sqrt[3]{(n+1)^2}+\sqrt[3]{n(n+1)}+\sqrt[3]{n^2}}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n(n+1)}+\sqrt[3]{n^2}}\right]=$$

$$=\sqrt[3]{e}\lim_{n\to\infty}\left[\sqrt[3]{n^2}\cdot\frac{n+1-n}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n(n+1)}+\sqrt[3]{n^2}}\right]=$$

$$=\sqrt[3]{e}\lim_{n\to\infty}\frac{\sqrt[3]{n^2}}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n(n+1)}+\sqrt[3]{n^2}}=\frac{\sqrt[3]{e}}{3}$$

Solution 3 by Kaushik Mahanta-India

By Stirling's approximation, we know:

$$\lim_{n\to\infty}(n+1)! = \sqrt{2\pi}\left(\frac{n+1}{e}\right)^{n+1};\quad (1)$$

$$\lim_{n\to\infty} n! = \sqrt{2\pi}\left(\frac{n}{e}\right)^{n};\quad (2)$$

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1;\quad (3)$$

$$\lim_{n\to\infty}\left[n^{\frac{2}{3}}\cdot\left(\frac{(n+1)^{\frac{2}{3}}}{\left((n+1)!\right)^{\frac{1}{3n+3}}}-\frac{n^{\frac{2}{3}}}{(n!)^{\frac{1}{3n}}}\right)\right]\overset{(1),(2)}{=}$$

$$=\lim_{n\to\infty}\left[n^{\frac{2}{3}}\cdot\left(\frac{(n+1)^{\frac{2}{3}}}{(2\pi(n+1))^{\frac{1}{2\cdot3(n+1)}}\cdot\left(\frac{n+1}{e}\right)^{\frac{n+1}{3(n+1)}}}-\frac{n^{\frac{2}{3}}}{(2\pi n)^{\frac{1}{2\cdot 3n}}\cdot\left(\frac{n}{e}\right)^{\frac{n}{3n}}}\right)\right]=$$

$$=\lim_{n\to\infty}\left[n^{\frac{2}{3}}\cdot\left(\frac{(n+1)^{\frac{2}{3}}}{(2\pi(n+1))^{\frac{1}{6(n+1)}}\cdot\left(\frac{n+1}{e}\right)^{\frac{1}{3}}}-\frac{n^{\frac{2}{3}}}{(2\pi n)^{\frac{1}{6n}}\cdot\left(\frac{n}{e}\right)^{\frac{1}{3}}}\right)\right]=$$

$$\left(\therefore \lim_{n\to\infty}(2\pi(n+1))^{\frac{1}{6(n+1)}}=1;\ \lim_{n\to\infty}(2\pi n)^{\frac{1}{6n}}=1\right)$$

$$=\lim_{n\to\infty}\left[n^{\frac{2}{3}}\left(\frac{(n+1)^{\frac{2}{3}}}{(n+1)^{\frac{1}{3}}}\cdot e^{\frac{1}{3}}-\frac{n^{\frac{2}{3}}}{n^{\frac{1}{3}}}\cdot e^{\frac{1}{3}}\right)\right]=e^{\frac{1}{3}}\lim_{n\to\infty}\left[n^{\frac{2}{3}}\left((n+1)^{\frac{1}{3}}-n^{\frac{1}{3}}\right)\right]=$$

$$=e^{\frac{1}{3}}\lim_{n\to\infty}\left[n^{\frac{2}{3}}\left(\frac{\left((n+1)^{\frac{1}{3}}-n^{\frac{1}{3}}\right)\left((n+1)^{\frac{2}{3}}+n^{\frac{2}{3}}+\left(n(n+1)\right)^{\frac{1}{3}}\right)}{(n+1)^{\frac{2}{3}}+n^{\frac{2}{3}}+\left(n(n+1)\right)^{\frac{1}{3}}}\right)\right]=$$

$$=e^{\frac{1}{3}}\lim_{n\to\infty}\frac{n^{\frac{2}{3}}(n+1-n)}{(n+1)^{\frac{2}{3}}+n^{\frac{2}{3}}+\left(n(n+1)\right)^{\frac{1}{3}}}=e^{\frac{1}{3}}\lim_{n\to\infty}\frac{n^{\frac{2}{3}}(n+1-n)}{n^{\frac{2}{3}}\left[(1+\frac{1}{n})^{\frac{2}{3}}+1+\left(1+\frac{1}{n}\right)^{\frac{1}{3}}\right]}=\frac{e^{\frac{1}{3}}}{3}$$

UP.312. If $m \in \mathbb{R}_+ = [0,\infty)$ and $ab, x, y, z \in \mathbb{R}_+^* = (0,\infty)$, then:

$$\frac{x^{m+1}}{(ax+bz)^{m+1}\cdot sec^{2m}\frac{\pi}{18}}+\frac{y^{m+1}}{(az+bx)^{m+1}\cdot csc^{2m}\frac{\pi}{9}}+\frac{z^{m+1}}{(ax+by)^{m+1}\cdot csc^{2m}\frac{2\pi}{9}}\ge\frac{3}{4^m\cdot(a+b)^{m+1}}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Daniel Văcaru-Romania

By generalized Radon Inequality, we have:

$$\frac{x^{m+1}}{(ax+bz)^{m+1}\cdot sec^{2m}\frac{\pi}{18}}+\frac{y^{m+1}}{(az+bx)^{m+1}\cdot csc^{2m}\frac{\pi}{9}}+\frac{z^{m+1}}{(ax+by)^{m+1}\cdot csc^{2m}\frac{2\pi}{9}}=$$

$$=\frac{\left(\frac{x}{ay+bz}\right)^{m+1}}{\left(sec^2\frac{\pi}{18}\right)^m}+\frac{\left(\frac{y}{az+bx}\right)^{m+1}}{\left(csc^2\frac{\pi}{9}\right)^m}+\frac{\left(\frac{z}{ax+by}\right)^{m+1}}{\left(csc^2\frac{2\pi}{9}\right)^m}\overset{g.R.i}{\ge}$$

$$\ge\frac{\left(\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by}\right)^{m+1}}{\left(sec^2\frac{\pi}{18}+csc^2\frac{\pi}{9}+csc^2\frac{2\pi}{9}\right)^m};\ (1)$$

But: $\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by}=\frac{x^2}{ayx+bzx}+\frac{y^2}{azy+bxy}+\frac{z^2}{axz+byz}\ge$

$$\overset{Bergstrom}{\ge}\frac{(x+y+z)^2}{(a+b)(xy+yz+zx)}\ge\frac{3}{a+b};\quad(2)$$

And $sec^2\frac{\pi}{18}+csc^2\frac{\pi}{9}+csc^2\frac{2\pi}{9}=12;\ (3)$

From $(1),(2),(3)$ **we get:**

$$\frac{x^{m+1}}{(ax+bz)^{m+1}\cdot sec^{2m}\frac{\pi}{18}}+\frac{y^{m+1}}{(az+bx)^{m+1}\cdot csc^{2m}\frac{\pi}{9}}+$$

$$+\frac{z^{m+1}}{(ax+by)^{m+1}\cdot csc^{2m}\frac{2\pi}{9}}\overset{(1)}{\ge}\frac{\left(\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by}\right)^{m+1}}{\left(sec^2\frac{\pi}{18}+csc^2\frac{\pi}{9}+csc^2\frac{2\pi}{9}\right)^m}\ge$$

$$\overset{(2),(3)}{\ge}\frac{\left(\frac{3}{a+b}\right)^{m+1}}{12^m}=\frac{3}{4^m\cdot(a+b)^{m+1}}$$

UP.313. Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, a_n, b_n \in \mathbb{R}_+^* = (0,\infty), \forall n \in \mathbb{N}^*$,

$$\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}} = a \in \mathbb{R}_+^*, b_n = \prod_{n=1}^{\infty} a_n \text{ . Find:}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Adrian Popa-Romania

Let be $x_n = \sqrt[n]{b_n} \Rightarrow \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \lim_{n\to\infty}(x_{n+1} - x_n)$

$$1)\ \lim_{n\to\infty}\frac{x_n}{n} = \lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n} =$$

$$= \lim_{n\to\infty}\frac{a_1\cdot a_2\cdot\ldots\cdot a_n\cdot a_{n+1}\cdot n^n}{a_1\cdot a_2\cdot\ldots\cdot a_n\cdot(n+1)^{n+1}} = \lim_{n\to\infty}\frac{a_{n+1}}{n+1}\cdot\left(\frac{n}{n+1}\right)^n = \frac{1}{e}\lim_{n\to\infty}\frac{a_{n+1}}{n+1}$$

$$\lim_{n\to\infty}\frac{a_n}{n} = \lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n!}}{n} = a\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{C-D'A}{=} a\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = a\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{a}{e}$$

$$2)\ \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} = \lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{n+1}{n} = \frac{a}{e^2}\cdot\frac{e^2}{a}\cdot 1 = 1$$

$$3)\ \lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right)^n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^n = \lim_{n\to\infty}\frac{b_{n+1}^{\frac{n}{n+1}}}{b_n} = \lim_{n\to\infty}\frac{(a_1\cdot a_2\cdot\ldots\cdot a_n\cdot a_{n+1})^{\frac{n}{n+1}}}{a_1\cdot a_2\cdot\ldots\cdot a_n} =$$

$$= \lim_{n\to\infty}\frac{a_{n+1}^{\frac{n}{n+1}}}{(a_1\cdot a_2\cdot\ldots\cdot a_n)^{\frac{1}{n+1}}} = \lim_{n\to\infty}\frac{\sqrt[n+1]{\frac{a_{n+1}^{n+1}}{(n+1)!}}\cdot\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n+1]{a_{n+1}}}}{\frac{\sqrt[n+1]{b_{n+1}\cdot a_{n+1}}}{\sqrt[n+1]{a_{n+1}}}} =$$

$$= \lim_{n\to\infty}\frac{\sqrt[n+1]{\frac{a_{n+1}^{n+1}}{(n+1)!}}\cdot\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n+1]{a_{n+1}}}}{\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}} = a\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)!}{b_{n+1}}} = a\lim_{n\to\infty}\sqrt[n]{\frac{n!}{b_n}} \overset{C-D'A}{=}$$

$$= a\lim_{n\to\infty}\frac{(n+1)!}{b_{n+1}}\cdot\frac{b_n}{n!} = a\lim_{n\to\infty}\frac{n}{a_n}\overset{(1)}{=} a\cdot\frac{a}{e} = \frac{a^2}{e}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \frac{a}{e^2}log\frac{a}{e^2} = \frac{a}{e^2}(2loga - 1)$$

UP.314. If $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ are sequences of strictly positive real numbers such that:

$$\lim_{n\to\infty}\frac{a_n}{(2n-1)!!} = a > 0;\ \lim_{n\to\infty}\frac{b_n}{(2n-1)!!} = b > 0.\text{ Then find:}$$

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!} - \sqrt[2n]{a_n\cdot b_n}\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by Marian Ursăraescu-Romania

$$L = \lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!} - \sqrt[2n]{a_n\cdot b_n}\right) =$$

$$= \lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right) + \lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!} - \sqrt[2n]{a_n\cdot b_n}\right);\quad (1)$$

$$L_1 = \lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right) = \lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot n\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} - 1\right);(2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} =$$

$$= \lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n = \frac{2}{e};\quad (3)$$

$$\lim_{n\to\infty} n\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} - 1\right) = \lim_{n\to\infty} n\cdot\frac{e^{log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)} - 1}{log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)}\cdot log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) =$$

$$= \lim_{n\to\infty} log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^n = \lim_{n\to\infty} log\left(\frac{\left(\sqrt[n+1]{(2n+1)!!}\right)^n}{(2n-1)!!}\right) =$$

$$= log\left(\lim_{n\to\infty}\frac{2n+1}{\sqrt[n+1]{(2n+1)!!}}\right) = log\left(\lim_{n\to\infty}\frac{2n-1}{\sqrt[n]{(2n-1)!!}}\right) \overset{(3)}{=} loge = 1; (4)$$

$$L_2 = \lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!} - \sqrt[2n]{a_n\cdot b_n}\right) = \lim_{n\to\infty}\frac{\sqrt[2n]{a_n\cdot b_n}}{n}\cdot\left(\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[2n]{a_n\cdot b_n}} - 1\right); \quad (5)$$

$$\lim_{n\to\infty}\frac{\sqrt[2n]{a_n\cdot b_n}}{n} = \lim_{n\to\infty}\sqrt[2n]{\frac{a_n\cdot b_n}{n^{2n}}} = \lim_{n\to\infty}\sqrt{\sqrt[n]{\frac{a_n\cdot b_n}{n^{2n}}}} = \sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{a_n\cdot b_n}{n^{2n}}}} =$$

$$\overset{C-D'A}{=} \sqrt{\lim_{n\to\infty}\frac{a_{n+1}\cdot b_{n+1}}{(n+1)^{2n+2}}\cdot\frac{n^{2n}}{a_n\cdot b_n}} =$$

$$= \sqrt{\lim_{n\to\infty}\frac{a_{n+1}}{(2n+1)!!}\cdot\frac{b_{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{a_n}\cdot\frac{(2n-1)!!}{b_n}\cdot\frac{(2n+1)^2}{(n+1)^2}\cdot\left(\frac{n}{n+1}\right)^{2n}} =$$

$$= \sqrt{a\cdot b\cdot\frac{1}{a}\cdot\frac{1}{b}\cdot 4\cdot\frac{1}{e^2}} = \frac{2}{e}; \quad (6)$$

$$\lim_{n\to\infty} n\left(\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[2n]{a_n\cdot b_n}} - 1\right) = \lim_{n\to\infty} n\left(\frac{e^{log\left(\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[2n]{a_n\cdot b_n}}\right)} - 1}{log\left(\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[2n]{a_n\cdot b_n}}\right)}\right) log\left(\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[2n]{a_n\cdot b_n}}\right) =$$

$$= \lim_{n\to\infty} nlog\left(\frac{\sqrt[2n]{\left((2n-1)!!\right)^2}}{\sqrt[2n]{a_n\cdot b_n}}\right) = \lim_{n\to\infty} nlog\left(\sqrt[2n]{\frac{\left((2n-1)!!\right)^2}{a_n\cdot b_n}}\right) =$$

$$= \frac{1}{2}\lim_{n\to\infty} log\left(\frac{(2n-1)!!}{a_n}\cdot\frac{(2n-1)!!}{b_n}\right) = \frac{1}{2}log\left(\frac{1}{ab}\right) = -\frac{1}{2}log(ab); (7)$$

From $(1),(2),\ldots,(7)$ we have:

$$L = \frac{2}{e} - \frac{2}{e}log\sqrt{ab} = \frac{2}{e}log\left(\frac{e}{\sqrt{ab}}\right)$$

Solution 2 by proposers

$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{a_n}\cdot\frac{2n+1}{n} = a\cdot\frac{1}{a}\cdot 2 = 2$ and similarly $\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = 2$

So, $\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}\cdot\left(\frac{n}{n+1}\right)^{n+1}=\frac{2}{e}$ and similarly

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\frac{2}{e}$$

Let: $v_n=\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[2n]{a_n\cdot b_n}}, \forall n\geq 2$

$$\lim_{n\to\infty}v_n=\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)!!}}{n+1}\cdot\frac{n}{\sqrt[2n]{a_n\cdot b_n}}\cdot\frac{n+1}{n}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{a_n}}\cdot\frac{\sqrt{n}}{\sqrt[2n]{b_n}}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}$$

$$=\sqrt{\frac{e}{2}}\cdot\sqrt{\frac{e}{2}}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D'A}{=}\frac{e}{2}\cdot\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}$$

$$=\frac{e}{2}\cdot\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{e}{2}\cdot 2\cdot e\cdot\frac{1}{e}=1$$

$\lim_{n\to\infty}\frac{v_n-1}{logv_n}=1$ and $\lim_{n\to\infty}v_n^n=\lim_{n\to\infty}\frac{(2n+1)!!}{\sqrt{a_n\cdot b_n}}\cdot\frac{1}{\sqrt[n+1]{(2n+1)!!}}=\lim_{n\to\infty}\frac{(2n-1)!!}{\sqrt{a_n\cdot b_n}}\cdot\frac{2n+1}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}=$

$$=\lim_{n\to\infty}\sqrt{\frac{(2n-1)!!}{a_n}}\cdot\sqrt{\frac{(2n-1)!!}{b_n}}\cdot 2\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot 2\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}$$

$$\overset{C-D'A}{=}\frac{2}{\sqrt{ab}}\cdot\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=\frac{2}{\sqrt{ab}}\cdot\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\left(\frac{n+1}{n}\right)^n=\frac{2}{\sqrt{ab}}\cdot\frac{1}{2}\cdot e=\frac{e}{\sqrt{ab}}$$

Let: $B_n=\sqrt[n+1]{(2n+1)!!}-\sqrt[2n]{a_n\cdot b_n}=\sqrt[2n]{a_n\cdot b_n}\cdot(v_n-1)=\sqrt[2n]{a_n\cdot b_n}\cdot\frac{v_n-1}{logv_n}\cdot logv_n$

$$=\frac{\sqrt[2n]{a_n\cdot b_n}}{n}\cdot\frac{v_n-1}{logv_n}\cdot logv_n^n=\sqrt{\frac{\sqrt[n]{a_n}}{n}\cdot\frac{\sqrt[n]{b_n}}{n}}\cdot\frac{v_n-1}{logv_n}\cdot logv_n^n, \forall n\geq 2$$

$$\lim_{n\to\infty}B_n=\sqrt{\frac{2}{e}\cdot\frac{2}{e}}\cdot 1\cdot log\frac{e}{\sqrt{ab}}=\frac{2}{e}\cdot\left(1-\frac{1}{2}log(ab)\right)=\frac{1}{e}\cdot(2-loga-logb)$$

UP.315. If $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ are strictly positive real numbers such that

$\lim_{n\to\infty}\frac{a_n}{n!}=a>0, \lim_{n\to\infty}\frac{b_n}{(2n-1)!!}=b>0$ then find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,(2n+1)!!}-\sqrt[n]{a_n\cdot b_n}\right)\cdot\frac{1}{n}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution by proposer

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)!}\cdot\frac{n!}{a_n}\cdot\frac{n+1}{n}=a\cdot\frac{1}{a}\cdot 1=1 \text{ and similary}$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{(2n+!)!!}\cdot\frac{(2n-1)!!}{b_n}\cdot\frac{2n+1}{n}=b\cdot\frac{1}{b}\cdot 2=2, \text{ so}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}\cdot\left(\frac{n}{n+1}\right)^{n+1}=1\cdot\frac{1}{e}=\frac{1}{e}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}\cdot\left(\frac{n}{n+1}\right)^{n+1}=2\cdot\frac{1}{e}=\frac{2}{e}$$

$$\text{Let: } u_n=\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{\sqrt[n]{a_n\cdot b_n}}, \forall n\geq 2 \text{ then}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{n}{\sqrt[n]{a_n}}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{n+1}\cdot\left(\frac{n+1}{n}\right)^2$$

$$=e\cdot\frac{e}{2}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D'A}{=}\frac{e^2}{2}\cdot\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\cdot\frac{(2n+!)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}$$

$$=\frac{e^2}{2}\cdot\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^n\cdot\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n\right)=\frac{e^2}{2}\cdot\frac{1}{e}\cdot 2\cdot\frac{1}{e}=1\Rightarrow\lim_{n\to\infty}\frac{u_n-1}{logu_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!\cdot(2n+1)!!}{a_n\cdot b_n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!\,(2n+1)!!}}$$

$$=\lim_{n\to\infty}\frac{n!}{a_n}\cdot\frac{(2n-1)!!}{b_n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{2n+1}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}$$

$$=\frac{1}{a}\cdot\frac{1}{b}\cdot e\cdot 2\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{C-D'A}{=}\frac{2e}{ab}\cdot\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}$$

$$=\frac{2e}{ab}\cdot\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\left(\frac{n+1}{n}\right)^n=\frac{2e}{ab}\cdot\frac{1}{2}\cdot e=\frac{e^2}{ab}.$$

$$\text{Let: } B_n=\left(\sqrt[n+1]{(n+1)!\,(2n+1)!!}-\sqrt[n]{a_n\cdot b_n}\right)\cdot\frac{1}{n}=\frac{\sqrt[n]{a_n\cdot b_n}}{n}\cdot(u_n-1)$$

$$=\frac{\sqrt[n]{a_n}}{n}\cdot\sqrt[n]{b_n}\cdot\frac{u_n-1}{logu_n}\cdot logu_n=\frac{\sqrt[n]{a_n}}{n}\cdot\frac{\sqrt[n]{b_n}}{n}\cdot\frac{u_n-1}{logu_n}\cdot logu_n^n, \forall n\geq 2$$

$$\text{So, }\lim_{n\to\infty}B_n=\frac{1}{e}\cdot\frac{2}{e}\cdot 1\cdot loge=\frac{2}{e^2}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru