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JP.316. If $x\in\mathbb{R}_+^*=(0,\infty)$, a,b,c are the lengths of the sides of ΔABC, with area F, altitudes h_a,h_b,h_c then:

$$\frac{6x-1}{h_a^2}+\left(\frac{2}{3x}-1\right)\frac{1}{h_b^2}+\frac{1}{h_c^2}\ge\frac{\sqrt{3}}{F}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Daniel Văcaru-Romania

We know that $2F=ah_a\Rightarrow\frac{1}{h_a}=\frac{a^2}{2F}\Rightarrow\frac{1}{h_a^2}=\frac{a^2}{4F^2}$ (and analogs)

$$\frac{6x-1}{h_a^2}+\left(\frac{2}{3x}-1\right)\frac{1}{h_b^2}+\frac{1}{h_c^2}=\frac{(6x-1)a^2+\left(\frac{2}{3x}-1\right)b^2+c^2}{4F^2}=$$

$$=\frac{\left(6xa^2+\frac{2}{3x}b^2\right)+c^2-a^2-b^2}{4F^2}\overset{Cos.Law}{=}$$

$$=\frac{6xa^2+\frac{2}{3x}b^2+2abcos\,C}{4F^2}\overset{AM-GM}{\ge}\frac{4ab+2abcos\,C}{4F^2}=\frac{ab(2+cosC)}{2F^2}=$$

$$=\frac{ab\left(1+2sin^2\frac{C}{2}\right)}{2F^2}=\frac{ab\left(cos^2\frac{C}{2}+3sin^2\frac{C}{2}\right)}{2F^2}\overset{AM-GM}{\ge}$$

$$\ge\frac{2ab\sqrt{3sin^2\frac{C}{2}\cdot cos^2\frac{C}{2}}}{2F^2}=\frac{2\sqrt{3}absin\frac{C}{2}cos\frac{C}{2}}{2F^2}=\frac{\sqrt{3}absin\,C}{2F^2}=\frac{\sqrt{3}F}{F^2}=\frac{\sqrt{3}}{F}$$

JP.317. In ΔABC the following relationship holds:

$$(a^3+b^3+c^3)\left(\frac{a}{4s^2-a^2}+\frac{b}{4s^2-b^2}+\frac{c}{4s^2-c^2}\right)\ge\frac{27\sqrt{3}}{32}\cdot F$$

Proposed by D.M. Bătineţu Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

$$\left(\sum_{cyc}a^3\right)\cdot\sum_{cyc}\frac{a}{4s^2-a^2}\overset{Radon}{\ge}\frac{8s^3}{9}\cdot\sum_{cyc}\frac{a}{4s^2-a^2}=\frac{8s^3}{9}\cdot\sum_{cyc}\frac{a^2}{4s^2a-a^3}$$

$$\overset{Bergstrom}{\geq}\frac{8s^3}{9}\cdot\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(4s^2a-a^3)}=\frac{8s^3\cdot 4s^2}{8s^3-\sum_{cyc}a^3}\geq\frac{32s^5}{8s^3-\frac{8s^3}{9}}=\frac{32s^5\cdot 9}{8s^3\cdot(9-1)}$$

$$=\frac{32\cdot 9s^5}{64s^3}=\frac{9s^2}{32}=\frac{9}{32}\cdot s\cdot s\overset{Mitrinovic}{\geq}\frac{9s(3\sqrt{3}r)}{32}=\frac{27\sqrt{3}}{32}\cdot sr=\frac{27\sqrt{3}}{32}\cdot F$$

q.e.d.

Observation: $a^3+b^3+c^3\overset{Radon}{\geq}\frac{(a+b+C)^3}{9}=\frac{(2s)^3}{9}=\frac{8s^3}{9}$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$a^3+b^3+c^3\overset{AM-GM}{\geq}3\sqrt[3]{(abc)^3}=3abc=3\cdot 4Rrs=12Rrs$$

$$\frac{a}{4s^2-a^2}+\frac{b}{4s^2-b^2}+\frac{c}{4s^2-c^2}=\frac{a^2}{4as^2-a^3}+\frac{b^2}{4bs^2-b^3}+\frac{c^2}{4cs^2-c^3}\overset{Bergstrom}{\geq}$$

$$\geq\frac{4s^2}{8s\cdot s^2-2s(s^2-6Rr-3r^2)}=\frac{2s}{4s^2-(s^2-6Rr-3r^2)}=\frac{2s}{3(s^2+2Rr+r^2)}\Rightarrow$$

$$(a^3+b^3+c^3)\left(\frac{a}{4s^2-a^2}+\frac{b}{4s^2-b^2}+\frac{c}{4s^2-c^2}\right)\geq\frac{12Rrs\cdot 2s}{3(s^2+2Rr+r^2)}=$$

$$=\frac{8Rrs^2}{s^2+2Rr+r^2}\overset{(1)}{\geq}\frac{27\sqrt{3}}{32}F=\frac{27\sqrt{3}}{32}sr$$

$$(1)\Leftrightarrow 8\cdot 32Rs\geq 27\sqrt{3}(s^2+2Rr+r^2)$$

But: $s\leq\frac{3\sqrt{3}}{2}R\ (Mitrinovic)\Rightarrow R\geq\frac{2}{3\sqrt{3}}s=\frac{6\sqrt{3}}{27}s$

$$8\cdot 32Rs\geq 8\cdot 32\cdot\frac{6\sqrt{3}}{2}s^2=\frac{512\sqrt{3}}{9}s^2\geq 27\sqrt{3}(s^2+2Rr+r^2)\Leftrightarrow$$

$$512s^2\geq 243(s^2+2Rr+r^2)\Leftrightarrow 269s^2\geq 243(2Rr+r^2);\ \ (2)$$

Other,

$$s^2\geq 16Rr-5r^2\ (Gerretsen)\Rightarrow$$

$$269s^2\geq 268(16Rr-5r^2)>243(16Rr-5r^2)\overset{(3)}{>}243(2Rr+r^2);$$

$$(3)\Leftrightarrow 16Rr-5r^2>2Rr+r^2\Leftrightarrow 14Rr>6r^2\Leftrightarrow R>\frac{3}{7}r\left(true\ by\ R\geq 2r>\frac{3}{7}r\right)\Rightarrow$$

$\Rightarrow(3)\Rightarrow(2)\Rightarrow(1)$ **true. Proved.**

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\left(\sum_{cyc} a^3\right)\cdot\sum_{cyc}\frac{a}{4s^2-a^2}=\left(\sum_{cyc} a\cdot a^2\right)\cdot\sum_{cyc}\frac{a}{(2s-a)(2s+a)}\overset{\substack{Chebyshev's\\ AM-GM}}{\geq}$$

$$\geq\frac{(\sum a)\cdot(\sum a^2)}{3}\cdot\sum\frac{a}{\frac{2s+a+2s-a}{2}}\geq\frac{2s\cdot 4\sqrt{3}S}{3}\cdot\frac{2s}{4s^2}=\frac{4\sqrt{3}S}{3}>\frac{27\sqrt{3}S}{32}$$

JP.318.If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, a,b,c –are the lengths of the sides of ΔABC with area F, altitudes h_a,h_b,h_c, then:

$$\frac{a^2}{(ax+by+cz)h_a}+\frac{b^2}{(ax+by+cz)h_b}+\frac{c^2}{(ax+by+cz)h_c}\geq\frac{2\sqrt{3}}{x+y+z}$$

Proposed by D.M.Bătineţu-Giurgiu,Neculai Stanciu-Romania

Solution 1 by Daniel Văcaru-Romania

We have:

$$\sum_{cyc}\left(\frac{a^2}{(ax+by+cz)h_a}\right)=\sum_{cyc}\left(\frac{a^3}{2F(ax+by+cz)}\right)=$$

$$=\frac{1}{2F}\sum_{cyc}\left(\frac{a^3}{ax+by+cz}\right)\overset{Holder}{\geq}\frac{1}{3\cdot 2F}\left(\frac{(a+b+c)^3}{\sum(ax+by+cz)}\right)=$$

$$=\frac{1}{6F}\cdot\frac{(a+b+c)^3}{(a+b+c)(x+y+z)}=\frac{(a+b+c)^2}{6F(x+y+z)}$$

But we have: $(a+b+c)^2\geq 3(ab+bc+ca)\geq 6F\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\right)$

But $A\to\frac{1}{sinA}$ is convexe function, and this is followed (Jensen) by

$$\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\geq\frac{3}{sin\left(\frac{A+B+C}{3}\right)}=\frac{6}{\sqrt{3}}=2\sqrt{3}$$

We obtain: $(a+b+c)^2\geq 12F\sqrt{3}$

$$\sum_{cyc}\left(\frac{a^2}{(ax+by+cz)h_a}\right)\geq\frac{(a+b+c)^2}{6F(x+y+z)}\geq\frac{12F\sqrt{3}}{6F(x+y+z)}=\frac{2\sqrt{3}}{x+y+z}$$

Solution 2 by Marin Chirciu-Romania

Using Hőlder Inequality, we have:

$$Lhs=\sum_{cyc}\left(\frac{a^2}{(ax+by+cz)h_a}\right)\ge\sum_{cyc}\left(\frac{a^2}{(ax+by+cz)\frac{2F}{a}}\right)=$$

$$=\frac{1}{2F}\sum_{cyc}\left(\frac{a^3}{ax+by+cz}\right)\ge\frac{1}{2F}\cdot\frac{(\sum a)^3}{3\sum(ax+by+cz)}=$$

$$=\frac{1}{2rs}\cdot\frac{(2s)^3}{3(\sum x)(\sum a)}=\frac{2s}{3r(x+y+z)}\overset{(1)}{\ge}\frac{2\sqrt{3}}{x+y+z}=Rhs$$

where $(1)\Leftrightarrow s\ge 3r\sqrt{3}$ (Mitrinović).

Equality holds when the triangle is equilateral.

Remark. The problem it can be developed.

If $x,y,z>0; n\in\mathbb{N}, n\ge 2$, in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^n}{(ax+by+cz)h_a}\ge\frac{2\sqrt{3}}{x+y+z}\left(\frac{2s}{3}\right)^{n-2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Hőlder Inequality, we get:

$$Lhs=\sum_{cyc}\frac{a^n}{(ax+by+cz)h_a}\ge\sum_{cyc}\frac{a^n}{(ax+by+cz)\frac{2F}{a}}=$$

$$=\frac{1}{2F}\sum_{cyc}\frac{a^{n+1}}{(ax+by+cz)}\ge\frac{1}{2F}\cdot\frac{(\sum a)^{n+1}}{3^{n-1}\sum(ax+by+cz)}=$$

$$=\frac{1}{2rs}\cdot\frac{(2s)^{n+1}}{3^{n-1}(\sum x)(\sum a)}=\frac{1}{2rs}\cdot\frac{(2s)^{n+1}}{3^{n-1}(x+y+z)\cdot 2s}\overset{(1)}{\ge}\frac{2\sqrt{3}}{x+y+z}\left(\frac{2s}{3}\right)^{n-2}=Rhs$$

where $(1)\Leftrightarrow 3\ge 3r\sqrt{3}$ (Mitrinović).

Equality holds if and only if the triangle is equilateral.

Note: For $n = 2$ we get the Problem JP.318 from 22-RMM-Autumn Edition 2021, proposed by D.M.Bătinețu-Giurgiu and Neculai Stanciu, Romania.

JP.319. In ΔABC the following relationship holds:

$$\sum_{cyc} a^3 \sum_{cyc} \frac{a}{4s^2 - a^2} \ge \frac{3\sqrt{3}}{2} \cdot S$$

D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution and generalizations by Marin Chirciu-Romania

Lemma.

In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a}{4s^2 - a^2} \ge \frac{2s}{3(s^2 + r^2 + 2Rr)}$$

Proof.

Using Bergstrőm Inequality, we get:

$$\sum_{cyc} \frac{a}{4s^2 - a^2} = \sum_{cyc} \frac{a^2}{4s^2 a - a^3} \ge \frac{(a+b+c)^2}{\sum(4s^2 a - a^3)} = \frac{\sum a^2 + 2\sum bc}{4s^2 \sum a - \sum a^3} =$$

$$= \frac{2(s^2 - r^2 - 4Rr) + 2(s^2 + r^2 + 2Rr)}{4s^2 \cdot 2s - 2s(s^2 - 3r^2 - 6Rr)} = \frac{4s^2}{6s(s^2 + r^2 + 2Rr)} = \frac{2s}{3(s^2 + r^2 + 2Rr)}$$

Let's solve the proposed problem.

Using lemma and the identity $\sum a^3 = 2s(s^2 - 3r^2 - 6Rr)$ we get:

$$Lhs = \sum_{cyc} a^3 \sum_{cyc} \frac{a}{4s^2 - a^2} \ge 2s(s^2 - 3r^2 - 6Rr) \cdot \frac{2s}{3(s^2 + r^2 + 2Rr)} \overset{(1)}{\ge} \frac{3\sqrt{3}}{2} \cdot S = Rhs$$

Where $(1) \Leftrightarrow 8s(s^2 - 3r^2 - 6Rr) \ge 9\sqrt{3} \cdot r(s^2 + r^2 + 2Rr)$

Which result from Mitrinović Inequality: $s \ge 3r\sqrt{3}$

We must show that:

$$8 \cdot 3r\sqrt{3}(s^2 - 3r^2 - 6Rr) \ge 9\sqrt{3} \cdot r(s^2 + r^2 + 2Rr) \Leftrightarrow$$

$8(s^2 - 3r^2 - 6Rr) \ge 3(s^2 + r^2 + 2Rr) \Leftrightarrow 5s^2 \ge 54Rr + 27r^2$ which result from

$$s^2 \geq 16Rr - 5r^2 \ (Gerretsen)$$

We must show:

$$5(16Rr - 5r^2) \geq 54Rr + 27r^2 \Leftrightarrow 80R - 25r \geq 54R + 27r \Leftrightarrow 26R \geq 52r \Leftrightarrow$$

$$R \geq 2r(Euler).$$

Equality holds if and only if the triangle is equilateral.

In ΔABC the following relationship holds:

$$\sum_{cyc} a^4 \sum_{cyc} \frac{a}{4s^2 - a^2} \geq \frac{9abc}{8}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a}{4s^2 - a^2} \geq \frac{2s}{3(s^2 + r^2 + 2Rr)}$$

Proof. Using Bergstrom Inequality, we get:

$$\sum_{cyc} \frac{a}{4s^2 - a^2} = \sum_{cyc} \frac{a^2}{4as^2 - a^3} \geq \frac{(a+b+c)^2}{\sum(4as^2 - a^3)} = \frac{\sum a^2 + 2\sum bc}{4s^2 \sum a - \sum a^3} =$$

$$= \frac{2(s^2 - r^2 - 4Rr) + 2(s^2 + r^2 + 4Rr)}{4s^2 \cdot 2s - 2s(s^2 - 3r^2 - 6Rr)} = \frac{4s^2}{6s(s^2 + r^2 + 2Rr)} = \frac{2s}{3(s^2 + r^2 + 2Rr)}$$

Let's solve the proposed problem:

Using lemma and the know identity: $\sum a^4 = 2[s^4 - s^2(8Rr + 6r^2) + r^2(4R + r)^2]$ we get:

$$Lhs = \sum_{cyc} a^4 \sum_{cyc} \frac{a}{4s^2 - a^2} \geq 2[s^4 - s^2(8Rr + 6r^2) + r^2(4R + r)^2] \cdot \frac{2s}{3(s^2 + r^2 + 2Rr)} \geq$$

$$\frac{\overset{(1)}{\geq} 9abc}{8} = Rhd \text{ where } (1) \Leftrightarrow$$

$$s^4 - s^2(8Rr + 6r^2) + r^2(4R + r)^2 \geq 27Rr(s^2 + r^2 + 2Rr) \Leftrightarrow$$

$$s^2(8s^2 - 48r^2 - 91Rr) + r^2(74R^2 + 37Rr + 8r^2) \geq 0.$$

We distinguish the cases:

Case 1. If $8s^2-48r^2-91Rr\geq 0$ the inequality is obviously.

Case 2. If $8s^2-48r^2-91Rr<0$ the inequality can be rewritten:

$r^2(74R^2+37Rr+8r^2)\geq s^2(8s^2-48r^2-91Rr)$ which result from Gerretsen

Inequality: $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$. We must prove that:

$$r^2(74R^2+37Rr+8r^2)\geq(4R^2+4Rr+3r^2)\left(91Rr+48r^2-8(16Rr-5r^2)\right)\Leftrightarrow$$

$$r(74R^2+37Rr+8r^2)\geq(4R^2+4Rr+3r^2)(-37R+88r)\Leftrightarrow$$

$74R^3-65R^2r-102Rr^2-128r^3\geq 0\Leftrightarrow(R-2r)(74R^2+86Rr+64r^2)\geq 0$ which result from $R\geq 2r$ $(Euler)$.

Equality holds if and only if the triangle is equilateral.

In ΔABC the following relationship holds:

$$\sum_{cyc}a^2\sum_{cyc}\frac{a}{4s^2-a^2}\geq\frac{3S}{2R}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{4s^2-a^2}\geq\frac{2s}{3(s^2+r^2+2Rr)}$$

Proof. Using Bergstrom Inequality, we get:

$$\sum_{cyc}\frac{a}{4s^2-a^2}=\sum_{cyc}\frac{a^2}{4as^2-a^3}\geq\frac{(a+b+c)^2}{\sum(4as^2-a^3)}=\frac{\sum a^2+2\sum bc}{4s^2\sum a-\sum a^3}=$$

$$=\frac{2(s^2-r^2-4Rr)+2(s^2+r^2+4Rr)}{4s^2\cdot 2s-2s(s^2-3r^2-6Rr)}=\frac{4s^2}{6s(s^2+r^2+2Rr)}=\frac{2s}{3(s^2+r^2+2Rr)}$$

Let's solve the proposed problem:

Using lemma and the know identity: $\sum a^2=2(s^2-r^2-4Rr)$ we get:

$$Lhs=\sum_{cyc}a^2\sum_{cyc}\frac{a}{4s^2-a^2}\geq 2(s^2-r^2-4Rr)\cdot\frac{2s}{3(s^2+r^2+2Rr)}\overset{(1)}{\geq}\frac{3S}{2R}=Rhd$$

where $(1)\Leftrightarrow 8R(s^2-r^2-4Rr)\geq 9r(s^2+r^2+2Rr)\Leftrightarrow$

$s^2(8R-9r) \geq r(32R^2+26Rr+9r^2)$ which result from Gerretsen Inequality:

$s^2 \geq 16Rr-5r^2$. We must show that:

$$(16Rr-5r^2)(8R-9r) \geq r(32R^2+26Rr+9r^2) \Leftrightarrow 16R^2-35Rr+6r^2 \geq 0 \Leftrightarrow$$

$(R-2r)(16R-3r) \geq 0$ which result from $R \geq 2r$ $(Euler)$.

Equality holds if and only if the triangle is equilateral.

In ΔABC the following relationship holds:

$$\sum_{cyc} a \sum_{cyc} \frac{a}{4s^2-a^2} \geq \frac{9}{8}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Bergstrom Inequality, we get:

$$\sum_{cyc} \frac{a}{4s^2-a^2} = \sum_{cyc} \frac{a^2}{4as^2-a^3} \geq \frac{(a+b+c)^2}{\sum(4as^2-a^3)} = \frac{\sum a^2+2\sum bc}{4s^2\sum a-\sum a^3} =$$

$$= \frac{2(s^2-r^2-4Rr)+2(s^2+r^2+4Rr)}{4s^2\cdot 2s-2s(s^2-3r^2-6Rr)} = \frac{4s^2}{6s(s^2+r^2+2Rr)} = \frac{2s}{3(s^2+r^2+2Rr)}$$

Let's solve the proposed problem:

Using lemma and the know identity: $\sum a = 2s$ we get:

$$Lhs = \sum_{cyc} a \sum_{cyc} \frac{a}{4s^2-a^2} \geq 2s \cdot \frac{2s}{3(s^2+r^2+2Rr)} \overset{(1)}{\geq} \frac{9}{8} = Rhd$$

where $(1) \Leftrightarrow 32s^2 \geq 27(s^2+r^2+2Rr) \Leftrightarrow 5s^2 \geq 27(r^2+2Rr)$, which result from

Gerretsen Inequality: $s^2 \geq 16Rr-5r^2$.

We must show that:

$$5(16Rr-5r^2) \geq 27(r^2+2Rr) \Leftrightarrow R \geq 2r \ (Euler).$$

Equality if and only if the triangle is equilateral.

JP.320. If in $\Delta ABC, D \in (BC), E \in (CA), F \in (AB)$ such that

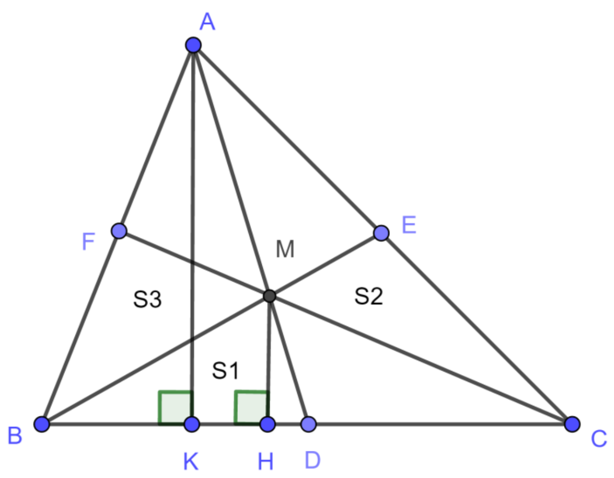
$$AD \cap BE \cap CF = \{M\}, \text{ then:}$$

$$\left(\frac{MD^2}{MA^2} + \frac{ME^2}{MB^2} + \frac{MF^2}{MC^2}\right)(a^8 + b^8 + c^8) \geq 64S^2$$

where S —area of ΔABC.

D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam



Let $S = [ABC];\ S_1 = [MBC];\ S_2 = [MAC];\ S_3 = [MAB]$

$$\Delta AHD \sim \Delta MHD \Rightarrow \frac{MA}{MD} = \frac{MA + MD}{MD} - 1 = \frac{AD}{MD} - 1 = \frac{AK \cdot BC}{MH \cdot BC} - 1 = \frac{S}{S_1} - 1 = \frac{S - S_1}{S_1}$$

$$\Rightarrow \frac{MD}{MA} = \frac{S_1}{S - S_1} = \frac{S}{S - S_1} - 1 \text{ (and analogs)}$$

$$\Omega = \frac{MD}{MA} + \frac{ME}{MB} + \frac{MF}{MC} = S\left(\frac{1}{S - S_1} + \frac{1}{S - S_2} + \frac{1}{S - S_3}\right) - 3 \overset{Bergstrom}{\geq}$$

$$\geq S \cdot \frac{(1 + 1 + 1)^2}{3S - (S_1 + S_2 + S_3)} - 3 = \frac{9S}{2S} - 3 = \frac{3}{2}$$

$$\frac{MD^2}{MA^2} + \frac{ME^2}{MB^2} + \frac{MF^2}{MC^2} \geq \frac{1}{3}\left(\frac{MD}{MA} + \frac{ME}{MB} + \frac{MF}{MC}\right)^2 = \frac{1}{3} \cdot \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

$$\left(\frac{MD^2}{MA^2} + \frac{ME^2}{MB^2} + \frac{MF^2}{MC^2}\right)(a^8 + b^8 + c^8) \geq \frac{3}{4}(a^8 + b^8 + c^8) \overset{AM-GM}{\geq} \frac{3}{4} \cdot 3\sqrt[3]{(abc)^8} =$$

$$=\frac{9}{4}\sqrt[3]{(4Rrs)^8}\overset{(*)}{\geq}64S^4=64(rs)^4$$

$$(*)\Leftrightarrow\left(\frac{9}{4}\right)^3\cdot(4Rrs)^8\geq 64^3\cdot(sr)^{12}\Leftrightarrow 9^3\cdot 4^5\cdot R^8\geq 4^9\cdot(sr)^4\Leftrightarrow$$

$$9^3\cdot R^8\geq(4sr)^4\Leftrightarrow 3^6\cdot R^8\geq(4sr)^4$$

Which is true because:

$$s\leq\frac{3\sqrt{3}}{2}R\ (Mitrinovic);r\leq\frac{R}{2}\ (Euler)\Rightarrow sr\leq\frac{3\sqrt{3}R^2}{4}\Rightarrow 4sr\leq 3\sqrt{3}R^2\Leftrightarrow$$

$$(4sr)^4\leq\left(3\sqrt{3}R^2\right)^4=3^4\cdot 3^2\cdot R^8=3^6\cdot R^8$$

Solution 2 by proposers

$$\left(\sum_{cyc}\left(\frac{MD}{MA}\right)^2\right)\cdot\left(\sum_{cyc}a^8\right)\overset{C-B-S}{\geq}\left(\sum_{cyc}\frac{MD}{MA}\cdot a^4\right)^2,\qquad(1)$$

Let: $x=[MBC],y=[MCA],z=[MAB]$ then

$$[MAB]=\frac{MA\cdot BU}{2},[MAD]=\frac{MD\cdot BU}{2},[MCA]=\frac{MA\cdot CV}{2},[MCD]=\frac{MD\cdot CV}{2}$$

$$\frac{[MBD]}{[MAB]}=\frac{MD}{MA}=\frac{[MCD]}{[MCA]}=\frac{[MBD]+[MCD]}{z+y}=\frac{[MBC]}{y+z}=\frac{x}{y+z}\text{ and analogs.}$$

$$\sum_{cyc}\frac{MD}{MA}\cdot a^4=\sum_{cyc}\frac{x}{y+z}\cdot a^4\overset{?}{\geq}8S^2,\quad(2)$$

From $(1),(2)$ we get

$$\left(\frac{MD^2}{MA^2}+\frac{ME^2}{MB^2}+\frac{MF^2}{MC^2}\right)(a^8+b^8+c^8)\geq 64S^2$$

JP.321.If $x,y,z>0$ such that $x^2+y^2+z^2\leq 12$ then

$$\sqrt{(x^3+1)(y^3+1)(z^3+1)}\leq 27$$

Proposed by George Apostolopoulos- Greece

Lemma.

If $x>0$ then $\sqrt{x^3+1}\leq\frac{x^2+2}{2}$

Proof: Using AM-GM inequality, we have:

$$\sqrt{x^3+1}=\sqrt{(x+1)(x^2-x+1)}\le\frac{(x+1)+(x^2-x+1)}{2}=\frac{x^2+2}{2}$$

Equality holds when $(x+1)=(x^2-x+1)\Leftrightarrow x=2$.

Let's solve the proposed problem.

Using lemma, we get:

$$Lhs=\sqrt{(x^3+1)(y^3+1)(z^3+1)}\le\frac{x^2+2}{2}\cdot\frac{y^2+2}{2}\cdot\frac{z^2+2}{2}=$$

$$=\frac{(x^2+2)(y^2+2)(z^2+2)}{8}=$$

$$=\frac{x^2y^2z^2+2(x^2y^2+y^2z^2+z^2x^2)+4(x^2+y^2+z^2)+8}{8}\le$$

$$\le\frac{64+2\cdot 48+4\cdot 12+8}{8}=27=Rhs,\text{ which result from:}$$

$$x^2+y^2+z^2\le 12,\ x^2y^2z^2\le\left(\frac{x^2+y^2+z^2}{3}\right)^3\le\left(\frac{12}{3}\right)^3=64\text{ and}$$

$$x^2y^2+y^2z^2+z^2x^2\le\frac{(x^2+y^2+z^2)^2}{3}=\frac{12^2}{3}=48$$

Equality holds when $x=y=z=2$.

Remark. The inequality it can be developed.

If $x,y,z>0$ such that $x^2+y^2+z^2\le 12$ then

$$\sqrt[3]{27(x^3+1)(y^3+1)(z^3+1)}\le 27$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma.

If $x>0$ then $\sqrt[3]{3(x^3+1)}\le\frac{x^2+5}{3}$

Proof: Using AM-GM inequality, we have:

$$\sqrt[3]{3(x^3+1)}=\sqrt[3]{3(x+1)(x^2-x+1)}=\frac{3+(x+1)+(x^2-x+1)}{3}=\frac{x^2+5}{3}$$

Equality holds when: $3=(x+1)=(x^2-x+1)\Leftrightarrow x=2$.

Let's solve the proposed problem. Using lemma, we get:

$$Lhs = \sqrt[3]{27(x^3+1)(y^3+1)(z^3+1)} \le \frac{x^2+5}{3}\cdot\frac{y^2+5}{3}\cdot\frac{z^2+5}{3} =$$

$$= \frac{(x^2+5)(y^2+5)(z^2+5)}{27} =$$

$$= \frac{x^2y^2z^2 + 5(x^2y^2+y^2z^2+z^2x^2) + 25(x^2+y^2+z^2) + 125}{27} \le$$

$$\le \frac{64 + 5\cdot 48 + 25\cdot 12 + 125}{27} = 27 = Rhs$$

which result from:

$$x^2+y^2+z^2 \le 12, x^2y^2z^2 \le \left(\frac{x^2+y^2+z^2}{3}\right)^3 \le \left(\frac{12}{3}\right)^3 = 64 \text{ and}$$

$$x^2y^2+y^2z^2+z^2x^2 \le \frac{(x^2+y^2+z^2)^2}{3} = \frac{12^2}{3} = 48$$

Equality holds when $x=y=z=2$.

If $x,y,z>0$ such that $x^2+y^2+z^2 \le 12$ then

$$\sqrt[4]{729(x^3+1)(y^3+1)(z^3+1)} \le 27$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma.

If $x>0$ then $\sqrt[4]{9(x^3+1)} \le \frac{x^2+8}{4}$

Proof: Using AM-GM inequality, we have:

$\sqrt[4]{9(x^3+1)} = \sqrt[4]{9(x+1)(x^2-x+1)} \le \frac{3+3+(x+1)+(x^2-x+1)}{4} = \frac{x^2+8}{4}$**, equality holds when:**

$$3 = (x+1) = (x^2-x+1) \Leftrightarrow x = 2.$$

Let's solve the proposed problem.

Using lemma, we get:

$$Lhs = \sqrt[4]{729(x^3+1)(y^3+1)(z^3+1)} \le$$

$$\le \frac{x^2+8}{4}\cdot\frac{y^2+8}{4}\cdot\frac{y^2+8}{4} = \frac{(x^2+8)(y^2+8)(z^2+8)}{64}$$

$$=\frac{x^2y^2z^2+8(x^2y^2+y^2z^2+z^2x^2)+64(x^2+y^2+z^2)+512}{64}\le$$

$$\le\frac{64+8\cdot 48+64\cdot 12+512}{64}=27=Rhs$$

which result from:

$$x^2+y^2+z^2\le 12, x^2y^2z^2\le\left(\frac{x^2+y^2+z^2}{3}\right)^3\le\left(\frac{12}{3}\right)^3=64 \text{ and}$$

$$x^2y^2+y^2z^2+z^2x^2\le\frac{(x^2+y^2+z^2)^2}{3}=\frac{12^2}{3}=48$$

Equality holds when $x=y=z=2$.

Remark. The inequality it can be generalizated.

If $x,y,z>0$ such that $x^2+y^2+z^2\le 12$ and $n\in\mathbb{N}, n\ge 2$ then

$$\sqrt[n]{3^{3(n-2)}(x^3+1)(y^3+1)(z^3+1)}\le 27$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma.

If $x>0$ and $n\in\mathbb{N}, n\ge 2$ then

$$\sqrt[n]{3^{n-2}(x^3+1)}\le\frac{x^2+3n-4}{n}$$

Proof: Using AM-GM inequality, we have:

$$\sqrt[n]{3^{n-2}(x^3+1)}=\sqrt[n]{3^{n-2}(x+1)(x^2-x+1)}\le$$

$$\le\frac{3+3+\cdots+3+(x+1)+(x^2-x+1)}{n}=$$

$$=\frac{x^2+3(n-2)+2}{n}=\frac{x^2+3n-4}{n}$$

Equality holds when: $3=(x+1)=(x^2-x+1)\Leftrightarrow x=2$.

Let's solve the proposed problem.

Using lemma, we get:

$$Lhs=\sqrt[n]{3^{3(n-2)}(x^3+1)(y^3+1)(z^3+1)}\le$$

$$\leq \frac{x^2+3n-4}{n}\cdot\frac{y^2+3n-4}{n}\cdot\frac{z^2+3n-4}{n}=$$

$$=\frac{(x^2+3n-4)(y^2+3n-4)(z^2+3n-4)}{n^3}=$$

$$=\frac{x^2y^2z^2+(3n-4)(x^2y^2+y^2z^2+z^2x^2)+(3n-4)^2(x^2+y^2+z^2)+(3n-4)^3}{n^3}\leq$$

$$\leq\frac{64+(3n-4)\cdot 48+(3n-4)^2\cdot 12+(3n-4)^3}{n^3}=$$

$$=\frac{[4+(3n-4)]^3}{n^3}=\frac{(3n)^3}{n^3}=27=Rhs$$

which result from:

$$x^2+y^2+z^2\leq 12, x^2y^2z^2\leq\left(\frac{x^2+y^2+z^2}{3}\right)^3\leq\left(\frac{12}{3}\right)^3=64 \text{ and}$$

$$x^2y^2+y^2z^2+z^2x^2\leq\frac{(x^2+y^2+z^2)^2}{3}=\frac{12^2}{3}=48$$

Equality holds when $x=y=z=2$.

Note.

For $n=2$ we get JP.321 din 22-RMM-Autumn Edition 2021, Proposed by George Apostolopoulos-Greece.

Solution 2 by Daniel Văcaru-Romania

We have:

$$\sqrt{x^3+1}=\sqrt{(x+1)(x^2-x+1)}\overset{AM-GM}{\leq}\frac{(x+1)+(x^2-x+1)}{2}=\frac{x^2+2}{2}\Rightarrow$$

$$\sqrt{(x^3+1)(y^3+1)(z^3+1)}\leq\frac{(x^2+2)(y^2+2)(z^2+2)}{8}=$$

$$=\frac{x^2y^2z^2+2(x^2y^2+y^2z^2+z^2x^2)+4(x^2+y^2+z^2)+8}{8}\Rightarrow$$

$$3\sqrt[3]{x^2y^2z^2}\leq x^2+y^2+z^2\Rightarrow 3\sqrt[3]{x^2y^2z^2}\leq 12\Rightarrow x^2y^2z^2\leq 64 \text{ and}$$

$$3(x^2y^2+y^2z^2+z^2x^2)\leq(x^2+y^2+z^2)^2\Rightarrow x^2y^2+y^2z^2+z^2x^2\leq 48$$

We obtain that:

$$\sqrt{(x^3+1)(y^3+1)(z^3+1)}\leq\frac{x^2y^2z^2+2(x^2y^2+y^2z^2+z^2x^2)+4(x^2+y^2+z^2)+8}{8}$$

$$\leq \frac{64+2\cdot 48+4\cdot 12+8}{8}=8+12+6+1=27$$

If $x,y,z>0; a>0$ such that $x^2+y^2+z^2\leq a$ then find the maximum of $E(x,y,z)=\sqrt{(x^3+1)(y^3+1)(z^3+1)}$

Proposed by Daniel Văcaru-Romania

Solution by proposer

We have:

$$\sqrt{x^3+1}=\sqrt{(x+1)(x^2-x+1)}\overset{AM-GM}{\leq}\frac{(x+1)+(x^2-x+1)}{2}=\frac{x^2+2}{2}\Rightarrow$$

$$\sqrt{(x^3+1)(y^3+1)(z^3+1)}\leq\frac{(x^2+2)(y^2+2)(z^2+2)}{8}=$$

$$=\frac{x^2y^2z^2+2(x^2y^2+y^2z^2+z^2x^2)+4(x^2+y^2+z^2)+8}{8}\Rightarrow$$

$$3\sqrt[3]{x^2y^2z^2}\leq x^2+y^2+z^2\Rightarrow 3\sqrt[3]{x^2y^2z^2}\leq a\Rightarrow x^2y^2z^2\leq\frac{a^3}{27};(1)\text{ and}$$

$$3(x^2y^2+y^2z^2+z^2x^2)\leq(x^2+y^2+z^2)^2\Rightarrow x^2y^2+y^2z^2+z^2x^2\leq\frac{a^2}{3};\ \ (2)$$

We obtain that:

$$\sqrt{(x^3+1)(y^3+1)(z^3+1)}\leq\frac{x^2y^2z^2+2(x^2y^2+y^2z^2+z^2x^2)+4(x^2+y^2+z^2)+8}{8}$$

$$\leq\frac{\frac{a^3}{27}+2\cdot\frac{a^2}{3}+4\cdot a+8}{8}=\frac{a^3+18a^2+108a}{216}+1$$

JP.322 Let $a,b,c>0$ such that $a+b+c=6$. Prove that:

$$(a^3+b^3+c^3+12)\left(\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\right)\geq 144$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

We have: $a^2(a-2)^2\geq 0\Leftrightarrow a^4-4a^3+4a^2\geq 0\Leftrightarrow(a^4+4a^2+4)-4(a^3+1)\geq 0$

$\Leftrightarrow 4(a^3+1)\leq(a^2+2)^2$. **So** $2\sqrt{a^3+1}\leq a^2+2\Leftrightarrow$

$$\frac{1}{2\sqrt{a^3+1}} \geq \frac{1}{a^2+2} \Leftrightarrow \frac{a^2}{\sqrt{a^3+1}} \geq 2 \cdot \frac{a^2}{a^2+2}$$

Similarly:

$$\frac{b^2}{\sqrt{b^3+1}} \geq 2 \cdot \frac{b^2}{b^2+2} \text{ and } \frac{c^2}{\sqrt{c^3+1}} \geq 2 \cdot \frac{c^2}{c^2+2}$$

Adding up these three inequalities, we have:

$$\frac{a^2}{\sqrt{a^3+1}} + \frac{b^2}{\sqrt{b^3+1}} + \frac{c^2}{\sqrt{c^3+1}} \geq 2\left(\frac{a^2}{a^2+2} + \frac{b^2}{b^2+2} + \frac{c^2}{c^2+2}\right)$$

Using the Cauchy-Schwartz inequality, we get:

$$\frac{a^2}{\sqrt{a^3+1}} + \frac{b^2}{\sqrt{b^3+1}} + \frac{c^2}{\sqrt{c^3+1}} \geq 2 \cdot \frac{(a+b+c)^2}{(a^2+b^2+c^2)+6} = \frac{72}{(a^2+b^2+c^2)+6}; \quad (*)$$

Now, we know that $a^3+b^3 \geq ab(a+b), b^3+c^3 \geq bc(b+c), c^3+a^3 \geq ca(c+a)$

So, $2(a^3+b^3+c^3) \geq ab(a+b)+bc(b+c)+ca(c+a)$ or

$$2(a^3+b^3+c^3)+(a^3+b^3+c^3) \geq$$

$$\geq a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+(a^3+b^3+c^3) \Leftrightarrow$$

$3(a^3+b^3+c^3) \geq a^2(a+b+c)+b^2(a+b+c)+c^2(a+b+c)$,namely

$$3(a^3+b^3+c^3) \geq (a^2+b^2+c^2)(a+b+c)$$

Since $a+b+c=6$ so $a^2+b^2+c^2 \leq \frac{a^3+b^3+c^3}{2}$. The inequality (*) becomes:

$$\frac{a^2}{\sqrt{a^3+1}} + \frac{b^2}{\sqrt{b^3+1}} + \frac{c^2}{\sqrt{c^3+1}} \geq \frac{72}{\frac{a^3+b^3+c^3}{2}+6} \Leftrightarrow$$

$$(a^3+b^3+c^3+12)\left(\frac{a^2}{\sqrt{a^3+1}} + \frac{b^2}{\sqrt{b^3+1}} + \frac{c^2}{\sqrt{c^3+1}}\right) \geq 144$$

Equality holds when $a=b=c=2$

JP.323. If $a, b, c > 0$ such that $a^2+b^2+c^2=12$ then:

$$\frac{a^4}{\sqrt{a^3+1}} + \frac{b^4}{\sqrt{b^3+1}} + \frac{c^4}{\sqrt{c^3+1}} \geq 16$$

Proposed by George Apostolopoulos- Greece

Solution 1 by Marin Chirciu-Romania

$$1)\ \frac{a^4}{\sqrt{a^3+1}}+\frac{b^4}{\sqrt{b^3+1}}+\frac{c^4}{\sqrt{c^3+1}}\geq 16$$

Lemma.

$$2)\ \text{If } a>0 \text{ then } \sqrt{a^3+1}\leq\frac{a^2+2}{2}$$

Proof: Using means inequality, we have:

$$\sqrt{a^3+1}=\sqrt{(a+1)(a^2-a+1)}\leq\frac{(a+1)+(a^2-a+1)}{2}=\frac{a^2+2}{2}$$

Equality for $(a+1)=(a^2-a+1)\Leftrightarrow a=2$.

Let's solve the proposed problem.

Using lemma, we get:

$$LHS=\sum\frac{a^4}{\sqrt{a^3+1}}\geq\sum\frac{a^4}{\frac{a^2+2}{2}}=2\sum\frac{a^4}{a^2+2}\geq 2\frac{(\sum a^2)^2}{\sum(a^2+2)}=16=RHD.$$

Equality holds when $a=b=c=2$.

Remark: The inequality it can be developed:

1) If $a,b,c>=0$ such that $a^2+b^2+c^2=12$ then

$$\frac{a^4}{\sqrt[3]{3(a^3+1)}}+\frac{b^4}{\sqrt[3]{3(b^3+1)}}+\frac{c^4}{\sqrt[3]{3(c^3+1)}}\geq 16$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma.

$$2)\ \text{If } a>0 \text{ then } \sqrt[3]{3(a^3+1)}\leq\frac{a^2+5}{2}$$

Proof: Using means inequality, we have:

$\sqrt[3]{3(a^3+1)}=\sqrt[3]{3(a+1)(a^2-a+1)}\leq\frac{3+(a+1)+(a^2-a+1)}{3}=\frac{a^2+5}{3}$ with equality for

$3=(a+1)=(a^2-a+1)$.

Let's solve the proposed problem.

Using lemma, we have:

$$LHS=\sum\frac{a^4}{\sqrt[3]{3(a^3+1)}}\ge\sum\frac{a^4}{\frac{a^2+5}{3}}=3\sum\frac{a^4}{a^2+5}\ge 3\frac{(\sum a^2)^2}{\sum(a^2+5)}=3\cdot\frac{12^2}{12+15}=16$$
$$=RHD.$$

Equality holds if and only if $a=b=c=2$.

3) If $a,b,c>0$ such that $a^2+b^2+c^2=12$ then

$$\frac{a^4}{\sqrt[4]{9(a^3+1)}}+\frac{b^4}{\sqrt[4]{9(b^3+1)}}+\frac{c^4}{\sqrt[4]{9(c^3+1)}}\ge 16$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma.

4) If $a>0$ then $\sqrt[4]{9(a^3+1)}\le\frac{a^2+8}{4}$

Proof: Using means inequality, we have:

$$\sqrt[4]{9(a^3+1)}=\sqrt[4]{3\cdot 3\cdot(a+1)(a^2-a+1)}\le$$
$$\le\frac{3+3+(a+1)+(a^2-a+1)}{4}=\frac{a^2+8}{4}$$

Equlity holds when $3=(a+1)=(a^2-a+1)\Leftrightarrow a=2$.

Let's solve the proposed problem.

Using lemma, we have:

$$LHS=\sum\frac{a^4}{\sqrt[4]{9(a^3+1)}}\ge\sum\frac{a^4}{\frac{a^2+8}{4}}=4\sum\frac{a^4}{a^2+8}=4\cdot\frac{12^2}{12+2}=16=RHD$$

Equality holds when $a=b=c=2$.

Remark: The inequality it can be developed.

5) If $a,b,c>0$ such that $a^2+b^2+c^2=12$ and $n\in\mathbb{N}, n=2$ then

$$\frac{a^4}{\sqrt[n]{3^{n-2}(a^3+1)}}+\frac{b^4}{\sqrt[n]{3^{n-2}(b^3+1)}}+\frac{c^4}{\sqrt[n]{3^{n-2}(c^3+1)}}\ge 16$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma.

6) If $a > 0$ and $n \in \mathbb{N}, n \geq 2$ then $\sqrt[n]{3^{n-2}(a^3+1)} \leq \frac{a^2+3n-4}{n}$

Proof: Using means inequality, we have

$$\sqrt[n]{3^{n-2}(a^3+1)} = \sqrt[n]{3\cdot 3\cdot \ldots\cdot 3(a+1)(a^2-a+1)} \leq$$

$$\leq \frac{3+3+\cdots+3+(a+1)+(a^2-a+1)}{n} = \frac{a^2+3(n-2)+2}{n}$$

$$= \frac{a^2+3n-4}{n}$$

Equality holds when $3 = (a+1) = (a^2-a+1) \Leftrightarrow a = 2$.

Let's solve the proposed problem.

Using lemma, we get:

$$LHS = \sum \frac{a^4}{\sqrt[n]{3^{n-2}(a^3+1)}} \geq \sum \frac{a^4}{\frac{a^2+3n-4}{n}} = n\sum \frac{a^4}{a^2+3n-4} \geq$$

$$\geq n\cdot \frac{(\sum a^2)^2}{\sum(a^2+3n-4)} = n\cdot \frac{12^2}{12+3(3n-4)} = RHS.$$

Equality holds if and only if $a = b = c = 2$.

Note.

For $n = 2$ we get JP.323 from 22-RMM-Autumn Edition 2021, Proposed by George Apostolopoulos, Greece.

Solution 2 by Daniel Văcaru-Romania

We have:

$$\sqrt{a^3+1} = \sqrt{(a+1)(a^2-a+1)} \leq \frac{(a+1)+(a^2-a+1)}{2} = \frac{a^2+2}{2}$$

Therefore,

$$\frac{a^4}{\sqrt{a^3+1}} + \frac{b^4}{\sqrt{b^3+1}} + \frac{c^4}{\sqrt{c^3+1}} \geq 2\left(\frac{a^4}{a^2+2} + \frac{b^4}{b^2+2} + \frac{c^4}{c^2+2}\right) \overset{Bergstrom}{\geq}$$

$$\geq 2\left(\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+6}\right) = 16$$

JP.324. Let $x,y,z>0$ such that $x^4+y^4+z^4=3$. Find the maximum value of expression:

$$P=\sqrt{\frac{yz}{7-2x}}+\sqrt{\frac{zx}{7-2y}}+\sqrt{\frac{xy}{7-2z}}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by proposer

Let $x^4+y^4+z^4=3$, and by AM-GM inequality, we have:

$$7-2x=3-2x+4=x^4+y^4+z^4-2x+4=$$

$$=(x^4-2x^2+1)+(x^2-2x+1)+x^2+(x^2+(y^4+z^4+1+1)+1+1)=$$

$$=(x^2-1)^2+(x-1)^2+x^2+(y^4+z^4+1+1)\geq x^2+4\sqrt[4]{(y^4\cdot z^4\cdot 1\cdot 1)}=x^2+4yz$$

$$\Rightarrow 7-2x\geq x^2+4yz\Leftrightarrow \frac{1}{7-2x}\leq\frac{1}{x^2+4yz}\Leftrightarrow\frac{yz}{7-2x}\leq\frac{yz}{x^2+4yz}\Leftrightarrow$$

$$\sqrt{\frac{yz}{7-2x}}\leq\sqrt{\frac{yz}{x^2+4yz}}$$

Similarly:

$$\sqrt{\frac{zx}{7-2y}}\leq\sqrt{\frac{zx}{y^2+4zx}}\text{ and }\sqrt{\frac{xy}{7-2z}}\leq\sqrt{\frac{xy}{z^2+4xy}}$$

Hence

$$P=\sqrt{\frac{yz}{7-2x}}+\sqrt{\frac{zx}{7-2y}}+\sqrt{\frac{xy}{7-2z}}\leq\sqrt{\frac{yz}{x^2+4yz}}+\sqrt{\frac{zx}{y^2+4zx}}+\sqrt{\frac{xy}{z^2+4xy}}\leq$$

$$\leq\sqrt{3\left(\frac{yz}{x^2+4yz}+\frac{zx}{y^2+4zx}+\frac{xy}{z^2+4xy}\right)};\ \ (1)$$

By Cauchy-Schwartz inequality, we have:

$$\frac{x^2}{x^2+4yz}+\frac{y^2}{y^2+4zx}+\frac{z^2}{z^2+4xy}\geq\frac{(x+y+z)^2}{x^2+4yz+y^2+4zx+z^2+4xy}\Leftrightarrow$$

$$\frac{x^2}{x^2+4yz}+\frac{y^2}{y^2+4zx}+\frac{z^2}{z^2+4xy}\geq\frac{(x+y+z)^2}{(x+y+z)^2+2(xy+yz+zx)};\ \ (2)$$

Using inequality: $xy + yz + zx \le \frac{(x+y+z)^2}{3} \overset{(2)}{\Rightarrow}$

$$\frac{x^2}{x^2+4yz} + \frac{y^2}{y^2+4zx} + \frac{z^2}{z^2+4xy} \ge \frac{(x+y+z)^2}{(x+y+z)^2 + 2\cdot\frac{(x+y+z)^2}{3}} =$$

$$= \frac{3(x+y+z)^2}{5(x+y+z)^2} = \frac{3}{5} \Leftrightarrow$$

$$\left(1 - \frac{x^2}{x^2+4yz}\right) + \left(1 - \frac{y^2}{y^2+4zx}\right) + \left(1 - \frac{z^2}{z^2+4xy}\right) \le 3 - \frac{3}{5} = \frac{12}{5} \Leftrightarrow$$

$$\frac{4yz}{x^2+4yz} + \frac{4zx}{y^2+4zx} + \frac{4xy}{z^2+4xy} \le \frac{12}{5} \Leftrightarrow$$

$$\frac{yz}{x^2+4yz} + \frac{zx}{y^2+4zx} + \frac{xy}{z^2+4xy} \le \frac{3}{5}; \quad (3)$$

From (1),(3) we get: $P \le \sqrt{3\cdot\frac{3}{5}} = \frac{3}{\sqrt{5}} \Rightarrow P_{Max} = \frac{3}{\sqrt{5}}$ **and equality occurs if and only if**

$$\begin{cases} x, y, z > 0; x^4 + y^4 + z^4 = 3 \\ x - 1 = y - 1 = z - 1 \\ x = y = z \end{cases} \Leftrightarrow x = y = z = 1.$$

The maximum value of expression P **is** $\frac{3}{\sqrt{5}}$ **for** $x = y = z = 1$.

JP.325. Let be a triangle ABC, A', B', C' the middles of the arches $\widehat{BC}, \widehat{CA}, \widehat{AB}$ made with the circumcircle. Prove that:

$$\frac{AB\cdot BC\cdot CA}{A'B'\cdot B'C'\cdot C'A'} \le \sqrt{\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right)}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Daniel Văcaru-Romania

By geometric consideration, we obtain $A'B' = 2R\cos\frac{A}{2}$ $(and\ analogs)$. **Then**

$$\frac{AB\cdot BC\cdot CA}{A'B'\cdot B'C'\cdot C'A'} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Again by geometry, $cos\frac{B-C}{2}=\frac{h_a}{w_a}$; $w_a=\frac{2bc}{b+c}cos\frac{A}{2}$; $h_a=\frac{bc}{2R}$; $cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$

$$sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}=\frac{r}{4R};cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}=\frac{s}{4R}$$

Then

$$\prod_{cyc}cos\left(\frac{A-B}{2}\right)=\prod_{cyc}\frac{h_a}{w_a}=\prod_{cyc}\frac{b+c}{4Rcos\frac{A}{2}}=\frac{\prod(b+c)}{(4R)^3\prod cos\frac{A}{2}}\geq\frac{8abc}{(4R)^2s}=$$

$$=\frac{2r}{R}=8\prod_{cyc}sin\frac{A}{2}$$

But all $cos\left(\frac{A-B}{2}\right)\leq 1\Rightarrow\prod_{cyc}cos\left(\frac{A-B}{2}\right)\leq 1\Rightarrow\prod_{cyc}cos\left(\frac{A-B}{2}\right)\leq\sqrt{\prod_{cyc}cos\left(\frac{A-B}{2}\right)}$

Therefore, we have:

$$8\prod_{cyc}sin\frac{A}{2}\leq\prod_{cyc}cos\left(\frac{A-B}{2}\right)\leq\sqrt{\prod_{cyc}cos\left(\frac{A-B}{2}\right)}$$

which is our inequality.

Solution 2 by proposer

$$m(\widehat{AB'})=B,m(\widehat{AC'})=C\Rightarrow m(\widehat{B'A'C'})=\frac{\pi-A}{2}$$

Applying sinus theorem, we have:

$$\frac{B'C'}{sin(\widehat{B'A'C'})}=2R\Rightarrow B'C'=2Rsin\left(\frac{\pi-A}{2}\right)=2Rcos\frac{A}{2};\ \ (1)$$

$$\overset{(1)}{\Rightarrow}A'B'\cdot B'C'\cdot C'A'=8R^3cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2};\ \ (2)$$

$$A'B'\cdot B'C'\cdot C'A'=8R^3sinA\ sinB\ sinC;\ \ (3)$$

From $(2),(3)\Rightarrow\frac{AB\cdot BC\cdot AC}{A'B'\cdot B'C'\cdot C'A'}=\frac{sinA\ sinB\ sinC}{cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}}=\frac{8sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}}{2};\ \ (4)$

From (4) we must show: $8sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\leq\sqrt{cos\left(\frac{A-B}{2}\right)cos\left(\frac{B-C}{2}\right)cos\left(\frac{C-A}{2}\right)};\ \ (5)$

But: $sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}=\frac{r}{4R};\ \ (6)$ and $cos\left(\frac{A-B}{2}\right)cos\left(\frac{B-C}{2}\right)cos\left(\frac{C-A}{2}\right)=\frac{s^2+r^2+2Rr}{8R^2};(7)$

From $(5),(6),(7)$ we must show that: $\frac{4r^2}{R^2}\leq\frac{s^2+r^2+2Rr}{8R^2}\Leftrightarrow 32r^2\leq s^2+r^2+2Rr$

$$\Leftrightarrow s^2 \geq 31r^2 - 2Rr; \ \ (8)$$

From $s^2 \geq 16Rr - 5r^2$; (9) we get: $31r^2 - 2Rr \leq 16Rr - 5r^2 \Leftrightarrow 18Rr \geq 36r^2 \Leftrightarrow R \geq 2r$(Euler).

JP.326 In acute $\Delta ABC, AD, BE, CF$ –altitudes and H –othocenter. Prove that:

$$\frac{HA}{HD} + \frac{HB}{HE} + \frac{HC}{HF} \geq 2\left(\left(\frac{R}{r}\right)^2 - 1\right)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by George Florin Şerban-Romania

$$\sin\widehat{HBD} = \sin\left(\frac{\pi}{2} - C\right) = \cos C = \frac{DH}{BH} = \frac{DH}{2R\cos B} \Rightarrow DH = 2R\cos B\cos C$$

$$\sum_{cyc}\frac{HA}{HD} = \sum_{cyc}\frac{2R\cos A}{2R\cos B\cos C} = \sum_{cyc}\frac{\cos A}{\cos B\cos C} = \frac{1}{\cos A\cos B\cos C}\cdot\sum_{cyc}\cos^2 A =$$

$$= \frac{\sum_{cyc}\cos^2 A}{\prod_{cyc}\cos A} = \frac{\frac{6R^2+4Rr+r^2-s^2}{2R^2}}{\frac{s^2-(2R+r)^2}{4R^2}} = \frac{12R^2+8Rr+2r^2-2s^2}{s^2-(2R+r)^2} \overset{(1)}{\geq} 2\left(\left(\frac{R}{r}\right)^2 - 1\right) = \frac{2R^2-2r^2}{r^2}$$

$$(1) \Leftrightarrow 12R^2r^2 + 8Rr^3 + 2r^4 - 2s^2r^2 \geq (2R^2 - 2r^2)s^2 - (2R+r)^2(2R^2 - 2r^2) \Leftrightarrow$$

$$12R^2r^2 + 8Rr^3 + 2r^4 + (2R+r)^2(2R^2 - 2r^2) \overset{(2)}{\geq} s^2(2R^2 - 2r^2 + 2r^2) = 2R^2s^2$$

$$2R^2s^2 \overset{Gerretsen}{\leq} 2R^2(4R^2 + 4Rr + 3r^2) \overset{(3)}{\leq} 12R^2r^2 + 8Rr^3 + 2r^4 + (2R+r)^2(2R^2 - 2r^2)$$

For $k = \frac{R}{r} \geq 2$ we get:

$$2k^2(4k^2 + 4k + 3) \leq 12k^2 + 8k + 2 + (2k+1)^2(2k^2 - 2) \Leftrightarrow$$

$$4k^4 + 4k^3 + 3k^2 \leq 4k^4 + 4k^3 + 3k^2 \ (true) \Rightarrow (3)true \Rightarrow (2)true \Rightarrow (1)true.$$

Solution 2 by proposer

In any ΔABC we have: $\frac{HA}{HD} + \frac{HB}{HE} + \frac{HC}{HF} = \tan A\tan B + \tan B\tan C + \tan C\tan A - 3$; (1)

But: $\tan A\tan B + \tan B\tan C + \tan C\tan A = \frac{s^2-r^2-4Rr}{s^2-(2R+r)^2}$; (2)

From $(1),(2) \Rightarrow$

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}=\frac{s^2-r^2-4Rr}{s^2-(2R+r)^2}-3=\frac{s^2-r^2-4Rr-3s^2+12R^2+12Rr+3r^2}{s^2-(2R+r)^2}=$$

$$=\frac{12R^2+8Rr+2r^2-2s^2}{s^2-(2R+r)^2};(3)$$

From: $s^2 \le 4R^2+4Rr+3r^2$ (Gerretsen); (4)

From $(3),(4)$ we have:

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}\ge\frac{12R^2+8Rr+2r^2-8R^2-8Rr-6r^2}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}\Rightarrow$$

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}\ge\frac{4R^2-4r^2}{2r^2}=2\left(\left(\frac{R}{r}\right)^2-1\right)$$

JP.327 Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\sin^2 A\cdot\cos\frac{B}{2}+\sin^2 B\cdot\cos\frac{C}{2}+\sin^2 C\cdot\cos\frac{A}{2}\le 3\sqrt{3}\left(\frac{1}{2}-\frac{r^3}{R^3}\right)$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Marian Ursărescu-Romania

We must show that:

$$(1-\cos^2 A)\cdot\cos\frac{B}{2}+(1-\cos^2 B)\cdot\cos\frac{C}{2}+(1-\cos^2 C)\cdot\cos\frac{A}{2}\le 3\sqrt{3}\left(\frac{1}{2}-\frac{r^3}{R^3}\right)\Leftrightarrow$$

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}-\left(\cos^2 A\cdot\cos\frac{B}{2}+\cos^2 B\cdot\cos\frac{C}{2}+\cos^2 C\cdot\cos\frac{A}{2}\right)\le\frac{3\sqrt{3}}{2}-\frac{3\sqrt{3}r^3}{R^3};\ (1)$$

But $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\le\frac{3\sqrt{3}}{2}(R.Kooistra);(2)$

From (1),(2) we must show that:

$$\cos^2 A\cdot\cos\frac{B}{2}+\cos^2 B\cdot\cos\frac{C}{2}+\cos^2 C\cdot\cos\frac{A}{2}\ge\frac{3\sqrt{3}r^3}{R^3};\ (3)$$

$$\frac{\cos^2 A}{\frac{1}{\cos\frac{B}{2}}}+\frac{\cos^2 B}{\frac{1}{\cos\frac{C}{2}}}+\frac{\cos^2 C}{\frac{1}{\cos\frac{A}{2}}}\overset{Bergstrom}{\ge}\frac{(\cos A+\cos B+\cos C)^2}{\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}};\ \ (4)$$

From (3),(4) we must show that:

$$\frac{(cosA+cosB+cosC)^2}{\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}}\ge\frac{3\sqrt{3}r^3}{R^3}\Leftrightarrow$$

$$\left(1+\frac{R}{r}\right)^2\ge\frac{3\sqrt{3}r^3}{R^3}\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}\right);\ (5)$$

But: $\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}=\frac{a^2+b^2+c^2}{2S};\ (6)$

Because: $2S\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}\right)=\sum_{cyc}\frac{bcsinA}{cos\frac{A}{2}}=2\sum_{cyc}bcsin\frac{A}{2}\Leftrightarrow$

$$\sum_{cyc}a^2\ge2\sum_{cyc}bc\cdot sin\frac{A}{2}\Leftrightarrow$$

$$a^2-2a\left(b\cdot sin\frac{C}{2}+c\cdot sin\frac{B}{2}\right)+b^2+c^2-2bc\cdot sin\frac{A}{2}\ge0,\text{ true because}$$

$$\Delta=-4\left(b\cdot cos\frac{C}{2}-c\cdot cos\frac{B}{2}\right)^2\le0$$

From (6) we have:

$$\sum_{cyc}\frac{1}{cos\frac{A}{2}}\le\frac{a^2+b^2+c^2}{2S}=\frac{2(s^2-r^2-4Rr)}{2sr}\overset{\substack{Gerretsen\\Euler}}{\le}\frac{9R^2}{2sr}\overset{Mitrinovic}{\le}\frac{9R^2}{2\cdot3\sqrt{3}r^2}=\frac{3R^2}{2\sqrt{3}r^2};\ (7)$$

From (5),(6) we must show:

$$\left(1+\frac{r}{R}\right)^2\ge\frac{3\sqrt{3}r^3}{R^3}\cdot\frac{3R^2}{2\sqrt{3}r^2}\Leftrightarrow\left(1+\frac{r}{R}\right)^2\ge\frac{9r}{2R}\Leftrightarrow1+\frac{2r}{R}+\frac{r^2}{R^2}\ge\frac{9r}{2R}\Leftrightarrow$$

$$2R^2+4Rr+2r^2\ge9Rr\Leftrightarrow(R-2r)(2R-r)\ge0\ true\ by\ R\ge2r(Euler)$$

Solution 2 by proposer

We have:

$$sin^2A\cdot cos\frac{B}{2}+sin^2B\cdot cos\frac{C}{2}+sin^2C\cdot cos\frac{A}{2}=$$

$$=(1-cos^2A)\cdot cos\frac{B}{2}+(1-cos^2B)\cdot cos\frac{C}{2}+(1-cos^2C)\cdot cos\frac{A}{2}=$$

$$=\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)-\left(\frac{\cos^2 A}{\sec\frac{B}{2}}+\frac{\cos^2 B}{\sec\frac{C}{2}}+\frac{\cos^2 C}{\sec\frac{A}{2}}\right)$$

It is well-known that in any triangle ABC holds:

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\le\frac{3\sqrt{3}}{2}$$

So, $\sin^2 A\cdot\cos\frac{B}{2}+\sin^2 B\cdot\cos\frac{C}{2}+\sin^2 C\cdot\cos\frac{A}{2}\le\frac{3\sqrt{3}}{2}-\left(\frac{\cos^2 A}{\sec\frac{B}{2}}+\frac{\cos^2 B}{\sec\frac{C}{2}}+\frac{\cos^2 C}{\sec\frac{A}{2}}\right)$

Using the Cauchy-Schwartz inequality, we have:

$$\frac{\cos^2 A}{\sec\frac{B}{2}}+\frac{\cos^2 B}{\sec\frac{C}{2}}+\frac{\cos^2 C}{\sec\frac{A}{2}}\ge\frac{(\cos A+\cos B+\cos C)^2}{\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}}$$

We know that $\cos A+\cos B+\cos C=1+\frac{r}{R}$, $R\ge 2r$**(Euler) and** $s\ge 3\sqrt{3}r$, **where**

s **—denotes the semiperimeter of the triangle** ABC. **So,**

$$\sin^2 A\cdot\cos\frac{B}{2}+\sin^2 B\cdot\cos\frac{C}{2}+\sin^2 C\cdot\cos\frac{A}{2}\le\frac{3\sqrt{3}}{2}-\frac{\left(1+\frac{r}{R}\right)^2}{\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}};(*)$$

Also, we have:

$$\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)^2+\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)^2+\left(\cos\frac{C}{2}-\cos\frac{A}{2}\right)^2\ge 0\Leftrightarrow$$

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\ge\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\Leftrightarrow$$

$$\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\ge\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\Leftrightarrow$$

$$\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}\ge\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\Leftrightarrow$$

$$\frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\left(\frac{C}{2}+\frac{A}{2}\right)}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\left(\frac{A}{2}+\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}\ge\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\Leftrightarrow$$

$$\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\frac{C}{2}\cos\frac{A}{2}+\cos\frac{c}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}\geq$$

$$\geq \sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\Leftrightarrow$$

$$\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq \sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\Leftrightarrow$$

$$\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\leq 2\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

So, the inequality (*) gives:

$$\sin^2A\cdot\cos\frac{B}{2}+\sin^2B\cdot\cos\frac{C}{2}+\sin^2C\cdot\cos\frac{A}{2}\leq\frac{3\sqrt{3}}{2}-\frac{\left(1+\frac{r}{R}\right)^2}{2\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)}$$

We know that: $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{4R+r}{s}$. **So,**

$$\sin^2A\cdot\cos\frac{B}{2}+\sin^2B\cdot\cos\frac{C}{2}+\sin^2C\cdot\cos\frac{A}{2}\leq\frac{3\sqrt{3}}{2}-\frac{\frac{(R+r)^2}{R^2}}{2\cdot\frac{4R+r}{s}}=$$

$$=\frac{3\sqrt{3}}{2}-\frac{s(R+r)^2}{2R^2\cdot(4R+r)}=3\sqrt{3}\left(\frac{1}{2}-\frac{r^3}{R^3}\right)$$

Equality holds if and only if the triangle ABC is equilateral.

JP.328 In ΔABC the following relationship holds:

$$4\leq \sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\leq\frac{2R}{r}$$

Proposed by George Apostolopoulos-Greece

Solution 1 by Marin Chirciu-Romania

$$4\leq \sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\leq\frac{2R}{r};(1)$$

Lemma:

In ΔABC the following relationship holds:

$$sec^2\frac{A}{2}+sec^2\frac{B}{2}+sec^2\frac{C}{2}=1+\left(\frac{4R+r}{s}\right)^2;(2)$$

Proof: We have:

$$\sum sec^2\frac{A}{2}=\sum\frac{1}{cos^2\frac{A}{2}}=\sum\frac{bc}{s(s-a)}=\frac{s^2+(4R+r)^2}{s^2}=1+\left(\frac{4R+r}{s}\right)^2 \text{ which result from:}$$

$$\sum\frac{bc}{s-a}=\frac{s^2+(4R+r)^2}{s}\Leftrightarrow\sum\frac{bc}{s(s-a)}=\frac{s^2+(4R+r)^2}{s^2}$$

Using lemma, LHS the inequality can be written:

$$1+\left(\frac{4R+r}{s}\right)^2\geq 4\Leftrightarrow 4R+r\geq s\sqrt{s}\ (Doucet\ inequality)$$

Equality holds if and only if the triangle is equilateral.

Using lemma, RHS the inequality can be written

$$1+\left(\frac{4R+r}{s}\right)^2\leq\frac{2R}{r}\Leftrightarrow s^2(2R-r)\geq r(4R+r)^2 \text{ which result from}$$

$$s^2\geq 16Rr-5r^2\ (Gerretsen\ inequality)$$

We must show that:

$$(16Rr-5r^2)(2R-r)\geq r(4R+r)^2\Leftrightarrow 8R^2-17Rr+2r^2\geq 0$$

$$\Leftrightarrow (R-2r)(8R-r)\geq 0 \text{ true from } R\geq 2r\ (Euler).$$

Equality holds if and only in the triangle is equilateral.

Remark:

In ΔABC the following relationship holds:

$$5-\frac{2r}{R}\leq sec^2\frac{A}{2}+sec^2\frac{B}{2}+sec^2\frac{C}{2}\leq 1+\frac{3R}{2r};(3)$$

Proposed by Marin Chirciu-Romania

For LHS of the inequality we have:

Using Lemma, we get:

$$1+\left(\frac{4R+r}{s}\right)^2=1+\frac{(4R+r)^2}{s^2}\geq 1+\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}=1+\frac{2(2R-r)}{R}=5-\frac{2r}{R} \text{ which result from}$$

Blundon-Gerretsen: $s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$

Equality holds if and only if the triangle is equilateral.

For RHS of the inequality we have:

Using lemma the inequality becomes:

$$1+\left(\frac{4R+r}{s}\right)^2 \le 1+\frac{3R}{2r} \Leftrightarrow \frac{(4R+r)^2}{s^2} \le \frac{3R}{2r} \Leftrightarrow 3Rs^2 \ge 2r(4R+r)^2$$

which result from Gerretsen inequality: $s^2 \ge 16Rr-5r^2$.

We must show that:

$$3R(16Rr-5r^2) \ge 2r(4R+r)^2 \Leftrightarrow 16R^2-31Rr-2r^2 \ge 0 \Leftrightarrow (R-2r)(16R+r) \ge 0$$
true by $R \ge 2r$ $(Euler)$.

Equality holds if and only if the triangle is equilateral.

Remark: The inequality (3) is much stronger that the inequality (1).

In ΔABC the following relationship holds:

$$4 \le 5-\frac{2r}{R} \le sec^2\frac{A}{2}+sec^2\frac{B}{2}+sec^2\frac{C}{2} \le 1+\frac{3R}{2r} \le \frac{2R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Marin Chirciu-Romania

See the inequality (3) and Euler inequality $R \ge 2r$

Equality holds if and only if the triangle is equilateral.

Solution 2 by Daniel Văcaru-Romania

We have:

$$sec^2\frac{A}{2}+sec^2\frac{B}{2}+sec^2\frac{C}{2} = \sum_{cyc}\left(1+tan^2\frac{A}{2}\right) \ge 3+\sum_{cyc} tan\frac{A}{2}tan\frac{B}{2} = 3+1 = 4$$

On the other hand, we have:

$$\sum_{cyc} sec^2\frac{A}{2} = 1+\frac{(4R+r)^2}{s^2} \overset{Gerretsen}{\le} 1+\frac{(4R+r)^2}{16Rr-5r^2} = \frac{16R^2+24Rr-4r^2}{16Rr-5r^2}$$

We prove that:

$$\frac{16R^2+24Rr-4r^2}{16Rr-5r^2} \le \frac{2R}{r} \Leftrightarrow 8R^2+12Rr-2r^2 \le 16R^2-5Rr \Leftrightarrow$$

$$8R^2-17Rr+2r^2\ge 0 \Leftrightarrow 8\left(\frac{R}{r}\right)^2-17\frac{R}{r}+2\ge 0.$$

We obtain:

$\frac{R}{r}\in\left(-\infty,\frac{1}{8}\right]\cup[2,\infty)$. But $R\ge 2r(Euler)\Rightarrow\frac{R}{r}\ge 2$, which proves the required inequality.

JP.329.In ΔABC the following relationship holds:

$$6\sqrt{3}\cdot r\le\frac{m_a}{\cos\frac{A}{2}}+\frac{m_b}{\cos\frac{B}{2}}+\frac{m_c}{\cos\frac{C}{2}}\le\frac{3\sqrt{6}}{2}\cdot R\sqrt{\frac{R}{r}}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Marin Chirciu-Romania

$$6\sqrt{3}\cdot r\le\frac{m_a}{\cos\frac{A}{2}}+\frac{m_b}{\cos\frac{B}{2}}+\frac{m_c}{\cos\frac{C}{2}}\le\frac{3\sqrt{6}}{2}\cdot R\sqrt{\frac{R}{r}};(1)$$

Lemma 1.

1) In ΔABC the following relationship holds:

$$\sum\frac{m_a}{\cos\frac{A}{2}}\ge\frac{27Rr}{s}$$

Proof: Using AM-GM inequality, we have:

$$\sum\frac{m_a}{\cos\frac{A}{2}}\ge 3\sqrt[3]{\prod\frac{m_a}{\cos\frac{A}{2}}}=3\sqrt[3]{\frac{m_am_bm_c}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}}\overset{(1)}{\ge}3\sqrt[3]{\frac{rs^2}{\frac{s}{4R}}}=3\sqrt[3]{4Rrs}\overset{(2)}{\ge}\frac{27Rr}{s},\text{where}$$

(1) result from $m_am_bm_c\ge rs^2$ and $m_a\ge\sqrt{s(s-a)}$; $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{4R}$.

Inequality $(2)\Leftrightarrow 3\sqrt[3]{4Rrs}\ge\frac{27Rr}{s}\Leftrightarrow 2s^2\ge 27Rr$ (Coșniță-Turtoiu, 1965),

true from Gerretsen $s^2\ge 16Rr-5r^2$ and Euler $R\ge 2r$.

Equality holds if and only if the triangle s equilateral.

Lemma 2.

2) In ΔABC the following relationship holds:

$$\sum \frac{m_a}{\cos\frac{A}{2}} \le \frac{3R(4R+r)}{s}$$

Proof: The triplets: (m_a, m_b, m_c) and $\left(\frac{1}{\cos\frac{A}{2}}, \frac{1}{\cos\frac{B}{2}}, \frac{1}{\cos\frac{C}{2}}\right)$ are inversely ordered.

With Chebyshev's inequality, we get:

$$\sum \frac{m_a}{\cos\frac{A}{2}} = \frac{1}{3}\left(\sum m_a\right)\left(\sum \frac{1}{\cos\frac{A}{2}}\right) \overset{(3)}{\le} \frac{1}{3}(4R+r)\cdot\frac{9R}{s} = \frac{3R(4R+r)}{s},$$

where (3) result from $\sum m_a = 4R + r$; $\sum \frac{1}{\cos\frac{A}{2}} \le \frac{9R}{s}$ true from lemma 3)

Lemma 3.

In ΔABC the following relationship holds:

$$\frac{1}{\cos\frac{A}{2}} + \frac{1}{\cos\frac{B}{2}} + \frac{1}{\cos\frac{C}{2}} \le \frac{9R}{s}$$

Proof:

Using the inequality $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \le \frac{(x+y+z)^2}{3xyz}$ for $x = \cos\frac{A}{2}$; $y = \cos\frac{B}{2}$; $z = \cos\frac{C}{2}$ we have:

$$\frac{1}{\cos\frac{A}{2}} + \frac{1}{\cos\frac{B}{2}} + \frac{1}{\cos\frac{C}{2}} \le \frac{\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)^2}{3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} \overset{Jensen}{\le} \frac{\left(\frac{3\sqrt{3}}{2}\right)^2}{3\cdot\frac{s}{4R}} = \frac{\frac{27}{4}}{\frac{3s}{4R}} = \frac{9R}{s}$$

Equality holds if and only if the triangle is equilateral.

Let's solve the proposed problem.

Using the abrove lemma's we get:

$$\frac{27Rr}{s} \le \sum \frac{m_a}{\cos\frac{A}{2}} \le \frac{3R(4R+r)}{s}$$

For LHS of the inequality, we get:

Using lemma 1 we have: $\sum \frac{m_a}{\cos\frac{A}{2}} \ge \frac{27Rr}{s} \overset{(4)}{\ge} 6\sqrt{3}\cdot r$, where (4)$\Leftrightarrow \frac{27Rr}{s} \ge 6\sqrt{3}\cdot r$

$\Leftrightarrow Rs \ge 6\sqrt{3}\cdot r^2$ true by Mitrinovic: $s \ge 3\sqrt{3}\cdot r$ and Euler: $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

For the RHS of the inequality, we get:

Using lemma 2) we have:

$$\sum \frac{m_a}{\cos\frac{A}{2}} \le \frac{3R(4R+r)}{s} \overset{(5)}{\le} \frac{3\sqrt{6}}{2} \cdot R\sqrt{\frac{R}{r}}$$

where (5)$\Leftrightarrow \frac{3R(4R+r)}{s} \le \frac{3\sqrt{6}}{2} \cdot R\sqrt{\frac{R}{r}} \Leftrightarrow 2r(4R+r)^2 \le 3Rs^2$ which result from

Gerretsen inequality: $s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$. We must show that:

$$2r(4R+r)^2 \le 3R \cdot \frac{r(4R+r)^2}{R+r} \Leftrightarrow R \ge 2r \ (Euler).$$

Equality holds if and only if the triangle is equilateral.

Remark: The inequality can be much stronger:

3) In ΔABC the following relationship holds:

$$\frac{27Rr}{s} \le \frac{m_a}{\cos\frac{A}{2}} + \frac{m_b}{\cos\frac{B}{2}} + \frac{m_c}{\cos\frac{C}{2}} \le \frac{3R(4R+r)}{s}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

See up lemma's.

Equality holds if and only if the triangle is equilateral.

Remark: The inequalities (4) are much strongers than (1)

4) In ΔABC the following relationship hods:

$$6\sqrt{3} \cdot r \le \frac{27Rr}{s} \le \sum \frac{m_a}{\cos\frac{A}{2}} \le \frac{3R(4R+r)}{s} \le \frac{3\sqrt{6}}{2} \cdot R\sqrt{\frac{R}{r}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

See the up proof's

Equality holds if and only if the triangle is equilateral

Remark: In the same class of problems:

5) In ΔABC the following relationship holds:

$$\frac{9R}{2} \le m_a sin\frac{A}{2} + m_b sin\frac{B}{2} + m_c sin\frac{C}{2} \le \frac{4R+r}{2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

We demonstrate the helpful results:

Lemma 1.

6) In ΔABC the following relationship holds:

$$\sum m_a sin\frac{A}{2} \ge \frac{9r}{2}$$

Proof:

Using AM-GM inequality, we have:

$$\sum m_a sin\frac{A}{2} \ge 3\sqrt[3]{\prod m_a sin\frac{A}{2}} = 3\sqrt[3]{m_a m_b m_c \prod sin\frac{A}{2}} \overset{(1)}{\ge} 3\sqrt[3]{rs^2 \cdot \frac{r}{4R}} = 3\sqrt[3]{\frac{r^2s^2}{4R}} \overset{(2)}{\ge} \frac{9r}{2} \text{ where}$$

(1) result from $m_a m_b m_c \ge rs^2$, which result from

$$m_a \ge \sqrt{s(s-a)} \text{ and } \prod sin\frac{A}{2} = \frac{r}{4R}$$

The inequality (2)$\Leftrightarrow 3\sqrt[3]{\frac{r^2s^2}{4R}} \le \frac{9r}{2} \Leftrightarrow 2s^2 \ge 27Rr$,

(Coșniță-Turtoiu, 1965), true from Gerretsen $s^2 \ge 16Rr - 5r^2$ and Euler $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Lemma 2.

7) In ΔABC the following relationship holds:

$$\sum m_a sin\frac{A}{2} \le \frac{4R+r}{2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Proof: The triplets: (m_a, m_b, m_c) **and** $\left(\frac{1}{sin\frac{A}{2}}, \frac{1}{sin\frac{B}{2}}, \frac{1}{sin\frac{C}{2}}\right)$ **are inversely ordered.**

With Chebyshev's Inequality, we get:

$$\sum m_a sin\frac{A}{2} \le \frac{1}{3}\sum m_a \sum sin\frac{A}{2} \overset{(3)}{\le} \frac{1}{3}(4R+r)\cdot\frac{3}{2} = \frac{4R+r}{2}$$

where (3) result from $\sum m_a \le 4R + r$ **and** $\sum sin\frac{A}{2} \le \frac{3}{2}$ **true from Jensen inequality for concave function** $t \to sin\,t$ **on** $(0,\pi)$.

Let's solve the proposed problem.

Using the up lemma's, we get:

$$\frac{9r}{2} \le \sum m_a sin\frac{A}{2} \le \frac{4R+r}{2}$$

Equality holds if and only if the triangle is equilateral.

8) In ΔABC the following relationship holds:

$$3\sqrt{3}\cdot r \le m_a tan\frac{A}{2} + m_b tan\frac{B}{2} + m_c tan\frac{C}{2} \le \frac{(4R+r)^2}{3s}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

We demonstrate the helpful results:

Lemma 1.

9) In ΔABC the following relationship holds

$$\sum m_a tan\frac{A}{2} \ge 3\sqrt{3}\cdot r$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Proof: Using AM-GM inequality, we have:

$$\sum m_a tan\frac{A}{2} \ge 3\sqrt[3]{\prod m_a tan\frac{A}{2}} = 3\sqrt[3]{m_a m_b m_c \prod tan\frac{A}{2}} \overset{(1)}{\ge} 3\sqrt[3]{rs^2\cdot\frac{r}{s}} = 3\sqrt[3]{r^2 s} \overset{(2)}{\ge} 3\sqrt{3}\cdot r,$$

where (1) result from $m_a m_b m_c \ge rs^2$ **and** $\prod tan\frac{A}{2} = \frac{r}{s}$.

The inequality (2)⇔ $3\sqrt[3]{r^2s} \geq 3\sqrt{3}\cdot r \Leftrightarrow s \geq 3\sqrt{3}\cdot r$ (Mitrinović)

Equality holds if and only if the triangle is equilateral.

Lemma 2.

10) In ΔABC the following relationship holds

$$\sum m_a tan\frac{A}{2} \leq \frac{(4R+r)^2}{3s}$$

Proof: The triplets: (m_a, m_b, m_c) and $\left(tan\frac{A}{2}, tan\frac{B}{2}, tan\frac{C}{2}\right)$ are inversely ordered.

With Chebyshev's inequality, we get:

$$\sum m_a tan\frac{A}{2} \leq \frac{1}{3}\sum m_a \sum tan\frac{A}{2} \overset{(3)}{\leq} \frac{1}{3}(4R+r)\frac{(4R+r)}{s} = \frac{(4R+r)^2}{3s}$$

where (3) result from $\sum m_a \leq 4R+r$ and $\sum tan\frac{A}{2} = \frac{4R+r}{s}$

Let's solve the proposed problem.

Using the up lemma's, we get: $3\sqrt{3}\cdot r \leq \sum m_a tan\frac{A}{2} \leq \frac{(4R+r)^2}{3s}$

Equality if and only if the triangle is equilateral.

Solution 2 by Daniel Văcaru-Romania

By CBS we have:

$$\frac{m_a}{cos\frac{A}{2}} + \frac{m_b}{cos\frac{B}{2}} + \frac{m_c}{cos\frac{C}{2}} \leq \sqrt{\sum_{cyc} m_a^2}\cdot\sqrt{\sum_{cyc} sec^2\frac{A}{2}} = \sqrt{\frac{3}{4}\sum_{cyc} a^2}\cdot\sqrt{\sum_{cyc} sec^2\frac{A}{2}} =$$

$$= R\sqrt{3}\cdot\sqrt{\sum_{cyc} sin^2 A}\cdot\sqrt{\sum_{cyc} sec^2\frac{A}{2}}.$$

But $\sum_{cyc} sin^2 A \leq \frac{9}{4} \Rightarrow \sqrt{\sum_{cyc} sin^2 A} \leq \frac{3}{2}$; (1) and $\sum_{cyc} sec^2\frac{A}{2} \leq \frac{2R}{r} \Rightarrow$

$$\sqrt{\sum_{cyc} sec^2\frac{A}{2}} \leq \sqrt{\frac{2R}{r}};\ (2)$$

It follows that

$$\frac{m_a}{\cos\frac{A}{2}}+\frac{m_b}{\cos\frac{B}{2}}+\frac{m_c}{\cos\frac{C}{2}}\le R\sqrt{3}\cdot\sqrt{\sum_{cyc}\sin^2A}\cdot\sqrt{\sum_{cyc}\sec^2\frac{A}{2}}\overset{(1),(2)}{\le}R\sqrt{3}\cdot\frac{3}{2}\cdot\sqrt{\frac{2R}{r}}=\frac{3\sqrt{6}}{2}\cdot R\sqrt{\frac{2R}{r}}$$

Again, we have: $\frac{m_a}{\cos\frac{A}{2}}=\frac{\sqrt{2(b^2+c^2)-a^2}}{2\cos\frac{A}{2}}=\frac{\sqrt{b^2+c^2+2bc\cdot\cos A}}{2\cos\frac{A}{2}}\overset{AM-GM}{\ge}\frac{\sqrt{2bc+2bc\cdot\cos A}}{2\cos\frac{A}{2}}=\sqrt{bc}$

We obtain:

$$\sum_{cyc}\frac{m_a}{\cos\frac{A}{2}}\ge\sum_{cyc}\sqrt{bc}=\sqrt{abc}\sum_{cyc}\frac{1}{\sqrt{a}}\overset{a\to\frac{1}{\sqrt{a}}-convex}{\ge}\sqrt{abc}\cdot\frac{3}{\sqrt{\frac{a+b+c}{3}}}=$$

$$=\sqrt{4Rrs}\cdot\frac{3\sqrt{3}}{\sqrt{2s}}=\sqrt{2Rr}\cdot3\sqrt{3}\overset{Euler}{\ge}\sqrt{4r^2}\cdot3\sqrt{3}=6\sqrt{3}r.$$

In conclusion, we have, indeed,

$$6\sqrt{3}\cdot r\le\frac{m_a}{\cos\frac{A}{2}}+\frac{m_b}{\cos\frac{B}{2}}+\frac{m_c}{\cos\frac{C}{2}}\le\frac{3\sqrt{6}}{2}\cdot R\sqrt{\frac{R}{r}}$$

JP.330. Let $a,b,c>0$ such that $abc=1$. Find the maximum value of the expression:

$$P=\sqrt{\frac{ab}{a^5+b^3-2a+6}}+\sqrt{\frac{bc}{b^5+c^3-2b+6}}+\sqrt{\frac{ca}{c^5+a^3-2c+6}}$$

Proposed by Hoang Le Nhat-Hanoi-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

For $a>0$ we have: $a^5-a^3-2a+2\ge0\Leftrightarrow a^3(a^2-1)-2(a-1)\ge0\Leftrightarrow$

$$(a-1)(a^4-a^3-2)\ge0\Leftrightarrow(a-1)(a^4-1a^3-1)\ge0\Leftrightarrow$$

$$(a-1)^2[(a+1)(a^2+1)+a^2+a+1]\ge0\ (true\ \forall a>0)$$

Equality for $a=1$. So, we have:

$$a^5-2a+2\ge a^3\Rightarrow a^5+b^3-2a+6\ge a^3+b^3+4=a^3+b^3+1+3\overset{AM-GM}{\ge}$$

$$\ge3\sqrt[3]{a^3\cdot b^3}+3=3ab+3=3(ab+1)$$

$$\Rightarrow a^5+b^3-2a+6\geq 3(ab+1)$$

Similarly: $a^5+b^3-2a+6\geq 3(bc+1)$ and $c^5+a^3-2c+6\geq 3(ac+1)$

Therefore,

$$P=\sum_{cyc}\sqrt{\frac{ab}{a^5+b^3-2a+6}}\leq\sum_{cyc}\sqrt{\frac{ab}{3(ab+1)}}\overset{abc=1}{=}\frac{1}{\sqrt{3}}\cdot\sum_{cyc}\sqrt{\frac{abc}{abc+c}}=$$

$$=\frac{1}{\sqrt{3}}\cdot\sum_{cyc}\sqrt{\frac{1}{a+1}}$$

Now, because $abc=1$ let $a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$; $(x,y,z>0)$

$$\Omega=\left(\sum_{cyc}\sqrt{\frac{1}{a+1}}\right)^2=\left(\sum_{cyc}\sqrt{\frac{1}{\frac{x}{y}+1}}\right)^2=\left(\sum_{cyc}\sqrt{\frac{y}{x+y}}\right)^2=$$

$$=\left(\sum_{cyc}\sqrt{\frac{y}{(x+y)(y+z)}}\cdot\sqrt{y+z}\right)^2\overset{BCS}{\leq}\left(\sum_{cyc}\frac{y}{(x+y)(y+z)}\right)\left(\sum_{cyc}(y+z)\right)=$$

$$=2\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{y}{(x+y)(y+z)}\right)=\frac{4(x+y+z)(xy+yz+zx)}{(x+y)(y+z)(z+x)}$$

On the other hand,

$$9(x+y)(y+z)(z+x)-8(x+y+z)(xy+yz+zx)=$$

$$=x(y-z)^2+y(z-x)^2+z(x-y)^2\geq 0\Rightarrow$$

$$9(x+y)(y+z)(z+x)\geq 8(x+y+z)(xy+yz+zx)\Rightarrow$$

$$\frac{(x+y+z)(xy+yz+zx)}{(x+y)(y+z)(z+x)}\leq\frac{9}{8}$$

So, we have:

$$\Omega\leq\frac{4\cdot 9}{8}=\frac{9}{2}\Rightarrow\left(\sum_{cyc}\sqrt{\frac{1}{a+1}}\right)^2\leq\frac{9}{2}\Rightarrow\sum_{cyc}\sqrt{\frac{1}{a+1}}\leq\frac{3}{\sqrt{2}}\Rightarrow$$

$$P\leq\frac{1}{\sqrt{3}}\cdot\sum_{cyc}\sqrt{\frac{1}{a+1}}\leq\frac{1}{\sqrt{3}}\cdot\frac{3}{\sqrt{2}}=\frac{3}{\sqrt{6}}=\frac{\sqrt{6}}{2}\Rightarrow$$

$$P_{min} = \frac{\sqrt{6}}{2} \Leftrightarrow \begin{cases} abc = 1 \\ a = b = c > 0 \end{cases} a = b = c = 1.$$

Solution 2 by proposer

We have: $a^5 - a^3 - 2a + 2 = a^4(a-1) + a^3(a-1) - 2(a-1) =$

$$= (a-1)(a^4 + a^3 - 2) = (a-1)[a^3(a-1) + 2a^2(a-1) + 2a(a-1) + 2(a-1)] =$$

$$= (a-1)^2(a^3 + 2a^2 + 2a + 2) \geq 0, \forall a > 0$$

$$\Rightarrow a^5 + b^3 - 2a + 6 \geq 0 \Rightarrow a^5 + b^3 - 2a + 6 \geq (a^3 + b^3 + 1) + 3 \overset{CBS}{\geq} 3 \cdot ab \cdot a + 3$$

$$= 3(ab+1)$$

$$\frac{ab}{a^5 + b^3 - 2a + 6} \leq \frac{ab}{3(ab+1)} \Leftrightarrow \sqrt{\frac{ab}{a^5 + b^3 - 2a + 6}} \leq \sqrt{\frac{ab}{3(ab+1)}} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{1 + \frac{1}{ab}}}$$

Similarly:

$$\sqrt{\frac{bc}{a^5 + b^3 - 2a + 6}} \leq \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{1 + \frac{1}{bc}}}$$

$$\sqrt{\frac{ca}{c^5 + a^3 - 2c + 6}} \leq \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{1 + \frac{1}{ca}}}$$

Hence,

$$P = \sqrt{\frac{ab}{a^5 + b^3 - 2a + 6}} + \sqrt{\frac{bc}{a^5 + b^3 - 2a + 6}} + \sqrt{\frac{ca}{c^5 + a^3 - 2c + 6}} \leq$$

$$\leq \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{1 + \frac{1}{ab}}} + \frac{1}{\sqrt{1 + \frac{1}{bc}}} + \frac{1}{\sqrt{1 + \frac{1}{ca}}}\right)$$

Let: $\frac{1}{ab} = x, \frac{1}{bc} = y, \frac{1}{ca} = z; (x, y, z > 0)$**, because** $abc = 1 \Rightarrow xyz = 1$ **and hence**

$$P \leq \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}}\right)$$

Because $xyz = 1$**, let** $x = \frac{m}{n}, y = \frac{n}{p}, z = \frac{p}{m}; (m, n, p > 0)$ **and hence**

$$P\le\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{1+\frac{m}{n}}}+\frac{1}{\sqrt{1+\frac{n}{p}}}+\frac{1}{\sqrt{1+\frac{p}{m}}}\right)=\frac{1}{\sqrt{3}}\left(\sqrt{\frac{n}{m+n}}+\sqrt{\frac{p}{p+n}}+\sqrt{\frac{m}{m+p}}\right);\ (1)$$

By C.B.S. we have:

$$\left(\sqrt{\frac{n}{m+n}}+\sqrt{\frac{p}{p+n}}+\sqrt{\frac{m}{m+p}}\right)^2=$$

$$=\left(\sqrt{\frac{n}{(n+m)(n+p)}}\cdot\sqrt{n+p}+\sqrt{\frac{p}{(p+n)(p+m)}}\sqrt{p+m}+\sqrt{\frac{m}{(m+p)(m+n)}}\cdot\sqrt{m+n}\right)^2=$$

$$\overset{BCS}{\le}\left(\frac{n}{(n+m)(n+p)}+\frac{p}{(p+m)(p+n)}+\frac{m}{(m+n)(m+p)}\right)(n+p+p+m+m+n)$$

$$=2(m+n+p)\cdot\frac{n(m+p)+p(m+n)+m(n+p)}{(m+n)(n+p)(p+m)}=$$

$$=\frac{4(mn+np+pm)(m+n+p)}{(m+n)(n+p)(p+m)};\ (2)$$

By AM-GM inequality, we have:

$$(m+n+p)(mn+np+pm)=(m+n)(n+p)(p+m)+mnp\le$$

$$\le(m+n)(n+p)(p+m)+\frac{(m+n)(n+p)(p+m)}{8}=\frac{9(m+n)(n+p)(p+m)}{8}$$

Then $\frac{(mn+np+pm)(m+n+p)}{(m+n)(n+p)(p+m)}\le\frac{9}{8}$ **and from (2) we get:**

$$\left(\sqrt{\frac{n}{m+n}}+\sqrt{\frac{p}{p+n}}+\sqrt{\frac{m}{m+p}}\right)^2\le 4\cdot\frac{9}{8}=\frac{9}{2}$$

$$(1)\Rightarrow P\le\frac{1}{\sqrt{3}}\left(\sqrt{\frac{n}{m+n}}+\sqrt{\frac{p}{p+n}}+\sqrt{\frac{m}{m+p}}\right)\le\frac{1}{\sqrt{3}}\cdot\frac{3}{\sqrt{2}}=\frac{\sqrt{6}}{2}\Rightarrow P_{min}=\frac{\sqrt{6}}{2}$$

Equality occurs if: $m=n=p\Leftrightarrow a=b=c=1$.

SP.316.In any ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\left(\frac{a}{s-a}\right)^{m+1}}+3m\ge 3(m+1)\sqrt{2}\,,m\in\mathbb{N}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by Marin Chirciu-Romania

Firstly, we prove that:

Lemma:

In any ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\left(\frac{a}{s-a}\right)^{m+1}}\ge 3\cdot 2^{m+1}\,,m\in\mathbb{N}$$

Proof: Using the means inequality, we get:

$$\sum_{cyc}\sqrt{\left(\frac{a}{s-a}\right)^{m+1}}\ge 3\sqrt[3]{\prod\left(\frac{a}{s-a}\right)^{m+1}}=3\sqrt[3]{\left[\frac{abc}{\prod(s-a)}\right]^{m+1}}=3\sqrt[3]{\left(\frac{4Rrs}{r^2s}\right)^{m+1}}=$$

$$=3\sqrt[3]{\left(\frac{4R}{r}\right)^{m+1}}\overset{Euler}{\ge}3\sqrt[3]{\left(\frac{4\cdot 2r}{r}\right)^{m+1}}=3\sqrt[3]{8^{m+1}}=2^{m+1}$$

Using lemma, we must show that:

$$3\cdot 2^{m+1}+3m\ge 3(m+1)\sqrt{2}\Leftrightarrow 2^{m+1}+m\ge(m+1)\sqrt{2}$$

We can prove with mathematical induction after $m\in\mathbb{N}$.

Let be the proposition: $P(m): 2^{m+1}+m\ge(m+1)\sqrt{2}$.

We have: $P(0): 2\ge\sqrt{2}$ true.

Suppose: $P(k): 2^{k+1}+k\ge(k+1)\sqrt{2}$ and we prove that $P(k+1)$ is true.

$$P(k): 2^{k+1}+k\ge(k+1)\sqrt{2}\Rightarrow P(k+1): 2^{k+2}+k+1\ge(k+2)\sqrt{2}\Leftrightarrow k\sqrt{2}\ge k-1$$

true for all $k\in\mathbb{N}$.Proved

Solution 2 by Daniel Văcaru-Romania

We could write LHS as:

$$\sum_{cyc}\sqrt{\left(\frac{a}{s-a}\right)^{m+1}}+3m=\sum_{cyc}\left(\sqrt{\left(\frac{a}{s-a}\right)^{m+1}}+m\right)=$$

$$=\sum_{cyc}\left(\sqrt{\left(\frac{a}{s-a}\right)^{m+1}}+\underbrace{1+1+\cdots+1}_{m-times}\right)\overset{AM-GM}{\geq}(m+1)\sum_{cyc}\sqrt{\frac{a}{s-a}}\overset{AM-GM}{\geq}$$

$$=3(m+1)\sqrt[6]{\frac{abc}{(s-a)(s-b)(s-c)}};\quad (1)$$

But:

$$\frac{abc}{(s-a)(s-b)(s-c)}=\frac{abc}{s^3-s^2(a+b+c)+s(ab+bc+ca)-abc}=$$

$$=\frac{abc}{s^3-2s+s(s^2+r^2+4Rr)-abc}=\frac{abc}{s^3-2s^3+s^3+sr^2+4Rrs-4Rrs}=$$

$$=\frac{abc}{sr^2}=\frac{4RS}{Sr}=\frac{4R}{Sr}\overset{Euler}{\geq}8\Rightarrow\sqrt[6]{\frac{abc}{(s-a)(s-b)(s-c)}}\geq\sqrt{2};\quad (2)$$

From (1) and (2) we obtain:

$$\sum_{cyc}\sqrt{\left(\frac{a}{s-a}\right)^{m+1}}+3m\geq 3(m+1)\sqrt{2}\,,m\in\mathbb{N}$$

SP.317. If $a,b,c,d,e\in\mathbb{R}_+^*=(0,\infty)$ and $a^2+b^2+c^2+d^2=e^2$, then

$$(a+c)(b+d)\leq e^2$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania

We have:

$$a+c\overset{AM-QM}{\leq}\sqrt{2(a^2+c^2)};\ (1)\text{ and }b+d\overset{AM-QM}{\leq}\sqrt{2(b^2+d^2)};\ (2)$$

Multiplying (1) and (2), we obtain

$$(a+c)(b+d)\leq\sqrt{2(a^2+c^2)(b^2+d^2)}\overset{GM-AM}{\leq}(a^2+c^2)+(b^2+d^2)=e^2\Rightarrow$$

$$(a+c)(b+d)\leq e^2$$

Solution 2 by Daniel Văcaru-Romania

We have:

$$(a+c)(b+d)=ab+ad+bc+cd\overset{GM-AM}{\le}\frac{a^2+c^2}{2}+\frac{a^2+d^2}{2}+\frac{c^2+d^2}{2}+\frac{c^2+b^2}{2}=$$

$$=a^2+b^2+c^2+d^2=e^2$$

Solution 3 by Abner Chinga Bazo-Lima-Peru

$a^2+b^2\ge 2ab$, equality occurs when $a=b$

$a^2+d^2\ge 2ad$, equality occurs when $a=d$

$b^2+c^2\ge 2bc$, equality occurs when $b=c$

$c^2+d^2\ge 2cd$, equality occurs when $c=d$

Therefore

$$2(a^2+b^2+c^2+d^2)\ge 2(ab+ad+bc+cd)$$

$$a^2+b^2+c^2+d^2\ge (a+c)(b+d)$$

$$(a+c)(b+d)\le e^2;\ \ a^2+b^2+c^2+d^2=e^2$$

Equality occurs when $a=b=c=d$

Solution 2 by proposers

Let be the matrix $A=\begin{pmatrix} a & b & c & d\\ d & a & b & c\end{pmatrix}$ and $A^t=\begin{pmatrix} a & d\\ b & a\\ c & b\\ d & c\end{pmatrix}$, then

$$A\cdot A^t=\begin{pmatrix} a & b & c & d\\ d & a & b & c\end{pmatrix}\cdot\begin{pmatrix} a & d\\ b & a\\ c & b\\ d & c\end{pmatrix}=\begin{pmatrix} a^2+b^2+c^2+d^2 & ad+ba+cb+dc\\ da+ab+bc+cd & a^2+b^2+c^2+d^2\end{pmatrix}=$$

$$=\begin{pmatrix} e^2 & (a+c)(b+d)\\ (a+c)(b+d) & e^2\end{pmatrix}\Rightarrow det(A\cdot A^t)$$

$$=e^4-(a+c)^2(b+d)^2,(1)$$

From Cauchy-Binet, we have: $det(A\cdot A^t)\ge 0\overset{(1)}{\Rightarrow} e^4-(a+c)^2(b+d)^2\ge 0$

$$\Leftrightarrow e^4\ge (a+c)^2(b+d)^2\Leftrightarrow (a+c)(b+d)\le e^2$$

SP.318. If $x, y \in \mathbb{R}_+^* = (0, \infty)$ and in triangle ABC, a, b, c are the lengths of the sides, h_a, h_b, h_c are the lengths of the altitudes, then

$$\frac{(2x-y)xa}{h_a} + \frac{(2y-x)yb}{h_b} + \frac{xyc}{h_c} \geq 2\sqrt{3}xy$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania

We have $\frac{a}{h_a} = \frac{a^2}{2F}$ (and analogs)

$$\frac{(2x-y)xa}{h_a} + \frac{(2y-x)yb}{h_b} + \frac{xyc}{h_c} = \frac{(2x-y)xa^2 + (2y-x)yb^2 + xyc^2}{2F} \overset{Cos.Law}{=}$$

$$= \frac{2(x^2a^2 + y^2b^2) + xy(c^2 - a^2 - b^2)}{2F} = \frac{x^2a^2 + y^2b^2 - abxycos\,C}{F} \overset{AM-GM}{\geq}$$

$$\geq \frac{2xyab - xyabcos\,C}{F} = \frac{xyab(2 - cosC)}{F} =$$

$$= \frac{xyab\left(3sin^2\frac{C}{2} + cos^2\frac{C}{2}\right)}{F} \overset{AM-GM}{\geq} \frac{2xyab\sqrt{3sin^2\frac{C}{2}cos^2\frac{C}{2}}}{F}$$

$$= \frac{xyab \cdot 2sin\frac{C}{2}cos\frac{C}{2}\sqrt{3}}{F} = \frac{xyab \cdot sinC\sqrt{3}}{F} = 2\sqrt{3}xy$$

Solution 2 by Marin Chirciu-Romania

Using $\frac{a}{h_a} = \frac{a^2}{2F}$ (and analogs) we get:

$$Lhs = \frac{(2x-y)xa}{h_a} + \frac{(2y-x)yb}{h_b} + \frac{xyc}{h_c} = \frac{(2x-y)xa^2 + (2y-x)yb^2 + xyc^2}{2F} =$$

$$== \frac{2(x^2a^2 + y^2b^2) + xy(c^2 - a^2 - b^2)}{2F} \overset{AM-GM}{\geq} \frac{2 \cdot 2xyab + xy(c^2 - a^2 - b^2)}{2F}$$

$$= \frac{xy(4ab + c^2 - a^2 - b^2)}{2F} \overset{(1)}{\geq} 2\sqrt{3}xy = Rhs$$

Where (1)$\Leftrightarrow$ $\frac{xy(4ab+c^2-a^2-b^2)}{2F} \geq 2\sqrt{3}xy \Leftrightarrow (4ab + c^2 - a^2 - b^2) \geq 4F\sqrt{3} \Leftrightarrow$

$$(4ab + c^2 - a^2 - b^2)^2 \geq 16F^2 \cdot 3 \Leftrightarrow$$

$$(4ab+c^2-a^2-b^2)^2 \overset{(2)}{\geq} \left(2\sum_{cyc} b^2c^2 - \sum_{cyc} a^4\right)\cdot 3 \Leftrightarrow$$

$$(4ab+c^2-a^2-b^2)^2 \geq 3(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4) \Leftrightarrow$$

$$c^4-2c^2(a^2+b^2-2ab)+(a^2+b^2-2ab)^2+a^4+b^4+3a^2b^2-2ab(a^2+b^2)\geq 0 \Leftrightarrow$$

$$c^4-2c^2(a^2+b^2-2ab)+(a^2+b^2-2ab)^2 \geq 0 \Leftrightarrow$$

$$\left[c^2-(a^2+b^2-2ab)\right]^2 \geq 2$$

Equality holds if $c^2=a^2+b^2-2ab$.

SP.319. If $(H_n)_{n\geq 1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$, then find:

$$\Omega = \lim_{n\to\infty} e^{-H_n}\cdot \sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{k!}}$$

Proposed by D.Bătineţu Giurgiu-Romania

Solution by Sergio Esteban-Argentina

By Stolz-Cesaro theorem:

$$\Omega = \lim_{n\to\infty} e^{-H_n}\cdot \sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{k!}} = \lim_{n\to\infty}\frac{1}{e^{H_n}}\cdot\sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{k!}} =$$

$$= \lim_{n\to\infty}\frac{e^{H_{n+1}}}{\left(e^{H_{n+1}}-e^{H_n}\right)\sqrt[n+1]{(n+1)!}} \overset{H_{n+1}=H_n+\frac{1}{n+1}}{\triangleq} \lim_{n\to\infty}\frac{e^{H_n+\frac{1}{n+1}}}{e^{H_n}\left(e^{\frac{1}{n+1}}-1\right)\sqrt[n+1]{(n+1)!}} =$$

$$= \lim_{n\to\infty}\frac{1}{\left(e^{\frac{1}{n+1}}-1\right)\sqrt[n+1]{(n+1)!}} = \lim_{n\to\infty}\frac{1}{\left(\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\right)\cdot\frac{1}{n+1}\cdot\sqrt[n+1]{(n+1)!}}$$

By Stirling's approximation:

$$\Omega = \lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}} = \lim_{n\to\infty}\frac{n+1}{\sqrt[2(n+1)]{2\pi(n+1)}\,\frac{n+1}{e}} = e$$

SP.320. If $x \in \mathbb{R}_+^* = (0,\infty)$ and in ΔABC, a,b,c –are lengths of the sides, h_a,h_b,h_c –are lengths of the altitudes, then:

$$\frac{(6x-1)a}{h_a}+\frac{\left(\frac{2}{3x}-1\right)b}{h_b}+\frac{c}{h_c}\geq 2\sqrt{3}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution 1 by Marian Ursărescu-Romania

We must show that:

$$\frac{(6x-1)a^2}{a\cdot h_a}+\frac{\left(\frac{2}{3x}-1\right)b^2}{b\cdot h_b}+\frac{c^2}{c\cdot h_c}\geq 2\sqrt{3}\Leftrightarrow$$

$$(6x-1)a^2+\left(\frac{2}{3x}-1\right)b^2+c^2\geq 4\sqrt{3}S;\ \ (1)$$

Theorem: If $m_1,m_2,m_3\in\mathbb{R}$ such that $m_1+m_2>0, m_2+m_3>0, m_3+m_1>0$ and $m_1m_2+m_2m_3+m_3m_1>0$ then

$$m_1a^2+m_2b^2+m_3c^2\geq 4S\sqrt{m_1m_2+m_2m_3+m_3m_1}$$

Proof: Let $\sqrt{m_1m_2+m_2m_3+m_3m_1}=m$, from $2S=bc\sin A$ and cosines law, we have:

$$2bc(m\cdot\sin A+m_1\cdot\cos A)\leq b^2(m_1+m_2)+c^2(m_1+m_3)\Leftrightarrow$$

$$\frac{b}{2c}(m_1+m_2)+\frac{c}{2b}(m_1+m_3)-(m\cdot\sin A+m_1\cdot\cos A)\geq 0;(i)$$

But: $\frac{b}{2c}(m_1+m_2)+\frac{c}{2b}(m_1+m_3)\geq\sqrt{(m_1+m_2)(m_1+m_3)}$; (ii) and

$$m\cdot\sin A+m_1\cdot\cos A\leq\sqrt{m^2+m_1^2};\ \ (iii)$$

From (i),(ii),(iii) we have:

$$m_1a^2+m_2b^2+m_3c^2\geq 4S\sqrt{m_1m_2+m_2m_3+m_3m_1}$$

In our case: $m_1=6x-1, m_2=\frac{2}{3x}-1, m_3=1$

$$m_1+m_2=2\left(3x-\frac{1}{3x}\right)-3>0;\ m_1+m_3=6x>0;\ m_2+m_3=\frac{2}{3x}>0$$

and $m_1m_2+m_2m_3+m_3m_1=3>0\Rightarrow$

$$(6x-1)a^2+\left(\frac{2}{3x}-1\right)b^2+c^2\geq\ 4\sqrt{3}S$$

Solution 2 by proposer

Let be F – area of the ΔABC.Then:

$$U=(6x-1)\cdot\frac{a}{h_a}+\left(\frac{2}{3x}-1\right)\cdot\frac{b}{h_b}+\frac{c}{h_c}=\frac{6x-1}{ah_a}\cdot a^2+\left(\frac{2}{3x}-1\right)\cdot\frac{b^2}{bh_b}+\frac{c^2}{ch_c}$$

$$=\frac{1}{2F}\cdot\left((6x-1)a^2+\left(\frac{2}{3x}-1\right)b^2+c^2\right)=\frac{1}{2F}\cdot V,\qquad(1)$$

Where $V=(6x-1)a^2+\left(\frac{2}{3x}-1\right)b^2+c^2$, (2)

By Oppenheimer inequality, we have:

$$va^2+ub^2+wc^2\geq 4\sqrt{vu+uw+wv}\cdot F,\forall u,v,w\epsilon\mathbb{R},\qquad(O)$$

If in (O) we let: $u=6x-1,v=\frac{2}{3x}-1,w=1$, we get:

$$V\geq 4\cdot\sqrt{(6x-1)\left(\frac{2}{3x}-1\right)+6x-1+\frac{2}{3x}-1}\cdot F$$

$$=4\cdot\sqrt{\frac{12x-2-18x^2+3x+18x^2-3x+2-3x}{3x}}\cdot F=4\cdot\sqrt{\frac{9x}{3x}}\cdot F=4\sqrt{3}F,\quad(3)$$

From $(1),(2),(3)$ we have: $U\geq\frac{1}{2F}\cdot 4\sqrt{3}F=2\sqrt{3}$

SP.321. Let a,b,c be the lengths of the sides of a triangle with circumradius R and iradius r. Prove that:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-4\left(\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}\right)+12\left(\frac{R}{2r}\right)^2\geq 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

We know that:

$$\frac{1}{a^2+b^2}\leq\frac{1}{4}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\Rightarrow\frac{c^2+a^2}{a^2+b^2}\leq\frac{1}{4}\left(1+\frac{c^2}{a^2}+\frac{a^2}{b^2}+\frac{c^2}{b^2}\right)\text{ and similarly}$$

$\frac{a^2+b^2}{b^2+c^2}\le\frac{1}{4}\left(1+\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{a^2}{c^2}\right)$ and $\frac{b^2+c^2}{c^2+a^2}\le\frac{1}{4}\left(1+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{b^2}{a^2}\right)$.

Adding up these inequalities, we have:

$$\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}$$
$$\le\frac{1}{4}\left(\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+3\right)$$

Also we know that:

$$\frac{a}{b}+\frac{b}{a}\le\frac{R}{r},\frac{b}{c}+\frac{c}{b}\le\frac{R}{r},\frac{c}{a}+\frac{a}{c}\le\frac{R}{r}.$$

So,

$$\frac{a^2}{b^2}+\frac{b^2}{a^2}\le\frac{R^2}{r^2}-2,\frac{b^2}{c^2}+\frac{c^2}{b^2}\le\frac{R^2}{r^2}-2,\frac{c^2}{a^2}+\frac{a^2}{c^2}\le\frac{R^2}{r^2}-2$$

Now,

$$\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}\le\frac{1}{4}\left(3\cdot\frac{R^2}{r^2}-6+\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+3\right)$$

or

$$4\left(\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}\right)\le 3\cdot\frac{R^2}{r^2}+\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-3$$
$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-4\left(\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}\right)+12\left(\frac{R}{2r}\right)^2\ge 3.$$

Equality holds if and only if the triangle is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{x+y}{y+z}+\frac{y+z}{z+x}+\frac{z+x}{x+y}\Leftrightarrow\frac{x+y}{y}+\frac{y+z}{z}+\frac{z+x}{x}-\frac{x+y}{y+z}-\frac{y+z}{z+x}-\frac{z+x}{x+y}\ge 3$$
$$\Leftrightarrow\frac{z(x+y)}{y(y+z)}+\frac{x(y+z)}{z(z+x)}+\frac{y(z+x)}{x(x+y)}\ge 3\to\textbf{true}$$
$$\because\frac{z(x+y)}{y(y+z)}+\frac{x(y+z)}{z(z+x)}+\frac{y(z+x)}{x(x+y)}\overset{A-G}{\ge}3\sqrt[3]{\frac{z(x+y)}{y(y+z)}\cdot\frac{x(y+z)}{z(z+x)}\cdot\frac{y(z+x)}{x(x+y)}}=3$$

$\therefore \frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq \frac{x+y}{y+z}+\frac{y+z}{z+x}+\frac{z+x}{x+y}$ and choosing $x=a^2, y=b^2, z=c^2$, we get

$$:\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}$$

$$\overset{?}{\geq} 3+4\left(\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}\right)-12\left(\frac{R}{2r}\right)^2$$

$$\Leftrightarrow \frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}+1 \overset{?}{\leq} \frac{R^2}{r^2}$$

$$\Leftrightarrow \frac{a^2+b^2+c^2-c^2}{b^2+c^2}+\frac{b^2+c^2+a^2-a^2}{c^2+a^2}+\frac{c^2+a^2+b^2-b^2}{a^2+b^2}+1 \overset{?}{\leq} \frac{R^2}{r^2}$$

$$\Leftrightarrow \left(\sum a^2\right)\sum \frac{1}{b^2+c^2}+1 \underset{(1)}{\overset{?}{\leq}} \frac{R^2}{r^2}+\frac{a^2}{c^2+a^2}+\frac{b^2}{a^2+b^2}+\frac{c^2}{b^2+c^2}$$

Now, $\frac{R^2}{r^2}+\frac{a^2}{c^2+a^2}+\frac{b^2}{a^2+b^2}+\frac{c^2}{b^2+c^2} \overset{\text{Bergstrom}}{\geq} \frac{R^2}{r^2}+\frac{(a+b+c)^2}{2\sum a^2} \overset{\text{Leibnitz}}{\geq} \frac{R^2}{r^2}$

$$+\frac{2s^2}{9R^2} \overset{\text{Gerretsen}}{\geq} \frac{R^2}{r^2}+\frac{27Rr+5Rr-10r^2}{9R^2}$$

$$\overset{\text{Euler}}{\geq} \frac{R^2}{r^2}+\frac{27Rr+10r^2-10r^2}{9R^2}=\frac{R^2}{r^2}+\frac{3r}{R}$$

$$\Rightarrow \frac{R^2}{r^2}+\frac{a^2}{c^2+a^2}+\frac{b^2}{a^2+b^2}+\frac{c^2}{b^2+c^2} \overset{(i)}{\geq} \frac{R^3+3r^3}{Rr^2}$$

$$\left(\sum a^2\right)\sum \frac{1}{b^2+c^2}+1 \overset{\text{Leibnitz}}{\leq} 9R^2\sum \frac{1}{b^2+c^2}+1 \overset{A-G}{\leq} 9R^2\sum \frac{1}{2bc}+1$$

$$=9R^2\left(\frac{2s}{8Rrs}\right)+1 \Rightarrow \left(\sum a^2\right)\sum \frac{1}{b^2+c^2}+1 \overset{(ii)}{\leq} \frac{9R+4r}{4r}$$

(i), (ii) $\Rightarrow$ in order to prove (1), it suffices to prove : $\frac{R^3+3r^3}{Rr^2} \geq \frac{9R+4r}{4r}$

$$\Leftrightarrow 4R^3+12r^3 \geq 9R^2r+4Rr^2$$

$$\Leftrightarrow 4t^3-9t^2-4t+12 \geq 0 \left(\text{where } t=\frac{R}{r}\right) \Leftrightarrow (t-2)\big((t-2)(4t+7)+8\big) \geq 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (1)$ is true

$$\therefore \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq 3+4\left(\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}\right)-12\left(\frac{R}{2r}\right)^2$$

$$\Rightarrow \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-4\left(\frac{a^2+b^2}{b^2+c^2}+\frac{b^2+c^2}{c^2+a^2}+\frac{c^2+a^2}{a^2+b^2}\right)+12\left(\frac{R}{2r}\right)^2\geq 3\ (Proved)$$

SP.322 Let a,b,c be the lengths of the sides of a triangle with circumradius R and iradius r. Prove that:

$$\frac{2r}{R}\leq\frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2}\leq\frac{R^2}{2r^2}-1$$

Proposed by George Apostolopoulos-Greece

Solution 1 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{a^2}{b^2+bc+c^2}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{2(a^2+b^2+c^2)+ab+bc+ca}\overset{\sum x^2\geq\sum xy}{\geq}\frac{4s^2}{3(a^2+b^2+c^2)}=$$

$$=\frac{4s^2}{6(s^2-4Rr-r^2)}$$

We need to show: $\frac{4s^2}{6(s^2-4Rr-r^2)}\geq\frac{2r}{R}\Leftrightarrow s^2(R-3r)+12Rr^2+3r^3\geq 0$

From $s^2\geq 16Rr-5r^2$ $(Gerretsen)$ we need to show that:

$$(16Rr-5r^2)(R-3r)+12Rr^2+3r^3\geq 0\Leftrightarrow 16R^2-41Rr+18r^2\geq 0\Leftrightarrow$$

$$(R-2r)(16R-9r)\geq 0(true)\ R\geq 2r(Euler).$$

Now,

$$\sum_{cyc}\frac{a^2}{b^2+bc+c^2}\overset{AM-GM}{\leq}\sum_{cyc}\frac{a^2}{2bc+bc}=\frac{1}{3}\sum_{cyc}\frac{a^2}{bc}=\frac{1}{3abc}\sum_{cyc}a^3=$$

$$=\frac{3abc+\sum a^3-3abc}{3abc}=\frac{3abc+(\sum a)(\sum a^2-\sum ab)}{3abc}=$$

$$=\frac{12Rrs+2s(2s^2-8Rr-2r^2-s^2-r^2-4Rr)}{3abc}=$$

$$=\frac{2s(s^2-6Rr-3r^2)}{12Rrs}=\frac{2(s^2-6Rr-3r^2)}{12Rr}\overset{Gerretsen}{\leq}$$

$$\leq \frac{4R^2+4Rr+3r^2-6Rr-3r^2}{6Rr} = \frac{2R-r}{3r}$$

Need to show:

$$\frac{2R-r}{3r} \leq \frac{R^2}{2r^2} - 1 = \frac{R^2-2r^2}{2r^2} \Leftrightarrow 4Rr - 2r^2 \leq 3R^2 - 6r^2 \Leftrightarrow$$

$$3R^2 - 4Rr - 4r^2 \geq 0 \Leftrightarrow (R-2r)(3R+2r) \geq 0\ (true) R \geq 2r\ (\text{Euler}).$$

Proved.

Solution 2 by Adrian Popa-Romania

$$b^2+bc+c^2 \overset{AM-GM}{\geq} 2bc+bc = 3bc \Rightarrow \sum_{cyc} \frac{a^2}{b^2+bc+c^2} \leq \sum_{cyc} \frac{a^2}{2bc+bc} =$$

$$= \sum_{cyc} \frac{a^3}{3abc} = \frac{a^3+b^3+c^3}{3abc} = \frac{2s(s^2-3r^2-6Rr)}{3abc} = \frac{2s^3-6sr^2}{12Rrs} - 1 \overset{(1)}{\leq} \frac{R^2}{2r^2} - 1$$

$$(1) \Leftrightarrow \frac{s^3-3sr^2}{6Rrs} \leq \frac{R^2}{2r^2} \Leftrightarrow \frac{s^2-3r^2}{3R} \leq \frac{R^2}{r} \Leftrightarrow s^2r - 3r^3 \leq 3R^3$$

$$\therefore s^2 \leq 4R^2+4Rr+3r^2\ (Gerretsen)$$

$$s^2r - 3r^3 \leq (4R^2+4Rr+3r^2)r - 3r^3 \leq 3R^3 \Leftrightarrow 3\left(\frac{R}{r}\right)^2 - \frac{4R}{r} - 4 \geq 0 \text{ true from}$$

$$R \geq 2r(Euler)$$

$$\sum_{cyc} \frac{a^2}{b^2+bc+c^2} \geq \sum_{cyc} \frac{2a^2}{3(b^2+c^2)} \geq \frac{2}{3} \cdot \frac{(a+b+c)^2}{2(a^2+b^2+c^2)} = \frac{4s^2}{3(a^2+b^2+c^2)} \overset{(2)}{\geq} \frac{2r}{R}$$

$$(2) \Leftrightarrow 2s^2R \geq 3r(2s^2-8Rr-2r^2) \Leftrightarrow$$

$$(3r-R)s^2 \overset{Gerretsen}{\leq} (3r-R)(4R^2+4Rr+3r^2) = 12R^2r - 4R^3 + 12Rr^2 - 4R^2r +$$

$$+9r^3 - 3r^2R \overset{(3)}{\leq} 12Rr^2 + 3r^3s^2$$

$$(3) \Leftrightarrow 4R^3 - 8R^2r + 3Rr^2 - 6r^3 \geq 0 \Leftrightarrow (R-2r)(4R^2+3r^2) \geq 0 \text{ true from}$$

$R \geq 2r\ (Euler)$. **Proved.**

Solution 3 by Ertan Yildirim-Turkey

Lemma 1. $a^3+b^3+c^3 = 2s(s^2-3r^2-6Rr)$

Lemma 2. $2s^2 \geq 27Rr$

Lemma 3. $a^2+b^2+c^2 \leq 9R^2$

$$Rhs: \sum_{cyc}\frac{a^2}{b^2+bc+c^2}\overset{AM-GM}{\le}\sum_{cyc}\frac{a^2}{2bc+bc}=\sum_{cyc}\frac{a^2}{3bc}=\sum_{cyc}\frac{a^3}{3abc}=\frac{1}{3abc}\sum_{cyc}a^3=$$

$$=\frac{1}{3\cdot 4Rrs}\cdot 2s(s^2-3r^2-6Rr)=\frac{1}{6Rr}\cdot(s^2-3r^2-6Rr)\overset{(1)}{\le}\frac{R^2}{2r^2}-1$$

$$(1)\Leftrightarrow s^2-3r^2-6Rr\le\frac{3R^3}{r}-6Rr\Leftrightarrow s^2\le 3r^2+\frac{3R^3}{r}$$

$$s^2\overset{Gerretsen}{\le}4R^2+4Rr+3r^2\overset{(2)}{\le}3r^2+\frac{3R^3}{r}$$

$$(2)\Leftrightarrow 4\left(\frac{R}{r}\right)^2+4\left(\frac{R}{r}\right)\le 3\left(\frac{R}{r}\right)^3 \text{ let } t=\frac{R}{r}\Rightarrow 4t^2+4t\le 3t^2\Leftrightarrow$$

$$0\le t(3t^2-4t-4)=t(3t+2)(t-2)\Rightarrow t=\frac{R}{r}\ge 2\ (true) R\ge 2r\ (Euler)$$

$$Lhs: \sum_{cyc}\frac{a^2}{b^2+bc+c^2}\overset{Bergstrom}{\ge}\frac{(a+b+c)^2}{2(a^2+b^2+c^2)+ab+bc+ca}\overset{\sum x^2\ge\sum xy}{\ge}\frac{4s^2}{3(a^2+b^2+c^2)}=$$

$$\ge\frac{4s^2}{3\cdot 9R^2}\overset{(3)}{\ge}\frac{2r}{R}$$

$$(3)\Leftrightarrow 2s^2\ge 27Rr\ (true)$$

Solution 4 by proposer

We have: $(a-b)^2\ge 0\Leftrightarrow a^2+b^2-2ab\ge 0\Leftrightarrow$

$$2a^2+4ab+4b^2-3a^2-3b^2-6ab\ge 0\Leftrightarrow(a^2+ab+b^2)\ge\frac{3}{4}(a+b)^2\Leftrightarrow$$

$\frac{1}{a^2+ab+b^2}\le\frac{4}{3}\cdot\frac{1}{(a+b)^2}$. Also we know that: $\frac{1}{(a+b)^2}\le\frac{1}{8}\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$.

So, $\frac{1}{a^2+ab+b^2}\le\frac{1}{6}\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\Leftrightarrow\frac{c^2}{a^2+ab+b^2}\le\frac{1}{6}\cdot\left(\frac{c^2}{a^2}+\frac{c^2}{b^2}\right)$ similarly $\frac{b^2}{c^2+ca+a^2}\le\frac{1}{6}\cdot\left(\frac{b^2}{c^2}+\frac{b^2}{a^2}\right)$

and $\frac{a^2}{b^2+bc+c^2}\le\frac{1}{6}\cdot\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)$.

Adding up these inequalities, we have:

$$\frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2}$$

$$\le\frac{1}{6}\cdot\left(\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)+\left(\frac{b^2}{c^2}+\frac{b^2}{a^2}\right)+\left(\frac{c^2}{a^2}+\frac{c^2}{b^2}\right)\right)$$

Now, will prove that: $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}$.

Consider the supstitutions $a = y + z, b = z + x, c = x + y$, where x, y, z are positive real numbers.

We know that: $\frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$ is the semiperimeter.

So, $\frac{R}{r} = \frac{(x+y)(y+z)(z+x)}{4xyz}$. We have:

$\frac{1}{(z+x)^2} + \frac{1}{(y+z)^2} \leq \frac{1}{4zx} + \frac{1}{4yz} = \frac{x+y}{4xyz}$ and multiplying by $(z+x)(y+z)$ both sides, we have

$\frac{y+z}{z+x} + \frac{z+x}{y+z} \leq \frac{(x+y)(y+z)(z+x)}{4xyz}$, namely $\frac{a}{b} + \frac{b}{a} \leq \frac{R}{r}$,similarly $\frac{b}{c} + \frac{c}{b} \leq \frac{R}{r}$ and $\frac{a}{c} + \frac{c}{a} \leq \frac{R}{r}$.

So, $\frac{a^2}{b^2} + \frac{a^2}{c^2} \leq \frac{R^2}{r^2} - 2$, then

$$\frac{a^2}{b^2+bc+c^2} + \frac{b^2}{c^2+ca+a^2} + \frac{c^2}{a^2+ab+b^2} \leq \frac{1}{6} \cdot \left(\left(\frac{R^2}{r^2} - 2\right) + \left(\frac{R^2}{r^2} - 2\right) + \left(\frac{R^2}{r^2} - 2\right)\right) = \frac{R^2}{2r^2} - 1.$$

Now, using Cauchy-Rogers inequality, we have:

$$\frac{a^2}{b^2+bc+c^2} + \frac{b^2}{c^2+ca+a^2} + \frac{c^2}{a^2+ab+b^2} \geq \frac{(a+b+c)^2}{2(a^2+b^2+c^2)+(ab+bc+ca)}$$

$$\geq \frac{4S^2}{2(a^2+b^2+c^2)+a^2+b^2+c^2} = \frac{2(2S^2)}{3(a^2+b^2+c^2)}$$

We know that: $2S^2 \geq 27Rr$ and $a^2+b^2+c^2 \geq 9R^2$. So,

$$\frac{a^2}{b^2+bc+c^2} + \frac{b^2}{c^2+ca+a^2} + \frac{c^2}{a^2+ab+b^2} \geq \frac{2r}{R}.$$

Equality holds if the triangle is equilateral.

Solution 5 and generalization by Marin Chirciu-Romania

For LHS using Bergtrom inequality, we have:

$$\sum_{cyc} \frac{a^2}{b^2+bc+c^2} \overset{Bergstrom}{\geq} \frac{(\sum a)^2}{\sum(b^2+bc+c^2)} = \frac{4s^2}{2\cdot 2(s^2-r^2-4Rr)+s^2+r^2+4Rr} =$$

$$= \frac{4s^2}{5s^2-3r^2-12Rr} \overset{(1)}{\geq} \frac{2r}{R}$$

Where (1)$\Leftrightarrow 2Rs^2 \geq r(5s^2-3r^2-12Rr) \Leftrightarrow s^2(2R-5r)+3r^2(4R+r) \geq 0$

We distinguish the cases:

(I) If $2R - 5r \geq 0$ inequality is obviously.

(II) If $2R - 5r < 0$ inequality it can be written as: $3r^2(4R+r) \geq s^2(5r-2R)$

Which follows from $s^2 \le 4R^2 + 4Rr + 3r^2$ $(Gerretsen)$.

It remains to prove that $3r^2(4R+r) \ge (4R^2+4Rr+3r^2)(5r-2R) \Leftrightarrow$

$4R^3 - 6R^2r - Rr^2 - 12r^3 \ge 0 \Leftrightarrow (R-2r)(4R^2+2Rr+3r^2) \ge 0$ which is true from

$R \ge 2r$ $(Euler)$

Equality holds if and only if triangle is equilateral.

For RHD we have:

$$\sum_{cyc}\frac{a^2}{b^2+bc+c^2} \overset{AM-GM}{\le} \sum_{cyc}\frac{a^2}{2bc+bc} = \frac{1}{3}\cdot\sum_{cyc}\frac{a^2}{bc} = \frac{1}{3}\cdot\frac{s^2-3r^2-6Rr}{2Rr} =$$

$$= \frac{s^2-3r^2-6Rr}{6Rr} \overset{(2)}{\le} \frac{R^2}{2r^2} - 1$$

Where (2)$\Leftrightarrow \frac{s^2-3r^2-6Rr}{6Rr} \le \frac{R^2}{2r^2} - 1 \Leftrightarrow r(s^2-3r^2-6Rr) \le 3R(R^2-2r^2)$

Which follows from $s^2 \le 4R^2 + 4Rr + 3r^2$ $(Gerretsen)$.

It remains to prove that

$$r(4R^2+4Rr+3r^2-3r^2-6Rr) \le 3R(R^2-2r^2) \Leftrightarrow$$

$3R^2 - 4Rr - 4r^2 \ge 0 \Leftrightarrow (R-2r)(3R+2r) \ge 0$ which is true from $R \ge 2r(Euler)$

Equality holds if and only if triangle is equilateral.

Remark. The inequality it can be developed.

In ΔABC the following relationship holds:

$$\frac{6r}{(\lambda+2)R} \le \sum_{cyc}\frac{a^2}{b^2+\lambda bc+c^2} \le \frac{3}{\lambda+2}\left(\frac{R^2}{2r^2}-1\right), \qquad \lambda > -2$$

Proposed by Marin Chirciu-Romania

Solution by proposer

For LHS using Bergtrom inequality, we have:

$$\sum_{cyc}\frac{a^2}{b^2+\lambda bc+c^2} \overset{Bergstrom}{\ge} \frac{(\sum a)^2}{\sum(b^2+\lambda bc+c^2)} = \frac{4s^2}{2\sum a^2 + \lambda\sum bc} =$$

$$= \frac{4s^2}{2\cdot 2(s^2-r^2-4Rr)+\lambda(s^2+r^2+4Rr)} = \frac{4s^2}{(\lambda+4)s^2+(\lambda-4)r^2+(4\lambda-16)Rr} \overset{(1)}{\ge}$$

$\overset{(1)}{\geq} \frac{6r}{(\lambda+2)R}$ where (1) $\Leftrightarrow$

$$\frac{4s^2}{(\lambda+4)s^2+(\lambda-4)r^2+(4\lambda-16)Rr} \geq \frac{6r}{(\lambda+2)R} \Leftrightarrow$$

$$2(\lambda+2)Rs^2 \geq 3r[(\lambda+4)s^2+(\lambda-4)r^2+(4\lambda-16)Rr] \Leftrightarrow$$

$$s^2[2(\lambda+2)R-3(\lambda+4)r]+3r^2[(4-\lambda)r+(16-4\lambda R] \geq 0$$

We distinguish the cases:

(I) If $2(\lambda+2)R-3(\lambda+4)r \geq 0$ inequality is obviously.

(II) If $2(\lambda+2)R-3(\lambda+4)r \leq 0$ the inequality it can be written as:

$3r^2[(4-\lambda)r+(16-4\lambda R] \geq s^2[3(\lambda+4)r-2(\lambda+2)R]$ **which follows from**

$$s^2 \leq 4R^2+4Rr+3r^2\ (Gerretsen).$$

It remains to prove that

$$3r^2[(4-\lambda)r+(16-4\lambda R] \geq (4R^2+4Rr+3r^2)[3(\lambda+4)r-2(\lambda+2)R] \Leftrightarrow$$

$$(4\lambda+8)R^3-(2\lambda+16)R^2r+(-9\lambda+6)Rr^2-(6\lambda+12)r^3 \geq 0 \Leftrightarrow$$

$(R-2r)[(4\lambda+8)R^2+6\lambda Rr+(3\lambda+6)r^2] \geq 0$ **which follows from** $R \geq 2r(Euler)$

and $[(4\lambda+8)R^2+6\lambda Rr+(3\lambda+6)r^2] > 0.$

Equality holds if and only if triangle is equilateral.

For RHD we have:

$$\sum_{cyc}\frac{a^2}{b^2+\lambda bc+c^2} \overset{AM-GM}{\leq} \sum_{cyc}\frac{a^2}{2bc+\lambda bc} = \frac{1}{2+\lambda}\cdot\sum_{cyc}\frac{a^2}{bc} = \frac{1}{2+\lambda}\cdot\frac{s^2-3r^2-6Rr}{2Rr} =$$

$$= \frac{s^2-3r^2-6Rr}{2(2+\lambda)Rr} \overset{(2)}{\leq} \frac{3}{\lambda+2}\left(\frac{R^2}{2r^2}-1\right)$$

Where (2)$\Leftrightarrow \frac{s^2-3r^2-6Rr}{6Rr} \leq \frac{R^2}{2r^2}-1 \Leftrightarrow r(s^2-3r^2-6Rr) \leq 3R(R^2-2r^2)$

Which follows from $s^2 \leq 4R^2+4Rr+3r^2\ (Gerretsen).$

It remains to prove that

$$r(4R^2+4Rr+3r^2-3r^2-6Rr) \leq 3R(R^2-2r^2) \Leftrightarrow$$

$3R^2-4Rr-4r^2 \geq 0 \Leftrightarrow (R-2r)(3R+2r) \geq 0$ **which is true from** $R \geq 2r(Euler)$

Equality holds if and only if triangle is equilateral.

Note:

For $\lambda = 1$ we get the Problem SP.322 from 22-RMM-Autumn Edition 2021, proposed by George Apostolopoulos-Greece.

SP.323. Let be $z_A, z_B, z_C \in \mathbb{C}^*$, different in pairs such that

$|z_A| = |z_B| = |z_C| = 1$. If $|z_A - z_B - z_C| + |z_B - z_C - z_A| +$

$+|z_C - z_A - z_B| = 6$, then ΔABC is an equilateral triangle.

Proposed by Marian Ursărescu-Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$|z_A| = |z_B| = |z_C| = 1$$

$$|z_A - z_B - z_C| + |z_B - z_C - z_A| + |z_C - z_A - z_B| = 6;\ \ (*)$$

We know that: $z_G = \frac{z_A+z_B+z_C}{3}$ then relation (*) is written as:

$$6 = |2z_A - 3z_G| + |2z_B - 3z_G| + |2z_C - 3z_G| \overset{CBS}{\leq}$$

$$\leq \sqrt{3} \cdot \sqrt{|2z_A - 3z_G|^2 + |2z_B - 3z_G|^2 + |2z_C - 3z_G|^2};\ \ (1)$$

$$|2z_A - 3z_G| = (2z_A - 3z_G) \cdot (2\overline{z_A} - 3\overline{z_G}) = 4 - 6z_A\overline{z_G} - 6z_G\overline{z_A} + 9z_G\overline{z_G}$$

$$\text{So, } l_1 = |2z_A - 3z_G|^2 + |2z_B - 3z_G|^2 + |2z_C - 3z_G|^2 =$$

$$= 12 - 6(z_A + z_B + z_C)\overline{z_G} - 6(\overline{z_A} + \overline{z_B} + \overline{z_C})z_G + 27z_G\overline{z_G} =$$

$$= 12 - 18z_G\overline{z_G} - 18z_G\overline{z_G} + 27z_G\overline{z_G} = 12 - 9z_G\overline{z_G}$$

$$\Rightarrow l_1 = 12 - 9|z_G|^2;\ \ (2)$$

From (1),(2) we have: $6 \leq \sqrt{3} \cdot \sqrt{12 - 9|z_G|^2} \Leftrightarrow$

$$36 \leq 3(12 - 9|z_G|^2) \Leftrightarrow 12 \leq 12 - 9|z_G|^2 \Leftrightarrow 0 \leq -9|z_G|^2 \Leftrightarrow |z_G| = 0 \Leftrightarrow G = O$$

So, triangle ABC is equilateral.

Solution 2 by proposer

Let $A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(0,1)$ and Ω – the middle of OH –(Euler point)

$$z_\Omega = \frac{z_O + z_H}{2} = \frac{z_1 + z_2 + z_3}{2} \Rightarrow A\Omega = |z_A - z_\Omega| = \left|z_1 - \frac{z_1 + z_2 + z_3}{2}\right| = \frac{|z_1 - z_2 - z_3|}{2}$$

$$|z_1 - z_2 - z_3| + |z_2 - z_1 - z_3| + |z_3 - z_1 - z_2| = 6 \Leftrightarrow$$

$$A\Omega + B\Omega + C\Omega = 3;\ \ (1)$$

Let A' –midle of $BC \Rightarrow \Omega A'^2 = \frac{2(B\Omega^2+C\Omega^2)-a^2}{4} \Rightarrow R^2 = 2(B\Omega^2 + C\Omega^2) - a^2$ and analogs.

$$A\Omega^2 + B\Omega^2 + C\Omega^2 = \frac{3R^2 + a^2 + b^2 + c^2}{4} \le \frac{3R^2 + 9R^2}{4} \le 3R^2$$

But: $(A\Omega + B\Omega + C\Omega)^2 \le 3(A\Omega^2 + B\Omega^2 + C\Omega^2) \le 9R^2$; (2)

From $(1),(2)$ equality when the ΔABC is equilateral.

SP.324. Find all functions $f: (0, +\infty) \to \mathbb{R}$ such that:

$$f(xy) \le xf(x) + yf(y) \le log(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$f(xy) \le xf(x) + yf(y) \le log(xy), \forall x, y > 0; \text{ (1)}$$

Take $x = y = 1 \Rightarrow f(1) \le 1 \cdot f(1) + 1 \cdot f(1) \le log1$

$$f(1) \le 2f(1) \le 0 \Rightarrow f(1) \ge 0 \text{ or } f(1) \le 0$$

Put $y = \frac{1}{x}$ in (1) we get: $f(1) \le xf(x) + \frac{1}{x}f\left(\frac{1}{x}\right) \le log1 \Leftrightarrow$

$$0 \le xf(x) + \frac{1}{x}f\left(\frac{1}{x}\right) \le 0 \Leftrightarrow f(x) = -\frac{1}{x^2}log\left(\frac{1}{x}\right)$$

In (1) put $y = 1$ to obtain: $f(x) \le xf(x) \le logx, \forall x > 0$; (2)

Replace $x \leftrightarrow \frac{1}{x}$ in *(2)* we get*:* $f\left(\frac{1}{x}\right) \le \frac{1}{x}f\left(\frac{1}{x}\right) \le log\left(\frac{1}{x}\right) \Leftrightarrow$

$$-x^2f(x) \le -xf(x) \le -logx \Leftrightarrow$$

$$logx \le xf(x) \le x^2f(x), x > 0; \ (3)$$

From (2) and (3) we get:

$$xf(x) = logx \Rightarrow f(x) = \frac{logx}{x}; \ \forall x > 0$$

Solution 2 by Remus Florin Stanca-Romania

$$f(xy) \le xf(x) + yf(y) \le log(xy), \forall x, y > 0$$

$$\text{Let } y = \frac{1}{x} \Rightarrow f(1) \le xf(x) + \frac{f\left(\frac{1}{x}\right)}{x} \le 0; \quad (1)$$

$$\text{Let } x = 1 \Rightarrow f(1) \le 2f(1) \le 0 \Rightarrow f(1) \ge 0 \text{ and } f(1) \le 0 \Rightarrow f(1) = 0 \overset{(1)}{\Rightarrow}$$

$$0\le xf(x)+\frac{f\left(\frac{1}{x}\right)}{x}\le 0\Rightarrow xf(x)+\frac{f\left(\frac{1}{x}\right)}{x}=0\Rightarrow f\left(\frac{1}{x}\right)=-x^2f(x);\ \ (2)$$

Put $\frac{1}{y}\to y\Rightarrow (xy)\le xf(x)-\frac{y^2f(y)}{y}\le log\left(\frac{x}{y}\right)\Rightarrow f\left(\frac{x}{y}\right)\le xf(x)-yf(y)\le log\left(\frac{x}{y}\right);(3)$

Put $x\to\frac{1}{x}\Rightarrow\frac{f\left(\frac{1}{x}\right)}{x}+yf(y)\le log\left(\frac{y}{x}\right)\overset{(2)}{\Rightarrow}yf(y)-xf(x)\le log\left(\frac{y}{x}\right)\overset{\cdot(-1)}{\Longrightarrow}$

$$xf(x)-yf(y)\ge log\left(\frac{x}{y}\right);\ \ (4)$$

From (3),(4) we get $xf(x)-yf(y)=log\left(\frac{x}{y}\right)$ and for $y=1$ we have:

$$xf(x)-f(1)=logx\Rightarrow xf(x)=logx\Rightarrow f(x)=\frac{logx}{x},\forall x>0$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$f(xy)\le xf(x)+yf(y)\le log(xy),\forall x,y>0\Leftrightarrow$$

$$f(xy)\le xf(x)+yf(y)\le logx+logy,\forall x,y>0\Leftrightarrow$$

$$xf(x)-logx\le -yf(y)+logy$$

$$x(x)-logx\le -(yf(y)-lofy)$$

Suppose $F(x)=xf(x)-logx;F(y)=yf(y)-logy,\forall x,y>0\Rightarrow F(x)\le -F(y)\Rightarrow$

$$F(x)+F(y)\le 0,\forall x,y>0$$

For $x=y$: $2F(x)\le 0\Rightarrow F(x)\le 0;\ \ (i)$

For $x=y=1$: $f(1)\le 2f(1)\le f(0)\Rightarrow f(1)=0$. So, $f(1)=0\Rightarrow F(1)=0$.

$$xf(x)-logx\le 0\Rightarrow f(x)\le\frac{logx}{x};\ \ (1)$$

$$F(x)=xf(x)-logx\Rightarrow\frac{F(x)}{x}=f(x)-\frac{logx}{x}$$

$$f(xy)\le xf(x)+yf(y)\le log(xy),\forall x,y>0$$

For $y=1$: $f(x)\le xf(x)\le logx,\forall x>0$

$$f\left(\frac{1}{x}\right)\le\frac{1}{x}f\left(\frac{1}{x}\right)\le -logx\Leftrightarrow f(x)-\frac{logx}{x}\ge f(x)+\frac{1}{x^2}f\left(\frac{1}{x}\right);\ \ (ii)$$

$$f\left(x\cdot\frac{1}{x}\right)\le xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\Leftrightarrow 0\le xf(x)+\frac{1}{x^2}f\left(\frac{1}{x}\right);\ \ (iii)$$

From (ii),(iii) we have $f(x)-\frac{logx}{x}\ge 0\Rightarrow f(x)\ge\frac{logx}{x};\ \ (2)$

From (1),(2) we get: $f(x)=\frac{logx}{x}$

Solution 4 by proposer

$$x=y=1\Rightarrow f(1)\le 2f(1)\le 0\Rightarrow f(1)\le 0 \text{ but } f(1)\ge 0\Rightarrow f(1)=0$$

$$y=1\Rightarrow f(x)\le xf(x)\le logx\Rightarrow f(x)\le\frac{logx}{x};\ \ (1)$$

$$y=\frac{1}{x}\Rightarrow f(1)\le xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\le log1\Rightarrow 0\le xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\le 0\Rightarrow$$

$$xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)=0\Rightarrow f\left(\frac{1}{x}\right)=-x^2f(x);\ \ (2)$$

In $(1)x\to\frac{1}{x}\Rightarrow f\left(\frac{1}{x}\right)\le\frac{log\left(\frac{1}{x}\right)}{\frac{1}{x}}\Rightarrow f\left(\frac{1}{x}\right)\le -xlogx;\ \ (3)$

From $(1),(2)\Rightarrow -x^2f(x)\le -xlogx\Rightarrow x^2f(x)\ge xlogx\Rightarrow f(x)\ge\frac{logx}{x};\ \ (4)$

From $(3),(4)\Rightarrow f(x)=\frac{logx}{x}$

SP.325. If $A,B\in M_2(\mathbb{C})$ are such that:

$$\det[(I_2-B)A+(A-I_2)B]=\det(A-B),\text{ then find:}$$

$$\Omega=(AB-BA)^n, n\in\mathbb{N}^*.$$

Proposed by Florică Anastase-Romania

Solution 1 by George Florin Şerban-Romania

$$det(C+xD)=detC+\big(TrC\cdot TrD+Tr(CD)\big)x+(detD)x^2$$

For $x=1$: $det(C+D)=detC+\big(TrC\cdot TrD+Tr(CD)\big)+(detD)$

$$det\big((I_2-B)A+(A-I_2)B\big)=det[(A-B)+(AB-BA)]=$$

$$=det(A-B)+Tr(A-B)\cdot Tr(AB-BA)-Tr(A-B)(AB-BA)+det(AB-BA)$$

$$=det(A-B)\Rightarrow det(AB-BA)=Tr(A-B)(AB-BA)$$

How $Tr(AB-BA)=0, Tr(AB)=Tr(BA)\Rightarrow Tr(AB-BA)=Tr(AB)-Tr(BA)=0$

Let $t=Tr[(A-B)(AB-BA)]=Tr(A^2B)-Tr(ABA)-Tr(BAB)+Tr(B^2A)=$

$$=Tr(BA^2)-Tr(ABA)-Tr(BAB)+Tr(AB^2)$$

$$t=Tr[(A-B)(AB-BA)]=Tr[(AB-BA)(A-B)]=$$

$$= Tr(ABA) + Tr(BAB) - Tr(BA^2) - Tr(AB^2) = -t \Rightarrow t = 0$$

Applying Hamilton Cayley Theorem, we get:

$$X^2 - (TrX)X + det(X)I_2 = O_2, \forall X \in M_2(\mathbb{C})$$

$$(AB - BA)^2 - tr(AB - BA)(AB - BA) + det(AB - BA)\, I_2 = O_2$$

$$det(AB - BA) = t = 0 \Rightarrow (AB - BA)^2 = O_2$$

From mathematical induction, we prove that:

$$P(n): (AB - BA)^n = O_2, \forall n \geq 2$$

$$(I). P(2): (AB - BA)^2 = O_2\ (true)$$

(II) **Suppose:** $P(k): (AB - BA)^k = O_2$ **then**

$$(AB - BA)^{k+1} = (AB - BA)^k \cdot (AB - BA) = O_2 \cdot (AB - BA) = O_2$$

Solution 2 by proposer

$$(I_2 - B)A + (A - I_2)B = A - BA + AB - B = A - B + AB - BA$$

From $tr[(A - B)(AB - BA)] = tr(A^2B - ABA - BAB + B^2A) = tr(A^2B) - tr(ABA) - tr(BAB) + tr(B^2A) = 0$ **and how** $tr(AB - BA) = 0$ **we get:**

$$tr[(A - B)(AB - BA)] = tr(A - B)tr(AB - BA)$$

$$det(A - B + AB - BA) = det(A - B) + det(AB - BA)\text{, then } det(AB - BA) = 0$$

From Hamilton Cayley theorem we have:

$$X^2 - (TrX)X + det(X)I_2 = O_2, \forall X \in M_2(\mathbb{C})$$

$$(AB - BA)^2 - tr(AB - BA)(AB - BA) + det(AB - BA)\, I_2 = O_2$$

From mathematical induction, we prove that:

$$P(n): (AB - BA)^n = O_2, \forall n \geq 2$$

$$(I). P(2): (AB - BA)^2 = O_2\ (true)$$

(II) **Suppose:** $P(k): (AB - BA)^k = O_2$ **then**

$$(AB - BA)^{k+1} = (AB - BA)^k \cdot (AB - BA) = O_2 \cdot (AB - BA) = O_2$$

So, $\Omega = (AB - BA)^n = O_2, n \in \mathbb{N}^*$.

tr(X)tr(Y)=tr(XY)⇔det(X+Y)=det(X)+det(Y).

SP.326. Let $x,y,z>0$ such that $xyz=1$. Find the minimum value of :

$$P=\frac{xy+yz+zx}{3}+\sum_{cyc}\frac{x^3}{(2y^2-yz+2z^2)^2}$$

Proposed by Hoang Le Nhat Tung-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\Omega=\sum_{cyc}\frac{x^3}{(2y^2-yz+2z^2)^2}=\sum_{cyc}\frac{\frac{x^4}{(2y^2-yz+2z^2)^2}}{x}=\sum_{cyc}\frac{\left(\frac{x^2}{2y^2-yz+2z^2}\right)^2}{x}\geq$$

$$\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}\frac{x^2}{2y^2-yz+2z^2}\right)^2}{x+y+z}$$

$$\sum_{cyc}\frac{x^2}{2y^2-yz+2z^2}=\sum_{cyc}\frac{(x^2)^2}{2x^2y^2-x^2yz+2x^2z^2}\overset{Bergstrom}{\geq}\frac{\left(\sum x^2\right)^2}{4\sum x^2y^2-xyz(x+y+z)}\overset{(1)}{\geq}1$$

$$(1)\Leftrightarrow\left(\sum x^2\right)^2\geq 4\sum x^2y^2-xyz(x+y+z)\Leftrightarrow$$

$$\sum x^4+2\sum x^2y^2\geq 4\sum x^2y^2-xyz(x+y+z)\Leftrightarrow$$

$$\sum x^4+xyz(x+y+z)\geq 2\sum x^2y^2$$

Which is clearly true. In fact, by Schur's inequality:

$$\sum x^4+xyz(x+y+z)\geq\sum xy(x^2+y^2)\overset{AM-GM}{\geq}\sum xy(2xy)=2\sum x^2y^2$$

So, we have:

$$\Omega\geq\frac{\left(\sum_{cyc}\frac{x^2}{2y^2-yz+2z^2}\right)^2}{x+y+z}\overset{(1)}{\geq}\frac{1}{x+y+z}$$

Much more,

$$xy+yz+zx\overset{AM-GM}{\geq}3\sqrt[3]{(xyz)^2}=3;\ (xyz=1)$$

$$(xy+yz+zx)^2\geq 3xyz(x+y+z)=3(x+y+z);\ (xyz=1)$$

$$\Rightarrow\frac{1}{x+y+z}\geq\frac{3}{(xy+yz+zx)^2}\Rightarrow$$

$$P=\Omega+\frac{xy+yz+zx}{3}\ge\frac{1}{x+y+z}+\frac{xy+yz+zx}{3}\ge$$

$$\ge\frac{3}{(xy+yz+zx)^2}+\frac{xy+yz+zx}{3}\overset{t=xy+yz+zx\ge3}{=}$$

$$=\frac{3}{t^2}+\frac{t}{9}+\frac{t}{9}+\frac{t}{9}\overset{AM-GM}{\ge}4\sqrt[4]{\frac{3t^3}{9^3t^2}}=4\sqrt[4]{\frac{9}{9^3}}=\frac{4}{3}\Rightarrow P\ge\frac{4}{3}$$

$$P_{min}=\frac{4}{3}\Leftrightarrow\begin{cases}x=y=z>0\\ xyz=1\end{cases}\Leftrightarrow x=y=z=1.$$

Solution 2 by proposer

$$P=\frac{xy+yz+zx}{3}+\sum_{cyc}\frac{x^3}{(2y^2-yz+2z^2)^2};\ (1)$$

By Schur's inequality, we have:

$$\sum x^4+xyz(x+y+z)\ge\sum xy(x^2+y^2);\ (2)$$

$$\sum xy(x^2+y^2)\overset{AM-GM}{\ge}\sum xy(2xy)=2\sum x^2y^2;\ (3)$$

From (2),(3) we get:

$$\sum x^4+xyz(x+y+z)\ge2\sum x^2y^2\Leftrightarrow\left(\sum x^2\right)^2\ge4\sum x^2y^2-xyz(x+y+z)\Leftrightarrow$$

$$\frac{(\sum x^2)^2}{4\sum x^2y^2-xyz(x+y+z)}\ge1;\ (4)$$

$$\sum_{cyc}\frac{x^3}{(2y^2-yz+2z^2)^2}=\sum_{cyc}\frac{\frac{x^4}{(2y^2-yz+2z^2)^2}}{x}=\sum_{cyc}\frac{\left(\frac{x^2}{2y^2-yz+2z^2}\right)^2}{x}\ge$$

$$\overset{Bergstrom}{\ge}\frac{\left(\sum_{cyc}\frac{x^2}{2y^2-yz+2z^2}\right)^2}{x+y+z};(5)$$

$$\sum_{cyc}\frac{x^2}{2y^2-yz+2z^2}=\sum_{cyc}\frac{(x^2)^2}{2x^2y^2-x^2yz+2x^2z^2}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(\sum x^2)^2}{4\sum x^2y^2-xyz(x+y+z)};\ (6)$$

From (4),(6) we get:

$$\sum_{cyc} \frac{x^2}{2y^2 - yz + 2z^2} \geq 1; \quad (7)$$

From (5),(7) we get:

$$\sum_{cyc} \frac{x^3}{(2y^2 - yz + 2z^2)^2} \geq \frac{1}{x+y+z} \Rightarrow$$

$$P = \frac{xy+yz+zx}{3} + \sum_{cyc} \frac{x^3}{(2y^2-yz+2z^2)^2} \geq \frac{xy+yz+zx}{3} + \frac{1}{x+y+z}; (8)$$

$$P \geq \left(\frac{1}{x+y+z} + \frac{xy+yz+zx}{9} + \frac{xy+yz+zx}{9}\right) + \frac{xy+yz+zx}{9} \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{1}{x+y+z} \cdot \frac{xy+yz+zx}{9} \cdot \frac{xy+yz+zx}{9}} + \frac{3\sqrt[3]{xy \cdot yz \cdot zx}}{9} =$$

$$= 3\sqrt[3]{\frac{(xy+yz+zx)^2}{81(x+y+z)}} + \frac{\sqrt[3]{(xyz)^2}}{3} \geq 3\sqrt[3]{\frac{3xyz(x+y+z)}{81(x+y+z)}} + \frac{1}{3}; (xyz = 1); (9)$$

$$\frac{xy+yz+zx}{3} + \frac{1}{x+y+z} \geq 3\sqrt[3]{\frac{3 \cdot 1}{81}} + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3}$$

So, from (8),(9) we have: $P \geq \frac{4}{3} \Rightarrow P_{min} = \frac{4}{3}$

Equality occurs if $\begin{cases} x = y = z > 0 \\ xyz = 1 \end{cases} \Leftrightarrow x = y = z = 1.$

Hence, the minimum value of expression P is $\frac{4}{3}$ when $x = y = z = 1$.

SP.327.If $a, b, c \geq 0, ab + bc + ca = 3$ then find:

$$min\Omega(a,b,c);\ \Omega(a,b,c) = \frac{1}{(a+b)^5} + \frac{1}{(b+c)^5} + \frac{1}{(c+a)^5}.$$

Proposed by Hoang Le Nhat- Hanoi- Vietnam

Solution by Marin Chirciu and Octavian Stroe-Romania

Lemma:

If $a,b,c\ge 0, ab+bc+ca=3$ then find $minf(a,b,c)$,

$$f(a,b,c)=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Proof:

We have:

$$f(a,b,c)=\sum\frac{1}{b+c}=\frac{\sum(a+b)(a+c)}{\prod(b+c)}=$$

$$=\frac{(a+b+c)^2+ab+bc+ca}{(a+b+c)(ab+bc+ca)-abc}=$$

$$=\frac{(a+b+c)^2+3}{3(a+b+c)-abc}\overset{x=a+b+c}{\cong}\frac{x^2+3}{3x-abc}\overset{abc\ge 0}{\ge}\frac{x^2+3}{3x}=\frac{x}{3}+\frac{1}{x};(1)$$

Another hand, we have:

$$\sum\frac{a}{b+c}=\sum a\sum\frac{1}{b+c}-3$$

We get:

$$f(a,b,c)=\frac{1}{a+b+c}\left(3+\sum\frac{a}{b+c}\right)\overset{Bergström}{\ge}\frac{1}{a+b+c}\left(3+\frac{(a+b+c)^2}{2\sum bc}\right)$$

$$=\frac{1}{x}\left(3+\frac{x^2}{2\cdot 3}\right)=\frac{x}{6}+\frac{3}{x};(2)$$

From $(1)+3\cdot(2)\Rightarrow 4f(a,b,c)\ge\frac{5x}{6}+\frac{10}{x}\Leftrightarrow f(a,b,c)\ge\frac{5x}{24}+\frac{5}{2x}.$

We get: $f(a,b,c)\ge\frac{5x}{24}+\frac{5}{2x}=\frac{5}{2}\left(\frac{x}{12}+\frac{1}{x}\right)\overset{AGM}{\ge}\frac{5}{2}\cdot\sqrt{\frac{x}{12}\cdot\frac{1}{x}}=\frac{5}{2\sqrt{3}},$

Equality for $\frac{x}{12}=\frac{1}{x}\Leftrightarrow x^2=12\Leftrightarrow x=2\sqrt{3}.$

We deduce that: $minf(a,b,c)=\frac{5}{2\sqrt{3}}$ **for** $a+b+c=2\sqrt{3}(ex: a=b=\sqrt{3}, c=0)$

Let solve the proposed problem.

Using Lemma and Hőlder, we get:

$$\Omega(a,b,c)=\frac{1}{(a+b)^5}+\frac{1}{(b+c)^5}+\frac{1}{(c+a)^5}\ge$$

$$\ge\frac{\left(\sum\frac{1}{b+c}\right)^5}{3^4}\ge\frac{\left(\frac{5}{2\sqrt{3}}\right)^5}{3^4}=3\left(\frac{5}{6\sqrt{3}}\right)^5$$

We deduce that: $min\Omega(a,b,c) = 3\left(\frac{5}{6\sqrt{3}}\right)^5$ $for\ a+b+c = 2\sqrt{3}$

$(ex.\ a = b = \sqrt{3}, c = 0)$

Remark: The problem can be dezvolted:

1) If $a,b,c \ge 0, ab+bc+ca = 3$ then find:

$$min\Omega(a,b,c);\ \Omega(a,b,c) = \frac{1}{(a+b)^n} + \frac{1}{(b+c)^n} + \frac{1}{(c+a)^n}, n \in \mathbb{N}^*$$

Proposed by Marin Chirciu and Octavian Stroe-Romania

Solution by proposers

Using Lema and Hőlder inequality, we get:

$$\Omega(a,b,c) = \frac{1}{(a+b)^n} + \frac{1}{(b+c)^n} + \frac{1}{(c+a)^n} \ge$$

$$\ge \frac{\left(\sum \frac{1}{b+c}\right)^n}{3^{n-1}} \ge \frac{\left(\frac{5}{2\sqrt{3}}\right)^n}{3^{n-1}} = 3\left(\frac{5}{6\sqrt{3}}\right)^n$$

We deduce that: $min\Omega(a,b,c) = 3\left(\frac{5}{6\sqrt{3}}\right)^n$ $for\ a+b+c = 2\sqrt{3}$

$(ex.\ a = b = \sqrt{3}, c = 0)$

Note: For $n = 5$ we get the Problem JP.297 from RMM number 20, Spring 2021, proposed by Hoang Le Nhat, Hanoi, Vietnam.

For $n = 1$ we get lemma.

SP.328. Let $a,b,c \in [1,3]$ such that $a+b+c = 6$. Find the maximum value of the expression:

$$P = a^6 + b^6 + c^6$$

Proposed by Hoang Le Nhat Tung-Vietnam

Solution 1 by Adrian Popa-Romania

$$P = max. \Rightarrow \begin{cases} a = max. \\ b = max. \\ c = max. \end{cases} \Rightarrow a = 3$$

If $b > 2 \Rightarrow c < 1$ false. So, $b_{max} = 2 \Rightarrow c = 1$.

$$P_{max} = 3^6 + 2^6 + 1^6 = 794.$$

Solution 2 by Florentin Vişescu-Romania

$$a, b, c \in [1,3], a + b + c = 6, \Omega = max\{a^6 + b^6 + c^6\}$$

From $a + b + c = 6 \Rightarrow c = 6 - a - b, c \in [1,3] \Rightarrow 1 < 6 - a - b < 3 \Rightarrow$

$$b - 5 < -a < b - 3 \Rightarrow 3 - b \le a \le 5 - b$$

How $b \in [1,3]$ we distinguish the cases:

(1) If $b \in [1,2] \Rightarrow -2 \le -b \le -1 \Rightarrow \begin{cases} 1 \le 3 - b \le 2 \\ 3 \le 5 - b \le 4 \end{cases} \Rightarrow a \in [3 - b, 3]$

(2) If $b \in [2,3] \Rightarrow -3 \le -b \le -2 \Rightarrow \begin{cases} 0 \le 3 - b \le 1 \\ 2 \le 5 - b \le 3 \end{cases} \Rightarrow a \in [1, 5 - b]$

Let be the function $f(a) = a^6 + b^6 + c^6 = a^6 + b^6 + (6 - a^6 - b^6)^6$

$$f'(a) = 6a^5 - 6(6 - a^6 - b^6)^5 =$$

$$= 6(a - 6 + a + b)\left[a^4 + a^3(6 - a - b) + a^2(6 - a - b)^2 + +a(6 - a - b)^3 + (6 - a - b)^4\right]$$

$$f'(a) = 0 \Leftrightarrow 2a - 6 + b = 0 \Leftrightarrow a = \frac{6 - b}{2}$$

We distingush the cases:

(i) If $\frac{6-b}{2} \in [3 - b, 3] \Leftrightarrow 6 - 2b \le 6 - b \le 6 \Leftrightarrow 2b \ge b \ge 0\ (true)$

(ii) If $\frac{6-b}{2} \in [1, 5 - b] \Leftrightarrow 2 \le 6 - b \le 10 - 2b \Leftrightarrow 0 \le 4 - b \le 8 - 2b \Leftrightarrow b \le 4 (true)$

(1) If $b \in [1,2]$

a	$3 - b$ $\qquad$ $\frac{6-b}{2}$ $\qquad$ 3
$f'(a)$	$- - - - - - - - - 0 + + + + + + + + +$
$f(a)$	$\searrow \searrow \searrow$ $f\left(\frac{6-b}{2}\right)$ $\nearrow \nearrow \nearrow$

$$f(3 - b) = (3 - b)^6 + b^6 + 3^6$$

$$f(3) = 3^6 + b^6 + (3 - b)^6$$

$$f\left(\frac{6 - b}{2}\right) = 2\left(\frac{6 - b}{2}\right)^6 + b^6$$

So, for $b \in [1,2] \Rightarrow\ 2\left(\frac{6-b}{2}\right)^6 + b^6 \le a^6 + b^6 + c^6 \le (3 - b)^6 + b^6 + 3^6$

(2) If $b \in [2,3]$

a	1	$\frac{6-b}{2}$	$5-b$
$f'(a)$	$- - - - - - - -$	0	$+ + + + + + + +$
$f(a)$	$\searrow \searrow \searrow$	$f\left(\frac{6-b}{2}\right)$	$\nearrow \nearrow \nearrow$

$$f(1) = 1 + b^6 + (5-b)^6$$

$$f(5-b) = (5-b)^6 + b^6 + 1$$

$$f\left(\frac{6-b}{2}\right) = 2\left(\frac{6-b}{2}\right)^6 + b^6$$

So, for $b \in [2,3] \Rightarrow 2\left(\frac{6-b}{2}\right)^6 + b^6 \le a^6 + b^6 + c^6 \le 1 + b^6 + (5-b)^6$

Let be the function $g(b) = 2\left(\frac{6-b}{2}\right)^6 + b^6$

(1) If $b \in [1,2]$

b	1		2
$g'(b)$	$- - - - - - - - - - - - - -$		
$g(b)$	$\searrow$	$\searrow$	$\searrow$

$$g(2) = 3 \cdot 2^6$$

(2) If $b \in [2,3]$

b	2		3
$g'(b)$	$- - - - - - - - - - - - - -$		
$g(b)$	$\searrow$	$\searrow$	$\searrow$

$$g(3) = 11 \cdot \frac{3^7}{2^5}$$

So, $3 \cdot 2^6 \le a^6 + b^6 + c^6 \le 11 \cdot \frac{3^7}{2^5}$

Let be the function $h(b) = (3-b)^6 + b^6 + 3^6, h: [1,2] \to \mathbb{R}$

$$h'(b) = 6b^5 - 6(3-b)^5 =$$

$$= 6(b - 3 + b)[b^4 + b^3(3-b) + b^2(3-b)^2 + b(3-b)^3 + (3-b)^4]$$

$$h'(b) = 0 \Leftrightarrow b = \frac{3}{2}$$

b	1 $\qquad\qquad$ $\frac{3}{2}$ $\qquad\qquad$ 2
$h'(b)$	$- - - - - - - - - 0 + + + + + + + + +$
$h(b)$	$\searrow \searrow \searrow \; h\left(\frac{3}{2}\right) \nearrow \nearrow \nearrow$

$$h(1) = 2^6 + 1^6 + 3^6$$

$$h(2) = 1^6 + 2^6 + 3^6$$

Let be the function $k(b) = (5-b)^6 + b^6 + 1, k: [2,3] \to \mathbb{R}$

$$k'(b) = 6b^5 - 6(5-b)^5 =$$

$$= 6(b - 5 + b)[b^4 + b^3(5-b) + b^2(5-b)^2 + b(5-b)^3 + (5-b)^4$$

$$k'(b) = 0 \Leftrightarrow b = \frac{5}{2}$$

b	2 $\qquad\qquad$ $\frac{5}{2}$ $\qquad\qquad$ 3
$k'(b)$	$- - - - - - - - - 0 + + + + + + + + +$
$k(b)$	$\searrow \searrow \searrow \; h\left(\frac{5}{2}\right) \nearrow \nearrow \nearrow$

$$k(2) = 3^6 + 2^6 + 1$$

$$k(3) = 2^6 + 3^6 + 1$$

So, $3 \cdot 2^6 \le a^6 + b^6 + c^6 \le 1 + 2^6 + 3^6$

Solution 3 by Marian Dincă-Romania

$$c = 6 - a - b \Rightarrow 1 \le 6 - a - b \le 3 \Rightarrow 3 \le a + b \le 5$$

$$P = a^6 + b^6 + (6 - a - b)^6 = f(a, b)$$

$$\frac{\partial f}{\partial a} = 6a^5 - 6(6-a-b)^5, \frac{\partial^2 f}{\partial a^2} = 30a^4 + 30(6-a-b)^4 > 0$$

$$\frac{\partial f}{\partial b} = 6b^5 - 6(6-a-b)^5, \frac{\partial^2 f}{\partial b^2} = 30b^4 + 30(6-a-b)^4 > 0$$

The expression is a separately convex function with:

$$(a, b) \in \{(1,1), (1,3)(3,3)\}$$

We evaluate the values: $f(1,3) = 1 + 3^6 + 2^6$ is the only solution, because for $(1,1)$ the sum: $a + b = 2 < 3$ and $(3,3)$ the sum: $a + b = 6 > 5$

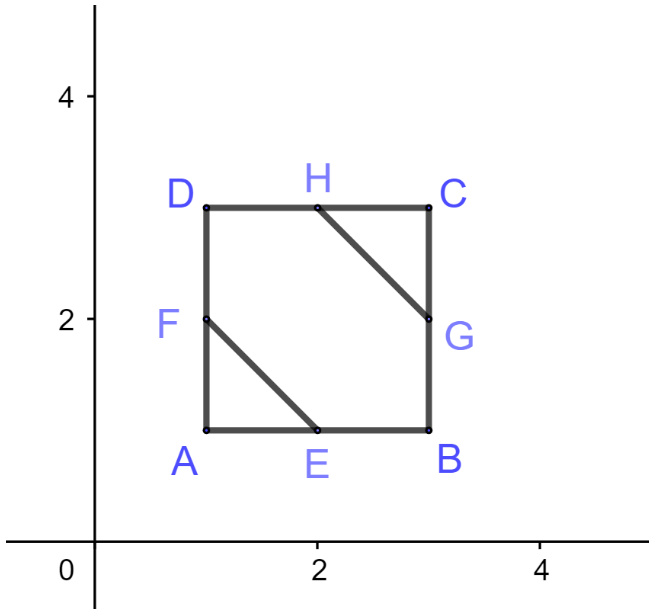
$$f:[1,3]\times[1,3]\cap\{a,b/3\le a+b\le 5\}\to\mathbb{R}$$

$[1,3]\times[1,3]\cap\{a,b/3\le a+b\le 5\}=EBGHDF$ where

$$E(2,1),B(3,1),G(3,2),H(2,3),D(1,3),F(1,2)$$

$f:M\to\mathbb{R}, f$ −convex and $M=EBGHDF$ −convex hexagon...

So, $maxf(a,b)\le\{f(E),f(B),f(G),f(H),f(D),f(F)\}=1+3^6+2^6$ because is simetric.



SP.329. Find:

$$\Omega=\lim_{n\to\infty}\frac{e^{\sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k}}}{\sqrt[n]{n!}}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Sergio Esteban-Argentina

$$i)\ \sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k}=-\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k};Let: S_n=\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k}$$

We have:

$$S_{n+1}=\sum_{k=1}^{n+1}\frac{(-1)^{k-1}\binom{n+1}{k}}{k}=\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\left(\binom{n}{k}+\binom{n}{k-1}\right)=$$

$$=\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n}{k}+\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n}{k-1}=S_n+\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n}{k-1}=$$

$$=S_n-\frac{1}{n+1}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}$$

Hence for all $m\ge 1, S_{m+1}-S_m=-\frac{1}{m+1}\sum_{k=1}^{m+1}(-1)^k\binom{m+1}{k}=$

$$=\frac{1}{m+1}-\frac{1}{m+1}\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}=\frac{1}{m+1}$$

Summing from $m=1,\dots,(n-1)$ yields $S_n-S_1=\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$

$$\sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k}=-H_n$$

Then by (i) we get:

$$\Omega=\lim_{n\to\infty}\frac{e^{\sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k}}}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{e^{-H_n}}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{e^{-(\gamma+\log n)}}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{e^{-\gamma}}{n\sqrt[n]{n!}}=0$$

Solution 2 by Florică Anastase-Romania

Let: $P(n): E(n)=\binom{n}{1}-\frac{1}{2}\binom{n}{2}+\frac{1}{3}\binom{n}{3}+\cdots+\frac{(-1)^n}{n}\binom{n}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$

$$P(1): E(1)=1, P(2): E(2)=2-\frac{1}{2}=1+\frac{1}{2}$$

$$P(n)\to P(n+1)$$

We prove that: $E(n+1)-E(n)=\frac{1}{n+1}$

$$E(n+1)-E(n)=\sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}\left[\binom{n+1}{i}-\binom{n}{i}\right]+\frac{(-1)^{n+1}}{n+1}\overset{\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}}{=}$$

$$=\sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}\binom{n}{i-1}+\frac{(-1)^{n+1}}{n+1}=\sum_{i=1}^{n}\frac{(-1)^{i-1}\cdot n!}{i!\,(n-i+1)!}+\frac{(-1)^{n+1}}{n+1}=$$

$$=\frac{1}{n+1}\sum_{i=1}^{n}\frac{(-1)^{i-1}\cdot(n+1)!}{i!\,(n-i+1)!}+\frac{(-1)^{n+1}}{n+1}=\frac{1}{n+1}\sum_{i=1}^{n}(-1)^{i-1}\binom{n+1}{i}+\frac{(-1)^{n+1}}{n+1}=$$

$$=-\frac{1}{n+1}\left[-\binom{n+1}{1}+\binom{n+1}{2}+\cdots+(-1)^n\binom{n+1}{n}\right]+\frac{(-1)^{n+1}}{n+1}=$$

$$=-\frac{1}{n+1}\left[\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}-(-1)^{n+1}\right]+\frac{(-1)^{n+1}}{n+1}=$$

$$=-\frac{1}{n+1}\left[(1-1)^{n+1}-1-(-1)^{n+1}\right]+\frac{(-1)^{n+1}}{n+1}=\frac{1}{n+1}+\frac{(-1)^n}{n+1}+\frac{(-1)^{n-1}}{n+1}=\frac{1}{n+1}$$

$$\Rightarrow\sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k}=-H_n$$

$$\Omega=\lim_{n\to\infty}\frac{e^{\sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k}}}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{e^{-H_n}}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{e^{-(\gamma+\log n)}}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{e^{-\gamma}}{n\sqrt[n]{n!}}=0$$

SP.330. Let ABC be a triangle with inradius r and circumradius R.Pove that:

$$\frac{48r}{R}\le\frac{\left(\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\right)^3}{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\le\frac{12R}{r}$$

Proposed by George Apostolopoulos-Greece

Solution by Marin Chirciu-Romania

Using the identity in any triangle: $\sum\tan\frac{A}{2}=\frac{4R+r}{s}$ the inequality becomes:

$$\frac{48r}{R}\cdot\frac{4R+r}{s}\le\left(\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\right)^3\le\frac{12R}{r}\cdot\frac{4r+r}{s}$$

$$\left(\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\right)^3\ge\frac{48r}{R}\cdot\frac{4R+r}{s}$$

Using AGM we get:

$$LHS=\left(\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\right)^3\ge 27\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}=\frac{27}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{27}{\frac{s}{4R}}$$

$$=\frac{108R}{r}\overset{(1)}{\ge}\frac{48r}{R}\cdot\frac{4R+r}{s}=LHD,$$

Where $(1) \Leftrightarrow 9R^2 \geq 4r(4R+r) \Leftrightarrow 9R^2 - 16Rr - 4r^2 \geq 0 \Leftrightarrow (R-2r)(9R+2r) \geq 0$

true by $R \geq 2r(Euler)$

Equality holds if and only if the triangle is equilateral.

$$\left(sec\frac{A}{2} + sec\frac{B}{2} + sec\frac{C}{2}\right)^3 \leq \frac{12R}{r} \cdot \frac{4r+r}{s}$$

Lemma: In any ΔABC: $sec\frac{A}{2} + sec\frac{B}{2} + sec\frac{C}{2} \leq \frac{9R}{s}$

Proof: Using the identity: $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{(x+y+z)^2}{3xyz}$, for $x = cos\frac{A}{2}; y = cos\frac{B}{2};$

$z = cos\frac{C}{2}$ we get:

$$\frac{1}{cos\frac{A}{2}} + \frac{1}{cos\frac{B}{2}} + \frac{1}{cos\frac{C}{2}} \leq \frac{\left(cos\frac{A}{2} + cos\frac{B}{2} + cos\frac{C}{2}\right)^2}{3cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}} \overset{Jensen}{\leq} \frac{\left(\frac{3\sqrt{3}}{2}\right)^2}{3 \cdot \frac{s}{4R}} = \frac{\frac{27}{4}}{\frac{3s}{4R}} = \frac{9R}{s}$$

Equality if and only if the triangle is equilateral.

$$LHS = \left(sec\frac{A}{2} + sec\frac{B}{2} + sec\frac{C}{2}\right)^3 \leq \left(\frac{9R}{s}\right)^3 \overset{(2)}{\leq} \frac{12R}{r} \cdot \frac{4R+r}{s} = RHS$$

Where $(2) \Leftrightarrow \left(\frac{9R}{s}\right)^3 \leq \frac{12R}{r} \cdot \frac{4R+r}{s} \Leftrightarrow 243R^2r \leq 4s^2(4R+r)$ true from:

$s^2 \geq 16Rr - 5r^2(Gerretsen)$.We must show that:

$243R^2r \leq 4(16Rr - 5r^2)(4R+r) \Leftrightarrow 13R^2 - 16Rr - 20r^2 \geq 0 \Leftrightarrow$

$(R-2r)(13R+10r) \geq 0$ true by $R \geq 2r(Euler)$

Equality if and only if the triangle is equilateral.

UP.316. If $(H_n)_{n\geq1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the armonic sequence, find:

$$\lim_{n\to\infty} e^{-2H_n} \cdot \sum_{k=2}^{n} \sqrt[k]{(2k-1)!!}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$L = \lim_{n\to\infty} e^{-2H_n} \cdot \sum_{k=2}^{n} \sqrt[k]{(2k-1)!!} = \lim_{n\to\infty} \frac{n^2}{e^{2H_n}} \cdot \frac{\sum_{k=2}^{n} \sqrt[k]{(2k-1)!!}}{n^2}; \quad (1)$$

$$\lim_{n\to\infty}\frac{n^2}{e^{2H_n}}=\left(\lim_{n\to\infty}\frac{n}{e^{H_n}}\right)^2\overset{L.C-S}{=}\left(\lim_{n\to\infty}\frac{n+1-n}{e^{H_{n+1}}-e^{H_n}}\right)^2=\left(\lim_{n\to\infty}\frac{1}{e^{H_n}\left(e^{\frac{1}{n+1}}-1\right)}\right)^2=$$

$$=\left(\lim_{n\to\infty}\frac{1}{e^{H_n}\left(\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}\cdot(n+1)}\right)}\right)^2=\left(\lim_{n\to\infty}\frac{1}{\frac{e^{H_n}}{n+1}}\right)^2=\left(\lim_{n\to\infty}\frac{1}{\frac{e^{H_n}}{n}\cdot\frac{n}{n+1}}\right)^2=$$

$$=\left(\lim_{n\to\infty}\frac{1}{\frac{e^{H_n}}{e^{\log n}}}\right)^2=\left(\lim_{n\to\infty}\frac{1}{e^{H_n-\log n}}\right)^2=\left(\lim_{n\to\infty}\frac{1}{e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n}}\right)^2=(e^{-\gamma})^2=e^{-2\gamma};\quad (2)$$

$$\lim_{n\to\infty}\frac{\sum_{k=2}^{n}\sqrt[k]{(2k-1)!!}}{n^2}\overset{L.C-S}{=}\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)!!}}{(n+1)^2-n^2}=\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)!!}}{2n+1}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{2n-1}=\lim_{n\to\infty}\frac{n}{2n-1}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{1}{2}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D'A}{=}$$

$$=\frac{1}{2}\cdot\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=\frac{1}{2}\cdot\lim_{n\to\infty}\frac{2n+!}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{1}{2}\cdot 2\cdot\frac{1}{e}=\frac{1}{e};\quad (3)$$

From $(1),(2),(3)$ we get:

$$\lim_{n\to\infty}e^{-2H_n}\cdot\sum_{k=2}^{n}\sqrt[k]{(2k-1)!!}=e^{-2\gamma}\cdot\frac{1}{e}=e^{-2\gamma-1}$$

Solution 2 by Samir HajAli-Damascus-Syria

$$\Omega=\lim_{n\to\infty}e^{-2H_n}\cdot\sum_{k=2}^{n}\sqrt[k]{(2k-1)!!}=\lim_{n\to\infty}\frac{\sum_{k=2}^{n}\sqrt[k]{(2k-1)!!}}{e^{2H_n}}\overset{L.C-S}{=}\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{e^{2H_n}-e^{2H_{n-1}}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{e^{2H_{n-1}+\frac{2}{n}}-e^{2H_{n-1}}}=\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{e^{2H_{n-1}}\left(e^{\frac{2}{n}}-1\right)}=\lim_{n\to\infty}\frac{\frac{2}{n}}{e^{\frac{2}{n}}-1}\times\lim_{n\to\infty}\frac{n\cdot\sqrt[n]{(2n-1)!!}}{2\cdot e^{2H_{n-1}}}=$$

$$=1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n\cdot(2n-1)!!}{2^n\cdot e^{2nH_{n-1}}}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}\cdot(2n+1)!!}{2^{n+1}\cdot e^{2(n+1)H_n}}\cdot\frac{2^n\cdot e^{2nH_{n-1}}}{n^n\cdot(2n-1)!!}=$$

$$=\lim_{n\to\infty}\left[\frac{2^n}{2^{n+1}}\cdot\frac{(n+1)^{n+1}}{n^n}\cdot\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{e^{2nH_{n-1}}}{e^{2(n+1)H_n}}\right]=$$

$$=\lim_{n\to\infty}\left[\frac{1}{2}\cdot\left(\frac{n+1}{n}\right)^n\cdot(n+1)(2n+1)\cdot\frac{e^{2nH_{n-1}}}{e^{2nH_n}\cdot e^{2H_n}}\right]=$$

$$=\frac{e}{2}\cdot\lim_{n\to\infty}\left[(n+1)(2n+1)\cdot\frac{e^{2nH_{n-1}}}{e^{2H_{n-1}}\cdot e^2\cdot e^{2H_n}}\right]=$$

$$\left(\therefore e^{2nH_n}=e^{2n\left(H_{n-1}+\frac{1}{n}\right)}=e^{2H_{n-1}+2}\right)$$

$$=\frac{e}{2}\cdot\lim_{n\to\infty}\left[(n+1)(2n+1)\cdot\frac{1}{e^2\cdot e^{2H_n}}\right]=\frac{e^{-1}}{2}\cdot\lim_{n\to\infty}\left[(n+1)(2n+1)\cdot\frac{1}{e^{2(logn+\gamma+\delta_n)}}\right]=$$

$$\left(\therefore H_n=logn+\gamma+\delta_n\ and\ \delta_n\underset{n\to 0}{\longrightarrow}0;\ \gamma-Euler-Mascheroni\ ct.\right)$$

$$=\frac{e^{-1}}{2}\cdot\lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{e^{2\gamma}\cdot e^{2\delta_n}\cdot e^{log(n^2)}}\right]=\frac{e^{-1}}{2e^{2\gamma}}\cdot\lim_{n\to\infty}\frac{1}{e^{2\delta_n}}\cdot\lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2}=$$

$$=\frac{e^{-1}}{2e^{2\gamma}}\cdot 1\cdot 2=\frac{e^{-1}}{e^{2\gamma}}=\frac{1}{e^{2\gamma+1}}$$

Solution 3 by Hemn Hsain-Cairo-Egypt

$$\left(\therefore H_n=logn+\gamma\ and\ \sqrt[k]{(2k-1)!!}\cong\left(\frac{(2k)^{2k}}{e^{2k}\cdot 2^k\cdot k!}\right)^{\frac{1}{k}}=\frac{2k}{e}\right.$$

$$\lim_{n\to\infty}e^{-2H_n}\cdot\sum_{k=2}^{n}\sqrt[k]{(2k-1)!!}=\lim_{n\to\infty}e^{-2logn-2\gamma}\cdot\sum_{k=2}^{n}\frac{2k}{e}=\lim_{n\to\infty}\frac{e^{-2\gamma}}{n^2}\cdot\frac{2}{e}\cdot\sum_{k=2}^{n}k=$$

$$=\lim_{n\to\infty}\frac{2e^{-2\gamma-1}(n^2+n)}{2n^2}=\lim_{n\to\infty}\frac{e^{-2\gamma-1}(n^2+n)}{n^2}=e^{-2\gamma-1}$$

UP.317. If $a,b\in\mathbb{R}$, find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^a\cdot\left((2n+1)!!\right)^b}-\sqrt[n]{n^a\cdot\left((2n-1)!!\right)^b}\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Adrian Popa-Romania

$$a_n=\sqrt[n]{n^a\cdot\left((2n-1)!!\right)^b}$$

$$1)\ \lim_{n\to\infty}\frac{a_n}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n^a\cdot\left((2n-1)!!\right)^b}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^a}{(n+1)^{n+1}}\cdot\frac{n^n}{n^a}\cdot\frac{\left((2n+1)!!\right)^b}{\left((2n-1)!!\right)^b}=$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{(2n+1)^b}{n+1}=\lim_{n\to\infty}\left[\left(1-\frac{1}{n+1}\right)^{-(n+1)}\right]^{\frac{-n}{n+1}}\cdot\frac{(2n+1)^b}{n+1}=$$

$$=\frac{1}{e}\lim_{n\to\infty}\frac{(2n+1)^b}{n+1}=\begin{cases}\frac{2}{e}, if\ b=1\\ +\infty, if\ b>1\\ 0, if\ b<1\end{cases}$$

We take $b=1$

$$2)\ \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)^a\cdot\left((2n+1)!!\right)^b}}{\sqrt[n]{n^a\cdot\left((2n-1)!!\right)^b}}=$$

$$=\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)^a\cdot\left((2n+1)!!\right)^b}{(n+1)^{n+1}}}\cdot\sqrt[n]{\frac{n^n}{n^a\cdot\left((2n-1)!!\right)^b}}\cdot\frac{n+1}{n}=\frac{e}{2}\cdot\frac{2}{e}\cdot 1=1$$

$$3)\ \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)^a\cdot\left((2n+1)!!\right)^b}}{\sqrt[n]{n^a\cdot\left((2n-1)!!\right)^b}}\right)^n=$$

$$=\lim_{n\to\infty}\frac{\left((n+1)^a(2n+1)!!\right)^{\frac{n}{n+1}}}{n^a(2n-1)!!}=$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^a\cdot(n+1)^{-\frac{a}{n+1}}\cdot\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{1}{\sqrt[n+1]{(2n+1)!!}}=$$

$$=\lim_{n\to\infty}\left(\sqrt[n+1]{n+1}\right)^{-a}\cdot(2n+1)\cdot\sqrt[n+1]{\frac{(n+1)^{n+1}}{(2n+1)!!}}\cdot\frac{1}{n+1}=2\cdot 2e=4e$$

$$\therefore\ \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{n+1}{2n+1}=2e$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^a\cdot\left((2n+1)!!\right)^b}-\sqrt[n]{n^a\cdot\left((2n-1)!!\right)^b}\right)=\begin{cases}\frac{2}{e}(log2+1), if\ b=1\\ +\infty, if\ b>1\\ 0, if\ b<1\end{cases}$$

UP.318.Find:

$$\Omega=\lim_{n\to\infty}\sqrt{n}\left(\frac{n+1}{\sqrt[2(n+1)]{(n+1)!}}-\frac{n}{\sqrt[2n]{n!}}\right)$$

Proposed by D.Bătineţu Giurgiu-Romania

Solution 1 by Sergio Esteban-Argentina

By Stirling's approximation:

$$\Omega=\lim_{n\to\infty}\sqrt{n}\left(\frac{n+1}{\sqrt[2(n+1)]{(n+1)!}}-\frac{n}{\sqrt[2n]{n!}}\right)=\lim_{n\to\infty}\sqrt{n}\left(\frac{n+1}{\sqrt{\frac{n+1}{e}}}-\frac{n}{\sqrt{\frac{n}{e}}}\right)=$$

$$=\sqrt{e}\cdot\lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n}}{\sqrt{n+1}}-\frac{n\sqrt{n}}{\sqrt{n}}\right)=\sqrt{e}\cdot\lim_{n\to\infty}\left(\sqrt{\frac{n(n+1)^2}{n+1}}-\sqrt{n^2}\right)=$$

$$=\sqrt{e}\cdot\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2}\right)=$$

$$=\sqrt{e}\cdot\lim_{n\to\infty}\frac{\left(\sqrt{n^2+n}-\sqrt{n^2}\right)\left(\sqrt{n^2+n}+\sqrt{n^2}\right)}{\left(\sqrt{n^2+n}-\sqrt{n^2}\right)}=\sqrt{e}\cdot\lim_{n\to\infty}\frac{n}{2n}=\frac{\sqrt{e}}{2}$$

Solution 2 by Marian Ursărescu-Romania

$$\Omega=\lim_{n\to\infty}\frac{n\sqrt{n}}{\sqrt[2n]{n!}}\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{n!}}{\sqrt[2n+2]{(n+1)!}}-1\right);\ \ (1)$$

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{n!}}=\lim_{n\to\infty}\sqrt{\frac{n}{\sqrt[n]{n!}}}=\sqrt{\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}}=\sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}}\overset{C-D'A}{=}\sqrt{\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}}=$$

$$=\sqrt{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n}=\sqrt{e};(2)$$

$$\lim_{n\to\infty} n\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{n!}}{\sqrt[2n+2]{(n+1)!}}-1\right)=\lim_{n\to\infty} n\left(e^{\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{n!}}{\sqrt[2n+2]{(n+1)!}}\right)}-1\right)=$$

$$=\lim_{n\to\infty} n\left(\frac{e^{\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{n!}}{\sqrt[2n+2]{(n+1)!}}\right)}-1}{\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{n!}}{\sqrt[2n+2]{(n+1)!}}\right)}\right)\cdot\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{n!}}{\sqrt[2n+2]{(n+1)!}}\right)=$$

$$=\lim_{n\to\infty} n\cdot\log\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{n!}}{\sqrt[2n+2]{(n+1)!}}\right)=\lim_{n\to\infty}\log\left[\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{\sqrt[2n]{n!}}{\sqrt[2n+2]{(n+1)!}}\right)^n\right]=$$

$$=\lim_{n\to\infty}\log\left[e\cdot\sqrt{\left(\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\right)^n}\right]=\lim_{n\to\infty}\log\left[e\cdot\sqrt{\frac{n!}{\left(\sqrt[n+1]{(n+1)!}\right)^n}}\right]=$$

$$=\lim_{n\to\infty}\log\left[e\cdot\sqrt{\frac{n!\cdot\sqrt[n+1]{(n+1)!}}{(n+1)!}}\right]=\lim_{n\to\infty}\log\left[e\cdot\sqrt{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}\right]=$$

$$=\lim_{n\to\infty}\log\left[e\cdot\sqrt{\frac{\sqrt[n]{n!}}{n}}\right]\overset{(2)}{=}\log\left(e\cdot\sqrt{\frac{1}{e}}\right)=\frac{1}{2};(3)$$

From $(1),(2),(3)$ we get

$$\Omega=\lim_{n\to\infty}\sqrt{n}\left(\frac{n+1}{\sqrt[2(n+1)]{(n+1)!}}-\frac{n}{\sqrt[2n]{n!}}\right)=\frac{\sqrt{e}}{2}$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\lim_{n\to\infty}\sqrt{n}\left(\frac{n+1}{\sqrt[2(n+1)]{(n+1)!}}-\frac{n}{\sqrt[2n]{n!}}\right)=\lim_{n\to\infty}\frac{n\sqrt{n}}{\sqrt[2n]{n!}}\left(\frac{\frac{n+1}{\sqrt[2n+2]{(n+1)!}}}{\frac{n}{\sqrt[2n]{n!}}}-1\right)=$$

$$=\sqrt{e}\lim_{n\to\infty} n\underbrace{\frac{\frac{\frac{n+1}{\sqrt[2n+2]{(n+1)!}}}{\frac{n}{\sqrt[2n]{n!}}}-1}{\log\left(\frac{\frac{n+1}{\sqrt[2n+2]{(n+1)!}}}{\frac{n}{\sqrt[2n]{n!}}}\right)}}_{\to 1}\cdot\log\left(\frac{\frac{n+1}{\sqrt[2n+2]{(n+1)!}}}{\frac{n}{\sqrt[2n]{n!}}}-1\right)=$$

$$=\sqrt{e}\lim_{n\to\infty}log\left(\left(\frac{n+1}{n}\right)^n\cdot\sqrt{\frac{n!}{\left(\sqrt[n+1]{(n+1)!}\right)^n}}\right)=$$

$$=\sqrt{e}\lim_{n\to\infty}log\left(\left(\frac{n+1}{n}\right)^n\cdot\sqrt{\frac{n!\cdot\sqrt[n]{n!}}{(n+1)!}}\right)=\sqrt{e}\lim_{n\to\infty}log\left(\left(\frac{n+1}{n}\right)^n\cdot\sqrt{\frac{\sqrt[n]{n!}}{n+1}}\right)=\frac{\sqrt{e}}{2}$$

$$\left(\therefore\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e\right)$$

$$\Omega=\lim_{n\to\infty}\sqrt{n}\left(\frac{n+1}{\sqrt[2(n+1)]{(n+1)!}}-\frac{n}{\sqrt[2n]{n!}}\right)=\frac{\sqrt{e}}{2}$$

UP.319. If $(H_n)_{n\ge1}, H_n=\sum_{k=1}^n\frac{1}{k}, (a_n)_{n\ge1}$ is sequence of real numbers strictly positive such that: $\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}=a\in\mathbb{R}_+^*=(0,\infty)$ then find:

$$\lim_{n\to\infty}e^{-3H_n}\cdot\sum_{k=2}^{n}\sqrt[k]{a_k}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania

Solution 1 by Marian Ursărescu-Romania

$$L=\lim_{n\to\infty}e^{-3H_n}\cdot\sum_{k=2}^{n}\sqrt[k]{a_k}=\lim_{n\to\infty}\frac{n^3}{e^{3H_n}}\cdot\frac{\sum_{k=2}^n\sqrt[k]{a_k}}{n^3};\quad (1)$$

$$\lim_{n\to\infty}\frac{n^3}{e^{3H_n}}=\left(\lim_{n\to\infty}\frac{n}{e^{H_n}}\right)^3\overset{L.C-S}{=}\left(\lim_{n\to\infty}\frac{n+1-n}{e^{H_{n+1}}-e^{H_n}}\right)^3=\left(\lim_{n\to\infty}\frac{1}{e^{H_n}(e^{H_{n+1}-H_n}-1)}\right)^3=$$

$$=\left(\lim_{n\to\infty}\frac{1}{e^{H_n}\left(\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}\cdot(n+1)}\right)}\right)^3=\left(\lim_{n\to\infty}\frac{1}{\frac{e^{H_n}}{n+1}}\right)^3=\left(\lim_{n\to\infty}\frac{1}{\frac{n}{n+1}\cdot\frac{e^{H_n}}{n}}\right)^3=$$

$$=\left(\lim_{n\to\infty}\frac{1}{e^{H_n-logn}}\right)^3=e^{-3\gamma};\quad (2)$$

$$\lim_{n\to\infty}\frac{\sum_{k=2}^{n}\sqrt[k]{a_k}}{n^3}\overset{L.C-S}{=}\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^3-n^3}=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{3n^2+3n+1}=$$

$$=\lim_{n\to\infty}\frac{(n+1)^2}{3n^2+3n+1}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2}=\frac{1}{3}\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^2}=\frac{1}{3}\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{2n}}}\overset{C-D'A}{=}$$

$$=\frac{1}{3}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{2n+2}}\cdot\frac{n^{2n}}{a_n}=\frac{1}{3}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^2a_n}\cdot\left(\frac{n}{n+1}\right)^{2n}=\frac{1}{3}\cdot a\cdot e^{-2};\quad(3)$$

From $(1),(2),(3)\Rightarrow L=\frac{1}{3}\cdot e^{-3\gamma}\cdot a\cdot e^{-2}=\frac{a}{3}\cdot e^{-3\gamma-2}$

Solution 2 by Samir HajAli-Damascus-Syria

$$a_n>0,\forall n\geq 1;\ \sum_{k=2}^{n}\sqrt[k]{a_k}-\text{is diverge, because}\lim_{k\to\infty}\sqrt[k]{a_k}=\lim_{k\to\infty}\frac{a_{k+1}}{a_k}\geq\lim_{k\to\infty}\frac{a_{k+1}}{k^2\cdot a_k}=a\neq 0$$

$$\Omega=\lim_{n\to\infty}\frac{\sum_{k=2}^{n}\sqrt[k]{a_k}}{e^{3H_n}}\overset{L.C-S}{=}\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{e^{3H_n-3H_{n-1}}}=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{e^{3H_{n-1}}\left(e^{\frac{3}{n}}-1\right)}=$$

$$=\lim_{n\to\infty}\frac{\frac{3}{n}}{e^{\frac{3}{n}}-1}\cdot\frac{n\sqrt[n]{a_n}}{3e^{3H_{n-1}}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n\cdot a_n}{3^n\cdot e^{3nH_{n-1}}}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}\cdot a_{n+1}}{3^{n+1}\cdot e^{3(n+1)H_n}}\cdot\frac{3^n\cdot e^{3nH_{n-1}}}{n^n\cdot a_n}=$$

$$=\frac{1}{3}\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}\cdot\frac{(n+1)^{n-2}\cdot(n+1)^3}{n^{n-2}}\cdot\frac{e^{3nH_{n-1}}}{e^{3(n+1)H_n}}=$$

$$=\frac{1}{3}\cdot a\cdot e\cdot\lim_{n\to\infty}(n+1)^3\cdot\frac{e^{3nH_{n-1}}}{e^{3H_n}\cdot e^{3nH_n}}=$$

$$\left(\because e^{3nH_n}=e^{3n\left(H_{n-1}+\frac{1}{n}\right)}=e^{3nH_{n-1}+3}\right)$$

$$=\frac{a}{3}\cdot e\cdot\lim_{n\to\infty}\frac{(n+1)^3}{e^3\cdot e^{3H_n}}=\frac{a}{3}\cdot\frac{1}{e^2}\cdot\frac{1}{e^{3\gamma}}\cdot\lim_{n\to\infty}\frac{(n+1)^3}{e^{\log n^3}\cdot e^{3\delta_n}}=$$

$$=\frac{a}{3e^2e^{-3\gamma}}\cdot\lim_{n\to\infty}\frac{(n+1)^3}{n^3}=\frac{a}{3}\cdot e^{-3\gamma-2}$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\lim_{n\to\infty}e^{-3H_n}\cdot\sum_{k=2}^{n}\sqrt[k]{a_k}=\lim_{n\to\infty}\frac{\sum_{k=2}^{n}\sqrt[k]{a_k}}{e^{3H_n}}\overset{L.C-S}{=}e^{-3\gamma}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{3n^2+3n+1}=$$

$$=e^{-3\gamma}\cdot\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{3n^2-3n+2}\overset{C-D'A}{=}e^{-3\gamma}\cdot\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}\cdot\frac{n^2}{3n^2+3n+5}\cdot\frac{n^2-3n+2}{3n^2+3n+5}=$$

$$= \frac{a \cdot e^{-3\gamma}}{3} \cdot \lim_{n\to\infty}\left(1 - 2 \cdot \frac{1 + \frac{1}{6n}}{n + 1 + \frac{5}{3n}}\right)^n = \frac{a \cdot e^{-3\gamma - 2}}{3}$$

UP.320. If $a, b, c \in \mathbb{R}, x_n = n!, y_n = (2n-1)!!, \forall n \in \mathbb{N}^*$, then find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^a \cdot x_{n+1}^b \cdot y_{n+1}^c} - \sqrt[n]{n^a \cdot x_n^b \cdot y_n^c}\right)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution by proposer

$\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{x_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$ and analogous $\lim_{n\to\infty}\frac{\sqrt[n]{y_n}}{n} = \frac{1}{e}$

Let: $u_n = \frac{\sqrt[n+1]{(n+1)^a \cdot x_{n+1}^b \cdot y_{n+1}^c}}{\sqrt[n]{n^a \cdot x_n^b \cdot y_n^c}}, \forall n \geq 2$ then $\lim_{n\to\infty} u_n = 1$, $\lim_{n\to\infty}\frac{u_n - 1}{log u_n} = 1$ and

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{(n+1)^a \cdot x_{n+1}^b \cdot y_{n+1}^c}{n^a \cdot x_n^b \cdot y_n^c} \cdot \frac{1}{\sqrt[n+1]{(n+1)^a \cdot x_{n+1}^b \cdot y_{n+1}^c}}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^a \cdot (n+1)^b \cdot (2n+1)^c \cdot \frac{1}{\left(\sqrt[n+1]{n+1}\right)^a} \cdot \frac{1}{\left(\sqrt[n+1]{x_{n+1}}\right)^b} \cdot \frac{1}{\left(\sqrt[n+1]{y_{n+1}}\right)^c}$$

$$= \lim_{n\to\infty}\left(\frac{2n+1}{n+1}\right)^c \cdot \left(\frac{n+1}{\sqrt[n+1]{x_{n+1}}}\right)^b \cdot \left(\frac{n+1}{\sqrt[n+1]{y_{n+1}}}\right)^c = \frac{2^a \cdot e^b \cdot e^c}{2^c} = e^{b+c}$$

$$B_n = \sqrt[n+1]{(n+1)^a \cdot x_{n+1}^b \cdot y_{n+1}^c} - \sqrt[n]{n^a \cdot x_n^b \cdot y_n^c} = \sqrt[n]{n^a \cdot x_n^b \cdot y_n^c} \cdot (u_n - 1)$$

$$= \sqrt[n]{n^a \cdot x_n^b \cdot y_n^c} \cdot \frac{u_n - 1}{log u_n} \cdot log u_n = \frac{\sqrt[n]{n^a \cdot x_n^b \cdot y_n^c}}{n} \cdot \frac{u_n - 1}{log u_n} \cdot log u_n^n$$

$$= \left(\sqrt[n]{n}\right)^a \cdot \left(\frac{n}{\sqrt[n]{x_n}}\right)^b \cdot \left(\frac{n}{\sqrt[n]{y_n}}\right)^c \cdot n^{b+c-1} \cdot \frac{u_n - 1}{log u_n} \cdot log u_n^n, \forall n \geq 2$$

Then: $\lim_{n\to\infty} B_n = 1 \cdot \frac{1}{e^b} \cdot \left(\frac{2}{e}\right)^c \cdot 1 \cdot log e^{b+c} \cdot \lim_{n\to\infty} n^{b+c-1} = \frac{(b+c)\cdot 2^c}{e^{b+c}} \cdot \lim_{n\to\infty} n^{b+c-1}$

$$=\begin{cases}0, \text{if } b+c<1\\ \dfrac{2^c}{e}, \text{if } b+c=1\\ \infty, \text{if } b+c>1\end{cases}$$

UP.321 Let $A_0A_1 \dots A_n$ be an Euclidean n-simplex.We'll use the following notations:

$-O, V, R, r$ the centre if it's circumscribed hypersphere, it's volume, it's circumradius and it's inradius,respectively.

$-O_i, R_i$ the centre and the radius of the hypersphere tangent to the circumscribed sphere of $A_0A_1 \dots A_n$ in the vertex A_i and to the hyperplane $A_0A_1 \dots A_{i-1}A_{i+1} \dots A_n$ simultaneously.

With the above notations, the following identity holds:

$$\sum_{i=0}^{n} \frac{1}{R_i} = \frac{n}{R} + \frac{1}{r}$$

Proposed by Vasile Jiglău-Romania

Solution by proposer

Let $A_0A_1A_2$ be an arbitrary triangle and denote by R_i the radius of the cercle which is tangent to the circumcircle of $A_0A_1A_2$ and to the side A_jA_k of the given triangle, simultaneously $(\{i,j,k\} = \{1,2,3\})$. In [1] the authors proved that:

$$\frac{1}{R_0} + \frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{R} + \frac{1}{r}$$

Where R and r are the circumradius and the inradius of $A_0A_1A_2$, respectively.

In the following we'll prove an extension of this identity to the Euclidean n-simplex.

Taking in the proposition enunciated below and in it's proof $n = 3$ one obtains the corresponding identity for tetrahedron.

Let $A_0A_1 \dots A_n$ be an Euclidean n-simplex.We'll use the following notations:

$-O, V, R, r$ the centre if it's circumscribed hypersphere, it's volume, it's circumradius and it's inradius,respectively.

$-O'_i, R'_i$ the centre and the radius of the hypersphere tangent to the circumscribed sphere of $A_0A_1 \dots A_n$ in the vertex A_i and to the hyperplane $A_0A_1 \dots A_{i-1}A_{i+1} \dots A_n$ simultaneously.

$-V_i$ the volume of the n-simplex $OA_0A_1 \dots A_{i-1}A_{i+1} \dots A_n$

$-V'_j$ the volume of the n-simplex $O'_jA_0A_1 \dots A_{j-1}A_{j+1} \dots A_n$

$-V"_i$ the volume of the (n-1)-simplex $A_0A_1 \dots A_{i-1}A_{i+1} \dots A_n$

Proposition: With the above notations, the following identity holds:

$$\sum_{i=0}^{n} \frac{1}{R_i} = \frac{n}{R} + \frac{1}{r}$$

Proof: Since the hypersphere having $-O'_i$ as centre defined above is tangent to the circumscribed hypersphere of the given n-simplex, it results that A_i, O'_i, O are collinear. We recall that the volume of $A_0A_1 \dots A_n$ is given by the formula $V = \frac{1}{n}h_iV"_i$, where h_i is the distance from the vertex A_i to the hyperplane $A_0A_1 \dots A_{i-1}A_{i+1} \dots A_n$.Projecting O, O'_i on the hyperplane $A_0A_1 \dots A_{j-1}A_{j+1} \dots A_n$, $(i \neq j)$ applying the Thales' theorem, then the precedent formula, we ramark that $\frac{V'_j}{V_j} = \frac{R_i}{R}$. We have:

$$V = \sum_{\substack{j=0 \\ j\neq i}}^{n} V'_j + V'_i = \sum_{\substack{j=0 \\ j\neq i}}^{n} \frac{R_i}{R} V_j + \frac{R_iV"_i}{n} = R_i\left(\frac{1}{R}\sum_{\substack{j=0 \\ j\neq i}}^{n} V_j + \frac{V"_i}{n}\right) \Rightarrow \frac{V}{R_i} = \frac{1}{R}\sum_{\substack{j=0 \\ j\neq i}}^{n} V_j + \frac{V"_i}{n}$$

Since the sum $\sum_{i=0}^{n}\sum_{\substack{j=0 \\ j\neq i}}^{n} V_j$ any V_j appears n times, we have $\frac{1}{R}\sum_{i=0}^{n}\sum_{\substack{j=0 \\ j\neq i}}^{n} V_j = \frac{1}{R}nV$

On the other hand $nV = r\sum_{i=1}^{n} V"_j$, therefore

$\sum_{i=1}^{n} \frac{V}{R_i} = \frac{nV}{R} + \frac{V}{r} \Rightarrow \sum_{i=0}^{n} \frac{1}{R_i} = \frac{n}{R} + \frac{1}{r}$, q.e.d.

Reference:

[1] I.Isaev, Y.Maltsev, A.Monastyreva-On some geometric relations of a triangle, Journal of Classical Geometry, volume 4.

UP.322. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}k\left[\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}\right]\binom{n}{k}}$$

Proposed by Marian Ursărescu-Romania

Solution by Sergio Esteban-Argentina

We will use the following identities:

$$i)\ \binom{x}{x}+\binom{x+1}{x}+\cdots+\binom{x+y}{x}=\binom{x+y+1}{x+1},\text{ put } x=k \text{ and } y=n-k$$

$$ii)\ k\binom{n}{k}=n\binom{n-1}{k-1}$$

$$iii)\ \sum_{k=0}^{r}\binom{m}{k}\binom{t}{r-k}=\binom{m+t}{r},\text{ put } m=n-1,t=n+1,r=n-1\Rightarrow$$

$$\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}k\left[\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}\right]\binom{n}{k}}=\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}k\binom{n+1}{k+1}\binom{n}{k}}=$$

$$=\lim_{n\to\infty}\sqrt[n]{n\sum_{k=1}^{n}\binom{n+1}{k+1}\binom{n-1}{k-1}}=\lim_{n\to\infty}\sqrt[n]{n\sum_{k=0}^{n}\binom{n-1}{k}\binom{n+1}{k+2}}=$$

$$=\lim_{n\to\infty}\sqrt[n]{n\binom{2n}{n-1}}=\lim_{n\to\infty}\sqrt[n]{\frac{n(2n)!}{(n-1)!\,(n+1)!}}\overset{\text{by Stirling's}}{=}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{2n}{e}\right)^{2}}{\left(\frac{n-1}{e}\right)^{\frac{n-1}{e}}\cdot\left(\frac{n+1}{e}\right)^{\frac{n+1}{e}}}=4$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}k\left[\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}\right]\binom{n}{k}}=4$$

UP.323 If $S_n=\sum_{k=1}^{n} log\left(cos\frac{\pi}{2^{k+2}}\right)$ then find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{n\cdot S_n}\right)^{\sum_{k=3}^{n} tan\frac{\pi}{k}}$$

Proposed by Florică Anastase-Romania

Solution by proposer

Let $S_n(x)=\sum_{k=1}^{n} log\left(cos\frac{x}{2^k}\right)$, with $cos\frac{x}{2}>0$ and from $sin2a=2sinacosa$ we get:

$$log(cosa)=log(sin2a)-log(sina)-log2$$

For $a=\frac{x}{2},\frac{x}{2^2},\dots,\frac{x}{2^n}$ we get: $S_n(x)=log(sinx)-nlog2-log\left(sin\frac{x}{2^n}\right)=log\left(\frac{sinx}{2^n sin\frac{x}{2^n}}\right)$

Then: $S_n=S_n\left(\frac{\pi}{4}\right)=log\left(\frac{\frac{1}{\sqrt{2}}}{2^n sin\frac{\pi}{2^{n+2}}}\right)=log\left(\frac{1}{\sqrt{2}\cdot 2^n sin\frac{\pi}{2^{n+2}}}\right)\xrightarrow[n\to\infty]{} log\left(\frac{2\sqrt{2}}{\pi}\right)$

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{n\cdot S_n}\right)^{\sum_{k=3}^{n} tan\frac{\pi}{k}}=e^{\lim\limits_{n\to\infty}\left(\sum_{k=3}^{n} tan\frac{\pi}{k}\right)log\sqrt[n]{n\cdot S_n}}$$

$$=e^{\lim\limits_{n\to\infty}\left(\sum_{k=3}^{n} tan\frac{\pi}{k}\right)\cdot\frac{log(n\cdot S_n)}{n}}=e^{\lim\limits_{n\to\infty}\frac{\left(\sum_{k=3}^{n} tan\frac{\pi}{k}\right)}{\sqrt{n}}\cdot\frac{log(n\cdot S_n)}{\sqrt{n}}},\quad (1)$$

$$\lim_{n\to\infty}\frac{\left(\sum_{k=3}^{n} tan\frac{\pi}{k}\right)}{\sqrt{n}}\overset{S-Cesaro}{=}\lim_{n\to\infty}\frac{tan\frac{\pi}{n+1}}{\sqrt{n+1}-\sqrt{n}}=\lim_{n\to\infty}\frac{tan\frac{\pi}{n+1}}{\frac{\pi}{n+1}}\cdot\frac{\pi}{(n+1)(\sqrt{n+1}-\sqrt{n})}$$
$$=0,\qquad (2)$$

$$\lim_{n\to\infty}\frac{log(n\cdot S_n)}{\sqrt{n}}=\lim_{n\to\infty}\frac{logn+logS_n}{\sqrt{n}}=\lim_{n\to\infty}\frac{logn}{\sqrt{n}}\overset{S-Cesaro}{=}\lim_{n\to\infty}\frac{log(n+1)-logn}{\sqrt{n+1}-\sqrt{n}}=$$
$$=\lim_{n\to\infty}\frac{log\left(1+\frac{1}{n}\right)^n}{n(\sqrt{n+1}-\sqrt{n})}=0,\qquad (3)$$

From $(1),(2),(3)$ we get: $\Omega=e^0=1$.

UP.324. For $n\in\mathbb{N}^*, n\geq 2, P_n=\prod_{k=1}^{n-1} sin\left(\frac{k\pi}{n}\right)$, find:

$$\Omega=\lim_{n\to\infty}\frac{n}{2}\cdot P_n\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{cos3x}{sin^n x}dx$$

Proposed by Florică Anastase-Romania

Solution 1 by Sergio Esteban-Argentina

i) We know that:

$$\frac{x^{2n}-1}{x-1}=\prod_{k=1}^{n-1}\left(x^2-2\cos\left(\frac{k\pi}{n}\right)+1\right)\Rightarrow$$

$$\lim_{x\to-1}\frac{x^{2n}-1}{x-1}=\lim_{x\to-1}\prod_{k=1}^{n-1}\left(x^2-2\cos\left(\frac{k\pi}{n}\right)+1\right)\Rightarrow$$

$$n=\prod_{k=1}^{n-1}2\left(1+\cos\left(\frac{k\pi}{n}\right)\right)=\prod_{k=1}^{n-1}2^2\cos^2\left(\frac{k\pi}{2n}\right)\Rightarrow\prod_{k=1}^{n-1}\cos\left(\frac{k\pi}{2n}\right)=\frac{\sqrt{n}}{2^{n-1}}$$

Analogously if $\lim\limits_{x\to-1}\Rightarrow\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{2n}\right)=\frac{\sqrt{n}}{2^{n-1}}$

By (i)$\Rightarrow\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)=\prod_{k=1}^{n-1}2\sin\left(\frac{k\pi}{2n}\right)\cos\left(\frac{k\pi}{2n}\right)=\frac{n}{2^{n-1}},\forall n\in\mathbb{N}^*,n\geq2\Rightarrow$

$$P_n=\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)=\frac{n}{2^{n-1}}$$

Now,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos3x}{\sin^n x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos^3x-3\cos x\sin^2x}{\sin^n x}dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos x\left(\frac{1}{\sin^n x}-4\sin^{2-n}x\right)dx\overset{u=\sin x\Rightarrow dx=\frac{du}{\cos x}}{\cong}$$

$$=\left[\frac{\csc^{n-1}x}{1-n}-\frac{4\csc^{n-3}x}{3-n}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}=-2^3\cdot\frac{2^n-3n+1}{8(n-3)(n-1)}$$

Finally, $\Omega=-\lim\limits_{n\to\infty}\left(\frac{n}{2}\cdot\frac{n}{2^{n-1}}\cdot\frac{2^n-3n+1}{(n-3)(n-1)}\right)=-1$

Solution 2 by Naren Bhandari-Bajura-Nepal

We have that $2i\sin x=e^{ix}-e^{-ix}$ with $i=\sqrt{-1}$ which follow that

$$P_n=\frac{1}{2^{n-1}}\left(\prod_{k=1}^{n-1}e^{\frac{ki\pi}{n}}\right)\left(\prod_{k=1}^{n-1}\left(1-e^{\frac{ki\pi}{n}}\right)\right)=\frac{L_n}{(2i)^{n-1}}\exp\left(\sum_{k=1}^{n-1}\frac{k\pi}{n}\right)=\frac{L_n}{2^{n-1}}$$

where L_n is the latter product to be evaluated by noticing the polynomial

$$F(X)=\prod_{k=1}^{n}\left(X-e^{-\frac{2ki\pi}{n}}\right)$$

whose zeros are non-trivial solution n-th roots or unity i.e.

$$f(X)=\sum_{k=1}^{n}x^k \text{ and } F(1)=\sum_{k=1}^{n-1}1^k=n\Rightarrow$$

$$P_n=\frac{n}{2^{n-1}}\Rightarrow\Omega=\lim_{n\to\infty}\frac{n\cdot P_n}{2}\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos 3x}{\sin^n x}dx=\lim_{n\to\infty}\frac{n^2}{2^n}\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{4\cos^3 x-\cos x}{\sin^n x}dx=$$

$$\overset{\sin x=u}{=}\lim_{n\to\infty}\frac{n^2}{2^n}\cdot\int_{\frac{1}{2}}^{1}\frac{1-u^2}{u^n}du==-\lim_{n\to\infty}\left(\frac{n^2}{2^n}\cdot\frac{2^n-3n+1}{n^2-4n+3}\right)=-1$$

UP.325 Let be $(a_n)_{n\geq 1},\left(f_n(x)\right)_{n\geq 1}; n\in\mathbb{N}, n\geq 7, x>1$

$$a_n=\left(\prod_{k=1}^{n}\binom{n}{k}\right)^2, f_n(x)=\int_{x}^{x^2}\frac{1}{\log\sqrt[n]{t}}dt$$

Then find:

$$\Omega_1=\lim_{x\to\infty}f_n(x) \text{ and } \Omega_2=\lim_{n\to\infty}\left(\frac{1}{a_n}\lim_{x\to 1}f_n(x)\right)$$

Proposed by Florică Anastase-Romania

Solution by proposer

Let $g:(1,\infty)\to\mathbb{R}, g(t)=\frac{1}{\log t}$ and $G:(1,\infty)\to\mathbb{R}, G'(t)=g(t)$

How $f_n(x)=G(x^2)-G(x)$ then f_n- differentiable

$$f'_n(x)=2x\cdot g(x)-g(x)=\frac{2x}{\log x^2}-\frac{1}{\log x}=\frac{x-1}{\log x}>0\Rightarrow f_n(x)\uparrow x\in(1,\infty)$$

How $\frac{1}{\log t}>\frac{1}{\log x^2},\forall t\in(x,x^2), x>1$ we have

$$f_n(x)=\int_x^{x^2}\frac{1}{log\sqrt[n]{t}}dt=n\int_x^{x^2}\frac{1}{logt}dt\ge n\int_x^{x^2}\frac{1}{logx^2}dt=\frac{n(x^2-x)}{2logx},\forall x>1$$

$$\Rightarrow \Omega_1=\lim_{x\to\infty}f_n(x)=\lim_{x\to\infty}\frac{n(x^2-x)}{2logx}\overset{L'H}{=}n\lim_{x\to\infty}\frac{2x-1}{2\cdot\frac{1}{x}}=n\lim_{x\to\infty}\frac{2x^2-x}{2}=+\infty$$

$$\frac{1}{tlogt}\le\frac{1}{logt}\le\frac{1}{t-1}+\frac{1}{3-t},\forall t\in(1,3)\quad (*)$$

$$\frac{1}{tlogt}\le\frac{1}{logt},\forall t\in(1,3)\Leftrightarrow\frac{1}{logt}\left(1-\frac{1}{t}\right)\ge 0,\forall t\in(1,3)\Leftrightarrow$$

$$\frac{t-1}{tlogt}\ge 0,\forall t\in(1,3)(true)$$

Now,

$$\frac{1}{logt}\le\frac{1}{t-1}+\frac{1}{3-t},\forall t\in(1,3)\Leftrightarrow\frac{1}{logt}\le\frac{2}{-t^2+4t-3},\forall t\in(1,3)\Leftrightarrow$$

$$logt+\frac{t^2-4t+3}{2}\ge 0,\forall t\in(1,3)$$

Let $h(t)=\ logt+\frac{t^2-4t+3}{2},t\in(1,3);h'(t)=\frac{(t-1)^2}{2}\ge 0,\forall t\in(1,3)\Rightarrow h(t)\uparrow t\in(1,3)$

From $(*)$ we have:

$$\int_x^{x^2}\frac{1}{tlogt}dt\le\int_x^{x^2}\frac{1}{logt}dt\le\int_x^{x^2}\left(\frac{1}{t-1}+\frac{1}{3-t}\right)dt$$

$$n\int_x^{x^2}\frac{1}{tlogt}dt\le f_n(x)\le n\int_x^{x^2}\left(\frac{1}{t-1}+\frac{1}{3-t}\right)dt$$

$$\int_x^{x^2}\frac{1}{tlogt}dt=log(logt)|_x^{x^2}=log2,\forall x\in\left(1,\sqrt{3}\right)\Rightarrow\lim_{x\to 1}\int_x^{x^2}\frac{1}{tlogt}dt=log2$$

$$\int_x^{x^2}\left(\frac{1}{t-1}+\frac{1}{3-t}\right)dt=log\left|\frac{t-1}{3-t}\right|\Bigg|_x^{x^2}=log\left(\frac{(x+1)(3-x)}{3-x^2}\right),\forall x\in\left(1,\sqrt{3}\right)$$

$$\lim_{x\to 1}\int_x^{x^2}\left(\frac{1}{t-1}+\frac{1}{3-t}\right)dt=log2$$

So, we have:

$$\lim_{x\to1} f_n(x) = nlog2$$

Let be:

$$b_n = \prod_{k=0}^{n}\binom{n}{k} = \prod_{k=1}^{n-1}\binom{n}{k} = \prod_{k=1}^{n-1}\frac{n!}{k!\,(n-k)!} = \frac{(n!)^{n-1}}{[1!\cdot 2!\cdot \ldots \cdot (n-1)!]^2}$$

$$2 < e_n \cong \frac{n+1}{\sqrt[n+1]{(n+1)!}} < 3; n \ge 6 \ (*)$$

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^n}{n!} = \frac{(n+1)^{n+1}}{(n+1)!} \overset{(*)}{>} 2^{n+1} \Rightarrow b_{n+1} > 2^{n+1}\cdot b_n$$

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^n}{n!} = \frac{(n+1)^{n+1}}{(n+1)!} \overset{(*)}{<} 3^{n+1} \Rightarrow b_{n+1} < 2^{n+1}\cdot b_n$$

Suppose: $\begin{cases} b_n > 2^{\frac{n^2}{2}} \\ b_{n+1} > 2^{n+1}\cdot b_n \end{cases} \Rightarrow b_{n+1} > 2^{n+1}\cdot 2^{\frac{n^2}{2}} = 2^{\frac{n^2+2n+2}{2}} > 2^{\frac{(n+1)^2}{2}}$

Suppose: $\begin{cases} b_n < 3^{n^2} \\ b_{n+1} < 3^{n+1}\cdot b_n \end{cases} \Rightarrow b_{n+1} < 3^{n+1}\cdot 3^{n^2} = 3^{n^2+n+1} < 3^{(n+1)^2}$

Therefore,

$$2^{n^2} < \left(\prod_{k=1}^{n}\binom{n}{k}\right)^2 < 3^{n^2}, \forall n \ge 7$$

$$\frac{nlog2}{3^{n^2}} < \frac{nlog2}{a_n} < \frac{nlog2}{2^{n^2}}$$

$$0 \le \frac{n}{2^{n^2}} = \frac{n}{2^n\cdot 2^n\cdot \ldots \cdot 2^n} < \frac{n}{2^n} = \frac{n}{(1+1)^n} =$$

$$= \frac{n}{1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}} = \frac{n}{1+n+\frac{n(n-1)}{2}+\cdots} < \frac{n}{\frac{n(n-1)}{2}} = 0$$

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{a_n}\lim_{x\to1} f_n(x)\right) = \lim_{n\to\infty}\left(\frac{nlog2}{a_n}\right) = 0$$

UP.326 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{n}{\log n}\right)^e\cdot e^{\int_0^e \log\left(\frac{\log(x+e)}{x^2+ne}\right)dx}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\frac{\beta-\alpha}{\int_\alpha^\beta \frac{1}{f(x)}dx}\le e^{\frac{1}{\beta-\alpha}\int_\alpha^\beta \log(f(x))dx}\le\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx,\alpha$$

$$<\beta\ (*)\ \text{means Integral Inequality}$$

For $x\in[0,e]\Rightarrow ne\le x^2+ne\le e^2+ne\Rightarrow\frac{\log n}{e^2+ne}\le\frac{\log(x+e)}{x^2+ne}\le\frac{\log(e+n)}{ne}$

Let $f:[0,e]\to\mathbb{R}, f(x)=\frac{\log(x+e)}{x^2+ne}$ we have:

$$\frac{e}{\int_0^e\frac{1}{f(x)}dx}=\frac{e}{\int_0^e\frac{x^2+ne}{\log(x+e)}dx}\ge\frac{e}{\int_0^e\frac{e^2+ne}{\log n}dx}=\frac{1}{e}\cdot\frac{\log n}{e+n},\qquad (1)$$

$$\frac{1}{e}\int_0^e f(x)dx=\frac{1}{e}\int_0^e\frac{\log(x+e)}{x^2+ne}dx\le\frac{1}{e}\int_0^e\frac{\log(e+n)}{ne}dx=\frac{1}{e}\cdot\frac{\log(e+n)}{n},\qquad (2)$$

From $(1),(2)$ we get:

$$\frac{1}{e}\cdot\frac{\log n}{e+n}\le e^{\frac{1}{e}\int_0^e\log\left(\frac{\log(x+e)}{x^2+ne}\right)dx}\le\frac{1}{e}\cdot\frac{\log(e+n)}{n}$$

$$\frac{1}{e}\cdot\frac{n}{e+n}\le\frac{n}{\log n}\cdot e^{\frac{1}{e}\int_0^e\log\left(\frac{\log(x+e)}{x^2+ne}\right)dx}\le\frac{1}{e}\cdot\frac{\log(e+n)}{\log n}$$

$$\frac{1}{e^e}\cdot\left(\frac{n}{e+n}\right)^e\le\left(\frac{n}{\log n}\right)^e\cdot e^{\int_0^e\log\left(\frac{\log(x+e)}{x^2+ne}\right)dx}\le\frac{1}{e^e}\cdot\left(\frac{\log(e+n)}{\log n}\right)^e$$

$$\lim_{n\to\infty}\frac{n}{e+n}=\lim_{n\to\infty}\frac{\log(e+n)}{\log n}=1$$

So,

$$\Omega=\lim_{n\to\infty}\left(\frac{n}{\log n}\right)^e\cdot e^{\int_0^e\log\left(\frac{\log(x+e)}{x^2+ne}\right)dx}=\frac{1}{e^e}$$

UP.327. If $(x_n)_{n\ge1}, x_n\in\mathbb{R}_+^*, \forall n\in\mathbb{N}^*$ satisfy $\lim\limits_{n\to\infty}(x_{n+1}-x_n)=x\in\mathbb{R}_+^*$, then find:

$$\lim_{n\to\infty}\left(x_{n+1}\sqrt[n+1]{n+1}-x_n\sqrt[n]{n}\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Marian Ursărescu-Romania

$$L=\lim_{n\to\infty}\left(x_{n+1}\sqrt[n+1]{n+1}-x_n\sqrt[n]{n}\right)=$$

$$=\lim_{n\to\infty}\left(x_{n+1}\sqrt[n+1]{n+1}-x_{n+1}\sqrt[n]{n}+x_{n+1}\sqrt[n]{n}-x_n\sqrt[n]{n}\right)=$$

$$=\lim_{n\to\infty}x_{n+1}\left(\sqrt[n+1]{n+1}-\sqrt[n]{n}\right)+\lim_{n\to\infty}\sqrt[n]{n}(x_{n+1}-x_n);\ \ (1)$$

$$\lim_{n\to\infty}\sqrt[n]{n}(x_{n+1}-x_n)=1\cdot x=x;(2)$$

$$\lim_{n\to\infty}x_{n+1}\left(\sqrt[n+1]{n+1}-\sqrt[n]{n}\right)=\lim_{n\to\infty}\frac{x_{n+1}}{n+1}\cdot(n+1)\cdot\left(\sqrt[n+1]{n+1}-\sqrt[n]{n}\right)$$

$$\lim_{n\to\infty}\frac{x_{n+1}}{n+1}=\lim_{n\to\infty}\frac{x_n}{n}\overset{LC-S}{=}\lim_{n\to\infty}\frac{x_{n+1}-x_n}{n+1-n}=x;\ \ (3)$$

Let be the function $f:[n,n+1]\to\mathbb{R}, f(x)=x^{\frac{1}{x}}$, from MVT we have:

$\exists c\in(n,n+1)$ such that $\frac{f(n+1)-f(n)}{n+1-n}=f'(c)\Leftrightarrow$

$$\sqrt[n+1]{n+1}-\sqrt[n]{n}=\frac{c^{\frac{1}{c}}(1-\log c)}{c^2}\Rightarrow$$

$$\lim_{n\to\infty}(n+1)\cdot\left(\sqrt[n+1]{n+1}-\sqrt[n]{n}\right)=\lim_{n\to\infty}(n+1)\cdot\frac{c^{\frac{1}{c}}(1-\log c)}{c^2}=0;\ \ (4)\text{ because}$$

From $c\in(n,n+1)\Rightarrow\lim\limits_{x\to\infty}\frac{x+1}{x}\cdot\frac{1-\log x}{x}=1\cdot0=0$

From (1),(2),(3),(4) we get $L=x$.

UP.328. Let $(\gamma_n)_{n\ge1}, \gamma_n=-\log n+\sum_{k=1}^{n}\frac{1}{k}$, with $\lim\limits_{n\to\infty}\gamma_n=\gamma$ (γ is Euler-Mascheroni constant), then find:

$$\lim_{n\to\infty}(\sin\gamma_n-\sin\gamma)\sqrt[n]{n!}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega = \lim_{n\to\infty}(sin\gamma_n - sin\gamma)\sqrt[n]{n!} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\cdot\frac{sin(H_n - logn) - sin\gamma}{\frac{1}{n}} =$$

$$= \frac{1}{2e}\cdot\lim_{n\to\infty}\frac{sin\left(\gamma + \frac{1}{2n} + o\left(\frac{1}{n^2}\right)\right) - sin\gamma}{\gamma + \frac{1}{2n} - \gamma} = \frac{cos\gamma}{2e}$$

Solution 2 by Marian Ursărescu-Romania

$$L = \lim_{n\to\infty}(sin\gamma_n - sin\gamma)\sqrt[n]{n!} = \lim_{n\to\infty}2sin\left(\frac{\gamma_n - \gamma}{2}\right)cos\left(\frac{\gamma_n + \gamma}{2}\right)\sqrt[n]{n!} =$$

$$= 2\lim_{n\to\infty}\frac{sin\left(\frac{\gamma_n - \gamma}{2}\right)}{\frac{\gamma_n - \gamma}{2}}\cdot(\gamma_n - \gamma)\cdot n\cdot\frac{\sqrt[n]{n!}}{n}\cdot cos\left(\frac{\gamma_n + \gamma}{2}\right); \quad (1)$$

$$\lim_{n\to\infty}\frac{sin\left(\frac{\gamma_n - \gamma}{2}\right)}{\frac{\gamma_n - \gamma}{2}} = 1; \quad (2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}; \quad (3)$$

Now, we use Cesaro-Stolz for $\frac{0}{0}$ case:

$$\lim_{n\to\infty}(\gamma_n - \gamma)\cdot n = \lim_{n\to\infty}\frac{\gamma_n - \gamma}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\gamma_{n+1} - \gamma_n}{\frac{1}{n+1} - \frac{1}{n}} =$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n+1} - log(n+1) + logn}{-\frac{1}{n(n+1)}} = \lim_{n\to\infty}\frac{1 - (n+1)log\left(1 + \frac{1}{n}\right)}{-\frac{1}{n}}$$

$$= \lim_{x\to 0}\frac{1 - \left(\frac{1}{x} + 1\right)log(1+x)}{-x} = \lim_{x\to 0}\frac{(1+x)log(1+x) - x}{x^2} \overset{L'H}{=}$$

$$= \lim_{x\to 0}\frac{log(1+x) + 1 - 1}{2x} = \frac{1}{2}; (4)$$

From (1),(2),(3),(4) we get

$$L = \frac{cos\gamma}{2e}$$

UP.329. If $(a_n)_{n\ge1}$ is a sequence of real positive numbers such that:

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^2a_n}=a\in\mathbb{R}_+^*,\text{ then find:}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(2n+1)!!}}-\sqrt[n]{\frac{a_nF_n}{(2n-1)!!}}\right)$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(2n+1)!!}}-\sqrt[n]{\frac{a_nF_n}{(2n-1)!!}}\right)\overset{LC-S}{=}\lim_{n\to\infty}\sqrt[n]{\frac{a_nF_n}{n^n(2n-1)!!}}\overset{C-D'A}{=}$$

$$\overset{C-D'A}{=}\lim_{n\to\infty}\frac{\frac{a_{n+1}F_{n+1}}{(n+1)^{n+1}(2n+1)!!}}{\frac{a_nF_n}{n^n(2n-1)!!}}=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{n^2}{(n+1)(2n+1)}\cdot\frac{a_{n+1}}{n^2a_n}\cdot\frac{F_{n+1}}{F_n}=\frac{\varphi}{2e}$$

$$\therefore (2n-1)!!=\frac{(2n)!}{2^n\cdot n!}$$

$$\therefore \lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\varphi,\varphi-Golden\,ratio.$$

UP. 330 For $n\in\mathbb{N},n\ge1$, F_n –Fibonacci numbers, prove that:

$$\frac{F_1}{3(F_1^2+F_2^2)^2}+\frac{F_2}{4(F_1^2+F_2^2+F_3^2)^2}+\cdots+\frac{F_n}{(n+2)(F_1^2+F_2^2+\cdots+F_{n+1}^2)^2}\ge\frac{(F_{n+2}-F_1)^2}{F_{n+2}^2(nF_{n+2}+F_n)}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$(i)\ \sum_{k=1}^{n}(k+2)F_k=nF_{n+2}+F_n$$

Proof:

$$F_{k+2}-F_{k+1}=F_k,\forall k>0;\ \ (1)$$

$$(k+2)F_k=2F_k+kF_k\overset{(1)}{\Rightarrow}kF_k=k(F_{k+2}-F_{k+1})=$$

$$=(k+2)F_{k+2}-(k+1)F_{k+1}-2(F_{k+2}-F_{k+1})-F_{k+1};\ \ (3)$$

$$\overset{(1)}{\Rightarrow} F_{k+1} = F_{k+3} - F_{k+2} \overset{(3)}{\Rightarrow}$$

$$\sum_{k=1}^{n} kF_k = \sum_{k=1}^{n}[(k+2)F_{k+2} - (k+1)F_{k+1}] - 2\sum_{k=1}^{n}(F_{k+2} - F_{k+1}) - \sum_{k=1}^{n}(F_{k+3} - F_{k+2}) =$$

$$= (n+2)F_{n+2} - 2F_2 - 2(F_{n+2} - F_2) - (F_{n+3} - F_3) =$$

$$= nF_{n+2} - F_{n+3} + F_3$$

$$\sum_{k=1}^{n} F_k = \sum_{k=1}^{n}(F_{k+2} - F_{k+1}) = F_{n+2} - F_2$$

$$\overset{(2)}{\Rightarrow} \sum_{k=1}^{n}(k+2)F_k = \sum_{k=1}^{n} kF_k + 2\sum_{k=1}^{n} F_k = 2(F_{n+2} - F_2) + (nF_{n+2} - F_{n+3} + F_3)$$

$$= (n+2)F_{n+2} - (F_{n+3} - F_3 - F_2) - 3F_2 =$$

$$= (n+2)F_{n+2} - F_{n+3} = (n+1)F_{n+2} - F_{n+1} = nF_{n+2} + F_n$$

$$(ii)\ \sum_{k=1}^{n} \frac{F_k}{F_1^2 + F_2^2 + \cdots + F_{k+1}^2} = 1 - \frac{1}{F_{n+2}}$$

Proof:

$$F_{i+2} - F_i = F_{i+1}, \forall k > 0 \Rightarrow F_{i+1} \cdot F_{i+2} - F_i \cdot F_{i+1} = F_{i+2}^2;\ \ (*)$$

Adding up relations $(*)$ for all $i \in 1, 2, \dots, k$, we get:

$$F_{k+1} \cdot F_{k+2} - F_1 \cdot F_2 = F_2^2 + F_3^2 + \cdots + F_{k+1}^2 \Rightarrow$$

$$F_1^2 + F_2^2 + \cdots + F_{k+1}^2 = F_{k+1} \cdot F_{k+2} \Rightarrow$$

$$\frac{F_k}{F_1^2 + F_2^2 + \cdots + F_{k+1}^2} = \frac{F_{k+2} - F_{k+1}}{F_{k+1} \cdot F_{k+2}} = \frac{1}{F_{k+1}} - \frac{1}{F_{k+2}} \Rightarrow$$

$$\sum_{k=1}^{n} \frac{F_k}{F_1^2 + F_2^2 + \cdots + F_{k+1}^2} = 1 - \frac{1}{F_{n+2}}$$

$$\frac{F_1}{3(F_1^2 + F_2^2)^2} + \frac{F_2}{4(F_1^2 + F_2^2 + F_3^2)^2} + \cdots + \frac{F_n}{(n+2)(F_1^2 + F_2^2 + \cdots + F_{n+1}^2)^2} =$$

$$= \frac{\left(\frac{F_1}{F_1^2 + F_2^2}\right)^2}{3F_1} + \frac{\left(\frac{F_2}{F_1^2 + F_2^2 + F_3^2}\right)^2}{4F_2} + \cdots + \frac{\left(\frac{F_n}{F_1^2 + F_2^2 + \cdots + F_{n+1}^2}\right)^2}{(n+2)F_n} \overset{Bergstrom}{\geq}$$

$$\geq \frac{\left(\sum_{k=1}^{n}\frac{F_k}{F_1^2+F_2^2+\cdots+F_{k+1}^2}\right)^2}{\sum_{k=1}^{n}(k+2)F_k}=\frac{\left(1-\frac{1}{F_{n+2}}\right)^2}{nF_{n+2}+F_n}=$$

$$=\frac{(F_{n+2}-F_1)^2}{F_{n+2}^2(nF_{n+2}+F_n)}=\frac{(F_{n+2}-F_1)^2}{F_{n+2}^2(nF_{n+2}+F_n)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru