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JP.331. In acute ΔABC with the lengths $BC = a, CA = b, AB = c$. Prove that:

$$\frac{a(b+c-a)}{b^2+c^2-a^2}+\frac{b(c+a-b)}{c^2+a^2-b^2}+\frac{c(a+b-c)}{a^2+b^2-c^2}\geq 3$$

Proposed by Hoang Le Nhat Tung-Hanoi –Vietnam

Solution 1 by proposer

$$\frac{a(b+c-a)}{b^2+c^2-a^2}+\frac{b(c+a-b)}{c^2+a^2-b^2}+\frac{c(a+b-c)}{a^2+b^2-c^2}\geq 3;\ \ (1)$$

We have $cosA = \frac{b^2+c^2-a^2}{2bc} > 0$ hence $b^2+c^2-a^2 > 0$ and similarly $c^2+a^2-b^2 > 0$;

$$a^2+b^2-c^2 > 0$$

$$\begin{cases} b+c > a \\ c+a > b \\ a+b > c \end{cases} \Leftrightarrow \begin{cases} b+c-a > 0 \\ c+a-b > 0 \\ a+b-c > 0 \end{cases} \Rightarrow$$

$$\frac{a(b+c-a)}{b^2+c^2-a^3}\geq 0; \frac{b(c+a-b)}{c^2+a^2-b^2}\geq 0; \frac{c(a+b-c)}{a^2+b^2-c^2}\geq 0$$

From AM-GM inequality, we have:

$$\frac{a(b+c-a)}{b^2+c^2-a^2}+\frac{b(c+a-b)}{c^2+a^2-b^2}+\frac{c(a+b-c)}{a^2+b^2-c^2}\geq$$

$$\geq 3\sqrt[3]{\frac{a(b+c-a)}{b^2+c^2-a^2}\cdot\frac{b(c+a-b)}{c^2+a^2-b^2}\cdot\frac{c(a+b-c)}{a^2+b^2-c^2}} =$$

$$= 3\sqrt[3]{\frac{abc(b+c-a)(c+a-b)(a+b-c)}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}};\ \ (2)$$

Let $0 < (b+c-a)(c+a-b) = c^2-(a-b)^2 \leq c^2$

$$0 < (c+a-b)(a+b-c) = a^2-(b-c)^2 \leq a^2$$

$$0 < (+b-c)(b+c-a) = b^2-(c-a)^2 \leq b^2$$

Hence, $0 \leq (b+c-a)^2(c+a-b)^2(a+b-c)^2 \leq a^2b^2c^2$ then

$$0 \leq (b+c-a)(c+a-b)(a+b-c) \leq abc \Leftrightarrow$$

$$abc(b+c-a)(c+a-b)(a+b-c) \geq (b+c-a)^2(c+a-b)^2(a+b-c)^2$$

We will to prove that:

$$(b+c-a)^2(c+a-b)^2 \geq (b^2+c^2-a^2)(c^2+a^2-b^2) \Leftrightarrow$$

$$[c^2-(a-b)^2]^2 \geq (c^2)^2-(a^2-b^2)^2 \Leftrightarrow$$

$$c^2-2c^2(a-b)^2+(a-b)^4 \geq c^4-(a-b)^2(a+b)^2 \Leftrightarrow$$

$2(a-b)^2(a^2+b^2-c^2) \geq 0$true because $(a-b)^2 \geq 0$ and$a^2+b^2 \geq c^2 \Rightarrow (3)$ is true.

Hence, $(b+c-a)^2(c+a-b)^2 \geq (b^2+c^2-a^2)(c^2+a^2-b^2)$ and similarly:

$$(c+a-b)^2(a+b-c)^2 \geq (c^2+a^2-b^2)(a^2+b^2-c^2)$$

$$(a+b-c)^2(b+c-a)^2 \geq (a^2+b^2-c^2)(b^2+c^2-a^2)$$

Therefore,

$$(b+c-a)^4(c+a-b)^4(a+b-c)^4 \geq \left(a^2+b^2-c^2\right)^2\left(b^2+c^2-a^2\right)^2\left(c^2+a^2-b^2\right)^2$$

Then

$$(b+c-a)^2(c+a-b)^2(a+b-c)^2 \geq \left(a^2+b^2-c^2\right)\left(b^2+c^2-a^2\right)\left(c^2+a^2-b^2\right); (4)$$

From (3),(4) it follows:

$$abc(b+c-a)(c+a-b)(a+b-c) \geq (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

Hence, $\frac{abc(b+c-a)(c+a-b)(a+b-c)}{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)} \geq 1;\ \ (5)$

From (2),(5) it follows: $\frac{a(b+c-a)}{b^2+c^2-a^2}+\frac{b(c+a-b)}{c^2+a^2-b^2}+\frac{c(a+b-c)}{a^2+b^2-c^2} \geq 3$

So, inequality (1) is true and we get the result.

Equality holds if and only if:

$$\begin{cases} \frac{a(b+c-a)}{b^2+c^2-a^2}=\frac{b(c+a-b)}{c^2+a^2-b^2}=\frac{c(a+b-c)}{a^2+b^2-c^2} \\ a-b=b-c=c-a=0 \end{cases} \Leftrightarrow a=b=c.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

ΔABC acute, hence

$$a+b-c>0, b+c-a>0, c+a-b>0, a^2+b^2-c^2>0,$$

$$b^2+c^2-a^2>0, c^2+a^2-b^2>0$$

We give: $a+b-c=x, b+c-a=y, c+a-b=z$ hence,

$$\frac{x+y}{2}=b, \frac{y+z}{2}=c, \frac{z+x}{2}=a$$

$$\frac{a(b+c-a)}{b^2+c^2-a^2}+\frac{b(c+a-b)}{c^2+a^2-b^2}+\frac{c(a+b-c)}{a^2+b^2-c^2}=$$

$$=\frac{\frac{1}{2}(x+z)y}{\left(\frac{x+y}{2}\right)^2+\left(\frac{y+z}{2}\right)^2-\left(\frac{z+x}{2}\right)^2}+\frac{\frac{1}{2}(x+y)z}{\left(\frac{z+x}{2}\right)^2+\left(\frac{y+z}{2}\right)^2-\left(\frac{x+y}{2}\right)^2}+$$

$$+\frac{\frac{1}{2}(y+z)x}{\left(\frac{x+y}{2}\right)^2+\left(\frac{z+x}{2}\right)^2-\left(\frac{y+z}{2}\right)^2}=$$

$$=\frac{(x+z)y}{y^2+xy+yz-zx}+\frac{(x+y)z}{z^2+yz+zx-xy}+\frac{(y+z)x}{x^2+xy+zx-yz}=$$

$$=\frac{x+z}{x+y+z-\frac{zx}{y}}+\frac{x+y}{x+y+z-\frac{xy}{z}}+\frac{y+z}{x+y+z-\frac{xy}{z}}\geq$$

$$\geq\frac{4(x+y+z)^2}{[(x+y)+(y+z)+(z+x)](x+y+z)-\frac{(x+z)xz}{y}-\frac{(x+y)xy}{z}-\frac{(y+z)yz}{x}}\geq 3$$

$$4(x+y+z)^2+3\left[\frac{(x+z)xz}{y}+\frac{(x+y)xy}{z}+\frac{(y+z)yz}{x}\right]\geq 6(x+y+z)^2 \text{ true},$$

Because:

$$3\left[\frac{(x+z)xz}{y}+\frac{(x+y)xy}{z}+\frac{(y+z)yz}{x}\right]=$$

$$=3\left[\left(\frac{x^2y}{z}+\frac{x^2z}{y}\right)+\left(\frac{y^2x}{z}+\frac{y^2z}{x}\right)+\left(\frac{z^2x}{y}+\frac{z^2y}{x}\right)\right]\geq 3[2(x^2+y^2+z^2)]=$$

$$=6(x^2+y^2+z^2)\geq 2(x+y+z)^2$$

Solution 3 by Rahim Shahbazov-Baku-Azerbaijan

It is enough to prove that:

$$\frac{abc(a+b-c)(b+c-a)(a+c-b)}{(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2)}\geq 1$$

$$abc(a+b-c)(b+c-a)(a+c-b)\geq\prod_{cyc}(b^2+c^2-a^2)$$

Lemma: $abc\geq(a+b-c)(b+c-a)(a+c-b)$

$$(x+y)(y+z)(z+x)\geq 8xyz$$

$$[(a+b-c)(b+c-a)(a+c-b)]^2\geq\geq\prod_{cyc}(b^2+c^2-a^2)$$

true from Gerretsen and $cosAcosBcosC \leq \frac{1}{2}\left(\frac{r}{R}\right)^2$

Solution 4 by Tran Hong-Dong Thap-Vietnam

$$\frac{a(b+c-a)}{b^2+c^2-a^2}+\frac{b(c+a-b)}{c^2+a^2-b^2}+\frac{c(a+b-c)}{a^2+b^2-c^2} \geq 3 \Leftrightarrow$$

$$\sum_{cyc} a(b+C-a)(a^2+c^2-b^2)(a^2+b^2-c^2) \geq 0 \Leftrightarrow$$

$$2\left(\sum a\right)\left[\sum a^5+abc\left(\sum a^2\right)-\sum ab(a^3+b^3)\right] \geq 0 \Leftrightarrow$$

$$2\left(\sum a\right)\frac{1}{2}\left[\sum (a-b)^2(a+b-c)^3\right] \geq 0 \Leftrightarrow$$

$$\left(\sum a\right)\left[\sum (a-b)^2(a+b-c)^3\right] \geq 0$$

Which is clearly true by:

$$a+b-c>0 \Rightarrow (a-b)^2(a+b-c)^3 \geq 0\ (and\ analogs)$$

Equality $\Leftrightarrow a=b=c$. Proved.

Solution 5 by Marin Chirciu-Romania

Lemma.

In acute ΔABC we have:

$$\sum_{cyc}\frac{a(b+c-a)}{b^2+c^2-a^2}=\frac{s^2-r^2-2Rr-4R^2}{s^2-(2R+r)^2}$$

Proof.

We have:

$$\sum_{cyc}\frac{a(b+c-a)}{b^2+c^2-a^2}=\sum_{cyc}\frac{a(2s-2a)}{2bccosA}=\sum_{cyc}\frac{a(s-a)}{bccosA}=\frac{\sum a^2(s-a)cosB\,cosC}{abc\prod cosA}=$$

$$=\frac{\frac{sr(s^2-r^2-2Rr-4R^2)}{R}}{4Rrs\cdot\frac{s^2-(2R+r)^2}{4R^2}}=\frac{s^2-r^2-2Rr-4R^2}{s^2-(2R+r)^2}$$

Which result from:

$$\sum_{cyc} a^2(s-a)\cos B\cos C = \frac{sr(s^2-r^2-2Rr-4R^2)}{R} \text{ and } \prod \cos A = \frac{s^2-(2R+r)^2}{4R^2}$$

Let's solve the proposed problem.

Using Lemma and $s^2 > (2R+r)^2$, (in acute) the inequality it can be written as:

$$\frac{s^2-r^2-2Rr-4R^2}{s^2-(2R+r)^2} \ge 3 \Leftrightarrow s^2-r^2-2Rr-4R^2 \ge 3s^2-3(2R+r)^2 \Leftrightarrow$$

$$8R^2+10Rr+2r^2 \ge 2s^2 \Leftrightarrow 4R^2+5Rr+r^2 \ge s^2 \text{ which result from}$$

$$s^2 \le 4R^2+4Rr+3r^2 (Gerretsen)$$

Remain to prove: $4R^2+5Rr+r^2 \ge 4R^2+4Rr+3r^2 \Leftrightarrow R \ge 2r\ (Euler)$.

JP.332. If $x_i > 1, \forall i = \overline{1,n}; n \in \mathbb{N}, n \ge 3$ then prove:

$$\frac{\log x_2}{\log^2(x_1^2x_2)} + \frac{\log x_3}{\log^2(x_1^2x_2^2x_3)} + \dots + \frac{\log x_n}{\log^2(x_1^2x_2^2\dots x_{n-1}^2x_n)} \le \frac{\log\sqrt[4]{x_2x_3\dots x_n}}{\log x_1 \cdot \log(x_1x_2x_3\dots x_n)}$$

Proposed by Florică Anastase-Romania

Solution 1 by Adrian Popa-Romania

$$x_i > 1 \Rightarrow \log x_i > 0$$

Denote: $\log x_i = a_i, a_i > 0$

$$\frac{\log x_2}{\log^2(x_1^2x_2)} + \frac{\log x_3}{\log^2(x_1^2x_2^2x_3)} + \dots + \frac{\log x_n}{\log^2(x_1^2x_2^2\dots x_{n-1}^2x_n)} \le \frac{\log\sqrt[4]{x_2x_3\dots x_n}}{\log x_1 \cdot \log(x_1x_2x_3\dots x_n)}$$

$$\Leftrightarrow \frac{a_2}{(2a_1+a_2)^2} + \frac{a_3}{(2a_1+2a_2+a_3)^2} + \dots + \frac{a_n}{(2a_1+2a_2+\dots+2a_{n-1}+a_n)^2} \le$$

$$\le \frac{a_2+a_3+\dots+a_n}{4a_1(a_1+a_2+\dots+a_n)}$$

$$\frac{a_1}{(2(a_1+a_2)-a_2)^2} + \frac{a_3}{(2(a_1+a_2+a_3)-a_3)^2} + \dots + \frac{a_n}{(2(a_1+a_2+\dots+a_n)-a_n)^2} \le$$

$$\le \frac{a_1+a_2+\dots+a_n-a_1}{4a_1(a_1+a_2+\dots+a_n)} = \frac{1}{4a_1} - \frac{1}{4(a_1+a_2+\dots+a_n)}$$

We must show that:

$$\frac{a_n}{(2a_1+2a_2+\dots+2a_{n-1}+a_n)^2} \le \frac{1}{4(a_1+a_2+\dots+a_{n-1})} - \frac{1}{4(a_1+a_2+\dots+a_n)}$$

$$\Leftrightarrow \frac{a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2} \le \frac{a_n}{4(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}$$

$$\Leftrightarrow 4(a_1+a_2+\cdots+a_{n-1})^2 + a_n(a_1+a_2+\cdots+a_{n-1}) + a_n^2 \ge$$

$$\ge 4(a_1+a_2+\cdots+a_{n-1})^2 + 4a_n(a_1+a_2+\cdots+a_n) \Rightarrow$$

$$a_n^2 \ge 0 \text{ true } \forall n \in \mathbb{N}.$$

$$\frac{a_2}{(2a_1+a_2)^2} + \frac{a_3}{(2a_1+2a_2+a_3)^2} + \cdots + \frac{a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2} \le$$

$$\le \frac{1}{4a_1} - \frac{1}{4(a_1+a_2)} + \frac{1}{4(a_1+a_2)} - \frac{1}{4(a_1+a_2+a_3)} + \cdots + \frac{1}{4(a_1+a_2+\cdots+a_{n-1})} -$$

$$-\frac{1}{4(a_1+a_2+\cdots+a_n)} = \frac{1}{4a_1} - \frac{1}{4(a_1+a_2+\cdots+a_n)}$$

Solution 2 by proposer

$$\frac{a_2}{a_1(a_1+a_2)} + \frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)} = \frac{a_2+a_3}{a_1(a_1+a_2+a_3)}$$

Suppose:

$$\frac{a_2}{a_1(a_1+a_2)} + \frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)} + \cdots + \frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}$$

$$= \frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)} \Rightarrow$$

$$\frac{a_2}{a_1(a_1+a_2)} + \frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)} + \cdots + \frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)} +$$

$$+\frac{a_{n+1}}{(a_1+a_2+\cdots+a_n)(a_1+a_2+\cdots+a_{n+1})} =$$

$$= \frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)} + \frac{a_{n+1}}{(a_1+a_2+\cdots+a_n)(a_1+a_2+\cdots+a_{n+1})} =$$

$$= \frac{(a_2+a_3+\cdots+a_n)^2 + a_1(a_2+a_3+\cdots+a_n) + a_{n+1}(a_2+a_3+\cdots+a_n) + a_1a_{n+1}}{(a_1+a_2+\cdots+a_n)a_1(a_1+a_2+\cdots+a_{n+1})} =$$

$$= \frac{a_2+a_3+\cdots+a_{n+1}}{a_1(a_1+a_2+\cdots+a_{n+1})}$$

From $(x+y)^2 \ge 4xy \Rightarrow \frac{1}{xy} \ge \frac{4}{(x+y)^2}$ **we have:**

$$\frac{a_2}{a_1(a_1+a_2)} \ge \frac{4a_2}{(2a_1+a_2)^2}$$

$$\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)} \ge \frac{4a_3}{(2a_1+2a_2+a_3)^2}$$

$$\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)} \ge \frac{4a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2}$$

Adding up relationships, we have:

$$\frac{4a_2}{(2a_1+a_2)^2}+\frac{4a_3}{(2a_1+2a_2+a_3)^2}+\cdots+\frac{4a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2} \le$$

$$\le \frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}+\cdots+\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}$$

$$= \frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)} \Rightarrow$$

$$\frac{a_2}{(2a_1+a_2)^2}+\frac{a_3}{(2a_1+2a_2+a_3)^2}+\cdots+\frac{a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2}$$

$$\le \frac{a_2+a_3+\cdots+a_n}{4a_1(a_1+a_2+\cdots+a_n)}$$

For: $a_1 = logx_1;\ a_2 = logx_2; \ldots; a_n = logx_n$ **we get:**

$$\frac{logx_2}{log^2(x_1^2x_2)}+\frac{logx_3}{log^2(x_1^2x_2^2x_3)}+\cdots+\frac{logx_n}{log^2(x_1^2x_2^2\ldots x_{n-1}^1x_n)} \le \frac{log\sqrt[4]{x_2x_3\ldots x_n}}{logx_1\cdot log(x_1x_2x_3\ldots x_n)}$$

JP.333. In ΔABC the following relationship holds:

$$\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a} \le \sqrt{(ab+bc+ca)\left(2+\frac{r}{2R}\right)}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution by proposer

Using some well-known formulas

$$r_a = \frac{S}{s-a};\ cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}};\ cosA+cosB+cosC = 1+\frac{r}{R}$$

We have:

$$\sum_{cyc}\frac{r_br_c}{bc} = \sum_{cyc}\frac{S^2}{bc(s-b)(s-c)} = \sum_{cyc}\frac{s(s-a)}{bc} = \sum_{cyc}cos^2\frac{A}{2} = \sum_{cyc}\frac{1+cosA}{2} = 2+\frac{r}{2R}$$

Hence,

$$2+\frac{r}{2R}=\sum_{cyc}\frac{r_br_c}{bc}\overset{BCS}{\geq}\frac{\left(\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}\right)^2}{ab+bc+ca}$$

Therefore,

$$\sqrt{(ab+bc+ca)\left(2+\frac{r}{2R}\right)}\geq\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}$$

JP.334. In ΔABC the following relationship holds:

$$\sqrt{\frac{a+b}{a-b+c}}+\sqrt{\frac{b+c}{b-c+a}}+\sqrt{\frac{c+a}{c-a+b}}\leq\frac{3R}{\sqrt{2}r}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by Eldeniz Hesenov-Georgia

$$\sqrt{\frac{a+b}{a-b+c}}+\sqrt{\frac{b+c}{b-c+a}}+\sqrt{\frac{c+a}{c-a+b}}=$$

$$=\frac{\sqrt{a+b}}{\sqrt{s-b}}+\frac{\sqrt{b+c}}{\sqrt{s-c}}+\frac{\sqrt{c+a}}{\sqrt{s-a}}\overset{BCS}{\leq}\sqrt{2s\cdot\sum_{cyc}\frac{1}{s-a}}=$$

$$=\sqrt{2s^2\cdot\frac{\sum(s-b)(s-c)}{s(s-a)(s-b)(s-c)}}=\sqrt{\frac{2s^2\sum(s-b)(s-c)}{s^2r^2}}\overset{\sum(s-b)(s-c)\leq\frac{sr}{3}}{\leq}$$

$$\leq\sqrt{\frac{2s^2}{3r^2}}=\frac{s}{r}\sqrt{\frac{2}{3}}\overset{(?)}{\leq}\frac{3R}{r\sqrt{2}}\Leftrightarrow 3\sqrt{3}R\geq 2s;(1)$$

$$27R^2\geq 3(a^2+b^2+c^2)\geq(a+b+c)^2=4s^2\Rightarrow(1)$$

Solution 2 and extensions by Marin Chirciu-Romania

$$Lhs=\left(\sum_{cyc}\sqrt{\frac{a+b}{a-b+c}}\right)^2\overset{BCS}{\leq}\sum_{cyc}(b+c)\cdot\sum_{cyc}\frac{1}{b+c-a}=4s\cdot\frac{4R+r}{2sr}=$$

$$=\frac{2(4R+r)}{r}\overset{(1)}{\leq}\frac{9R^2}{2r^2}=Rhd$$

Where $(1)\Leftrightarrow\frac{2(4R+r)}{r}\leq\frac{9R^2}{2r^2}\Leftrightarrow 9R^2\geq 4r(4R+r)\Leftrightarrow 9R^2-16Rr-4r^2\geq 0$

$\Leftrightarrow (R-2r)(9R+2r)\geq 0$ true from $R\geq 2r$ $(Euler)$.

Remark. Let's find extensions inequality:

2) In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{b+c}{b+c-a}}\geq 3\sqrt{2}$$

Proposed by Marin Chirciu-Romania

Proof. Using AM-GM inequality, we have:

$$\prod_{cyc}(b+c)=2s(s^2+r^2+2Rr)\text{ and }\prod_{cyc}(b+c-a)=8r^2s$$

Hence,

$$Lhs=\sum_{cyc}\sqrt{\frac{b+c}{b+c-a}}\geq 3\sqrt[3]{\prod_{cyc}\sqrt{\frac{b+c}{b+c-a}}}=3\sqrt[6]{\prod_{cyc}\frac{b+c}{b+c-a}}=$$

$$=3\sqrt[6]{\frac{\prod(b+c)}{\prod(b+c-a)}}=3\sqrt[6]{\frac{2s(s^2+r^2+2Rr)}{8r^2s}}=3\sqrt[6]{\frac{s^2+r^2+2Rr}{4r^2}}\overset{(2)}{\geq}3\sqrt{2}=Rhd$$

$$(2)\Leftrightarrow 3\sqrt[6]{\frac{s^2+r^2+2Rr}{4r^2}}\geq 3\sqrt{2}\Leftrightarrow\frac{s^2+r^2+2Rr}{4r^2}\geq 8\Leftrightarrow s^2\geq 31r^2-2Rr$$

Which result from $s^2\geq 16Rr-5r^2(Gerretsen)$

We must to prove $16Rr-5r^2\geq 31r^2-2Rr\Leftrightarrow R\geq 2r(Euler)$

3) In ΔABC the following relationship holds:

$$3\sqrt{2}\leq\sum_{cyc}\sqrt{\frac{b+c}{b+c-a}}\leq\frac{3R}{r\sqrt{2}}$$

Proof. See up relationships.

Equality holds if and only if the triangle is equilateral.

Solution 3 by Soumitra Mandal-Chandar Nagore-India

We know that:

$$\sum_{cyc}\frac{1}{s-a}=\frac{4R+r}{sr}\text{ and }R\ge 2r\ (Euler)$$

$$\sum_{cyc}\sqrt{\frac{a+b}{a-b+c}}\overset{BCS}{\le}\sqrt{\left(\sum_{cyc}(a+b)\right)\left(\sum_{cyc}\frac{1}{a-b+c}\right)}=$$

$$=\sqrt{2(a+b+c)\left(\sum_{cyc}\frac{1}{2(s-a)}\right)}=\sqrt{2s\cdot\sum_{cyc}\frac{1}{s-a}}=$$

$$=\sqrt{2s\cdot\frac{4R+r}{sr}}=\sqrt{2\cdot\frac{4R+r}{r}}\le\sqrt{2\cdot\frac{4R+\frac{R}{2}}{r}}=3\sqrt{\frac{R}{r}}=3\sqrt{2\left(\frac{R}{2r}\right)}\le$$

$$\le 3\sqrt{\frac{R}{r}\left(\frac{R}{2r}\right)}=\frac{3R}{\sqrt{2}r}$$

JP.335 If $a,b,c>0$ such that $ab+bc+ca\le 3$ then prove:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\ge\frac{15}{4}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by proposer

Applying some well-known result, we have:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}=$$

$$=\frac{3}{4}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\sum_{cyc}\left(\frac{1}{4a^2}+\frac{1}{(b+c)^2}\right)\ge\frac{3}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\sum_{cyc}\frac{1}{a(b+c)}\ge$$

$$\ge\frac{3}{4}\cdot\frac{9}{ab+bc+ca}+\frac{9}{a(b+c)+b(c+a)+c(a+b)}=\frac{45}{4(ab+bc+ca)}\ge\frac{15}{4}$$

Solution 2 by George Florin Şerban-Romania

Applying Iranian inequality:

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{1}{(a+b)^2}\right)\ge\frac{9}{4},\sum_{cyc}\frac{1}{(a+b)^2}\ge\frac{9}{4\sum_{cyc}ab}\ge\frac{9}{4\cdot 3}=\frac{3}{4}$$

$$\sum_{cyc}\frac{1}{a^2}=\sum_{cyc}\left(\frac{1}{a}\right)^2\overset{\sum x^2\ge\sum xy}{\ge}\sum_{cyc}\frac{1}{ab}\overset{Bergstrom}{\ge}\frac{(1+1+1)^2}{\sum_{cyc}ab}=\frac{9}{\sum_{cyc}ab}\ge\frac{9}{3}=3$$

Therefore,

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\ge\frac{15}{4}$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

First, for $a,b>0$ we have:

$$\frac{1}{2a^2}+\frac{1}{2b^2}+\frac{1}{(a+b)^2}\ge\frac{5}{4ab};(and\ analogs)$$

$$\Leftrightarrow\frac{1}{2a^2}+\frac{1}{2b^2}+\frac{1}{(a+b)^2}-\frac{5}{4ab}\ge 0$$

$$\Leftrightarrow\frac{2a^4+2b^4-2a^2b^2-ab(a^2+b^2)}{4a^2b^2(a+b)^2}\ge 0$$

$$\Leftrightarrow\frac{(a-b)^2(2a^2+3ab+2b^2)}{4a^2b^2(a+b)^2}\ge 0(\text{true for } a,b>0)$$

$$\Rightarrow\sum_{cyc}\left(\frac{1}{2a^2}+\frac{1}{2b^2}+\frac{1}{(a+b)^2}\right)\ge\frac{5}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\overset{Bergstrom}{\ge}$$

$$\ge\frac{5}{4}\cdot\frac{9}{ab+bc+ca}\overset{ab+bc+ca\le 3}{\ge}\frac{5}{4}\cdot\frac{9}{3}=\frac{15}{4}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ we have:

$$\frac{1}{4a^2}+\frac{1}{(b+c)^2}\ge\frac{1}{a(b+c)};\frac{1}{4b^2}+\frac{1}{(c+a)^2}\ge\frac{1}{b(c+a)};\frac{1}{4c^2}+\frac{1}{(a+b)^2}\ge\frac{1}{c(a+b)}$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}=$$

$$=\frac{3}{4}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\left(\frac{1}{4a^2}+\frac{1}{(b+c)^2}\right)+\left(\frac{1}{4b^2}+\frac{1}{(c+a)^2}\right)+\left(\frac{1}{4c^2}+\frac{1}{(a+b)^2}\right)\ge$$

$$\geq \frac{3}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\frac{1}{a(b+c)}+\frac{1}{b(c+a)}+\frac{1}{c(a+b)} \geq$$

$$\geq \frac{3}{4}\cdot\frac{9}{ab+bc+ca}+\frac{9}{2(ab+bc+ca)} \geq \frac{3}{4}\cdot\frac{9}{3}+\frac{9}{2\cdot 3}=\frac{15}{4}$$

Solution 5 by Marian Dincă-Romania

$$\frac{1}{a^2}+\frac{1}{(b+c)^2}=\frac{1}{4a^2}+\frac{1}{4a^2}+\frac{1}{4a^2}+\frac{1}{4a^2}+\frac{1}{(b+c)^2}=\frac{3}{4a^2}+\frac{1}{(2a)^2}+\frac{1}{(b+c)^2} \Rightarrow$$

$$\sum_{cyc}\left(\frac{1}{a^2}+\frac{1}{(b+c)^2}\right)=\frac{3}{4}\sum_{cyc}\frac{1}{a^2}+\sum_{cyc}\left(\frac{1}{(2a)^2}+\frac{1}{(b+c)^2}\right) \overset{\substack{AM-GM \\ \sum x^2 \geq \sum xy}}{\geq}$$

$$\geq \frac{3}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+2\sum_{cyc}\sqrt{\frac{1}{(2a)^2}\cdot\frac{1}{(b+c)^2}} \geq$$

$$\geq \frac{3}{4}\cdot\frac{9}{ab+bc+ca}+\sum_{cyc}\frac{1}{a(b+c)} \geq \frac{3}{4}\cdot\frac{9}{3}+\frac{9}{\sum a(b+c)} \geq \frac{9}{4}+\frac{9}{6}=\frac{15}{4}$$

JP.336 For all positive integers $n>3$ prove that:

$$\frac{\sqrt{2n+1}-1}{2}<\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}<\frac{\sqrt{2n}}{2}$$

Proposed by Nguyen Hung Viet-Hanoi-Vietnam

Solution by proposer

We have clearly:

$$S=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}<$$

$$<\frac{1}{\sqrt{0}+\sqrt{1}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-2}+\sqrt{2n-1}}$$

Hence,

$$2S<1+\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-2}+\sqrt{2n-1}}+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}=$$

$$=1+\frac{\sqrt{1}-\sqrt{2}}{-1}+\frac{\sqrt{2}-\sqrt{3}}{-1}+\cdots+\frac{\sqrt{2n-1}-\sqrt{2n}}{-1}=\sqrt{2n}; \quad (1)$$

On the other hand,

$$S=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}>$$

$$>\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\cdots+\frac{1}{\sqrt{2n}+\sqrt{2n+1}}$$

$$2S>\frac{\sqrt{1}-\sqrt{2}}{-1}+\frac{\sqrt{2}-\sqrt{3}}{-1}+\cdots+\frac{\sqrt{2n}-\sqrt{2n+1}}{-1}=\sqrt{2n+1}-1;\ \ (2)$$

From (1),(2) it follows that conclusion.

Solution 2 by Daniel Văcaru-Romania

We have:

$$\frac{1}{\sqrt{2n-2}+\sqrt{2n-1}}>\frac{1}{\sqrt{2n-1}+\sqrt{2n}}>\frac{1}{\sqrt{2n}+\sqrt{2n+1}}\Rightarrow$$

$$\begin{cases}\dfrac{2}{\sqrt{2n-1}+\sqrt{2n}}<\dfrac{1}{\sqrt{2n-2}+\sqrt{2n-1}}+\dfrac{1}{\sqrt{2n-1}+\sqrt{2n}}\\ \dfrac{2}{\sqrt{2n-1}+\sqrt{2n}}>\dfrac{1}{\sqrt{2n-1}+\sqrt{2n}}+\dfrac{1}{\sqrt{2n}+\sqrt{2n+1}}\end{cases}$$

With values from 1 to n and summing, we obtain:

$$\sum_{k=1}^{n}\left(\frac{1}{\sqrt{2k-1}+\sqrt{2k}}+\frac{1}{\sqrt{2k}+\sqrt{2k+1}}\right)<$$

$$<2\left(\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}\right)<$$

$$<\sum_{k=1}^{n}\left(\frac{1}{\sqrt{2k-2}+\sqrt{2k-1}}+\frac{1}{\sqrt{2k-1}+\sqrt{2k}}\right)$$

Calculating the two sums from LHS and RHS, we obtain:

$$\sqrt{2n+1}-1<2\left(\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}\right)<\sqrt{2n}$$

Therefore,

$$\frac{\sqrt{2n+1}-1}{2}<\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}<\frac{\sqrt{2n}}{2}$$

JP.337. If $a_i, b_i \in (0,1); p, q \in \mathbb{N}^*, n \geq 2$ then prove:

$$\sum_{i=1}^{n} log_{a_i} \sqrt[n]{\frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}}} + \sum_{i=1}^{n} log_{b_i} \sqrt[n]{\frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}}} \geq \left(\sqrt{p} + \sqrt{q}\right)^2$$

Proposed by Florică Anastase-Romania

Solution 1 by Marian Ursărescu-Romania

We must show:

$$\sum_{i=1}^{n} log_{a_i} \frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} + \sum_{i=1}^{n} log_{b_i} \frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}} \geq n\left(\sqrt{p} + \sqrt{q}\right)^2; \ \ (1)$$

$$\frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} \leq \sqrt{a_i^{2p} \cdot b_i^{2q}} = a_i^p \cdot b_i^q, a_i \in (0,1)$$

$$\Rightarrow log_{a_i} \frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} \geq log_{a_i}\left(a_i^p \cdot b_i^q\right) = p + q log_{a_i} b_i; \ \ (2)$$

Now, $\frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}} \leq \sqrt{a_i^{2q} \cdot b_i^{2p}} = a_i^q \cdot b_i^p, b_i \in (0,1)$

$$\Rightarrow \ log_{b_i} \frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}} \geq log_{b_i}\left(a_i^q \cdot b_i^p\right) = p + q log_{b_i} a_i; \ \ (3)$$

From $(2)\&(3)$ we have:

$$\sum_{i=1}^{n} log_{a_i} \frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} + \sum_{i=1}^{n} log_{b_i} \frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}} \geq \sum_{i=1}^{n} 2p + \sum_{i=1}^{n} q\left(log_{a_i} b_i + \frac{1}{log_{b_i} a_i}\right) \geq$$

$$\geq 2np + 2nq; \ \ (4)$$

From $(1)\&(4)$ we must show: $2(p+q) \geq \left(\sqrt{p} + \sqrt{q}\right)^2 \Leftrightarrow \left(\sqrt{p} - \sqrt{q}\right)^2 \geq 0$ true. Proved.

Solution 2 by Adrian Popa-Romania

$$\frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} \leq \sqrt{a_i^{2p} \cdot b_i^{2q}} = a_i^p \cdot b_i^q, a_i \in (0,1)$$

$$\Rightarrow log_{a_i} \frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}} \geq log_{a_i}\left(a_i^p \cdot b_i^q\right) = p + q log_{a_i} b_i$$

$$\Rightarrow log_{a_i}\sqrt[n]{\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}}\ge \frac{1}{n}(p+qlog_{a_i}b_i)$$

$$\sum_{i=1}^{n} log_{a_i}\sqrt[n]{\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}}\ge p+\frac{q}{n}\sum_{i=1}^{n} log_{a_i}b_i;\ \ (1)$$

Similarly

$$\sum_{i=1}^{n} log_{b_i}\sqrt[n]{\frac{2a_i^{2q}\cdot b_i^{2p}}{a_i^{2q}+b_i^{2p}}}\ge q+\frac{p}{n}\sum_{i=1}^{n} log_{b_i}a_i;\ \ (2)$$

From $(1)\&(2)$ we must show:

$$p+\frac{q}{n}\sum_{i=1}^{n} log_{a_i}b_i+q+\frac{p}{n}\sum_{i=1}^{n} log_{b_i}a_i\ge p+q+2\sqrt{pq}\Leftrightarrow$$

$$\frac{q}{n}\sum_{i=1}^{n} log_{a_i}b_i+\frac{p}{n}\sum_{i=1}^{n} log_{b_i}a_i\ge 2\sqrt{pq}$$

But

$$\frac{q}{n}\ log_{a_i}b_i+\frac{p}{n}\ log_{b_i}a_i \overset{AM-GM}{\ge} 2\sqrt{\frac{pq}{n^2}log_{a_i}b_i\cdot log_{b_i}a_i}=\frac{2}{n}\sqrt{pq}$$

Therefore,

$$\frac{q}{n}\sum_{i=1}^{n} log_{a_i}b_i+\frac{p}{n}\sum_{i=1}^{n} log_{b_i}a_i\ge n\cdot\frac{2}{n}\sqrt{pq}=2\sqrt{pq}$$

Solution 3 by Ravi Prakash-New Delhi-India

Suppose $a,b\in(0,1)\Leftrightarrow\Rightarrow a^{2p},a^{2q},b^{2p},b^{2q}\in(0,1)$

Let $G_1=\sqrt{a^{2p}b^{2q}}; G_2=\sqrt{a^{2q}b^{2p}}; H_1=\frac{2a^{2p}b^{2q}}{a^{2p}+b^{2q}}; H_2=\frac{2a^{2q}b^{2p}}{a^{2q}+b^{2p}}$

Note that $G_1,G_2,H_1,H_2\in(0,1)$, we have:

$0<H_1\le G_1\le 1$ also $loga<0\Rightarrow\frac{logH_1}{loga}\ge\frac{logG_1}{loga}\Rightarrow log_aH_1\ge log_aG_1$

$$\Rightarrow log_aH_1\ge plog_aa+qlog_ab$$

Similarly: $log_bH_2\ge plog_ba+qlog_bb$

$$log_aH_1+log_bH_2\ge p+q+q\frac{logb}{loga}+p\frac{loga}{logb}\ge p+q+2\sqrt{pq}=\left(\sqrt{p}+\sqrt{q}\right)^2$$

Thus,

$$\sum_{i=1}^{n} log_{a_i}\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}+\sum_{i=1}^{n} log_{b_i}\frac{2a_i^{2q}\cdot b_i^{2p}}{a_i^{2q}+b_i^{2p}}\geq n\left(\sqrt{p}+\sqrt{q}\right)^2\Leftrightarrow$$

$$\sum_{i=1}^{n} log_{a_i}\sqrt[n]{\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}}+\sum_{i=1}^{n} log_{b_i}\sqrt[n]{\frac{2a_i^{2q}\cdot b_i^{2p}}{a_i^{2q}+b_i^{2p}}}\geq\left(\sqrt{p}+\sqrt{q}\right)^2$$

Solution 4 by proposer

$$\log_{a_i}\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}\overset{HM-GM}{\geq}\log_{a_i}\sqrt{a_i^{2p}\cdot b_i^{2q}}=\log_{a_i}\left(a_i^p\cdot b_i^q\right)=p+q\log_{a_i}b_i$$

$$\log_{b_i}\frac{2a_i^{2q}\cdot b_i^{2p}}{a_i^{2q}+b_i^{2p}}\overset{HM-GM}{\geq}\log_{b_i}\sqrt{a_i^{2q}\cdot b_i^{2p}}=\log_{a_i}\left(a_i^q\cdot b_i^p\right)=p+q\log_{b_i}a_i$$

Adding, we get:

$$\log_{a_i}\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}+\log_{b_i}\frac{2a_i^{2q}\cdot b_i^{2p}}{a_i^{2q}+b_i^{2p}}\geq 2p+q\left(\log_{a_i}b_i+\log_{b_i}a_i\right)\overset{AM-GM}{\geq}2p+2q$$

$$\sum_{i=1}^{n}\left(\log_{a_i}\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}+\log_{b_i}\frac{2a_i^{2q}\cdot b_i^{2p}}{a_i^{2q}+b_i^{2p}}\right)\geq 2np+2nq\Leftrightarrow$$

$$\sum_{i=1}^{n}\log_{a_i}\sqrt[n]{\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}}+\sum_{i=1}^{n}\log_{b_i}\sqrt[n]{\frac{2a_i^{2q}\cdot b_i^{2p}}{a_i^{2q}+b_i^{2p}}}\geq 2p+2q\geq\left(\sqrt{p}+\sqrt{q}\right)^2$$

JP.338. In $\Delta ABC, P, Q\in Int(\Delta ABC)$ such that:

$\beta\overrightarrow{AB}+\gamma\overrightarrow{BP}+\overrightarrow{PC}=0$ and $\overrightarrow{AQ}+\alpha\overrightarrow{QB}+\overrightarrow{BC}=0, \alpha,\beta,\gamma\in\mathbb{R}; \alpha,\gamma\neq 1$.

Prove that A, P, Q are collinears if and only if $\alpha+\gamma=\beta+1$.

Proposed by Florică Anastase -Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$\beta\overrightarrow{AB}+\gamma\overrightarrow{BP}+\overrightarrow{PC}=0;\ \ (1)$$

$$\beta\left(\overrightarrow{AP}+\overrightarrow{PB}\right)+\gamma\overrightarrow{BP}+\overrightarrow{PC}=0\Leftrightarrow\beta\overrightarrow{AP}+(\beta-\gamma)\overrightarrow{PB}+\overrightarrow{PC}=0$$

So, P is perycenter of $(A,\beta),(B,\gamma-\beta),(C,1)$

If $\beta+\gamma-\beta+1\neq 0, \gamma+1\neq 0$

$$\overrightarrow{AQ}+\alpha\overrightarrow{QB}+\overrightarrow{BC}=0;$$

$$\overrightarrow{AQ}+\alpha\overrightarrow{QB}+\overrightarrow{BQ}+\overrightarrow{QC}=0 \Leftrightarrow \overrightarrow{AQ}+(\alpha-1)\overrightarrow{QB}+\overrightarrow{QC}=0; \quad (2)$$

So, Q is centroid of $(A,1),(B,1-\alpha),(C,1)$

If $1-\alpha+1+1\neq 0 \Rightarrow \alpha\neq 3$.

Suppose: $\alpha+\gamma=\beta+1$ and let us prove A,P,Q are collinear.

$\alpha+\gamma=\beta+1 \Leftrightarrow \alpha-1=\beta-\gamma$ substituted in (2):

$$\begin{cases}\overrightarrow{AQ}+(\beta-\gamma)\overrightarrow{QB}+\overrightarrow{QC}=0\\ \beta\overrightarrow{AP}+(\beta-\gamma)\overrightarrow{PB}+\overrightarrow{PC}=0\end{cases}\Rightarrow$$

$$\overrightarrow{AQ}-\beta\overrightarrow{AP}+(\beta-\gamma)\left(\overrightarrow{QB}-\overrightarrow{PB}\right)+\left(\overrightarrow{QC}-\overrightarrow{PC}\right)=0$$

$$\overrightarrow{AQ}-\beta\overrightarrow{AP}+(\beta-\gamma)\left(\overrightarrow{QB}+\overrightarrow{BP}\right)+\left(\overrightarrow{QC}+\overrightarrow{CP}\right)=0$$

$$\overrightarrow{AQ}-\beta\overrightarrow{AP}+(\beta-\gamma)\overrightarrow{QP}+\overrightarrow{QP}=0$$

$$\overrightarrow{AQ}-\beta\overrightarrow{AP}+(\beta-\gamma+1)\overrightarrow{QP}=0; (\alpha+\gamma=\beta+1 \Rightarrow \alpha=\beta-\gamma+1)$$

$$\overrightarrow{AQ}-\beta\overrightarrow{AP}+\alpha\overrightarrow{QP}=0 \Rightarrow \overrightarrow{AQ}-\beta\overrightarrow{AP}+\alpha\left(\overrightarrow{AQ}+\overrightarrow{AP}\right)=0$$

$(1+\alpha)\overrightarrow{AQ}+(\alpha-\beta)\overrightarrow{AP}=0 \Rightarrow (1+\alpha)\overrightarrow{AQ}=(\beta-\alpha)\overrightarrow{AP} \Leftrightarrow A,P,Q$ are collinear.

Now, suppose A,P,Q are collinear, we show that $\alpha+\gamma=\beta+1$

From (1) we have:

$$\beta\overrightarrow{AP}+(\beta-\gamma)\overrightarrow{PB}+\overrightarrow{PC}=0 \Leftrightarrow -\beta\overrightarrow{PA}+(\beta-\gamma)\overrightarrow{PB}+\overrightarrow{PC}=0$$

So, $-\beta\overrightarrow{QA}+(\beta-\gamma)\overrightarrow{QB}+\overrightarrow{QC}=(-\beta+\beta-\gamma+1)\overrightarrow{QP}$

$$-\beta\overrightarrow{QA}+(\beta-\gamma)\overrightarrow{QB}+\overrightarrow{QC}=(1-\gamma)\overrightarrow{QP}$$

From (2): $-\overrightarrow{QA}+(\alpha-1)\overrightarrow{QB}+\overrightarrow{QC}=0$

$$\Rightarrow (-\beta+1)\overrightarrow{QA}+(\beta-\gamma-\alpha+1)\overrightarrow{QB}=(1-\gamma)\overrightarrow{QP}$$

$$(\beta-\gamma-\alpha+1)\overrightarrow{QB}=(1-\gamma)\overrightarrow{QP}-(-\beta+1)\overrightarrow{QA}$$

$$(\beta-\gamma-\alpha+1)\overrightarrow{QB}=(1-\gamma)\overrightarrow{QP}+(\beta-1)\overrightarrow{QA}$$

$$(\beta-\gamma-\alpha+1)\overrightarrow{QB}=(1-\gamma)\overrightarrow{QP}+(\beta-1)\left(\overrightarrow{QP}+\overrightarrow{PA}\right)=$$

$$=(\beta-\gamma)\overrightarrow{QP}+(\beta-1)\overrightarrow{PA}=(\beta-\gamma)\left(\overrightarrow{QA}+\overrightarrow{AP}\right)+(\beta-1)\overrightarrow{PA}=$$

$$=(\beta-\gamma-\beta+1)\overrightarrow{AP}+(\beta-\gamma)\overrightarrow{QA};$$

$$\left(\overrightarrow{AP}=\lambda\overrightarrow{QP}\Rightarrow\overrightarrow{AP}=\lambda\left(\overrightarrow{QA}+\overrightarrow{AP}\right)\Rightarrow\lambda(1-\lambda)\overrightarrow{QP}=\lambda\overrightarrow{QA}\right)$$

$$(\beta-\gamma-\alpha+1)\overrightarrow{QB}=\lambda(1-\gamma)\overrightarrow{QP}+\frac{(\beta-\gamma)(1-\lambda)}{\lambda}\overrightarrow{AP}=$$

$$=\lambda(1-\gamma)\overrightarrow{QP}+(\beta-\gamma)(1-\lambda)\overrightarrow{QP}$$

$$\Rightarrow(\beta-\gamma-\alpha+1)\overrightarrow{QB}=\left(\lambda(1-\beta)+(\beta-\gamma)\right)\overrightarrow{QP}$$

$\Rightarrow Q,B,P$ are collinear, so A,Q,B,P are collinears, this is impossible because Q,P inside of triangle, from (*) it must be $\beta-\gamma-\alpha+1=0\Leftrightarrow\alpha+\gamma=\beta+1$.

Solution 2 by proposer

$$\overrightarrow{AQ}+\alpha\overrightarrow{QB}+\overrightarrow{BC}=0\Leftrightarrow\left(\overrightarrow{AQ}+\overrightarrow{QB}+\overrightarrow{BC}\right)=(\alpha-1)\overrightarrow{BQ}\Leftrightarrow\overrightarrow{AC}=(\alpha-1)\overrightarrow{BQ}$$

$$\overrightarrow{AQ}=\overrightarrow{AB}+\overrightarrow{BQ}=\overrightarrow{AB}+\frac{1}{\alpha-1}\overrightarrow{AC};\ (1)$$

$$\beta\overrightarrow{AB}+\gamma\overrightarrow{BP}+\overrightarrow{PC}=0\Leftrightarrow\beta\overrightarrow{AB}+\gamma\left(\overrightarrow{BA}+\overrightarrow{AP}\right)+\overrightarrow{PC}=0\Leftrightarrow$$

$$(\beta-\gamma)\overrightarrow{AB}+\gamma\overrightarrow{AP}+\overrightarrow{PC}=0\Leftrightarrow$$

$$(\beta-\gamma)\overrightarrow{AB}+(\gamma-1)\overrightarrow{AP}+\overrightarrow{AC}=0\Leftrightarrow$$

$$\overrightarrow{AP}=-\frac{1}{\gamma-1}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right)=\frac{1}{1-\gamma}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right);\ (2)$$

From (1) and (2) A,P,Q are collinear if and only if exist $\lambda\in\mathbb{R}$ such that

$$\overrightarrow{AP}=\lambda\overrightarrow{AQ}\Leftrightarrow\frac{1}{1-\gamma}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right)=\lambda\left(\overrightarrow{AB}+\frac{1}{\alpha-1}\overrightarrow{AC}\right)\Leftrightarrow$$

$$\begin{cases}\dfrac{\beta-\gamma}{1-\gamma}=\lambda\\ \dfrac{1}{1-\gamma}=\dfrac{\lambda}{\alpha-1}\end{cases}\Leftrightarrow\begin{cases}\beta-\gamma=\lambda(1-\gamma)\\ \alpha-1=\lambda(1-\gamma)\end{cases}\Leftrightarrow\alpha+\gamma=\beta+1$$

JP.339 Solve in real numbers the system:

$$\begin{cases}11(x^4-y^4)+4xy(x^2+y^2)+x=0\\ 2(x^4-y^4)-22xy(x^2+y^2)+y=0\end{cases}$$

Proposed by Florică Anastase-Romania

Solution by proposer:

$$\begin{cases}11(x^4-y^4)+4xy(x^2+y^2)+x=0\\2(x^4-y^4)-22xy(x^2+y^2)+y=0\end{cases}$$

$$\begin{cases}\dfrac{x}{x^2+y^2}=-11(x^2-y^2)-4xy\\\dfrac{y}{x^2+y^2}=-2(x^2-y^2)+22xy\end{cases}$$

Let be $z\in\mathbb{C}, z=x+yi$ with $\frac{1}{z}=\frac{x-yi}{x^2+y^2}$ and $z^2=x^2-y^2+2xyi$

From system we get:

$$\frac{x-yi}{x^2+y^2}=-11(x^2-y^2)-4xy+2(x^2-y^2)-22xyi=(-11+2i)(x^2-y^2+2xyi)$$

$$\Leftrightarrow\frac{1}{z}=(-11+2i)z^2\Leftrightarrow z^3=\frac{1}{(1-2i)^3}\Leftrightarrow z=\frac{1+2i}{5}$$

Therefore, $x=\frac{1}{5}$ și $y=\frac{2}{5}$

JP.340 Prove that :

$$sin10^\circ=\frac{1}{4}-\frac{\sqrt{3}}{4}tan10^\circ+\frac{1}{4}tan^2 10^\circ-\frac{\sqrt{3}}{4}tan^3 10^\circ$$

Proposed by Pedro Henrique O. Pantoja-Natal-Brazil

Solution by proposer

Simplifying the expression of the statement, we have:

$$\frac{1}{4}-\frac{\sqrt{3}}{4}tan10^\circ+\frac{1}{4}tan^2 10^\circ-\frac{\sqrt{3}}{4}tan^3 10^\circ=\frac{(1+tan^2 10^\circ)(1-\sqrt{3}tan10^\circ)}{4}$$

Then $4sin10^\circ=(1+tan^2 10^\circ)(1-\sqrt{3}tan10^\circ);(*)$

It is sufficient to prove that $(*)$ is true. Observe that:

$$tan100^\circ-tan10^\circ=\frac{sin(100^\circ-10^\circ)}{cos100^\circ\cdot cos10^\circ}=\frac{sin90^\circ}{\frac{1}{2}(cos110^\circ+cos90^\circ)}=-\frac{2}{cos70^\circ}=$$

$$=-\frac{2}{cos(60^\circ+10^\circ)}=-\frac{2}{cos60^\circ cos10^\circ-sin60^\circ sin10^\circ}=$$

$$= \frac{4}{\sqrt{3}sin10^\circ - cos10^\circ} = \frac{4cos10^\circ}{\sqrt{3}tan10^\circ - 1}$$

However, $tan100^\circ = -\frac{1}{tan10^\circ}$

Therefore, $\left(tan10^\circ + \frac{1}{tan10^\circ}\right)\left(1 - \sqrt{3}tan10^\circ\right) = 4cos10^\circ$

Hence, $(1 + tan^2 10^\circ)\left(1 - \sqrt{3}tan10^\circ\right) = 4sin10^\circ$

JP.341 Find all positive integers n such that: $N = \frac{2^{2n} - n^2 - 1}{n!}$ is a perfect square.

Proposed by Pedro Henrique O. Pantoja-Natal-Brazil

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

N is perfect square if, first of all, is positive integer number.

For $n \geq 5$: $2^{2n} - 1 \equiv 3, 5 \ (mod\ 10)$

$n^2 \equiv 0, 1, 4, 5, 6, 9 \ (mod\ 10)$

Being that $n! \equiv 0 \ (mod\ 10) \Rightarrow 2^{2n} - 1 \equiv n^2 \ (mod\ 10)$

$\Rightarrow n \equiv 0 \ (mod\ 10), n \equiv 5 \ (mod\ 10)$ .Contradiction!

So, $n < 5$. Proving $n = 1, 2, 3, 4$ we get:

$N = 2, \frac{11}{2}, 9, \frac{239}{24}$. Therefore, $n = 3$.

Solution 2 by proposer

Suppose there is $m \in \mathbb{Z}$ such that $m^2 = N \Leftrightarrow n^2 + n!\,m^2 + 1 = 2^{2n}$.

First note that $n \in \mathbb{N}$ is odd, in fact, if n for even the left side will be odd and the even right, a contradiction.

If $n \geq 5 \Rightarrow n!\,m^2 \equiv 0 \ (mod\ 5)$, and $n^2 \equiv 0, 14 \ (mod\ 5)$ then

$n^2 + n!\,m^2 + 1 \equiv 0, 1, 2 \equiv (mod\ 5)$.

On the other hand,

$2^{2n} = 4^n \equiv (-1)^n = \pm 1 (mod\ 5)$

Thus we conclude that n must be even, a contradiction. Therefore,

$n < 5 \Rightarrow n = 1$ or $n = 3$.

Testing $n = 1$ does not work, and if $n = 3$ then

$$9 + 6m^2 = 64 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3.$$

Answer: $n = 3$.

JP.342. Let be $ABCDA'B'C'D'$ cube with length side 1 and $M \in BC$, $N \in DD'$, $P \in A'B'$. Find minimum perimeter of $\Delta\, MNP$.

Proposed by Florentin Vișescu-Romania

Solution 1 by proposer

Let be $M(x,1,0), N(1,0,z), P(0,y,1), x,y,z \in \mathbb{R}$.

$$MN = \sqrt{(x-1)^2 + 1 + z^2}, MP = \sqrt{x^2 + (y-1)^2 + 1}, NP = \sqrt{y^2 + (z-1)^2 + 1}.$$

$$P_{\Delta\, MNP} = MN + MP + PN =$$

$$= \sqrt{(x-1)^2 + z^2 + 1} + \sqrt{(y-1)^2 + x^2 + 1} + \sqrt{(z-1)^2 + y^2 + 1} \geq$$

$$\geq \sqrt{(x+y+z-3)^2 + (x+y+z)^2 + 9} \text{ (Minkowski)}$$

Equality holds if and only if $x = y = z$.

Let $f(t) = \sqrt{(t-3)^2 + t^2 + 9} = \sqrt{2t^2 - 6t + 18}, f: \mathbb{R} \to \mathbb{R}$.

$$f'(t) = \frac{4t-6}{2\sqrt{2t^2 - 6t + 18}} = \frac{2t-3}{\sqrt{2t^2 - 6t + 18}}.$$

$$f'(t) = 0 \Rightarrow t = \frac{3}{2}.$$

For $t \in \left(-\infty, \frac{3}{2}\right), f'(t) < 0, f$ –decreasing on $\left(-\infty, \frac{3}{2}\right)$.

For $t \in \left(\frac{3}{2}, \infty\right), f'(t) > 0, f$ –increasing on $\left(\frac{3}{2}, \infty\right)$.

Then $t = \frac{3}{2}$ minimum point $f(t) \geq f\left(\frac{3}{2}\right), \forall\, t \in \mathbb{R}$.

$$f\left(\frac{3}{2}\right) = \sqrt{2 \cdot \frac{9}{4} - 6 \cdot \frac{3}{2} + 18} = \sqrt{\frac{18 + 4 \cdot 18 - 2 \cdot 18}{4}} = \sqrt{\frac{3 \cdot 18}{4}} = \frac{3}{2}\sqrt{6}.$$

So $P_{\Delta\, MNP} \geq \frac{3\sqrt{6}}{2}$

Equality holds if and only if $x = y = z = \frac{1}{2}$

M, N, P midpoints of $[BC], [DD']$ and $[A'B']$ respectively.

Solution 2 by Ravi Prakash-New Delhi-India

Let coordinate of M, N, P be $M(1, y, 0), N(0, 1, z), P(x, 0, 1), 0 \le x, y, z \le 1$

$$MN = \sqrt{1^2 + (1-y)^2 + z^2} = |\vec{i} + (1-y)\vec{j} + \vec{k}|$$

$$NP = \sqrt{x^2 + 1^2 + (1-z)^2} = |x\vec{j} + \vec{k} + (1-z)\vec{i}|$$

$$PM = \sqrt{(1-x)^2 + y^2 + 1^2} = |(1-x)\vec{k} + y\vec{i} + \vec{j}|$$

$$MN + NP + PM = |\overrightarrow{MN}| + |\overrightarrow{NP}| + |\overrightarrow{PM}| \ge |\overrightarrow{MN} + \overrightarrow{NP} + \overrightarrow{PM}| =$$

$$= |(1 + 1 - z + y)\vec{i} + (1 - y + x + 1)\vec{j} + (z + 1 + 1 - x)\vec{k}|$$

$$\Rightarrow (MN + NP + PM)^2 \ge (2 - z + y)^2 + (2 - y + x)^2 + (2 - x + z)^2 =$$

$$= 12 + (y - z)^2 + (x - y)^2 + (z - x)^2$$

For $MN + NP + PM$ to be least, we must $y - z = x - y = z - x = 0$

$\Rightarrow x = y = z$. Thus,

$$MN + NP + PM \ge 3\sqrt{1 + (1-x)^2 + x^2} = 3\sqrt{2}\sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} \ge \frac{3\sqrt{3}}{\sqrt{2}}$$

Equality when $x = y = z = \frac{1}{2}$.

Least perimeter of ΔMNP is $\frac{3\sqrt{6}}{2}$ for $M\left(1, \frac{1}{2}, 0\right), N\left(0, 1, \frac{1}{2}\right), P\left(\frac{1}{2}, 0, 1\right)$

Solution 3 by Adrian Popa-Romania

$$M \in (BC): MB = x \Rightarrow MC = 1 - x$$

$$N \in (DD'): ND = y \Rightarrow ND' = 1 - y$$

$$P \in (A'B'): PA' = z \Rightarrow PB' = 1 - z$$

$$P_{MNP} = PM + MN + NP$$

$$PM^2 = 2 + (1-z)^2 + 1; MN^2 = (1-x)^2 + y^2 + 1;\ \ PN^2 = z^2 + (1-y)^2 + 1$$

$$P_{MNP} = min \Rightarrow (PM + MN + NP)^2 = min \Rightarrow$$

$$PM^2 + MN^2 + NP^2 + 2PM \cdot MN + 2PM \cdot NP + 2MN \cdot NP = min$$

$$PM^2 + MN^2 + NP^2 \ge PM \cdot MN + PM \cdot NP + MN \cdot NP$$

Equality holds when $PM \equiv MN \equiv NP$ hence,

$$(PM^2+MN^2+NP^2+2PM\cdot MN+2PM\cdot NP+2MN\cdot NP)_{min}=9PM^2$$

In this case $\Delta PP'M\equiv\Delta MM'N\equiv\Delta PP'N$ hence, $P'M\equiv M'N\equiv P'N$.

$$(PM+MN+NP)^2=PM^2+MN^2+NP^2+2PM^2+2PN^2+2MN^2=$$

$$=3(MN^2+PN^2+PM^2)=6(x^2+y^2+z^2-x-y-z+3)$$

Let be $f(x,y,z)=x^2+y^2+z^2-2x-2y-2z$

$$\frac{\partial f}{\partial x}=2x-1=0\Rightarrow x=\frac{1}{2}$$

$$\frac{\partial f}{\partial y}=2y-1=0\Rightarrow y=\frac{1}{2}$$

$$\frac{\partial f}{\partial z}=2z-1=0\Rightarrow z=\frac{1}{2}$$

Therefore,

$$f_{min}=\frac{9}{4}\Rightarrow P^2_{min}=\frac{27}{2}\Rightarrow P_{min}=\frac{3\sqrt{6}}{2}$$

JP.343 In acute ΔABC, g_a –Gergonne's cevian the following relationship holds:

$$max\{g_a^2\cdot cosA, g_b^2\cdot cosB, g_c^2\cdot cosC\}\ge r^2\left(1+\frac{r}{R}\right)\left(\frac{43}{9}-\frac{8R}{9r}\right)$$

Proposed by Radu Diaconu-Romania

Solution 1 by proposer

Using the following relationship:

$$g_a^2=(s-a)^2+2rh_a\ (and\ analogs)$$

$$a^2+b^2+c^2=2s^2-2r^2-8Rr$$

$h_a+h_b+h_c\ge 9r$ and $s^2\ge 27r^2$ we have:

$$g_a^2+g_b^2+g_c^2=3s^2-2s(a+b+c)+a^2+b^2+c^2+2r(h_a+h_b+h_c)\Leftrightarrow$$

$$g_a^2+g_b^2+g_c^2=a^2+b^2+c^2-s^2+2r(h_a+h_b+h_c)\Leftrightarrow$$

$$g_a^2+g_b^2+g_c^2=s^2-2r^2-8Rr+2r(h_a+h_b+h_c)\Leftrightarrow$$

$$g_a^2+g_b^2+g_c^2=s^2-2r^2-8Rr+18r^2=s^2-8Rr+16r^2\ge 43r^2-8Rr.$$

WLOG, suppose: $a \le b \le c \Rightarrow cosA \ge cosB \ge cosC$ **and**

$(s-a)^2 \ge (s-b)^2 \ge (s-c)^2; 2rh_a \ge 2rh_b \ge 2rh_c$ **hence**

$$g_a^2 \ge g_b^2 \ge g_c^2$$

Applying Chebyshev's Inequality, it follows that:

$$g_a^2 \cdot cosA + g_b^2 \cdot cosB + g_c^2 \cdot cosC \ge \frac{1}{3}(cosA + cosB + cosC)(g_a^2 + g_b^2 + g_c^2) \ge$$

$$\ge \frac{1}{3}\left(1+\frac{r}{R}\right)(43r^2 - 8Rr) = r^2\left(1+\frac{r}{R}\right)\left(\frac{43}{3} - \frac{8R}{3r}\right)$$

Therefore,

$$max\{g_a^2 \cdot cosA, g_b^2 \cdot cosB, g_c^2 \cdot cosC\} \ge \frac{g_a^2 \cdot cosA + g_b^2 \cdot cosB + g_c^2 \cdot cosC}{3}$$

$$\ge r^2\left(1+\frac{r}{R}\right)\left(\frac{43}{9} - \frac{8R}{9r}\right)$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Since in ΔABC **(acute):** $CosA, cosB, cosC > 0 \Rightarrow$

$$max\{g_a^2 \cdot cosA, g_b^2 \cdot cosB, g_c^2 \cdot cosC\} \ge \frac{1}{3}(g_a^2 \cdot cosA + g_b^2 \cdot cosB + g_c^2 \cdot cosC) \overset{g_a \ge h_a}{\ge}$$

$$\ge \frac{1}{3}(h_a^2 cosA + h_b^2 cosB + h_c^2 cosC) = \frac{4S^2}{3}\left(\frac{cosA}{a^2} + \frac{cosB}{b^2} + \frac{cosC}{c^2}\right) =$$

$$= \frac{bc(b^2+c^2-a^2) + ac(a^2+c^2-b^2) + ab(a^2+b^2-c^2)}{2} =$$

$$= \frac{1}{3 \cdot 8 \cdot R^2}\left[\sum bc(b^2+c^2) - abc\sum a\right] =$$

$$= \frac{(a+b+c)^2(ab+bc+ca) - 2(ab+bc+ca)^2 - 2(a+b+c)abc}{24R^2} =$$

$$= \frac{(s^2+4Rr+r^2)(2s^2-8Rr-2r^2) - 16Rrs^2}{24R^2} =$$

$$= \frac{(s^2+4Rr+r^2)(s^2-4Rr-r^2) - 8Rrs^2}{12R^2} =$$

$$= \frac{s^4 - 8Rrs^2 - (4Rr+r^2)^2}{12R^2} \overset{(1)}{\ge} r^2\left(1+\frac{r}{R}\right)\left(\frac{43}{9} - \frac{8R}{9r}\right) = \frac{r(R+r)(43r-8R)}{9R}$$

$$(1) \Leftrightarrow 3[s^4 - 8Rrs^2 - (4Rr+r^2)^2] \ge 4Rr(R+r)(43r-8R)$$

$$\Leftrightarrow 3s^4 - 24Rrs^2 - 3(4Rr + r^2)^2 - 4Rr(R + r)(43r - 8R) \geq 0$$

But: $s^2 \geq 16Rr - 5r^2 = 8Rr + r(8R - 5r) \overset{R\geq 2r}{\geq} 8Rr + 11r^2 > 8Rr$

$$\Rightarrow 3s^2(s^2 - 8Rr) - 3(4Rr + r^2)^2 - 4Rr(R + r)(43r - 8R) \geq$$

$$\geq 3(16Rr - 5r^2)(8Rr - 5r^2) - 3(4Rr + r^2)^2 - 4Rr(R + r)(43r - 8R) \geq 0; (2)$$

$$(2) \overset{t=\frac{R}{r}\geq 2}{\Longleftrightarrow} 3(16t - 5)(8t - 5) - 3(4t + 1)^2 - 4t(t + 1)(43 - 8t) \geq 0$$

$$\Leftrightarrow 32t^3 + 196t^2 - 556t + 72 \geq 0 \Leftrightarrow 4(t - 2)(8t^2 + 65t - 9) \geq 0$$

true because $t \geq 2 \Rightarrow 4(t - 2) \geq 0, 8t^2 + 65t - 9 = 8t^2 + 56t + 9(t - 1) \overset{t\geq 2}{>}$

$> 9(t - 1) \geq 9 > 0 \Rightarrow (2) \Rightarrow (1)$ is true.

JP.344. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$.

Prove that:

$$(3a^5 - 3a + 2b^3 + 34)(3b^5 - 3b + 2c^3 + 34)(3c^5 - 3c + 2a^3 + 34) \geq 6^6$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by proposer

We have: $\begin{cases} 3a^5 - 3a + 6 \geq 6a^2 \\ 2b^3 + 1 \geq 3b^2 \end{cases} \Rightarrow$

$$3a^5 - 3a + 2b^3 + 34 \geq 3(2a^2 + b^2 + 9) = 3\left(2a^2 + b^2 + 3(ab + bc + ca)\right) \Rightarrow$$

$$3a^5 - 3a + 2b^3 + 34 \geq 3(a + b)(2a + b + 3c) =$$

$$= 3(a + b)[(a + c) + (a + c) + (b + c)] \geq$$

$$\geq 3(a + b) \cdot 3\sqrt[3]{(a + c)^2(b + c)} = 9\sqrt[3]{(a + b)^3(a + c)^2(b + c)}$$

Similarly:

$$3b^5 - 3b + 2c^3 + 34 \geq 9\sqrt[3]{(b + c)^3(b + a)^2(a + b)}$$

$$3c^5 - 3c + 2a + 34 \geq 9\sqrt[3]{(c + a)^3(b + c)^2(a + b)}$$

Hence

$$(3a^5 - 3a + 2b^3 + 34)(3b^5 - 3b + 2c^3 + 34)(3c^5 - 3c + 2a + 34) \geq$$

$$\geq 9^3(a + b)^2(b + c)^2(c + a)^2$$

$$(a+b)(b+c)(c+a) \geq \frac{8}{9}(a+b+c)(ab+bc+ca) \geq \frac{8}{9}\sqrt{3(ab+bc+ca)} \cdot 3 = 8$$

Therefore,

$$(3a^5 - 3a + 2b^3 + 34)(3b^5 - 3b + 2c^3 + 34)(3c^5 - 3c + 2a + 34) \geq 6^6$$

Solution 2 by Rustam Tahmazov-Baku-Azerbaijan

$$2a^5 + (a^5 + 4) - 3a \overset{AM-GM}{\geq} 2a^5 + 5a - 3a = 2a^5 + 2a \overset{AM-GM}{\geq} 4a^3$$

$$LHS \geq \prod_{cyc}(4a^3 + 2b^3 + 30) \geq 6^6$$

$$LHS = (2a^3 + 2 + 2 + 2a^3 + 2b^3 + 2 + 24) \cdot (2 + 2b^3 + 2 + 2c^3 + 2 + 2b^3 + 24) \cdot$$

$$\cdot (2 + 2 + 2c^3 + 2 + 2c^3 + 2a^3 + 24) \overset{Holder}{\geq} [2(a+b+c) + 2(ab+bc+ca) + 24]^3$$

$$= \left(2\sum a + 3a\right)^3 \geq 6^6$$

$$(a+b+c)^2 \geq 3(ab+bc+ca) = 9 \Rightarrow a+b+c \geq 3$$

$$LHS \geq (2 \cdot 3 + 30) = 36^6 = 6^6$$

Solution 3 by Eldeniz Hesenov-Georgia

$$x^5 - 2x + 4 \geq x^3 + 2; \ \ (1)$$

$$x^5 - x^3 - 2x + 2 \geq 0 \Leftrightarrow (x-1)^2(x^3 + 2x^2 + 2x + 2) \geq 0$$

$$LHS = \prod_{cyc}\left(2a^5 + (a^5 - 2a + 4) - a + 3 + 30 + 2b^3\right) \geq$$

$$\geq \prod_{cyc}(2a^5 + (a^3 + 1) + 30 - a + 2b^3) \overset{AM-GM}{\geq}$$

$$\geq \prod_{cyc}(2a^5 + 2a + 30 + 2b^3) \overset{AM-GM}{\geq}$$

$$\geq \prod_{cyc}(2 \cdot 2a^3 + 2b^3 + 30) = \prod_{cyc}(2a^3 + 2b^3 + 2a^3 + 2 + 2 + 2 + 24) \overset{Holder}{\geq}$$

$$\geq \left(2\sum a + 2\sum ab + 24\right)^3 \geq \left(2\sum a + 30\right)^3 = (3 \cdot 2 + 30)^3 = 6^6$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ and $ab + bc + ca = 3, a + b + c \geq 2$ consider:

$$\left(3a^5-3a+2b^3+34\right)\left(3b^5-3b+2c^3+34\right)\left(3c^5-3c+2a+34\right)\geq$$

$$\geq\left(2a^5+2a+2b^3+30\right)\left(2b^5+2b+2c^3+30\right)\left(2c^5+2c+2a^3+30\right)\geq$$

$$\geq(4a^3+2b^3+30)(4b^3+2c^3+30(4c^3+2a^3+30)=$$

$$=2^3(a^3+a^3+b^3+15)(b^3+b^3+c^3+15)(c^3+c^3+a^3+15)\geq$$

$$\geq 2^3(9(a+b+c)+9)^3\geq 2^3+18^3=6^6$$

JP.345 If $a,b,c\in\mathbb{C}$; $|a|=|b|=|c|=3$ then:

$$\sum_{cyc}|a+3|+3\sum_{cyc}\left|a^2+1\right|+\sum_{cyc}\left|a^3+3\right|\geq 18$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$6=2\cdot 3=2\cdot|a|=2\cdot|-a|=|2\cdot(-a)|=|-2a|=$$

$$=\left|a^3+3-a(a^2+1)-(a+3)\right|\leq$$

$$\leq\left|a^3+3\right|+|a|\cdot\left|a^2+1\right|+|a+3|=$$

$$=\left|a^3+3\right|+3\left|a^2+1\right|+|a+3|$$

$$|a+3|+3\left|a^2+1\right|+\left|a^3+3\right|\geq 6$$

By summing:

$$\sum_{cyc}|a+3|+3\sum_{cyc}\left|a^2+1\right|+\sum_{cyc}\left|a^3+3\right|\geq 6+6+6=18$$

SP.331. ΔABC has inradius r, circumradius R, sides lengths $a=BC$, $b=AC$, $c=AB$, and altitudes h_a,h_b,h_c from the vertices A,B,C,respectively. Prove that:

$$\frac{9r^2}{R}\leq\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c\leq\frac{9R}{4}$$

Proposed by George Apostolopoulos-Greece

Solution 1 by proposer

Since $bc = 2R \cdot h_a, ca = 2R \cdot h_b, ab = 2R \cdot h_c$, the right inequality becomes

$$2\left(\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}\right) \le 9R^2$$

We know that in any triangle we have $a^2+b^2+c^2 \le 9R^2$

So it suffices to prove that:

$$2\left(\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}\right) \le a^2+b^2+c^2 \Leftrightarrow$$

$$\left(2a^2-\frac{2ab^2}{a+b}\right)+\left(2b^2-\frac{2bc^2}{b+c}\right)+\left(2c^2-\frac{2ca^2}{c+a}\right) \ge a^2+b^2+c^2 \Leftrightarrow$$

$$\frac{2a^3}{c+a}+\frac{2b^3}{a+b}+\frac{2c^3}{b+c} \ge a^2+b^2+c^2$$

Using Cauchy-Schwarz inequality, we have:

$$\frac{2a^3}{c+a}+\frac{2b^3}{a+b}+\frac{2c^3}{b+c} = 2\left(\frac{a^4}{ca+a^2}+\frac{b^4}{ab+b^2}+\frac{c^4}{bc+c^2}\right) \ge$$

$$\ge \frac{2(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)+(ab+bc+ca)} \ge \frac{2(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)+(a^2+b^2+c^2)} = a^2+b^2+c^2$$

Namely,

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c \le \frac{9}{4}R$$

For the left inequality, we have (AM-GM):

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c \ge 3\sqrt[3]{\frac{(abc)(h_ah_bh_c)}{(a+b)(b+c)(c+a)}}$$

We know that $abc = 4Rrs, R > -2r\ (Euler), s = \frac{a+b+c}{2} \ge 3\sqrt{3}r.$

Also, we know that:

$$\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)^3 \ge \frac{27}{h_ah_bh_c};\ \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} = \frac{1}{r}$$

So,

$$\left(\frac{1}{r}\right)^3 \geq \frac{27}{h_a h_b h_c} \Leftrightarrow h_a h_b h_c \geq 27r^3.$$

Now,

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c \geq \frac{3\sqrt[3]{(4Rrs)(27r^3)}}{\sqrt[3]{(a+b)(b+c)(c+a)}} \geq$$

$$\geq \frac{3\sqrt[3]{4(2r)r(3\sqrt{3}r)27r^3}}{\frac{(a+b)+(b+c)+(c+a)}{3}} = \frac{3\sqrt[3]{8\cdot 3\sqrt{3}\cdot 27r^6}}{\frac{45}{3}} = \frac{3\cdot 2\cdot \sqrt{3}\cdot 3\cdot 3r^2}{45} = \frac{27\sqrt{3}r^2}{25}$$

We know that $2s \leq 3\sqrt{3}R$, so

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c \geq \frac{9r^2}{R}$$

Namely,

$$\frac{9r^2}{R} \leq \frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c \leq \frac{9R}{4}$$

Equality holds if and only if the triangle ABC is equilateral.

Solution 2 by Marian Ursărescu-Romania

For the left hand:

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c \geq 3\sqrt[3]{\frac{abc\cdot h_a h_b h_c}{(a+b)(b+c)(c+a)}}$$

We must show:

$$\sqrt[3]{\frac{abc\cdot h_a h_b h_c}{(a+b)(b+c)(c+a)}} \geq \frac{3r^2}{R};\ (1)$$

But $abc = 4Rrs, h_a h_b h_c = \frac{2s^2r^2}{R}$ and $(a+b)(b+c)(c+a) = 2s(s^2+r^2+2Rr)$; (2)

From $(1)\&(2)$ we must show that:

$$\sqrt[3]{\frac{4Rrs\cdot 2s^2r^2}{2R(s^2+r^2+2Rr)}} \geq \frac{3r^2}{R} \Leftrightarrow \frac{4s^2r^3}{s^2+r^2+2Rr} \geq \frac{27r^6}{R^3}$$

$$\Leftrightarrow \frac{4s^2}{s^2+r^2+2Rr} \geq \frac{27r^3}{R^3};\ \ (3)$$

$$s^2+r^2+2Rr\overset{Gerretsen}{\leq}4R^2+6Rr+4r^2\overset{Euler}{\leq}4R^2+3R^2+R^2=8R^2;(4)$$

From $(3)\&(4)$ **we must show:**

$\frac{s^2}{2R^2}\geq\frac{27r^3}{R^3}\Leftrightarrow s^2R\geq 2\cdot 27r^3$ **true because** $s^2\geq 27r^2$ $(Mitrinovic), R\geq 2r$ $(Euler)$

$$\Rightarrow s^2R\geq 2\cdot 27r^3$$

For the right hand, we have:

$h_a=csinB=\frac{bc}{2R}$ **and similarly, then we must show:**

$$\frac{1}{2R}\left(\frac{bc^2}{b+c}+\frac{ca^2}{c+a}+\frac{ab^2}{a+b}\right)\leq\frac{9R}{4}\Leftrightarrow$$

$$\frac{2bc^2}{b+c}+\frac{2ca^2}{c+a}+\frac{2ab^2}{a+b}\leq 9R^2;(5)$$

But $\frac{2bc}{b+c}\leq\frac{b+c}{2}$ **and similarly;** (6)

From $(5)\&(6)$ **we must show:**

$$\frac{(b+c)c}{2}+\frac{(a+c)a}{2}+\frac{(a+b)b}{2}\leq 9R^2\Leftrightarrow$$

$$\frac{a^2+b^2+c^2+ab+bc+ca}{2}\leq 9R^2;\ (7)$$

But $ab+bc+ca\leq a^2+b^2+c^2;\ (8)$

From $(7)\&(8)$ **we get:**

$a^2+b^2+c^2\leq 9R^2$ **which is true.**

Solution 3 by Alex Szoros-Romania

$$\frac{1}{(b+c)^2}\leq\frac{1}{4bc}\Rightarrow\frac{1}{a^2(b+c)^2}\leq\frac{1}{4a^2bc}\Rightarrow$$

$$\sum_{cyc}\frac{1}{a^2(b+c)^2}\leq\sum_{cyc}\frac{1}{4a^2bc}=\frac{1}{4abc}\sum_{cyc}\frac{1}{a}=\frac{1}{4abc}\sum_{cyc}\frac{bc}{abc}=\frac{1}{4(abc)^2}\sum_{cyc}bc\leq$$

$$\leq\frac{1}{4(abc)^2}\sum_{cyc}a^2$$

$$\Rightarrow\sum_{cyc}\frac{1}{a^2(b+c)^2}\leq\frac{9R^2}{4(4RrF)^2}=\frac{9}{64F^2}\Rightarrow\sum_{cyc}\frac{4F^2}{a^2(b+c)^2},=\frac{9}{16}$$

$$\sum_{cyc}\left(\frac{2F}{a}\right)^2\frac{1}{(b+c)^2}\le\frac{9}{16}\Leftrightarrow$$

$$\sum_{cyc}\left(\frac{h_a}{b+c}\right)^2\le\frac{9}{16},\forall\Delta ABC;\ (1)$$

$$\sum_{cyc}\left(\frac{c}{b+c}\right)h_a=\sum_{cyc}\left(\frac{h_a}{b+c}\right)c\le\sqrt{\left(\sum_{cyc}\left(\frac{h_a}{b+c}\right)\right)^2\left(\sum_{cyc}c^2\right)^2}=\frac{9R}{4};(2)$$

$$\sum_{cyc}\frac{ch_a}{b+c}\ge 3\sqrt[3]{\frac{abch_ah_bh_c}{(a+b)(b+c)(c+a)}}=\frac{3\cdot 2}{\sqrt[3]{(a+b)(b+c)(c+a)}}$$

$$\Rightarrow\sum_{cyc}\frac{ch_a}{b+c}\ge\frac{6S}{\sqrt[3]{(a+b)(b+c)(c+a)}};\ (3)$$

$$\frac{6S}{\sqrt[3]{(a+b)(b+c)(c+a)}}\ge\frac{9r^2}{R};\ (4)$$

$$(4)\Leftrightarrow\frac{2RS}{3r^2}\ge\sqrt[3]{(a+b)(b+c)(c+a)}\Leftrightarrow$$

$$\frac{2Rrs}{3r^2}\ge\sqrt[3]{(a+b)(b+c)(c+a)}\Leftrightarrow\left(\frac{2Rs}{3r}\right)^3\ge(a+b)(b+c)(c+a)$$

$$\Leftrightarrow\left(\frac{2Rs}{3r}\right)^3\ge(a+b+c)(ab+bc+ca)-abc$$

$$\Leftrightarrow\left(\frac{2Rs}{3r}\right)^3\ge 2s(s^2+r^2+4Rr)-4Rrs\Leftrightarrow\frac{4s^2R^3}{27r^3}\ge s^2+r^2+2Rr$$

$$\Leftrightarrow\frac{4s^2R^3}{27r^3}.=\frac{4R^3(16Rr-5r^2)}{27r^3}\ge 4R^2+4Rr+3r^2+r^2+2Rr\ge s^2+r^2+2Rr$$

$$4R^3(16Rr-5r^2)\ge 27r^3(4R^2+6Rr+4r^2)$$

$$\Leftrightarrow 2R^3(16Rr-5r^2)\ge 27r^3(2R^2+3Rr+2r^2)$$

$$(R\ge 2r\Rightarrow R^2\ge 4r^2)$$

$$\Leftrightarrow 2R(16Rr-5r^2)\ge\frac{27}{4}r(2R^2+3Rr+2r^2)$$

$$\Leftrightarrow 8R(16Rr-5r^2)\ge 27r(2R^2+3Rr+2r^2)$$

$$\Leftrightarrow 128R^2r-40Rr^2\ge 54R^2r+81Rr^2+54r^3$$

$$\Leftrightarrow 74R^2r - 121Rr^2 - 54r^3 \ge 0$$

$$\Leftrightarrow 74\left(\frac{R}{r}\right)^2 - 121\left(\frac{R}{r}\right) - 54 \ge 0; t = \frac{R}{r} \ge 2 \Rightarrow$$

$$74t^2 - 121t - 54 \ge 0 \Leftrightarrow (t-2)(74t+27) \ge 0 \text{ true.}$$

Solution 4 by Agayev Seddredin-Azerbaijan

$$\frac{c}{b+c}\cdot h_a + \frac{a}{c+a}\cdot h_b + \frac{b}{a+b}\cdot h_c = \frac{c}{b+c}\frac{2F}{a} + \frac{a}{c+b}\frac{2F}{b} + \frac{b}{c+a}\frac{2F}{c} =$$

$$= 2F\left(\frac{c}{a(b+c)} + \frac{b}{b(c+a)} + \frac{c}{c(a+b)}\right) \ge$$

$$\ge 2F\cdot 3\sqrt[3]{\frac{1}{(a+b)(b+c)(c+a)}} \ge \frac{18F}{2(a+b+c)} = \frac{9F}{2s} = \frac{9sr}{2s} = \frac{9r}{2} \ge \frac{9r^2}{R}$$

$$\frac{c}{b+c}\cdot h_a + \frac{a}{c+a}\cdot h_b + \frac{b}{a+b}\cdot h_c = 2F\left(\frac{c}{a(b+c)} + \frac{b}{b(c+a)} + \frac{c}{c(a+b)}\right) \le$$

$$\le 2F\left(\frac{c}{2a\sqrt{bc}} + \frac{a}{2b\sqrt{ac}} + \frac{b}{2c\sqrt{ab}}\right) = 2F\left(\frac{\sqrt{bc}}{ab} + \frac{\sqrt{ac}}{bc} + \frac{\sqrt{ab}}{ca}\right) \le$$

$$\le F\left(\frac{b+c}{2ab} + \frac{a+c}{2bc} + \frac{a+b}{2ac}\right) = F\frac{a^2+b^2+c^2+ab+bc+ca}{2abc} \le$$

$$\le F\frac{18R^2}{2abc} = \frac{abc}{4R}\frac{18R^2}{2abc} = \frac{9R}{4}$$

$$(ab+bc+ca \le a^2+b^2+c^2 \le 9R^2)$$

SP.332. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \le \frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

Proposed by George Apostolopoulos-Greece

Solution 1 by proposer

We know that $\frac{1}{a+b} \le \frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)$

So, $\frac{c^2}{a+b} \le \frac{1}{4}\left(\frac{c^2}{a}+\frac{c^2}{b}\right)=\frac{1}{4}\left(c\cdot\frac{c}{a}+c\cdot\frac{c}{b}\right)$

And similarly: $\frac{a^2}{b+c} \le \frac{1}{4}\left(a\cdot\frac{a}{b}+a\cdot\frac{a}{c}\right)$ and $\frac{b^2}{c+a} \le \frac{1}{4}\left(b\cdot\frac{b}{c}+b\cdot\frac{b}{a}\right)$

Adding up these three last inequalities, we get:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \le \frac{1}{4}\left[\left(a\cdot\frac{a}{b}+a\cdot\frac{a}{c}\right)+\left(b\cdot\frac{b}{c}+b\cdot\frac{b}{a}\right)+\left(c\cdot\frac{c}{a}+c\cdot\frac{c}{b}\right)\right];\ (*)$$

Now, we will prove that: $\frac{a}{b}+\frac{b}{a} \le \frac{R}{r}$

Consider the substitutions $a=y+z, b=z+x, c=x+y$, where x,y,z are positive real numbers. We know that:

$$\frac{R}{r}=\frac{abc}{4(s-a)(s-b)(s-c)}=\frac{(x+y)(y+z)(z+x)}{4xyz}$$

We have: $\frac{1}{(z+x)^2}+\frac{1}{(y+z)^2} \le \frac{1}{4zx}+\frac{1}{4yz}=\frac{x+y}{4xyz}$

And multiplying by $(z+x)(y+z)$ both sides, we get:

$$\frac{y+z}{z+x}+\frac{z+x}{y+z} \le \frac{(x+y)(y+z)(z+x)}{4xyz}$$

Namely $\frac{a}{b}+\frac{b}{a} \le \frac{R}{r}$. Similarly: $\frac{b}{c}+\frac{c}{b} \le \frac{R}{r}$ and $\frac{c}{a}+\frac{a}{c} \le \frac{R}{r}$. So,

$$\frac{a}{b}+\frac{b}{a} \le \frac{R}{r} \Leftrightarrow \frac{a^2}{b^2}+\frac{b^2}{a^2} \le \frac{R^2}{r^2}-2$$

$$\frac{b}{c}+\frac{c}{b} \le \frac{R}{r} \Leftrightarrow \frac{b^2}{c^2}+\frac{c^2}{b^2} \le \frac{R^2}{r^2}-2$$

$$\frac{c}{a}+\frac{a}{c} \le \frac{R}{r} \Leftrightarrow \frac{c^2}{a^2}+\frac{a^2}{c^2} \le \frac{R^2}{r^2}-2$$

Applying the Cauchy-Schwartz inequality, we have:

$$(a^2+b^2)\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right) \ge \left(a\cdot\frac{a}{b}+b\cdot\frac{b}{a}\right)^2 \Rightarrow \frac{a^2}{b^2}+\frac{b^2}{a^2} \le \sqrt{a^2+b^2}\cdot\sqrt{\frac{R^2}{r^2}-2}$$

And similarly: $\frac{b^2}{c^2}+\frac{c^2}{b^2} \le \sqrt{b^2+c^2}\cdot\sqrt{\frac{R^2}{r^2}-2}$

$$\frac{c^2}{a^2}+\frac{a^2}{c^2} \le \sqrt{c^2+a^2}\cdot\sqrt{\frac{R^2}{r^2}-2}$$

From the $(*)$ inequality, we get:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{1}{4}\sqrt{\frac{R^2}{r^2}-2}\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

But:

$$\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)^2\le 3(a^2+b^2+b^2+c^2+c^2+a^2)$$

$$=6(a^2+b^2+c^2)\Leftrightarrow$$

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\le\sqrt{6}\cdot\sqrt{a^2+b^2+c^2}$$

It is well-known that $a^2+b^2+c^2\le 9R^2$. So

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\le 3\sqrt{6}R$$

Namely,

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{1}{4}\sqrt{\frac{R^2-2r^2}{r^2}}\cdot 3\sqrt{6}R=\frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

Equality holds if and only if the triangle ABC is equilateral.

Solution 2 by Tran Hong-Dong Thap-Vietnam

By AM-GM inequality, we have:

$$b+c\ge 2\sqrt{bc},a+b\ge 2\sqrt{ab},c+a\ge 2\sqrt{ca}$$

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{a^2}{2\sqrt{bc}}+\frac{b^2}{2\sqrt{ca}}+\frac{c^2}{2\sqrt{ab}}=\frac{a^2\sqrt{a}+b^2\sqrt{b}+c^2\sqrt{c}}{2\sqrt{abc}}=$$

$$=\frac{a\cdot a\sqrt{a}+b\cdot b\sqrt{b}+c\cdot c\sqrt{c}}{2\sqrt{abc}}\overset{BCS}{\le}\frac{\sqrt{(a^2+b^2+c^2)(a^3+b^3+c^3)}}{2\sqrt{abc}}=$$

$$=\frac{1}{2}\sqrt{\frac{(a^2+b^2+c^2)(a^3+b^3+c^3)}{abc}}=\frac{1}{2}\sqrt{\frac{2(s^2-4Rr-r^2)\cdot 2s(s^2-6Rr-3r^2)}{4Rrs}}=$$

$$=\frac{1}{2}\sqrt{\frac{(s^2-4Rr-r^2)(s^2-6Rr-3r^2)}{Rr}}\overset{(1)}{\le}\frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

$$(1)\Leftrightarrow\frac{(s^2-4Rr-r^2)(s^2-6Rr-3r^2)}{Rr}\le\frac{27R^2(R^2-2r^2)}{2r}$$

Which is true because: $\frac{s^2-6Rr-3r^2}{R} \le \frac{3(R^2-2r^2)}{r}$; (2)

$$\Leftrightarrow r(s^2-6Rr-3r^2) \le 3R(R^2-2r^2)$$

Because: $2r \le R \Rightarrow r \le \frac{R}{2}$; (3)

$$s^2 \le 4R^2+4Rr+3r^2 \Rightarrow s^2-6Rr-3r^2 \le 4R^2-2Rr \overset{(4)}{\le} 6(R^2-2r^2)$$

$(4) \Leftrightarrow 2R^2+2Rr-12r^2 \ge 0 \Leftrightarrow 2(R-2r)(R+3r) \ge 0$ true by $R \ge 2r(Euler)$.

From $(3)\&(4) \Rightarrow (2)$ is true.

$$s^2-4Rr-r^2 < 9R^2; (5)$$

$\Leftrightarrow \frac{a^2+b^2+c^2}{2} < 9R^2$ ture by: $a^2+b^2+c^2 \le 9R^2 \Rightarrow$

$$\frac{a^2+b^2+c^2}{2} < a^2+b^2+c^2 < 9R^2$$

From $(2)\&(5) \Rightarrow (1)$ is true.

Solution 3 by Marian Ursărescu-Romania

We must to show: $\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)^2 \le \frac{27}{8}\frac{R^2}{r^2}(R^2-2r^2)$; (1)

From BCS inequality, we have:

$$\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)^2 \le (a^4+b^4+c^4)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right); \quad (2)$$

From $(1)\&(2)$ we must show:

$$(a^4+b^4+c^4)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right) \le \frac{27}{8}\frac{R^2}{r^2}(R^2-2r^2); (3)$$

But $a^4+b^4+c^4 \le 54r^3(R-r)$; (4)

$$(a+b)^2 \ge 4ab \Rightarrow \frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2} \le \frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) \le \frac{1}{4}\frac{1}{4r^2}$$

$$= \frac{1}{16r^2}(Leuenberger); \quad (5)$$

From $(3),(4)\&(5)$ we must show:

$$54R^3(R-r)\frac{1}{16r^2} \le \frac{27}{8}\frac{R^2}{r^2}(R^2-2r^2) \Leftrightarrow R(R-r) \le R^2-2r^2$$

$$\Leftrightarrow R^2-Rr \le R^2-2r^2 \Leftrightarrow Rr \ge 2r^2 \Leftrightarrow R \ge 2r(Euler)$$

Solution 4 and extensions by Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

Proposed by George Apostolopoulos-Greece

Solution by Marin Chirciu

Lemma. 2) In ΔABC the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Proof. We have:

$$\sum\frac{a^2}{b+c}=\frac{\sum a^2(a+b)(a+c)}{\prod(b+c)}=\frac{4s^4(s^2-3r^2-4Rr)}{2s(s^2+r^2+2Rr)}=\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Which follows from the identities:

$$\sum a^2(a+b)(a+c)=4s^4(s^2-3r^2-4Rr)$$

$$\prod(b+c)=2s(s^2+r^2+2Rr)$$

Let's get back to the main problem.

The following inequality is much stronger by the main problem.

3) In ΔABC the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{Rs}{2r}$$

Using Lemma, we get:

$$\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}\le\frac{Rs}{2r}\Leftrightarrow 4r(s^2-3r^2-4Rr)\le R(s^2+r^2+2Rr)\Leftrightarrow$$

$$R(s^2+r^2+2Rr)\ge 4r(s^2-3r^2-4Rr)\Leftrightarrow s^2(R-4r)+r(2R^2+17Rr+12r^2)\ge 0$$

We distinguish the cases:

Case 1) If $(R-4r)\ge 0$ is obviously true.

Case 2) If $(R-4r)<0$, inequality it can be write:

$r(2R^2+17Rr+12r^2)\ge s^2(4r-R)$ **which is true from**

$$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$$

It remains to prove:

$$r(2R^2+17Rr+12r^2)\ge(4R^2+4Rr+3r^2)(4r-R)\Leftrightarrow$$

$$2R^2r+17Rr^2+12r^3\ge-4R^3-4R^2r-3Rr^2+16R^2+16Rr^2+12r^3$$

$$\Leftrightarrow 4R^3-10R^2r+4Rr^2\ge 0\Leftrightarrow 2R^2-5Rr+2r^2\ge 0\Leftrightarrow(R-2r)(2R-r)\ge 0$$

true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

It is suffices to prove:

$\frac{Rs}{2r}\le\frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}\Leftrightarrow 2s^2\le 27(R^2-2r^2)$ which follows from

$$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$$

It remains to prove:

$$2(4R^2+4Rr+3r^2)\le 27(R^2-2r^2)\Leftrightarrow 8R^2+8Rr+6r^2\le 27R^2-54r^2$$

$\Leftrightarrow 19R^2-8Rr-60r^2\ge 0\Leftrightarrow(R-2r)(19R-30r)\ge 0$ true from $R\ge 2r(Euler)$.

Remark. The inequality can be developed.

4) In ΔABC the following relationship holds:

$$s\le\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{Rs}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. In ΔABC the following relationship holds:

$$\sum\frac{a^2}{b+c}=\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Proof. We have:

$$\sum\frac{a^2}{b+c}=\frac{\sum a^2(a+b)(a+c)}{\prod(b+c)}=\frac{4s^4(s^2-3r^2-4Rr)}{2s(s^2+r^2+2Rr)}=\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Which follows from the identities:

$$\sum a^2(a+b)(a+c)=4s^4(s^2-3r^2-4Rr)$$

$$\prod(b+c)=2s(s^2+r^2+2Rr)$$

Let's get back to the main problem.

For RHD, using lemma, we have:

$$\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}\le\frac{Rs}{2r}\Leftrightarrow 4r(s^2-3r^2-4Rr)\le R(s^2+r^2+2Rr)\Leftrightarrow$$

$$R(s^2+r^2+2Rr)\ge 4r(s^2-3r^2-4Rr)\Leftrightarrow s^2(R-4r)+r(2R^2+17Rr+12r^2)\ge 0$$

We distinguish the cases:

Case 1) If $(R-4r)\ge 0$ is obviously true.

Case 2) If $(R-4r)<0$, inequality it can be write:

$r(2R^2+17Rr+12r^2)\ge s^2(4r-R)$ which is true from

$$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$$

It remains to prove:

$$r(2R^2+17Rr+12r^2)\ge(4R^2+4Rr+3r^2)(4r-R)\Leftrightarrow$$

$$2R^2r+17Rr^2+12r^3\ge -4R^3-4R^2r-3Rr^2+16R^2+16Rr^2+12r^3$$

$$\Leftrightarrow 4R^3-10R^2r+4Rr^2\ge 0\Leftrightarrow 2R^2-5Rr+2r^2\ge 0\Leftrightarrow(R-2r)(2R-r)\ge 0$$

true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

For Lhs, using lemma, we have:

$$\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}\ge s\Leftrightarrow 2(s^2-3r^2-4Rr)\ge s^2+r^2+2Rr\Leftrightarrow$$

$\Leftrightarrow s^2\ge 10Rr+7r^2$ which follows from $s^2\ge 16Rr-5r^2(Gerretsen)$.

It remains to prove: $16Rr-5r^2\ge 10Rr+7r^2\Leftrightarrow R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

SP.333. Let $x,y,z>0$ be positive real numbers such that $x+y+z=3$.

Find the minimum value of expression:

$$P=\frac{x}{2\sqrt{y}+\sqrt{z}}+\frac{y}{2\sqrt{z}+\sqrt{x}}+\frac{z}{2\sqrt{x}+\sqrt{y}}+\frac{(x+y)(y+z)(z+x)}{16}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by proposer

By Cauchy-Schwarz inequality, we have:

$$\frac{x}{2\sqrt{y}+\sqrt{z}}+\frac{y}{2\sqrt{z}+\sqrt{x}}+\frac{z}{2\sqrt{x}+\sqrt{y}}=\frac{x^2}{2x\sqrt{y}+x\sqrt{z}}+\frac{y^2}{2y\sqrt{z}+y\sqrt{x}}+\frac{z^2}{2z\sqrt{x}+z\sqrt{y}}\geq$$

$$\geq\frac{(x+y+z)^2}{2x\sqrt{y}+x\sqrt{z}+2y\sqrt{z}+y\sqrt{x}+2z\sqrt{x}+z\sqrt{y}}=$$

$$=\frac{3^2}{2\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)+\left(y\sqrt{x}+z\sqrt{y}+x\sqrt{z}\right)}$$

Hence

$$\frac{x}{2\sqrt{y}+\sqrt{z}}+\frac{y}{2\sqrt{z}+\sqrt{x}}+\frac{z}{2\sqrt{x}+\sqrt{y}}\geq\frac{9}{2\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)+\left(y\sqrt{x}+z\sqrt{y}+x\sqrt{z}\right)};(1)$$

By CBS inequality, we have:

$$\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2=\left(\sqrt{x}\cdot\sqrt{xy}+\sqrt{y}\cdot\sqrt{yz}+\sqrt{z}\cdot\sqrt{zx}\right)^2\leq$$

$$\leq(x+y+z)(xy+yz+zx)=3(xy+yz+zx)$$

$$x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\leq\sqrt{3(xy+yz+zx)}$$

And similarly:

$$y\sqrt{x}+z\sqrt{y}+x\sqrt{z}\leq\sqrt{3(xy+yz+zx)};\quad(2)$$

From (1),(2) we get:

$$\frac{x}{2\sqrt{y}+\sqrt{z}}+\frac{y}{2\sqrt{z}+\sqrt{x}}+\frac{z}{2\sqrt{x}+\sqrt{y}}\geq\frac{9}{3\sqrt{3(xy+yz+zx)}}=\frac{3}{\sqrt{3(xy+yz+zx)}};(3)$$

We have inequality:

$(x+y)(y+z)(z+x)\geq\frac{8}{9}(x+y+z)(xy+yz+zx)$, **and** $x+y+z=3$ **then**

$$(x+y)(y+z)(z+x)\geq\frac{8(xy+yz+zx)}{3}$$

Hence, let in (3), $t=\sqrt{3(xy+yz+zx)}>0$

By Cauchy-Schwartz inequality, we get:

$$P\geq\frac{3}{t}+\frac{\frac{8t^2}{9}}{16}=\frac{3}{t}+\frac{t^2}{18}=\frac{3}{2t}+\frac{3}{2t}+\frac{t^2}{18}\geq3\sqrt[3]{\frac{3}{2t}\cdot\frac{3}{2t}\cdot\frac{t^2}{18}}=3\sqrt[3]{\frac{1}{8}}=\frac{3}{2}$$

Therefore, $P_{min}=\frac{3}{2}$ **and equality occurs if** $\begin{cases}x+y+z=3\\x=y=z>0\\xy+yz+zx=3\end{cases}\Leftrightarrow x=y=z=1.$

Hence, the minimum value of expression P is $\frac{3}{2}$ for $x=y=z=1$.

Solution 2 by Tran Hong-Dong Thap-Vietnam

By BCS inequality, we have:

$$2\sqrt{y}+\sqrt{z}=\sqrt{y}+\sqrt{y}+\sqrt{z}\le\sqrt{3(y+y+z)}=\sqrt{3(2y+z)}$$

Similarly:

$$2\sqrt{z}+\sqrt{x}\le\sqrt{3(2z+x)}$$

$$2\sqrt{x}+\sqrt{y}\le\sqrt{3(2x+y)}$$

$$\Rightarrow\frac{x}{2\sqrt{y}+\sqrt{z}}+\frac{y}{2\sqrt{z}+\sqrt{x}}+\frac{z}{2\sqrt{x}+\sqrt{y}}\ge\frac{x}{\sqrt{3(2y+z)}}+\frac{y}{\sqrt{3(2z+x)}}+\frac{z}{\sqrt{3(2x+y)}}=$$

$$\overset{\varphi(t)=\frac{1}{\sqrt{t}},t>0}{=}x\varphi\big(3(2y+z)\big)+y\varphi\big(3(2z+x)\big)+z\varphi\big(3(2x+y)\big)=$$

$$=3\left[\frac{x}{3}\varphi\big(3(2y+z)\big)+\frac{y}{3}\varphi\big(3(2z+x)\big)+\frac{z}{3}\varphi\big(3(2x+y)\big)\right]\overset{Jensen}{\ge}$$

$$\ge 3\varphi(3xy+3yz+3zx)=\frac{3}{\sqrt{3(xy+yz+zx)}}=\frac{\sqrt{3}}{\sqrt{xy+yz+zx}};(1)$$

On the other hand,

$$9(x+y)(y+z)(z+x).=8(x+y+z)(xy+yz+zx);(2)$$

$$\Leftrightarrow x(y-z)^2+y(z-x)^2+z(x-y)^2\ge 0,(true\forall x,y,z>0)$$

$$\overset{(2)}{\Rightarrow}\frac{(x+y)(y+z)(z+x)}{16}\ge\frac{8(x+y+z)(xy+yz+zx)}{9\cdot 16}=$$

$$=\frac{8\cdot 3(xy+yz+zx)}{9\cdot 16}=\frac{xy+yz+zx}{6}$$

Let $t=\sqrt{xy+yz+zx}\overset{(1),(2)}{\Rightarrow}$

$$P=\frac{x}{2\sqrt{y}+\sqrt{z}}+\frac{y}{2\sqrt{z}+\sqrt{x}}+\frac{z}{2\sqrt{x}+\sqrt{y}}+\frac{(x+y)(y+z)(z+x)}{16}\ge$$

$$\ge\frac{\sqrt{3}}{t}+\frac{t^2}{6}=\frac{\sqrt{3}}{2t}+\frac{\sqrt{3}}{2t}+\frac{t^2}{6}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{\sqrt{3}\cdot\sqrt{3}\cdot t^2}{2t\cdot 2t\cdot 6}}=\frac{3}{2}\Rightarrow$$

$$P\ge\frac{3}{2}\Rightarrow P_{min}=\frac{3}{2}\Leftrightarrow\begin{cases}t=\sqrt{3}\\x=y=z\end{cases}\Leftrightarrow\begin{cases}xy+yz+zx=3\\x=y=z\end{cases}\Leftrightarrow x=y=z=1.$$

Solution 3 by Mohamed Amine Ben Ajiba-Morocco

By AM-GM inequality, we have:

$$2\sqrt{y}+\sqrt{z}\le (y+1)+\frac{z+1}{2}=\frac{2y+z+3}{2}$$

$$\Rightarrow \sum_{cyc}\frac{x}{2\sqrt{y}+\sqrt{z}}\ge 2\sum_{cyc}\frac{x}{2y+z+3}\overset{Bergstrom}{\ge}\frac{2(\sum x)^2}{\sum x(2y+z+3)}=\frac{6}{3+\sum xy};\ \ (1)$$

$$\frac{(x+y)(y+z)(z+x)}{16}=\frac{(3-x)(3-y)(3-z)}{16}=\frac{3\sum xy-xyz}{16}\overset{(\sum xy)^2\ge 3xyz\sum x}{\ge}$$

$$\ge\frac{27\sum xy-(\sum xy)^2}{144};\ \ (2)$$

From (1),(2) we get: $P\ge\frac{6}{3+a}+\frac{27a-a^2}{144};(a=\sum xy\le 3)$

$$f(a)=\frac{6}{3+a}+\frac{27a-a^2}{144}$$

$$f'(a)=\frac{-2a^3-15(3-a)(3+a)-144(3-a)-54}{144(3+a)^2}<0$$

$$\Rightarrow f\downarrow\Rightarrow a\le 3\Rightarrow P\ge f(a)\ge f(3)=\frac{3}{2}$$

Equality when $x=y=z=1$.

SP.334 Let x,y,z be a positive real numbers such that $x+y+z=1$. Prove that:

$$(3x^2+1)(3y^2+1)(3z^2+1)\ge 27(xy+z)(yz+x)(zx+y)$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by proposer

By CBS inequality, we have:

$$x^2+y^2+z^2=\frac{x^2+y^2}{2}+\frac{y^2+z^2}{2}+\frac{z^2+x^2}{2}\ge xy+yz+zx\ge xy+yz+zx\Leftrightarrow$$

$$x^2+y^2+z^2+2(xy+yz+zx)\ge 3(xy+yz+zx)\Leftrightarrow$$

$$(x+y+z)^2\ge 3(xy+yz+zx)\Leftrightarrow$$

$$1^2 \geq 3(xy+yz+zx) \Leftrightarrow$$

$$3x+1 \geq 3(x^2+xy+yz+zx) = 3(x+y)(x+z)$$

Similarly: $3y+1 \geq 3(y+z)(y+x)$

$3z^2+1 \geq 3(z+x)(z+y)$. Hence

$$(3x^2+1)(3y^2+1)(3z^2+1) \geq 27(x+y)^2(y+z)^2(z+x)^2; \ (1)$$

We have: $x+y+z=1$ then

$$\begin{cases} yz+x = yz+x(x+y+z) = (x+y)(x+z) \\ zx+y = zx+y(x+y+z) = (y+z)(y+x) \\ xy+z = xy+z(x+y+z) = (z+x)(z+y) \end{cases}$$

$$\Rightarrow (xy+z)(yz+x)(zx+y) = (x+y)^2(y+z)^2(z+x)^2; \ (2)$$

From (1),(2) we get:

$$(3x^2+1)(3y^2+1)(3z^2+1) \geq 27(xy+z)(yz+x)(zx+y)$$

Equality occurs if $\begin{cases} x+y+z=1 \\ x=y=z>0 \end{cases} \Leftrightarrow x=y=z=\frac{1}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

$$xyz>0; x+y+z=1 \Rightarrow \prod(3x^2+1) \geq 27\prod(xy+z)$$

$$3x^2+1 = 3x^2+\frac{1}{3}+\frac{1}{3}+\frac{1}{3} \overset{Bergstrom}{\geq} \frac{(3x+1+1+1)^2}{(3+3+3+3)} = \frac{3(x+1)^2}{4}$$

$$3x^2+1 \geq \frac{3}{4}[(x+y)+(x+z)]^2 \overset{AM-GM}{\geq} 3(x+y)(x+z) = 3(yz+x)$$

Therefore, $\prod(3x^2+1) \geq 27\prod(xy+z)$

Equality holds for $x=y=z=\frac{1}{3}$.

Solution 3 by Eldeniz Hesenov-Georgia

$$xyz>0; x+y+z=1 \Rightarrow$$

$$3x^2+1 = \frac{9x^2+3}{3} = \frac{9x^2+1+1+1}{3} \overset{Bergstrom}{\geq} \frac{(3x+1+1+1)^2}{12} =$$

$$= \frac{3}{4}(x+1)^2 = \frac{3}{4}(x+y+z+x)^2; (1)$$

$$LHS = \sum\frac{3}{4}(x+y+z+x)^2 \overset{AM-GM}{\geq} 27\prod(xy+z)$$

Solution 4 by Abdul Aziz-Semarang-Indonesia

$$(x+y+z)^2 \geq 3(xy+yz+zx) \Leftrightarrow$$

$$1 \geq 3(xy+yz+zx) \Leftrightarrow$$

$$3x^2+1 \geq 3(x+y)(x+z) = 3(x+yz)$$

Similarly: $3y^2+1 \geq 3(y+x)(y+z) = 3(y+xz)$

$$3z^2+1 \geq 3(z+x)(z+y) = 3(z+xy)$$

Therefore, $\prod(3x^2+1) \geq 27\prod(xy+z)$

Equality holds for $x=y=z=\frac{1}{3}$.

Solution 5 by Marin Chirciu-Romania

Lemma. If $x,y,z>0$ such that $x+y+z=1$ then:

$$3x^2+1 \geq 3(x+yz)$$

Proof. We have: $3x^2+1 \geq 3(x+y)(x+z);\ (1)$

$$\Leftrightarrow 3x^2+1 \geq 3(x^2+xy+yx+zx) \Leftrightarrow 3x^2+1 \geq 3x^2+3(xy+yz+zx)$$

$\Leftrightarrow 1 \geq 3(xy+yz+zx)$ which follows from

$1 = x+y+z = (x+y+z)^2 \geq 3(xy+yz+zx)$, equality when $x=y=z=\frac{1}{3}$.

From $x+y+z=1$, we get $(x+y)(x+z) = x+yz;\ (2)$ from:

$$(x+y)(x+z) = x^2+xy+yz+zx = x(x+y+z) = yz =$$

$$= x\cdot 1+yz = x+yz$$

From (1),(2) it follows that:

$3x^2+1 \geq 3(x+y)(x+z)$ and $(x+y)(x+z) = x+yz$, thus

$$3x^2+1 \geq 3(x+yz)$$

Let's get back to the main problem.

Using Lemma, we get: $\prod(3x^2+1) \geq 27\prod(xy+z)$

Equality holds for $x=y=z=\frac{1}{3}$.

Solution 6 by Tran Hong-Dong Thap-Vietnam

Let $p = x+y=z;\ 0<q=xy+yz+zx \leq \frac{p^2}{3} = \frac{1}{3} \Rightarrow 27q \leq 9$

$$0 < r = xyz \le \frac{(x+y+z)^3}{27} = \frac{1}{27}. \text{ So,}$$

$$\prod(3x^2+1) \ge 27\prod(xy+z)$$

$$\Leftrightarrow 3\sum x^2 - 18\sum x^2y^2 - 27xyz\sum x^2 - 27xyz + 1 \ge 0$$

$$\Leftrightarrow 3(1-2q) - 18(q^2-2r) - 27r(1-2q) - 27r + 1 \ge 0$$

$$\Leftrightarrow -18q^2 - 6q + 54qr - 18r + 4 \ge 0$$

$$\Leftrightarrow -9q^2 - 3q + 27qr - 9r + 2 \ge 0$$

$$\Leftrightarrow (27q-9)r \ge 9q^2 + 3q - 2$$

$$\Leftrightarrow (9-27q)r \le 2 - 9q^2 - 3q; \ \ (1)$$

$$\text{But} \begin{cases} 9-27q \ge 0 \\ 0 < r \le \frac{1}{27} \end{cases} \Rightarrow (9-27q)r \le (9-27q)\frac{1}{27} \overset{(2)}{\le} 2 - 9q^2 - 3q;$$

$$(2) \Leftrightarrow \frac{1}{3} - q \le 2 - 9q^2 - 3q \Leftrightarrow 9q^2 + 2q - \frac{5}{3} \le 0 \Leftrightarrow$$

$$27q^2 + 6q - 5 \le 0 \Leftrightarrow (3q-1)(9q+5) \le 0 \text{ true by } 0 < q \le \frac{1}{3} \Rightarrow$$

$$3q - 1 \le 0; 9q + 5 > 0 \Rightarrow (2) \Rightarrow (1) \text{ is true.}$$

SP.335. Let $x, y, z > 0$ positive real numbers such that

$$\left(\sqrt{x^3}+\sqrt{y^3}\right)\left(\sqrt{y^3}+\sqrt{z^3}\right)\left(\sqrt{z^3}+\sqrt{x^3}\right) = 8$$

Prove that:

$$x+y+z \ge \sqrt[3]{xyz(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)}$$

Proposed by Hong Le Nhat Tung-Hanoi-Vietnam

Solution 1 by proposer

By Holder's inequality, we have:

$$\left(\sqrt{x^3}+\sqrt{y^3}\right)\left(\sqrt{x^3}+\sqrt{y^3}\right)(1+1) \ge \left(\sqrt{x\cdot x\cdot 1}+\sqrt{y\cdot y\cdot 1}\right)^3$$

$$\Rightarrow 2\left(\sqrt{x^3}+\sqrt{y^3}\right)^2 \ge (x+y)^3$$

Similarly: $2\left(\sqrt{y^3}+\sqrt{z^3}\right)^2 \geq (y+z)^3$

$2\left(\sqrt{z^3}+\sqrt{x^3}\right)^2 \geq (z+x)^3$. Hence

$$8\left(\sqrt{x^3}+\sqrt{y^3}\right)^2\left(\sqrt{y^3}+\sqrt{z^3}\right)^2\left(\sqrt{z^3}+\sqrt{x^3}\right)^2 \geq (x+y)^3(y+z)^3(z+x)^3 \Leftrightarrow$$

$$8\cdot 8^2 \geq (x+y)^3(y+z)^3(z+x)^3 \Leftrightarrow$$

$$(x+y)(y+z)(z+x) \leq 8$$

By CBS inequality we have:

$$\sqrt[3]{xyz(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)} =$$

$$= \sqrt[3]{z(x^2+xy+y^2)x(y^2+yz+z^2)y(z^2+zx+x^2)} \leq$$

$$\leq \frac{z(x^2+xy+y^2)+x(y^2+yz+z^2)+y(z^2+zx+x^2)}{3} \leq$$

$$\leq \frac{xy(x+y+z)+yz(xy+z)+zx(x+y+z)}{3} = \frac{(x+y+z)(xy+yz+zx)}{3}$$

Hence

$$\sqrt[3]{xyz(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)} \leq \frac{(x+y+z)(xy+yz+zx)}{3};\ \ (1)$$

We have: $(x+y)(y+z)(z+x) \overset{AM-GM}{\geq} 2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zx} = 8xyz \Rightarrow$

$$xyz \leq \frac{(x+y)(y+z)(z+x)}{8} \Leftrightarrow$$

$$(x+y)(y+z)(z+x)+xyz \leq (x+y)(y+z)(z+x)+\frac{(x+y)(y+z)(z+x)}{8} \Leftrightarrow$$

$$(x+y+z)(xy+yz+zx) \leq \frac{9(x+y)(y+z)(z+x)}{8} \leq \frac{9\cdot 8}{8} \Leftrightarrow$$

$$(x+y+z)(xy+yz+zx) \leq 9$$

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq xy+yz+zx+2(xy+yz+zx)$$
$$= 3(xy+yz+zx) \Rightarrow$$

$$x+y+z \geq \sqrt{3(xy+yz+zx)}$$

Hence $9 \geq (x+y+z)(xy+yz+zx) \geq \sqrt{3(xy+yz+zx)}(xy+yz+zx) =$

$$\sqrt{3(xy+yz+zx)^3} \Rightarrow$$

$$81 \geq 3(xy+yz+zx)^3 \Leftrightarrow (xy+yz+zx)^3 \leq 27 \Leftrightarrow xy+yz+zx \leq 3; \quad (2)$$

From (1),(2) we get:

$$\sqrt[3]{xyz(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)} \leq \frac{3(x+y+z)}{3} = x+y+z \Rightarrow$$

$$x+y+z \geq \sqrt[3]{xyz(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)}$$

Equality occurs if $\begin{cases} \left(\sqrt{x^3}+\sqrt{y^3}\right)\left(\sqrt{y^3}+\sqrt{z^3}\right)\left(\sqrt{z^3}+\sqrt{x^3}\right) = 8 \\ x = y = z > 0 \end{cases} \Leftrightarrow x = y = z = 1.$

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

$$xyz\prod_{cyc}(x^2+xy+y^2) = \prod_{cyc} z(x^2+xy+y^2) = \prod_{cyc}(x^2z+xyz+y^2z) \overset{AM-GM}{\leq}$$

$$\overset{AM-GM}{\leq} \left(\frac{\sum(x^2z+xyz+y^2z)}{3}\right)^3 = \frac{(\sum x)^3(\sum xy)^3}{27}$$

$$\Rightarrow \sqrt[3]{xyz\prod_{cyc}(x^2+xy+y^2)} \leq \frac{1}{3}\left(\sum xy\right)\left(\sum x\right)$$

It is suffice to prove that: $\sum xy \leq 3$**. We have:**

$$8 = \prod_{cyc}\left(\sqrt{x^3}+\sqrt{y^3}\right) \overset{AM-GM}{\geq} 8\sqrt{(xyz)^3} \Rightarrow 1 \geq xyz$$

$$\prod_{cyc}\left(\sqrt{x^3}+\sqrt{y^3}\right) = 9 \Rightarrow \left(\sum \sqrt{x^3}\right)\left(\sum \sqrt{(xyz)^3}\right) = 8 + \sqrt{(xyz)^3} \leq 9$$

$$3\sum \sqrt{x^3}\sqrt{y^3} \leq \left(\sum \sqrt{x^3}\right)^2 \Rightarrow 9 \geq \sqrt{3}\sqrt{\left(\sum \sqrt{(xy)^3}\right)^3}$$

By Power Mean Inequality $\frac{1}{3}\sum xy \leq \left(\frac{1}{3}\sum \sqrt{(xy)^3}\right)^{\frac{2}{3}} \leq 1 \Rightarrow \sum xy \leq 3$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$x+y+z \geq \sqrt[3]{xyz(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)}$$

$$\Leftrightarrow \frac{x+y+z}{\sqrt[3]{xyx}} \geq \sqrt[3]{(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)}$$

From AM-GM inequality: $\frac{x+y+z}{3} \geq \sqrt[3]{xyz} \Rightarrow \frac{x+y+z}{\sqrt[3]{xyz}} \geq 3$**. Let us prove that:**

$$\sqrt[3]{(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)} \le 3$$

$$\Leftrightarrow \frac{2x^2+2y^2+2z^2+xy+yz+zx}{3} \le 3$$

$$\Leftrightarrow 2x^2+2y^2+2z^2+xy+yz+zx \le 9$$

Let be the function: $f(x,y,z) = \left(\sqrt{x^3}+\sqrt{y^3}\right)\left(\sqrt{y^3}+\sqrt{z^3}\right)\left(\sqrt{z^3}+\sqrt{x^3}\right)$, the local maximum of $f(x,y,z) = 8$ is occurred when $x = y = z = 1$, so

$$\frac{2x^2+2y^2+2z^2+xy+yz+zx}{9} \le 1$$

$$8 = \prod_{cyc}\left(\sqrt{x^3}+\sqrt{y^3}\right) \overset{AM-GM}{\ge} 8\sqrt{(xyz)^3} \Rightarrow 1 \ge xyz$$

$$\frac{1}{3}\sum xy \le \left(\frac{1}{3}\sum\sqrt{(xy)^3}\right)^{\frac{2}{3}} \le 1 \Rightarrow \sum xy \le 3$$

SP.336. Let x, y, z be a positive real numbers such that

$$(x^6+y^6)(y^6+z^6)(z^6+x^6) = 8$$

Prove that:

$$(3x^2-4xy+3y^2)(3y^2-4yz+3z^2)(3z^2-4zx+3x^2) \ge 8$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by proposer

By CBS inequality we have:

$$\sqrt[3]{\frac{x^6+y^6}{2}} = \sqrt[3]{\frac{(x^2+y^2)(x^4-x^2y^2+y^4)}{2}} =$$

$$= \sqrt[3]{\frac{x^2+y^2}{2}\cdot\frac{x^2-xy\sqrt{3}+y^2}{2-\sqrt{3}}\cdot\frac{x^2+xy\sqrt{3}+y^2}{2+\sqrt{3}}} \le$$

$$\frac{\frac{x^2+y^2}{2}+\frac{x^2-xy\sqrt{3}+y^2}{2-\sqrt{3}}+\frac{x^2+xy\sqrt{3}+y^2}{2+\sqrt{3}}}{3} = \frac{\frac{9}{2}x^2-6xy+\frac{9}{2}y^2}{3} =$$

$$=\frac{3x^2-4xy+3y^2}{2}\Rightarrow 3x^2-4xy+3y^2\ge 2\sqrt[3]{\frac{x^6+y^6}{2}}$$

Similarly: $3y^2-4yz+3z^2\ge 2\sqrt[3]{\frac{y^6+z^6}{2}}$

$3z^2-4zx+3x^2\ge \sqrt[3]{\frac{z^6+x^6}{2}}$. Hence,

$$(3x^2-4xy+3y^2)(3y^2-4yz+3z^2)(3z^2-4zx+3x^2)$$

$$\ge 8\sqrt[3]{\frac{(x^6+y^6)(y^6+z^6)(z^6+x^6)}{8}}\Leftrightarrow$$

$$(3x^2-4xy+3y^2)(3y^2-4yz+3z^2)(3z^2-4zx+3x^2)\ge 8\sqrt[3]{\frac{8}{8}}=8$$

Equality holds if $\begin{cases}(x^6+y^6)(y^6+z^6)(z^6+x^6)=8\\ x=y=z>0\end{cases}\Leftrightarrow x=y=z=1.$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Lemma. If $x,y>0$ then $3x^2-4xy+3y^2\ge 2\sqrt[3]{\frac{x^6+y^6}{2}}$; (1)

Proof. $(1)\Leftrightarrow (3x^2-4xy+3y^2)^3\ge 4(x^6+y^6), t=\frac{x}{y}>0\Rightarrow$

$$(3t^2-4t+3)^3\ge 4(t^6+1)\Leftrightarrow$$

$$23t^6-108t^5+225t^4-280t^3+225t^2-108t+23\ge 0\Leftrightarrow$$

$$(t-1)^4(23t^2-16t+23)\ge 0\Leftrightarrow 23(t-1)^4\left[\left(t-\frac{8}{23}\right)^2+\frac{304}{23}\right]\ge 0$$

Which is true for $t>0$. Equality holds$\Leftrightarrow t=1\Leftrightarrow x=y$. Now,

$$\Omega_1=\prod_{cyc}(3x^2-4xy+3y^2)\ge 8\sqrt[3]{\frac{(x^6+y^6)(y^6+z^6)(z^6+x^6)}{8}}=8$$

Equality holds$\Leftrightarrow \begin{cases}x=y=z\\(x^6+y^6)(y^6+z^6)(z^6+x^6)=8\end{cases}\Leftrightarrow x=y=z=1.$

Solution 3 by Ravi Prakash-New Delhi-India

We first show that: $(3x^2-4xy+3y^2)^3\ge 4(x^6+y^6),\forall x,y>0$; (1)

Let us denote: $x = rcos\theta, y = rsin\theta, r > 0, \theta \in \left(0, \frac{\pi}{2}\right)$, then (1) becomes:

$$\left(3 - 2sin(2\theta)\right)^3 \geq 4(sin^6\theta + cos^6\theta) \Leftrightarrow$$

$$\left(3 - 2sin(2\theta)\right)^3 \geq 4[(sin^2\theta + cos^2\theta)^3 - 3sin^2\theta cos^2\theta(sin^2\theta + cos^2\theta)] \Leftrightarrow$$

$$(3 - 2sin2\theta)^3 \geq 4\left(1 - \frac{3}{4}sin^2 2\theta\right) = 4 - 3sin^2 2\theta \Leftrightarrow$$

$$23 - 54sin2\theta + 36sin^2 2\theta - 8ain^3 2\theta \geq 0 \Leftrightarrow$$

$$(1 - sin2\theta)(8sin^2 2\theta - 31sin2\theta + 23) \geq 0 \Leftrightarrow$$

$$(1 - sin2\theta)^2(23 - 8sin2\theta) \geq 0$$

Equality in (1) if and only if $sin2\theta = 1 \Leftrightarrow \theta = \frac{\pi}{4} \Leftrightarrow x = y$.

In view of (1):

$$\prod_{cyc}(3x^2 - 4xy + 3y^2)^3 \geq 4^3(x^6 + y^6)(y^6 + z^6)(z^6 + x^6) = 4^3 \cdot 8$$

$$(3x^2 - 4xy + 3y^2)(3y^2 - 4yz + 3z^2)(3z^2 - 4zx + 3x^2) \geq 8$$

Equality when $x = y = z = 1$.

SP.337 Let $x, y, z > 0$.

1) If $xy + yz + zx \leq 3\left(2\sqrt{3} - 3\right)$ then

$$\sqrt{\frac{xy + yz + zx}{3}} + 1 \leq \sqrt[3]{(x+1)(y+1)(z+1)}.$$

2) If $xy + yz + zx > 3\left(2\sqrt{3} - 3\right)$ then

$$\sqrt{xy + yz + zx} + 1 < \sqrt{(x+1)(y+1)(z+1)}.$$

Proposed by Florentin Vișescu-Romania

Solution by proposer

Let $xy + yz + zx = k > 0,\ z = \frac{k - xy}{x + y}$

Because $x, y, z > 0 \Rightarrow k - xy > 0$, or $x < \frac{k}{y}$.

Let $f(x)=(x+1)(y+1)(z+1)=(x+1)(y+1)\left(\frac{k-xy}{x+y}+1\right), f:\left(0,\frac{k}{y}\right)\to\mathbb{R}.$

$$f'(x)=(1-y^2)\left(\frac{(x+y)^2-(y^2+k)}{(x+y)^2}\right)=$$

$$=\frac{(1-y)^2\left(x+y-\sqrt{y^2+k}\right)\left(x+y+\sqrt{y^2+k}\right)}{(x+y)^2}.$$

1. If $y=1\Rightarrow f'(x)=0\Rightarrow f(x)=\text{ct}=2(x+1)\left(\frac{k-x}{x+1}+1\right)=2(k+1).$

2. If $y<1\Rightarrow 1-y^2>0.$

$$f'(x)=0\Rightarrow x+y-\sqrt{y^2+k}=0\Rightarrow x=\sqrt{y^2+k}-y>0.$$

We show that: $\sqrt{y^2+k}-y<\frac{k}{y}\Leftrightarrow \sqrt{y^2+k}<\frac{y^2+k}{y}\Leftrightarrow y<\sqrt{y^2+k}\Leftrightarrow y^2<y^2+k\Leftrightarrow$

$0<k$, **true.**

x	0 $\quad$ $\sqrt{y^2+k}-y$ $\quad$ $\frac{k}{y}$
$f'(x)$	$- - - - - - - - - -0 + + + + + + + + + +$
$f(x)$	↘ ↘ ↘ ↘ ↘ ↘ 1 ↗ ↗ ↗ ↗ ↗ ↗

$$f\left(\sqrt{y^2+k}-k\right)=(y+1)\left(\sqrt{y^2+k}-y+1\right)^2.(1)$$

Then $f(x)\ge(y+1)\left(\sqrt{y^2+k}-y+1\right)^2, \forall x\in\left(0,\frac{k}{y}\right), y<1.$

Let $g(y)=(y+1)\left(\sqrt{y^2+k}-y+1\right)^2, g:(0,1)\to\mathbb{R}.$

$$g'(y)=\left(\sqrt{y^2+k}-y+1\right)\cdot\frac{3y^2-3y\sqrt{y^2+k}-\sqrt{y^2+k}+k+2y}{\sqrt{y^2+k}}.$$

$$g'(y)=0\Rightarrow 3y^2-3y\sqrt{y^2+k}-\sqrt{y^2+k}+k+2y=0\Leftrightarrow$$

$$3y^2+2y+k=\sqrt{y^2+k}(3y+1)\Leftrightarrow 6y^3+3y^2-3y^2k-2ky+k^2-k=0\Leftrightarrow$$

$$(2y+1-k)(3y^2-k)=0\Rightarrow y_1=\frac{k-1}{2}\text{sau}y_2=\sqrt{\frac{k}{3}}.$$

a)If $k\le 1\Rightarrow y_1=\frac{k-1}{2}\le 0$ **and** $0<y_2=\sqrt{\frac{k}{3}}<1.$

y	0 $\qquad\qquad\qquad \sqrt{k/3} \qquad\qquad\qquad 1$
$g'(y)$	$- - - - - - - - - - 0 + + + + + + + + + +$
$g(y)$	↘ ↘ ↘ ↘ $\quad 2 \quad$ ↗ ↗ ↗ ↗

$$g\left(\sqrt{\frac{k}{3}}\right) = \left(\sqrt{\frac{k}{3}} + 1\right) \cdot \left(\sqrt{\frac{k}{3} + k} - \sqrt{\frac{k}{3}} + 1\right)^2 = \left(\sqrt{\frac{k}{3}} + 1\right)^3 . (2)$$

Then $f(x) \geq \left(\sqrt{\frac{k}{3}} + 1\right)^3, \forall\, x \in \left(0; \frac{k}{y}\right), y < 1, k \leq 1.$

b) If $k \in (1; 3] \Rightarrow 0 < \frac{k-1}{2} \leq \sqrt{\frac{k}{3}} \leq 1.$

We show that:

$$\frac{k-1}{2} \leq \sqrt{\frac{k}{3}} \Leftrightarrow 3k^2 - 6k + 3 \leq 4k \Leftrightarrow 3k^2 - 10k + 3 \leq 0 \Leftrightarrow k \in \left[\frac{1}{3}; 3\right], \text{true}.$$

y	0	$\frac{k-1}{2}$		$\sqrt{\frac{k}{3}}$	1
$g'(y)$	$+ + + +$	0	$- - - - - - - -$	0	$+ + + +$
$g(y)$	↗ ↗ ↗ ↗	3	↘ ↘ ↘ ↘	2	↗ ↗ ↗

$$\lim_{\substack{y \to 0 \\ y > 0}} g(y) = \left(\sqrt{k} + 1\right)^2 . (3)$$

$$g\left(\sqrt{\frac{k}{3}}\right) = \left(\sqrt{\frac{k}{3}} + 1\right)^3 . (2)$$

Compare $\left(\sqrt{k} + 1\right)^2$ with $\left(\sqrt{\frac{k}{3}} + 1\right)^3$.

$$\left(\sqrt{k} + 1\right)^2 < \left(\sqrt{\frac{k}{3}} + 1\right)^3 \Leftrightarrow k + 2\sqrt{k} + 1 < \frac{k}{3}\sqrt{\frac{k}{3}} + 3\frac{k}{3} + 3\sqrt{\frac{k}{z}} + 1 \Leftrightarrow k > 3\left(2\sqrt{3} - 3\right).$$

Then$f(x)\geq\left(\sqrt{\frac{k}{3}}+1\right)^3$, $\forall x\in\left(0,\frac{k}{y}\right), y<1, k\in\left(1;3\left(2\sqrt{3}-3\right)\right]$

and $f(x)>\left(\sqrt{k}+1\right)^2, \forall x\in\left(0;\frac{k}{y}\right), y<1, k\in\left(3\left(2\sqrt{3}-3\right);3\right]$.

c) If $k>3\ \Rightarrow\ 1<\sqrt{\frac{k}{3}}<\frac{k-1}{2}$.

y	0 $\qquad\qquad\qquad\qquad$ 1
$g'(y)$	$+++++++++++++++++$
$g(y)$	3 ↗ ↗ ↗ ↗ ↗ ↗ ↗

$$\lim_{\substack{y\to 0\\ y>0}} g(y)=\left(\sqrt{k}+1\right)^2. (3)$$

Then $f(x)>\left(\sqrt{k}+1\right)^2, \forall x\in\left(0;\frac{k}{y}\right), y<1, k\in(3,\infty)$.

3)If $y>1\Rightarrow 1-y^2<0$.

x	0 $\qquad$ $\sqrt{y^2+k}-y$ $\qquad$ $\frac{k}{y}$
$f'(x)$	$++++++++0---------$
$f(x)$	↗ ↗ ↗ ↗ ↗ ↗ 4 ↘ ↘ ↘ ↘ ↘

$\lim_{\substack{x\to 0\\ x>0}} f(x)=(y+1)\left(\frac{k}{y}+1\right). (4), \lim_{x\to\frac{k}{y}} f(x)=\left(\frac{k}{y}+1\right)(y+1). (4')$

Then $f(x)>(y+1)\left(\frac{k}{y}+1\right), \forall x\in\left(0;\frac{k}{y}\right), y>1$.

Let $h(y)=(y+1)\left(\frac{k}{y}+1\right), h:(1;\infty)\to\mathbb{R}. h'(y)=\frac{k}{y}+1-\frac{k}{y^2}(y+1)$

$$=\frac{k}{y}+1-\frac{k}{y}-\frac{k}{y^2}=\frac{y^2-k}{y^2}. h'(y)=0\Rightarrow y^2=k\Rightarrow y=\sqrt{k}.$$

a) If $k\in(0;1]\ \Rightarrow y=\sqrt{k}\leq 1$.

y	0 $\qquad\qquad\qquad\qquad$ ∞
$h'(y)$	$+++++++++++++++++$
$h(y)$	5 ↗ ↗ ↗ ↗ ↗ ↗ ↗

$$\lim_{\substack{y\to 1\\ y>1}} h(y) = 2(k+1).\ (5)$$

Then $f(x) > 2(k+1), \forall\, x \in \left(0;\frac{k}{y}\right), y > 1, k \in (0,1]$.

b) If $k > 1 \Rightarrow y = \sqrt{k} > 1$

y	1 $\qquad\qquad$ $\sqrt{k}$ $\qquad\qquad$ ∞
$h'(y)$	$- - - - - - - - 0 + + + + + + + +$
$h(y)$	$\searrow \searrow \searrow \searrow \searrow \searrow$ 6 $\nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow$

$$h(\sqrt{k}) = (\sqrt{k}+1)^2.\ (6)$$

Then $f(x) > (\sqrt{k+1})^2, \forall\, x \in \left(0;\frac{k}{y}\right), y > 1, k > 1$.

I showed that:

for $k \in (0,1] \left(\sqrt{\frac{k}{3}}+1\right)^3 < 2(k+1) \Leftrightarrow \frac{k}{3}\sqrt{\frac{k}{3}} + 3\cdot\frac{k}{3} + 3\sqrt{\frac{k}{3}} + 1 < 2k+2 \Leftrightarrow$

$$\frac{k}{3}\sqrt{\frac{k}{3}} - 3\frac{k}{3} + 3\sqrt{\frac{k}{3}} - 1 < 0 \Leftrightarrow \left(\sqrt{\frac{k}{3}}-1\right)^3 < 0 \Leftrightarrow k < 3, \text{true}.$$

In conclusion:

For $xy+yz+zx \le 3(2\sqrt{3}-3) \Rightarrow \sqrt{\frac{xy+yz+zx}{3}} + 1 \le \sqrt[3]{(x+1)(y+1)(z+1)}$.

For $xy+yz+zx > 3(2\sqrt{3}-3) \Rightarrow \sqrt{xy+yz+zx} + 1 < \sqrt{(x+1)(y+1)(z+1)}$.

SP.338 If $t \in [0,2\pi); n \in \mathbb{N}$ then:

$$|1+\cos nt + i\sin nt| + |1+\cos 2nt + i\sin 2nt| +$$
$$+|1+\cos 3nt + i\sin 3nt| \ge 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$2|cos\theta| = \sqrt{(1+cos2\theta)^2 + sin^2 2\theta} = |1+cos2\theta + isin2\theta| = \left|1+e^{2i\theta}\right|$$

$$LHS=\frac{1}{2}\left[2\left|1+e^{2i\theta}\right|+\left|1+e^{4i\theta}\right|+\left|1+e^{8i\theta}\right|+\left|1+e^{10i\theta}\right|\right]=$$

$$=\frac{1}{2}\left[2\left|1+e^{2i\theta}\right|+\left|e^{2i\theta}\right|\left|1+e^{4i\theta}\right|+\left|1+e^{6i\theta}\right|+\left|1+e^{8i\theta}\right|\left|e^{2i\theta}\right|+\left|1+e^{10i\theta}\right|\right]=$$

$$=\frac{1}{2}\left[\left|1+e^{2i\theta}\right|+\left|-e^{2i\theta}-e^{6i\theta}\right|+\left|1+e^{6i\theta}\right|+\left|1+e^{2i\theta}\right|+\left|-e^{2i\theta}-e^{-10i\theta}\right|\right.$$
$$\left.+\left|1+e^{10i\theta}\right|\right]\geq$$

$$\geq\frac{1}{2}\left[\left|1+e^{2i\theta}-e^{2i\theta}-e^{6i\theta}+1+e^{6i\theta}\right|+\left|1+e^{2i\theta}-e^{2i\theta}-e^{10i\theta}+1+e^{10i\theta}\right|\right]=2$$

$$|1+cos(nt)+isin(nt)|+|1+cos(2nt)+isin(2n)|+|1+cos(3nt+isin(3nt)|=$$

$$=\left|1+e^{int}\right|+\left|e^{int}\right|\left|1+e^{2nit}\right|+\left|1+e^{3nit}\right|\overset{e^{int}=1}{=}$$

$$=\left|1+e^{int}\right|+\left|-e^{int}-3e^{3int}\right|+\left|1+e^{3nit}\right|\geq 1+e^{int}-e^{int}-e^{3int}+1+e^{3int}=2$$

Solution 2 by Florentin Vişescu-Romania

$$|1+\cos nt+i\sin nt|+|1+\cos 2nt+i\sin 2nt|+$$
$$+|1+\cos 3nt+i\sin 3nt|\geq 2\Leftrightarrow$$

$$\left|2cos^2\frac{nt}{2}+2isin\frac{nt}{2}cos\frac{nt}{2}\right|+\left|2cos^2nt+2isinntcosnt\right|$$
$$+\left|2cos^2\frac{3nt}{2}+2isin\frac{3nt}{2}cos\frac{3nt}{2}\right|\geq 2$$

$$2\left|cos\frac{nt}{2}\right|+2|cosnt|+2\left|cos\frac{3nt}{2}\right|\geq 2$$

$$\left|cos\frac{nt}{2}\right|+|cosnt|+\left|cos\frac{3nt}{2}\right|\geq 1\overset{\frac{nt}{2}=\alpha}{\Longrightarrow}|cos\alpha|+|cos2\alpha|+|cos3\alpha|\geq 1$$

$$|cos\alpha|+\left|2cos^2\alpha-1\right|+\left|4cos^3\alpha-3cos\alpha\right|\geq 1$$

Let us denote $cos\alpha=x\in[-1,1]\Rightarrow|x|+\left|2x^2-1\right|+|x|\left|4x^2-3\right|\geq 1$

Case 1) $x\in\left[0,\frac{\sqrt{2}}{2}\right]:x-2^2+1-4x^3+x\geq 1\Leftrightarrow-2x(2x^2+x-2)\geq 0$ **true from**

$$2x^2+x-2\leq 0,\forall x\in\left[0,\frac{\sqrt{2}}{2}\right]$$

Case 2) $x\in\left[\frac{\sqrt{2}}{2},\frac{\sqrt{3}}{2}\right]:x+2x^2-1-4x^3+3x\geq 1\Leftrightarrow(x-1)(x+1)(2x-1)\leq 0$ **true.**

Case 3) $x\in\left[\frac{\sqrt{3}}{2},1\right]:x+2x^2-1+4x^3-3x-1\geq 0\Leftrightarrow 2x^3+x^2-x-1\geq 0$

Let $f(x)=2x^3+x^2-x-1, f'(x)=6x^2+2x-1\geq 0, \forall x\in\left[\frac{\sqrt{3}}{2},1\right]\Rightarrow$

$$f(x)\geq f\left(\frac{\sqrt{3}}{2}\right)>0$$

Solution 3 by Marian Dincă-Romania

$$\text{Let } cos(nt)+isin(nt)=z, z\in\mathbb{C}, |z|=1=|-z|$$

$$|1+cos(nt)+isin(nt)|+|1+cos(2nt)+isin(2nt)|+|1+cos(3nt)+isin(3nt)|$$

$$=|1+z|+\left|1+z^2\right|+\left|1+z^3\right|=|1+z|+|-z|\left|1+z^2\right|+\left|1+z^3\right|=$$

$$=|1+z|+\left|-z-z^3\right|+\left|1+z^3\right|\geq\left|1+z-z-3z+1+z^3\right|=|2=2$$

Use triangle inequality: $|x|+|y|+|z|\geq|x+y+z|, \forall x,y,z\in\mathbb{C}$

Solution 4 by proposer

$$\text{Let } a=\cos t+i\sin t\in\mathbb{C}; |a|=1\Rightarrow|a^n|=1$$

$$a^n=\cos nt+i\sin nt;$$

$$a^{2n}=\cos 2nt+i\sin 2nt; a^{3n}=\cos 3nt+i\sin 3nt$$

Inequality asked can be written:

$$|1+a^n|+\left|1+a^{2n}\right|+\left|1+a^{3n}\right|\geq 2$$

$$2=2\cdot 1=2\cdot|a^n|=2|-a^n|=|-2a^n|=$$

$$=\left|a^{3n}+1-a^n(a^{2n}+1)-(a^n+1)\right|\leq\left|a^{3n}+1\right|+|a^n|\cdot\left|a^{2n}+1\right|+|a^n+1|=$$

$$=\left|a^{3n}+1\right|+1\cdot\left|a^{2n}+1\right|+|a^n+1|$$

Hence: $|1+a^n|+\left|1+a^{2n}\right|+\left|1+a^{3n}\right|\geq 2$

SP.339. Solve for real numbers:

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by proposer

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7;\ \ (1)$$

Let $\begin{cases} x^3-2x^2+2x\ge 0 \\ 4x-3x^4\ge 0\end{cases} \Leftrightarrow \begin{cases} x(x^2-2x+2)\ge 0 \\ x(3x^3-4)\ge 0\end{cases} \Leftrightarrow 0\le x\le \sqrt[3]{\frac{4}{3}}$

We have: $x^2-x+1=\left(x^2-x+\frac{1}{4}\right)+\frac{3}{4}=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\ge\frac{3}{4}>0$

Hence, by AM-GM inequality we have:

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=$$

$$\sqrt{x(x^2-2x+2)}+3\sqrt[3]{(x^2-x+1)\cdot 1\cdot 1}+2\sqrt[4]{x(4-3x^3)\cdot 1\cdot 1}\le$$

$$\le\frac{x+x^2-2x+2}{2}+x^2-2x+2+1+1+\frac{2(x+(4-3x^3)+1+1)}{4}$$

Hence,

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}\le\frac{-3x^3+3x^2-2x+14}{2};\quad (2)$$

From (1),(2) we get:

$$\frac{x^4-3x^3}{2}+7\le\frac{-3x^3+3x^2-2x+14}{2}\Leftrightarrow x^4-3x^2+2x\le 0\Leftrightarrow$$

$$x(x+2)(x-1)^2\le 0;(3)$$

Because $x\ge 0$ then $x(x+2)\ge 0,(x-1)^2\ge 0$ then $x(x+2)(x-1)^2\ge 0;(4)$

From (3),(4) it follows that: $x(x+2)(x-1)^2=0$

Equality $x(x+2)(x-1)^2=0$ holds for:

$$\begin{cases} x=x^2-2x+2 \\ x^2-x+1=1 \\ x=4-3x^3=1 \\ x(x+2)(x-1)^2=0\end{cases}\Leftrightarrow\begin{cases}(x-1)(x-2)=0 \\ x(x-1)=0 \\ 3x^3+x-4=0 \\ x(x+2)(x-1)^2=0\end{cases}\Leftrightarrow x=1$$

Hence, the solution of equation is $S=\{1\}$.

Solution 2 by Agayev Seddredin-Azerbaijan

$$\sqrt{x^3-2x^2+2x}=\sqrt{x(x^2-2x+2)}\le\frac{x+x^2-2x+2}{2}=\frac{x^2-x+2}{2}$$

$$\sqrt[3]{x^2-x+1}=\sqrt[3]{(x^2-x+1)\cdot 1\cdot 1}\le\frac{x^2-x+1+1+1}{3}=\frac{x^2-x+3}{3}$$

$$\sqrt[4]{4x-3x^4}=\sqrt[4]{(4x-3x^4)\cdot 1\cdot 1\cdot 1}\le\frac{4x-3x^4+1+1+1}{4}=\frac{3+4x-3x^4}{4}$$

$$\frac{x^4-3x^3}{2}+7\le\frac{x^2-x+2}{2}+3\frac{x^2-x+3}{3}+2\frac{3+4x-3x^4}{4}$$

$$\Rightarrow 4x^4-3x^3-3x^2-x+3\le 0$$

$$\Rightarrow (x-1)^2(4x^2+5x+3)\le 0\Rightarrow x=1.$$

Solution 3 by Nikos Ntorvas-Greece

It must be: $x^3-2x^2+2x\ge 0, 4x-3x^4\ge 0\Leftrightarrow x(x^2-2x+2)\ge 0, x(3x^3-4)\le 0\Leftrightarrow$

$$x((x-1)^2+1)\ge 0, 3x\left(x^3-\frac{4}{3}\right)\le 0\Leftrightarrow x\in\left[0,\sqrt[3]{\frac{4}{3}}\right]$$

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7;\ \ (1)$$

$$I=\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}\le$$

$$\le\frac{x+2+x^2-2x}{2}+(x^2-x+1)+1+1+\frac{2(x+(4-3x^3)+1+1)}{4}\Leftrightarrow$$

$$I\le\frac{-3x^3+3x^2-2x+14}{2}$$

$$(1)\Leftrightarrow\frac{x^4-3x^3}{2}+7\le\frac{-3x^3+3x^2-2x+14}{2}\Leftrightarrow x^4-3x^2+2x\le 0\Leftrightarrow$$

$$x(x^3-3x+2)\le 0\Leftrightarrow x(x+2)(x-1)^2\le 0$$

But $x(x+2)(x-1)^2\ge 0$ because $x\in\left[0,\sqrt[3]{\frac{4}{3}}\right]$

It must be: $x(x+2)(x-1)^2=0\Leftrightarrow x_1=-2, x_2=0, x_3=1$

For $x_1=-2$: $-2\notin\left[0,\sqrt[3]{\frac{4}{3}}\right]$. False!

For $x_2=0$: from (1) we have: $3=7$. False!

For $x_3=1$: from (1) we have that: $6=6$. True!

The unique solution of (1) is $x=1$.

SP.340. Find all pairs of integers (x,y) such that

$$x^4-2x^2-y^2-5y-3=0$$

Proposed by George Apostolopoulos-Messsolonghi-Greece

Solution 1 by proposer

We have:

$$x^4-2x^2-y^2-5y-3=0$$

$$x^4+(x^2y-x^2y)+(x^2-3x^2)-y-y^2-3y-3=y$$

$$(x^4-x^2y-3x^2)+(x^2y-y^2-3y)+(x^2-y-3)=y$$

$$y=(x^2-y-3)(x^2+y+1).$$

Denote: $t=x^2-y-3$ **then** $x^2=t+y+3$ **and**

$$y=t(t+2y+4)\Leftrightarrow y(1-2t)=t^2+4t$$

So, $y=\frac{t^2+4t}{-2t+1}$; $\left(if\ t=\frac{1}{2}\Rightarrow y\cdot 0=\frac{1}{4}+2\Rightarrow t\neq\frac{1}{2}\right)$

$$y=\frac{t^2+4t}{-2t+1}=-\frac{t}{2}-\frac{9}{4}+\frac{9}{4}\cdot\frac{1}{-2t+1}$$

$$4y=-2t-9+\frac{9}{-2t+1}$$

Since y **is an integer, we see that** $-2t+1$ **divides** 9**, so**

$$-2t+1\in\{\pm 1,+3,\pm 9\}$$

We get $t\in\{-,1,1,2,\}$ **and hence**

$$(x,y)\in\{(1,-1),(-1,-1),(1,-4),(-1,-4)\}$$

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

$$x^4-2x^2+1=y^2+5y+4$$

$$\left(2(x^2-1)\right)^2=(2y)^2+2\cdot 2y\cdot 5+5^2-9$$

$$\left(2(x^2-1)\right)^2-(2y+5)^2=-9$$

$$(2y+5)^2-\left(2(x^2-1)\right)^2=9$$

$$(2y+5+2x^2-2)(2y+5-2x^2+5)=9$$

For $x_{1,2}=\pm 1\Rightarrow y_1=-1, y_2=-4$

Let $x\neq 1$**, distinguish the following cases:**

$$1)\ \begin{cases}2y+5+2x^2-2=9\\2y+5-2x^2+2=1\end{cases}\Rightarrow 4y+10=10\Rightarrow y=0\Rightarrow$$

$$2x^2 = 6(\text{no solution in } \mathbb{Z})$$

$$2)\ \begin{cases} 2y+5-2x^2-2=-1 \\ 2y+5-2x^2+2=-9 \end{cases} \Rightarrow 4y+10=-10 \Rightarrow y=-5 \Rightarrow$$

$$2x^2 = 6\ (\text{no solution in } \mathbb{Z})$$

Solution 3 by Ravi Prakash-New Delhi-India

$$x^4-2x^2-y^2-5y-3=0 \Leftrightarrow (x^2-1)^2=y^2+5y+4;(1)$$

$$y^2+5y+4-(x^2-1)^2=0 \Rightarrow y_{1,2}=\frac{1}{2}\left(-5\pm\sqrt{9+4(x^2-1)^2}\right)$$

For $y\in\mathbb{Z}, 9+4(x^2-1)^2$ must be a perfect square of an odd integer. Thus,

$$t^2=4(x^2-1)^2+9 \Rightarrow t^2-(x^2-1)^2=9 \text{ this is possible if}$$

$$t^2=25,(2x^2-2)^2=16 \Rightarrow t=\pm5, 2x^2=\pm4 \Rightarrow t=\pm5, 2x^2\in\{-1,3\} -\text{impossible.}$$

$$\text{Next, } t^2=9,(x^2-1)^2=0 \Rightarrow t=\pm3, x=\pm1$$

$$\text{When } x=1, y^2+5y+4=0 \overset{(1)}{\Rightarrow} y\in\{-4,-1\}$$

Therefore,

$$S=\{(1,-1),(1,-4),(-1,-1),(-1,-4)\}$$

SP.341. Let a,b,c be positive real numbers such that

$abc+ab+bc+ca=4$. Find the maximum value of expression:

$$T=\frac{1}{\sqrt{2a^5+b^3-2a^2+26}}+\frac{1}{\sqrt{2b^5+c^3-2b^2+26}}+\frac{1}{\sqrt{2c^5+a^3-2c^2+26}}$$

Proposed by Hoang Le Nhat-Tung-Hanoi-Vietnam

Solution 1 by proposer

$$a^5-a^2-3a+3=a^4(a-1)+a^3(a-1)+a^2(a-1)-3(a-1)=$$

$$=(a-1)(a^4+a^3+a^2-3)=(a-1)(a^3+2a^2+3a+3)\geq 0;\forall a>0$$

$$\text{Hence, } a^5-a^2-3a+3\geq 0 \Rightarrow a^5-a^2+3\geq 3a \Rightarrow$$

$$2a^5+b^3-2a^2+26\geq b^3+2+6a+18\geq 3b+6a+18$$

$$2a^5+b^3-2a^2+26\geq 3(b+2a+6) \Leftrightarrow$$

$$\frac{1}{\sqrt{2a^5+b^3-2a^2+26}} \leq \frac{1}{\sqrt{3(b+2a+6)}}; \quad (1)$$

Using BCS inequality, we have:

$$\frac{1}{\sqrt{9(b+2a+6)}} \leq \frac{1}{2}\left(\frac{1}{9}+\frac{1}{b+2a+6}\right) \overset{(1)}{\Rightarrow}$$

$$\frac{1}{\sqrt{2a^5+b^3-2a^2+26}} \leq \frac{\sqrt{3}}{2}\left(\frac{1}{9}+\frac{1}{b+2a+6}\right)$$

Similarly: $\frac{1}{\sqrt{2b^5+c^3-2b^2+26}} \leq \frac{\sqrt{3}}{2}\left(\frac{1}{9}+\frac{1}{c+2b+6}\right)$

$\frac{1}{\sqrt{2c^5+a^3-2c^2+26}} \leq \frac{\sqrt{3}}{2}\left(\frac{1}{9}+\frac{1}{a+2c+6}\right)$. **Hence,**

$$T \leq \frac{\sqrt{3}}{2}\left(\frac{1}{3}+\frac{1}{b+2a+6}+\frac{1}{c+2b+6}+\frac{1}{a+2c+6}\right); \quad (2)$$

We have: $abc+ab+bc+ca=4 \Leftrightarrow$

$$abc+2ab+2bc+2ca+4a+4b+4c+8=$$

$$=ab+2a+2b+4+bc+2b+2c+4+ca+2c+2a+4 \Leftrightarrow$$

$$(a+2)(b+2)(c+2)=(a+2)(b+2)+(b+2)(c+2)+(c+2)(a+2) \Leftrightarrow$$

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+1}=1$$

Hence, using inequality $\frac{1}{x+y+z} \leq \frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$; $\forall x,y,z>0$ **we have:**

$$\frac{1}{b+2a+6}+\frac{1}{c+2b+6}+\frac{1}{a+2c+6}=$$

$$=\frac{1}{(a+2)+(a+2)+(b+2)}+\frac{1}{(b+2)+(b+2)+(c+2)}$$

$$+\frac{1}{(c+2)+(c+2)+(a+2)} \leq$$

$$\leq \frac{1}{9}\left(\frac{1}{a+2}+\frac{1}{a+2}+\frac{1}{b+2}\right)+\frac{1}{9}\left(\frac{1}{b+2}+\frac{1}{b+2}+\frac{1}{c+2}\right)+\frac{1}{9}\left(\frac{1}{c+2}+\frac{1}{c+2}+\frac{1}{a+2}\right)$$

$$=\frac{1}{3}\left(\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\right)=\frac{1}{3} \Rightarrow$$

$$\frac{1}{b+2a+6}+\frac{1}{c+2b+6}+\frac{1}{a+2c+6} \leq \frac{1}{3}; \quad (3)$$

From (2),(3) it follows that:

$$T \le \frac{\sqrt{3}}{2}\left(\frac{1}{3}+\frac{1}{3}\right)=\frac{\sqrt{3}}{2}\cdot\frac{2}{3}=\frac{1}{\sqrt{3}} \Rightarrow T_{max}=\frac{1}{\sqrt{3}}$$

Equality occurs if $\begin{cases} a=b=c>0 \\ abc+ab+bc+ca=4 \end{cases} \Leftrightarrow a=b=c=1.$

Hence, the maximum value of expression T is $\frac{1}{\sqrt{3}}$ when $a=b=c=1$.

Solution 2 by Tran Hong-Dong Thap-Vietnam

- $a,b,c>0,\ abc+ab+bc+ca=4 \leftrightarrow \frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}=1;\ (1)$
- $2a^5-2a^2+3 \overset{(2)}{\ge} 3a^2 \leftrightarrow 2a^5-5a^2+3\ge 0$

  $\leftrightarrow (a-1)^2(2a^3+4a^2+6a+3)\ge 0$ (true for $a>0$)
- $b^3 \overset{(3)}{\ge} 3b-2 \leftrightarrow b^3-3b+2\ge 0 \leftrightarrow (b-1)^2(b+2)\ge 0$ (true for $a>0$)

$$\to 2a^5+b^3-2a^2+26=2a^5-2a^2+3+b^3+23 \overset{\text{By }(2),(3)}{\ge} 3a^2+3b-2+23$$

$$=3a^2+3b+21 \overset{a^2\ge 2a-1}{\ge} 3(2a-1)+3b+21=6a+3b+18$$

$$=3(2a+b+6)$$

$\to 2a^5+b^3-2a^2+26\ge 3(2a+b+6)$; (and analogs)

$$\to T=\frac{1}{\sqrt{2a^5+b^3-2a^2+26}}+\frac{1}{\sqrt{2b^5+c^3-2b^2+26}}+\frac{1}{\sqrt{2c^5+a^3-2c^2+26}} \overset{B.C.S}{\le}$$

$$\sqrt{3}.\sqrt{\frac{1}{2a^5+b^3-2a^2+26}+\frac{1}{2b^5+c^3-2b^2+26}+\frac{1}{2c^5+a^3-2c^2+26}}$$

$$\le \sqrt{3}.\sqrt{\frac{1}{3(2a+b+6)}+\frac{1}{3(2b+c+6)}+\frac{1}{3(2c+a+6)}}=$$

$$=\sqrt{\frac{1}{2a+b+6}+\frac{1}{2b+c+6}+\frac{1}{2c+a+6}}=$$

$$=\sqrt{\sum\frac{1}{a+2+a+2+b+2}}\overset{\text{Cauchy-Schwarz}}{\leq}\sqrt{\frac{1}{9}\left(\sum\frac{3}{a+2}\right)}=$$

$$=\sqrt{\frac{1}{3}\sum\frac{1}{a+2}}\overset{\text{By }(1)}{\leq}\sqrt{\frac{1}{3}.1}=\frac{1}{\sqrt{3}};$$

So, $P\leq\frac{1}{\sqrt{3}}\to P_{max}=\frac{1}{\sqrt{3}}\leftrightarrow\begin{cases}abc+ab+bc+ca=4\\ a=b=c=1\end{cases}\leftrightarrow a=b=c=1.$

SP.342. Let a,b,c be positive real numbers such that $a+b+c+1=4abc$.

Find the maximum value of expression:

$$S=\frac{1}{\sqrt[3]{2a^5-2a^3+b^2+26}}+\frac{1}{\sqrt[3]{2b^5-2b^3+c^2+26}}+\frac{1}{\sqrt[3]{2c^5-2c^3+a^2+26}}$$

Proposed by Hoang Le Nhat-Tung-Hanoi-Vietnam

Solution 1 by proposer

$$2a^5-2a^3-4a+a=2(a^5-a^3-2a+2)=2[a^4(a-1)+a^3(a-1)-2(a-1)]=$$

$$=2(a-1)^2(a^3+2a^2+2a+2)\geq 0;\forall a>0$$

$$\Rightarrow 2a^5-2a^3-4a+a>0\Rightarrow$$

$$\Rightarrow 2a^5-2a^3+b^2+26\geq 4a+21+(b^2+1)\geq 4a+21+2b$$

$$\frac{1}{\sqrt[3]{2a^5-2a^3+b^2+26}}\leq\frac{1}{\sqrt[3]{4a+2b+21}}=9\sqrt[3]{\frac{1}{27}\cdot\frac{1}{27}\cdot\frac{1}{4a+2b+21}}\leq$$

$$\leq 3\left(\frac{2}{27}+\frac{1}{4a+2b+21}\right)$$

Similarly, $\frac{1}{\sqrt[3]{2b^5-2b^3+c^2+26}}\leq 3\left(\frac{2}{27}+\frac{1}{4b+2c+21}\right)$

$$\frac{1}{\sqrt[3]{2c^5-2c^3+a^2+26}}\leq 3\left(\frac{2}{27}+\frac{1}{4c+2a+21}\right)$$

Hence, $S\leq 3\left(\frac{2}{9}+\frac{1}{4a+2b+21}+\frac{1}{4b+2c+21}+\frac{1}{4c+2a+21}\right);\ (1)$

$$a+b+c+1=4abc\Leftrightarrow 2(a+b+c+1)=8abc$$

$$4ab+2a+2b+1+4bc+2b+2c+1+4ca+2c+2a+1$$
$$=8abc+4ab+4bc+4ca+2a+2b+2c+1$$
$$(2a+1)(2b+1)+(2b+1)(2c+1)+(2c+1)(2a+1)$$
$$=(2a+1)(2b+1)(2c+1)$$
$$\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}=1;\ \ (2)$$

Hence, using inequality $\frac{1}{x+y+z}\le\frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right),\forall x,y,z>0$ **and from (2) we get:**

$$\frac{1}{4a+2b+21}+\frac{1}{4b+2c+21}+\frac{1}{4c+2a+21}=$$
$$=\frac{1}{4a+2b+3+9+9}+\frac{1}{4b+2c+3+9+9}+\frac{1}{4c+2a+3+9+9}\le$$
$$\le\frac{1}{9}\left(\frac{2}{3}+\frac{1}{4a+2b+3}+\frac{1}{4b+2c+3}+\frac{1}{4c+2a+3}\right)\le$$
$$\le\frac{1}{9}\left(\frac{2}{3}+\frac{1}{2(2a+1)+2b+1}+\frac{1}{2(2b+1)+2c+1}+\frac{1}{2(2c+1)+2a+1}\right)$$
$$\le\frac{1}{9}\left(\frac{2}{3}+\frac{1}{3}\left(\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\right)\right)=\frac{\frac{1}{9}\left(\frac{2}{3}+\frac{1}{3}\cdot 1\right)1}{9}$$

From (1) we have:

$$S\le 3\left(\frac{2}{9}+\frac{1}{9}\right)=3\cdot\frac{1}{3}=1\Rightarrow S_{Max}=1.$$

Equality occurs if $\begin{cases}a+b+c+1=4abc\\ a=b=c>0\end{cases}\Leftrightarrow a=b=c=1.$

Therefore,

$$S_{Max}=1 \text{ when } a=b=c=1.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2a^5-2a^3+4\ge 4a\Leftrightarrow(a-1)^2(2a^3+4a^2+4a+4)\ge 0$$
$$b^2+1\ge 2b\Leftrightarrow(b-1)^2\ge 0$$
$$2a^5-2a^3+b^2+26\ge 4a+2b+21(and\ analogs)$$
$$\Rightarrow S=\sum\frac{1}{\sqrt[3]{2a^5-2a^3+b^2+26}}\le\sum\sqrt[3]{\frac{1}{4a+2b+21}}\overset{Holder}{\le}\sqrt[3]{9\sum\frac{1}{4a+2b+21}}=$$

$$=\sqrt[3]{9\sum\frac{1}{2(2a+7)+(2b+7)}}\overset{CBS}{\leq}\sqrt[3]{\sum\left(\frac{2}{2a+7}+\frac{1}{2b+7}\right)}=\sqrt[3]{3\sum\frac{1}{2a+7}}$$

- $4abc=\sum a+1\overset{AM-GM}{\geq}4\sqrt[4]{abc}\Rightarrow abc\geq 1\Rightarrow\sum a\geq 3$ and $\sum ab\geq 3\Rightarrow$

$$\sum\frac{1}{2a+7}=\frac{\sum(2a+7)(2b+7)}{\prod(2a+7)}=\frac{4\sum ab+28\sum a+147}{8abc+28\sum ab+98\sum a+343}\leq$$

$$\left(abc\geq 1,\sum a\geq 3,\sum ab\geq 3\right)$$

$$\leq\frac{4\sum ab+28\sum a+147}{3(4\sum ab+28\sum a+147)}=\frac{1}{3}\Rightarrow max\{S\}=1.$$

Equality holds when $a=b=c=1$.

SP.343 If $a,b,c\in\mathbb{C};|a|=|b|=|c|=5$ then:

$$\sum_{cyc}|a+5|+5\sum_{cyc}\left|a^{10}+1\right|+\sum_{cyc}\left|a^{11}+5\right|\geq 30$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$10=2|a|=2|-a|=|-2a|=\left|a^{11}+5-a(a^{10}+1)-(a+5)\right|\leq$$

$$\leq\left|a^{11}+5\right|+|a|\cdot\left|a^{10}+1\right|+|a+5|=\left|a^{11}+5\right|+5\left|a^{10}+1\right|+|a+5|$$

$$|a+5|+5\left|a^{10}+1\right|+\left|a^{11}+5\right|\geq 10$$

By summing:

$$\sum_{cyc}|a+5|+5\sum_{cyc}\left|a^{10}+1\right|+\sum_{cyc}\left|a^{11}+5\right|\geq 10+10+10=30$$

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

$$|a+5|+5\left|a^{10}+1\right|+\left|a^{11}+5\right|=|a+5|+|a|\left|a^{10}+1\right|+\left|a^{11}+5\right|=$$

$$=|a+5|+\left|a^{11}+a\right|+\left|a^{11}+5\right|\geq\left|(a+5)+(a^{11}+a)-(a^{11}+5)\right|=$$

$=2|a|=10$. Therefore,

$$\sum_{cyc}|a+5|+5\sum_{cyc}\left|a^{10}+1\right|+\sum_{cyc}\left|a^{11}+5\right|\geq 10+10+10=30$$

SP.344 If $n \in \mathbb{N}, n \geq 2$ prove that:

$$\frac{n}{n+2}+\int_0^1 \left(\tan^{-1}(x^n)\right)^2 dx \geq 2\int_0^1 \tan^{-1}(x^n)\sqrt[n]{\tan^{-1}x}\,dx$$

Proposed by Florică Anastase-Romania

Solution 1 by Mohammad Rostami-Kabul-Afghanistan

$$|\tan^{-1}x|<\frac{\pi}{2}\Rightarrow (\tan^{-1})^{\frac{n}{2}}\geq \tan^{-1}x, \forall n \in \mathbb{N}, n\geq 2 \Rightarrow$$

$$\tan^{-1}x \geq (\tan^{-1})^{\frac{2}{n}} \Rightarrow \tan^{-1}x \geq \left(\sqrt[n]{\tan^{-1}x}\right)^2 \Rightarrow \int_0^1 \tan^{-1}x\,dx \geq \int_0^1 \left(\sqrt[n]{\tan^{-1}x}\right)^2 dx$$

$$\left[x\tan^{-1}x-\frac{1}{2}\log(x^2+1)\right]_0^1 \geq \int_0^1 \left(\sqrt[n]{\tan^{-1}x}\right)^2 dx \Rightarrow$$

$$\frac{\pi}{4}-\frac{1}{2}\log 2 \geq \int_0^1 \left(\sqrt[n]{\tan^{-1}x}\right)^2 dx\,;(I)$$

$$\frac{n}{n+2}\geq \frac{\pi}{4}-\frac{1}{2}\log 2 \Leftrightarrow \frac{4n}{n+2}\geq \pi-2\log 2 \Leftrightarrow 4n \geq (\pi-\log 4)(n+2) \Leftrightarrow$$

$$4n \geq \frac{175}{100}(n+2) \Leftrightarrow 400n \geq 175n+350 \Leftrightarrow 225n \geq 350(true), \forall n \geq 2;(II)$$

$$(I),(II): \frac{n}{n+2}\geq \frac{\pi}{4}-\frac{1}{2}\log 2 \geq \int_0^1 \left(\sqrt[n]{\tan^{-1}x}\right)^2 dx \Rightarrow$$

$$\frac{n}{n+2}\geq \int_0^1 \left(\sqrt[n]{\tan^{-1}x}\right)^2 dx\,;\ (III)$$

$$\frac{n}{n+2}\geq \int_0^1 \left(\sqrt[n]{\tan^{-1}x}\right)^2 dx \Rightarrow$$

$$\frac{n}{n+2}+\int_0^1 (\tan^{-1}(x^n))^2 dx \geq \int_0^1 (\tan^{-1}(x^n))^2 dx+\int_0^1 \left(\sqrt[n]{\tan^{-1}x}\right)^2 dx\,;(IV)$$

$$\left(\tan^{-1}(x^n)-\sqrt[n]{\tan^{-1}x}\right)^2 \geq 0 \Rightarrow$$

$$(tan^{-1}(x^n)^2+\left(\sqrt[n]{tan^{-1}x}\right)^2-2tan^{-1}(x^n)\sqrt[n]{tan^{-1}x}\geq 0\Leftrightarrow$$

$$tan^{-1}(x^n)^2+\left(\sqrt[n]{tan^{-1}x}\right)^2\geq 2tan^{-1}(x^n)\sqrt[n]{tan^{-1}x}$$

$$\int_0^1\left(tan^{-1}(x^n)\right)^2dx+\int_0^1\left(\sqrt[n]{tan^{-1}x}\right)^2dx\geq 2\int_0^1\left(tan^{-1}(x^n)\sqrt[n]{tan^{-1}x}\right)dx;(V)$$

$$(IV),(V):\frac{n}{n+2}+\int_0^1(tan^{-1}(x^n))^2dx\geq\int_0^1\left(tan^{-1}(x^n)\right)^2dx+\int_0^1\left(\sqrt[n]{tan^{-1}x}\right)^2dx\geq$$

$$\geq 2\int_0^1\left(tan^{-1}(x^n)\sqrt[n]{tan^{-1}x}\right)dx$$

Therefore,

$$\frac{n}{n+2}+\int_0^1\left(tan^{-1}(x^n)\right)^2dx\geq 2\int_0^1 tan^{-1}(x^n)\sqrt[n]{tan^{-1}x}dx$$

Solution 2 by proposer

$$tanx\geq x,\forall x\in[0,1]\Rightarrow tan^{-1}x\leq x,\forall x\in[0,1]$$

$$\left(\sqrt[n]{x}-tan^{-1}(x^n)\right)^2\geq 0$$

$$\sqrt[n]{x^2}-2\sqrt[n]{x}tan^{-1}(x^n)+\left(tan^{-1}(x^n)\right)^2\geq 0$$

$$\sqrt[n]{x^2}+\left(tan^{-1}(x^n)\right)^2\geq 2\sqrt[n]{x}tan^{-1}(x^n)\geq 2\sqrt[n]{tan^{-1}x}\,tan^{-1}(x^n)$$

$$\int_0^1\sqrt[n]{x^2}dx+\int_0^1\left(tan^{-1}(x^n)\right)^2dx\geq 2\int_0^1\sqrt[n]{tan^{-1}x}\,tan^{-1}(x^n)dx$$

$$\frac{n}{n+2}+\int_0^1\left(tan^{-1}(x^n)\right)^2dx\geq 2\int_0^1 tan^{-1}(x^n)\sqrt[n]{tan^{-1}x}dx\,;n\in\mathbb{N},n\geq 2$$

SP.345 Prove that in any triangle ABC,

$$\left(\frac{b+c-a}{a}\right)^2+\left(\frac{c+a-b}{b}\right)^2+\left(\frac{a+b-c}{c}\right)^2+\frac{8r}{R}\geq 7$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by proposer

Lemma 1. For all non-negative real numbers x,y,z then

$$x^2+y^2+z^2+2xyz+1\geq 2(xy+yz+zx)$$

Darij Grinberg

Proof. Since $(y-1)(z-1)\cdot(z-1)(x-1)(x-1)(y-1)\geq 0$ so we can assume $(y-1)(z-1)\geq 0$. Then we have:

$$x^2+y^2+z^2+2xyz+1-2(xy+yz+zx)=(x-1)^2+(y-z)^2+2x(y-1)(z-1)\geq 0$$

Lemma 2. In any triangle ABC we always have:

$$(a)\frac{h_ah_b}{r_ar_b}+\frac{h_bh_c}{r_br_c}+\frac{h_ch_a}{r_cr_a}=2\left(2-\frac{r}{R}\right)$$

$$(b)\frac{h_ah_bh_c}{r_ar_br_c}=\frac{2r}{R}$$

Proof.$(a)\sum_{cyc}\frac{h_ah_b}{r_ar_b}=\sum_{cyc}\frac{4(s-a)(s-b)}{ab}=\sum_{cyc}4sin^2\frac{C}{2}=\sum_{cyc}2(1-cosC)=2\left(2-\frac{r}{R}\right)$

$$(b)\frac{h_ah_bh_c}{r_ar_br_c}=\frac{8(s-a)(s-b)(s-c)}{abc}=\frac{2r}{R}$$

Come back to the main problem.

We apply lemma 1 for $(x,y,z)=\left(\frac{h_a}{r_a},\frac{h_b}{r_b},\frac{h_c}{r_c}\right)$ and use lemma 2 to obtain:

$$\left(\frac{h_a}{r_a}\right)^2+\left(\frac{h_b}{r_b}\right)^2+\left(\frac{h_c}{r_c}\right)^2+\frac{4r}{R}+1\geq 4\left(2-\frac{r}{R}\right)$$

Note that $\frac{h_a}{r_a}=\frac{b+c-a}{a}$

Therefore this inequality is equivalent to the desired result and we are done.

Solution 2 by Tran Hong-Dong Thap-Vietnam

Lemma: If a,b,c are non-negative numbers, then

$$a^2+b^2+c^2+2abc+1\ge 2(ab+bc+ca);\ (1)$$

Proof. Among the numbers $1-a, 1-b, 1-c$ there are always two of them with the same sign; let us say $(1-b)(1-c)\ge 0$. We have:

$$a^2+b^2+c^2+2abc+1-2(ab+bc+ca)=$$
$$=(a-1)^2+(b-c)^2+2a+2abc-2(ab+ac)=$$
$$=(a-1)^2+(b-c)^2+2a(1-b)(1-c)\ge 0\text{, which is clearly true.}$$

Equality holds when $a=b=c$. Now, let us denote:

$$x=\frac{b+c-a}{a}, y=\frac{a+c-b}{b}, z=\frac{a+b-c}{c}\Rightarrow x,y,z>0$$

$$\frac{1}{2+x}+\frac{1}{2+y}+\frac{1}{2+z}=1\Rightarrow xy+yz+zx+xyz=4$$

$$\text{Now, }\sum\left(\frac{b+c-a}{a}\right)^2+\frac{8r}{R}\ge 7\Leftrightarrow x^2+y^2+z^2+4xyz\ge 7\Leftrightarrow$$

$x^2+y^2+z^2+2xyz\ge 7-2xyz;\ (2)$. Using Lemma with $a=x, b=y, c=z\Rightarrow$

$$x^2+y^2+z^2+2xyz+1\ge 2(xy+yz+zx)\Rightarrow$$
$$x^2+y^2+2xyz\ge 2(xy+yz+zx)-1;\ (3)$$

From (2),(3) we need to prove:

$$2(xy+yz+zx)-1=7-2xyz\Leftrightarrow xy+yz+zx=4(true)\Rightarrow(2)-true.$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=x+y, b=y+z, c=z+x; x,y,z>0$. We have:

$$r=\sqrt{\frac{xyz}{x+y+z}}, R=\frac{(x+y)(y+z(z+x)}{4\sqrt{xyz(x+y+z)}}$$

$$\sum\left(\frac{b+c-a}{a}\right)^2+\frac{8r}{R}\ge 7;(*)\Leftrightarrow\sum\frac{4z^2}{(x+y)^2}+\frac{32xyz}{(x+y)(y+z)(z+x)}\ge 7\Leftrightarrow$$

$$4\sum z^2(y+z)^2(z+x)^2+32xyz\prod(x+y)\ge 7\prod(x+y)^2$$

- $\sum z^2(y+z)^2(z+x)^2=\sum x^6+2\sum x^5y+\sum x^4y^2+4xyz\sum x^3+2xyz\sum x^2y+3(xyz)^2$
- $xyz\prod(x+y)=xyz\sum x^2y+2(xyz)^2$
- $\prod(x+y)^2=\sum x^4y^2+6xyz\sum x^2y+2\sum x^3y^3+2xyz\sum x^3+10(xyz)^2$

$$(*) \Leftrightarrow 4\sum x^6 + 8\sum x^5y + 2xyz\sum x^3 + 6(xyz)^2$$
$$\ge 3\sum x^4y^2 + 2xyz\sum x^2y + 14\sum x^3y^3$$

By Schur's inequality, we have:

$$\sum x^2(x^2-y^2)(x^2-z^2) \ge 0 \Leftrightarrow 2\sum x^6 + 6(xyz)^2 \ge 2\sum x^4y^2; (1)$$
$$\sum x^4(x-y)(x-z) \ge 0 \Leftrightarrow 2\sum x^6 + 2xyz\sum x^3 \ge 2\sum x^5y; (2)$$

From (1),(2) it follows that:

$$4\sum x^6 + 8\sum x^5y + 2xyz\sum x^3 + 6(xyz)^2 \ge 10\sum x^5y + 2\sum x^4y^2 \overset{(1)}{\ge}$$
$$\ge 3\sum x^4y^2 + 2\sum x^3y^2z + 14\sum x^3y^3$$
$$(1) \Leftrightarrow 10\sum x^5y \ge \sum x^4y^2 + 2\sum x^3y^2z + 7\sum x^3y^3$$

Which is true from Murihead inequality: $[5,1,0]$ majories $[4,2,0], [3,2,1], [3,3,0]$.

UP.331 If $a,b,c \in (0,1), n \in \mathbb{N}, n \ge 2$ then prove:

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right) \ge \sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

Proposed by Florică Anastase-Romania

Solution 1 by Adrian Popa-Romania

$$\sqrt[n]{sina} = \sqrt[n]{1\cdot 1\cdot \ldots \cdot 1\cdot sina} \overset{AM-GM}{\le} \frac{1+1+\cdots+1+sina}{n} = \frac{n-1+sina}{n} \Rightarrow$$
$$1-\sqrt[n]{sina} \ge 1-\frac{n-1+sina}{n} = \frac{1-sina}{n}$$

We need to prove that:

$$\sum_{cyc}\frac{1-sina}{n} \ge \sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}; \quad (*)$$

We must to prove:

$$\frac{1-sina}{2n}+\frac{1-sinb}{2n}\geq\frac{1-sinasinb}{2n+1-sinasinb}\Leftrightarrow$$

$$1-sina+1-sinb\geq 1-sinasinb\Leftrightarrow$$

$$1-sina-sinb+sinasinb\geq 0\Leftrightarrow$$

$(1-sina)(1-sinb)\geq 0$. So,

$$\frac{1-sina}{2n}+\frac{1-sinb}{2n}\geq\frac{1-sinasinb}{2n+1-sinasinb};(1)$$

$$\frac{1-sinb}{2n}+\frac{1-sinc}{2n}\geq\frac{1-sinbsinc}{2n+1-sinbsinc};(2)$$

$$\frac{1-sinc}{2n}+\frac{1-sina}{2n}\geq\frac{1-sincsina}{2n+1-sincsina};(3)$$

Adding these three relations, we get:

$$\sum_{cyc}\frac{1-sina}{n}\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

Solution by proposer

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right)\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

$$3-\sum_{cyc}\sqrt[n]{sina}\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

$$3-\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}\geq\sum_{cyc}\sqrt[n]{sina}$$

$$\sum_{cyc}\left(1-\frac{1-sinasinb}{2n+1-sinasinb}\right)\geq\sum_{cyc}\sqrt[n]{sina}$$

$$\sum_{cyc}\frac{2n}{2n+1-sinasinb}\geq\sum_{cyc}\sqrt[n]{sina}$$

$$\sum_{cyc}\frac{1}{2n+1-sinasinb}\geq\frac{1}{2n}\sum_{cyc}\sqrt[n]{sina};\quad(1)$$

Let be the function $f:[0,1]\to\mathbb{R}$,

$$f(x)=4n-(2n+1)\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right)+\alpha x\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right),\alpha\in[0,1]$$

$$f'(x)=-\frac{2n+1}{n\sqrt[n]{x^{n-1}}}+\alpha\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right)+\frac{\alpha}{n}\sqrt[n]{x}$$

$$f''(x)=\frac{2n+1}{n^2\cdot x\sqrt[n]{x^{n-1}}}+\frac{(n+1)\alpha}{n^2\sqrt[n]{x^{n-1}}}$$

$$f''(x)>0\Rightarrow f'_{[0,1]}-increasing; f'(1)=-2+\alpha\left(\sqrt[n]{\alpha}+1\right)-\frac{1-\alpha}{n}$$

$$\alpha\in[0,1],\alpha\left(\sqrt[n]{\alpha}+1\right)\le 2\Rightarrow f'(1)\le 0\Rightarrow f_{[0,1]}-decreasing\Rightarrow f(x)\ge f(1),\forall x\in[0,1]$$

$$f(1)=4n-(2n+1)\left(\sqrt[n]{\alpha}+1\right)+\alpha\left(\sqrt[n]{\alpha}+1\right)\ge 0,\forall\alpha\in[0,1]$$

Let $g(\alpha)=4n-(2n+1)\left(\sqrt[n]{\alpha}+1\right)+\alpha\left(\sqrt[n]{\alpha}+1\right)\ge 0,\forall\alpha\in[0,1]$

$$g'(\alpha)=\frac{-(2n+1)+\sqrt[n]{\alpha^{n-1}}\left((n+1)\sqrt[n]{\alpha}+1\right)}{n\sqrt[n]{\alpha^{n-1}}}\le 0,,\forall\alpha\in[0,1]\Rightarrow g-decreasing$$

$$f(1)=g(\alpha)\ge g(1)=0$$

We have: $f(\beta)=4n-(2n+1)\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right)+\alpha\beta\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right)\ge 0\Leftrightarrow$

$$4n\ge(2n+1-\alpha\beta)\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right)\Leftrightarrow\frac{1}{2n+1-\alpha\beta}\ge\frac{\sqrt[n]{\alpha}+\sqrt[n]{\beta}}{4n}$$

Therefore,

$$\frac{1}{2n+1-sinasinb}\ge\frac{\sqrt[n]{sina}+\sqrt[n]{sinb}}{4n}\ (and\ analogs);\ \ (2)$$

From $(1),(2)$ ***we get***

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right)\ge\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

UP.332 Let $(x_n)_{n\ge1},(y_n)_{n\ge1}$ be sequences of positive real numbers such that:

$$x_1>1,x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}};\ y_1>0,y_{n+1}=\frac{(n+1)n^n y_n}{y_n^n+n^n(n-1)}$$

Find: $\lim\limits_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}$

Proposed by Florică Anastase-Romania

Solution 1 by Adrian Popa-Romania

$$n=1\Rightarrow x_2=\frac{1+0\cdot x_1^1}{1\cdot x_1^0}=1$$

$$n=2\Rightarrow x_3=\frac{1+x_2^2}{2\cdot x_2^1}=1$$

Suppose that $x_n=1,\forall n\in\mathbb{N}$, then $x_{n+1}=\frac{1+(n-1)\cdot 1^n}{n\cdot 1^{n-1}}=1$.

So, $x_n=1,\forall n\in\mathbb{N}$. Now,

$$n=1\Rightarrow y_2=\frac{2\cdot 1^1\cdot y_1}{y_1^1+1^1\cdot 0}=\frac{2y_1}{y_1}=2$$

$$n=2\Rightarrow y_3=\frac{3\cdot 2^2\cdot 2}{2^2+2^2\cdot 1}=\frac{24}{8}=3$$

Suppose that $y_n=n,\forall n\in\mathbb{N}$, then we have:

$$y_{n+1}=\frac{(n+1)\cdot n^n\cdot n}{n^n+n^n(n-1)}=\frac{(n+1)n^{n+1}}{n^n(1+n-1)}=n+1.$$

So, $y_n=n,\forall n\in\mathbb{N}$.

Therefore,

$$\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}=\lim_{n\to\infty}\left(\frac{1+n}{n}\right)^{\sqrt{n}}=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{\sqrt{n}}{n}}=e^0=1$$

Solution 2 by proposer

$$x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}\overset{AM-GM}{\geq}\frac{\sqrt[n]{x_n^{n(n-1)}}}{x_n^{n-1}}=1\Rightarrow x_n\geq 1,\forall n\in\mathbb{N};\ \ (1)$$

$$x_{n+1}-x_n=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}-x_n=\frac{1+(n-1)x_n^n-nx_n^n}{nx_n^{n-1}}=\frac{1-x_n^n}{nx_n^{n-1}}\overset{(1)}{\leq}0,\forall n\in\mathbb{N}\Rightarrow$$

$(x_n)_{n\geq 1}$ –decreasing; (2) and $x_n\in(1,x_1)$.

From (1),(2) we get $(x_n)_{n\geq 1}$ –convergent; (3)

$$y_{n+1}=\frac{(n+1)n^ny_n}{y_n^n+n^n(n-1)}\Leftrightarrow\frac{y_{n+1}}{n+1}=\frac{n^ny_n}{y_n^n+n^n(n-1)}\Leftrightarrow$$

$$\frac{n+1}{y_{n+1}}=\frac{y_n^n+n^n(n-1)}{n^ny_n}=\frac{y_n^{n-1}}{n^n}+\frac{n-1}{y_n}=\frac{1}{n\left(\frac{n}{y_n}\right)^{n-1}}+\frac{n-1}{n}\cdot\frac{n}{y_n}\overset{\frac{n}{y_n}=x_n}{=}$$

$$=\frac{1}{nx_n^{n-1}}+\frac{(n-1)x_n}{n}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}=x_{n+1}$$

So,

$$\frac{n}{y_n}=x_n,\forall n\in\mathbb{N}\Rightarrow y_n=\frac{n}{x_n}\underset{n\to\infty}{\overset{(3)}{\Longrightarrow}}\lim y_n=+\infty$$

$$\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}=\lim_{n\to\infty}\left(1+\frac{x_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}=\lim_{n\to\infty}\left[\left(1+\frac{x_n}{y_n}\right)^{\frac{y_n}{x_n}}\right]^{\frac{\sqrt{n}}{y_n}}=e^{\lim\limits_{n\to\infty}\left(\frac{n}{y_n}\cdot\frac{1}{\sqrt{n}}\right)}=$$

$$=e^{\lim\limits_{n\to\infty}\frac{x_n}{\sqrt{n}}}\overset{(3)}{=}1$$

UP.333. If $x_p=a_p+ib_p,p=\overline{1,4}$ are roots of the equation:

$$x^4-2(k+1)x^3+2(k+1)^2x^2-2(k^2+1)(k+1)x+(k^2+1)^2=0,$$

$k\in R^*$ then prove:

$$\sum_{p=1}^{4}arctg\frac{a_p}{|b_p|}=\pi+2\left(\frac{k-|k|}{k}\right)arctgk.$$

Proposed by Florentin Vişescu-Romania

Solution by proposer

How $k\in R^*.\exists!\ \alpha\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)\setminus\{0\}$, such that $k=tg\alpha$ then

$$x^4-2(tg\alpha+1)x^3+2(tg\alpha+1)^2x^2-2(tg^2\alpha+1)(tg\alpha+1)x+(tg^2\alpha+1)^2=0$$

or

$$x^4-2\left(\frac{sin\alpha+cos\alpha}{cos\alpha}\right)x^3+2\left(\frac{sin\alpha+cos\alpha}{cos\alpha}\right)^2x^2-2\frac{1}{cos^2\alpha}\left(\frac{sin\alpha+cos\alpha}{cos\alpha}\right)x+\frac{1}{cos^4\alpha}=0.$$

Hence

$$(xcos\alpha)^4-2(xcos\alpha)^3(cos\alpha+sin\alpha)+2(xcos\alpha)^2(cos\alpha+sin\alpha)^2$$

$$-2xcos\alpha(cos\alpha+sin\alpha)+1=0\ sau\ (xcos\alpha)^2-2xcos\alpha(cos\alpha+sin\alpha)+$$

$$+2(cos\alpha+sin\alpha)^2-2\frac{cos\alpha+sin\alpha}{xcos\alpha}+\frac{1}{(xcos\alpha)^2}=0.$$

Denote $xcos\alpha+\frac{1}{xcos\alpha}=y$ we get:

$$(xcos\alpha)^2+\frac{1}{(xcos\alpha)^2}=y^2-2.$$

Hence $y^2-2(cos\alpha+sin\alpha)y+4sin\alpha cos\alpha=0$

Or $(y-2sin\alpha)(y-2cos\alpha)=0.$

So $xcos\alpha+\frac{1}{xcos\alpha}=2cos\alpha$ or $xcos\alpha+\frac{1}{xcos\alpha}=2sin\alpha$, Hence

$x^2cos^2\alpha-2xcos^2\alpha+1=0$ or $x^2cos^2\alpha-2xcos\alpha sin\alpha+1=0$ with roots

$x_{1,2}=\frac{cos\alpha\pm isin\alpha}{cos\alpha}\in C\backslash R$ or $x_{3,4}=\frac{sin\alpha\pm icos\alpha}{cos\alpha}\in C\backslash R.$

Therefore, $x_{1,2}=1\pm itg\alpha\ si\ x_{3,4}=tg\alpha\pm i,\alpha\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)\backslash\{0\}$

$$x_{1,2}=1\pm ki\ si\ x_{3,4}=k\pm i,k\in R^*.$$

$$\sum_{p=1}^{4}arctg\frac{a_p}{|b_p|}=2arctg\frac{1}{|k|}+2arctgk=2\left(arctg\frac{1}{|k|}+arctgk\right)$$

$$=2\left(\frac{\pi}{2}-arctg|k|+arctgk\right)=$$

$$=\pi+2(arctgk-arctg|k|)=\begin{cases}\pi,k>0\\ \pi+4arctgk,k<0\end{cases}=\pi+2\left(\frac{k-|k|}{k}\right)arctgk.$$

UP.334. Let be $n\in N^*\ si\ A_n\in M_{8n}(Q)$, such that

$$det(A_n^4+2A_n^2(1-n^2-k)+(1+n^2+k)^2I_{8n})=0,\forall\ k=\overline{1,2n}.\ \textbf{Find:}$$

$$\lim_{n\to\infty}det\left(\frac{1}{n}A_n\right).$$

Proposed by Florentin Vişescu-Romania

Solution by proposer

Let be $x^4+2x^2(1-n^2-k)+(1+n^2+k)^2=0,k=\overline{1,2n}.$

Denote $x^2=y$ we get:

$$y^2+2y(1-n^2-k)+(1+n^2+k)^2=0,k=\overline{1,2n}.$$

$$\Delta = 4(1-n^2-k)^2 - 4(1+n^2+k)^2 = -16(n^2+k), k = \overline{1,2n}.$$

$$y_{1,2} = \frac{-2(1-n^2-k) \pm 4i\sqrt{n^2+k}}{2} = \left(\sqrt{n^2+k} \pm i\right)^2.$$

Hence

$$x^2 = \left(\sqrt{n^2+k}+i\right)^2 \Rightarrow x_{1,2} = \pm\left(\sqrt{n^2+k}+i\right)$$

$$x^2 = \left(\sqrt{n^2+k}-i\right)^2 \Rightarrow x_{3,4} = \pm\left(\sqrt{n^2+k}-i\right).$$

How $k = \overline{1,2n} \Rightarrow \sqrt{n^2+k} \in R\backslash Q.$

$$det(A_n^4 + 2A_n^2(1-n^2-k) + (1+n^2+k)^2 I_{8n})$$
$$= det(A_n - x_1 I_{8n})(A_n - x_2 I_{8n})(A_n - x_3 I_{8n})(A_n - x_4 I_{8n})$$
$$= det(A_n - x_1 I_{8n}) det(A_n - x_2 I_{8n})$$
$$det(A_n - x_3 I_{8n}) det(A_n - x_4 I_{8n}) = 0 .$$

So, at least one is accepted x_1, x_2, x_3, x_4 **root of characteristic polynomial** $A_n\left(p_{A_n}(X)\right)$.

How $p_{A_n}(X) \in Q[X]$ **then:**

All the numbers x_1, x_2, x_3, x_4 **are roots of** $p_{A_n}(X), k = \overline{1,2n}$.

How that roots are different two by two, we have:

$$p_{A_n}(X) = \prod_{k=1}^{2n}(X^4 + 2X^2(1-n^2-k) + (1+n^2+k)^2) = det(A_n) = p_{A_n}(0)$$

$$\prod_{k=1}^{2n}(1+n^2+k)^2 = \left(\prod_{k=1}^{2n}(1+n^2+k)\right)^2.$$

$$\lim_{n\to\infty} det\left(\frac{1}{n}A_n\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right)^{8n} det(A_n) = \lim_{n\to\infty}\left(\frac{1}{n}\right)^{8n}\left(\prod_{k=1}^{2n}(1+n^2+k)\right)^2 =$$

$$\left(\lim_{n\to\infty}\left(\frac{1}{n}\right)^{4n}\prod_{k=1}^{2n}(1+n^2+k)\right)^2 = \left(\lim_{n\to\infty}\prod_{k=1}^{2n}\frac{1+n^2+k}{n^2}\right)^2.$$

$$\lim_{n\to\infty}\prod_{k=1}^{2n}\frac{1+n^2+k}{n^2} = \lim_{n\to\infty}\prod_{k=1}^{2n}\left(1+\frac{k+1}{n^2}\right) = \lim_{n\to\infty} e^{ln\left(\prod_{k=1}^{2n}\left(1+\frac{k+1}{n^2}\right)\right)} = \lim_{n\to\infty} e^{\sum_{k=1}^{2n} ln\left(1+\frac{k+1}{n^2}\right)}$$

Now,

Let be the functions $f,g:[0;\infty)\to R, f(x)=\ln(x+1)-x\ si\ g(x)=\ln(x+1)-x+\frac{x^2}{2}$.

Hence $f'(x)=\frac{-x}{x+1}\le 0,\forall\in[0;\infty)$

$$g'(x)=\frac{x^2}{x+1}\ge 0,\forall\in[0;\infty).$$

So, f **−decreasing function and** g **−inceasing function. Hence**

$$f(x)\le f(0)=0,\forall\in[0;\infty)\ si\ g(x)\ge g(0)=0,\forall\in[0;\infty).$$

Then, we have: $x-\frac{x^2}{2}\le\ln(x+1)\le x,\forall\in[0;\infty)$.

For $x=\frac{k+1}{n^2}\in[0;\infty)$ **it follows that:**

$\frac{k+1}{n^2}-\frac{\left(\frac{k+1}{n^2}\right)^2}{2}\le\ln\left(\frac{k+1}{n^2}+1\right)\le\frac{k+1}{n^2}$. **Hence**

$$\sum_{k=1}^{2n}\left(\frac{k+1}{n^2}-\frac{\left(\frac{k+1}{n^2}\right)^2}{2}\right)\le\sum_{k=1}^{2n}\left(\ln\left(\frac{k+1}{n^2}+1\right)\right)\le\sum_{k=1}^{2n}\left(\frac{k+1}{n^2}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\left(\frac{k+1}{n^2}-\frac{\left(\frac{k+1}{n^2}\right)^2}{2}\right)\le\lim_{n\to\infty}\sum_{k=1}^{2n}\left(\ln\left(\frac{k+1}{n^2}+1\right)\right)\le\lim_{n\to\infty}\sum_{k=1}^{2n}\left(\frac{k+1}{n^2}\right)$$

Hence

$$\lim_{n\to\infty}\left(\frac{\frac{(2n+1)(2n+2)}{2}-1}{n^2}-\frac{\frac{(2n+1)(2n+2)(4n+3)}{6}-1}{2n^4}\right)$$

$$\le\lim_{n\to\infty}\sum_{k=1}^{2n}\left(\ln\left(\frac{k+1}{n^2}+1\right)\right)\le\lim_{n\to\infty}\left(\frac{\frac{(2n+1)(2n+2)}{2}-1}{n^2}\right)$$

Hence

$$2\le\lim_{n\to\infty}\sum_{k=1}^{2n}\left(\ln\left(\frac{k+1}{n^2}+1\right)\right)\le 2\Rightarrow\lim_{n\to\infty}\sum_{k=1}^{2n}\left(\ln\left(\frac{k+1}{n^2}+1\right)\right)=2.$$

Therefore,

$$\lim_{n\to\infty}\prod_{k=1}^{2n}\frac{1+n^2+k}{n^2}=e^2\ deci, \lim_{n\to\infty} det\left(\frac{1}{n}A_n\right)=e^4.$$

UP.335 If $a,b,c\in\left(0,\frac{\pi}{2}\right), a+b+c=\pi$ then prove:

$$\sum_{i=1}^{n}\int_{i}^{i+1}\frac{dx}{(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)}\le\frac{3n}{(n+1)\pi^2}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)=$$

$$(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^a tana+e^b tanbx+e^c tancx^2)\overset{BCS}{\ge}$$

$$\ge\left(\sqrt{ae^{a+tana}tana}+\sqrt{be^{b+tanb}tanb}+\sqrt{ce^{c+tanc}tanc}\right)^2x^2\ge$$

$$\ge\left(\sqrt{atana(1+a)(1+tana)}+\sqrt{btanb(1+b)(1+tanb)}\right.$$

$$\left.+\sqrt{ctanc(1+c)(1+tanc)}\right)^2x^2$$

$$\ge(atana+btanb+ctanc)^2x^2=\left((a+b+c)\left(\frac{atana+btanb+ctanc}{a+b+c}\right)\right)^2x^2\ge$$

$$\ge\left((a+b+c)tan\left(\frac{a^2+b^2+c^2}{a+b+c}\right)\right)^2x^2\ge$$

$$\ge\left((a+b+c)tan\left(\frac{a+b+c}{3}\right)\right)^2x^2=\frac{\pi^2}{3}x^2$$

Hence, $\frac{1}{(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)}\le\frac{3}{\pi^2}\cdot\frac{1}{x^2}$

Therefore,

$$\sum_{i=1}^{n}\int_{i}^{i+1}\frac{dx}{(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^c tancx^2+e^b tanbx+e^a tana)}\le$$

$$\le\frac{3}{\pi^2}\cdot\sum_{i=1}^{n}\int_{i}^{i+1}\frac{1}{x^2}dx=\frac{3}{\pi^2}\cdot\sum_{i=1}^{n}\left(-\frac{1}{x}\right)\Bigg|_{i}^{i+1}=\frac{3}{\pi^2}\cdot\sum_{i=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{3n}{(n+1)\pi^2}$$

So,

$$\sum_{i=1}^{n}\int_{i}^{i+1}\frac{dx}{(ae^{tana}x^2+be^{tanb}x+ce^{tanc})(e^{c}tancx^2+e^{b}tanbx+e^{a}tana)}\le\frac{3n}{(n+1)\pi^2}$$

UP.336 If $0<a<b<\frac{\pi}{2}$ then prove:

$$\frac{(b-a)3\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-sin4(a+b)}}<cot2a-cot2b+\frac{\pi}{4}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=1-cos4x-\text{continuous}$$

$$g:(0,\infty)\to(0,\infty), g(x)=\frac{1}{\sqrt[3]{x}}=x^{-\frac{1}{3}}, g'(x)=-\frac{1}{3}x^{-\frac{4}{3}}<0, g''(x)=\frac{4}{9}x^{-\frac{7}{3}}>0\Rightarrow$$

g –convex function.

Applying Jensen integral inequality, we get:

$$g\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\le\frac{1}{b-a}\int_a^b g\big(f(x)\big)dx\Leftrightarrow$$

$$\frac{1}{\sqrt[3]{\frac{1}{b-a}\int_a^b f(x)dx}}\le\frac{1}{b-a}\int_a^b\frac{dx}{\sqrt[3]{f(x)}}\Leftrightarrow$$

$$\frac{1}{\sqrt[3]{\frac{1}{b-a}\int_a^b(1-cos4x)dx}}\le\frac{1}{b-a}\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}\Leftrightarrow$$

$$\frac{b-a}{\sqrt[3]{\frac{1}{b-a}\left(b-a-\frac{sin4b-sin4a}{4}\right)}}\le\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}\Leftrightarrow$$

$$\frac{b-a}{\sqrt[3]{1-\frac{1}{4}\cdot\frac{sin4b-sin4a}{b-a}}}\le\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}\Leftrightarrow$$

$$\frac{b-a}{\sqrt[3]{1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a}}}\le\int_a^b\frac{dx}{\sqrt[3]{1-\cos 4x}}$$

$$u(t)=\frac{\sin t}{t}, t\in\left(0,\frac{\pi}{2}\right)\Rightarrow u'(t)=\frac{t\cos t-\sin t}{t^2}$$

$$v(t)=t\cos t-\sin t\Rightarrow v'(t)=-t\sin t<0,\forall t\in\left(0,\frac{\pi}{2}\right)\Rightarrow v(t)<v(0)=0$$

$$\Rightarrow u'(t)<0\Rightarrow u(t)=\frac{\sin t}{t}\text{ –decreasing }\Rightarrow\frac{\sin 2(a+b)}{2(a+b)}<\frac{\sin 2(b-a)}{2(b-a)}\Rightarrow$$

$$1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a}<1-\frac{1}{2}\cdot\frac{\sin 2(b+a)\cos 2(a+b)}{b+a}; (*)$$

$$\frac{b-a}{\sqrt[3]{1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a}}}=$$

$$=\frac{b-a}{\sqrt[3]{1-\frac{\sin 2(b-a)}{2(b-a)}\cdot\cos 2(a+b)}}\overset{(*)}{\ge}\frac{b-a}{\sqrt[3]{1-\frac{\sin 2(b+a)}{2(b+a)}\cdot\cos 2(a+b)}}=$$

$$=\frac{b-a}{\sqrt[3]{1-\frac{\sin 4(a+b)}{4(a+b)}}}=\frac{(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-\sin 4(a+b)}}$$

Hence,

$$\frac{(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-\sin 4(a+b)}}\le\int_a^b\frac{dx}{\sqrt[3]{1-\cos 4x}};\quad (1)$$

$$1+\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\overset{AGM}{\ge}3\sqrt[3]{\frac{1}{\sin^2 x\cos^2 x}}=\frac{6}{\sqrt[3]{2(4\sin^2 x\cos^2 x)}}=$$

$$=\frac{6}{\sqrt[3]{2\sin^2 2x}}=\frac{6}{\sqrt[3]{1-\cos 4x}}$$

Hence,

$$6\int_a^b\frac{dx}{\sqrt[3]{1-\cos 4x}}<\int_a^b dx+\int_a^b\frac{1}{\cos^2 x}dx+\int_a^b\frac{1}{\sin^2 x}dx\Leftrightarrow$$

$$\int_a^b \frac{dx}{\sqrt[3]{1-cos4x}} < \frac{1}{6}[(b-a)+(tanb-tana)+(cota-cotb)] <$$

$$< \frac{1}{6}(tanb-tana)\left(1+\frac{1}{tana\ tanb}\right)+\frac{\pi}{12}=$$

$$=\frac{1}{6}\left(\frac{sinb}{cosb}-\frac{sina}{cosa}\right)\cdot\frac{1+tana\ tanb}{tana\ tanb}+\frac{\pi}{12}=$$

$$=\frac{1}{6}\cdot\frac{sin(b-a)}{cosa\ cosb}\cdot\frac{cosa\ cosb+sina\ sinb}{sina\ sinb}+\frac{\pi}{12}=$$

$$=\frac{1}{6}\cdot\frac{4sin(b-a)cos(b-a)}{4sina\ sinb\ cosa\ cosb}+\frac{\pi}{12}=\frac{1}{6}\cdot\frac{2sin(2b-2a)}{sin2a\ sin2b}+\frac{\pi}{12}=$$

$$=\frac{1}{3}\cdot\frac{sin2b\cos 2a-sin2a\ cos2b}{sin2a\ sin2b}+\frac{\pi}{12}=\frac{1}{3}\left(\frac{cos2a}{sin2a}-\frac{cos2b}{sin2b}\right)+\frac{\pi}{12}=$$

$=\frac{1}{3}(cot2a-cot2b)+\frac{\pi}{12}$. Hence,

$$3\int_a^b \frac{dx}{\sqrt[3]{1-cos4x}} < cot2a-cot2b+\frac{\pi}{4};\ \ (2)$$

From (1), (2) it follows that:

$$\frac{(b-a)3\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-sin4(a+b)}} < cot2a-cot2b+\frac{\pi}{4}$$

UP.337. If $0 < a \le b$ then:

$$\int_a^b\int_a^b\int_a^b \frac{yzdxdydz}{3x^2+2y^2+z^2} \le \frac{(b-a)^2(b+a)}{12}\cdot log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Abdul Hannan-Tezpur-India

$$0 < \frac{yz}{3x^2+2y^2+z^2} \overset{AGM}{\le} \frac{yz}{6\sqrt[6]{x^6y^4z^2}} = \frac{\sqrt[3]{yz^2}}{6x} \overset{AGM}{\le} \frac{\frac{y+z+z}{3}}{6x} = \frac{y+2z}{18x}$$

Therefore,

$$\int_a^b\int_a^b\int_a^b \frac{yzdxdydz}{3x^2+2y^2+z^2} \le \frac{1}{18}\left[\int_a^b\int_a^b\int_a^b \frac{y}{x}dxdydz + \int_a^b\int_a^b\int_a^b \frac{2z}{x}dxdydz\right] =$$

$$= \frac{1}{18}\left[\left(\int_a^b \frac{dx}{x}\right)\left(\int_a^b ydy\right)\left(\int_a^b dz\right) + \left(\int_a^b \frac{dx}{x}\right)\left(\int_a^b dy\right)\left(\int_a^b 2zdz\right)\right] =$$

$$= \frac{1}{18}\left[log\left(\frac{b}{a}\right)\left(\frac{b^2-a^2}{2}\right)(b-a) + log\left(\frac{b}{a}\right)(b-a)(b^2-a^2)\right] =$$

$$= \frac{1}{18}\cdot\frac{3}{2}\cdot log\left(\frac{b}{a}\right)(b^2-a^2)(b-a) = \frac{(b-a)^2(b+a)}{12}\cdot log\left(\frac{b}{a}\right)$$

Solution 2 by proposer

First, we prove that if $x, y, z > 0$ then:

$$\frac{18yz}{3x^2+2y^2+z^2} \le \frac{y+2z}{x} \Leftrightarrow 18xyz \le (y+2z)(3x^2+2y^2+z^2)$$

$$(y+2z)(3x^2+2y^2+z^2) =$$

$$= 3x^2y + 2y^3 + yz^2 + 6x^2z + 4zy^2 + 2z^3 \ge$$

$$\ge 18\cdot\sqrt[18]{(x^2y)^3\cdot(y^3)^2\cdot yz^2\cdot(x^2z)^6\cdot(zy^2)^4\cdot(z^3)^2} =$$

$$= 18\sqrt[18]{(xyz)^{18}} = 18xyz$$

By integration:

$$\int_a^b\int_a^b\int_a^b \frac{18yzdxdydz}{3x^2+2y^2+z^2} \le \int_a^b \frac{1}{x}dx\cdot\int_a^b y\,dy\cdot\int_a^b dz +$$

$$+2\int_a^b \frac{1}{x}dx\cdot\int_a^b dy\cdot\int_a^b z\,dz$$

$$18\int_a^b\int_a^b\int_a^b \frac{yzdxdydz}{3x^2+2y^2+z^2} \le (\log b - \log a)\cdot\frac{b^2-a^2}{2}\cdot(b-a) +$$

$$+2(\log b - \log a)\cdot(b-a)\cdot\frac{b^2-a^2}{2} =$$

$$= \frac{3}{2}\log\left(\frac{b}{a}\right)\cdot(b^2-a^2)\cdot(b-a) = \frac{3}{2}(b-a)^2(b+a)\log\left(\frac{b}{a}\right)$$

$$\int_a^b\int_a^b\int_a^b \frac{yzdxdydz}{3x^2+2y^2+z^2} \le \frac{(b-a)^2(b+a)}{12}\cdot\log\left(\frac{b}{a}\right)$$

UP.338 Let a, b, c be positive real numbers such that $a^2+b^2+c^2=12$.

Prove that:

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ba}\right)\left(\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\right)\geq 12$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

We have: $\frac{1}{\sqrt{a^3+1}}=\frac{1}{\sqrt{(a+1)(a^2-a+1)}}\geq\frac{1}{\frac{(a+1)+(a^2-a+1)}{2}}=\frac{2}{a^2+2}$. So

$\frac{a^2}{\sqrt{a^3+1}}\geq\frac{2a^2}{a^2+2}$ and similarly: $\frac{b^2}{\sqrt{b^3+1}}\geq\frac{2b^2}{b^2+2}$; $\frac{c^2}{\sqrt{c^3+1}}\geq\frac{2c^2}{c^2+1}$.

Adding up these last inequalities, we obtain:

$$\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\geq 2\left(\frac{a^2}{a^2+2}+\frac{b^2}{b^2+2}+\frac{c^2}{c^2+2}\right)$$

Applying the Cauchy-Schwarz inequality, we have:

$$\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\geq 2\frac{(a+b+c)^2}{(a^2+b^2+c^2)+6}=2\frac{(a+b+c)^2}{18}=$$

$$=\frac{a^2+b^2+c^2+2(ab+bc+ca)}{9}\geq\frac{(ab+bc+ca)+2(ab+bc+ca)}{2}=$$

$$=\frac{ab+bc+ca}{3}$$

Now, we will prove that $ab+bc+ca\geq\frac{3abc(a^2+b^2+c^2)}{a^3+b^3+c^3}$ or

$$(a^3+b^3+c^3)(ab+bc+ca)\geq 3abc(a^2+b^2+c^2)$$

We have:

$$(a^3+b^3+c^3)(ab+bc+ca)-3abc(a^2+b^2+c^2)=$$

$$=a^4b+ab^4+b^4c+bc^4+c^4a+ca^4-2a^3bc-2ab^3c-2abc^3=$$

$$=ab(a^3-c^3)+ca(a^3-b^3)+ab(b^3-c^3)+bc(b^3-a^3)+ca(c^3-b^3)$$
$$+bc(c^3-a^3)=$$

$$=(a^3-b^3)(ac-bc)+(b^3-c^3)(ab-ca)+(c^3-a^3)(bc-ab)=$$

$$=c(a-b)^2(a^2+ab+b^2)+a(b-c)^2(b^2+bc+c^2)+b(b-c)^2(c^2+ca+a^2)\geq 0$$

So, $$\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\ge\frac{\frac{3abc(a^2+b^2+c^2)}{a^3+b^3+c^3}}{3}=\frac{12abc}{a^3+b^3+c^3}\Leftrightarrow$$

$$\frac{a^3+b^3+c^3}{abc}\left(\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\right)\ge 12$$

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ba}\right)\left(\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\right)\ge 12$$

Equality holds when $a=b=c=12$.

Solution 2 by Rustam Tahmazov-Baku-Azerbaijan

$$Lhs=\frac{1}{abc}(a^3+b^3+c^3)\sum\frac{a^2}{\sqrt{a^3+1}}\ge 12\Leftrightarrow$$

$$(a^3+b^3+c^3)\sum\frac{a^2}{\sqrt{a^3+1}}\ge 12abc$$

$$\sum\frac{a^2}{\sqrt{a^3+1}}=\sum\frac{a^2}{\sqrt{(a+1)(a^2-a+1)}}\overset{AM-GM}{\ge}\sum\frac{2a^2}{(a+1)+(a^2-a+1)}=$$

$$=\sum\frac{2a^2}{a^2+2}\overset{CBS}{\ge}2\frac{(a+b+c)^2}{\sum a^2+6}=\frac{(a+b+c)^2}{9};(1)$$

By power-mean inequality:

$$\sqrt[3]{a^3+b^3+c^3}\ge\frac{a+b+c}{\sqrt[3]{9}};\ \sqrt[3]{(a^3+b^3+c^3)^2}\ge\frac{a^2+b^2+c^2}{\sqrt[3]{9}}$$

$$\Rightarrow a^3+b^3+c^3\ge\frac{(a+b+c)(a^2+b^2+c^2)}{3}=4(a+b+c);(2)$$

From $(1)\&(2)$ we get:

$$Lhs\ge 4(a+b+c)\cdot\frac{(a+b+c)^2}{9}\ge 12abc$$

$\Rightarrow (a+b+c)^3\ge 27abc$ true from AM-GM.

UP.339 Prove that for any positive real numbers a,b,c

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{1}{2}(a+b+c)\ge\frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by proposer

Lemma 1. For any positive real numbers a, b, c then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a^2+b^2+c^2)(a+b+c)}{ab+bc+ca}$$

Proof. The inequality is equivalent to

$$(ab+bc+ca)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\geq(a^2+b^2+c^2)(a+b+c)$$

Or $\frac{a^3c}{b}+\frac{b^3a}{c}+\frac{c^3b}{a}\geq a^2b+b^2c+c^2a$

This follows from the AM-GM inequality as follows

$$\frac{a^3c}{b}+\frac{b^3a}{c}\geq 2a^2b,$$

$$\frac{b^3a}{c}+\frac{c^3b}{a}\geq 2b^2c,$$

$$\frac{c^3b}{a}+\frac{a^3c}{b}\geq 2c^2a.$$

Using the above lemma, it's enough to show that

$$\frac{(a^2+b^2+c^2)(a+b+c)}{ab+bc+ca}+\frac{1}{2}(a+b+c)\geq\frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

This is equivalent to

$$\frac{(a^2+b^2+c^2)(a+b+c)}{ab+bc+ca}-\frac{3(a^2+b^2+c^2)}{a+b+c}\geq\frac{3(a^2+b^2+c^2)}{2(a+b+c)}-\frac{1}{2}(a+b+c)$$

$$\frac{(a^2+b^2+c^2)[(a+b+c)^2-3(ab+bc+ca)]}{(ab+bc+ca)(a+b+c)}\geq\frac{3(a^2+b^2+c^2)-(a+b+c)^2}{2(a+b+c)}$$

$$\frac{(a^2+b^2+c^2)(a^2+b^2+c^2-ab-bc-ca)}{ab+bc+ca}\geq a^2+b^2+c^2-ab-bc-ca$$

$$(a^2+b^2+c^2-ab-bc-ca)\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)\geq 0$$

The last inequality is clearly true and we are done.

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c)=\left(\frac{a^2}{b}-2a+b\right)+\left(\frac{b^2}{c}-2b+c\right)+\left(\frac{c^2}{a}-2c+a\right)=$$

$$=\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}$$

$$\frac{9(a^2+b^2+c^2)}{2(a+b+c)}-\frac{3(a+b+c)}{2}=\frac{9(a^2+b^2+c^2)-3(a+b+c)^2}{2(a+b+c)}=$$

$$=\frac{6(a^2+b^2+c^2-ab-bc-ca)}{2(a+b+c)}=\frac{3[(a-b)^2+(b-c)^2+(c-a)^2]}{2(a+b+c)}$$

Therefore,

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{1}{2}(a+b+c)\ge\frac{9(a^2+b^2+c^2)}{3(a+b+c)}\Leftrightarrow$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}\ge\frac{3[(a-b)^2+(b-c)^2+(c-a)^2]}{2(a+b+c)}\Leftrightarrow$$

$$\left(\frac{1}{b}-\frac{3}{2(a+b+c)}\right)(a-b)^2+\left(\frac{1}{c}-\frac{3}{2(a+b+c)}\right)(b-c)^2$$

$$+\left(\frac{1}{a}-\frac{3}{2(a+b+c)}\right)(c-a)^2\ge 0\Leftrightarrow$$

$$\frac{2(a+c)-b}{2b(a+b+c)}(a-b)^2+\frac{2(a+b)-c}{2c(a+b+c)}(b-c)^2+\frac{2(b+c)-a}{2a(a+b+c)}(c-a)^2\ge 0\Leftrightarrow$$

$$S=\sum\frac{2(a+c)-b}{2b(a+b+c)}(a-b)^2\ge 0;(1)$$

Let us denote: $S_a=\frac{2(a+c)-b}{2b(a+b+c)}, S_b=\frac{2(a+b)-c}{2c(a+b+c)}, S_c=\frac{2(b+c)-a}{2a(a+b+c)}$

If $a\ge b\ge c$ **then** $S_a>0, S_c>0$

$$S_a+2S_b=\frac{2(a+c)-b}{2b(a+b+c)}+2\cdot\frac{2(b+c)-a}{2a(a+b+c)}>0\Leftrightarrow$$

$$a(2a+2b-c)+2c(2b+2c-a)\ge 0\Leftrightarrow$$

$2a^2+2ab+2bc+4c^2-3ac\ge 0$ **true by** $2a^2+2c^2=2(a^2+c^2)\ge 4ac>3ac\Rightarrow$

$$2a^2+2ab+2bc+4c^2>2a^2+2c^2>3ac$$

Similarly: $S_c+2S_b>0\overset{S.O.S.}{\Longrightarrow}S\ge 0;(2)$

If $a\le b\le c$ **then** $S_b>0$

$$S_a + S_b = \frac{2(a+c)-b}{2b(a+b+c)} + \frac{2(a+b)-c}{2c(a+b+c)} > 0 \Leftrightarrow$$

$$a(2a+2b-c) + c(2b+2c-a) > 0 \Leftrightarrow$$

$$2a^2 + 2c^2 - 2ac + 2ab + 2bc \geq 0 \text{true by } 2a^2 + 2b^2 \geq 4ac > 2ac \Rightarrow$$

$$2a^2 + 2c^2 - 2ac + 2ab + 2bc > 2a^2 + 2c^2 - 2ac > 0$$

Similarly: $S_b + S_c > 0 \overset{S.O.S.}{\Longrightarrow} S \geq 0; (3) \Rightarrow S \geq 0, \forall a,b,c > 0 \Rightarrow (1)$ is true.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c > 0$, we have $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{1}{2}(a+b+c) \geq \frac{9}{2} \cdot \frac{a^2+b^2+c^2}{a+b+c}$

$$(a+b+c)\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{1}{2}(a+b+c)\right) \geq \frac{9}{2}(a^2+b^2+c^2)$$

$$2(a+b+c)\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) + (a+b+c)^2 \geq 9(a^2+b^2+c^2)$$

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} + \frac{a^2c}{b} + \frac{c^2b}{a} + \frac{b^2a}{c} + ab + bc + ca \geq 3(a^2+b^2+c^2)$$

and since $\left(\frac{a^3}{b} + ab\right) + \left(\frac{b^3}{c} + bc\right) + \left(\frac{c^3}{a} + ca\right) \geq 2(a^2+b^2+c^2)$

We will show $\frac{a^2c}{b} + \frac{c^2b}{a} + \frac{b^2a}{c} \geq a^2 + b^2 + c^2$

Iff $a^3c^2 + c^3b^2 + b^3a^2 \geq a^3bc + b^3ca + c^3ab$

Select $a \leq b \leq c; a, ax, axy, x, y \geq 1$

$$a^3(axy)^2 + (axy)^2 + (axy)^2(ax)^2 + (ax)^3a^2$$
$$\geq a^3(axaxy) + (ax)^3(aaxy) + (axy)^3(aax)$$

Iff $x^2y^2 + x^5y^3 + x^3 \geq x^2y + x^4y + x^4y^3$

$$y^2 + x^3y^3 + x \geq y + x^2y + x^2y^3$$

$$y(y-1) + x^2y^3(x-1) \geq x(xy-1) \geq xy(x-1) \text{true.}$$

$$y(y-1) + (x-1)(x^2y^3 - xy) = (x-1)xy(xy^2-1) \geq 0 \text{true.}$$

Solution 4 by Abdul Hannan-Tezpur-India

Claim: $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}$; (1)

Proof: (1)$\Leftrightarrow \left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)(ab+bc+ca)\geq(a^2+b^2+c^2)(a+b+c)$

$$\Leftrightarrow \sum a^3+\sum a^2c+\sum\frac{a^3c}{b}\geq\sum a^3\sum a^2b+\sum a^2c$$

$$\Leftrightarrow \sum\frac{a^3c}{b}\geq\sum a^2b\Leftrightarrow\sum a^4c^2\geq a^3b^2c$$

Which is true by$\sum x^2\geq\sum xy,\ \ x=a^2c, y=b^2a, z=c^2b$

So, it is enough to prove that

$$\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}+\frac{a+b+c}{2}\geq\frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

$$\Leftrightarrow (a+b+c)^2\left(\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{1}{2}\right)\geq\frac{9(a^2+b^2+c^2)}{2}$$

Let $k=\frac{a^2+b^2+c^2}{ab+bc+ca}\geq 1$**. To prove:**

$$(k+2)\left(k+\frac{1}{2}\right)\geq\frac{9k}{2}\Leftrightarrow(k-1)^2\geq 0 \text{which is true.}$$

Solution 5 by Marin Chirciu-Romania

Lemma. If $a,b,c>0$ **then** $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}$

Proof. We have: $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}\Leftrightarrow$

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)(ab+bc+ca)\geq(a+b+c)(a^2+b^2+c^2)\Leftrightarrow$$

$$\sum a^3+\sum ab^2+\sum\frac{ab^3}{c}\geq\sum a^3+\sum ab^2+\sum a^2b\Leftrightarrow$$

$$\sum\frac{ab^3}{c}\geq\sum a^2b\Leftrightarrow\sum a^2b^4\geq abc\sum a^2b\Leftrightarrow$$

$$a^2(b^2-ca)^2+b^2(c^2-ab)^2+c^2(a^2-bc)^2\geq 0$$

Equality holds if and only if $a=b=c$**.**

Let's get back to the main problem.

$$LHS=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+n(a+b+c)\geq\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}+n(a+b+c)\geq$$

$$\overset{(1)}{\geq}3(n+1)\frac{a^2+b^2+c^2}{a+b+c}=RHS,$$

$$(1)\Leftrightarrow\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}+n(a+b+c)\geq 3(n+1)\frac{a^2+b^2+c^2}{a+b+c}$$

$$\Leftrightarrow\frac{\sum a\sum a^2}{\sum bc}+n\sum a\geq 3(n+1)\frac{\sum a^2}{\sum a}\Leftrightarrow\sum a\left(\frac{\sum a^2}{\sum bc}+n\right)\geq 3(n+1)\frac{\sum a^2}{\sum a}\Leftrightarrow$$

$$\left(\sum a\right)^2\left(\frac{\sum a^2}{\sum bc}+n\right)\geq 3(n+1)\sum a^2\Leftrightarrow$$

$$\frac{(\sum a)^2}{\sum a^2}\left(\frac{\sum a^2}{\sum bc}+n\right)\geq 3(n+1)\Leftrightarrow\frac{\sum a^2+2\sum bc}{\sum a^2}\left(\frac{\sum a^2}{\sum bc}+n\right)\geq 3(n+1)$$

$$\Leftrightarrow\left(1+\frac{2}{t}\right)(t+n)\geq 3(n+1)\Leftrightarrow(t+2)(t+n)\geq 3(n+1)t$$

$\Leftrightarrow t^2-(2n+1)t+2n\geq 0$, which follows from $t\geq 1\geq 2n$, where

$t=\frac{\sum a^2}{\sum bc}$, equality for $t=1\Leftrightarrow\sum a^2=\sum bc\Leftrightarrow$

$$(a-b)^2+(b-c)^2+(c-a)^2=0\Leftrightarrow a=b=c.$$

Note. For $n=\frac{1}{2}$, we obtain Problem UP.339,RMM, proposed by Nguyen Viet Hung-Hanoi-Vietnam.

UP.340 If $0<a\leq b; f:[a,b]\to[1,\infty); f$ continuous, then:

$$3(b-a)^2\int_a^b f(x)\,dx\leq 2(b-a)^3+\left(\int_a^b f(x)\,dx\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

We prove by induction that: $x_1,x_2,\dots,x_n\in[1,\infty); n\in\mathbb{N}^*$implies:

$$x_1+x_2+\cdots+x_n\leq n-1+x_1x_2\dots x_n$$

Checking: $n=1; x_1\leq 1-1+x_1\Leftrightarrow x_1\leq x_1$

$n=2: x_1+x_2\leq 1+x_1x_2\Leftrightarrow(x_1-1)(x_2-1)\geq 0$. True.

$P(k): x_1+x_2+\cdots+x_k\leq k-1+x_1x_2\dots x_k$ (1)

Suppose that it's true:

$P(k+1): x_1+x_2+\cdots x_k+x_{k+1}\leq k+x_1x_2\dots x_kx_{k+1}$ (to prove)

$$x_1+x_2+\cdots+x_k+x_{k+1}\overset{P(k)}{\leq} k-1+x_1x_2\ldots x_k+x_{k+1}$$

Remains to prove that:

$$k-1+x_1x_2\ldots x_k+x_{k+1}\leq k+x_1x_2\ldots x_kx_{k+1}$$

$$x_1x_2\ldots x_kx_{k+1}-x_1x_2\ldots x_k-x_{k+1}+1\geq 0$$

$$x_1x_2\ldots x_k(x_{k+1}-1)-(x_{k+1}-1)\geq 0$$

$(x_{k+1}-1)(x_1x_2\ldots x_k-1)\geq 0$ **which is true because**

$$x_{k+1}\geq 1;x_1x_2\ldots x_k\geq 1$$

$$P(k)\to P(k+1)$$

In (1) we take: $x_1\to f(x);x_2\to f(y);x_3\to f(z)$

$$f(x)+f(y)+f(z)\leq 2+f(x)f(y)f(z)\ \ (2)$$

By integration in (2):

$$\int_a^b\int_a^b\int_a^b\big(f(x)+f(y)+f(z)\big)\,dx\,dy\,dz\leq\int_a^b\int_a^b\int_a^b 2dx\,dy\,dz+$$

$$+\int_a^b\int_a^b\int_a^b f(x)f(y)f(z)\,dx\,dy\,dz$$

$$3\int_a^b f(x)\,dx\int_a^b\int_a^b dy\,dz\leq 2(b-a)^3+\left(\int_a^b f(x)\,dx\right)^3$$

$$3(b-a)^2\int_a^b f(x)\,dx\leq 2(b-a)^3+\left(\int_a^b f(x)\,dx\right)^3$$

Equality holds for $a=b$ or $f(x)\equiv 1$.

Solution 2 by Ravi Prakash-New Delhi-India

Let $g(x)=f(x)-1$, then $g(x)\geq 0,\forall x\in[a,b]$.

Also, $g(x)$ –is continuous function on $[a,b]$. Let:

$$A=\int_a^b g(x)dx\Rightarrow\int_a^b f(x)dx=(b-a)+A$$

Consider

$$2(b-a)^3+\left(\int_a^b f(x)dx\right)^3-3(b-a)^2\int_a^b f(x)dx=2(b-a)^3+$$

$$+[(b-a)^3+3(b-a)^2A+3(b-a)A^2+a^3-3(b-a)^3-3(b-a)^2A]=$$

$$=3(b-a)A^2+A^3\geq 0,(A\geq 0)$$

Therefore,

$$3(b-a)^2\int_a^b f(x)\,dx\leq 2(b-a)^3+\left(\int_a^b f(x)\,dx\right)^3$$

Solution 3 by Mohammad Rostami-Kabul-Afghanistan

$$t=\int_a^b dx=b-a\geq 0,u=\int_a^b f(x)dx>0$$

$$f(x)\geq 1\Rightarrow\int_a^b f(x)dx\geq\int_a^b dx\Rightarrow\int_a^b f(x)dx\geq b-a\Rightarrow u\geq t\Rightarrow u-t\geq 0$$

$$\begin{cases}f(x)\geq 1\\ 0<a\leq b\end{cases}\Rightarrow 2t+u\geq 0\Rightarrow t+u\geq -t\Rightarrow -t(u-t)\leq(u-t)(u+t)\Rightarrow$$

$$-tu+t^2\leq u^2-t^2\Rightarrow 2t^2\leq u^2+tu\Rightarrow 3t^2\leq u^2+tu+t^2\Rightarrow$$

$$3t^2(u-t)\leq(u-t)(u^2+tu+t^2)\Rightarrow 3t^2u-3t^3\leq u^3-t^3\Rightarrow$$

$$3t^2u\leq 2t^3+u^3\Leftrightarrow 3t^2u\leq 2t^3+u^3$$

Therefore,

$$3(b-a)^2\int_a^b f(x)\,dx\leq 2(b-a)^3+\left(\int_a^b f(x)\,dx\right)^3$$

UP.341. Find:

$$\Omega=\lim_{n\to\infty}\left(\int_0^{\frac{\pi}{3}}\frac{\sin x}{\cos x\,(1+\cos^n x)}dx\right)$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\int_0^{\frac{\pi}{3}}\frac{\sin x}{\cos x\,(1+\cos^n x)}dx=\int_0^{\frac{\pi}{3}}\frac{\sin x\cdot\cos^{n-1}x}{\cos^n x\,(1+\cos^n x)}dx=-\frac{1}{n}\int_0^{\frac{\pi}{3}}\frac{(\cos^n x)'}{\cos^n x\,(1+\cos^n x)}dx=$$

$$y=\cos^n x\,;dy=(\cos^n x)'dx$$

$$x=0\Rightarrow y=1;x=\frac{\pi}{3}\Rightarrow y=\left(\frac{1}{2}\right)^n$$

$$=-\frac{1}{n}\int_1^{\frac{1}{2^n}}\frac{dy}{y(y+1)}=\frac{1}{n}\int_{\frac{1}{2^n}}^1\left(\frac{1}{y}-\frac{1}{y+1}\right)dy=\frac{1}{n}\left(\ln 1-\ln\frac{1}{2^n}-\ln\frac{1}{2}+\ln\left(1+\frac{1}{2^n}\right)\right)=$$

$$=\frac{1}{n}\left(\ln 2+\ln\frac{1+\frac{1}{2^n}}{\frac{1}{2^n}}\right)=\frac{1}{n}\cdot\ln\left(\frac{2^n+1}{2}\right)$$

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1+2^n}{2}\right)\overset{CESARO-STOLZ}{=}$$

$$=\lim_{n\to\infty}\frac{\ln\left(\frac{1+2^{n+1}}{2}\right)-\ln\left(\frac{1+2^n}{2}\right)}{n+1-n}=\lim_{n\to\infty}\ln\left(\frac{1+2^{n+1}}{1+2^n}\right)=\ln 2$$

UP.342. Prove that if $0<a\le b$ then:

$$\left(\int_a^b\frac{\log x}{x}dx\right)^2\ge\left(\int_{\frac{a+b}{2}}^b\frac{\log x}{x}dx+\int_{\sqrt{ab}}^b\frac{\log x}{x}dx\right)\left(\int_a^{\frac{a+b}{2}}\frac{\log x}{x}dx+\int_a^{\sqrt{ab}}\frac{\log x}{x}dx\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohammad Rostami-Kabul-Afganistan

$$(a-b)^2\ge 0\Rightarrow a^2-2ab+b^2\ge 0\Rightarrow a^2+2ab+b^2\ge 4ab\Rightarrow (a+b)^2\ge 4ab\Rightarrow$$

$$a+b\ge 2\sqrt{ab}\Rightarrow\frac{a+b}{2}\ge\sqrt{ab};(I)$$

$$a\le b\Rightarrow a+b\le 2b\Rightarrow\frac{a+b}{2}\le b;(II)$$

$$a\le b\Rightarrow a^2\le ab\Rightarrow a\le\sqrt{ab};\quad(III)$$

$$\int_a^b\frac{\log x}{x}dx=\underbrace{\int_a^{\sqrt{ab}}\frac{\log x}{x}dx}_{A}+\underbrace{\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx}_{B}+\underbrace{\int_{\frac{a+b}{2}}^b\frac{\log x}{x}dx}_{C}=A+B+C$$

$$\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx+\int_{\sqrt{ab}}^{b}\frac{\log x}{x}dx=\underbrace{\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx}_{C}+\underbrace{\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx}_{B}+\underbrace{\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx}_{C}=C+B+C$$

$$=2C+B$$

$$\int_{a}^{\frac{a+b}{2}}\frac{\log x}{x}dx+\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx=\underbrace{\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx}_{A}+\underbrace{\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx}_{B}+\underbrace{\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx}_{A}=A+B+A$$

$$=2A+B$$

$$\left(\int_{a}^{b}\frac{\log x}{x}dx\right)^2\ge\left(\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx+\int_{\sqrt{ab}}^{b}\frac{\log x}{x}dx\right)\left(\int_{a}^{\frac{a+b}{2}}\frac{\log x}{x}dx+\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx\right)\Leftrightarrow$$

$$(A+B+C)^2\ge(2C+B)(2A+B)\Leftrightarrow$$

$$A^2+B^2+C^2+2AB+2BC+2CA\ge 4AC+2CB+2BA+B^2\Leftrightarrow$$

$$A^2+C^2+2AC\ge 4AC\ \Leftrightarrow (A-C)^2\ge 0$$

Solution 2 by Nassim Nicholas Taleb-New York-USA

$$\text{Let } G[x]=\int\frac{\log x}{x}dx$$

$$\big(G(b)-G(a)\big)^2\ge\left(\left(G(b)-G\left(\frac{a+b}{2}\right)\right)+\left(G(b)-G(\sqrt{ab})\right)\right)\cdot$$

$$\cdot\left(\left(G\left(\frac{a+b}{2}\right)-G(a)\right)+G(\sqrt{ab})-G(a)\right)\cdot\left(G(a)+G(b)-G\left(\frac{a+b}{2}\right)-G(\sqrt{ab})\right)^2$$

$$\ge 0$$

Solution 3 by proposer

$$0<a\le\sqrt{ab}\le\frac{a+b}{2}\le b$$

Denote:

$$u=\int_a^{\sqrt{ab}}\frac{\log x}{x}dx;\ v=\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx;\ w=\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx$$

$$\int_{\sqrt{ab}}^{b}\frac{\log x}{x}dx=\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx+\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx=v+w$$

$$\int_a^{\frac{a+b}{2}}\frac{\log x}{x}dx=\int_a^{\sqrt{ab}}\frac{\log x}{x}dx+\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx=u+v$$

Inequality to prove can be written:

$$(u+v+w)^2\ge(w+v+w)(u+v+u)$$

$$(u+v+w)^2\ge(v+2w)(v+2u)$$

$$u^2+v^2+w^2+2uv+2uw+2vw\ge v^2+2uv+2vw+4wu$$

$$u^2+w^2-2uw\ge 0\Leftrightarrow(u-w)^2\ge 0$$

Equality holds for $a=b$.

UP.343 Let a,b,c be positive real numbers such that $a^2+b^2+c^2=3$.

Prove that:

$$2(a^4+b^4+c^4)-(a^3+b^3+c^3)\ge 3abc$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

We will prove that:

Lemma. If a,b,c are real numbers, then

$$2(a+b+c)(a^2+b^2+c^2)\ge 3(a^3+b^3+c^3+3abc)$$

Proof. We have:

$$2(a+b+c)(a^2+b^2+c^2)-3(a^3+b^3+c^3)=$$

$$=2a^3+2a(b^2+c^2)+2b^3+2b(c^2+a^2)+2c^3+2c(a^2+b^2)-3a^3-3b^3-3c^3-9abc=$$

$$=[-a^3+a^2(b+c)-abc]+[-b^3+b^2(c+a)-abc]+[-c^3+c^2(a+b)-abc]=$$

$$=[a^2(b+c)-a(b+c)^2+bc(b+c)]+[b^2(c+a)-b(c+a)^2+ca(c+a)]+$$

$$+[c^2(a+b)-c(a+b)^2+ab(a+b)]=-a(a-b)(a-c)-b(b-a)(b-c)-$$

$$-c(c-a)(c-b)+(b+c)(a-b)(a-c)+(c+a)(b-a)(b-c)+$$

$$+(a+b)(c-a)(c-b)=(b+c-a)(a-b)(a-c)+(c+a-b)(b-a)(b-c)+$$
$$+(a+b-c)(c-a)(c-b)$$

Putting $b+c-a=x, c+a-b=y, a+b-c=z$ we have:

$$\frac{1}{4}[x(x-y)(x-z)+y(y-x)(y-z)+z(z-x)(z-y)]\geq 0$$

Because we have the well-known Schur's inequality. The lemma is completely solved.

Now, $a^3+b^3+c^3=3$, so $3(a^3+b^3+c^3)\leq 2(a+b+c)\cdot 3$

$a^3+b^3+c^3+3abc\leq 2(a+b+c)$. Also we have the well-known inequality:

$(a+b+c)^2\leq 3(a^2+b^2+c^2)\Leftrightarrow a+b+c\leq 3$. So,

$a^3+b^3+c^3+3abc\leq 6$. Also we have:

$(a^2+b^2+c^2)^2\leq 3(a^4+b^4+c^4)$, so $a^4+b^4+c^4\geq 3\Leftrightarrow 2(a^4+b^4+c^4)\geq 6$.

So, $2(a^4+b^4+c^4)\geq a^3+b^3+c^3+3abc\Leftrightarrow$

$2(a^4+b^4+c^4)-(a^3+b^3+c^3)\geq 3abc$.

Equality holds when $a=b=c=1$.

UP.344 Let a,b,c be non-negative real numbers, no two of which are zero. Prove that:

$$\frac{a}{a^2+2(b+c)^2}+\frac{b}{b^2+2(c+a)^2}+\frac{c}{c^2+2(a+b)^2}\geq\frac{1}{a+b+c}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by proposer

By Cauchy-Schwarz inequality we have

$$\sum_{cyc}\frac{a}{a^2+2(b+c)^2}\geq\frac{(a+b+c)^2}{\sum(a^3+2a(b+c)^2)}$$

It suffices to show that $\frac{(a+b+c)^2}{\sum(a^3+2a(b+c)^2)}\geq\frac{1}{a+b+c}$. This is equivalent to

$$(a+b+c)^3\geq\sum_{cyc}(a^3+2a(b+c)^2)$$

Or

$$3(a+b)(b+c)(c+a) \geq 2a(b^2+c^2)+2b(c^2+a^2)+2c(a^2+b^2)+12abc$$

$$a(b^2+c^2)+b(c^2+a^2)+c(a^2+2v) \geq 6abc$$

$a(b-c)^2+b(c-a)^2+c(a-b)^2 \geq 0$ which is obvious and we are done.

The equality holds for $a=b=c$ or $(a,b,c)=(x,0,0)$ with $x>0$ and it's permutations.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum \frac{a}{a^2+2(b+c)^2} \overset{BCS}{\geq} \frac{(\sum a)^2}{\sum(a^3+2a(b+c)^2)} \overset{(1)}{\geq} \frac{1}{\sum a}$$

$$(1) \Leftrightarrow \left(\sum a\right)^3 \geq \sum a^3+2\sum a^2b+12abc \Leftrightarrow \sum a^2b \geq 6abc$$

$$\Leftrightarrow \sum a(b-c)^2 \geq 0(true).$$

Solution 3 by Rohan Shinde-Bangalore-India

First of all note that by Murihead inequality, since $\{2,1,0\}$ majories $\{1,1,1\}$

$$a^2b+b^2a+a^2c+ac^2+b^2c+bc^2 \geq 6abc$$

$$6abc-(a^2b+b^2a+a^2c+ac^2+b^2c+bc^2)+(a+b+c)^3 \leq (a+b+c)^3$$

$$\frac{1}{6abc-(a^2b+b^2a+a^2c+ac^2+b^2c+bc^2)+(a+b+c)^3} \geq \frac{1}{(a+b+c)^3};(1)$$

$$\frac{a}{a^2+2(b+c)^2}+\frac{b}{b^2+2(a+c)^2}+\frac{c}{c^2+2(a+b)^2}=$$

$$=\frac{a^2}{a^3+2a(b+c)^2}+\frac{b^2}{b^3+2b(a+c)^2}+\frac{c^2}{c^3+2c(a+b)^2} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(a+b+c)^2}{a^3+b^3+c^3+2a(b+c)^2+2b(c+a)^2+2c(a+b)^2}=$$

$$=\frac{(a+b+c)^2}{(a+b+c)^3+6abc-(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)} \overset{(1)}{\geq}$$

$$\geq \frac{(a+b+c)^2}{(a+b+c)^3}=\frac{1}{a+b+c}$$

Solution 4 by Marian Ursărescu-Romania

We must show that:

$$\frac{a}{a+b+c}\cdot\frac{1}{a^2+2(b+c)^2}+\frac{b}{a+b+c}\cdot\frac{1}{b^2+2(c+a)^2}+\frac{c}{a+b+c}\cdot\frac{1}{c^2+2(a+b)^2} \geq \frac{1}{(a+b+c)^2};(1)$$

Let $f:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{x}, f'(x)=-\frac{1}{x^2}, f''(x)=\frac{2}{x^3}>0\Rightarrow f$ −convex and from Jensen:

$$p_1f(x_1)+p_2f(x_2)+p_3f(x_3)\ge f(p_1x_1+p_2x_2+p_3x_3), p_1+p_2+p_3=1$$

$$\Rightarrow\sum_{cyc}\frac{a}{a+b+c}\cdot\frac{1}{a^2+2(b+c)^2}\ge$$

$$\ge\frac{a+b+c}{a^3+b^3+c^3+2a(b+c)^2+2b(c+a)^2+2c(a+b)^2};(2)$$

From (1),(2) we have:

$$\frac{a+b+c}{a^3+b^3+c^3+2a^2b+2ab^2+2b^2c+2bc^2+2c^2a+2ca^2+12abc}\ge\frac{1}{(a+b+c)^3}$$

$$\Leftrightarrow (a+b+c)^3\ge a^3+b^3+c^3+2ab(a+b)+2bc(b+c)+2ca(c+a)+12abc$$

$$\Leftrightarrow ab(a+b)+bc(b+c)+ca(c+a)\ge 6abc, \text{ which is true because:}$$

$$ab(a+b)+bc(b+c)+ca(c+a)\ge 2\left(ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}\right)\ge 6\sqrt[3]{a^3b^3c^3}=6abc$$

Solution 5 by Tran Hong-Dong Thap-Vietnam

- $\frac{a}{a^2+2(b+c)^2}+\frac{b}{b^2+2(c+a)^2}+\frac{c}{c^2+2(a+b)^2}\ge\frac{1}{a+b+c};$

$$\leftrightarrow (a+b+c)\left(\sum a(b^2+2(c+a)^2)(c^2+2(a+b)^2)\right)$$

$$-(a^2+2(b+c)^2)(b^2+2(c+a)^2)(c^2+2(a+b)^2)\ge 0;$$

$$\leftrightarrow 6\sum ab(a^4+b^4)+6\sum a^2b^2(a^2+b^2)+4abc(a^3+b^3+c^3)$$

$$-30a^2b^2c^2-9abc(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)\ge 0;(1)$$

By AM-GM we have:

- $3\left(a^5b+a^4c^2+b^2c^4\right)\ge 3.3.\sqrt[3]{a^9b^3c^6}=9abc(a^2c);(2)$
- $3\left(b^5c+b^4a^2+c^2a^4\right)\ge 9abc(b^2a);(3)$
- $3\left(c^5a+c^4b^2+a^2b^4\right)\ge 9abc(c^2b);(4)$
- $3\left(b^5a+b^4c^2+a^2c^4\right)\ge 9abc(b^2c);(5)$
- $3\left(c^5b+c^4a^2+b^2a^4\right)\ge 9abc(c^2a);(6)$
- $3\left(a^5c+a^4b^2+c^2b^4\right)\ge 9abc(a^2b);(7)$

$$\overset{(3)+(4)+(5)+(6)+(7)}{\Rightarrow}\quad 3\sum ab(a^4+b^4)+6\sum a^2b^2(a^2+b^2)\ge 9abc\sum(a^2b+ab^2);(8)$$

- $3\sum ab(a^4+b^4)=$

$$3(a^5b+ab^5+b^5c+c^5b+a^5c+c^5a)\overset{AM-GM}{\geq}3.6.\sqrt[6]{a^{12}b^{12}c^{12}}=$$

$$18a^2b^2c^2;(9)$$

- $4abc(a^3+b^3+c^3)\overset{AM-GM}{\geq}4abc.3.abc=12abc;(10)$

$$\overset{(9)+(10)}{\Rightarrow}3\sum ab(a^4+b^4)+4abc(a^3+b^3+c^3)\geq 18abc+12abc$$

$$=30abc;(11)$$

$$\overset{(10)+(11)}{\Rightarrow}6\sum ab(a^4+b^4)+6\sum a^2b^2(a^2+b^2)+4abc(a^3+b^3+c^3)$$

$$\geq 30a^2b^2c^2+9abc(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2);$$

$\rightarrow(1)$ is true. Proved.

Solution 6 by Marin Chirciu-Romania

Using Bergstrom Inequality, we have:

$$LHS=\sum\frac{a}{a^2+2(b+c)^2}=\sum\frac{a^2}{a^3+2a(b+c)^2}\geq\frac{(\sum a)^2}{\sum[a^3+2a(b+c)^2]}\overset{(1)}{\geq}\frac{1}{\sum a}=RHS$$

$$(1)\Leftrightarrow\frac{(\sum a)^2}{\sum[a^3+2a(b+c)^2]}\geq\frac{1}{\sum a}\Leftrightarrow\left(\sum a\right)^3\geq\sum a^3+2\sum a(b+c)^2$$

$$\Leftrightarrow\sum a^3+3\prod(b+c)\geq\sum a^3+2\sum a(b+c)^2$$

$$\Leftrightarrow\sum a^3+3\left[\sum bc(b+c)+2abc\right]\geq\sum a^3+2\left[\sum bc(b+c)+6abc\right]$$

$$\Leftrightarrow\sum bc(b+c)\geq 6abc\Leftrightarrow 3\sum bc(b+c)+6abc\geq 2\sum bc(b+c)+12abc$$

$$\Leftrightarrow\sum bc(b+c)\geq 6abc\Leftrightarrow\sum a(b-c)^2\geq 0,(true).$$

Equality holds when $a=b=c$.

Remark. Inequality can be developed.

If $a,b,c>0$ and $\lambda\geq 2$ then:

$$\sum_{cyc}\frac{a}{a^2+\lambda a(b+c)^2}\geq\frac{9}{(1+4\lambda)(a+b+c)}$$

Proposed by Marin Chirciu-Romania

Solution.

Using Bergstrom Inequality, we have:

$$LHS = \sum \frac{a}{a^2 + \lambda(b+c)^2} = \sum \frac{a^2}{a^3 + \lambda a(b+c)^2}$$

$$\geq \frac{(\sum a)^2}{\sum[a^3 + \lambda a(b+c)^2]} \overset{(1)}{\geq} \frac{9}{(1+4\lambda)\sum a} = RHS$$

$$(1) \Leftrightarrow \frac{(\sum a)^2}{\sum[a^3 + \lambda a(b+c)^2]} \geq \frac{9}{(1+4\lambda)\sum a}$$

$$\Leftrightarrow (1+4\lambda)\left(\sum a\right)^3 \geq 9\sum a^3 + 9\lambda \sum a(b+c)^2$$

$$\Leftrightarrow (1+4\lambda)\sum a^3 + 3(1+4\lambda)\prod(b+c) \geq 9\sum a^3 + 9\lambda\sum a(b+c)^2$$

$$\Leftrightarrow (1+4\lambda)\sum a^3 + 3(1+4\lambda)\left[\sum bc(b+c) + 2abc\right]$$

$$\geq 9\sum a^3 + 9\lambda\left[\sum bc(b+c) + 6abc\right]$$

$\Leftrightarrow (4\lambda - 8)\sum a^3 + 3(\lambda+1)\sum bc(b+c) \geq (30\lambda - 6)abc$, which is true from Schur's inequality: $\sum a^3 \geq \sum bc(b+c) - 3abc, \lambda \geq 2$.

It is enough to prove that:

$$(4\lambda - 8)\left(\sum bc(b+c) - 3abc\right) + 3(\lambda+1)\sum bc(b+c) \geq (30\lambda - 6)abc$$

$\Leftrightarrow (7\lambda - 5)\sum bc(b+c) \geq (42\lambda - 30)abc \Leftrightarrow (7\lambda - 5)\sum bc(b-c)^2 \geq 0$, which follows from: $\sum bc(b-c)^2 \geq 0$ and $(7\lambda - 5) \geq 0$ true, because $\lambda \geq 2$.

Equality holds when $a = b = c$.

Note.For $\lambda = 2$ we obtain Problem UP.344 from R.M.M.-2021, proposed by Nguyen Viet Hung-Hanoi-Vietnam.

UP.345 Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$.

Prove that: $(a+b)(b+c)(c+a) - 2abc \leq 6$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

We will prove that:

Lemma. If a,b,c are real numbers, then:

$$2(a+b+c)(a^2+b^2+c^2)\geq 3(a^3+b^3+c^3+3abc)$$

Proof. We have: $2(a+b+c)(a^2+b^2+c^2)-3(a^3+b^3+c^3+3abc)=$

$$=2a^3+2a(b^2+c^2)+2b^3+2b(c^2+a^2)+2c^3+2c(a^2+b^2)-3a^3-3b^3-3c^3-9abc=$$

$$=[-a^3+a^2(b+c)-abc]+[-b^3+b^2(c+a)-abc]+[-c^3+c^2(a+b)-abc]=$$

$$=[a^2(b+c)-a(b+c)^2+bc(b+c)]+[b^2(c+a)-b(c+a)^2+ca(c+a)+$$

$$+[c^2(a+b)-c(a+b)^2+ab(a+b)]=$$

$$=-a(a-b)(a-c)-b(b-a)(b-c)-c(c-a)(c-b)+(b+c)(a-b)(a-c)+$$

$$+(c+a)(b-a)(b-c)+(a+b)(c-a)(c-b)=$$

$$(b+c-a)(a-c)(b-c)+(c+a-b)(b-a)(b-c)+(a+b-c)(c-a)(c-b)$$

We putting: $b+c-a=x, c+a-b=y, a+b-c=z$ **then we have:**

$$\frac{1}{4}[x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y)]\geq 0$$

Because we have the well-known Schur's inequality. The lemma is completely solved.

Now, $a^2+b^2+c^2=3$, **so** $3(a^3+b^3+c^3+3abc)\leq 2(a+b+c)\cdot 3$

$a^3+b^3+c^3+3abc\leq 2(a+b+c)$. **Also we have the well-known inequality**

$(a+b+c)^2\leq 3(a^2+b^2+c^2)\Leftrightarrow a+b+c\leq 3$. **So,**

$$a^3+b^3+c^3+3abc\leq 6;\ \ (1)$$

Also we have:

$$a^3+b^3+c^3+3abc\geq ab(a+b)+bc(b+c)+ca(c+a)$$

So, $a^3+b^3+c^3+3abc\geq (a+b)(b+c)(c+a)-2abc;\ \ (2)$

From $(1)\&(2)$ **we have:** $(a+b)(b+c)(c+a)-2abc\leq 6$

Equality holds when $a=b=c=1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum a\overset{BCS}{\leq}\sqrt{3\sum a^2}\Rightarrow 9\geq\left(\sum a\right)\left(\sum a^2\right)=\sum a^3+\sum a^2b\overset{Schur's}{\geq}$$

$$\geq 2\sum a^2b - 3abc \geq 2\sum a^2b - \frac{1}{2}\sum a^2b \Rightarrow 9 \geq \frac{3}{2}\sum a^2b \Rightarrow \sum a^2b \leq 6.$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$3 = a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} = \frac{p^2}{3} \Rightarrow 0 < p \leq 3$$

Now, $(a+b)(b+c)(c+a) - 2abc \leq 6 \Leftrightarrow pq - r - 2r \leq 6 \Leftrightarrow pq \leq 6 + 3r$

$$\Leftrightarrow p\left(\frac{p^2-3}{2}\right) - 6 \leq 3r \Leftrightarrow \frac{p^3-3p}{6} - 2 \leq r; (1)$$

On the other hand, by Schur's inequality, we have:

$$r \geq \frac{p(4q-p^2)}{9} = \frac{p(p^2-6)}{9}; (2)$$

From (1),(2) we need to prove: $\frac{p(p^2-6)}{9} \geq \frac{p^3-3p}{6} - 2 \Leftrightarrow \frac{p^3}{18} + \frac{p}{6} - 2 \leq 0 \Leftrightarrow$

$\frac{1}{18}(p-3)(p^3+3p+12) \leq 0$**, which is true by** $0 < p \leq 3 \Rightarrow (1)$ **is true.**

Solution 4 by Ruxandra Daniela Tonilă-Romania

$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \Leftrightarrow 3 \geq \frac{(a+b+c)^2}{3} \Leftrightarrow 9 \geq (a+b+c)^2$$

$$\Leftrightarrow 3 \geq a+b+c \Leftrightarrow a^2+b^2+c^2 \geq a+b+c; (1)$$

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$$

$$(a+b)(b+c)(c+a) - 2abc = (a+b+c)(ab+bc+ca) - 3abc$$

So, we need to prove that:$(a+b+c)(ab+bc+ca) - 3abc \leq 6$

$$(1): \begin{cases} a+b+c \leq a^2+b^2+c^2 \\ ab+bc+ca \leq a^2+b^2+c^2 \end{cases} \Rightarrow (a+b+c)(ab+bc+ca) \leq (a^2+b^2+c^2)^2$$

$$\Leftrightarrow (a+b+c)(ab+bc+ca) \leq 9; (2)$$

$$a+b+c \leq a^2+b^2+c^2 \Leftrightarrow \frac{a+b+c}{3} \leq 1 \Rightarrow 1 \geq \frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

$$\Rightarrow abc \leq 1 \Rightarrow 3abc \leq 3; (3)$$

From (1),(2), we get:

$$(a+b+c)(ab+bc+ca) - 3abc \leq 9 - 3 \Leftrightarrow$$

$$(a+b+c)(ab+bc+ca) - 3abc \leq 6 \Leftrightarrow$$

$$(a+b)(b+c)(c+a) - 2abc \leq 6$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru