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JP.346 Find all values of k such that the following inequality:

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{kab}{a+b}\geq\left(1+\frac{k}{4}\right)(a+b)$$

holds for all positive real numbers a,b.

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by Alex Szoros-Romania

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{kab}{a+b}\geq\left(1+\frac{k}{4}\right)(a+b);(*)\Leftrightarrow\frac{a^2}{b}+\frac{b^2}{a}-(a+b)+k\left(\frac{ab}{a+b}-\frac{a+b}{4}\right)\geq 0$$

Let be the function $f:\mathbb{R}\to\mathbb{R}, f(k)=\left(\frac{ab}{a+b}-\frac{a+b}{4}\right)k+\frac{a^2}{b}+\frac{b^2}{a}-(a+b)$

$$f'(k)=\frac{ab}{a+b}-\frac{a+b}{4}=\frac{4ab-(a+b)^2}{4(a+b)}\Rightarrow f'(k)=-\frac{(a-b)^2}{4(a+b)}\leq 0,\forall a,b>0\Rightarrow$$

f −decreasing; (1)

But $f(16)=\frac{a^2}{b}+\frac{b^2}{a}-(a+b)-\frac{16(a-b)^2}{4(a+b)}=\frac{(a+b)(a-b)^2}{ab}-\frac{4(a-b)^2}{a+b}$

$$f(16)=(a-b)^2\left(\frac{a+b}{ab}-\frac{4}{a+b}\right)=(a-b)^2\left[\frac{(a+b)^2-4ab}{ab(a+b)}\right]=\frac{(a-b)^4}{ab(a+b)}\geq 0,$$

$\forall a,b>0\Rightarrow f(16)\geq 0;(2)$

From (1),(2) it follows that: $k\leq 16\Rightarrow f(k)\geq f(16)\geq 0$, so $f(k)\geq 0,\forall k\leq 16$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{kab}{a+b}\geq\left(1+\frac{k}{4}\right)(a+b)\Leftrightarrow\frac{a^2}{b}+\frac{b^2}{a}-(a+b)+k\left(\frac{ab}{a+b}-\frac{a+b}{4}\right)\geq 0$$

$$\Leftrightarrow\frac{(a-b)^2(a+b)}{ab}\geq\frac{k(a-b)^2}{4(a+b)}$$

For $a=b\Rightarrow(*)$ holds for all $k\Rightarrow(*)\Leftrightarrow(a+b)^2\geq\frac{kab}{4}$

If $k\leq 0\Rightarrow(*)$ holds for all $a,b>0$. Let $k>0$ and let $x=\frac{a}{b}$, then

$(*)\Leftrightarrow x^2+\left(2-\frac{k}{4}\right)x+1\geq 0$, with $\Delta=\left(2-\frac{k}{4}\right)^2-4=\frac{k(k-16)}{16}$

$k\leq 0\Rightarrow\Delta\leq 0\Rightarrow(*)$ holds for all $a,b>0$.

If $k\leq 16$, and if $0<x<\frac{1}{2}\left(-2+\frac{k}{4}+\sqrt{\Delta}\right)>0\Rightarrow x^2+\left(2-\frac{k}{4}\right)x+1<0.$

Therefore, $k \leq 16$.

Solution 3 by proposer

The inequality is equivalent to

$$\frac{a^2}{b}+\frac{b^2}{c}-(a+b)+\frac{kab}{a+b}-\frac{k(a+b)}{4}\geq 0$$

$$\frac{a^3+b^3-ab(a+b)}{ab}-\frac{k[(a+b)^2-4ab]}{4(a+b)}\geq 0$$

$$\left(\frac{a+b}{ab}-\frac{k}{4(a+b)}\right)(a-b)^2\geq 0$$

$$\frac{a+b}{ab}\geq\frac{k}{4(a+b)}$$

$$\frac{4(a+b)^2}{ab}\geq k$$

This is true for all positive real numbers a, b if and only if

$$\min_{a,b>0}\frac{4(a+b)^2}{ab}\geq k$$

Because $4(a+b)^2\geq 16ab$ and the equality happens when $a=b$, it follows that

$$\min_{a,b>0}\frac{4(a+b)^2}{ab}=16$$

So, $k\leq 16$.

JP.347 Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\frac{2(a+b)(a+c)}{b+c}+\frac{2(b+c)(b+a)}{c+a}+\frac{2(c+a)(c+b)}{a+b}>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by proposer

We rewrite the inequality as

$$\sum_{cyc}\frac{2(a^2+ab+bc+ca)}{b+c}>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}$$

Or equivalent to

$$\sum_{cyc}\left(\frac{2(a^2+bc)}{b+c}+2a\right)>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}$$

$$\sum_{cyc}\frac{2(a^2+bc)}{b+c}>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}-2(a+b+c)$$

$$\sum_{cyc}\frac{2(a^2+bc)}{b+c}>\frac{[(a+b+c)^2-(ab+bc+ca)]^2}{2(ab+bc+ca)(a+b+c)}$$

$$\sum_{cyc}\frac{2(a^2+bc)}{b+c}>\frac{(a^2+b^2+c^2+ab+bc+ca)^2}{2(ab+bc+ca)(a+b+c)}$$

Now, we use the Cauchy-Schwarz inequality to obtain

$$\sum_{cyc}\frac{a^2+bc}{b+c}=\sum_{cyc}\frac{(a^2+bc)^2}{(a^2+bc)(b+c)}\geq\frac{(a^2+b^2+c^2+ab+bc+ca)^2}{\sum(a^2+bc)(b+c)}=$$

$$=\frac{(a^2+b^2+c^2+ab+bc+ca)^2}{2\sum bc(b+c)}\geq\frac{(a^2+b^2+c^2+ab+bc+ca)^2}{2(ab+bc+ca)(a+b+c)}$$

Note that equality can't happen. The proof is completed.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{2(a+b)(a+c)}{b+c}+\frac{2(b+c)(b+a)}{c+a}+\frac{2(c+a)(c+b)}{a+b}$$

$$>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c};(*)$$

$$(*)\Leftrightarrow 2\sum_{cyc}\frac{a^2}{b+c}+2\sum_{cyc}a+2\sum_{cyc}\frac{bc}{b+c}>\frac{(\sum a)^3}{\sum ab}+\frac{\sum ab}{\sum a}\Leftrightarrow$$

$$2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{a^2}{b+c}\right)+2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{bc}{b+c}\right)$$

$$>\left(\sum_{cyc}a^3\right)+\frac{(\sum ab)^2}{\sum a}\Leftrightarrow$$

$$2\sum_{cyc}\frac{a^3b}{b+c}+2\sum_{cyc}\frac{a^3c}{b+c}+2abc\sum_{cyc}\frac{a}{b+c}+2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+2abc\sum_{cyc}\frac{b}{b+c}$$

$$+2abc\sum_{cyc}\frac{c}{b+c}+2\sum_{cyc}\frac{(bc)^2}{b+c}>\left(\sum_{cyc}a\right)^3+\frac{(\sum ab)^2}{\sum a}\Leftrightarrow$$

$$2\sum_{cyc}a^3+2abc\sum_{cyc}\frac{a}{b+c}+2\sum_{cyc}a^2b+6abc+6abc+2\sum_{cyc}\frac{(bc)^2}{b+c}$$

$$>\sum_{cyc}a^3+3\sum_{cyc}a^2b+6abc+\frac{(\sum ab)^2}{\sum a}\Leftrightarrow$$

$$\left(\sum_{cyc}a^3+3abc\right)+2\sum_{cyc}\frac{(bc)^2}{b+c}+2abc\sum_{cyc}\frac{a}{b+c}+3abc>\sum_{cyc}a^2b+\frac{(\sum ab)^2}{\sum a};(1)$$

which is true, because $\sum a^3+3abc\ge\sum a^2b(Schur's)$ and $2\sum\frac{b^2c^2}{b+c}\ge\frac{(\sum bc)^2}{\sum a}(BCS)$.

Solution 3 by Tran Hong-Dong Thap-Vietnam

We have: $a+b>0, b+c>0, c+a>0$.

$$\sum_{cyc}\frac{2(a+b)(b+c)}{b+c}=2\sum_{cyc}\frac{\left((a+b)(b+c)\right)^2}{(a+b)(b+c)(c+a)}=2\frac{\sum\left((a+b)(b+c)\right)^2}{(a+b)(b+c)(c+a)}=$$

$$=\frac{2\sum\left((a+b)(a+c)\right)^2}{(a+b+c)(ab+bc+ca)-abc}>\frac{2\sum\left((a+b)(a+c)\right)^2}{(a+b+c)(ab+bc+ca)}$$

We need to prove that:

$$\frac{2\sum\left((a+b)(a+c)\right)^2}{(a+b+c)(ab+bc+ca)}>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}\Leftrightarrow$$

$$2\sum_{cyc}\left((a+b)(a+c)\right)^2>(a+b+c)^4+(ab+bc+ca)^2\Leftrightarrow$$

$$2[(a^2+ab+bc+ca)^2+(b^2+ab+bc+ca)^2+(c^2+ab+bc+ca)^2]$$

$$>(a+b+c)^4+(ab+bc+ca)^2\Leftrightarrow$$

$$2[a^4+b^4+c^4+2(a^2+b^2+c^2)(ab+bc+ca)+3(ab+bc+ca)^2]$$

$$>\left(a^2+b^2+c^2+2(ab+bc+ca)\right)^2+(ab+bc+ca)^2\Leftrightarrow$$

$$2\left[\sum_{cyc} a^4 + 2\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) + 3\left(\sum_{cyc} ab\right)^2\right] >$$

$$> \left(\sum_{cyc} a^2\right)^2 + 4\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) + 5\left(\sum_{cyc} ab\right)^2 \Leftrightarrow$$

$$2\sum_{cyc} a^4 + \left(\sum_{cyc} ab\right)^2 > \left(\sum_{cyc} a^2\right)^2 \Leftrightarrow \sum_{cyc} a^4 + \left(\sum_{cyc} ab\right)^2 > 2\sum_{cyc} a^2b^2 \Leftrightarrow$$

$$\sum_{cyc} a^4 + 2abc(a+b+c) > \sum_{cyc} a^2b^2$$

Which is true, because: $x^2 + y^2 + z^2 \geq xy + yz + zx$. Hence,

$$\sum_{cyc} (a^2)^2 \geq \sum_{cyc} a^2b^2 \Rightarrow \sum_{cyc} a^4 + 2abc(a+b+c) > \sum_{cyc} a^2b^2$$

JP.348 If $a, b, c > 0$ then:

$$\left(\frac{a^4}{b^4} + \frac{b^4}{c^4} + \frac{c^4}{a^4}\right)\left(\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3}\right) \geq \left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

We will prove:

$$\sum_{cyc} \frac{a^4}{b^4} \geq \sum_{cyc} \frac{a^2}{b^2} \quad (1)$$

$$\sum_{cyc} \frac{a^3}{b^3} \geq \sum_{cyc} \frac{a^2}{b^2} \quad (2)$$

(1) $\Leftrightarrow \sum_{cyc} a^8c^4 \geq \sum_{cyc} a^6b^2c^4$ (by multiplying with $(abc)^4$)

$$\sum_{cyc} a^8c^4 = \frac{1}{12}\sum_{cyc} 12a^8c^4 = \frac{1}{12}\sum_{cyc}(8a^8c^4 + 2a^8c^4 + 2a^8c^4) =$$

$$= \frac{1}{12}\sum_{cyc}(8a^8c^4 + 2b^8a^4 + 2c^8b^4) \geq$$

$$\overset{AM-GM}{\geq}\frac{1}{12}\sum_{cyc}12\sqrt[12]{(a^8c^4)^8\cdot(b^8a^4)^2\cdot(c^8b^4)^2}=\sum_{cyc}\sqrt[12]{a^{72}b^{24}c^{48}}=\sum_{cyc}a^6b^2c^4$$

(2) $\Leftrightarrow \sum_{cyc} a^6c^3 \geq \sum_{cyc} a^5bc^3$ **(by multiplying with** $(abc)^3$**)**

$$\sum_{cyc}a^6c^3=\frac{1}{9}\sum_{cyc}9a^6c^3=\frac{1}{9}\sum_{cyc}(7a^6c^3+a^6c^3+a^6c^3)=$$

$$=\frac{1}{9}\sum_{cyc}(7a^6c^3+b^6a^3+c^6b^3)\overset{AM-GM}{\geq}\frac{1}{9}\sum_{cyc}9\sqrt[9]{(a^6c^3)^7\cdot b^6a^3\cdot c^6b^3}=$$

$$=\sum_{cyc}\sqrt[9]{a^{45}\cdot b^9\cdot c^{27}}=\sum_{cyc}a^5bc^3$$

By multiplying (1); (2):

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right)\geq\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Equality holds for $a=b=c$.

Solution 2 by Tan Tey Tan-China

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right)\geq\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(3\cdot\sqrt[3]{\prod_{cyc}\frac{a^3}{b^3}}\right)=\left(\sum_{cyc}\frac{a^4}{b^4}\right)\cdot 3=$$

$$=\left(\sum_{cyc}\frac{a^4}{b^4}\right)(1+1+1)\geq\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\left(\frac{a}{b}\right)^4\overset{CBS}{\geq}\frac{1}{3}\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2,\quad \sum_{cyc}\left(\frac{a}{b}\right)^3\overset{AM-GM}{\geq}3\prod_{cyc}\left(\frac{a}{b}\right)=3$$

Therefore,

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right)\geq\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}\right)\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\right)\geq\frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}{9}\geq$$

$$\geq\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)=\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2$$

Solution 5 by Samar Das-India

$$\sum_{cyc}\left(\frac{a}{b}\right)^4\geq 3\sqrt[3]{\prod_{cyc}\left(\frac{a}{b}\right)^4}=3;(1);\sum_{cyc}\left(\frac{a}{b}\right)^4\geq 3\sqrt[3]{\prod_{cyc}\left(\frac{a}{b}\right)^3}=3;(2)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^2\geq 3\sqrt[3]{\prod_{cyc}\left(\frac{a}{b}\right)^2}=3;(3)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^4=\sum_{cyc}\left(\frac{a}{b}\right)^{2+2}\geq\frac{1}{3}\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)^2=\frac{1}{3}\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2;(4)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^3=\frac{1}{3}\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right);(5);\ \sum_{cyc}\frac{a}{b}\geq 3;(6)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^4\cdot\sum_{cyc}\left(\frac{a}{b}\right)^3-\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2\geq\frac{1}{3}\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2\cdot\frac{1}{3}\sum_{cyc}\left(\frac{a}{b}\right)-\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2=$$

$$=\frac{1}{9}\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2\left[\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)-9\right]=\frac{1}{9}\sum_{cyc}\left(\frac{a}{b}\right)^2\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)-9\right)$$

Therefore,

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right)\geq\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Solution 6 by Tran Hong-Dong Thap-Vietnam

Let $x=\frac{a}{b}>0, y=\frac{b}{c}>0, z=\frac{c}{a}>0\Rightarrow xyz=1$

$$x^4+y^4+z^4=x^2\cdot x^2+y^2\cdot y^2+z^2\cdot z^2 \overset{Chebyshev's}{\geq} \frac{1}{3}\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x^2\right) \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{3}\cdot 3\cdot \sqrt[3]{(xyz)^2}\cdot \sum_{cyc} x^2 = \sum_{cyc} x^2 ;(1)$$

$$x^3+y^3+z^3=x\cdot x^2+y\cdot y^2+z\cdot z^2 \overset{Chebyshev's}{\geq} \frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right) \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{3}\cdot 3\cdot \sum_{cyc} x^2 = \sum_{cyc} x^2 ;(2)$$

From (1),(2) it follows that:

$$\left(\sum_{cyc} x^4\right)\left(\sum_{cyc} x^3\right) \geq \left(\sum_{cyc} x^2\right)^2$$

Therefore,

$$\left(\sum_{cyc} \frac{a^4}{b^4}\right)\left(\sum_{cyc} \frac{a^3}{b^3}\right) \geq \left(\sum_{cyc} \frac{a^2}{b^2}\right)^2$$

Equality holds for $a=b=c$.

JP.349 Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a} \geq \frac{3}{2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

By AM-GM inequality, we have:

$$\frac{a^6}{a^2+b}+\frac{a^2+b}{4}+\frac{1}{2} \geq 3\sqrt[3]{\frac{a^6(a^2+b)}{8(a^2+b)}}=\frac{3a^2}{2}$$

$$\frac{b^6}{b^2+c}+\frac{b^2+c}{4}+\frac{1}{2} \geq \frac{3b^2}{2};\ \frac{c^6}{c^2+a}+\frac{c^2+a}{4}+\frac{1}{2} \geq \frac{3c^2}{2}$$

Adding up these inequalities, we get:

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}+\frac{a^2+b^2+c^2}{4}+\frac{3}{4}+\frac{3}{2}\geq\frac{3}{2}(a^2+b^2+c^2)$$

It ia well-known that $(a+b+c)^2\leq 3(a^2+b^2+c^2)$, so $a^3+b^3+c^3\geq 3$. So,

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}\geq\frac{5}{4}(a^2+b^2+c^2)-\frac{9}{4}\geq\frac{15}{4}-\frac{9}{4}=\frac{3}{2}$$

Namely

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}\geq\frac{3}{2}$$

Equality holds when $a=b=c=1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}a^2\overset{C-S}{\geq}\frac{(\sum a)^2}{3};(1)$$

$$\sum_{cyc}\frac{a^6}{a^2+b}\overset{C-S}{\geq}\frac{(\sum a^3)^2}{\sum a^2+\sum a}\overset{Chebyshev's}{\geq}\frac{(\sum a^2)^2(\sum a)^2}{9(\sum a^2+3)}\overset{(1)}{\geq}\frac{(\sum a^2)^2}{\sum a^2+\sum a^2}=\frac{\sum a^2}{2}\overset{(1)}{\geq}\frac{3}{2}$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$a+b+c=3\Rightarrow t=a^2+b^2+c^2\overset{C-S}{\geq}\frac{(a+b+c)^2}{3}=3$$

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}=\frac{\sum(a^2)^3}{a^2+b}\overset{Holder}{\geq}\frac{(\sum a^2)^3}{3(\sum a^2+\sum a)}=\frac{t^3}{3(t+3)}$$

We need to prove:

$$\frac{t^3}{3(t+3)}\geq\frac{3}{2}\Leftrightarrow 2t^3-9t-27\geq 0\Leftrightarrow (t-3)(2t^2+6t+9)\geq 0$$

which is true, because $t\geq 3$. Equality holds for $t=3\Leftrightarrow a=b=c=1$.

Solution 4 by Sanong Huayrerai Nakon Pathom-Thailand

For $a,b,c>0$ and $a+b+c=3$ we have:

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}\geq\frac{(a^2+b^2+c^2)^3}{3(a^2+b^2+c^2+a+b+c)}\geq\frac{3}{2}$$

Iff $2(a^2+b^2+c^2)^3\geq 9(a^2+b^2+c^2+3)$

Iff $(a^2+b^2+c^2)^3+(a^2+b^2+c^2)^3\geq 9(a^2+b^2+c^2)+9\cdot 3$, which is true because $(a^2+b^2+c^2)^3=(a^2+b^2+c^2)^2(a^2+b^2+c^2)\geq 9(a^2+b^2+c^2)$ and

$(a^2+b^2+c^2)^3=(a^2+b^2+c^2)^2(a^2+b^2+c^2)\geq 9\cdot 3$, because $a+b+c=3\Rightarrow$
$a^2+b^2+c^2\geq 3$.

JP.350 For $1\leq a,b,c\leq \frac{2\sqrt{3}}{3}$, prove that:

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2}+(a+b+c)^2-3(a+b+c)\leq 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

First, will prove that $\sqrt{4-3a^2}\leq -3a^2+3a+1$. We have:

$$-3a^2+3a+1-\sqrt{4-3a^2}=3a(1-a)+\frac{1-(4-3a^2)}{1+\sqrt{4-3a^2}}=$$

$$=3a(1-a)+\frac{3a^2-3}{1+\sqrt{4-3a^2}}=3a(1-a)+\frac{3(a-1)(a+1)}{1+\sqrt{4-3a^2}}=$$

$$=3a(1-a)-3(1-a)\cdot\frac{a+1}{1+\sqrt{4-3a^2}}=3a(1-a)\left(\frac{a+1}{1+\sqrt{4-3a^2}}-1\right)=$$

$$=3a(a-1)\cdot\frac{a+1-1-\sqrt{4-3a^2}}{1+\sqrt{4-3a^2}}=3a(a-1)\left(\frac{a-\sqrt{4-3a^2}}{1+\sqrt{4-3a^2}}\right)=$$

$$=3a(a-1)\frac{a^2-4+3a^2}{\left(a+\sqrt{4-3a^2}\right)\left(1+\sqrt{4-3a^2}\right)}=$$

$$=3a(a-1)\cdot 4\cdot\frac{a^2-1}{\left(a+\sqrt{4-3a^2}\right)\left(a+\sqrt{4-3a^2}\right)}=\frac{12a(a-1)^2(a+1)}{\left(a+\sqrt{4-3a^2}\right)\left(1+\sqrt{4-3a^2}\right)}\geq 0$$

Equality holds when $a=1$. Similarly:

$$\sqrt{4-3b^2}\leq -3b^2+3b+1,\sqrt{4-3c^2}\leq -3c^2+3c+1$$

So,

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2}\leq -3(a^2+b^2+c^2)+3(a+b+c)+3$$

We know that $a^2+b^2+c^2\geq\frac{1}{3}(a+b+c)^2$. So,

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2}\leq -(a+b+c)^2+3(a+b+c)+3, \text{ namely}$$

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2}+(a+b+c)^2-3(a+b+c)\leq 3$$

Equality holds when $a=b=c=1$.

Solution 2 by Tran Hong-Dong Thap-Vietnam

Because: $1 \le a,b,c \le \frac{2\sqrt{3}}{3} \Rightarrow 3 \le a+b+c \le 2\sqrt{3},; 4-3a^2, 4-3b^2, 4-3c^2 \ge 0$

Now, $\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2} \overset{C-S}{\le} \sqrt{3[12-(a^2+b^2+c^2)]} \overset{\sum a^2 \ge \frac{(\sum a)^2}{3}}{\le}$

$$\le \sqrt{3(12-(\sum a)^2)} = \sqrt{3(12-t^2)}$$

We need to prove:

$$\sqrt{3(12-t^2)}+t^2-3t \le 3; (\because 3 \le t \le 2\sqrt{3}) \Leftrightarrow$$

$$\sqrt{3(12-t^2)} \le 3t+3-t^2 \Leftrightarrow$$

$$3(12-t^2) \le (3t+3-t^2)^2; (\because\ 3 \le t \le 2\sqrt{3} \Rightarrow 3t+3-t^2 \ge 6\sqrt{3}-9>0) \Leftrightarrow$$

$t^4-6t^3+6t^2+18t-27 \ge 0$, which is true because: $t \ge 3 \Rightarrow (t-3)^2 \ge 0 \Rightarrow$

$$t^2-3 \ge 6 > 0.$$

JP.351 In ΔABC the following relationship holds:

$$\prod_{cyc} sin^2A \ge 4\prod_{cyc} cosA - 5\prod_{cyc} cos^2A$$

Proposed by Cristian Miu-Romania

Solution 1 by proposer

We may assume that ΔABC is acute.

Now, it is enough to prove that these exists a triangle with sides $sin2A, sin2B, sin2C$,

because $sin^2 2A = sin^2 2B + sin^2 2C - 2sinBsinCcos(\pi-2A)$

This equality follows from $a^2 = b^2+c^2-2bccosA$ using law of sinus and changing

$$A \to \pi-2A, B \to \pi-2B, C \to \pi-2C.$$

Let us R_o, r_o, F_o —the radius of the circumscribed circle, inscribed circle and area of triangle, $sin2A = 2R_o sin2A \Rightarrow R_o = \frac{1}{2}$.

$$F_o = \frac{sin2Bsin2Csin(\pi-2A)}{2} = \frac{sin2Asin2Bsin2C}{2}$$

$$r_o = \frac{F_o}{s_o} = 2cosAcosBcosC$$

Because: $\sum sin2A = 4\prod sinA$.

Now, we want write Gerretsen inequality: $s^2 \geq 16Rr - 5r^2$ and obtain:

$$\left(\frac{1}{2}\sum sin2A\right)^2 \geq 16\prod cosA - 20\prod cos^2A$$

Hence,

$$\prod_{cyc} sin^2A \geq 4\prod_{cyc} cosA - 5\prod_{cyc} cos^2A$$

Because: $\sum sin2A = 4\prod sinA$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

If ΔABC is an obtuse triangle, thus $\prod cosA \leq 0 \Rightarrow$

$$\prod_{cyc} sin^2A \geq 0 \geq 4\prod_{cyc} cosA - 5\prod_{cyc} cos^2A$$

Let ΔABC be an acute triangle.

Let $X = \pi - 2A; Y = \pi - 2B; Z = \pi - 2C; \sum X = \pi$

$$\prod_{cyc} sin^2A \geq 4\prod_{cyc} cosA - 5\prod_{cyc} cos^2A \Leftrightarrow \prod_{cyc} cos^2\frac{X}{2} \geq 4\prod_{cyc} sin\frac{X}{2} - 5\prod_{cyc} sin^2\frac{X}{2} \Leftrightarrow$$

$$\left(\frac{s}{4R}\right)^2 \geq 4\left(\frac{r}{4R}\right) - 5\left(\frac{r}{4R}\right)^2 \Leftrightarrow s^2 \geq 16Rr - 5r^2 (Gerretsen)$$

Therefore,

$$\prod_{cyc} sin^2A \geq 4\prod_{cyc} cosA - 5\prod_{cyc} cos^2A$$

JP.352 If $a, b, c \in \mathbb{C}; |a| = |b| = |c| = 1$ then:

$$3|a+b+c| + 2(|a-b| + |b-c| + |c-a|) \geq 9$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$LHS = \sum_{cyc}(|a+b+c| + (|a-b| + |c-a|)) \geq \sum_{cyc}|a+b+c+a-b+a-c| =$$

$$= 3\sum_{cyc}|a| = 9$$

Therefore,

$$3|a+b+c|+2(|a-b|+|b-c|+|c-a|)\geq 9$$

Solution 2 by Samar Das-India

$$a,b,c\in\mathbb{C};|a|=|b|=|c|=1$$

Let $a=cos\theta+isin\theta; b=cos\alpha+isin\alpha; c=cos\beta+isin\beta$

$$a+b+c=cos\theta+isin\theta+cos\alpha+isin\alpha+cos\beta+isin\beta$$

$$|a+b+c|=\sqrt{(cos\alpha+cos\beta+cos\gamma)^2+(sin\alpha+sin\beta+sin\gamma)^2}=$$

$$=\sqrt{3+2\big(cos(\theta-\alpha)+cos(\alpha-\beta)+cos(\beta-\theta)\big)}=$$

$$=\sqrt{3-2\left(sin^2\left(\frac{\theta-\alpha}{2}\right)+sin^2\left(\frac{\alpha-\beta}{2}\right)+sin^2\left(\frac{\beta-\theta}{2}\right)\right)};(1)$$

$$|a-b|=\sqrt{(cos\theta-cos\alpha)^2+(sin\theta-sin\alpha)^2}=\sqrt{2-2cos(\theta-\alpha)}$$

$$=2\left|sin\left(\frac{\theta-\alpha}{2}\right)\right|;(2)$$

$$|b-c|=2\left|sin\left(\frac{\alpha-\beta}{2}\right)\right|;(3)\quad |c-a|=2\left|sin\left(\frac{\beta-\theta}{2}\right)\right|;(4)$$

Now, $3|a+b+c|+2|a-b|+2|b-c|+2|c-a|=$

$$=3\sqrt{9-2\left(sin^2\left(\frac{\theta-\alpha}{2}\right)+sin^2\left(\frac{\alpha-\beta}{2}\right)+sin^2\left(\frac{\beta-\theta}{2}\right)\right)}+$$

$$+4\left(\left|sin\left(\frac{\theta-\alpha}{2}\right)\right|+\left|sin\left(\frac{\alpha-\beta}{2}\right)\right|+\left|sin\left(\frac{\beta-\theta}{2}\right)\right|\right)=9$$

Solution 3 by proposer

$$|a+b+c|+|a-b|+|c-a|\geq$$

$$\geq|a+b+c+a-b|+|c-a|=$$

$$=|2a+c|+|c-a|=|2a+c|+|a-c|\geq$$

$$\geq|2a+c+a-c|=3|a|=3$$

$$|a+b+c|+|a-b|+|c-a|\geq 3\quad \textbf{(1)}$$

Analogous:

$$|a+b+c|+|c-a|+|b-c|\geq 3\quad \textbf{(2)}$$

$$|a+b+c|+|a-b|+|b-c|\geq 3 \quad (3)$$

By adding (1); (2); (3):

$$3|a+b+c|+2(|a-b|+|b-c|+|c-a|)\geq 9$$

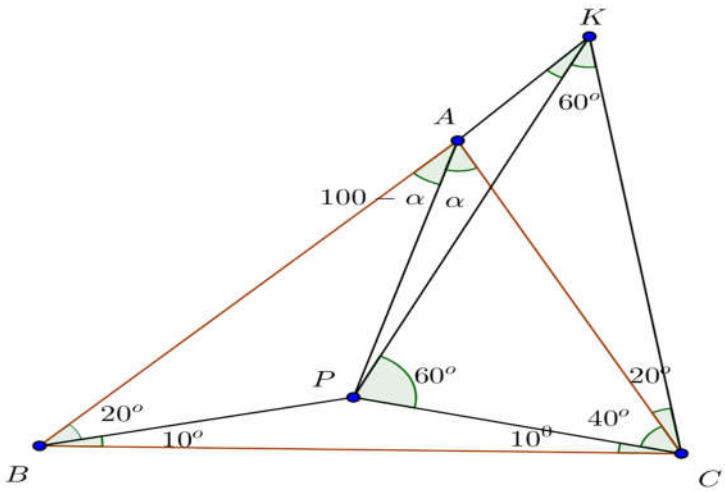
JP.353 In $\Delta ABC, P\in Int(\Delta ABC), \mu(\widehat{ABP})=20^o,$

$$\mu(\widehat{PBC})=\mu(\widehat{PCB})=10^o, \mu(\widehat{PCA})=40^0.$$

Prove that: $|AP|+|BC|=\sqrt{3}|AB|$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution 1 by proposer



Let $K\in[BA), |PK|=|PB|\Rightarrow \Delta PKC$ −is equilateral triangle.

$$|CK|=|CA|, |CA|=|CP|, \alpha=70^o, \mu(\widehat{BAP})=30^o$$

In $\Delta APB, \Delta APC$ applying sines law, we have:

$$\frac{|AP|}{sin20^o}=\frac{|AB|}{sin50^o};(1)\Rightarrow\frac{|AP|}{|AB|}=\frac{sin20^o}{sin50^o}$$

$$\frac{|AP|}{sin40^o}=\frac{|PC|}{sin70^o};(2)\Rightarrow\frac{|AP|}{|PC|}=2sin20^o;(3)(PB=PC)$$

In $\Delta PBC: \frac{|BC|}{sin20^o}=\frac{|PC|}{sin10^o}\Rightarrow |BC|=2cos10^o;(4)$

$$|AB|+|BC|=2|PB|sin20^o+2|PB|cos10^o=$$

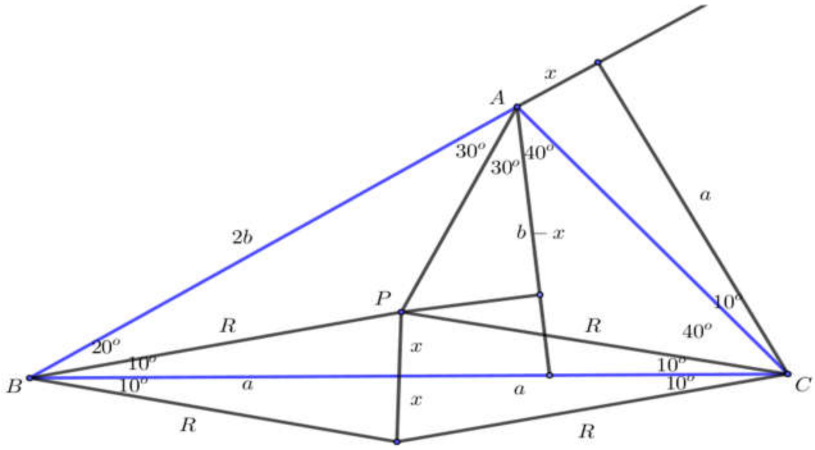
$$=2|PB|(sin20^o+sin80^o)=2|PB|\cdot 2sin50^o cos30^o=2\sqrt{3}|PB|sin50^o;(5)$$

$$\overset{(1)}{\Rightarrow} sin50^o = \frac{|AB|}{|AP|} sin20^o = \frac{|AB|}{2|PB| sin20^o} sin20^o = \frac{|AB|}{2|PB|}; (6)$$

$$\overset{(5),(6)}{\Longrightarrow} |AP| + |BC| = 2\sqrt{3}|PB| \cdot \frac{|AB|}{2|PB|}$$

$$|AP| + |BC| = \sqrt{3}|AB|$$

Solution 2 by Eltac Qafarli-Baku-Azerbaijan



$$(2b + x)^2 + a^2 = 8 \cdot 4a^2; |AP| = \frac{2(b - x)}{\sqrt{3}}$$

$$2b + x = a\sqrt{3} \Rightarrow 3b = \sqrt{3}a + b - x$$

$$\sqrt{3}b = a + \frac{b - x}{\sqrt{3}}$$

$$\frac{\sqrt{3}|AB|}{2} = \frac{|BC|}{2} + \frac{|AP|}{2} \Rightarrow \sqrt{3}|AB| = |BC| + |AP|$$

JP.354 In acute $\Delta ABC, O$ −circumcenter, $F, K \in (AB), M, L(BC), E, N \in (CA)$ $\overline{FOE}, \overline{MON}, \overline{LOK}$ −are the antiparallels. Let ρ_a, ρ_b, ρ_c −inradii of $AFE, \Delta\Delta BLK, \Delta CMN$. Prove that: $\rho_a + \rho_b + \rho_c = R$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by proposer

Let u_a −semiperimeter of ΔAFE

$$2u_a = R(cotB + cotC) + R\left(\frac{sinB + sinC}{sinBsinC}\right) =$$

$$= R\frac{sinA}{sinBsinC} + R\frac{sinB + sinC}{sinBsinC} = R\frac{\frac{a+b+c}{2R}}{\frac{bc}{R^2}} = \frac{2s}{2R}\frac{4R^2}{bc}$$

$$2u_a = \frac{4R^2s}{bc} \Rightarrow u_a = \frac{2R^2s}{bc}; \quad (1)$$

$$[AFE] = \frac{1}{2}\frac{R}{sinC}\frac{R}{sinB}sinA = \frac{R^2}{2\frac{c}{2R}\frac{b}{2R}}\frac{a}{2R} = \frac{aR^2 \cdot 4R^2}{4Rbc} = \frac{a}{bc}R^3; (2)$$

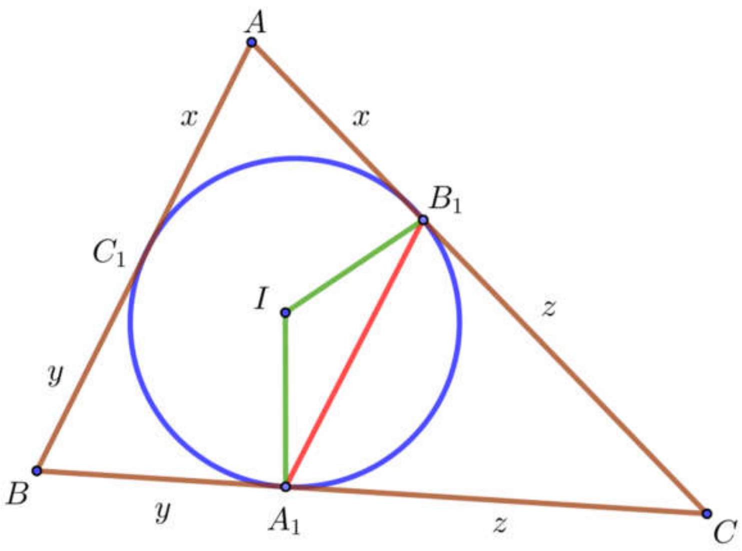
From (1),(2) it follows that: $\frac{a}{bc}R^3 = \frac{2R^2s}{bc}\rho_a, \left(\rho_a = \frac{aR}{2s}\right)$

Therefore,

$$\rho_a + \rho_b + \rho_c = \frac{R}{2s}(a+b+c) = R$$

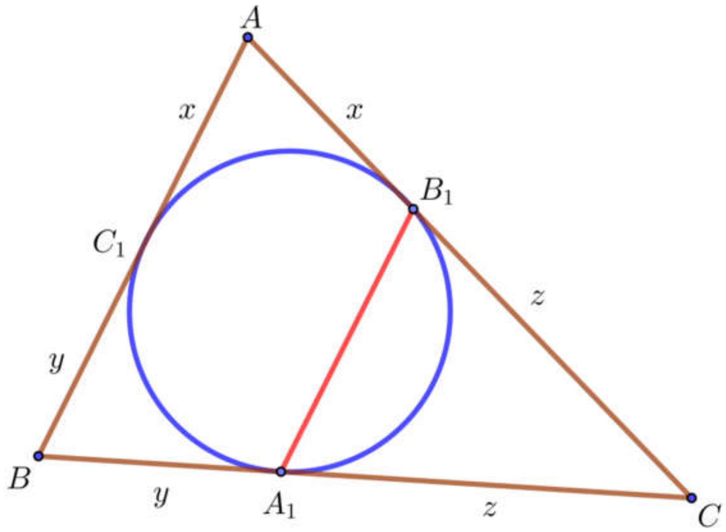
JP.355 In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2 + \left(\frac{BC}{B_1C_1}\right)^2 + \left(\frac{CA}{C_1A_1}\right)^2 \geq \frac{6R}{r}$$



Proposed by Marian Ursărescu-Romania

Solution 1 by George Florin Şerban-Romania



$$AB_1 = AC_1 = x; BC_1 = A_1B = y; B_1C = A_1C = z;$$

$$y + z = a;\ x + z = b; x + y = c \Rightarrow x + y + z = s; z = s - c; y = s - b; x = s - a$$

ΔB_1CA_1 —isosceles, from law of sines: $\frac{A_1B_1}{sinC} = \frac{B_1C}{sin\left(\frac{\pi - C}{2}\right)} \Rightarrow \frac{A_1B_1}{sinC} = \frac{s-c}{cos\frac{C}{2}}$

$$A_1B_1 = \frac{(s-c)sinC}{cos\frac{C}{2}} = \frac{2(s-c)sin\frac{C}{2}cos\frac{C}{2}}{cos\frac{C}{2}} = 2(s-c)sin\frac{C}{2}$$

$$\frac{AB}{A_1B_1} = \frac{c}{2(s-c)sin\frac{C}{2}} \Rightarrow \frac{AB}{A_1B_1} \cdot \frac{BC}{B_1C_1} \cdot \frac{AC}{A_1C_1} =$$

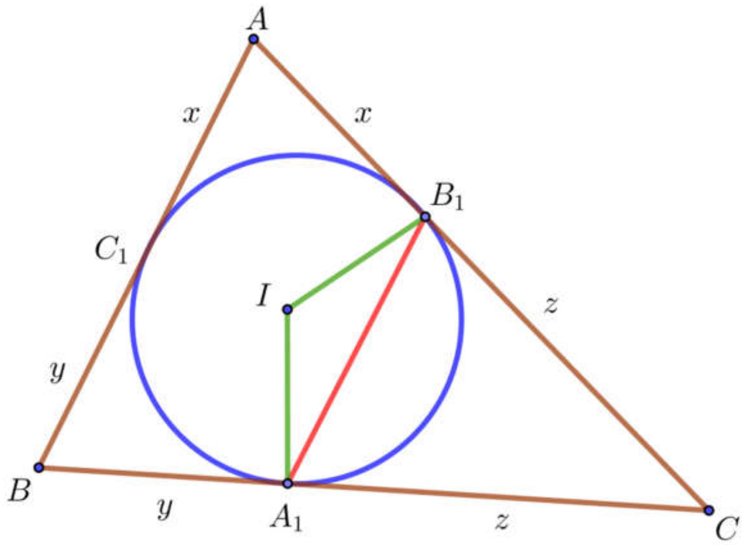
$$= \frac{abc}{8(s-a)(s-b)(s-c)sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}} = \frac{4RF}{\frac{8F^2}{s} \cdot \frac{r}{4R}} =$$

$$= \frac{4RF \cdot 4Rs}{8F^2r} = \frac{2R^2}{r^2}$$

$$\left(\frac{AB}{A_1B_1}\right)^2 + \left(\frac{BC}{B_1C_1}\right)^2 + \left(\frac{CA}{C_1A_1}\right)^2 \overset{AM-GM}{\geq} 3\sqrt[3]{\left(\frac{AB}{A_1B_1}\right)^2 \cdot \left(\frac{BC}{B_1C_1}\right)^2 \cdot \left(\frac{CA}{C_1A_1}\right)^2} =$$

$$= 3\sqrt[3]{\frac{4R^4}{r^4}} \overset{(*)}{\geq} \frac{6R}{r}; (*) \Leftrightarrow \frac{4R^4}{r^4} \geq \frac{8R^3}{r^3} \Leftrightarrow R \geq 2r(Euler).$$

Solution 2 by Alex Szoros-Romania



IA_1CB −inscriptible, then $\mu(\angle A_1IB_1) = \pi - \mu(C)$

From Law of cosines in ΔA_1IB_1: $A_1B_1^2 = r^2 + r^2 - 2r^2 \cdot cos(\pi - C)$

$$A_1B_1^2 = 2r^2(1 + cosC) \Rightarrow \frac{AB^2}{A_1B_1^2} = \frac{c^2}{2r^2(1 + cosC)} \Rightarrow$$

$$\sum_{cyc}\left(\frac{AB}{A_1B_1}\right)^2 = \frac{1}{2r^2}\sum_{cyc}\frac{c^2}{1 + cosC} \geq \frac{1}{2r^2}\cdot\frac{(\sum c)^2}{3 + \sum cosC}$$

$$\sum_{cyc}\left(\frac{AB}{A_1B_1}\right)^2 \geq \frac{4s^2}{2r^2\left(3 + 1 + \frac{r}{R}\right)} \Rightarrow \sum_{cyc}\left(\frac{AB}{A_1B_1}\right)^2 \geq \frac{2s^2R}{r^2(4R + r)}; (1)$$

It is enough to prove that: $\frac{2s^2R}{r^2(4R+r)} \geq \frac{6R}{r}; (2)$

$$(2) \Leftrightarrow \frac{s^2}{r(4R + r)} \geq 3 \Leftrightarrow s^2 \geq 3r(4R + r)(Doucet)$$

From (1),(2) it follows that:

$$\left(\frac{AB}{A_1B_1}\right)^2 + \left(\frac{BC}{B_1C_1}\right)^2 + \left(\frac{CA}{C_1A_1}\right)^2 \geq \frac{6R}{r}$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

IA_1CB −inscriptible, then $\mu(\angle A_1IB_1) = \pi - \mu(C)$

$$sin(A_1IM) = sin\left(\frac{\pi - C}{2}\right) = \frac{A_1M}{r} = \frac{A_1B_1}{2r}$$

$A_1B_1 = 2r \cdot cos\frac{C}{2}$;and analogs.

$$\sum_{cyc}\left(\frac{BC}{B_1C_1}\right)^2 = \frac{1}{4r^2}\sum_{cyc}\frac{a^2}{cos^2\frac{A}{2}} = \frac{(4R)^2}{4r^2}\sum_{cyc}sin^2\frac{A}{2} = \frac{4R^2}{r^2}\sum_{cyc}\frac{1 - cosA}{2} =$$

$$= \frac{2R^2}{r^2}\left(2 - \frac{r}{R}\right)$$

$$\Rightarrow \sum_{cyc}\left(\frac{BC}{B_1C_1}\right)^2 = \frac{4R^2}{r^2} - \frac{2R}{r} \overset{(*)}{\geq} \frac{6R}{r}; (*) \Leftrightarrow \frac{4R^2}{r^2} \geq \frac{8R}{r} \Leftrightarrow R \geq 2r(Euler).$$

Therefore,

$$\left(\frac{AB}{A_1B_1}\right)^2 + \left(\frac{BC}{B_1C_1}\right)^2 + \left(\frac{CA}{C_1A_1}\right)^2 \geq \frac{6R}{r}$$

Solution 4 by proposer

$$\mu(B_1\widehat{A_1C_1}) = \frac{\pi}{2} - \frac{A}{2}$$

$$\text{In } \Delta A_1B_1C_1: \begin{cases} B_1C_1 = 2rsin\left(\frac{\pi}{2} - \frac{A}{2}\right) = 2rcos\frac{A}{2} \\ r = 4Rsin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2} \end{cases} \Rightarrow$$

$$B_1C_1 = 8Rsin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}cos\frac{A}{2} = 4RsinAsin\frac{B}{2}sin\frac{C}{2}$$

$$\frac{BC}{B_1C_1} = \frac{a}{2asin\frac{B}{2}sin\frac{C}{2}} = \frac{1}{2sin\frac{B}{2}sin\frac{C}{2}}$$

We must to show:

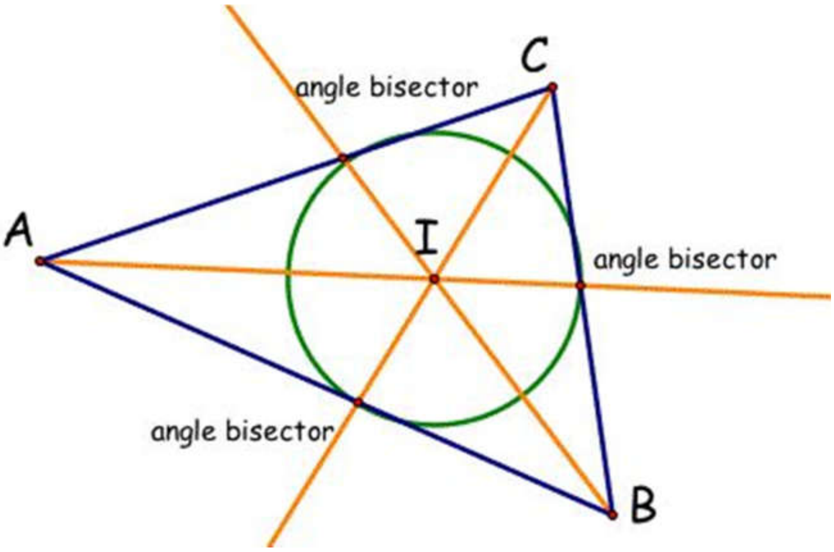
$$\frac{1}{4}\sum_{cyc}\frac{1}{sin^2\frac{B}{2}sin^2\frac{C}{2}} \geq \frac{6R}{r}; (1)$$

$$\text{But } \sum_{cyc}\frac{1}{sin^2\frac{B}{2}sin^2\frac{C}{2}} = \frac{8R(2R - r)}{r^2}; (2)$$

From (1),(2) it follows that: $\frac{2R(2R-r)}{r^2} \geq \frac{6R}{r} \Leftrightarrow 2R - r \geq 3r \Leftrightarrow 2R \geq 4r \Leftrightarrow R \geq 2r(Euler)$

JP.356 In $\Delta ABC, I$ −incenter and R_a, R_b, R_c −circumradii in $\Delta IBC, \Delta IAB, \Delta IAC$. Prove that:

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2 \geq 1$$



Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\mu(\widehat{BIC})=\pi-\frac{B}{2}-\frac{C}{2}=\frac{\pi}{2}+\frac{A}{2}\Rightarrow$$

$$2R_a=\frac{a}{\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)}=\frac{a}{\cos\frac{A}{2}}=\frac{2R\sin A}{\cos\frac{A}{2}}=4R\sin\frac{A}{2}\Rightarrow R_a=2R\sin\frac{A}{2}(and\ analogs)$$

$$\frac{R_a}{a}=\frac{2R\sin\frac{A}{2}}{2R\sin A}=\frac{\sin\frac{A}{2}}{\sin A}=\frac{1}{2\cos\frac{A}{2}}\Rightarrow\left(\frac{R_a}{a}\right)^2=\frac{1}{4\cos^2\frac{A}{2}}$$

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2=\frac{1}{4}\sum_{cyc}\frac{1}{\cos^2\frac{A}{2}}$$

But $\sum_{cyc}\frac{1}{\cos^2\frac{A}{2}}=1+\left(\frac{4R+r}{s}\right)^2$ and $(4R+r)^2\geq 3s^2(Doucet)\Rightarrow$

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2\geq 1$$

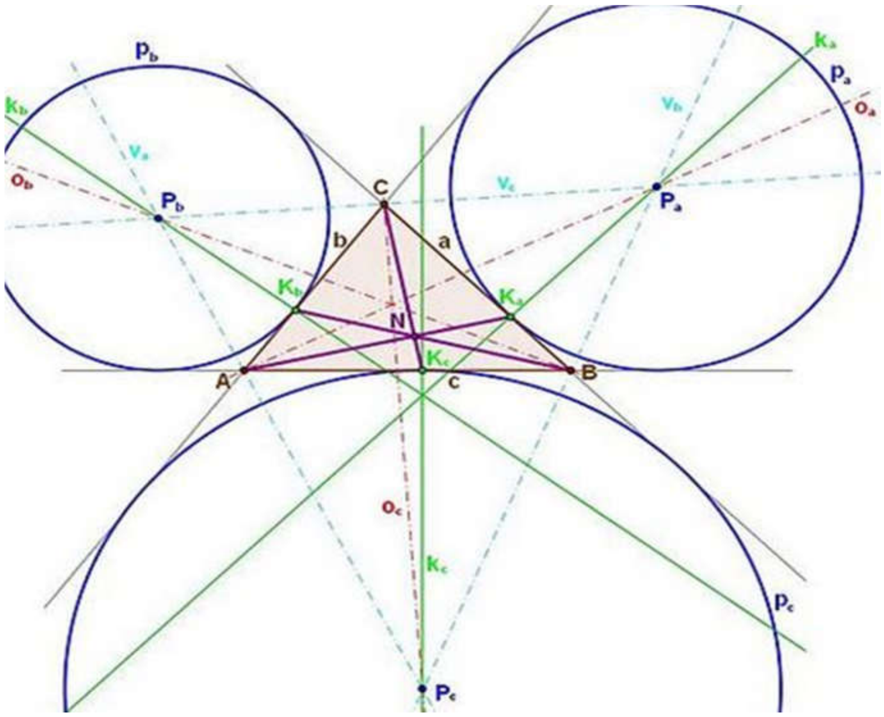
Solution 2 by Daniel Văcaru-Romania

BY Sine Law in $\Delta IBC, a = 2R_a sin(\widehat{BIC}) = 2R_a sin\left(\pi - \frac{B}{2} - \frac{C}{2}\right) =$

$$= 2R_a sin\left(\frac{B+C}{2}\right) = 2R_a cos\frac{A}{2} \Rightarrow \frac{R_a}{a} = \frac{1}{2cos\frac{A}{2}} \Rightarrow$$

$$\left(\frac{R_a}{a}\right)^2 = \sum_{cyc} \frac{1}{4cos^2\frac{A}{2}} \overset{Bergstrom}{\geq} \frac{9}{4\sum cos^2\frac{A}{2}} = \frac{9}{4\left(2+\frac{r}{2R}\right)} \overset{Euler}{\geq} 1$$

JP.357 In $\Delta ABC, N_a$ –Nagel's point. Prove that if N_a lies on incircle if and only if $s^2 + 4r^2 = 16Rr$



Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

If that inscribed circle of ΔABC passes through point $N_a \Leftrightarrow IN_a = r \Leftrightarrow IN_a^2 = r^2; (1)$

$$\overrightarrow{IN_a} = \frac{(s-a)\overrightarrow{IA} + (s-b)\overrightarrow{IB} + (s-c)\overrightarrow{IC}}{s} \Leftrightarrow$$

$$IN^2 = \frac{\sum(s-a)^2 IA^2 + \sum bc(s-a)\overrightarrow{IA}\cdot\overrightarrow{IB}}{s^2}; (2)$$

$$\sum(s-a)^2IA^2=\sum(s-a)^2(bc-4Rr)=\sum bc(s-a)^2-4Rr\sum(s-a)^2=$$

$$=abc\sum\frac{(s-a)^2}{a}-4Rr\sum(s-a)^2=4Rrs\cdot\frac{s(s^2+r^2-12Rr)}{4Rr}-4Rr$$

$$s^2-2r^2-8Rr=s^2(s^2+r^2-12Rr)-4Rr(s^2-2r^2-8Rr)=$$

$$s^2(s^2+r^2-12Rr)-4Rrs^2+4Rr(2r^2+8Rr)=$$

$$=s^2(s^2+r^2-16Rr)+8Rr^2(4R+r);(3)$$

$$\sum(s-a)(s-b)\overrightarrow{IA}\cdot\overrightarrow{IB}=\sum(s-a)(s-b)\left(\frac{IA^2+IB^2-AB^2}{2}\right)=$$

$$=\sum(s-a)(s-b)\left(\frac{bc-4Rr+ac-4Rr-c^2}{2}\right)=$$

$$=\sum(s-a)(s-b)\left(\frac{2c(s-c)-8Rr}{2}\right)=$$

$$=\sum(s-a)(s-b)(c(s-c)-4Rr)=$$

$$=(s-a)(s-b)(s-c)\cdot 2s-4Rr\sum(s-a)(s-b)=$$

$$=2s^2r^2-4Rr\big(r(4R+r)\big)=2s^2r^2-4Rr^2(4R+r);(4)$$

From (2),(3),(4) it follows that:

$$IN^2=s^2+r^2-16Rr+\frac{8Rr^2(4R+r)}{s^2}+4r^2-\frac{8Rr^2(4R+r)}{s^2}=s^2+5r^2-16Rr;(5)$$

From (1),(5) it follows that: $s^2+4r^2=16Rr$.

Solution 2 by Daniel Văcaru-Romania

We know that: $N_aI=3IG\Rightarrow 3IG=r\Rightarrow r^2=9GI^2\Rightarrow r^2=s^2+5r^2-16Rr\Rightarrow$

$16Rr=s^2+4r^2$. **Or, else:**

$$N_aM^2=\frac{s-a}{s}MA^2+\frac{s-b}{s}MB^2+\frac{s-c}{s}MC^2+4r^2-4Rr;(*)$$

We obtain $N_aI^2=\frac{s-a}{s}IA^2+\frac{s-b}{s}IB^2+\frac{s-c}{s}IC^2+4r^2-4Rr$ **and**

$$AI=\frac{bc}{s}\cos\frac{A}{2},AI^2=\frac{b^2c^2}{s^2}\cdot\frac{s(s-a)}{s^2}\Rightarrow\frac{s-a}{s}AI^2=\frac{bc(s-a)^2}{s^2}$$

$$\sum_{cyc}\frac{s-a}{s}IA^2=\sum_{cyc}\frac{bc(s-a)^2}{s^2}=\frac{s^2\sum ab-6abcs+abc(a+b+c)}{s^2}=$$

$$=\frac{s^2\sum ab-4abcs}{s^2}=\frac{s^2(s^2+r^2+4Rr)-16Rrs^2}{s^2}=s^2+r^2-12Rr$$

Again in $(*)$, we obtain

$$N_aI^2=s^2+r^2-12Rr+4R^2-4Rr=s^2+5r^2-16Rr.$$

But $N_aI=r\Rightarrow r^2=s^2+5r^2-16Rr.$

Therefore, $s^2+4r^2=16Rr.$

JP.358 If $x,y,z>0, xyz=1$ then in ΔABC the following relationship holds:

$$a^4\left(y+z+\frac{1}{x}\right)+b^4\left(z+1+\frac{1}{y}\right)+c^4\left(x+1+\frac{1}{z}\right)\ge 1296r^4$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Şerban-Romania

$$a^4\left(y+z+\frac{1}{x}\right)+b^4\left(z+1+\frac{1}{y}\right)+c^4\left(x+1+\frac{1}{z}\right)=$$

$$(a^4y+b^4z+c^4x)+(a^4+b^4+c^4)+\left(\frac{a^4}{x}+\frac{b^4}{y}+\frac{c^4}{z}\right)\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{(abc)^4\cdot xyz}+\sum_{cyc}a^4+3\sqrt[3]{\frac{(abc)^4}{xyz}}=3\sqrt[3]{(abc)^4}+\sum_{cyc}a^4+3\sqrt[3]{(abc)^4}\ge$$

$$\ge 6\sqrt[3]{(4Rrs)^4}+\frac{(\sum a)^4}{3^3}=6\sqrt[3]{2^8(Rrs)^4}+\frac{16s^4}{27}\overset{\substack{R\ge 2r\\ s\ge 3\sqrt{3}r}}{\ge}6\sqrt[3]{2^{12}\cdot 3^6\cdot r^{12}}+16\cdot 27r^4$$

$$=1296r^4$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}a^4\left(y+z+\frac{1}{x}\right)\overset{AM-GM}{\ge}3\sqrt[3]{(abc)^4\prod_{cyc}\left(y+1+\frac{1}{x}\right)}\overset{Holder}{\ge}$$

$$\ge 3\sqrt[3]{(4Rrs)^4}\left(\sqrt[3]{xyz}+1+\frac{1}{\sqrt[3]{xyz}}\right)\overset{\substack{R\ge 2r\\ s\ge 3\sqrt{3}r}}{\ge}9\sqrt[3]{4^4\cdot(2r)^4\cdot\left(3\sqrt{3}r\right)^4\cdot r^4}=1296r^4$$

Solution 3 by Samar Das-India

$$T=a^4\left(y+z+\frac{1}{x}\right)+b^4\left(z+1+\frac{1}{y}\right)+c^4\left(x+1+\frac{1}{z}\right)\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{a^4b^4c^4\left(y+1+\frac{1}{x}\right)\left(z+1+\frac{1}{y}\right)\left(x+1+\frac{1}{z}\right)}=$$

$$=3(abc)^{\frac{4}{3}}\cdot\sqrt[3]{\left(yz+y+1+\frac{z}{x}+\frac{1}{xy}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\left(x+\frac{1}{z}+1\right)}=$$

$$=3(abc)^{\frac{4}{3}}\cdot\sqrt[3]{\left(\frac{2}{x}+2+y+\frac{1}{y}+2z+\frac{z}{x}\right)\left(x+\frac{1}{z}+1\right)}=$$

$$=3(abc)^{\frac{4}{3}}\cdot\sqrt[3]{6+3\sum_{cyc}\left\{\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2+2\right\}+\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)}\geq$$

$$\geq 3(abc)^{\frac{4}{3}}\cdot\sqrt[3]{6+18+3\sqrt[3]{\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}}}=9(abc)^{\frac{4}{3}}=9(4Rrs)^{\frac{4}{3}};$$

$$\because cosA+cosB+cosC=1+\frac{r}{R}\leq\frac{3}{2};$$

$$tan\frac{A}{2}\leq tan30^\circ=\frac{1}{\sqrt{3}};\ r=(s-a)tan\frac{A}{2}\Rightarrow tan\frac{A}{2}=\frac{r}{s-a}$$

$$tan30^\circ\geq\frac{r}{s-\frac{2}{3}s}\Rightarrow s\geq 3\sqrt{3};$$

Therefore,

$$T\geq 9(4Rrs)^{\frac{4}{3}}=9\left(4r\cdot 3\sqrt{3}r\cdot 2r\right)^{\frac{4}{3}}=1296R^4$$

Solution 4 by Tran Hong-Dong Thap-Vietnam

$$\frac{(a^2)^2}{\frac{1}{y+1+\frac{1}{x}}}+\frac{(b^2)^2}{\frac{1}{z+1+\frac{1}{y}}}+\frac{(c^2)^2}{\frac{1}{x+1+\frac{1}{z}}}\overset{CBS}{\geq}\frac{(a^2+b^2+c^2)^2}{\frac{1}{y+1+\frac{1}{x}}+\frac{1}{z+1+\frac{1}{y}}+\frac{1}{x+1+\frac{1}{z}}}=$$

$$=\frac{(a^2+b^2+c^2)^2}{\frac{x}{xy+x+1}+\frac{y}{yz+y+1}+\frac{z}{xz+z+1}}\overset{xyz=1}{=}$$

$$=\frac{(a^2+b^2+c^2)^2}{\frac{x}{xy+x+xyz}+\frac{y}{yz+y+1}+\frac{yz}{xyz+yz+y}}\overset{xyz=1}{=}$$

$$=\frac{(a^2+b^2+c^2)^2}{\frac{1+y+yz}{1+y+yz}}=(a^2+b^2+c^2)^2\overset{Ionescu-Weitzenbock}{\geq}\left(4\sqrt{3}F\right)^2=$$

$$=16\cdot 3\cdot 27\cdot r^4=1296r^4$$

JP.359 Find all n natural numbers such that:

$$\sqrt[3]{\frac{n+27}{(n+8)(n+1)}}\in\mathbb{Q}$$

Proposed by George Florin Şerban-Romania

Solution by proposer

$$(n+27)(n+8)(n+1)=(n+27)(n^2+9n+8)=n^3+36n^2+251n+216$$

is perfect cube. We want to prove:

$$(n+6)^3<n^3+36n^2+251n+216<(n+12)^3$$

$$n^3+18n^2+108n+216\leq n^3+36n^2+251n+216\Leftrightarrow$$

$$18n^2+143n\geq 0\Leftrightarrow n(18n+143)>0 \text{ true for all } n\in\mathbb{N}.$$

On the other hand $n^3+36n^2+251n+216<n^3+36n^2+432n+1728$**, true.**

If $n^3+36n^2+251n+216=(n+7)^3\Leftrightarrow$

$$n^3+36n^2+251n+216=n^3+21n^2+147n+343\Leftrightarrow 15n^2+104n=127$$

$$\Leftrightarrow n(15n+104)=127, 127 - \text{prime number}, n=1 \text{ is false.}$$

If $(n+27)(n+8)(n+1)=(n+8)^3, (n+27)(n+1)=(n+8)^2, n+8\neq 0$

$$n^2+28n+27=n^2+16n+64\Leftrightarrow 12n=37\Leftrightarrow n=\frac{27}{12}\notin\mathbb{N} \text{ false.}$$

If $n^3+36n^2+251n+216=(n+9)^3\Leftrightarrow$

$$n^3+36n^2+251n+216=n^3+27n^2+243n+729\Leftrightarrow n(9n+8)=3^3\cdot 19$$

$$9n+8\in\{9,19,27,57,171,513\}\Leftrightarrow 9n\in\{1,11,19,49,163,505\}, \text{ false.}$$

If $n^3+36n^2+251n+216=(n+10)^3 \Leftrightarrow$

$$n^3+36n^2+251n+216=n^3+30n^2+300n+100 \Leftrightarrow 6n^2-49=784 \Leftrightarrow$$

$$36n^2-294n=4704 \Leftrightarrow (n-24,5)^2=5304,25 \Leftrightarrow$$

$6n-24,5=\sqrt{5304,25}\notin\mathbb{N}$, false.

If $n^3+36n^2+251n+216=(n+11)^3 \Leftrightarrow$

$$n^3+36n^2+251n+216=n^3+33n^2+363n+1331 \Leftrightarrow$$

$n(3n-112)=5\cdot 223, 3n-112\in\{1,5,223,1115\}$,

$$3n\in\{113,117,335,1227\}\Rightarrow n\in\left\{\frac{113}{3},39,\frac{335}{3},409\right\}\cap\mathbb{N}=\{39,409\}$$

If $n=39\Rightarrow n+1=40$ which isn't perfect cube.

If $n=409\Rightarrow n+1=410, 7^3<343<410<8^3<512$, which isn't perfect cube.

So, $n=0$.

JP.360 If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\sum_{cyc}\frac{\tan^2 x}{\tan^3 x+\cot x}+\sum_{cyc}\frac{\cot^2 x}{\cot^3 x+\tan x}\ge 2\sum_{cyc}\frac{1}{\tan^2 x+\cot^2 x}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{\tan^2 x}{\tan^3 x+\cot x}+\frac{\cot^2 x}{\cot^3 x+\tan x}\overset{BERGSTROM}{\ge}\frac{(\tan x+\cot x)^2}{\tan^3 x+\cot^3 x+\tan x+\cot x}$$

$$=\frac{(\tan x+\cot x)^2}{(\tan x+\cot x)(\tan^2 x-\tan x\cot x+\cot^2 x+1)}=$$

$$=\frac{\tan x+\cot x}{\tan^2 x+\cot^2 x}\overset{AM-GM}{\ge}\frac{2\sqrt{\tan x\cot x}}{\tan^2 x+\cot^2 x}=$$

$$=\frac{2}{\tan^2 x+\cot^2 x}$$

By summing:

$$\sum_{cyc}\frac{\tan^2 x}{\tan^3 x+\cot x}+\sum_{cyc}\frac{\cot^2 x}{\cot^3 x+\cot x}\ge 2\sum_{cyc}\frac{1}{\tan^2 x+\cot^2 x}$$

Equality holds for: $x = y = z = \frac{\pi}{4}$.

Solution 2 by Daniel Văcaru-Romania

We have $\frac{\tan^2 x}{\tan^3 x+\cot x} = \frac{\tan^2 x}{\tan^3 x+\frac{1}{\tan x}} = \frac{\tan^3 x}{\tan^4 x+1}$ and $\frac{\cot^2 x}{\cot^3 x+\tan x} = \frac{\frac{1}{\tan^2 x}}{\frac{1}{\tan^3 x}+tanx} = \frac{tanx}{\tan^4 x+1}$.

It follows that: $\frac{\tan^2 x}{\tan^3 x+\cot x} + \frac{\cot^2 x}{\cot^3 x+\tan x} = \frac{\tan^3 x+\tan x}{\tan^4 x+1}$. On the other hand, with calculus, we find $\frac{1}{\tan^2 x+\cot^2 x} = \frac{\tan^2 x}{\tan^4 x+1}$. We must to prove that:

$$\frac{\tan^3 x + \tan x}{\tan^4 x + 1} \ge \frac{2\tan^2 x}{\tan^4 x + 1}; (*)$$

But is clear that $\tan^3 x + \tan x = \tan x\,(\tan^2 x + 1) \overset{AM-GM}{\ge} 2\tan x$, which proves $(*)$.

Writing $(*)$ for x, y, z and adding, we obtain:

$$\sum_{cyc} \frac{\tan^2 x}{\tan^3 x + \cot x} + \sum_{cyc} \frac{\cot^2 x}{\cot^3 x + \tan x} \ge 2\sum_{cyc} \frac{1}{\tan^2 x + \cot^2 x}$$

SP.346 Determine all functions $f: (0,\infty) \to \mathbb{R}$ such that:

$$f(xy) \le xf(x) + yf(y) \le log(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

For $x = y = 1 \Rightarrow f(1) \le 2f(1) \le 0 \Rightarrow f(1) \ge 0$ and how $f(1) \ge 0 \Rightarrow f(1) = 0$.

For $y = 1 \Rightarrow f(x) \le xf(x) \le logx \Rightarrow f(x) < -\frac{logx}{x}; (1)$

For $y = \frac{1}{x} \Rightarrow f(1) \le xf(x) + \frac{1}{x}f\left(\frac{1}{x}\right) \le log1 \Rightarrow$

$$0 \le xf(x) + \frac{1}{x}f\left(\frac{1}{x}\right) \le 0 \Rightarrow xf(x) + \frac{1}{x}f\left(\frac{1}{x}\right) = 0 \Rightarrow f\left(\frac{1}{x}\right) = -x^2 f(x); (2)$$

In (1) we take: $x \to \frac{1}{x} \Rightarrow f\left(\frac{1}{x}\right) \le \frac{log\left(\frac{1}{x}\right)}{\frac{1}{x}} \Rightarrow 1/x) \le -xlogx; (3)$

From (2),(3) it follows that: $-x^2 f(x) \le -xlogx \Leftrightarrow x^2 f(x) \ge xlogx \Rightarrow f(x) \ge \frac{logx}{x}; (4)$

From (1),(4) we get: $f(x) = \frac{logx}{x}; \forall x > 0$.

Solution 2 by Ravi Prakash-New Delhi-India

$$f:(0,\infty)\to\mathbb{R}; f(xy)\le xf(x)+yf(y)\le log(xy),\forall x,y>0$$

Put $x=y=1$, we get $f(1)\le f(1)+f(1)\le 0\Leftrightarrow f(1)\le 0$ but $f(1)\ge 0\Rightarrow f(1)=0$.

For $y=\frac{1}{x}, f(1)\le xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\le log1; (1)$

$$\Leftrightarrow 0\le xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\le 0\Leftrightarrow xf(x)=\frac{1}{x}f\left(\frac{1}{x}\right)$$

Replacing x by $\frac{1}{x}$, y by $\frac{1}{y}$, we get: $f\left(\frac{1}{xy}\right)\le\frac{1}{x}f\left(\frac{1}{x}\right)+\frac{1}{y}f\left(\frac{1}{y}\right)\le log\left(\frac{1}{xy}\right)\Rightarrow$

$$-xyf(xy)\le -xf(x)-yf(y)\le -log(xy)\Leftrightarrow$$

$$log(xy)\le xf(x)+yf(y(\le xyf(xy)$$

Put $y=1\Rightarrow logx\le xf(x)+0\le xf(x)\Rightarrow\frac{logx}{x}\le f(x),\forall x>0; (2)$

Also, from (1), putting $y=1\Rightarrow f(x)\le xf(x)\le logx\Rightarrow f(x)\le\frac{1}{x}logx; (3)$

From (2),(3) it follows that $f(x)=\frac{1}{x}logx,\forall x>0$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$f(xy)\le xf(x)+yf(y)\le log(xy),\forall x,y>0; (*)$$

$$x=y=1\overset{(*)}{\Rightarrow}f(1)\le 2f(1)\le 0\Rightarrow 0\le f(1)\le 0\Rightarrow f(1)=0$$

$$y=\frac{1}{x}\overset{(*)}{\Rightarrow}0\le xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\le 0\Rightarrow xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)=0\Rightarrow$$

$$f\left(\frac{1}{x}\right)=-x^2f(x);\forall x>0$$

$$y=1\overset{(*)}{\Rightarrow}f(x)\le -xlogx\Leftrightarrow xf(x)\le logx\Rightarrow f(x)\le\frac{logx}{x};\forall x>0; (1)$$

$$x=\frac{1}{x}\overset{(*)}{\Rightarrow}f\left(\frac{1}{x}\right)\le -xlogx\Leftrightarrow -x^2f(x)\le -xlogx\Rightarrow f(x)\ge\frac{logx}{x};\forall x>0; (2)$$

From (1),(2) it follows that $f(x)=\frac{logx}{x};\forall x>0$, which is true when we replace f in (*).

Therefore,

$$f(x)=\frac{logx}{x};\forall x>0.$$

SP.347 If $A, B \in M_3(\mathbb{R})$ such that $tr((AB-BA)^2)=0$. Prove that:

$$det((AB-BA)^2+AB-BA+I_3)=\left(1-det(AB-BA)\right)^2$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\text{Let } C=AB-BA, P_C(X)=X^3-trCX^2+trC^*X-detC$$

$$trC=tr(AB-BA)=0, trC^*=\frac{1}{2}((trC)^2-trC^2)=0 \Rightarrow$$

$$P_C(X)=C^3-detC$$

$$P_C(X)=(X-\lambda_1)(X-\lambda_2)(X-\lambda_3), \lambda_1,\lambda_2,\lambda_3 \in \mathbb{C}$$

$$\text{Le } f(x)=x^2+x+1 \Rightarrow det(C^2+C+I_3)=f(C)=f(\lambda_1)f(\lambda_2)f(\lambda_3)$$

$$\text{Let } \varepsilon \in \mathbb{C}, \varepsilon^3=1, \varepsilon^2+\varepsilon+1=0, \varepsilon+\bar{\varepsilon}=-1, \varepsilon\cdot\bar{\varepsilon}=1, \varepsilon^2=\bar{\varepsilon}$$

$$\begin{cases} P_C(\varepsilon)=(\varepsilon-\lambda_1)(\varepsilon-\lambda_2)(\varepsilon-\lambda_3) \\ P_C(\bar{\varepsilon})=(\bar{\varepsilon}-\lambda_1)(\bar{\varepsilon}-\lambda_2)(\bar{\varepsilon}-\lambda_3) \end{cases} \Rightarrow$$

$$P_C(\varepsilon)\cdot P_C(\bar{\varepsilon})=\left(1+\lambda_1+\lambda_1^2\right)\left(1+\lambda_2+\lambda_2^2\right)\left(1+\lambda_3+\lambda_3^2\right)=f(\lambda_1)f(\lambda_2)f(\lambda_3); (2)$$

$$\text{From (1),(2) it follows that: } det(C^2+C+I_3)=P_C(\varepsilon)\cdot P_C(\bar{\varepsilon}); (3)$$

$$P_C(\varepsilon)=1-detC, P_C(\bar{\varepsilon})=1-detC \Rightarrow P_C(\varepsilon)\cdot P_C(\bar{\varepsilon})=(1-detC)^2; (4)$$

$$\text{From (3),(4) it follows that } det(C^2+C+I_3)=(1-detC)^2$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } AB-BA=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \overset{not}{=} C$$

$$\text{Now, } Tr(C)=Tr(AB-BA)=Tr(AB)-Tr(BA)=0 \Rightarrow a_{11}+a_{22}+a_{33}=0$$

$$\text{Next, } Tr(C^2)=a_{11}^2+a_{12}a_{21}+a_{13}a_{31}+a_{22}^2+a_{12}a_{21}+a_{23}a_{32}+a_{33}^2+a_{13}a_{31}+ a_{23}a_{32}$$

$$0=(a_{11}+a_{22}+a_{33})^2+2(a_{12}a_{21}-a_{22}a_{11})+2(a_{13}a_{31}-a_{11}a_{33})$$

$$2Tr(C^*)=(a_{11}+a_{22}+a_{33})^2=0 \Rightarrow Tr(C^*)=0$$

Characteristic polynomial of C is:

$$det(\lambda I_3-C)=\lambda^3-\left(Tr(C)\right)\lambda^2+Tr(C^*)\lambda-det(C) \Rightarrow$$

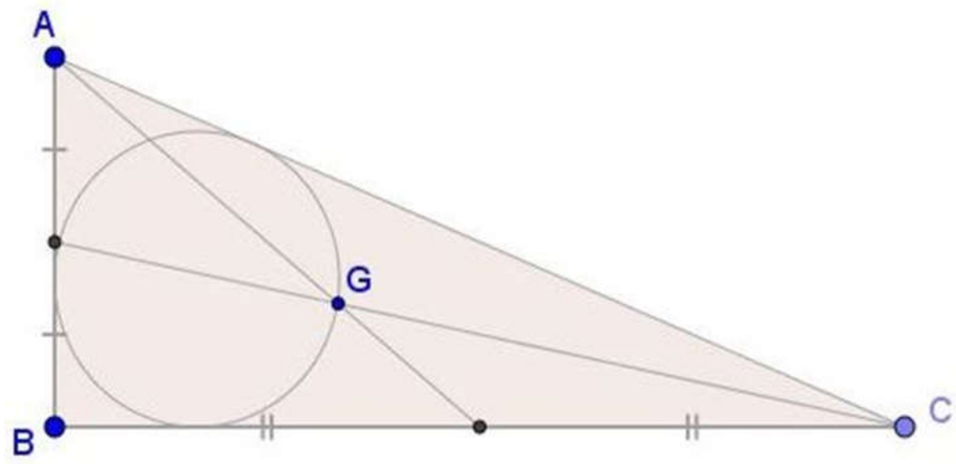
$$det(\lambda I_3-C)=\lambda^3-det(C)$$

Now, $det(C^2+C+I_3)=det\left((\omega I_3-C)(\omega^2 I_3-C)\right)=det(\omega I_3-C)|^2=$

$$=\left|\omega^3-det(C)\right|^2=\left(1-det(C)\right)^2$$

SP.348 Prove that if G-centroid in ΔABC lies on incircle then:

$$s^2=16Rr+4r^2$$



Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

The inscribed circle of ΔABC passes through to G if and only if $IG=r\Leftrightarrow IG^2=r^2$; (1)

$$\overrightarrow{IG}=\frac{\overrightarrow{IA}+\overrightarrow{IB}+\overrightarrow{IC}}{3}\Leftrightarrow IG^2=\frac{\sum IA^2+2\sum\overrightarrow{IA}\cdot\overrightarrow{IB}}{9};(2)$$

$$\sum IA^2=s^2+r^2-8Rr;(3)$$

$$\sum\overrightarrow{IA}\cdot\overrightarrow{IB}=\sum\frac{IA^2+IB^2-AB^2}{2}=\sum\frac{bc-4Rr+ac-4Rr-c^2}{2}=$$

$$=\sum\frac{2c(s-c)-8Rr}{2}=\sum c(s-c)-12Rr=8Rr+2r^2-12Rr=2r^2-4Rr;(4)$$

From (2),(3),(4) it follows that:

$$IG^2=\frac{s^2+r^2-8Rr+4r^2-8Rr}{9}=\frac{s^2+5r^2-16Rr}{9};(5)$$

From (1),(5) it follows that $\frac{s^2+5r^2-16Rr}{9}=r^2\Leftrightarrow s^2=16Rr+4r^2$

Solution 2 by Daniel Văcaru-Romania

We know that: $9GI^2 = s^2 + 5r^2 - 16Rr$; $(*)$, It follows that ΔABC prove that inscribed circle of ΔABC passes through to G if and only if $GI = r$ and we have equivalency

$$9r^2 = 9GI^2 = s^2 + 5r^2 - 16Rr \Leftrightarrow s^2 = 16Rr + 4r^2.$$

To prove $(*)$, we use Leibniz relationship, i.e. for all point M and ΔABC, we have

$MA^2 + MB^2 + MC^2 = \frac{a^2+b^2+c^2}{3} + 3MG^2$ and $AI = \frac{b+c}{2s}AD$, where AD is bisectors from A.

We have: $AI = \frac{b+c}{2s} \cdot \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{bc}{s}\cos\frac{A}{2} \Rightarrow AI^2 = \frac{b^2c^2}{s^2} \cdot \frac{s(s-a)}{bc} = \frac{bc(s-a)}{s} \Rightarrow$

$$\sum_{cyc} AI^2 = \frac{s(ab+bc+ca) - 3abc}{s} = ab + bc + ca - 12Rr = s^2 + r^2 - 8Rr$$

We obtain: $9IG^2 = 3(s^2 + r^2 - 8Rr) - (2s^2 - 2r^2 - 8Rr) = s^2 + 5r^2 - 16Rr$

SP.349 If $a \in \left(0, \frac{\pi}{2}\right)$ then prove:

$$(sina)^{\sqrt{log_{sina}cosa}} + (cosa)^{\sqrt{log_{cosa}sina}} \le \sqrt{2}$$

Proposed by Ionuţ Florin Voinea-Romania

Solution by proposer

$$a \in \left(0, \frac{\pi}{2}\right) \Rightarrow sina, cosa \in (0,1) \Rightarrow log_{sina}cosa > 0, log_{cosa}sina > 0$$

$$(sina)^{\sqrt{log_{sina}cosa}} \overset{AM-GM}{\le} (sina)^{\frac{1}{2}+\frac{1}{2}log_{sina}cosa} = \sqrt{sina}(sina)^{log_{sina}\sqrt{cosa}}$$

$$= \sqrt{sinacosa}$$

Similarly, $(cosa)^{\sqrt{log_{cosa}sina}} \le \sqrt{sinacosa}$

So, $(sina)^{\sqrt{log_{sina}cosa}} + (cosa)^{\sqrt{log_{cosa}sina}} \le 2\sqrt{sinacosa}$

But $\sqrt{sinacosa} \le \frac{sin^2a+cos^2a}{2} = \frac{1}{2}$, therefore

$$(sina)^{\sqrt{log_{sina}cosa}} + (cosa)^{\sqrt{log_{cosa}sina}} \le \sqrt{2}$$

Equality holds if $a = \frac{\pi}{4}$.

SP.350 If $x,y,z>0, xy+yz+zx=1$ and $\lambda\ge\frac{2}{3}$, then:

$$\frac{1}{x^2(x^2+\lambda)}+\frac{1}{y^2(y^2+\lambda)}+\frac{1}{z^2(z^2+\lambda)}\ge\frac{27}{3\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Using identity: $\frac{1}{x^2(x^2+\lambda)}=\frac{1}{\lambda}\left(\frac{1}{x^2}-\frac{1}{x^2+\lambda}\right)$, inequality can be written as:

$$\frac{1}{\lambda}\sum\left(\frac{1}{x^2}-\frac{1}{x^2+\lambda}\right)\ge\frac{27}{3\lambda+1}\Leftrightarrow\sum\left(\frac{1}{x^2}-\frac{1}{x^2+\lambda}\right)\ge\frac{27\lambda}{3\lambda+1}\Leftrightarrow$$

$$\sum\frac{1}{x^2}-\sum\frac{1}{x^2+\lambda}\ge\frac{27\lambda}{3\lambda+1}\Leftrightarrow\sum\frac{1}{x^2}+\frac{1}{\lambda}\sum\frac{x^2}{x^2+\lambda}\ge\frac{27\lambda}{3\lambda+1}+\frac{3}{\lambda}$$

$$\Leftrightarrow\sum\frac{1}{x^2}+\frac{1}{\lambda}\sum\frac{x^2}{x^2+\lambda}\ge\frac{27\lambda^2+9\lambda+3}{\lambda(3\lambda+1)}$$

Which follows from $\sum\frac{1}{x^2}\ge 9,(1)$ and $\sum\frac{x^2}{x^2+\lambda}\ge\frac{3}{3\lambda+1},(2)$, where

$$\sum\frac{1}{x^2}\ge\sum\frac{1}{xy}\ge\frac{9}{\sum xy}=\frac{9}{1}$$

Equality holds if and only if $x=y=z=\frac{1}{\sqrt{3}}$.

$$\sum\frac{x^2}{x^2+\lambda}\ge\frac{(\sum x)^2}{\sum(x^2+\lambda)}=\frac{\sum x^2+2\sum yz}{\sum x^2+3\lambda}=\frac{\sum x^2+2\cdot 1}{\sum x^2+3\lambda}\overset{(3)}{\ge}\frac{3}{3\lambda+1}$$

$$(3)\Leftrightarrow\frac{\sum x^2+2\cdot 1}{\sum x^2+3\lambda}\ge\frac{3}{3\lambda+1}\Leftrightarrow(3\lambda-2)\sum x^2\ge(3\lambda-2)$$

Which follows from $\sum x^2\ge\sum yz=1$ and $\lambda\ge\frac{2}{3}\Leftrightarrow(3\lambda-2)\ge 0$.

From (1),(2), it follows that:

$$\sum\frac{1}{x^2}+\frac{1}{\lambda}\sum\frac{x^2}{x^2+\lambda}\ge 9+\frac{1}{\lambda}\cdot\frac{3}{3\lambda+1}=\frac{27\lambda^2+9\lambda+1}{\lambda(3\lambda+1)}$$

Equality holds if and only if $x=y=z=\frac{1}{\sqrt{3}}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{1}{x^2(x^2+\lambda)}=\frac{1}{\lambda}\sum_{cyc}\frac{(x^2+\lambda)-x^2}{x^2(x^2+\lambda)}=\frac{1}{\lambda}\sum_{cyc}\frac{1}{x^2}-\frac{1}{\lambda^2}\sum_{cyc}\frac{(x^2+\lambda)-x^2}{x^2+\lambda}$$

$$\sum_{cyc}\frac{1}{x^2(x^2+\lambda)}=\frac{1}{\lambda}\sum_{cyc}\frac{1}{x^2}+\frac{1}{\lambda^2}\sum_{cyc}\frac{x^2}{x^2+\lambda}-\frac{3}{\lambda^2}$$

$$\sum_{cyc}\frac{1}{x^2}=\sum_{cyc}\left(\frac{1}{x}\right)^2\geq\sum_{cyc}\left(\frac{1}{x}\cdot\frac{1}{y}\right)=\sum_{cyc}\frac{1}{xy}\overset{CBS}{\geq}\frac{9}{\sum xy}=9;(1)$$

$$\sum_{cyc}\frac{x^2}{x^2+\lambda}\overset{CBS}{\geq}\frac{(x+y+z)^2}{(x^2+y^2+z^2)+3\lambda}\overset{(2)}{\geq}\frac{3}{3\lambda+1}$$

$$(2)\Leftrightarrow 3\left(\lambda-\frac{2}{3}\right)\sum_{cyc}x^2\geq 9\lambda-2(3\lambda+1)\sum_{cyc}xy=3\lambda-2$$

Which is true because: $\lambda\geq\frac{2}{3}$ and $\sum x^2\geq\sum xy=1$

From (1),(2) it follows that:

$$\sum_{cyc}\frac{1}{x^2(x^2+\lambda)}\geq\frac{9}{\lambda}+\frac{3}{\lambda^2(3\lambda+1)}-\frac{3}{\lambda^2}=\frac{27}{3\lambda+1}$$

Solution 3 by Michael Sterghiou-Greece

$$\sum_{cyc}\frac{1}{x^2(x^2+\lambda)}\geq\frac{27}{3\lambda+1};(1)$$

Let $(p,q,r)=\left(\sum_{cyc}x;\sum_{cyc}xy;xyz\right);q=1$ and $p^2\geq 3q\Rightarrow p\geq\sqrt{3}$

$$r\leq\left(\frac{q}{3}\right)^{\frac{3}{2}}=\frac{\sqrt{3}}{9};(1)\Rightarrow\sum_{cyc}\frac{\frac{1}{x^2}}{x^2+\lambda}\overset{CBS}{\geq}\frac{\left(\sum\frac{1}{x}\right)^2}{\left(\sum x^2\right)+3\lambda}\overset{(*)}{\geq}3\lambda+1$$

$$\left(\sum_{cyc}\frac{1}{x}\right)^2=\frac{1}{r^2},\sum_{cyc}x^2=p^2-2;(q=1)$$

So, it suffices to prove that: $27r^2(p^2-2+3\lambda)\leq 3\lambda+1$ or

$$27p^2r^2+(3\lambda-2)\cdot 27r^2\leq 3\lambda+1;(2)$$

But $q^2\geq 3pr\Rightarrow p^2r^2\leq\frac{1}{9}$, so (2) becomes the stronger inequality:

$$3+(3\lambda-2)\cdot 27r^2\leq 3\lambda+1\Leftrightarrow 27r^2\leq\frac{3\lambda-2}{3\lambda-2}=1 \text{ as } 3\lambda-2>0.$$

If $\lambda=\frac{2}{3}$ we have equality, which is true $\left(r\leq\frac{\sqrt{3}}{9}\right)$.

SP.351 If $x,y\in\left(0,\frac{\pi}{2}\right)$; $\sqrt[3]{1+\tan x}+\sqrt[3]{1+\tan y}=2\sqrt[3]{2}$ then:

$$\sqrt[3]{1-\tan x}+\sqrt[3]{1-\tan y}\le 4-2\sqrt[3]{2}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by proposer

$$\text{Denote } z=\sqrt[3]{1+\tan x}+\sqrt[3]{1-\tan x}$$

$$z^3=\left(\sqrt[3]{1+\tan x}\right)^3+\left(\sqrt[3]{1-\tan x}\right)^3+$$

$$+3\sqrt[3]{1+\tan x}\cdot\sqrt[3]{1-\tan x}\left(\sqrt[3]{1+\tan x}+\sqrt[3]{1-\tan x}\right)$$

$$z^3=1+\tan x+1-\tan x+3\sqrt[3]{1-\tan^2 x}\cdot z$$

$$\frac{z^3-2}{3z}=\sqrt[3]{1-\tan^2 x}\le 1\Rightarrow\frac{z^3-2}{3z}\le 1$$

$$z^3-2\le 3z\Rightarrow z^3-3z\le 0$$

$$z^3-z-2(z+1)\le 0\Rightarrow z(z-1)(z+1)-2(z+1)\le 0$$

$$(z+1)(z^2-z-2)\le 0$$

$$(z+1)\left(z^2-1-(z+1)\right)\le 0$$

$$(z+1)\left((z-1)(z+1)-(z+1)\right)\le 0$$

$$(z+1)^2(z-2)\le 0\Rightarrow z-2\le 0\Rightarrow z\le 2$$

$$\sqrt[3]{1+\tan x}+\sqrt[3]{1-\tan x}\le 2\ \ (1)$$

Analogous:

$$\sqrt[3]{1+\tan y}+\sqrt[3]{1-\tan y}\le 2\ \ \ (2)$$

By adding (1); (2):

$$\sqrt[3]{1-\tan x}+\sqrt[3]{1-\tan y}+\sqrt[3]{1+\tan x}+\sqrt[3]{1+\tan y}\le 4$$

$$\sqrt[3]{1-\tan x}+\sqrt[3]{1-\tan y}+2\sqrt[3]{2}\le 4$$

$$\sqrt[3]{1-\tan x}+\sqrt[3]{1-\tan y}\le 4-2\sqrt[3]{2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=\sqrt[3]{1+\tan x}; b=\sqrt[3]{1+\tan y}\Rightarrow 1-\tan x=2-a^3; 1-\tan y=2-b^3$

So, we need to prove that: $\sqrt[3]{2-a^3}+\sqrt[3]{2-b^3}\le 4-2\sqrt[3]{2}, \forall a,b\in\left(1,2\sqrt[3]{2}-1\right)$

and $a+b=2\sqrt[3]{3}$

We have: $\sqrt[3]{2-a^3}\le 2-a \Leftrightarrow 2-a^3\le(2-a)^3 \Leftrightarrow 6(1-a)^2\ge 0$ which is true.

Hence,

$$\sqrt[3]{2-a^3}\le 2-a \text{ and } \sqrt[3]{2-b^3}\le 2-b$$

Therefore,

$$\sqrt[3]{2-a^3}+\sqrt[3]{2-b^3}\le 4-(a+b)\le 4-2\sqrt[3]{2}$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

Let us denote: $a=\sqrt[3]{1+\tan x}; b=\sqrt[3]{1-\tan x}; c=\sqrt[3]{1+\tan y}; d=\sqrt[3]{1-\tan y}$

$$\Rightarrow a^3+b^3=2; c^3+d^3=2\Rightarrow a,c<1<b,d$$

Other, $\sqrt[3]{1+\tan x}+\sqrt[3]{1+\tan y}=2\sqrt[3]{2}\Leftrightarrow b+d=2\sqrt[3]{2}$.

Inequality becomes as: $a+c\le 4-2\sqrt[3]{2}; (1)$

If $a,c<0$ then: $0<4-2\sqrt[3]{2}\Leftrightarrow 2\sqrt[3]{2}<4\Leftrightarrow 16<64(true)$.

If $0\le a,c<1$ then: $(1)\Leftrightarrow a+c\le 4-(b+d)\Leftrightarrow a+b+c+d\le 4$, which is true by Holder's inequality:

$$\frac{a^3}{1}+\frac{b^3}{1}+\frac{c^3}{1}+\frac{d^3}{1}\ge\frac{(a+b+c+d)^3}{4^2}\Rightarrow(a+b+c+d)^3\le 4^3\Rightarrow a+b+c+d\le 4$$

If $a<0<c<1$ then: $a+b+c+d<c+b+d<1+2\sqrt[3]{2}<4\Rightarrow(1)$ is true.

If $c<0<a<1$ then: $a+b+c+d<a+b+d<1+2\sqrt[3]{2}<4\Rightarrow(1)$ is true.

SP.352 Let $(x_n)_{n\ge1},(y_n)_{n\ge1}$ be sequences of real numbers with

$$x_1=0, y_1=1,$$

$$x_{n+1}=\frac{ax_n+by_n}{a+b}, y_{n+1}=\frac{cx_n+dy_n}{c+d}, \forall n\ge 1, a,b,c,d>0, ad\ne bc.$$

Prove that if $(z_n)_{n\ge1}, z_n=y_n-x_n$, then $(z_n)_{n\ge1}$ –geometric progression, and if $q<1, q$ –ratio of progression, then $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n$.

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

$$z_{n+1} = y_{n+1} - x_{n+1} = \frac{ad - bc}{(a+b)(c+d)}(y_n - x_n) = \frac{ad - bc}{(a+b)(c+d)} z_n$$

We deduce that $(z_n)_{n\geq 1}$ −geometric progression with $q = \frac{ad-bc}{(a+b)c+d)}$ ratio, (1).

We get, $z_n = z_1 q^{n-1} = 1 \cdot q^{n-1} = q^{n-1}$, where $q = \frac{ad-bc}{(a+b)c+d)}$.

From $z_n = y_n - x_n$ and $z_n = q^{n-1}$, then $y_n = x_n + q^{n-1}$ which replacing in $x_{n+1} = \frac{ax_n + by_n}{a+b}$, it follows $x_{n+1} = x_n + \frac{b}{a+b} q^{n-1}$, and then for $1, 2, \dots, n-1$ values, we get: $x_n = x_1 + \frac{b}{a+b}(1 + q + q^2 + \dots + q^{n-1}) = 0 + \frac{b}{a+b} \cdot \frac{1-q^n}{1-q} = \frac{b}{a+b} \cdot \frac{1-q^n}{1-q}$.

How $q \in (0,1)$, it follows that $(x_n)_{n\geq 1}, x_n = \frac{b}{a+b} \cdot \frac{1-q^n}{1-q}$ is convergent, and then

$\lim\limits_{n\to\infty} x_n = \frac{b}{a+b} \cdot \frac{1}{1-q}$ and replacing $q = \frac{ad-bc}{(a+b)c+d)}$, we get:

$$\lim_{n\to\infty} x_n = \frac{b(c+d)}{ac + 2bc + bd}.$$

Using $y_n = x_n + q^{n-1}, q \in 0,1), (x_n)_{n\geq 1}$ −convergent, then $(y_n)_{n\geq 1}$ −convergent.

Hence, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$.

Solution 2 by Remus Florin Stanca-Romania

$$z_n = y_n - x_n = \frac{cx_{n-1} + dy_{n-1}}{c+d} - \frac{ax_{n-1} + by_{n-1}}{a+b} =$$

$$= \frac{acx_{n-1} + ady_{n-1} + bcx_{n-1} + bdy_{n-1} - acx_{n-1} - bcy_{n-1} - adx_{n-1} - bdy_{n-1}}{(a+b)(c+d)} =$$

$$= \frac{adz_{n-1} - bcz_{n-1}}{(a+b)(c+d)} = z_n \Rightarrow \frac{z_n}{z_{n-1}} = \frac{ad - bc}{(a+b)(c+d)} = q$$

$$y_{n+1} = \frac{cx_n + dy_n}{c+d} = \frac{c}{c+d}x_n + \frac{d}{c+d}y_n + \frac{c}{c+d}y_n - \frac{c}{c+d}y_n \Rightarrow$$

$$y_{n+1} - y_n = -\frac{c}{c+d}z_n \Rightarrow \sum_{k=1}^{n}(y_{k+1} - y_k) = -\frac{c}{c+d}\sum_{k=1}^{n} z_k \Rightarrow$$

$$y_{n+1} - y_1 = -\frac{c}{c+d}z_1 \cdot \frac{q^n - 1}{q - 1}; z_1 = 1 \Rightarrow y_{n+1} = 1 - \frac{c}{c+d} \cdot \frac{q^n - 1}{q - 1}; q < 1 \Rightarrow$$

$$\lim_{n\to\infty} y_n = 1 - \frac{c}{c+d}\cdot\frac{1}{1-q};(1)$$

$$x_{n+1} = \frac{ax_n + by_n}{a+b} = \frac{ax_n}{a+b} + \frac{bx_n}{a+b} + \frac{by_n}{a+b} - \frac{bx_n}{a+b} \Rightarrow x_{n+1} - x_n = \frac{b}{a+b}z_n \Rightarrow$$

$$\sum_{k=1}^{n}(x_{k+1} - x_k) = \frac{b}{a+b}\cdot\frac{q^n - 1}{q-1} \Rightarrow \lim_{n\to\infty} x_n = \frac{b}{a+b}\cdot\frac{1}{1-q};(2)$$

We need to prove that:

$$1 - \frac{c}{c+d}\cdot\frac{q^n-1}{q-1} = \frac{b}{a+b}\cdot\frac{1}{1-q} \Leftrightarrow \frac{1}{1-q}\left(\frac{c}{c+d} + \frac{b}{a+b}\right) = 1 \Leftrightarrow$$

$$\frac{ac+2bc+bd}{(a+b)(c+d)} = 1 - q \Leftrightarrow 1 - \frac{ad-bc}{(a+b)(c+d)} = \frac{ac+2bc+bd}{(a+b)(c+d)} \Leftrightarrow$$

$\frac{ac+ad+bc+bd-ad+bc}{(a+b)(c+d)} = \frac{ac+2bc+bd}{(a+b)(c+d)}$, **which is true.**

Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

SP.353 Let $\lambda > 0$ fixed. Solve for real numbers:

$$\begin{cases} \lambda x = \sqrt{\lambda^2 y^2 - 1} + \sqrt{\lambda^2 z^2 - 1} \\ \lambda y = \sqrt{\lambda^2 z^2 - 1} + \sqrt{\lambda^2 x^2 - 1} \\ \lambda z = \sqrt{\lambda^2 x^2 - 1} + \sqrt{\lambda^2 y^2 - 1} \end{cases}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Conditions: $x, y, z \geq \frac{1}{\lambda}, (1)$ **and denote** $\begin{cases} a = \sqrt{\lambda^2 x^2 - 1} \\ b = \sqrt{\lambda^2 y^2 - 1} \\ c = \sqrt{\lambda^2 z^2 - 1} \end{cases}, a, b, c \geq 0 \Rightarrow \begin{cases} \lambda x = b + c \\ \lambda y = c + a \\ \lambda z = a + b \end{cases}$

$$\Leftrightarrow \begin{cases} \lambda^2 x^2 = b^2 + c^2 + 2bc \\ \lambda^2 y^2 = c^2 + a^2 + 2ca \\ \lambda^2 z^2 = a^2 + b^2 + 2ab \end{cases} \Leftrightarrow \begin{cases} a^2 + 1 = b^2 + c^2 + 2bc \\ b^2 + 1 = c^2 + a^2 + 2ca \\ c^2 + 1 = a^2 + b^2 + 2ab \end{cases} \Leftrightarrow$$

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 3 \Leftrightarrow (a+b+c)^2 = 3 \Leftrightarrow a + b + c = \sqrt{3}.$$

From $a = \sqrt{3} - (b+c), a = \sqrt{\lambda^2 x^2 - 1}$ **and** $b + c = \lambda x$, **then**

$$\sqrt{\lambda^2x^2-1}=\sqrt{3}-\lambda x\Leftrightarrow \lambda^2x^2-1=3+\lambda^2x^2-2\lambda\sqrt{3}x\Leftrightarrow x=\frac{2}{\lambda\sqrt{3}}.$$

Similarly, $y=z=\frac{2}{\lambda\sqrt{3}}$.

$$(x,y,z)=\left(\frac{2}{\lambda\sqrt{3}},\frac{2}{\lambda\sqrt{3}},\frac{2}{\lambda\sqrt{3}}\right)$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\lambda x=\sqrt{\lambda^2y^2-1}+\sqrt{\lambda^2z^2-1}$$

As $RHS\ge 0,\lambda x\ge 0\Rightarrow x\ge 0$. Similarly, $y\ge 0,z\ge 0$.

$$\lambda(x-y)=\sqrt{\lambda^2y^2-1}-\sqrt{\lambda^2x^2-1}=\frac{\lambda^2(y^2-x^2)}{\sqrt{\lambda^2y^2-1}+\sqrt{\lambda^2x^2-1}}\Rightarrow$$

$\frac{\lambda(x+y)}{\sqrt{\lambda^2y^2-1}+\sqrt{\lambda^2x^2-1}}=-1$ (not possible as RHS<0) or $x=y$

Similarly, $y=z,z=x$. Thus, $\lambda x=2\sqrt{\lambda^2x^2-1}\Rightarrow\lambda^2x^2=4(\lambda^2x^2-1)$

$$\Rightarrow x=\frac{2}{\sqrt{3}\lambda}$$

Therefore, $x=y=z=\frac{2}{\sqrt{3}\lambda}$

Solution 3 by Aggeliki Papasyropoulou-Greece

$\sqrt{\lambda^2x^2-1}>0;\sqrt{\lambda^2y^2-1}>0;\ \sqrt{\lambda^2z^2-1}>0$ because if $\sqrt{\lambda^2x^2-1}=0\Rightarrow$

$$\lambda y=\sqrt{\lambda^2z^2-1};\lambda z=\sqrt{\lambda^2y^2-1}\Leftrightarrow$$

$$\lambda^2y^2=\lambda^2z^2-1;\lambda^2z^2=\lambda^2y^2-1\Rightarrow\lambda^2y^2+\lambda^2z^2=\lambda^2z^2+\lambda^2y^2-2\Leftrightarrow 0=-2$$

Let $a=\sqrt{\lambda^2x^2-1};b=\sqrt{\lambda^2y^2-1};c=\sqrt{\lambda^2z^2-1}$ and we need to solve the system:

$$\begin{cases}\sqrt{a^2+1}=b+c;(1)\\ \sqrt{b^2+1}=c+a;(2)\\ \sqrt{c^2+1}=a+b;(3)\end{cases}\Rightarrow a^2+1+b^2+1+c^2+1=(b+c)^2+(c+a)^2+(a+b)^2$$

$$\sum_{cyc}a^2+3=2\sum_{cyc}a^2+2\sum_{cyc}ab\Leftrightarrow a^2+b^2+c^2+2ab+2bc+2ca=3\Leftrightarrow$$

$$(a+b+c)^2=3\Leftrightarrow a+b+c=\sqrt{3}$$

So, $(1)\Leftrightarrow\sqrt{a^2+1}=\sqrt{3}-a$. Let $f(x)=x+\sqrt{x^2+1},x>0;f\nearrow$

$$(2) \Leftrightarrow \sqrt{b^2+1} = \sqrt{3} - b;\ \ (3) \Leftrightarrow \sqrt{c^2+1} = \sqrt{3} - c.$$

So, $a = b = c$ and we solve the equation $\sqrt{a^2+1} = \sqrt{3} - a; \sqrt{3} > a > 0 \Leftrightarrow$

$$a^2 + 1 = 3 + a^2 - 2a\sqrt{3} \Leftrightarrow a = \frac{\sqrt{3}}{3} \text{ and } b = c = \frac{\sqrt{3}}{3}$$

So, we solve equation: $\frac{\sqrt{3}}{3} = \sqrt{\lambda^2 x^2 - 1} \Leftrightarrow \lambda^2 x^2 - 1 = \frac{1}{3} \Leftrightarrow$

$$\lambda^2 x^2 = \frac{4}{3} \Leftrightarrow x^2 = \frac{4}{3\lambda^2}; (x > 0) \Leftrightarrow x = \frac{2\sqrt{3}}{3\lambda}$$

Therefore,

$$x = y = z = \frac{2\sqrt{3}}{3\lambda}$$

SP.354 If $x, y, z \geq 1$ then:

$$y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + log(x^{xy} \cdot y^{yz} \cdot z^{zx})$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ruxandra Daniela Tonilă-Romania

$$y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + log(x^{xy} \cdot y^{yz} \cdot z^{zx})$$

$$\Leftrightarrow y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + y \cdot logx^x + z \cdot logy^y + x \cdot logz^z$$

Let $a = x^x; b = y^y; c = z^z; a, b, c \geq 1$

$$xc + zb + ya - x - y - z - yloga - zlogb - xlogc \geq 0$$

$$y(a - 1 - loga) + z(b - 1 - logb) + x(c - 1 - logc) \geq 0$$

Since $x, y, z \geq 1$, to prove that inequality holds, we have to prove that:

$$\begin{cases} a - 1 - loga \geq 0 \\ b - 1 - logb \geq 0 \\ c - 1 - logc \geq 0 \end{cases} \Leftrightarrow \begin{cases} a \geq 1 + loga \\ b \geq 1 + logb \\ c \geq 1 + logc \end{cases} \Leftrightarrow \begin{cases} e^{loga} \geq 1 + loga \\ e^{logb} \geq 1 + logb \\ e^{logc} \geq 1 + logc \end{cases}$$

Let $x' = loga; y' = logb; z' = logc \Leftrightarrow \begin{cases} e^{x'} \geq 1 + x' \\ e^{y'} \geq 1 + y' \\ e^{z'} \geq 1 + z' \end{cases}$ which is true $\forall x', y', z' \in \mathbb{R}$

Therefore,

$$y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + log(x^{xy} \cdot y^{yz} \cdot z^{zx})$$

Solution 2 by Ravi Prakash-New Delhi-India

For $x \geq 1, logx \geq 0; x^x = e^{xlogx} \geq 1 + logx$

$$y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + xy \cdot logx + yz \cdot logy + zx \cdot logz =$$
$$x + y + z + log(x^{xy} \cdot y^{yz} \cdot z^{zx})$$

Solution 3 by Nikos Ntorvas-Greece

$$\sum_{cyc} y \cdot x^x \geq \sum_{cyc} x + \sum_{cyc} xy \cdot logx, \forall x, y, z \geq 1 \Leftrightarrow$$
$$\sum_{cyc} y \cdot x^x \geq \sum_{cyc} x + \sum_{cyc} y \cdot logx^x$$
$$\sum_{cyc} y(x^x - logx^x) - \sum_{cyc} x \geq 0$$
$$\sum_{cyc} y(x^x - logx^x - 1) \geq 0, \forall x, y, z \geq 1$$

We need to prove that: $f(t) = t^t - logt^t - 1 \geq 0, \forall t \geq 1$.

$$f'(t) = (1 + logt)\left(e^{tlogt} - 1\right) > 0, \forall t > 1$$

- $t \geq 1 \Rightarrow 1 + logt \geq 1$
- $t \geq 1 \Rightarrow tlogt \geq 0 \Rightarrow e^{tlogt} - 1 \geq 0$

So, f increasing for $t \geq 1$ and we have that:

$$t \geq 1 \Leftrightarrow f(t) \geq f(1) \Leftrightarrow f(t) \geq 0, \forall t \geq 1.$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$loge^{xz^z} + loge^{zy^y} + loge^{yx^x} \geq loge^x + loge^y + loge^z + log(x^{xy} \cdot y^{yz} \cdot z^{zx})$$
$$loge^{xz^z + zy^y + yx^x} \geq log(e^{x+y+z} \cdot x^{xy} \cdot y^{yz} \cdot z^{zx})$$
$$e^{xz^z + zy^y + yx^x} \geq e^{x+y+z} \cdot x^{xy} \cdot y^{yz} \cdot z^{zx}$$

$e^{xz^z - x} \cdot e^{yx^x - y} \cdot e^{zy^y - z} \geq x^{xy} \cdot y^{yz} \cdot z^{zx}$, which is true

Because $e^{xz^z - x} \geq z^{zx} \Leftrightarrow e^{x(z^z - 1)} \geq z^{zx} \Leftrightarrow e^{z^z - 1} \geq z^z$.

Analogously: $e^{yx^x - y} \geq x^{xy}$ and $e^{zy^y - z} \geq y^{yz}$.

Solution 5 by proposer

It is known that $e^t \geq 1 + t; t \geq 0$

Replace $t \to x \ln x\,; x \geq 1$

$$e^{x \ln x} \geq 1 + x \ln x \Rightarrow e^{\ln x^x} \geq 1 + \ln x^x$$

$$x^x \geq 1 + \ln x^x \quad (1)$$

Multiplying (1) with y:

$$y \cdot x^x \geq y + y \ln x^x \quad (2)$$

Analogous with (2):

$$z \cdot y^y \geq z + z \ln y^y \quad (3)$$

$$x \cdot z^z \geq x + x \ln z^z \quad (4)$$

By adding (2); (3); (4):

$$y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + z \ln(y^y) +$$

$$+ x \ln(z^z) + y \ln x^x = x + y + z + \ln(x^{xy}) + \ln(y^{yz}) + \ln(z^{zx}) =$$

$$= x + y + z + \ln(x^{xy} \cdot y^{yz} \cdot z^{zx})$$

Equality holds for $x = y = z = 1$.

Solution 6 by Lazaros Zachariadis-Thessaloniki-Greece

$$xy \cdot logx = y \cdot logx^x \leq y(x^x - 1) = y \cdot x^x - y$$

So, $xy \cdot logx + y \leq y \cdot x^x$ or $logx^{xy} + y \leq y \cdot x^x$. Thus.

$$\sum_{cyc} y \cdot x^x \geq \sum_{cyc} y + \sum_{cyc} logx^{xy}$$

Therefore,

$$y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + log(x^{xy} \cdot y^{yz} \cdot z^{zx})$$

SP.355 Let I_a, I_b, I_c and r_a, r_b, r_c denote the excenters and exradii of the triangle ABC, respectively. Let ρ_a –be the radius of the circle that lies inside and touches internally the excircle opposite A and touches the sides I_aI_b, I_aI_c of triangle $I_aI_bI_c$ externally. Let ρ_b, ρ_c be defined similarly. Prove that:

$$\frac{\rho_a}{r_a} + \frac{\rho_b}{r_b} + \frac{\rho_c}{r_c} \leq 1$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by proposer

The following equalities can be obtained easily.

$$\sin\left(\frac{B+C}{4}\right)=\frac{\rho_a}{r_a-\rho_a};\sin\left(\frac{C+A}{4}\right)=\frac{\rho_b}{r_b-\rho_b};\sin\left(\frac{A+B}{4}\right)=\frac{\rho_c}{r_c-\rho_c};(1)$$

$$\frac{\rho_a}{r_a}=\frac{\sin\left(\frac{\pi-A}{4}\right)}{1+\sin\left(\frac{\pi-A}{4}\right)};\frac{\rho_b}{r_b}=\frac{\sin\left(\frac{\pi-B}{4}\right)}{1+\sin\left(\frac{\pi-B}{4}\right)};\frac{\rho_c}{r_c}=\frac{\sin\left(\frac{\pi-C}{4}\right)}{1+\sin\left(\frac{\pi-C}{4}\right)};(2)$$

Moreover we have:

$$1+\sin\left(\frac{\pi-A}{4}\right)=2\cos^2\left(\frac{\pi+A}{8}\right);1+\sin\left(\frac{\pi-B}{4}\right)=2\cos^2\left(\frac{\pi+B}{8}\right);$$

$$1+\sin\left(\frac{\pi-C}{4}\right)=2\cos^2\left(\frac{\pi-C}{8}\right);(3)$$

Substituing (3) into (2) we get:

$$\frac{\rho_a}{r_a}+\frac{\rho_b}{r_b}+\frac{\rho_c}{r_c}=\frac{2\cos^2\left(\frac{\pi+A}{8}\right)-1}{2\cos^2\left(\frac{\pi+A}{8}\right)}+\frac{2\cos^2\left(\frac{\pi+B}{8}\right)-1}{2\cos^2\left(\frac{\pi+B}{8}\right)}+\frac{2\cos^2\left(\frac{\pi+C}{8}\right)-1}{2\cos^2\left(\frac{\pi+C}{8}\right)}=$$

$$=3-\frac{1}{2}\left[\frac{1}{\cos^2\left(\frac{\pi+A}{8}\right)}+\frac{1}{\cos^2\left(\frac{\pi+B}{8}\right)+\cos^2\left(\frac{\pi+C}{8}\right)}\right];(4)$$

Let $f(x)=\frac{1}{\cos^2 x}, f''(x)>0\rightarrow f$ –convex function, using Jensen's inequality, (4) becomes

$$\frac{1}{\cos^2\left(\frac{\pi+A}{8}\right)}+\frac{1}{\cos^2\left(\frac{\pi+B}{8}\right)+\cos^2\left(\frac{\pi+C}{8}\right)}\ge\frac{3}{\cos^2\left(\frac{3\pi+A+B+C}{8\cdot 3}\right)}=\frac{3}{\cos^2\frac{\pi}{6}}$$

$$\frac{\rho_a}{r_a}+\frac{\rho_b}{r_b}+\frac{\rho_c}{r_c}\le 3-\frac{1}{2}\cdot 4=1$$

SP.356 If $a,b,c>0$ such that $abc=1$ then:

$$\frac{(1+ab)^3}{(c+a)(a+b)}+\frac{(1+bc)^3}{(a+b)(b+c)}+\frac{(1+ca)^3}{(b+c)(c+a)}\ge\frac{54}{(a^2+b^2+c^2)^2}$$

Proposed by Pedro Pantoja-Natal-Brazil

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{(1+ab)^3}{(c+a)(a+b)}+\frac{(1+bc)^3}{(a+b)(b+c)}+\frac{(1+ca)^3}{(b+c)(c+a)}\ge\frac{54}{(a^2+b^2+c^2)^2}$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^2\left(\sum_{cyc}(1+ab)^3(b+c)\right)\ge 54\prod_{cyc}(a+b)$$

$$\because\sum_{cyc}a^2\overset{CBS}{\ge}\frac{1}{3}\left(\sum_{cyc}a\right)^2\Rightarrow$$

$$\left(\sum_{cyc}a^2\right)^2\ge\frac{1}{9}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a\right)^3\overset{AM-GM}{\ge}\frac{1}{3}\left(\sum_{cyc}a\right)^3=$$

$$=\frac{1}{24}\left(\sum_{cyc}(a+b)\right)^3\overset{AM-GM}{\ge}\frac{9}{8}\prod_{cyc}(a+b);(1)$$

$$\sum_{cyc}(1+ab)^3(b+c)\overset{AM-GM}{\ge}\sum_{cyc}\left(2\sqrt{ab}\right)^3\left(2\sqrt{bc}\right)\overset{abc=1}{=}16\sum_{cyc}\sqrt{a^2b^3}\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge}16\cdot 3\cdot\sqrt[6]{\prod_{cyc}a^2b^3}\overset{abc=1}{=}48;(2)$$

$$(1),(2)\Rightarrow\left(\sum_{cyc}a^2\right)^2\left(\sum_{cyc}(1+ab)^3(b+c)\right)\ge 54\prod_{cyc}(a+b)$$

Therefore,

$$\sum_{cyc}\frac{(1+ab)^3}{(c+a)(a+b)}\ge\frac{54}{(\sum a^2)^2}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$a^2+b^2+c^2\ge 3\cdot\sqrt[3]{(abc)^2}=3$$

$$\sum_{cyc}\frac{(1+ab)^3}{(a+b)(a+c)}\overset{Holder}{\ge}\frac{(3+ab+bc+ca)^3}{3\sum\left((a+b)(b+c)\right)}\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\geq} \frac{\left(3+3\cdot\sqrt[3]{(abc)^2}\right)^3}{3\sum\left((a+b)(aa+c)\right)} = \frac{(3+3)^3}{3\sum\left((a+b)(a+c)\right)} = \frac{6^3}{3\sum\left((a+b)(a+c)\right)} =$$

$$= \frac{72}{\sum\left((a+b)(a+c)\right)} \overset{(1)}{\geq} \frac{54}{(a^2+b^2+c^2)^2}$$

$$(1) \Leftrightarrow 4(a^2+b^2+c^2)^2 \geq 3\sum_{cyc}\left((a+b)(a+c)\right) \Leftrightarrow$$

$$4(a^2+b^2+c^2)^2 \geq 3\left(a^2+b^2+c^2+3(ab+bc+ca)\right)$$

But: $ab+bc+ca \leq a^2+b^2+c^2 \Rightarrow$

$$3\left(a^2+b^2+c^2+3(ab+bc+ca)\right) \leq 3\left(a^2+b^2+c^2+3(a^2+b^2+c^2)\right) =$$

$$= 12(a^2+b^2+c^2);$$

We need to prove that:

$$4(a^2+b^2+c^2)^2 \geq 12(a^2+b^2+c^2) \Leftrightarrow a^2+b^2+c^2 \geq 3(true).$$

Solution 3 by Aggeliki Papasyropoulou-Greece

$$\frac{(1+ab)^3}{(c+a)(a+b)} + \frac{(1+bc)^3}{(a+b)(b+c)} + \frac{(1+ca)^3}{(b+c)(c+a)} \geq \frac{54}{(a^2+b^2+c^2)^2}; (1)$$

$$a+b+c \geq 3\sqrt[3]{abc}\ (AM-GM) \Leftrightarrow a+b+c=3; (abc=1); (2)$$

$$ab+bc+ca \geq 3\sqrt[3]{(abc)^2} = 3 \Rightarrow ab+bc+ca \geq 3; (3)$$

$$P = \frac{(1+ab)^3}{(c+a)(a+b)} + \frac{(1+bc)^3}{(a+b)(b+c)} + \frac{(1+ca)^3}{(b+c)(c+a)};$$

$$(p,q,r) = (\textstyle\sum a, \sum ab, \prod a); p \geq 3; q \geq 3; r = 1$$

$$P\cdot[(c+a)(a+b)+(a+b)(b+c)+(b+c)(c+a)](1+1+1) \geq$$

$$\geq [(1+ab)+(1+bc)+(1+ca)] \Leftrightarrow$$

$$P\cdot[\textstyle\sum a^2 + 3\sum ab]\cdot 3 \geq (3+\sum ab)^3 \Leftrightarrow$$

$$P \geq \frac{(3+q)^3}{3(p^2-2q+3q)} = \frac{(q+3)^3}{3(p^2+q)}$$

So, it's enough to prove that:

$$\frac{(q+3)^3}{3(p^2+q)} \geq \frac{54}{(p^2-2q)^2} \Leftrightarrow (p^2-2q)^2(q+3)^3 \geq 3q\cdot 3^3(p^2+q); (4)$$

$$q \geq 3 \Rightarrow q+3 \geq 6; p \geq 3 \text{ and } p^2 \geq 3q; (5)$$

$$(4) \Rightarrow LHS \geq (p^2-2q)^2 6^3 \geq 3 \cdot 2 \cdot 3^3 (p^2+q) \Leftrightarrow$$

$$4(p^2-2q)^2 \geq 3p^2+3q$$

$$(5) \Rightarrow 3q+3p^2 \leq p^2+3p^2 = 4p^2$$

So, we have to prove that:

$$p^2-2q \geq p \Leftrightarrow a^2+B62+c^2 \geq a+b+c; (6)$$

$$CBS: 3(a^2+b^2+c^2) \geq (a+b+c)^2; (7)$$

From (6),(7) we must to prove that:

$$\frac{(a+b+c)^2}{3} \geq a+b+c \Leftrightarrow a+b+c \geq 3; \text{ true.}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{(1+ab)^3}{(c+a)(a+b)} + \frac{(1+bc)^3}{(a+b)(b+c)} + \frac{(1+ca)^3}{(b+c)(c+a)} \geq \frac{3(1+ab)(1+bc)(1+ca)}{\sqrt[3]{(a+b)^2(b+c)^2(c+a)^2}} \geq$$

$$\geq \frac{54}{(a^2+b^2+c^2)^2}$$

$$(1+a)(1+b)(1+c)(a^2+b^2+c^2)^2 \geq 18\sqrt[3]{(a+b)^2(b+c)^2(c+a)^2}$$

$$(1+a)(1+b)(1+c)(a^2+b^2+c^2) \geq \frac{18((a+b)^2+(b+c)^2+(c+a)^2)}{3} =$$

$$= 12(a^2+b^2+c^2+ab+bc+ca)$$

$$8(a^2+b^2+c^2)^2 \geq 12(a^2+b^2+c^2+ab+bc+ca)$$

$$24(a^2+b^2+c^2) \geq 12(a^2+b^2+c^2+ab+bc+ca)$$

$$2(a^2+b^2+c^2) \geq a^2+b^2+c^2+ab+bc+ca \text{ true.}$$

Solution 5 by proposer

$$\left(\sum_{cyc}(b+c)\right)\left(\sum_{cyc}(a+c)\right)\left(\sum_{cyc}\frac{(1+bc)^3}{(a+b)(b+c)}\right) \geq \left(\sum_{cyc}(1+bc)\right)^3$$

$$\sum_{cyc}\frac{(1+bc)^3}{(a+b)(b+c)} \geq \frac{(3+ab+bc+ca)^3}{4(a+b+c)^2}$$

$$(1+1+1+ab+bc+ca)^3 \overset{AM-GM}{\geq} \left(6\sqrt[3]{(abc)^2}\right)^3 = 216; (*)$$

Note that $a^2+b^2+c^2 \ge a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}(a+b+c)$, by Murihead's inequality since

$(2,0,0) \succ \left(\frac{4}{3},\frac{1}{3},\frac{1}{3}\right)$. However $abc=1$, then $\frac{1}{(a+b+c)^2} \ge \frac{1}{(a^2+b^2+c^2)^2}$; $(**)$

Therefore, from $(*)$, $(**)$ we have:

$$\sum_{cyc}\frac{(1+bc)^3}{(a+b)(b+c)} \ge \frac{54}{(a^2+b^2+c^2)^2}$$

Equality holds if and only of $a=b=c=1$.

SP.357 Let $S(n)$ be the sum of the digits of the positive integer n. Determine all pairs of positive integers (a,b), $a\ge b$ such that the equation $S(a^2)=(b-2018)^4$ has only finite solutions in positive integers.

Proposed by Pedro Pantoja-Natal-Brazil

Solution by proposer

i) If $a>b$ the equation has infinite solutions: $a=10^{9k^4}-1$,

$b=2018+3k, k\in\mathbb{N}$.

Indeed: $a^2=\underbrace{99\ldots.9}_{9k^4-1}8\underbrace{000\ldots0}_{9k^4-1}1$, then $s(a^2)=9(9k^4-1)+8+1=81k^4$

On the other hand $(b-2018)^4=(3k^4)=81k^4$.

ii) If $a=b$. Note that $S(a^2)=(a-2018)^4\le a^2 \Rightarrow$

$(a-2018)^2\le a \Rightarrow a^2-4037a+2018^2\le 0 \Rightarrow 1974<a<2064$, so we conclude that the number a has 4 digits.

If $a\ge 2021$ then $(a-2018)^4\ge 81$ and $S(a^2)<36(9+9+9+9)$.

If $a\le 2015$ then $(a-2018)^4\ge 81$ and $S(a^4)<36(9+9+9+9)$.

Therefore, $a\in\{2015,\ldots,2021\}$, $a\ne 2018$. Testing all these possibilities in the statement equation, we obtain as the only solution $a=2020$.

Cheking: $a^2=4080400 \Rightarrow S(a^2)=16$ and $(2020-2018)^4=2^4=16$.

Answer: $(a,b)=(2020,2020)$.

SP.358 If $x, y, z > 0$ then:

$$4\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+3\geq 6\sqrt[3]{xyz}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$4\sum_{cyc}\frac{x^3}{(y+1)(z+1)}\overset{Holder}{\geq} 4\cdot\frac{(x+y+z)^3}{3\sum(y+1)(z+1)}=4\cdot\frac{(x+y+z)^3}{3(\sum xy+2\sum x+3)}\geq$$

$$\overset{3\sum xy\leq(\sum x)^2}{\geq} 4\cdot\frac{(\sum x)^3}{3\left(\frac{1}{3}(\sum x)^2+2\sum x+3\right)}=4\cdot\frac{(\sum x)^3}{(\sum x)^2+6\sum x+9}=4\cdot\frac{(\sum x)^3}{(\sum x+3)^2}$$

$$\rightarrow 4\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+3\geq 4\cdot\frac{(\sum x)^3}{(\sum x+3)^2}+3;\ \ (1)$$

Let us denote:$t=\sum x\overset{AM-GM}{\geq} 3\cdot\sqrt[3]{xyz}>0\rightarrow 2t\geq 6\cdot\sqrt[3]{xyz};\ \ (2)$

From $(1),(2)$ we need to prove that:

$$4\cdot\frac{t^3}{(t+3)^2}+3\geq 2t\Leftrightarrow 4t^3+3(t+3)^2\geq 2t(t+3)^2\Leftrightarrow$$

$$4t^3+3(t^2+6t+9)\geq 2t(t^2+6t+9)\Leftrightarrow 2t^3-9t^2+27\geq 0$$

$\Leftrightarrow (t-3)^2(2t+3)\geq 0$, which is true because $t>0\rightarrow 2t+3>0$.

Equality holds when $x=y=z$ and $t=3\Leftrightarrow x=y=z=1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM-GM: $\sum x\geq 3\sqrt[3]{xyz}$. So, it is suffices to prove that:

$$4\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+3\geq 2\sum_{cyc}x\Leftrightarrow$$

$$4\sum_{cyc}x^3(x+1)+3\prod_{cyc}(x+1)\geq 2\left(\sum_{cyc}x\right)\prod_{cyc}(x+1)\Leftrightarrow$$

$$4\sum_{cyc}x^4+4\sum_{cyc}x^3+3xyz+3\sum_{cyc}x+3\sum_{cyc}xy+3\geq$$

$$\geq 2xyz\sum_{cyc}x+2\left(\sum_{cyc}x\right)^2+2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)+2\sum_{cyc}x\Leftrightarrow$$

$$4\sum_{cyc}x^4+4\sum_{cyc}x^3+\sum_{cyc}x+3\geq$$

$$\geq 2xyz\sum_{cyc}x+2\sum_{cyc}x^2+\sum_{cyc}xy+2\sum_{cyc}x^2y+3xyz$$

We have:

$$2\sum_{cyc}x^4\overset{Murihead}{\geq}2\sum_{cyc}x^2yz=2xyz\sum_{cyc}x;\ \ (1)$$

$$4\sum_{cyc}x^3\overset{Murihead}{\geq}2\sum_{cyc}x^2y;\ \ (2)$$

$$2\sum_{cyc}x^4+\sum_{cyc}x=\sum_{cyc}(x^4+x^4+x)\overset{AM-GM}{\geq}3\sum_{cyc}x^3;\ \ (3)$$

$$\sum_{cyc}x^3\overset{AM-GM}{\geq}3xyz;\ \ (4)$$

$$2\sum_{cyc}x^3+3=\sum_{cyc}(x^3+x^3+1)\overset{AM-GM}{\geq}3\sum_{cyc}x^2\geq 2\sum_{cyc}x^2+\sum_{cyc}xy;\ \ (5)$$

From $(1)\to(5)$ it follows that:

$$4\sum_{cyc}x^4+4\sum_{cyc}x^3+\sum_{cyc}x+3\geq 2xyz\sum_{cyc}x+2\sum_{cyc}x^2+\sum_{cyc}xy+2\sum_{cyc}x^2y+3xyz$$

Therefore,

$$4\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+3\geq 6\sqrt[3]{xyz}$$

Solution 3 by proposer

$$\frac{x^3}{(y+1)(z+1)}+\frac{y+1}{8}+\frac{z+1}{8}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{x^3}{(y+1)(z+1)}\cdot\frac{y+1}{8}\cdot\frac{z+1}{8}}=$$

$$=3\sqrt[3]{\frac{x^3}{4^3}}=\frac{3x}{4}$$

$$\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+\sum_{cyc}\frac{y+1+z+1}{8}\ge\sum_{cyc}\frac{3x}{4}$$

$$\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+3\cdot\frac{2}{8}+\frac{1}{8}\sum_{cyc}(y+z)\ge\frac{3}{4}\sum_{cyc}x$$

$$\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+\frac{3}{4}\ge 3\sum_{cyc}x-\frac{1}{4}\sum_{cyc}x$$

$$4\sum_{cyc}\frac{x^3}{(y+1)(z+1)}+3\ge 3\sum_{cyc}x-\sum_{cyc}x=2\sum_{cyc}x=2(x+y+z)\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge}2\cdot 3\sqrt[3]{xyz}=6\sqrt[3]{xyz}$$

Equality holds if and only if $x=y=z=1$.

SP.359 If $A\in M_3(\mathbb{R}), p\in\mathbb{R}^*$ such that $det(A^2-pA+p^2I_3)=0$.

Prove that:

$$2det(A^2+p^2I_3)\ge(detA+p^3)^2$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$det(A^2-pA+p^2I_3)=det(A+p\varepsilon)\cdot det(A+\overline{p\varepsilon})=0,\text{ where }\varepsilon^2+\varepsilon+1=0,$$

$$\varepsilon^3=1,\varepsilon^2=-1-\varepsilon.$$

Let $f(x)=det(A+xI_3)=detA+a_1x+a_2x^2+x^3\to$

$$f(p\varepsilon)=0\to detA+a_1p\varepsilon+a_2p^2\varepsilon^2+p^3\varepsilon^3=0\to$$

$$detA+a_1p\varepsilon+a_2p^2(-1-\varepsilon)+p^3=0\to$$

$$detA-a_2p^2+p\varepsilon(a_1-a_2p)=0\to a_1=pa_2;detA-a_2p^2+p^3=0$$

$$a_2=\frac{detA+p^3}{p^2},a_1=\frac{detA+p^3}{p}\to$$

$$f(x)=detA+\frac{detA+p^3}{p}x+\frac{detA+p^3}{p^2}x^2+x^3$$

$$f(pi)=detA+(detA+p^3)i-(detA+p^3)-p^3i=p^3+detAi,$$

$$f(-pi)=p^3-detAi\to f(pi)f(-pi)=p^6+(detA)^2\to$$

$$det(A+pi)\cdot det(A-pi)=p^6+(detA)^2\to$$

$$det(A^2+p^2I_3)=p^6+(detA)^2\ge\frac{(p^3+detA)^2}{2}$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $P(\lambda)=det(\lambda I_3-A)=\lambda^3-(TrA)\lambda^2+(TrA^*)\lambda-detA$ be the characteristic polynomial of A.

Now, $0=det(A^2-pA+p^2I_3)=det(A+\omega pI_3)(A+\omega^2pI_3)=$

$$=|det(A+\omega pI_3)|^2\Rightarrow det(A+\omega pI_3)=0\Rightarrow -det(-A-\omega pI_3)=0$$

$$\Rightarrow P(-\omega p)=0\Rightarrow(-\omega p)^3-(TrA)\omega^2p^2+(TrA^*)(-\omega p)-detA=0$$

$$\Rightarrow(p^3+detA)-\frac{1}{2}[(TrA)p^2+(TrA^*)p]+\frac{\sqrt{3}}{2}i[(TrA)p^2-(TrA^*)p]=0$$

$p^3+detA=\frac{1}{2}(TrA)p^2+(TrA^*)p$ **and** $(TrA)p^2=(TrA^*)p$

$$p^3+detA=\frac{1}{2}[(TrA)p^2+(TrA)p^2]=(TrA)p^2=(TrA^*)p$$

Now,

$$2det(A^2+p^2I_3)=2det(A-ipI_3)(A+ipI_3)=|2det(A-ipI_3)|^2=$$

$$=2|-det(ipI_3-A)|^2=2\left|-ip^3+(TrA)p^2+(TrA^*)pi-detA\right|=$$

$$=2[(TrA)p^2-p^3\}^2+2[(TrA)p^2-detA]^2=2(detA)^2+2(p^3)^2=$$

$$=(detA+p^3)^2+(detA-p^3)^2\ge(detA+p^3)^2$$

Solution 3 by Ruxandra Daniela Tonilă-Romania

Let $x^2-px+p^2=0;\Delta=-3p^2;x_{1,2}=\frac{p(1\pm i\sqrt{3})}{2}$

$$det(A^2-pA+p^2I_3)=det\left[\left(A-p\cdot\frac{1+i\sqrt{3}}{2}I_3\right)\left(A-p\cdot\frac{1-i\sqrt{3}}{2}I_3\right)\right]=$$

$$=det\left(A-p\cdot\frac{1+i\sqrt{3}}{2}I_3\right)\cdot det\left(A-p\cdot\frac{1-i\sqrt{3}}{2}I_3\right)=0;(1)$$

Let $P_A(x)=det(A-xI_3)=x^3-(TrA)x^2+(TrA^*)x-detA$ be the characteristic polynomial. From (1) it follows that:

$$P_A\left(p\cdot\frac{1+i\sqrt{3}}{2}\right)\cdot P_A\left(p\cdot\frac{1-i\sqrt{3}}{2}\right)=0;(2)$$

Let $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$ be the eigenvalues for A.

$$\overset{(2)}{\Rightarrow}\begin{cases}\lambda_1=p\cdot\dfrac{1+i\sqrt{3}}{2}\ or\lambda_1=p\cdot\dfrac{1-i\sqrt{3}}{2}\in\mathbb{C}\\ A\in M_3(\mathbb{R})\Leftrightarrow detA,TrA,TrA^*\in\mathbb{R}\quad\Rightarrow\lambda_2=\overline{\lambda_1}\\ TrA=\lambda_1+\lambda_2+\lambda_3\\ TrA^*=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1\end{cases}$$

Therefore, the eigenvalues for A are $\lambda_1=p\cdot\frac{1+i\sqrt{3}}{2},\lambda_2=p\cdot\frac{1-i\sqrt{3}}{2},\lambda_3\in\mathbb{R}$.

$$\begin{cases}TrA=p\cdot\dfrac{1+i\sqrt{3}+1-i\sqrt{3}}{2}+\lambda_3=p+\lambda_3\\ detA=p^2\cdot\dfrac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}\cdot\lambda_3=p^3\lambda_3\\ TrA^*=p^2\cdot\dfrac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}+\lambda_3p\cdot\dfrac{1+i\sqrt{3}+1-i\sqrt{3}}{2}=p^2+\lambda_3p\end{cases}$$

$$\Rightarrow P_A(x)=x^3-(p+\lambda_3)x^2+(p^2+p\lambda_3)x-p^2\lambda_3$$

$$det(A^2+p^2I_3)=det[(A+ipI_3)(A-ipI_3)]=P_A(-ip)P_A(ip);(3)$$

$$P_A(ip)=(ip)^3-(p+\lambda_3)(ip)^2+(p^2+p\lambda_3)ip-p^2\lambda_3=p^3+ip^2\lambda_3;(4)$$

$$P_A(-ip)=(-ip)^3-(p+\lambda_3)(-ip)^2+(p^2+p\lambda_3)(-ip)-p^2\lambda_3=p^3-ip^2\lambda_3;(5)$$

From (3),(4),(5) it follows that:

$$det(A^2+p^2I_3)=(p^3+ip^2\lambda_3)(p^3-ip^2\lambda_3)$$

$$detA+p^3=p^2\lambda_3+p^3;(7)$$

We must show that: $2det(A^2+p^2I_3)\geq(detA+p^3)^2\overset{(6;7)}{\Longleftrightarrow}$

$$2(p^6+p^4\lambda_3^2)\geq(p^2\lambda_3+p^3)^2\Leftrightarrow$$

$$2p^6+2p^4\lambda_3^2\geq p^4\lambda_3^2+p^6+2p^5\lambda_3|:p^4$$

$$p^2+\lambda_3^2-2p\lambda_3\geq 0\Leftrightarrow(p-\lambda_3)^2\geq 0,\text{ true }\forall p,\lambda_3\in\mathbb{R}.$$

Therefore,

$$2det(A^2+p^2I_3)\geq(detA+p^3)^2.$$

SP.360 $z_1,z_2z_3\in\mathbb{C}^*$ **different in pairs such that** $|z_1|=|z_2|=|z_3|$. **If**

$$\sum_{cyc}\left|\frac{2z_1-z_2-z_3}{(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2|}\right|=\frac{1}{|z_1-z_2|}+\frac{1}{|z_2-z_3|}+\frac{1}{|z_3-z_1|},$$

then z_1,z_2,z_3 **are affixes on equilateral triangle.**

Proposed by Marian Ursărescu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } A(z_1), B(z_2), C(z_3); \Delta ABC \subset C(O,R); |z_1| = |z_2| = |z_3| = R$$

$$|z_1 - z_2| = AB = c; |z_2 - z_3| = BC = a; |z_1 - z_3| = CA = b$$

$$|2z_1 - z_2 - z_3| = 2\left|z_1 - \frac{z_2 + z_3}{2}\right| = 2m_a$$

$$\big|(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|\big| =$$

$$= |(a + b + c)z_1 - (az_1 + bz_2 + cz_3)| =$$

$$2s\left|z_1 - \frac{az_1 + bz_2 + cz_3}{a + b + c}\right| = 2s \cdot AI$$

Now,

$$\sum_{cyc}\left|\frac{2z_1 - z_2 - z_3}{(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|}\right| = \frac{1}{|z_1 - z_2|} + \frac{1}{|z_2 - z_3|} + \frac{1}{|z_3 - z_1|}$$

$$\Leftrightarrow \frac{1}{s}\sum_{cyc} m_a \cdot sin\,\frac{A}{2} \overset{Lascu}{\geq} 4R\sum_{cyc}\frac{b + c}{a} \cdot cos\,\frac{A}{2} sin\,\frac{A}{2} = \sum_{cyc}\frac{b + c}{a} \cdot a = \sum_{cyc} ab$$

Equality holds if and only if triangle is equilateral.

Therefore,

$$\sum_{cyc}\left|\frac{2z_1 - z_2 - z_3}{(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|}\right| = \frac{1}{|z_1 - z_2|} + \frac{1}{|z_2 - z_3|} + \frac{1}{|z_3 - z_1|},$$

then z_1, z_2, z_3 are affixes on equilateral triangle.

Solution 2 by proposer

$$A(z_1), B(z_2), C(z_3), \Delta ABC \in C(o, R), |z_1| = R, |z_1 - z_2| = AB = c, |z_2 - z_3| = BC = a$$

$$|z_2 - z_3| = AC = b \rightarrow \sum_{cyc}\frac{2\left|z_1 - \frac{z_2 + z_3}{2}\right|}{|b(z_1 - z_2) + c(z_1 - z_3)|} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leftrightarrow$$

$$\sum_{cyc}\frac{\left|z_1 - \frac{z_2 + z_3}{2}\right|}{\left|\frac{z_1(b + c) - bz_2 - cz_3}{a + b + c}\right|} = \frac{a + b + c}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \leftrightarrow$$

$$\sum_{cyc}\frac{\left|z_1 - \frac{z_2 + z_3}{2}\right|}{\left|z_1 - \frac{az_1 + bz_2 + cz_3}{a + b + c}\right|} = s\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \leftrightarrow \sum_{cyc}\frac{m_a}{AI} = s\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right); (1)$$

$$But\sum_{cyc}\frac{m_a}{AI}\ge s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right);(2)$$

From (1),(2) equality holds if and only if ΔABC equilateral.

$$AI=\frac{r}{sin\frac{A}{2}}\cdot m_a\ge\frac{b+c}{2}cos\frac{A}{2}\to\sum_{cyc}\frac{m_a}{AI}\ge\sum_{cyc}\frac{\frac{b+c}{2}cos\frac{A}{2}}{rsin\frac{A}{2}}=$$

$$=\frac{1}{2}\sum_{cyc}(b+c)sin\frac{A}{2}sin\frac{B}{2}=\frac{1}{4r}\sum_{cyc}\frac{(b+c)a}{2R}=\frac{1}{8Rr}\sum_{cyc}a(b+c)=\frac{1}{4Rr}\sum_{cyc}bc$$

$$=s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

UP.346 Solve for real numbers:

$$2\int_0^x\frac{x^2\cdot e^{\arctan x}}{\sqrt{1+x^2}}dx=1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\Omega=\int_0^x\frac{x^2\cdot e^{\arctan x}}{\sqrt{1+x^2}}dx=\int_0^x x\cdot e^{\arctan x}\cdot\left(\sqrt{1+x^2}\right)'dx=$$

$$=xe^{\arctan x}\sqrt{1+x^2}-\int_0^x\left(e^{\arctan x}+\frac{x}{1+x^2}e^{\arctan x}\right)\sqrt{1+x^2}dx=$$

$$=xe^{\arctan x}\sqrt{1+x^2}-\int_0^x e^{\arctan x}\sqrt{1+x^2}dx-\int_0^x\frac{x\sqrt{1+x^2}}{1+x^2}e^{\arctan x}dx=$$

$$=xe^{\arctan x}\sqrt{1+x^2}-\int_0^x e^{\arctan x}\sqrt{1+x^2}dx-\int_0^x e^{\arctan x}\left(\sqrt{1+x^2}\right)'dx=$$

$$=xe^{\arctan x}\sqrt{1+x^2}-\int_0^x e^{\arctan x}\sqrt{1+x^2}-$$

$$-e^{\arctan x}\cdot\sqrt{1+x^2}+1+\int_0^x\frac{\sqrt{1+x^2}}{1+x^2}e^{\arctan x}dx=$$

$$=xe^{\arctan x}\sqrt{1+x^2}-e^{\arctan x}\sqrt{1+x^2}+1-\int_0^x e^{\arctan x}\left(\sqrt{1+x^2}-\frac{1}{\sqrt{1+x^2}}\right)dx=$$

$$\Omega = e^{\arctan x}\cdot\sqrt{1+x^2}(x-1)+1-\int_0^x e^{\arctan x}\cdot\frac{1+x^2-1}{\sqrt{1+x^2}}dx =$$

$$= e^{\arctan x}\sqrt{1+x^2}(x-1)+1-\int_0^x \frac{x^2e^{\arctan x}dx}{\sqrt{1+x^2}}$$

$$\Omega = e^{\arctan x}\cdot\sqrt{1+x^2}\cdot(x-1)+1-\Omega$$

$$2\Omega = e^{\arctan x}\cdot\sqrt{1+x^2}\cdot(x-1)+1$$

$$2\Omega = 1 \Rightarrow e^{\arctan x}\cdot\sqrt{1+x^2}\cdot(x-1)+1=1$$

$$e^{\arctan x}\cdot\sqrt{1+x^2}\cdot(x-1)=0$$

$$x-1=0$$

$$x=1.$$

Solution 2 by Surjeet Singhania-India

Let:

$$f(x)=\int_0^x \frac{t^2e^{tan^{-1}t}}{\sqrt{1+t^2}}dt=\int_0^{tan^{-1}x} e^t tan^2 t\cdot sec\, t\, dt =$$

$$= e^t sec\, t\cdot tan\, t|_0^{tan^{-1}x}-\int_0^{tan^{-1}x} e^t(tan\, t+sec^2 t)sec\, t\, dt =$$

$$= xe^{tan^{-1}x}\sqrt{1+x^2}-\int_0^{tan^{-1}x} e^t(tan\, t\cdot sec\, t+sec\, t)dt-f(x)$$

Now,

$$f(x)=\frac{xe^{tan^{-1}x}\sqrt{1+x^2}}{2}-\frac{e^{tan^{-1}x}\sqrt{1+x^2}}{2}+\frac{1}{2}$$

Given that: $f(x)=\frac{1}{2}\to\frac{(x-1)e^{tan^{-1}x}\sqrt{1+x^2}}{2}=0.$

Since $e^{tan^{-1}x}\sqrt{1+x^2}>0\to(x-1)\underbrace{e^{tan^{-1}x}\sqrt{1+x^2}}_{\neq 0}=0\to x=1.$

There is only real solution at $x=1$.

UP.347 Solve for complex numbers:

$$\begin{cases}\frac{|x|^2}{3}+\frac{|y|^2}{5}=\frac{|x+y|^2}{8}\\ 10x+y=7+14i\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{|x|^2}{3}+\frac{|y|^2}{5}-\frac{|x+y|^2}{8}=0\Rightarrow 40|x|^2+24|y|^2-15|x+y|^2=0$$

$$40x\cdot\overline{x}+24y\cdot\overline{y}-15(x+y)\left(\overline{x+y}\right)=0$$

$$40x\overline{x}+24y\overline{y}-15(x+y)(\overline{x}+\overline{y})=0$$

$$40x\overline{x}+24y\overline{y}-15(x\overline{x}+x\overline{y}+y\overline{x}+y\overline{y})=0$$

$$40x\overline{x}+24y\overline{y}-15x\overline{x}-15x\overline{y}-15\overline{x}y-15y\overline{y}=0$$

$$25x\overline{x}+9y\overline{y}-15x\overline{y}-15\overline{x}y=0$$

$$5x(5\overline{x}-3\overline{y})-3y(5\overline{x}-3\overline{y})=0$$

$$(5\overline{x}-3\overline{y})(5x-3y)=0$$

$$(5x-3y)\cdot\left(\overline{5x-3y}\right)=0$$

$$|5x-3y|^2=0\Rightarrow 5x-3y=0\Rightarrow x=\frac{3y}{5}$$

$$10x+y=6+12i\Rightarrow 10\cdot\frac{3y}{5}+y=7+14i$$

$$6y+y=7+14i\Rightarrow 7y=7+14i\Rightarrow y=1+2i$$

$$x=\frac{3}{5}y=\frac{3}{5}(1+2i)=\frac{3}{5}+\frac{6}{5}i$$

$$\text{Solution}\begin{cases}x=\frac{3}{5}+\frac{6}{5}i\\ y=1+2i\end{cases}$$

Solution 2 by Daniel Văcaru-Romania

Consider $x=x_1+ix_2$ and $y=y_1+iy_2$; $x_1,x_2,y_1,y_2\in\mathbb{R}$. Then $|x|^2=x_1^2+x_2^2$, $|y|^2=y_1^2+y_2^2$ and $|x+y|^2=(x_1+y_1)^2+(x_2+y_2)^2$. Then the first equation is the condition for equality in Bergstrom inequality, proving that $\frac{x_1}{3}=\frac{y_1}{5}\overset{not}{=}k$ and $\frac{x_2}{3}=\frac{y_2}{5}\overset{not}{=}l$. Writing the equality for $10x+y=7+14i$ in terms of real and imaginary parts, we obtain $30k+5k=7\Rightarrow k=\frac{1}{5}$ and $30l+5l=14\Rightarrow l=\frac{2}{5}$.

Then $x_1=\frac{3}{5}$ and $y_1=1$; $x_2=\frac{6}{5}$; $y_2=2$. We obtain $x=\frac{3+6i}{5}$ and $y=1+2i$.

Solution 3 by Bangalore Mathematical Institute-group of solving

Using Bergstrom inequality:

$$\frac{a^2}{x}+\frac{b^2}{y}\geq\frac{(a+b)^2}{x+y}, \forall a,b,x.y\in\mathbb{R}_+$$

$$\frac{|z_1|^2}{3}+\frac{|z_2|^2}{5}\geq\frac{(|z_1|+|z_2|)^2}{8}$$

By triangle inequality: $|z_1+z_2|\geq|z_1|+|z_2|$, we have:

$$\frac{|z_1|^2}{3}+\frac{|z_2|^2}{5}\geq\frac{(|z_1|+|z_2|)^2}{8}\geq\frac{|z_1+z_2|^2}{8}$$

But given that: $\frac{|z_1|^2}{3}+\frac{|z_2|^2}{5}=\frac{|z_1+z_2|^2}{8}$

So, because if equality z_1 is collinear with z_2 and also $\frac{|z_1|}{|z_2|}=\frac{3}{5}$, we get:

$$\begin{cases} z_1=3\lambda e^{i\theta} \\ z_2=5\lambda e^{i\theta} \end{cases}$$

Substituting in $10z_1+z_2=7+14i$.

Comparing real and imaginary parts, we get:

$$\lambda\cos\theta=\frac{1}{5};\ \ \lambda\sin\theta=\frac{2}{5}\Rightarrow\lambda=1,\qquad\theta=\tan^{-1}(2).$$

$$\text{So, }\begin{cases} z_1=3e^{i\theta} \\ z_2=5e^{i\theta} \end{cases}$$

UP.348 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ and ABC is a triangle with the area F, then:

$$\frac{(x+3y)(y+4z+3x)}{(y+3z)(z+3x)}\cdot a^4+\frac{(y+3z)(z+4x+3y)}{(z+3x)(x+3y)}\cdot b^4$$
$$+\frac{(z+3x)(x+4y+3z)}{(x+3y)(y+3z)}\cdot c^4\geq 32F^2$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution 1 by proposer

Let be $m=x+3y, n=y+3z, p=z+3x$, then the inequality from enunciation becomes:

$$\sum_{cyc}\frac{m(n+p)}{np}\cdot a^4=\sum_{cyc}\frac{mn+mp}{np}\cdot a^4=\sum_{cyc}\frac{u+v}{w}\cdot a^4\ge$$

$$\overset{Bătineţu-Giurgiu}{\ge} 32\cdot F^2$$

So, $u=mn, v=mp, w=np$ and Bătinețu-Giurgiu inequality is:

$$\sum_{cyc}\frac{u+v}{w}a^4\ge 32F^2$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{(x+3y)(y+4z+3x)}{(y+3z)(z+3x)}\cdot a^4\overset{AM-GM}{\ge}3\sqrt[3]{\prod_{cyc}\frac{(x+3y)(y+4z+3x)}{(y+3z)(z+3x)}\cdot a^4}=$$

$$=3\sqrt[3]{\left(\prod_{cyc}\frac{(y+3z)+(z+3x)}{x+3y}\right)\cdot(4RF)^4}\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge}3\sqrt[3]{(4F)^4\left(\prod_{cyc}\frac{2\sqrt{(y+3z)(z+3x)}}{z+3y}\right)\cdot R^4}\overset{Euler-Mitrinovic}{\ge}$$

$$\ge 3\sqrt[3]{(4F)^4\cdot 8\cdot\frac{4s^2}{27}\cdot 4r^2}=32F^2$$

Therefore,

$$\sum_{cyc}\frac{(x+3y)(y+4z+3x)}{(y+3z)(z+3x)}\cdot a^4\ge 32F^2$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Tahiland

$$\sum_{cyc}\frac{(x+3y)(y+4z+3x)}{(y+3z)(z+3x)}\cdot a^4=\sum_{cyc}\left(\frac{x+3y}{y+3z}+\frac{x+3y}{z+3x}\right)\cdot a^4=$$

$$=\sum_{cyc}\left(\frac{x+3y}{y+3z}\right)\cdot a^4+\sum_{cyc}\left(\frac{x+3y}{z+3x}\right)\cdot a^4\ge 2(a^4+b^4+c^4)\ge$$

$$\ge\frac{2}{3}(a^2+b^2+c^2)^2\ge\frac{2}{3}\left(4\sqrt{3}F\right)^2=32F^2$$

For $a,b,c;x,y,z>0$ we have:

$$\frac{x}{y}\cdot a+\frac{y}{z}\cdot b+\frac{z}{x}\cdot c\geq a+b+c\Leftrightarrow x^2za+y^2xb+z^2yc\geq xyz(a+b+c)$$

$xza(x-y)+yxb(y-z)+zyc(z-x)\geq 0$, which is true from

$$\frac{1}{3}(xza+yxb+zyc)\big((x-y)+(y-z)+(z-x)\big)\geq 0$$

UP.349 If $m\geq 0;x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{y+z}{x\cdot h_a^{m+1}}+\frac{z+x}{y\cdot h_b^{m+1}}+\frac{x+y}{z\cdot h_c^{m+1}}\geq\frac{2}{\left(\sqrt{F}\right)^{m+1}}\left(\sqrt[4]{3}\right)^{3-m}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

We have:

$$U=\sum_{cyc}\frac{y+z}{x\cdot h_a^{m+1}}=\sum_{cyc}\frac{y+z}{x\cdot(ah_a)^{m+1}}\cdot a^{m+1}=\frac{1}{2^{m+1}\cdot F^{m+1}}\cdot\sum_{cyc}\frac{y+z}{x}\cdot a^{m+1}\quad (1)$$

Let be $u,v,w\in\mathbb{R}_+^*$ such that $u^{m+1}=x,v^{m+1}=y,w^{m+1}=z$ then relation (1) becomes:

$$U=\frac{1}{2^{m+1}\cdot F^{m+1}}\cdot\sum_{cyc}\frac{v^{m+1}+w^{m+1}}{u^{m+1}}a^{m+1}\geq$$

$$\geq\frac{1}{2^{m+1}F^{m+1}}\cdot\frac{1}{2^m}\sum_{cyc}\left(\frac{v+w}{u}\cdot a\right)^{m+1}\overset{Radon}{\geq}\frac{1}{2^{2m+1}\cdot F^{m+1}\cdot 3^m}\left(\sum_{cyc}\frac{v+w}{u}\cdot a\right)^{m+1}\geq$$

$$\geq\frac{1}{2^{2m+1}\cdot 3^m\cdot F^{m+1}}\cdot 2^{m+1}\left(\sum_{cyc}\frac{\sqrt{vw}}{u}a\right)^{m+1}\geq$$

$$=\frac{1}{6^m\cdot F^{m+1}}\left(3\cdot\sqrt[3]{\prod_{cyc}\left(\frac{\sqrt{vw}}{u}\cdot a\right)}\right)^{m+1}=\frac{3}{2^mF^{m+1}}\left(\sqrt[3]{abc}\right)^{m+1}=$$

$$=\frac{3}{2^m\cdot F^{m+1}}\left(\sqrt[3]{(abc)^2}\right)^{\frac{m+1}{2}}\geq\frac{3}{2^mF^{m+1}}\left(\frac{4\sqrt{3}F}{3}\right)^{\frac{m+1}{2}}=$$

$$=\frac{3}{2^mF^{m+1}}\cdot 2^{m+1}\cdot\frac{F^{\frac{m+1}{2}}}{3^{\frac{m+1}{4}}}=\frac{2}{F^{\frac{m+1}{2}}}\cdot 3^{1-\frac{m+1}{4}}=$$

$$=\frac{2}{\left(\sqrt{F}\right)^{m+1}}\cdot 3^{\frac{3-m}{4}}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\prod_{cyc} h_a=\prod_{cyc}\frac{2F}{a}=\frac{2F^2}{R};\ \ R^2\ge\frac{4F}{3\sqrt{3}}(Euler-Mitrinovic)$$

$$\prod_{cyc} h_a\le\sqrt[4]{3^3}\cdot\sqrt{F^3};(1)$$

$$\sum_{cyc}\frac{y+z}{x\cdot h_a^{m+1}}\overset{AM-GM}{\ge}3\sqrt[3]{\prod_{cyc}\frac{y+z}{x\cdot h_a^{m+1}}}=3\sqrt[3]{\frac{\prod(y+z)}{x}\cdot\frac{1}{\left(\prod h_a\right)^{m+1}}}\overset{AM-GM}{\ge}$$

$$\overset{\substack{AM-GM\\(1)}}{\ge}3\sqrt[3]{\frac{8}{\left(\sqrt[4]{3^3}\cdot\sqrt{F^3}\right)^{m+1}}}=\frac{2}{\sqrt{F^{m+1}}}\cdot\left(\sqrt[4]{3}\right)^{3-m}$$

UP.350 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ΔABC with the area F the following inequality holds:

$$\sum_{cyc}\frac{y+z}{x+\sqrt{(x+2y)(x+2z)}}a^2\ge 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

We have

$$\sum_{cyc}\frac{y+z}{x+\sqrt{(x+2y)(x+2z)}}a^2\ge\sum_{cyc}\frac{y+z}{x+\frac{x+2y+x+2z}{2}}a^2=$$

$$=\sum_{cyc}\frac{y+z}{2x+y+z}a^2=\sum_{cyc}\frac{2(x+y+z)-(2x+y+z)}{2x+y+z}a^2=$$

$$=2(x+y+z)\cdot\sum_{cyc}\frac{a^2}{2x+y+z}-\sum_{cyc}a^2\overset{Bergstrom}{\ge}$$

$$\geq 2\cdot\frac{(a+b+c)^2}{\sum_{cyc}(2x+y+z)}(x+y+z)a^2=\sum_{cyc}a^2=$$

$$=2(x+y+z)\cdot\frac{(a+b+c)^2}{4(x+y+z)}-\sum_{cyc}a^2=\frac{1}{2}(a+b+c)^2-\sum_{cyc}a^2=$$

$$=\frac{1}{2}\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)-\sum_{cyc}a^2=\sum_{cyc}ab-\frac{1}{2}\sum_{cyc}a^2 \quad \textbf{(1)}$$

But $a^2+b^2+c^2=2(s^2-r^2-4Rr), ab+bc+ca=s^2+r^2+4Rr$ and then relationship (1) becomes:

$$\sum_{cyc}\frac{y+z}{x+\sqrt{(x+2y)(x+2z)}}a^2\geq s^2+r^2+4Rr-(s^2-r^2-4Rr)=$$

$$=2(r^2+4Rr)=2r(4R+r)\overset{Doucet}{\geq}2r\sqrt{3}s=2\sqrt{3}sr=2\sqrt{3}F$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have: $\sqrt{(x+2y)(x+2z)}\overset{AM-GM}{\leq}\frac{(x+2y)+(x+2z)}{2}=x+y+z$

Thus,

$$\sum_{cyc}\frac{y+z}{x+\sqrt{(x+2y)(x+2z)}}\cdot a^2\geq\sum_{cyc}\frac{y+z}{(x+y)+(y+z)}\cdot a^2\overset{Tsintsifas}{\geq}2\sqrt{3}F$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{x+y}{z+\sqrt{(z+2x)(z+2y)}}\geq\frac{x+y}{z+x+y+z}=\frac{x+y}{(z+x)+(z+y)}$$

Hence,

$$\sum_{cyc}\frac{y+z}{x+\sqrt{(x+2y)(x+2z)}}\cdot a^2\geq\sum_{cyc}\frac{y+z}{(x+y)+(y+z)}\cdot a^2=$$

$$=\frac{m}{n+p}a^2+\frac{n}{p+m}\cdot b^2+\frac{p}{n+m}\cdot c^2\geq\frac{1}{2}(a^2+b^2+c^2)\geq\frac{1}{2}\left(4\sqrt{3}F\right)=2\sqrt{3}F$$

Which is true, because:

$$m(m+p)(m+n)b^2+n(n+m)(n+p)a^2+p(p+m)(p+n)c^2\geq$$

$$\geq\frac{1}{2}(m+n)(n+p)(p+m)(a^2+b^2+c^2)$$

$$a^2(m+n)(n+p)\big(2n-(m+p)\big)+b^2(m+n)(m+p)\big(2m-(n+p)\big)+$$

$$+c^2(p+m)(p+n)\big(2p-(m+n)\big)\ge 0$$

$$\frac{\big(a^2(m+n)(n+p)+b^2(m+n)(m+p)+c^2(p+m)(p+n)\big)(2m+2n+2p-(2m+2n+2p)}{3}\ge 0$$

UP.351 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ and ABC is a triangle, then:

$$\frac{(x+2y)(2x+y+3z)\cdot a^2}{(y+z)(z+2x)\cdot h_a^2}+\frac{(y+2z)(2y+z+3x)\cdot b^2}{(z+2x)(x+2y)\cdot h_b^2}+$$

$$+\frac{(z+2x)(2z+x+3y)\cdot c^2}{(x+2y)(y+2z)\cdot h_c^2}\ge 8$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

Let be $u=x+2y, v=y+2z, w=z+2x$, then we have:

$$\sum_{cyc}\frac{(x+2y)(2x+y+3z)a^2}{(y+2z)(z+2x)h_a^2}=\sum_{cyc}\frac{u(v+w)a^2}{v\cdot w\cdot h_a^2}=$$

$$=\sum_{cyc}\frac{uv+wu}{vw}\cdot\frac{a^4}{a^2h_a^2}=\frac{1}{4F^2}\cdot\sum_{cyc}\frac{uv+wu}{vw}a^4=$$

$$=\frac{1}{4F^2}\cdot\sum_{cyc}\frac{m+p}{n}a^4\overset{Bătinețu-Giurgiu}{\ge}\frac{1}{4F^2}\cdot 32F^2=8$$

So, $m=uv, n=vw, p=wu$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\prod_{cyc}\frac{a}{h_a}=\prod_{cyc}\frac{a^2}{2F}=\frac{2R^2}{F}\overset{Mitrinovic-Euler}{\ge}\frac{8}{3\sqrt{3}}$$

$$\prod_{cyc}\frac{a^2}{h_a^2}\ge\left(\frac{4}{3}\right)^3;(1)$$

Thus,

$$\sum_{cyc}\frac{(x+2y)(2x+y+3z)\cdot a^2}{(y+z)(z+2x)\cdot h_a^2}\overset{AM-GM}{\ge}3\cdot\sqrt[3]{\prod_{cyc}\frac{(x+2y)(2x+y+3z)\cdot a^2}{(y+z)(z+2x)\cdot h_a^2}}=$$

$$=3\sqrt[3]{\left(\prod_{cyc}\frac{(z+2x)+(y+2z)}{x+2y}\right)\left(\prod_{cyc}\frac{a^2}{h_a^2}\right)}\overset{Cesaro,(1)}{\geq}3\sqrt[3]{8\cdot\left(\frac{4}{3}\right)^3}=8$$

UP.352 If $x,y,z>0$ and ΔABC have the semiperimeter s the following relationship holds:

$$\frac{(y+z)a}{x\cdot h_a(s-a)}+\frac{(z+x)b}{y\cdot h_b(s-b)}+\frac{(x+y)c}{z\cdot h_c(s-c)}\geq\frac{12\sqrt{3}}{s}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum_{cyc}\frac{y+z}{x\cdot h_a(s-a)}\geq 3\sqrt[3]{\frac{(y+z)(z+x)(x+y)\cdot abc}{xyz\cdot h_ah_bh_c\cdot(s-a)(s-b)(s-c)}};(1)$$

$$\frac{(x+y)(y+z)(z+x)}{xyz}\geq\frac{2\sqrt{yz}\cdot 2\sqrt{zx}\cdot 2\sqrt{xy}}{xyz}=8;(2)$$

From $(1),(2)$ we must to show that:

$$\sum_{cyc}\frac{y+z}{x\cdot h_a(s-a)}\geq 6\cdot\sqrt[3]{\frac{abc}{h_ah_bh_c\cdot(s-a)(s-b)(s-c)}}\Leftrightarrow$$

$$\frac{abc}{h_ah_bh_c\cdot(s-a)(s-b)(s-c)}\geq\frac{8\cdot 3\sqrt{3}}{s^3};(3)$$

But: $abc=4Rrs, h_ah_bh_c=\frac{2s^2r^2}{R}$ and $(s-a)(s-b)(s-c)=sr^2;(4)$

From $(3),(4)$ we must to show:

$$\frac{4Rrs\cdot R}{2s^2r^2\cdot sr62}\geq\frac{8\cdot 3\sqrt{3}}{s^3}\Leftrightarrow R^2s\geq 4\cdot 3\sqrt{3}r^3$$

$\Leftrightarrow R^2\geq 4r^2$, which is true because $R\geq 2r(Euler)$ and $s\geq 3\sqrt{3}r(Mitriovic)$.

Solution 2 by proposers

We have:

$$\sum_{cyc}\frac{(y+z)a}{x\cdot h_a(s-a)}\ge 2\sum_{cyc}\frac{\sqrt{yz}\cdot a}{x\cdot h_a(s-a)}=2\sum_{cyc}\frac{\sqrt{yz}\cdot a^2}{x(ah_a)(s-a)}=$$

$$=\frac{2}{2F}\cdot\sum_{cyc}\frac{\sqrt{yz}\cdot a^2}{x(s-a)}\ge\frac{1}{F}\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{yz}\cdot a^2}{x(s-a)}}=$$

$$=\frac{3}{F}\cdot\sqrt[3]{\frac{(abc)^2}{(s-a)(s-b)(s-c)}}\overset{Carliz}{\ge}\frac{3}{F}\cdot\frac{4\sqrt{3}F}{3}\cdot\frac{1}{\sqrt[3]{(s-a)(s-b)(s-c)}}=$$

$$=4\sqrt{3}\cdot\frac{\sqrt[3]{s}}{\sqrt[3]{s(s-a)(s-b)(s-b)}}=\frac{4\sqrt{3}\sqrt[3]{s}}{\sqrt[3]{F^2}}=\frac{4\sqrt{3}\sqrt[3]{s}}{\sqrt[3]{s^2r^2}}=$$

$$=\frac{4\sqrt{3}}{\sqrt[3]{s}\cdot\sqrt[3]{r^2}}\overset{Mitrinovic}{\ge}\frac{4\sqrt{3}}{\sqrt[3]{s\cdot\frac{s^2}{27}}}=\frac{12\sqrt{3}}{s}$$

UP.353 Let $(x_n)_{n\ge1}$ be a sequence of real numbers with $x_1=\sqrt[3]{a^3-a}, a\ge 2$ and $x_{n+1}=\sqrt[3]{a^3-a+x_n}, n\ge 1$. Prove that $(x_n)_{n\ge1}$ –convergent and find $\Omega_1=\lim_{n\to\infty}x_n, \Omega_2=\lim_{n\to\infty}\{x_n\}$, where $\{*\}$ –fractional part.

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Let $P(n): x_{n+1}>x_n, \forall n\ge 1$. We have: $P(1): x_2>x_1\Leftrightarrow a^3>a$ true for $a>1$.

Next, $P(k)\Rightarrow P(k+1): x_{k+1}>x_k, k\ge 1\Rightarrow \sqrt[3]{a^3-a+x_k}>\sqrt[3]{a^3-a+x_{k-1}}$ is true.

From $P(1)$ –true and $P(k)\Rightarrow P(k+1), \forall k\ge 1\Rightarrow P(n)$ –true for all $n\ge 1$.

Let $P(n): x_n<a, n\ge 1$. We have: $\sqrt[3]{a^3-a}<a\Leftrightarrow a^3-a<a^3\Leftrightarrow a>0$ true.

Next, $P(k)\Rightarrow P(k+1): x_k<a\Rightarrow x_{k+1}<a\Leftrightarrow\sqrt[3]{a^3-a+x_k}<a\Leftrightarrow a^3-a+x_k<a^3$

$\Leftrightarrow x_k<a$ true.

From $P(1)$ true and $P(k)\Rightarrow P(k+1), \forall k\ge 1\Rightarrow P(n)$ is true for all $n\ge 1$.

Because sequence $(x_n)_{n\ge1}$ –bounded and increasing, by Weierstrass theorem, it follows that $(x_n)_{n\ge1}$ –is convergent $\Rightarrow \exists l\in\mathbb{R}$ such that $l=\lim_{n\to\infty}x_n$. We have:

$x_{n+1} = \sqrt[3]{a^3 - a + x_n} \Leftrightarrow l = \sqrt[3]{a^3 - a + l^3} \Leftrightarrow l^3 = a^3 - a + l \Leftrightarrow$

$(l - a)(l^2 + al + a^2 - 1) = 0 \Leftrightarrow l = a$, because $l^2 + al + a^2 - 1 = 0$ has

$\Delta = 4 - 3a^2 < 0$, so for $a \geq 2$, has not real roots.

Hence, $\Omega_1 = \lim_{n\to\infty} x_n = a$. Now,

$x_1 = \sqrt[3]{a^3 - a} > a - 1$ and $(x_n)_{n\geq1}$ −increasing, so $x_n > a - 1, \forall n \geq 1$ and from

$(x_n)_{n\geq1}$ − bounded, we have: $x_n < a, \forall n \geq 1$. It follows that:

$a - 1 \leq x_n < a, \forall n \geq 1 \Rightarrow [x_n] = a - 1$, hence $\{x_n\} = x_n - [x_n] = x_n - (a - 1)$ and

$\Omega_2 = \lim_{n\to\infty}\{x_n\} = \lim_{n\to\infty}\left(x_n - (a - 1)\right) = a - (a - 1) = 1.$

Therefore, $\Omega_2 = \lim_{n\to\infty}\{x_n\} = 1$

Solution 2 by Ravi Prakash-New Delhi-India

We first show that: $1 \leq x_n \leq a, \forall n \geq 1$

For $n = 1$: $x_1 = \sqrt[3]{a^3 - a} \geq 1 (\because a \geq 2)$ and $x_1^3 \leq a^3 \to x_1 \leq a \to 1 \leq x_1 \leq a$.

Next, assume that $1 \leq x_n \leq a$.

$x_{n+1}^3 = a^3 - a + x_n \geq 1 \to x_{n+1} \geq 1$ also $x_{n+1}^3 = a^3 + (x_n - a) \leq a^3$. Thus

$1 \leq x_{n+1} \leq a$. Hence, $1 \leq x_n \leq a, \forall n \geq 1$

Next, we show that: $x_{n+1} \geq x_n, \forall n \geq 1$. We have:

$x_2^3 = (a^3 - a) + x_1 = x_1^3 + x_1 \to x_2^3 \geq x_1^3 \to x_2 \geq x_1 (\because x_1 \geq 1)$

Next, assume that $x_{k+1} \geq x_k$ for some $k \geq 1$. Now,

$x_{k+2}^3 - x_{k+1}^3 = x_{k+1} - x_k \geq 0 \to x_{k+2}^3 \geq x_{k+1}^3 \to x_{k+2} \geq x_{k+1}$. Thus,

$x_{n+1} \geq x_n, \forall n \geq 1$. So, $(x_n)_{n\geq1}$ is an bounded increasing sequence, then

$(x_n)_{n\geq1}$ −converges.

Let $\lim_{n\to\infty} x_n = l, l \geq 1$. Now, $\lim_{n\to\infty} x_{n+1}^3 = \lim_{n\to\infty}(a^3 - a + x_n) \to l^3 = a^3 - a + l$

$(l - a) + (l - a)(l^2 + a^2 + la) = 0 \to (l - a)(l^2 + a^2 + la + 1) = 0 \to l = a$

$\to \lim_{n\to\infty} x_n = a; (1)$

We next show that: $\lim_{n\to\infty}[x_n] = [a]$.

For $0 < \epsilon < 1$, there exists a positive integer numbers m such that $|x_n - a| < \epsilon, \forall n \geq m$.

$$\rightarrow -\epsilon < x_n - a < \epsilon, \forall n \geq m \rightarrow 0 \leq x_n - a \leq \epsilon, \forall n \geq m$$

$$\rightarrow a \leq x_n < a + \epsilon, \forall n \geq m \rightarrow [a] \leq [x_n] \leq [a], \forall n \geq m \rightarrow$$

$$\lim_{n\to\infty} [x_n] = [a]; (2)$$

From $(1), (2)$, we get $\lim_{n\to\infty} (x_n - [x_n]) = a - [a] \rightarrow \lim_{n\to\infty} \{x_n\} = \{a\}$

UP.354 Find:

$$\Omega = \lim_{n\to\infty} \left(1 + \left(\sum_{k=1}^{n-1} \frac{k}{n} sin \frac{k\pi}{n}\right)^{-1}\right)^n$$

Proposed by Florică Anastase-Romania

Solution 1 by Narendra Bhandari-Bajura-Nepal

Clearly, $\frac{k\pi}{n} > 0$ for all $k \in [1, n-1]$ so due to Taylor series around $x = 0$ we have:

$sin\, x \leq x, \forall x \geq 0$. Choose $x = \frac{k\pi}{n}$ we have

$$sin\left(\frac{k\pi}{n}\right) \leq \frac{k\pi}{n} \Rightarrow S(n) = \sum_{k=1}^{n-1} \frac{k}{n} sin\left(\frac{k\pi}{n}\right) \leq \pi \sum_{k=1}^{n-1} \frac{k^2}{n^2} = \frac{\pi}{6} \cdot \frac{(n-1)(2n-1)}{n}$$

Since the partial sum of the later sum diverges as $n \rightarrow \infty$ so

$S(n) \rightarrow \infty$ (due to comparison test). Therefore, $L = \lim_{n\to\infty}(1 + (S(n)^{-1})^n = 1^\infty$

And hence we have:

$$L = \lim_{n\to\infty} exp(1 + (S(n)^{-1})^n = exp\left(\lim_{n\to\infty} n\left(\sum_{k=1}^{n-1} \frac{k}{n} sin\left(\frac{k\pi}{n}\right)\right)^{-1}\right)$$

Since latter expression is Riemann integrable function on $[0, 1]$, (reads may prove it) so the limit reduces to:

$$L = exp\left(\int_0^1 x \cdot sin(\pi x)\, dx\right)^{-1} \stackrel{IBP}{=} e^{(\pi^{-1})^{-1}} = e^\pi$$

Solution 2 by proposer

$$\forall n \in \mathbb{N}^*, n \geq 2: \sum_{k=1}^{n-1} \frac{k}{n} sin \frac{k\pi}{n} = \frac{1}{2} cot \frac{\pi}{2n}$$

Let be $z = cos\frac{\pi}{n} + isin\frac{\pi}{n}$, then:

$$\sum_{k=1}^{n-1} ksin\frac{k\pi}{n} = Im(z + 2z^2 + 3z^3 + \cdots (n-1)z^{n-1}), z^n = -1 \Rightarrow$$

$$z + 2z^2 + 3z^3 + \cdots (n-1)z^{n-1} = \frac{(n-1)z^{n+1} - nz^n + z}{(z-1)^2} = \frac{(1-n)z + n + z}{(z-1)^2} =$$

$$= \frac{n-(n-2)z}{1 - 2sin^2\frac{\pi}{2n} + 2isin\frac{\pi}{2n}cos\frac{\pi}{2n} - 1} = \frac{n-(n-2)z}{-4sin^2\frac{\pi}{2n}\left(cos\frac{\pi}{n} + isin\frac{\pi}{n}\right)} =$$

$$= \frac{n-2}{4sin^2\frac{\pi}{2n}} - \frac{n}{4sin^2\frac{\pi}{2n}}\left(cos\frac{\pi}{n} - isin\frac{\pi}{n}\right) \Rightarrow$$

$$\sum_{k=1}^{n-1} ksin\frac{k\pi}{n} = Im\left(\sum_{k=1}^{n-1} kz^k\right) = \frac{nsin\frac{\pi}{n}}{4sin^2\frac{\pi}{2n}} = \frac{n}{2}cot\frac{\pi}{2n} \Rightarrow \sum_{k=1}^{n-1} \frac{k}{n}sin\frac{k\pi}{n} = \frac{1}{2}cot\frac{\pi}{2n}$$

$$\lim_{n\to\infty} log\left(1 + \left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^n = \lim_{n\to\infty} nlog\left(1 + 2tan\frac{\pi}{2n}\right) =$$

$$= \lim_{n\to\infty} \frac{log\left(1 + 2tan\frac{\pi}{2n}\right)}{2tan\frac{\pi}{2n}} \cdot \frac{2tan\frac{\pi}{2n}}{\frac{\pi}{2n}} \cdot \frac{\pi}{2n} \cdot n = \pi$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(1 + \left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^n = e^{\pi}$$

UP.355 Find:

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}}\left(\frac{cotx}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{tan(2nx)}}\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by Narendra Bhandari-Bajura-Nepal

Since:

$$A(n) = \sum_{k=1}^{n-1}\frac{k}{n}sin\left(\frac{k\pi}{n}\right) = \sum_{k=1}^{n-1}\left(\frac{n-k}{n}\right)sin\left(\frac{k\pi}{n}\right) \Rightarrow A(n) = \frac{1}{2}\sum_{k=1}^{n-1}sin\left(\frac{k\pi}{n}\right)$$

Now,

$$\sum_{k=1}^{n-1} sin\left(\frac{k\pi}{n}\right) = \mathfrak{F}\left(\sum_{k=1}^{n-1} e^{\frac{k\pi}{n}}\right) = \mathfrak{F}\left(\frac{1-e^{i\pi}}{1-e^{\frac{i\pi}{n}}}\right) = \frac{\mathfrak{F}\left(ie^{-\frac{i\pi}{2n}}\right)}{sin\left(\frac{\pi}{2n}\right)} = \frac{1}{2}cos\left(\frac{\pi}{2n}\right)$$

So we are now supposed to find:

$$L = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}} \frac{cot\, x}{cot\left(\frac{\pi}{2n}\right)}\right)^{\frac{1}{tan(2nx)}} = \lim_{n\to\infty}(\mathfrak{B}(n))^{\frac{1}{tan(2nx)}}$$

We note that as $x \to \frac{\pi}{2n}, \mathfrak{B}(n) \to 1$ then limit attains the form of 1^∞ so

$$log\mathfrak{B}(n) = \lim_{x\to\frac{\pi}{2n}} \frac{1}{tan(2nx)} log\left(\frac{cot\, x}{cot\left(\frac{\pi}{2n}\right)}\right)$$

As $\mathfrak{B}(n)$ attains $\frac{0}{0}$ form so due to L-hopitals rule we have

$$log\mathfrak{B}(n) = -\lim_{x\to\frac{\pi}{2n}} \frac{csc\, x\cdot sec\, x}{2n\cdot sec^2(2nx)} = -\frac{csc\left(\frac{\pi}{2n}\right)\cdot sec\left(\frac{\pi}{2n}\right)}{2n}$$

Therefore,

$$L = \lim_{n\to\infty} exp\left(-\frac{csc\left(\frac{\pi}{2n}\right)}{2n}\right) = e^{-\pi^{-1}} = \frac{1}{\sqrt[\pi]{e}}$$

Solution by proposer

$$\forall n \in \mathbb{N}^*, n \geq 2: \sum_{k=1}^{n-1} \frac{k}{n} sin\frac{k\pi}{n} = \frac{1}{2}cot\frac{\pi}{2n}$$

Let be $z = cos\frac{\pi}{n} + isin\frac{\pi}{n}$, then:

$$\sum_{k=1}^{n-1} ksin\frac{k\pi}{n} = Im(z + 2z^2 + 3z^3 + \cdots (n-1)z^{n-1}), z^n = -1 \Rightarrow$$

$$z + 2z^2 + 3z^3 + \cdots (n-1)z^{n-1} = \frac{(n-1)z^{n+1} - nz^n + z}{(z-1)^2} = \frac{(1-n)z + n + z}{(z-1)^2} =$$

$$= \frac{n-(n-2)z}{1 - 2sin^2\frac{\pi}{2n} + 2isin\frac{\pi}{2n}cos\frac{\pi}{2n} - 1} = \frac{n-(n-2)z}{-4sin^2\frac{\pi}{2n}\left(cos\frac{\pi}{n} + isin\frac{\pi}{n}\right)} =$$

$$=\frac{n-2}{4sin^2\frac{\pi}{2n}}-\frac{n}{4sin^2\frac{\pi}{2n}}\left(cos\frac{\pi}{n}-isin\frac{\pi}{n}\right)\Rightarrow$$

$$\sum_{k=1}^{n-1}ksin\frac{k\pi}{n}=Im\left(\sum_{k=1}^{n-1}kz^k\right)=\frac{nsin\frac{\pi}{n}}{4sin^2\frac{\pi}{2n}}=\frac{n}{2}cot\frac{\pi}{2n}\Rightarrow\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}=\frac{1}{2}cot\frac{\pi}{2n}$$

$$\Omega=\lim_{x\to\frac{\pi}{2n}}\left(\frac{cotx}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{tan(2nx)}}=\lim_{x\to\frac{\pi}{2n}}\left(\frac{cotx}{2}\cdot\left(\frac{1}{2}cot\frac{\pi}{2n}\right)^{-1}\right)^{\frac{1}{tan(2nx)}}$$

$$=\lim_{x\to\frac{\pi}{2n}}\left(\frac{cotx}{cot\frac{\pi}{2n}}\right)^{cot(2nx)}=\lim_{x\to\frac{\pi}{2n}}\left(1+\frac{cotx-cot\frac{\pi}{2n}}{cot\frac{\pi}{2n}}\right)^{cot(2nx)}=$$

$$=\lim_{x\to\frac{\pi}{2n}}\left(1+\frac{cotx-cot\frac{\pi}{2n}}{cot\frac{\pi}{2n}}\right)^{\frac{cot\frac{\pi}{2n}}{cotx-cot\frac{\pi}{2n}}\cdot\frac{\left(cotx-cot\frac{\pi}{2n}\right)cot(2nx)}{cot\frac{\pi}{2n}}}=$$

$$=e^{\lim\limits_{x\to\frac{\pi}{2n}}cot(2nx)\cdot\frac{sin\left(\frac{\pi}{2n}-x\right)}{sinx\cdot sin\frac{\pi}{2n}}\cdot tan\frac{\pi}{2n}}=e^{-\frac{1}{nsin\frac{\pi}{n}}}=$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}}\left(\frac{1}{2}(cotx)\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{tan(2nx)}}\right)=\lim_{n\to\infty}e^{-\frac{\frac{\pi}{n}}{sin\frac{\pi}{n}}\cdot\frac{1}{\pi}}=\frac{1}{\sqrt[\pi]{e}}$$

UP.356 Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\because\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)=\frac{n}{2^{n-1}},\forall n\in\mathbb{N},n\geq 3$$

$$(1+z)^m=\sum_{l=0}^{m}\binom{m}{l}z^l,m\in\mathbb{N}^*;(1)$$

Let $z=cos\frac{2k\pi}{n}+isin\frac{2k\pi}{n},k=\overline{1,n}\Rightarrow 1+z=1+cos\frac{2k\pi}{n}+isin\frac{2k\pi}{n}=$

$$=2cos\frac{k\pi}{n}\left(cos\frac{k\pi}{n}+isin\frac{k\pi}{n}\right)\Rightarrow$$

$$2^m cos^m\frac{k\pi}{n}\left(cos\frac{mk\pi}{n}+isin\frac{mk\pi}{n}\right)=\sum_{l=0}^{m}\binom{m}{l}\left(cos\frac{2lk\pi}{n}+isin\frac{2lk\pi}{n}\right)\Rightarrow$$

$$2^m cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=\sum_{l=0}^{m}\binom{m}{l}cos\frac{2lk\pi}{n},k=\overline{1,n}\Rightarrow$$

$$2^m\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=\sum_{l=0}^{m}\binom{m}{l}\sum_{k=1}^{n}cos\frac{2ik\pi}{n}=$$

$$=\binom{m}{0}\sum_{k=1}^{n}1+\sum_{i=1}^{n}\binom{m}{i}\sum_{k=1}^{n}cos\frac{2lk\pi}{n};(2)$$

Now,

$$\because\sum_{k=1}^{n}a^{k-1}cos(k\theta)=\frac{a^{n+1}cos(n\theta)-a^n cos(n+1)\theta+cos\theta-a}{a^2-2acos\theta+1};a=1,\theta=\frac{2l\pi}{n}\Rightarrow$$

$$\sum_{k=1}^{n}cos\frac{2lk\pi}{n}=\frac{cos2l\pi-cos\frac{(n+1)2l\pi}{n}+cos\frac{2l\pi}{n}-1}{2-2cos\frac{2l\pi}{n}}=0,\forall l=\overline{1,m};m<n;(3)$$

From $(2),(3)$ it follows that:

$$2^m\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=n\binom{m}{0}\Rightarrow\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=\frac{n}{2^m}$$

For $m=n-1$, it follows that:

$$\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)=\frac{n}{2^{n-1}}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)}=\lim_{n\to\infty}\sqrt[n]{\frac{n}{2^{n-1}}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{n+1}{2^n}\cdot\frac{2^{n-1}}{n}=\frac{1}{2}$$

UP.357 If $f:\left[0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=-\int_0^x log(cosy)\,dy$

Prove that:

$$\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} f(x)=-\int_0^{\frac{\pi}{2}} log(siny)\,dy$$

Proposed by Florică Anastase-Romania

Solution 1 by Hussain Reza Zadah-Afghanistan

$$I=\int_0^1 \frac{logy}{\sqrt{1-y^2}}dy \overset{y=sint}{=} \int_0^{\frac{\pi}{2}} \frac{log(sint)}{\sqrt{1-sin^2t}}cost\,dt=\int_0^{\frac{\pi}{2}} log(cost)\,dt=\int_0^{\frac{\pi}{2}} log(sint)\,dt$$

$$I=\int_0^{\frac{\pi}{2}} log(sint)\,dt\to 2I=\int_0^{\frac{\pi}{2}} log(sin2t)\,dt-\frac{\pi}{2}log2$$

$$\lim_{x\to\frac{\pi}{2}} f(x)=-\int_0^{\frac{\pi}{2}} log(siny)\,dy=\frac{\pi}{2}log2$$

Solution 2 by Dawid Bialek-Poland

$$f(x)=-\int_0^x log(cosy)\,dy \overset{y\to\frac{\pi}{2}-y}{=} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}-x} log\left(cos\left(\frac{\pi}{2}-x\right)\right)dy=-\int_{\frac{\pi}{2}-x}^{\frac{\pi}{2}} log(siny)\,dy$$

$$\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} f(x)=-\int_{\frac{\pi}{2}-\frac{\pi}{2}}^{\frac{\pi}{2}} log(siny)\,dy=-\int_0^{\frac{\pi}{2}} log(siny)\,dy$$

Solution 3 by proposer

$$\because \prod_{k=1}^{n-1} sin\frac{k\pi}{n}=\frac{\sqrt{n}}{2^{n-1}}$$

Let: $x_k, k=1,2,..,2n$ the roots of the unity.

$$x_k=cos\frac{k\pi}{2n}+isin\frac{k\pi}{2n}, k=1,2,\dots,2n$$

$$x^{2n}-1=\prod_{k=1}^{2n}(x-x_k) \overset{x_{1,2}=\pm1-roots}{\triangleq} (x^2-1)\prod_{k=1}^{n-1}(x-x_k)(x-\overline{x_k})$$

$$= (x^2 - 1)\prod_{k=1}^{n-1}\left(x^2 - 2x\cos\frac{k\pi}{n} + 1\right)$$

$$\Rightarrow x^{2n-2} + x^{2n-4} + \cdots + x^2 + 1 = \prod_{k=1}^{n-1}\left(x^2 - 2x\cos\frac{k\pi}{n} + 1\right) \overset{x=1}{\Longrightarrow}$$

$$n = \prod_{k=1}^{n-1}\left(2 - 2\cos\frac{k\pi}{n}\right) = \prod_{k=1}^{n-1}\left(4\sin^2\frac{k\pi}{2n}\right)$$

$$n = 2^{2(n-1)} \cdot \sin^2\frac{\pi}{2n} \cdot \sin^2\frac{2\pi}{2n} \cdot \ldots \cdot \sin^2\frac{(n-1)\pi}{2n}$$

$$2^{n-1} \cdot \sin\frac{\pi}{2n} \cdot \sin\frac{2\pi}{2n} \cdot \ldots \cdot \sin\frac{(n-1)\pi}{2n} = \sqrt{n} \Rightarrow \prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{\sqrt{n}}{2^{n-1}}$$

$$\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = \frac{1}{2}\int_0^{\pi}\log(\sin x)\,dx = \frac{\pi}{2}\int_0^1 \log(\sin \pi x)\,dx =$$

$$= \lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=1}^{n-1}\log\left(\sin\frac{k\pi}{n}\right) = \lim_{n\to\infty}\frac{\pi}{2n}\log\left(\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}\right) =$$

$$= \lim_{n\to\infty}\frac{\pi}{2n}\log\left(\frac{\sqrt{n}}{2^{n-1}}\right) = \frac{\pi}{2}\lim_{n\to\infty}\frac{\log\sqrt{n} - (n-1)\log 2}{n} = -\frac{\pi}{2}\log 2; (1)$$

$$\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = \frac{1}{2}\int_0^{\pi}\log(\sin x)\,dx = \frac{\pi}{2}\int_0^1 \log(\sin \pi x)\,dx =$$

$$= \lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=1}^{n-1}\log\left(\sin\frac{k\pi}{n}\right) = \lim_{n\to\infty}\frac{\pi}{2n}\log\left(\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}\right) =$$

$$= \lim_{n\to\infty}\frac{\pi}{2n}\log\left(\frac{n}{2^{n-1}}\right) = \frac{\pi}{2}\lim_{n\to\infty}\frac{\log n - (n-1)\log 2}{2n} = -\frac{\pi}{2}\log 2$$

$$f(x) = -\int_0^x \log(\cos y)\,dy \overset{y=\frac{\pi}{2}-t}{=} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}-x} \log(\sin t)\,dt =$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}-x}\left(\log 2 + \log\left(\sin\frac{t}{2}\right) + \log\left(\cos\frac{t}{2}\right)\right)dt =$$

$$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\frac{x}{2}} log(sinu)\,du + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\frac{x}{2}} log(cosu)\,du - xlog2 =$$

$$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\frac{x}{2}} log(cosu)\,du - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{4}+\frac{x}{2}} log(cosu)\,du - xlog2$$

$$= 2f\left(\frac{\pi}{4}-\frac{x}{2}\right) - 2f\left(\frac{\pi}{4}+\frac{x}{2}\right) - xlog2$$

Therefore,

$$\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} f(x) = \frac{\pi}{2}log2 = -\int_0^{\frac{\pi}{2}} log(siny)\,dy$$

UP.358 If $0 < a \le b < \frac{\pi}{2}$ then:

$$\left(\int_a^b \frac{sinx}{x}dx\right)^2 + \left(\int_a^b \frac{cosx}{x}dx\right)^2 \le log^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$A = \int_a^b \frac{sinx}{x}dx\,, B = \int_a^b \frac{cosx}{x}dx,\ u = |u|(cost + isint), t \in [0, 2\pi),$$

$$A = |u|cost, B = |u|sint, |u|^2 = A^2 + B^2 = A\cdot A + B\cdot B =$$

$$= |u|cost\cdot A + |u|sint\cdot B =$$

$$= |u|\left(cost\int_a^b \frac{sinx}{x}dx + sint\int_a^b \frac{cosx}{x}dx\right) \le$$

$$\leq |u|\int_a^b \sqrt{\left(\frac{sinx}{x}\right)^2+\left(\frac{cosx}{x}\right)^2}\,dx = |u|\int_a^b \frac{1}{x}dx = |u|log\left(\frac{b}{a}\right)$$

$$|u|^2 \leq |u|log\left(\frac{b}{a}\right) \to |u| \leq log\left(\frac{b}{a}\right) \to |u|^2 \leq log^2\left(\frac{b}{a}\right) \to A^2+B^2 \leq log^2\left(\frac{b}{a}\right)$$

$$\left(\int_a^b \frac{sinx}{x}dx\right)^2+\left(\int_a^b \frac{cosx}{x}dx\right)^2 \leq log^2\left(\frac{b}{a}\right)$$

Solution 2 by Gabriel Brehuescu-Romania

Applying BCS inequality, we get:

$$\left(\int_a^b \frac{sinx}{x}dx\right)^2 = \left(\int_a^b \left(\frac{sinx}{\sqrt{x}}\right)\cdot\frac{1}{\sqrt{x}}dx\right)^2 \leq \int_a^b \left(\frac{sinx}{\sqrt{x}}\right)^2 dx\cdot\int_a^b \left(\frac{1}{\sqrt{x}}\right)^2 dx$$
$$= \int_a^b \frac{sin^2x}{x^2}dx\cdot\int_a^b \frac{dx}{x};(1)$$

$$\left(\int_a^b \frac{cosx}{x}dx\right)^2 = \left(\int_a^b \left(\frac{cosx}{\sqrt{x}}\right)\cdot\frac{1}{\sqrt{x}}dx\right)^2 \leq \int_a^b \left(\frac{cosx}{\sqrt{x}}\right)^2 dx\cdot\int_a^b \left(\frac{1}{\sqrt{x}}\right)^2 dx$$
$$= \int_a^b \frac{cos^2x}{x^2}dx\cdot\int_a^b \frac{dx}{x};(2)$$

From $(1),(2)$ we get:

$$\left(\int_a^b \frac{sinx}{x}dx\right)^2+\left(\int_a^b \frac{cosx}{x}dx\right)^2 \leq \int_a^b \frac{dx}{x}\cdot\int_a^b \left(\frac{sin^2x+cos^2x}{x}\right)dx = \left(\int_a^b \frac{dx}{x}\right)^2$$
$$= log^2\left(\frac{b}{a}\right)$$

Solution 3 by Mohammad Rostami-Kabul-Afghanistan

$$\begin{cases} U = \int_a^b \frac{sinx}{x}dx \\ V = \int_a^b \frac{cosx}{x}dx \end{cases} \to \begin{cases} W = |W|(cos\theta + isin\theta) \\ |w|^2 = U^2+V^2 \\ U = |W|cos\theta \\ V = |W|sin\theta \end{cases}$$

$$|W|^2 = U^2 + V^2 = |W| \cdot U\cos\theta + |W| \cdot V\sin\theta =$$

$$= |W|\left(\cos\theta \int_a^b \frac{\sin x}{x}dx + \sin\theta \int_a^b \frac{\cos x}{x}dx\right) \le$$

$$\le |W| \int_a^b \sqrt{\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\cos x}{x}\right)^2}\, dx = |W| \int_a^b \frac{1}{x}dx = |W|\log\left(\frac{b}{a}\right)$$

$$\Rightarrow |W|^2 \le |W|\log^2\left(\frac{b}{a}\right) \Rightarrow |W| \le \log\left(\frac{b}{a}\right) \Rightarrow |W|^2 \le \log^2\left(\frac{b}{a}\right)$$

$$U^2 + V^2 \le \log^2\left(\frac{b}{a}\right)$$

Therefore,

$$\left(\int_a^b \frac{\sin x}{x}dx\right)^2 + \left(\int_a^b \frac{\cos x}{x}dx\right)^2 \le \log^2\left(\frac{b}{a}\right)$$

Solution 4 by Ravi Prakash-New Delhi-India

Let $A = \int_a^b \frac{\sin x}{x}dx\,, B = \int_a^b \frac{\cos x}{x}dx$

Now, $A^2 + B^2 = |B + iA| = \left|\int_a^b \frac{1}{x}(\cos x + i\sin x)dx\right|^2 = \left|\int_a^b \frac{e^{ix}}{x}dx\right|^2 \le$

$$\le \left(\int_a^b \frac{1}{x}|e^{ix}|dx\right)^2 = \left(\int_a^b \frac{1}{x}dx\right)^2 = \log^2\left(\frac{b}{a}\right)$$

Therefore,

$$\left(\int_a^b \frac{\sin x}{x}dx\right)^2 + \left(\int_a^b \frac{\cos x}{x}dx\right)^2 \le \log^2\left(\frac{b}{a}\right)$$

UP.359 Find:

$$\Omega = \int_0^\infty \frac{\tan^{-1}x}{x^3 + 1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by proposer

$$\Omega(a) = \int_0^\infty \frac{\tan^{-1}ax}{x^3 + 1}dx, a > 0 \to \Omega'(a) = \int_0^\infty \frac{x}{(x^3 + 1)(1 + a^2x^2)}dx$$

$$\because b = a^2, \frac{x}{(x + 1)(x^2 - x + 1)(1 + bx^2)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1} + \frac{Dx + E}{1 + bx^2} \to$$

$$A=-\frac{1}{3(b+1)},B=-\frac{2b-1}{3(b^2-b+1)},C=\frac{b+1}{3(b^2-b+1)},D=\frac{b^3}{b^3+1},E=-\frac{b}{b^3+1}$$

Then, we have:

$$\int\frac{A}{x+1}dx=Alog(x+1)+K$$

$$\int\frac{Bx+C}{x^2-x+1}dx=\frac{B}{2}log(x^2-x+1)+\frac{2C+B}{\sqrt{3}}tan^{-1}\frac{2x-1}{\sqrt{3}}+K$$

$$\int\frac{Dx+E}{bx^2+1}dx=\frac{D}{2b}log(bx^2+1)+\frac{E}{\sqrt{b}}tna^{-1}\sqrt{b}x+K$$

We make the notations:

$$P(x)=\int\frac{x}{(x+1)(x^2-x+1)(1+bx^2)}dx=$$

$$=\int\frac{A}{x+1}dx+\int\frac{Bx+C}{x^2-x+1}dx+\int\frac{Dx+E}{1+bx^2}dx$$

We have:

$$\int_0^\infty\frac{x}{(x^3+1)(1+bx^2)}dx=\lim_{x\to\infty}P(x)-P(0)$$

We can write: $P(x)=Q(x)+R(x)$**, where:**

$$Q(x)=\frac{2C+B}{\sqrt{3}}tan^{-1}\frac{2x-1}{\sqrt{3}}+\frac{E}{\sqrt{b}}tna^{-1}\sqrt{b}x$$

$$R(x)=Alog(x+1)+\frac{B}{2}log(x^2-x+1)+\frac{D}{2b}log(bx^2+1)$$

Let us denote: $\Delta_1=\lim_{x\to\infty}Q(x)-Q(0)$**, we have:**

$$\lim_{x\to\infty}Q(x)=\frac{2C+B}{\sqrt{3}}\cdot\frac{\pi}{2}+\frac{E}{\sqrt{b}}\cdot\frac{\pi}{2};Q(0)=-\frac{2C+B}{\sqrt{3}}\cdot\frac{\pi}{6}$$

So, $\Delta_1=\frac{2C+B}{\sqrt{3}}\cdot\frac{\pi}{2}+\frac{E}{\sqrt{b}}\cdot\frac{\pi}{2}+\frac{2C+B}{\sqrt{3}}\cdot\frac{\pi}{6}$**, but we have:** $2C+B=\frac{b+1}{b^3+1};E=-\frac{b}{b^3+1}\to$

$$\Delta_1=\frac{4\pi\sqrt{3}-9\pi\sqrt{b}+4\pi\sqrt{3}b}{18(b^3+1)}$$

We make the notation: $\Delta_2(x)=\lim_{x\to\infty}R(x)-R(0)$**, where**

$$R(x)=Alog(x+1)+\frac{B}{2}log(x^2-x+1)+\frac{D}{2b}log(bx^2+1),R(0)=0.$$

$$R(x)=Alog\frac{x+1}{x}+\frac{B}{2}log\frac{x^2-x+1}{x^2}+\frac{D}{2b}log\frac{bx^2+1}{bx^2}+\left(A+B+\frac{D}{b}\right)log+\frac{D}{2b}logb$$

But $A+B+\frac{D}{b}=0$, so $\lim_{x\to\infty}R(x)=\frac{D}{2b}logb=\frac{b^2}{2(b^3+1)}logb\to\Delta_2=\frac{b^2logb}{2(b^3+1)}$

$$\int_0^\infty\frac{x}{(x^3+1)(1+bx^2)}dx=\Delta_1+\Delta_2=\frac{4\pi\sqrt{3}-9\pi\sqrt{b}+4\pi\sqrt{3}b+9b^2logb}{18(b^3+1)}$$

$$\int_0^\infty\frac{x}{(x^3+1)(1+a^2x^2)}dx=\frac{4\pi\sqrt{3}-9a\pi+4\pi\sqrt{3}a^2+18a^4loga}{18(a^6+1)}$$

$$\Omega=\Omega(1)=\int_0^1\Omega'(a)\,da,\because\lim_{\substack{a\to0\\a>0}}\Omega(a)=0$$

$$\Omega=\frac{2\pi\sqrt{3}}{9}\int_0^1\frac{1}{a^6+1}da+\frac{\pi}{2}\int_0^1\frac{a}{a^6+1}da+\frac{2\pi\sqrt{3}}{9}\int_0^1\frac{a^2}{a^6+1}da+\int_0^1\frac{a^4loga}{a^6+1}da\,;(1)$$

$$\Omega_1=\int_0^1\frac{1}{a^6+1}da=\frac{1}{6}\left(\pi+\sqrt{3}log\left(2+\sqrt{3}\right)\right);(2)$$

$$\Omega_2=\int_0^1\frac{a}{a^6+1}da=\frac{1}{18}\left(\pi\sqrt{3}+3log2\right);(3)$$

$$\Omega_3=\int_0^1\frac{a^2}{a^6+1}da=\frac{\pi}{12}$$

We calculate the fourth integral:

$$\Omega_4=\int_0^1\frac{x^4logx}{x^6+1}dx$$

For this, we consider the function: $f(x)=\frac{x^4}{x^6+1}$. We develop the function in power series.

We have for $x\in(0,1)$: $f(x)=x^4-x^{10}+x^{16}-x^{22}+x^{28}-x^{34}+x^{40}-x^{46}+\cdots$

$$\Omega_4=\int_0^1f(x)dx=-\frac{1}{5^2}+\frac{1}{11^2}-\frac{1}{17^2}+\frac{1}{23^2}-\frac{1}{29^2}+\frac{1}{35^2}-\frac{1}{41^2}+\cdots$$

$$=-\left(\frac{1}{5^2}+\frac{1}{17^2}+\frac{1}{29^2}+\frac{1}{41^2}+\cdots\right)+\left(\frac{1}{11^2}+\frac{1}{23^2}+\frac{1}{35^2}+\frac{1}{47^2}+\cdots\right)$$

Now, we will use the trigamma function, which is defined by the relationship:

$$\Psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}\to\Omega_4=\frac{1}{144}\left(-\Psi_1\left(\frac{5}{12}\right)+\Psi_1\left(\frac{11}{12}\right)\right)$$

But, we have the equality: $-\Psi_1\left(\frac{5}{12}\right)+\Psi_1\left(\frac{11}{12}\right)=4\pi^2\sqrt{3}-80G$, where G is Catalan's constant. So we have:

$$\Omega_4=\frac{1}{36}\pi^2\sqrt{3}-\frac{5}{9}G;\ (5)$$

Replacing the relationships (2),(3),(4),(5) in the relation (1), we obtain:

$$\Omega=\frac{1}{18}\pi^2\sqrt{3}+\frac{1}{9}\pi log\left(2+\sqrt{3}\right)-\frac{1}{12}\pi log2-\frac{5}{9}G.$$

Solution 2 by Syed Shahabudeen-India

$$\Omega=\int_0^\infty\frac{tan^{-1}x}{x^3+1}dx\overset{x=\frac{1-y}{1+y}}{=}2\int_{-1}^1\frac{tan^{-1}\left(\frac{1-y}{1+y}\right)}{6y^2+2}\cdot\frac{dy}{(1+y)^2}=$$

$$=2\int_{-1}^1\frac{\left(\frac{\pi}{4}-tan^{-1}x\right)(1+y)}{6y^2+2}dy=$$

$$=\frac{\pi}{2}\int_{-1}^1\frac{1+y}{6y^2+2}dy-2\int_{-1}^1\frac{1+y}{6y^2+2}tan^{-1}y\,dy$$

$$A=\int_{-1}^1\frac{1+y}{6y^2+2}dy=\frac{1}{3}\int_0^1\frac{1}{y^2+\frac{1}{3}}dy=\frac{\pi}{3\sqrt{3}}$$

$$B=\int_{-1}^1\frac{1+y}{6y^2+2}tan^{-1}y\,dy=2\int_0^1\frac{ytan^{-1}y}{6y^2+2}dy=$$

$$=2\left(\frac{\pi}{4}log2-\frac{1}{12}\int_0^1\frac{log(3y^2+1)}{y^2+1}dy\right)$$

$$I(a)=\int_0^1\frac{log(ay^2+1)}{y^2+1}dy$$

$$\frac{dI}{da}=\int_0^1\frac{y^2}{(ay^2+1)(y^2+1)}dy=\frac{1}{a-1}\int_0^1\left(\frac{1}{y^2+1}-\frac{1}{ay^2+1}\right)dy=$$

$$=\frac{1}{a-1}\left(\frac{\pi}{4}-\frac{tan^{-1}\sqrt{a}}{\sqrt{a}}\right)=\frac{\pi}{4(a-1)}-\frac{tan^{-1}\sqrt{a}}{(a-1)\sqrt{a}}\rightarrow$$

$$I(a)=\frac{\pi}{4}log(a-1)-\int\frac{tan^{-1}\sqrt{a}}{(a-1)\sqrt{a}}da$$

$$\int\frac{\tan^{-1}\sqrt{a}}{(a-1)\sqrt{a}}da\overset{m=\sqrt{a}}{=}\frac{\pi}{2}\int\frac{dm}{m^2-1}+2\int\frac{\tan^{-1}\left(\frac{m-1}{m+1}\right)}{m^2-1}dm=$$

$$=\frac{\pi}{4}\log\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}\right)+Ti_2\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}\right)+C$$

$$I(a)=\frac{\pi}{4}\log(a-1)-\frac{\pi}{4}\log\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}\right)-Ti_2\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}\right)+C$$

Ti_2 —Inverse tangent integral. When $a=0\to I=0$

$$I(0)=-Ti_2(-1)+C=0\to C=-G;(G-Catalan\ ct.)$$

$$I(3)=\frac{\pi}{4}\log 2+\frac{\pi}{4}\log(2+\sqrt{3})-Ti_2\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)-G$$

Finally we'll get:

$$\Omega=\frac{\pi^2}{6\sqrt{3}}-\frac{\pi}{6}\log 2+\frac{\pi}{12}\log 2+\frac{\pi}{12}\log(2+\sqrt{3})-\frac{1}{3}Ti_2\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)-\frac{G}{3}$$

UP.360 Find $x,y>0$ such that:

$$\begin{cases}x+y=\frac{2}{3}\\ \frac{(x+1)^2}{3x^2-2x+1}+\frac{(y+1)^2}{3y^2-2y+1}=\frac{16}{3}\end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Christos Tsifakis-Greece

$$x+y=\frac{2}{3}\Leftrightarrow\begin{cases}3x=2-3y\\ x+1=\frac{5}{3}-y\end{cases}\Leftrightarrow\begin{cases}3x^2-2x=-3xy\\ x+1=\frac{5}{3}-y\end{cases}\Leftrightarrow\begin{cases}3x^2-2x+1=1-3xy\\ x+1=\frac{5}{3}-xy\end{cases}$$

$$\Leftrightarrow\begin{cases}3x^2-2x+1=1-(2-3xy)y\\ x+1=\frac{5}{3}-y\end{cases}\Leftrightarrow\begin{cases}3x^2-2x+1=3y^2-2y+1\\ x+1=\frac{5}{3}-y\end{cases}$$

So,

$$\begin{cases} x+y=\frac{2}{3} \\ \frac{(x+1)^2}{3x^2-2x+1}+\frac{(y+1)^2}{3y^2-2y+1}=\frac{16}{3} \end{cases} \Leftrightarrow \begin{cases} x+y=\frac{2}{3} \\ \frac{\left(\frac{5}{3}-y\right)^2}{3y^2-2y+1}+\frac{(y+1)^2}{3y^2-2y+1}=\frac{16}{3} \end{cases} \Leftrightarrow$$

$$\begin{cases} x+y=\frac{2}{3} \\ 3\left(\frac{5}{3}-y\right)^2+3(y+1)^2=16(3y^2-2y+1) \end{cases} \Leftrightarrow$$

$$\begin{cases} x+y=\frac{2}{3} \\ 3\left(\frac{25}{9}-\frac{10}{3}y+y^2\right)+3(y^2+2y+1)=48y^2-32y+16 \end{cases} \Leftrightarrow$$

$$\begin{cases} x+y=\frac{2}{3} \\ 3y^2-2y+\frac{1}{3}=0 \end{cases} \Leftrightarrow \begin{cases} x+y=\frac{2}{3} \\ y=\frac{1}{3} \end{cases} \Leftrightarrow x=y=\frac{1}{3}$$

Solution 2 by Ravi Prakash-New Delhi-India

Put $x=\frac{1}{3}+t; y=\frac{1}{3}-t;\ -\frac{1}{3}<t<\frac{1}{3}$

$$\frac{(x+1)^2}{3x^2-2x+1}=\frac{\left(\frac{4}{3}+t\right)^2}{3\left(\frac{1}{3}+t\right)^2-2\left(\frac{1}{3}+t\right)+1}=\frac{\left(\frac{4}{3}+t\right)^2}{3\left(t^2+\frac{2}{3}t+\frac{1}{9}\right)-\frac{2}{3}-2t+1}=$$

$$=\frac{\left(\frac{4}{3}+t\right)^2}{3t^2+\frac{2}{3}}$$

$$\frac{(y+1)^2}{3y^2-2y+1}=\frac{\left(\frac{4}{3}-t\right)^2}{3\left(\frac{1}{3}-t\right)^2-2\left(\frac{1}{3}-t\right)+1}=\frac{\left(\frac{4}{3}-t\right)^2}{3\left[\frac{1}{9}-\frac{2}{3}t+t^2\right]-\frac{2}{3}+2t+1}=$$

$$=\frac{\left(\frac{4}{3}-t\right)^2}{3t^2+\frac{2}{3}}$$

Then, it follows that:

$$\frac{\left(\frac{4}{3}+t\right)^2+\left(\frac{4}{3}-t\right)^2}{3t^2+\frac{2}{3}}=\frac{16}{3}\Rightarrow 2\cdot\frac{16}{9}+2t^2=16t^2+\frac{32}{9}\Rightarrow 14t^2=0$$

$t=0$. Thus, $x=y=\frac{1}{3}$.

Solution 3 by proposer

First we prove: $\frac{x^2+2x+1}{3x^2-2x+1}\le\frac{4(3x+1)}{3}$; (1)

$$3(x^2+2x+1)\le 4(3x+1)(3x^2-2x+1)$$

$$3x^2+6x+3\le 36x^3-12x^2+4x+4$$

$$36x^3-15x^2-2x+1\ge 0$$

$$36x^3-12x^2-3x^2+x-3x+1\ge 0$$

$$12x^2(3x-1)-x(3x-1)-(3x-1)\ge 0$$

$$(3x-1)(12x^2-x-1)\ge 0$$

$$(3x-1)[(12x^2-4x)+(3x-1)]\ge 0$$

$$(3x-1)^2(4x+1)\ge 0$$

Equality holds for $x=\frac{1}{3}$. Analogous:

$$\frac{y^2+2y+1}{3y^2-2y+1}\le\frac{4(3y+1)}{3};(2)$$

By adding (1),(2):

$$\frac{x^2+2x+1}{3x^2-2x+1}+\frac{y^2+2y+1}{3y^2-2y+1}\le\frac{4}{3}(3x+1+3y+1)=$$

$$=\frac{4}{3}(3(x+y)+2)=\frac{4}{3}\left(3\cdot\frac{2}{3}+2\right)=\frac{16}{3}.$$

Equality holds for $x=y=\frac{1}{3}$.

Solution: $x=y=\frac{1}{3}$.

Solution 4 by Fayssal Abdelli-Bejaia-Algerie

$$\begin{cases} x+y=\frac{2}{3};(1) \\ \frac{(x+1)^2}{3x^2-2x+1}+\frac{(y+1)^2}{3y^2-2y+1}=\frac{16}{3};(2) \end{cases}$$

From $(1) \to y = \frac{2}{3} - x$

$$(2) \to \frac{(x+1)^2}{3x^2 - 2x + 1} + \frac{\left(\frac{2}{3} - x + 1\right)^2}{3\left(\frac{2}{3} - x\right)^2 - 2\left(\frac{2}{3} - x\right) + 1} = \frac{16}{3}$$

$$\frac{x^2 + 2x + 1}{3x^2 - 2x + 1} + \frac{x^2 - \frac{10}{3}x + \frac{25}{9}}{3\left(\frac{4}{9} + x^2 - \frac{4}{3}x\right) - \frac{4}{3} + 2x + 1} = \frac{16}{3}$$

$$\frac{x^2 + 2x + 1}{3x^2 - 2x + 1} + \frac{x^2 - \frac{10}{3}x + \frac{25}{9}}{3x^2 - 2x + 1} = \frac{16}{3}$$

$$\frac{2x^2 - \frac{4}{3}x + \frac{34}{9}}{3x^2 - 2x + 1} = \frac{16}{3} \to 42x^2 - 28x + \frac{14}{3} = 0$$

$$3x^2 - 2x + \frac{1}{3} = 0 \to x = \frac{1}{3} \to y = \frac{1}{3}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

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