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SOLUTIONS

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JP.361 Find $x, y, z > 0$ such that:

$$\begin{cases} x^3y + y^3z + z^3x = xyz(x + y + z) \\ 2x + 3y + 5z = 10 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$x^3y + y^3z + z^3x = \left(\frac{4}{7}x^3y + \frac{1}{7}y^3z + \frac{2}{7}z^3x\right) + (\frac{4}{7}\left(y^3z + \frac{1}{7}z^3x + \frac{2}{7}x^3y\right) +$$

$$+\left(\frac{4}{7}z^3x + \frac{1}{7}x^3y + \frac{2}{7}y^3z\right) \overset{AM-GM}{\geq} 7 \cdot \frac{1}{7}\sqrt[7]{(x^3y)^4(y^3z)(z^3x)^2} +$$

$$+7 \cdot \frac{1}{7}\sqrt[7]{(y^3z)^4(z^3x)(x^3y)^2} + 7 \cdot \frac{1}{7}\sqrt[7]{(z^3x)^4(x^3y)(y^3z)^2} =$$

$$= \sqrt[7]{x^{14}y^7z^7} + \sqrt[7]{x^7y^{14}z^7} + \sqrt[7]{x^7y^7z^{14}} =$$

$$= x^2yz + xy^2z + xyz^2 = xyz(x + y + z)$$

Equality holds for $x^3y = y^3z = z^3x \Leftrightarrow x = y = z$.

$$2x + 3y + 5z = 10 \Rightarrow 10x = 10 \Rightarrow x = 1.$$

Solution: $x = y = z = 1$.

Solution 2 by AsmatQatea-Afghanistan

$$x^3y + y^3z + z^3x = xyz(x + y + z)|:(xyz)$$

$$\rightarrow \frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} = x + y + z$$

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} \overset{Bergstrom}{\geq} x + y + z$$

Equality holds when $x = y = z \rightarrow 2x + 3x + +5x = 10 \rightarrow x = 1$.

So, $x = y = z = 1$.

Solution 3 by AmritAwasthi-India

$$\sum_{cyc} x^3y = \sum_{cyc} x^2yz \Leftrightarrow \sum_{cyc} x^2y(x - z) = 0$$

The left summand becomes zero when $x = y = z$. Let $x = y = z = a$.

Therefore, putting in the next equation we get,

$$2x + 3y + 5z = 10 \rightarrow 2a + 3a + 5a = 10 \rightarrow$$

$$10a = a \rightarrow a = 1.$$ So, $x = y = z = 1$.

Solution 4 by Abdul Aziz-Semarang-Indonesia

$$x^3y + x^2yz + xyz^2 \overset{AM-GM}{\geq} 3x^2yz$$

$$y^3z + y^2zx + yzx^2 \overset{AM-GM}{\geq} 3y^2zx$$

$$z^3x + z^2xy + zxy^2 \overset{AM-GM}{\geq} 3z^2xy$$

Thus,

$$x^3y + y^3z + z^3x \geq xyz(x + y + z)$$

Since: $x^3y + y^3z + z^3x = xyz(x + y + z)$, then: $x = y = z$.

Since, $2x + 3y + 5z = 10$, we have:

$$x = y = z = 1.$$

Solution 5 by Daniel Văcaru-Romania

$$\begin{cases} x^3y + y^3z + z^3x = xyz(x + y + z); (1) \\ 2x + 3y + 5z = 10; (2) \end{cases}$$

We have: $x + y + z \overset{(1)}{=} \frac{x^3y+y^3z+z^3x}{xyz} = \frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} \overset{Bergstrom}{\geq} \frac{(x+y+z)^2}{x+y+z} = x + y + z$

$x = y = z$. From (2), we obtain $x = y = z = 1$.

JP.362. Let ABC be a triangle with inradius r, and circumradius R. Equilateral triangles with AB, $BC$and CA, are drawn externally to triangle ABC. Let K, L and M be the centroids of the equilateral triangles, respectively. Prove that:

$$\frac{2r}{R} \leq \frac{[KLM]}{[ABC]} \leq \left(\frac{R}{2r}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let a, b, c be the lengths of sides BC, CA, AB, respectively. Then, we have:

$$AK=\frac{2}{3}\cdot\frac{c\sqrt{3}}{2}=\frac{c\sqrt{3}}{3},AM=\frac{2}{3}\cdot\frac{b\sqrt{3}}{2}=\frac{b\sqrt{3}}{3}$$

We use Law of cosines on ΔAKH:

$$KM^2=AK^2+AM^2-2AK\cdot AMcos(\angle KAM)=\frac{c^2}{3}+\frac{b^2}{3}-\frac{2}{3}bccos(cos30^\circ+A+30^\circ)$$

$$=\frac{b^2+c^2}{3}-\frac{2}{3}bc(cos60^\circ cosA-sin60^\circ sinA)=$$

$$=\frac{b^2+c^2}{3}-\frac{2}{3}bc\left(\frac{1}{2}\cdot\frac{b^2+c^2-a^2}{2bc}-\frac{\sqrt{3}}{2}\cdot\frac{a}{2R}\right)=$$

$$=\frac{a^2+b^2+c^2}{6}+\frac{\sqrt{3}}{3}\cdot\frac{abc}{2R}$$

So, $KM^2=\frac{a^2+b^2+c^2}{6}+\frac{2\sqrt{3}}{3}\cdot[ABC]$.Similarly, we have:

$KL^2=\frac{a^2+b^2+c^2}{6}+\frac{2\sqrt{3}}{3}\cdot[ABC]$**and** $LM^2=\frac{a^2+b^2+c^2}{6}+\frac{2\sqrt{3}}{3}\cdot[ABC]$.

Namely, $KL=KL=LM$.SoΔKLM is equilateral. Now,

$$[KLM]=\frac{KM^2\cdot\sqrt{3}}{4}=\frac{\sqrt{3}}{24}(a^2+b^2+c^2)+\frac{1}{2}[ABC]$$

We know that: $a^2+b^2+c^2\le 9R^2$and $4\sqrt{3}[ABC]\le a^2+b^2+c^2\le 9R^2$.

Namely, $[KLM]\le\frac{\sqrt{3}}{24}(9R^2)+\frac{1}{2}\left(\frac{3\sqrt{3}}{4}\right)=\frac{3\sqrt{3}}{4}R^2$. So,

$$\frac{[KLM]}{[ABC]}\le\frac{\frac{3\sqrt{3}}{4}R^2}{rs}\le\frac{\frac{3\sqrt{3}}{4}R^2}{r(3\sqrt{3}r)}=\frac{R^2}{4r^2}=\left(\frac{R}{2r}\right)^2$$

Also, we have: $[KLM]=\frac{\sqrt{3}}{24}(a^2+b^2+c^2)+\frac{1}{2}[ABC]\ge\frac{\sqrt{3}}{24}(36r^2)+\frac{1}{2}(rs)$.

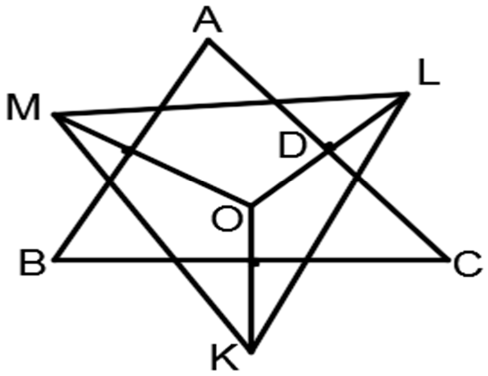
We know that: $[KLM]\ge\frac{3\sqrt{3}}{2}r^2+\frac{1}{2}r\left(3\sqrt{3}r\right)=3\sqrt{3}r^2$. Namely,

$$\frac{[KLM]}{[ABC]}\ge\frac{3\sqrt{3}r^2}{rs}\ge\frac{3\sqrt{3}r^2}{\frac{3\sqrt{3}}{2}R}=\frac{2R}{r}$$

Therefore,

$$\frac{2r}{R}\le\frac{[KLM]}{[ABC]}\le\left(\frac{R}{2r}\right)^2$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let O be the circumcenter of ΔABC and D the midpoint of AC.

The medians of equilateral triangles passing through the midpoints of AB, BC, CA intersect at O.

$$\text{We have}: DL=\frac{b}{2}\tan\frac{\pi}{6}=\frac{\sqrt{3}}{6}b \text{ and } OD=R\cos B\rightarrow DL$$

$$=R\cos B+\frac{\sqrt{3}}{6}b \text{ (And analogs)}$$

$$\rightarrow [KLM]=\sum[MOL]=\frac{1}{2}\sum OL.OM.\sin MOL$$

$$=\frac{1}{2}\sum\left(R\cos B+\frac{\sqrt{3}}{6}b\right)\left(R\cos C+\frac{\sqrt{3}}{6}c\right)\sin(\pi-A)=$$

$$=\frac{R^2}{2}\sum\cos B.\cos C.\sin A+\frac{\sqrt{3}R}{12}\sum(c\cos B+b\cos C)\sin A+\frac{1}{24}\sum bc\sin A$$

$$\text{Now,}\sum\cos B.\cos C.\sin A=\left(\prod\cos A\right)\left(\sum\tan A\right)=\left(\prod\cos A\right)\left(\prod\tan A\right)$$

$$=\prod\sin A=\frac{F}{2R^2}.$$

$$c\cos B+b\cos C=\frac{c^2+a^2-b^2}{2a}+\frac{a^2+b^2-c^2}{2a}=a\rightarrow$$

$$\sum(c\cos B+b\cos C)\sin A=\frac{1}{2R}\sum a^2.$$

$$\text{and}:\sum bc\sin A=\frac{1}{2R}\sum abc=\frac{3.4RF}{2R}=6F\rightarrow$$

$$[KLM]=\frac{R^2}{2}\cdot\frac{F}{2R^2}+\frac{\sqrt{3}R}{12}\cdot\frac{1}{2R}\sum a^2+\frac{1}{24}.6F=\frac{1}{2}F+\frac{\sqrt{3}}{24}\sum a^2.$$

$$Now,\quad [KLM]\overset{Ionescu\,Weitzenbock}{\gtrsim}\frac{1}{2}F+\frac{\sqrt{3}}{24}.4\sqrt{3}F=F\to\frac{[KLM]}{[ABC]}\geq 1\overset{Euler}{\gtrsim}\frac{2r}{R}\ (1).$$

$$And\ [KLM]\overset{Leibniz}{\lesssim}\frac{1}{2}F+\frac{\sqrt{3}}{24}.9R^2\to\frac{[KLM]}{[ABC]}\leq\frac{1}{2}+\frac{3\sqrt{3}R^2}{8sr}\leq$$

$$\overset{Mitrinovic}{\lesssim}\frac{1}{2}+\frac{3\sqrt{3}R^2}{8(3\sqrt{3}r)r}\overset{Euler}{\lesssim}\frac{1}{2}\left(\frac{R}{2r}\right)^2+\frac{1}{2}\left(\frac{R}{2r}\right)^2=\left(\frac{R}{2r}\right)^2\to\frac{[KLM]}{[ABC]}\leq\left(\frac{R}{2r}\right)^2\quad(2)$$

$$(1),(2)\to\frac{2r}{R}\leq\frac{[KLM]}{[ABC]}\leq\left(\frac{R}{2r}\right)^2$$

JP.363. Let a,b,c be positive real numbers with $a^2+b^2+c^2=12$.

Prove that:

$$\frac{a^4}{\sqrt{a^3+1}}+\frac{b^4}{\sqrt{b^3+1}}+\frac{c^4}{\sqrt{c^3+1}}\geq 16$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

We have: $a^2(a-2)^2\geq 0$, with equality for $a=2$. So, $a^2(a^2-4a+4)\geq 0\Leftrightarrow$

$$a^4+4a^2+4-4a^3-4=(a^2+2)^2-4(a^3+1)\geq 0\Leftrightarrow 4(a^3+1)\leq(a^2+2)^2\Rightarrow$$

$2\sqrt{a^3+1}\leq a^2+2$. Now, $\frac{1}{2\sqrt{a^3+1}}\geq\frac{1}{a^2+2}\Leftrightarrow\frac{a^4}{2\sqrt{a^3+1}}\geq\frac{a^4}{a^2+2}$.

Similarly,

$$\frac{a^4}{2\sqrt{b^3+1}}\geq\frac{b^4}{b^2+2};\frac{c^4}{2\sqrt{c^3+1}}\geq\frac{c^4}{c^2+2}$$

Adding up these inequalities, we have:

$$\frac{1}{2}\left(\frac{a^4}{\sqrt{a^3+1}}+\frac{b^4}{\sqrt{b^3+1}}+\frac{c^4}{\sqrt{c^3+1}}\right)\geq\frac{a^4}{a^2+2}+\frac{b^4}{b^2+2}+\frac{c^4}{c^2+2}$$

Applying Cauchy-Schwarz inequality, we get:

$$\frac{a^4}{\sqrt{a^3+1}}+\frac{b^4}{\sqrt{b^3+1}}+\frac{c^4}{\sqrt{c^3+1}}\geq 2\cdot\frac{(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)+6}=\frac{2\cdot 12^2}{12+6}=16$$

Equality holds when $a = b = c = 2$.

Solution 2 by Angel Plaza-Spain

By Jensen's inequality, by doing $a^2 = x, b^2 = y, c^2 = z$, the inequality becomes:

$$\frac{x^2}{\sqrt{x^{\frac{3}{2}}+1}} + \frac{y^2}{\sqrt{y^{\frac{3}{2}}+1}} + \frac{z^2}{\sqrt{z^{\frac{3}{2}}+1}} \geq 16$$

For x, y, z positive real numbers with $x + y + z = 12$.

Function $f(x) = \frac{x^2}{\sqrt{x^{\frac{3}{2}}+1}}$ is convex for $x \in (0, 12)$ because

$$f''(x) = \frac{10x^{\frac{3}{2}} + 5x^3 + 32}{16\left(x^{\frac{3}{2}}+1\right)^{\frac{5}{2}}} > 0$$

Therefore, by Jensen's inequality, we get:

$$\frac{x^2}{\sqrt{x^{\frac{3}{2}}+1}} + \frac{y^2}{\sqrt{y^{\frac{3}{2}}+1}} + \frac{z^2}{\sqrt{z^{\frac{3}{2}}+1}} \geq 3 \cdot \frac{4^2}{\sqrt{4^{\frac{3}{2}}+1}} = 16$$

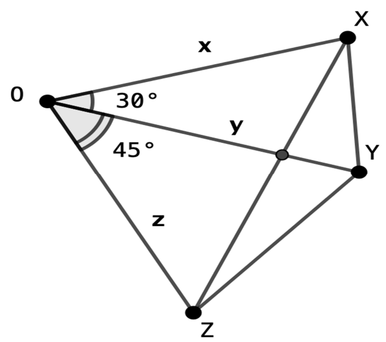
JP.364 If $x, y, z > 0$ then:

$$\sqrt{x^2 - xy\sqrt{3} + y^2} + \sqrt{y^2 - yz\sqrt{2} + z^2} = \sqrt{x^2 + z^2 - \frac{xz(\sqrt{6}-\sqrt{2})}{2}} \Leftrightarrow \frac{2\sqrt{2}}{x} + \frac{2}{z} = \frac{\sqrt{2}+\sqrt{6}}{y}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Denote $OX = x, OY = y; OZ = z$



$$m(\widehat{XOY})=30^\circ, m(\widehat{YOZ})=45^\circ, m(\widehat{XOZ})=75^\circ$$

$$Y\in Int(\widehat{XOZ})$$

$$XY=\sqrt{x^2+y^2-2xy\cos 30^\circ}=\sqrt{x^2+y^2-2xy\cdot\frac{\sqrt{3}}{2}}=\sqrt{x^2+y^2-xy\sqrt{3}}$$

$$YZ=\sqrt{y^2+z^2-2yz\cos 45^\circ}=\sqrt{y^2+z^2-2yz\cdot\frac{\sqrt{2}}{2}}=\sqrt{y^2+z^2-yz\sqrt{2}}$$

$$XZ=\sqrt{x^2+z^2-2xz\cos 75^\circ}=\sqrt{x^2+z^2-2xz(\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ)}=$$

$$=\sqrt{x^2+z^2-2xz\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right)}=\sqrt{x^2+z^2-\frac{xz(\sqrt{6}-\sqrt{2})}{2}}$$

$$\sqrt{x^2-xy\sqrt{3}+y^2}+\sqrt{y^2-yz\sqrt{2}+z^2}=\sqrt{x^2+z^2-\frac{xz(\sqrt{6}-\sqrt{2})}{2}}\Leftrightarrow$$

$$\Leftrightarrow XY+YZ=XZ\Leftrightarrow X,Y,Z-\text{colineare}$$

$$\Leftrightarrow [XOY]+[YOZ]=[XOZ]\Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}xy\sin 30^\circ+\frac{1}{2}yz\sin 45^\circ=\frac{1}{2}xz\sin 75^\circ$$

$$\Leftrightarrow xy\sin 30^\circ+yz\sin 45^\circ=xz\sin(30^\circ+45^\circ)$$

$$\Leftrightarrow xy\cdot\frac{1}{2}+yz\cdot\frac{\sqrt{2}}{2}=xz(\sin 30^\circ\cos 45^\circ+\sin 45^\circ\cos 30^\circ)$$

$$\Leftrightarrow \frac{xy}{2}+\frac{yz\sqrt{2}}{2}=xz\left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\right)\Leftrightarrow xy+yz\sqrt{2}=\frac{xz(\sqrt{2}+\sqrt{6})}{2}$$

$$\Leftrightarrow 2xy+2\sqrt{2}yz=xz(\sqrt{2}+\sqrt{6})\Leftrightarrow \frac{2}{z}+\frac{2\sqrt{2}}{x}=\frac{\sqrt{2}+\sqrt{6}}{y}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$x^2-xy\sqrt{3}+y^2=x^2-2\cos\frac{\pi}{6}xy+y^2=\left(x\cos\frac{\pi}{6}-y\right)^2+x^2\sin^2\frac{\pi}{6}=$$

$$=\left|\left(x\cos\frac{\pi}{6}-y\right)+x\sin\frac{\pi}{6}i\right|^2$$

$$\to\sqrt{x^2-xy\sqrt{3}+y^2}=\left|\left(x\cos\frac{\pi}{6}-y\right)+\left(\sin\frac{\pi}{6}\right)xi\right|$$

Next,

$$\sqrt{y^2 - yz\sqrt{2} + z^2} = \left|\left(y - \cos\frac{\pi}{4}z\right) + \left(\sin\frac{\pi}{4}\right)zi\right|$$

$$x^2 + z^2 - \frac{\sqrt{6}-\sqrt{2}}{2}xz = x^2 + z^2 - 2\left(\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}\right)zx =$$

$$= \left(x\cos\frac{\pi}{6} - z\cos\frac{\pi}{4}\right)^2 + \left(x\sin\frac{\pi}{6} + z\sin\frac{\pi}{4}\right)^2 =$$

$$= \left|\left(x\cos\frac{\pi}{6} - z\cos\frac{\pi}{4}\right) + \left(x\sin\frac{\pi}{6} + z\sin\frac{\pi}{4}\right)i\right|^2$$

$$\to \sqrt{x^2 + z^2 - \frac{1}{2}(\sqrt{6}-\sqrt{2})xz} = \left|\left(x\cos\frac{\pi}{6} - z\cos\frac{\pi}{4}\right) + \left(x\sin\frac{\pi}{6} + z\sin\frac{\pi}{4}\right)i\right|$$

Now,

$$\sqrt{x^2 - xy\sqrt{3} + y^2} + \sqrt{y^2 - yz\sqrt{2} + z^2} = \sqrt{x^2 + z^2 - \frac{1}{2}(\sqrt{6}-\sqrt{2})xz}$$

$$\Leftrightarrow \left|\left(x\cos\frac{\pi}{6} - y\right) + \left(\sin\frac{\pi}{6}\right)xi\right| + \left|\left(y - \cos\frac{\pi}{4}z\right) + \left(\sin\frac{\pi}{4}\right)zi\right| =$$

$$= \left|\left(x\cos\frac{\pi}{6} - z\cos\frac{\pi}{4}\right) + \left(x\sin\frac{\pi}{6} + z\sin\frac{\pi}{4}\right)i\right| \Leftrightarrow \frac{x\cos\frac{\pi}{6} - y}{y - z\cos\frac{\pi}{4}} = \frac{x\sin\frac{\pi}{6}}{z\sin\frac{\pi}{4}}$$

$$\Leftrightarrow xz\left(\cos\frac{\pi}{6}\sin\frac{\pi}{4}\right) - yz\sin\frac{\pi}{4} = xy\sin\frac{\pi}{6} - xz\sin\frac{\pi}{6}\cos\frac{\pi}{6}$$

$$\Leftrightarrow xz\left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) = \frac{yz}{\sqrt{2}} + \frac{xy}{2z}$$

$$\Leftrightarrow \frac{1+\sqrt{3}}{2\sqrt{2}} \cdot \frac{1}{y} = \frac{1}{x\sqrt{2}} + \frac{1}{2z} \Leftrightarrow \frac{\sqrt{6}+\sqrt{2}}{y} = \frac{\sqrt{2}}{x} + \frac{2}{z}$$

Therefore,

$$\frac{2\sqrt{2}}{x} + \frac{2}{z} = \frac{\sqrt{2}+\sqrt{6}}{y}$$

JP365. If in ΔABC exists the relationship:

$$\frac{4}{w_a} = \frac{1}{r} + \frac{1}{r_a} \text{ then prove that } AH \geq 2r.$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\frac{4}{w_a}=\frac{1}{r}+\frac{1}{r_a}\Leftrightarrow\frac{4}{\frac{2bc}{b+c}cos\frac{A}{2}}=\frac{s+s-a}{F}\Leftrightarrow\frac{4F(b+c)}{2bc\cdot cos\frac{A}{2}}=2s-a\Leftrightarrow$$

$$\frac{4F(b+c)}{2bc\cdot sin\frac{A}{2}cos\frac{A}{2}}\cdot sin\frac{A}{2}=b+c\Leftrightarrow\frac{4F}{bcsin\frac{A}{2}}\cdot sin\frac{A}{2}=1$$

$$\frac{4F}{2F}\cdot sin\frac{A}{2}=1\Rightarrow sin\frac{A}{2}=\frac{1}{2}\Rightarrow\frac{A}{2}=30^{\circ}\Rightarrow A=60^{\circ}$$

$$AH=2R\cdot sinA=R\Rightarrow R\ge 2r(Euler)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{4}{w_a}=\frac{1}{r}+\frac{1}{r_a}\leftrightarrow\frac{2(b+c)}{bc.\cos\frac{A}{2}}=\frac{1}{r}\left(1+\frac{s-a}{s}\right)=\frac{b+c}{rs}\leftrightarrow 2rs=bc.\cos\frac{A}{2}=\frac{4srR}{a}.\cos\frac{A}{2}$$

$$\leftrightarrow a=2R\cos\frac{A}{2}\leftrightarrow 4R\cos\frac{A}{2}\sin\frac{A}{2}=2R\cos\frac{A}{2}\leftrightarrow\sin\frac{A}{2}=\frac{1}{2}.$$

$$Now, we\ know\ that: AH=2R|\cos A|=2R\left|1-2\sin^2\frac{A}{2}\right|=2R\left|1-2.\frac{1}{4}\right|=R\overset{Euler}{\geq}2r$$

$$Therefore,\qquad AH\ge 2r.$$

JP.366. In acute ΔABC the following relationship holds:

$$cosA+\sqrt{cosAcosB}+\sqrt[3]{cosAcosBcosC}<2$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$cosA+\sqrt{cosAcosB}+\sqrt[3]{cosAcosBcosC}=$$

$$=cosA+\frac{1}{2}\sqrt{cosA\cdot 4cosB}+\frac{1}{4}\sqrt[3]{cosA\cdot 4cosB\cdot 16cosC}\le$$

$$\le coaA+\frac{1}{4}(cosA+4cosB)+\frac{1}{12}(cosA+4cosB+16cosC)=$$

$$=\frac{1}{12}(12cosA+3cosA+12cosB+cosA+4cosB+16cosC)=$$

$$=\frac{16}{12}(cosA+cosB+cosC)=\frac{4}{3}(cosA+cosB+cosC);(1)$$

$$cosA + cosB + cosC \leq \frac{3}{2}; (2)$$

From (1),(2) it follows that:

$$cosA + \sqrt{cosAcosB} + \sqrt[3]{cosAcosBcosC} < \frac{4}{3} \cdot \frac{3}{2} < 2$$

Equality holds if and only if triangle is equilateral.

JP.367. $a, b, c \in \mathbb{C}^*$ −different in pairs, $A(a), B(b), C(c);\ |a| = |b| = |c| = 1$.

If

$(ab)^3 + (bc)^3 + (ca)^3 = 3(abc)^2$, then ΔABC is equilateral.

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$A(a), B(b), C(c);\ |a| = |b| = |c| = 1; \Delta ABC \subset C(0,1) \Rightarrow |a|^2 = 1 \Rightarrow$$

$$a \cdot \overline{a} = 1; b \cdot \overline{b} = 1; c \cdot \overline{c} = 1$$

$$\left(\overline{ab}\right)^3 + \left(\overline{bc}\right)^3 + (\overline{ca})^3 = 3\left(\overline{abc}\right)^2$$

$$\frac{1}{a^3} \cdot \frac{1}{b^3} + \frac{1}{b^3} \cdot \frac{1}{c^3} + \frac{1}{c^3} \cdot \frac{1}{a^3} = \frac{3}{a^2b^2c^2} \Leftrightarrow a^3 + b^3 + c^3 = 3abc \Leftrightarrow$$

$$a^3 + b^3 + c^3 - 3abc = 0 \Leftrightarrow$$

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

- If $a + b + c = 0 \Rightarrow a_H = a_O \Rightarrow H = O \Rightarrow \Delta ABC$ −equilateral.
- If $a^2 + b^2 + c^2 = ab + bc + ca \Rightarrow \Delta ABC$ −equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$|a| = |b| = |c| = 1 \to \Delta ABC \in C(O, R = 1).$$

$$(*) \leftrightarrow \left(\overline{ab}\right)^3 + \left(\overline{bc}\right)^3 + (\overline{ca})^3 = 3\left(\overline{abc}\right)^2 \leftrightarrow \frac{1}{(ab)^3} + \frac{1}{(bc)^3} + \frac{1}{(ca)^3} = \frac{3}{(abc)^2}$$

$$\leftrightarrow a^3 + b^3 + c^3 = 3abc \leftrightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

$$\leftrightarrow a + b + c = 0 \ or \ a^2 + b^2 + c^2 - ab - bc - ca = 0$$

$$If\ a + b + c = 0 \to z_H = z_O \to H \equiv O \to \Delta ABC\ is\ equilateral.$$

$$If\ a^2 + b^2 + c^2 - ab - bc - ca = 0 \leftrightarrow \frac{b - a}{c - a} = \frac{c - b}{a - b}$$

$$\leftrightarrow \Delta ABC \sim \Delta BCA \to \Delta ABC \text{ is equilateral.}$$

$$Therefore, (ab)^3 + (bc)^3 + (ca)^3 = 3(abc)^2 \to \Delta ABC \text{ is equilateral.}$$

Solution 3 by Daoudi Abdessattar-Tunisia

$$A(a), B(b), C(c), (a-b)(b-c)(c-a) \neq 0 \text{ and } |a| = |b| = |c| = 1$$

$$Suppose\ that: (ab)^3 + (bc)^3 + (ca)^3 = 3(abc)^2;\ (1)$$

$$(1) \Leftrightarrow \frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2} = 3 \Leftrightarrow \sum_{cuc} e^{i(\alpha+\beta-2\gamma)} = 3, (a = e^{i\alpha}, b = e^{i\beta}, c = e^{i\gamma}, \alpha, \beta, \gamma$$
$$\in [-\pi, \pi]$$

$$\Leftrightarrow \begin{cases} \sum cos(\alpha+\beta-2\gamma) = 3 \\ \sum sin(\alpha+\beta-2\gamma) = 0 \end{cases} \Leftrightarrow \alpha+\beta = 2\gamma \Leftrightarrow ab = c^2, bc = a^2, ca = b^2$$

$$ab = c^2 \to ab^3 = (bc)^2 = a^4 \to b^3 = a^3 \text{ and similarly,} \quad a^3 = c^3.$$

$$a, b, c \text{ are roots of equation } z^3 = e^{i\theta}, -\pi < \theta < \pi$$

$$\to \{a, b, c\} \in \left\{e^{\frac{i\theta}{3}}, e^{\frac{i\theta}{3}+\frac{2i\pi}{3}}, e^{\frac{i\theta}{3}-\frac{2i\pi}{3}}\right\} \to AB = BC = CA.$$

JP.368. $a, b, c \in \mathbb{C}^*$ −different in pairs, $A(a), B(b), C(c);\ |a| = |b| = |c| = 1.$

If $|a-b|\left(\frac{1}{a}+\frac{1}{b}\right) + |b-c|\left(\frac{1}{b}+\frac{1}{c}\right) + |c-a|\left(\frac{1}{c}+\frac{1}{a}\right) = 0$, then ΔABC is equilateral.

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$A(a), B(b), C(c);\ |a| = |b| = |c| = 1; \Delta ABC \subset C(0,1)$$

$$|a|^2 = 1 \Rightarrow a \cdot \bar{a} = 1 \Rightarrow a \cdot \bar{a} = b \cdot \bar{b} = c \cdot \bar{c} = 1.$$

$$\frac{|a-b| + |a-c|}{a} + \frac{|b-c| + |b-a|}{b} + \frac{|c-a| + |c-b|}{c} = 0 \Leftrightarrow$$

$$\overline{\frac{|a-b| + |a-c|}{a} + \frac{|b-c| + |b-a|}{b} + \frac{|c-a| + |c-b|}{c}} = 0$$

$$(\beta+\gamma)a + (\alpha+\gamma)b + (\alpha+\beta)c = 0$$

$$(\alpha+\beta+\gamma-\alpha)a + (\alpha+\beta+\gamma-\beta)b + (\alpha+\beta+\gamma-\gamma)c = 0 \Leftrightarrow$$

$$(\alpha+\beta+\gamma)(a+b+c) = \alpha a + \beta b + \gamma c \Leftrightarrow$$

$$a+b+c=\frac{\alpha a+\beta b+\gamma c}{\alpha+\beta+\gamma}\Leftrightarrow H=I\Rightarrow \Delta ABC \text{ —equilateral.}$$

Solution 2 by Mohamed Amine-Ben Ajiba-Tanger-Morocco

$$|a|=|b|=|c|=1\to \Delta ABC\in C(O,R=1).$$

$$Let\ x=|b-c|=BC, y=|c-a|=CA, z=|a-b|=AB.$$

$$(*)\leftrightarrow z\left(\frac{1}{\overline{a}}+\frac{1}{\overline{b}}\right)+x\left(\frac{1}{\overline{b}}+\frac{1}{\overline{c}}\right)+y\left(\frac{1}{\overline{c}}+\frac{1}{\overline{a}}\right)=0$$

$$\leftrightarrow z(a+b)+x(b+c)+y(c+a)=0$$

$$\leftrightarrow (x+y+z)(a+b+c)=xa+yb+zc$$

$$\leftrightarrow a+b+c=\frac{xa+yb+zc}{x+y+z}\leftrightarrow z_H=z_I\leftrightarrow H\equiv I$$

$$then\ \Delta ABC\ is\ equilateral.$$

JP.369. Find $x,y,z\in\left(0,\frac{\pi}{2}\right)$ such that:

$$\frac{cos(5x)}{cosx}+\frac{cos(5y)}{cosy}+\frac{cos(5z)}{cosz}+\frac{15}{4}=0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\frac{cos(5x)}{cosx}+\frac{cos(5y)}{cosy}+\frac{cos(5z)}{cosz}+\frac{15}{4}=\frac{15}{4}+\sum_{cyc}\frac{cos(5x)}{cosx}=$$

$$=5\left(\frac{3}{4}+\sum_{cyc}\frac{cos(5x)}{5cosx}\right)=5\sum_{cyc}\left(\frac{cos(5x)}{5cosx}+\frac{1}{4}\right)=$$

$$=5\sum_{cyc}\left(\frac{16cos^5x-20cos^3x+5cosx}{5cosx}+\frac{1}{4}\right)=$$

$$=5\sum_{cyc}\left(\frac{16cos^4x-20cos^2x+5}{5}+\frac{1}{4}\right)=5\sum_{cyc}\frac{64cos^4x-80cos^2x+20+5}{20}=$$

$$=\frac{5}{20}\sum_{cyc}(64cos^4x-80cos^2x+25)=\frac{1}{4}\sum_{cyc}(8cos^2x-5)^2=0$$

$$\rightarrow \begin{cases} 8cos^2x - 5 = 0 \\ 8cos^2y - 5 = 0 \\ 8cos^2y - 5 = 0 \end{cases} \rightarrow \begin{cases} cos^2x = \frac{5}{8} \\ cos^2y = \frac{5}{8} \\ cos^2z = \frac{5}{8} \end{cases} \rightarrow \begin{cases} cosx = \pm\sqrt{\frac{5}{8}} \\ cosy = \pm\sqrt{\frac{5}{8}} \\ cosz = \pm\sqrt{\frac{5}{8}} \end{cases}$$

Therefore,

$$x, y, z \in \left\{cos^{-1}\left(\sqrt{\frac{5}{8}}\right); \pi - cos^{-1}\left(\sqrt{\frac{5}{8}}\right)\right\}$$

Solution 2 by Daniel Văcaru-Romania

$$\frac{cos(5x)}{cosx} + \frac{cos(5y)}{cosy} + \frac{cos(5z)}{cosz} + \frac{15}{4}; (*)$$

Using $\cos 2x = 2\cos^2 x - 1, \cos 3x = 4\cos^3 x - 3\cos x$ and

$\cos(x + y) = \cos x \cos y - \sin x \sin y$, we obtain

$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$ and

$$\frac{\cos 5x}{\cos x} = 16\cos^4 x - 20\cos^2 x + 5; (1)$$

We write $(*)$ as $\sum\left(16\cos^4 x - 20\cos^2 x + \frac{25}{4}\right) = 0 \Leftrightarrow$

$\sum\left(4\cos^2 x - \frac{5}{2}\right)^2 = 0$. For $x, y, z \in \left(0, \frac{\pi}{2}\right)$ we obtain: $\cos^2 x = \frac{5}{8}$

$$\rightarrow \cos x = \frac{\sqrt{5}}{2\sqrt{2}} \rightarrow x = y = z = \cos^{-1}\left(\frac{\sqrt{5}}{2\sqrt{2}}\right) \in \left(0, \frac{\pi}{2}\right)$$

JP.370. In ΔABC the following relationship holds:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} + \sqrt{\frac{a^2 + b^2}{2}} + \sqrt{\frac{b^2 + c^2}{2}} + \sqrt{\frac{c^2 + a^2}{2}} \leq 6\sqrt{3}R$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

First we prove that $\forall a, b > 0$ then:

$$\sqrt{ab} + \sqrt{\frac{a^2 + b^2}{2}} \le a + b; \quad (1)$$

Denote $u^2 = \frac{a}{b} \to a = u^2 b$

$$\sqrt{u^2 b \cdot b} + \sqrt{\frac{u^4 b^2 + b^2}{2}} \le u^2 b + b, \qquad u + \sqrt{\frac{u^4 + 1}{2}} \le u^2 + 1$$

$$\sqrt{\frac{u^4 + 1}{2}} \le (u^2 - u + 1)^2, \quad \frac{u^4 + 1}{2} \le (u^2 - u + 1)^2$$

$$u^4 + 1 \le 2(u^4 + u^2 + 1 - 2u^3 + 2u^2 - 2u)$$

$$2u^4 + 2u^2 + 2 - 4u^3 + 4u^2 - 4u - u^4 - 1 \ge 0$$

$$u^4 - 4u^3 + 6u^2 - 4u + 1 \ge 0, \qquad (u - 1)^4 \ge 0$$

By (1) it follows that:

$$\sum_{cyc} \left(\sqrt{ab} + \sqrt{\frac{a^2 + b^2}{2}} \right) \le \sum_{cyc} (a + b) = 2(a + b + c) =$$

$$= 2 \cdot 2s = 4s \overset{Mitrinovic}{\le} 4 \cdot \frac{3\sqrt{3}R}{2} = 6\sqrt{3}R$$

Solution 2 by Daniel Văcaru-Romania

By CBS: $\sum_{i=1}^{n} a_i b_i \le \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}$, we have:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} + \sqrt{\frac{a^2 + b^2}{2}} + \sqrt{\frac{b^2 + c^2}{2}} + \sqrt{\frac{c^2 + a^2}{2}} =$$

$$= 1 \cdot \sqrt{ab} + 1 \cdot \sqrt{bc} + 1 \cdot \sqrt{ca} + 1 \cdot \sqrt{\frac{a^2 + b^2}{2}} + 1 \cdot \sqrt{\frac{b^2 + c^2}{2}} + 1 \cdot \sqrt{\frac{c^2 + a^2}{2}} \overset{CBS}{\le}$$

$$\leq \sqrt{6\left(ab+bc+ca+\frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}\right)}=$$

$$=\sqrt{6(a^2+b^2+c^2+ab+bc+ca)} \leq \sqrt{12}\cdot\sqrt{a^2+b^2+c^2}$$

But:

$a^2+b^2+c^2 \leq 9R^2 \rightarrow \sqrt{a^2+b^2+c^2} \leq 3R$. **We have:**

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}+\sqrt{\frac{a^2+b^2}{2}}+\sqrt{\frac{b^2+c^2}{2}}+\sqrt{\frac{c^2+a^2}{2}} \leq 2\sqrt{3}\sqrt{a^2+b^2+c^2}=6\sqrt{3}R$$

JP.371 Solve for real positive numbers the equation

$$x^{log3}+x^{log4}+x^{log5}=x^{log6}$$

Proposed by D.M. Bătineţu-Giurgiu ,Neculai Stanciu-Romania

Solution 1 by proposers

It is well known that $x^{loga}=a^{logx}$ **for any real positive number** a**.**

So, the given equation becomes $3^{logx}+4^{logx}+5^{logx}=6^{logx}$

, then we denote $logx=t$**, and the equation becomes**

$$3^t+4^t+5^t=6^t \Leftrightarrow \left(\frac{3}{6}\right)^t+\left(\frac{4}{6}\right)^t+\left(\frac{5}{6}\right)^t=1,$$

and since LHS is decreasing, it results that we have unique solution.

We note that $t=3$**.**

Therefore $logx=3$**and the solution is** $x=e^3$**.**

Solution 2 by Ravi Prakash-New Delhi-India

Let $y=x^{\log 3} \rightarrow \log y_1=\log 3\cdot\log x=\log(3^{\log x}) \rightarrow y_1=3^{\log x} \rightarrow x^{\log 3}=3^{\log x}$

Similarly for other three expressions. The equation becomes:

$$3^y+4^y+5^y=6^y, where\ y=\log x, (x>0); (1)$$

Now, we can rewrite as:

$$\left(\frac{1}{2}\right)^y+\left(\frac{2}{3}\right)^y+\left(\frac{5}{6}\right)^y=1; (2)$$

Let $f(t)=\left(\frac{1}{2}\right)^t+\left(\frac{2}{3}\right)^t+\left(\frac{5}{6}\right)^t, t\in\mathbb{R}$

$$f'(t)=-\left(\frac{1}{2}\right)^t\log 2-\left(\frac{2}{3}\right)^t\log\left(\frac{3}{2}\right)-\left(\frac{5}{6}\right)^t\log\left(\frac{6}{5}\right)<0,\forall t\in\mathbb{R}\to$$

f –is strictly decreasing on $\mathbb{R}$. Thus, equation (2) and (1) have at most one solution.

So, $y=3$ is the only solution of (1), thus

$\log x=3\to x=3^b$,where b is base of the logarithm in the question.

Solution 3 by Daniel Văcaru-Romania

$$(*): x^{log3}+x^{log4}+x^{log5}=x^{log6}$$

We have $x^{\log a}=\left(10^{\log x}\right)^{\log a}=\left(10^{\log a}\right)^{\log x}=a^{\log x}; (1)$

Taking account of (1), we have:

$$(*)\Leftrightarrow 3^{\log x}+4^{\log x}+5^{\log x}=6^{\log x}; (2)$$

Setting $\log x\overset{not}{\triangleq}t$, we obtain equation $3^t+4^t+5^t=6^t\Leftrightarrow\left(\frac{3}{6}\right)^t+\left(\frac{4}{6}\right)^t+\left(\frac{5}{6}\right)^t=1$

But: $f:\mathbb{R}\to\mathbb{R}_+^*, f(t)=\left(\frac{3}{6}\right)^t+\left(\frac{4}{6}\right)^t+\left(\frac{5}{6}\right)^t$ is decreasing. Then equation $f(t)=1$ has an unique solution, $t=1\Leftrightarrow\log x=3\Leftrightarrow x=1000$.

JP.372. If in ΔABC: $a^2+b^2=2c^2$ then:

$$2am_a+m_c^2\cdot\sqrt{\frac{ab}{m_am_b}}\le\frac{\sqrt{3}}{2}(a^2+b^2+c^2)$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$a^2m_a^2=a^2\cdot\frac{2(b^2+c^2)-a^2}{4}=\frac{2a^2b^2+2a^2c^2-a^4}{4}=$$

$$=\frac{2a^2b^2+a^2(a^2+b^2)-a^4}{4}=\frac{2a^2b^2+a^4+a^2b^2-a^4}{4}=\frac{3a^2b^2}{4}\quad (1)$$

$$b^2m_b^2=b^2\cdot\frac{2(a^2+c^2)-b^2}{4}=\frac{2a^2b^2+2b^2c^2-b^4}{4}=$$

$$=\frac{2a^2b^2+b^2(a^2+b^2)-b^4}{4}=\frac{2a^2b^2+a^2b^2+b^4-b^4}{4}=\frac{3a^2b^2}{4}\quad (2)$$

By (1); (2) $\Rightarrow a^2m_a^2=b^2m_b^2\Rightarrow am_a=bm_b$ (3)

$$\frac{c^4}{m_c^4}=\frac{c^4}{\left(\frac{2(a^2+b^2)-c^2}{4}\right)^2}=\frac{16c^4}{4(a^2+b^2)^2+c^4-4c^2(a^2+b^2)}=$$

$$=\frac{4\cdot(2c^2)^2}{4(a^2+b^2)^2+\left(\frac{a^2+b^2}{2}\right)^2-(a^2+b^2)\cdot 2(a^2+b^2)}=$$

$$=\frac{4(a^2+b^2)^2}{2(a^2+b^2)+\frac{(a^2+b^2)^2}{4}}=\frac{4}{2+\frac{1}{4}}=\frac{16}{9}\quad (4)$$

$$\frac{a^2b^2}{m_a^2m_b^2}=\frac{a^2b^2}{\frac{2(b^2+c^2)-a^2}{4}\cdot\frac{2(a^2+c^2)-b^2}{4}}=$$

$$=\frac{16a^2b^2}{\left(2\left(b^2+\frac{a^2+b^2}{2}\right)-a^2\right)\left(2\left(a^2+\frac{a^2+b^2}{2}\right)-b^2\right)}=$$

$$=\frac{16a^2b^2}{(2b^2+a^2+b^2-a^2)(2a^2+a^2+b^2-b^2)}=$$

$$=\frac{16a^2b^2}{9a^2b^2}=\frac{16}{9}\quad (5)$$

By (4); (5) $\Rightarrow \frac{c^4}{m_c^4}=\frac{a^2b^2}{m_a^2m_b^2}\Rightarrow\frac{c^2}{m_c^2}=\frac{ab}{m_am_b}\Rightarrow$

$$\Rightarrow c=m_c\sqrt{\frac{ab}{m_am_b}}\quad (6)$$

$$am_a+bm_b+cm_c\overset{CBS}{\le}\sqrt{(a^2+b^2+c^2)(m_a^2+m_b^2+m_c^2)}$$

By (3); (6):

$$am_a+am_a+m_c\cdot m_c\sqrt{\frac{ab}{m_am_b}}\le\sqrt{(a^2+b^2+c^2)\cdot\frac{3}{4}(a^2+b^2+c^2)}$$

$$2am_a+m_c^2\sqrt{\frac{ab}{m_am_b}}\le\frac{\sqrt{3}}{2}(a^2+b^2+c^2)$$

Equality holds for $a=b=c$.

JP.373 If $a,b,c<0$; $a+b+c=3$; F_n – Fibonacci numbers; L_n – Lucas numbers; P_n – Pell numbers; $n\in\mathbb{N}$; $n\geq 2$ then:

$$\frac{a^2(P_n-F_n)(P_n-L_n)}{F_nL_n}+\frac{b^2(F_n-L_n)(F_n-P_n)}{L_nP_n}+\frac{c^2(L_n-P_n)(L_n-F_n)}{P_nF_n}\geq 9$$

Proposed by Daniel Sitaru – Romania

Solution by proposer:

Lemma: If $x,y,z>0$ different in pairs then:

$$\frac{xy}{(z-x)(z-y)}+\frac{yz}{(x-y)(x-z)}+\frac{zx}{(y-z)(y-x)}=1 \quad (1)$$

Proof:

$$\frac{xy}{(z-x)(z-y)}+\frac{yz}{(x-y)(x-z)}+\frac{zx}{(y-z)(y-x)}=$$

$$=\frac{-xy(x-y)-yz(y-z)-zx(z-x)}{(x-y)(y-z)(z-x)}=$$

$$=\frac{-x^2y+xy^2-y^2z+yz^2-z^2x+zx^2}{(xy-xz-y^2+yz)(z-x)}=$$

$$=\frac{xy^2+yz^2+zx^2-x^2y-y^2z-z^2x}{xy^2+yz^2+zx^2-x^2y-y^2z-z^2x}=1$$

We take in (1): $x=F_n$; $y=L_n$; $z=P_n$

$$\frac{F_nL_n}{(P_n-F_n)(P_n-L_n)}+\frac{L_nP_n}{(F_n-L_n)(F_n-P_n)}+\frac{P_nF_n}{(L_n-P_n)(L_n-F_n)}=1 \quad (2)$$

$$\frac{a^2(P_n-F_n)(P_n-L_n)}{F_nL_n}+\frac{b^2(F_n-L_n)(F_n-P_n)}{L_nP_n}+\frac{c^2(L_n-P_n)(L_n-F_n)}{P_nF_n}=$$

$$=\frac{a^2}{\frac{F_nL_n}{(P_n-F_n)(P_n-L_n)}}+\frac{b^2}{\frac{L_nP_n}{(F_n-L_n)(F_n-P_n)}}+\frac{c^2}{\frac{P_nF_n}{(L_n-P_n)(L_n-F_n)}}\geq$$

$$\overset{BERGSTROM}{\geq}\frac{(a+b+c)^2}{\frac{F_nL_n}{(P_n-F_n)(P_n-L_n)}+\frac{L_nP_n}{(F_n-L_n)(F_n-P_n)}+\frac{P_nF_n}{(L_n-P_n)(L_n-F_n)}}\overset{(2)}{=}$$

$$=\frac{(a+b+c)^2}{1}=3^2=9$$

Equality holds for $a=b=c$.

JP.374 Solve for complex numbers:

$$57x^6-180x^5+234x^4-159x^3+60x^2-12x+1=0$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$57x^6-180x^5+234x^4-159x^3+60x^2-12x+1=0$$

$$(64x^6-8x^6+x^6)+(12x^5-192x^5)+(240x^4-6x^4)+$$

$$+(x^3-160x^3)+60x^2-12x+1=0$$

$$(64x^6-192x^5+240x^4-160x^3+60x^2-12x+1)+$$

$$+x^6+(x^3-6x^4+12x^5-8x^6)=0$$

$$\binom{6}{6}(2x)^6-\binom{6}{5}(2x)^5+\binom{6}{4}(2x)^4-\binom{6}{3}(2x)^3+\binom{6}{2}(2x)^2-$$

$$-\binom{6}{1}2x+\binom{6}{0}+x^6+x^3(1-6x+12x^2-8x^3)=0$$

$$(1-2x)^6+x^6+x^3(1-2x)^3=0$$

$$\left(\frac{1-2x}{x}\right)^6+\left(\frac{1-2x}{x}\right)^3+1=0$$

$$\left(\frac{1}{x}-2\right)^6+\left(\frac{1}{x}-2\right)^3+1=0$$

$$\frac{1}{x}-2=y\Rightarrow y^6+y^3+1=0$$

We denote $y^3=t;t^2+t+1=0,t_{1,2}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

$$t_1=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3};t_2=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

$$y^3=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$$y_k=\cos\frac{\frac{2\pi}{3}+2k\pi}{3}+i\sin\frac{\frac{2\pi}{3}+2k\pi}{3};k\in\{0,1,2\}$$

$$y^3=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

$$y'_k = \cos\frac{\frac{4\pi}{3}+2k\pi}{3} + i\sin\frac{\frac{4\pi}{3}+2k\pi}{3}; k\in\{0,1,2\}$$

Solutions:

$$x_k = \frac{1}{y_k+2}; x'_k = \frac{1}{y'_k+2}; k\in\{0,1,2\}$$

JP. 375 Let a, b, c be positive real numbers such that $a+b+c=3$.

Prove that:

$$9(a^2+b^2+c^2)-2(a^3+b^3+c^3)\geq 21$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by proposer

It is well-known the identity

$$(-a+b+c)^3+(a-b+c)^3+(a+b-c)^3=(a+b+c)^3-24abc$$

So $(3-2a)^3+(3-2b)^3+(3-2c)^3=27-24abc$

Using the AM-GM inequality, we have:

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc} \text{ so } abc\leq 1$$

Now,

$$(27-54a+36a^2-8a^3)+(27-54b+36b^2-8b^3)+$$
$$+(27-54c+36c^2-8c^3)\geq 27-24$$

Or

$$-54(a+b+c)+36(a^2+b^2+c^2)-8(a^3+b^3+c^3)\geq 3-3\cdot 27 \text{ so}$$
$$36(a^2+b^2+c^2)-8(a^3+b^3+c^3)\geq 3-3\cdot 27+3\cdot 54=84$$

Namely $9(a^2+b^2+c^2)-2(a^3+b^3+c^3)\geq 21$

Equality holds when $a=b=c=1$.

Solution 2 by George Florin Șerban-Romania

$$p=\sum_{cyc}a=3, \sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab=9-2q; q=\sum_{cyc}ab$$

$$\sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3\prod_{cyc}(a+b) = p(p^2-3q)+3r =; \ (r=abc)$$

$$= 3(9-3q)+3r = 27-9q+3r$$

$$9\sum_{cyc} a^2 - 2\sum_{cyc} a^3 = 9(9-2q)-2(27-9q+3r) = 27-6r \overset{(*)}{\geq} 21$$

$$(*) \Leftrightarrow 6r \leq 6 \rightarrow r \leq 1; (**)$$

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = \left(\frac{3}{3}\right)^3 = 1 \rightarrow r \leq 1 \rightarrow (**) \rightarrow (*) \text{ true.}$$

Solution 3 by Samir Cabiyev-Azerbaijan

$$(a+b+c)(a^2+b^2+c^2) = a^3+b^3+c^3+ab^2+ac^2+ba^2+bc^2+$$

$$a^2c+b^2c+(abc+abc+abc)-3abc =$$

$$= a^3+b^3+c^3+ab(a+b+c)+bc(a+b+c)+ca(a+b+c)-3abc =$$

$$= a^3+b^3+c^3+(a+b+c)(ab+bc+ca)-3abc$$

$$a^2+b^2+c\&^2 = (a+b+c)^2-2(ab+bc+ca)$$

Else, $9(a^2+b^2+c^2)-2(a^3+b^3+c^3) =$

$$= 9(a+b+c)^2-18(ab+bc+ca)-$$

$$-2\Big((a+b+c)(a^2+b^2+c^2)-(a+b+c)(ab+bc+ca)-3abc\Big) =$$

$$= 9\cdot 3^2-18(ab+bc+ca)-6(a^2+b^2+c^2-(ab+bc+ca)-abc) =$$

$$= 81-18(ab+bc+ca)-6((a+b+c)^2-3(ab+bc+ca)-abc) =$$

$$= 81-54-18(ab+bc+ca)+18(ab+bc+ca)-6abc =$$

$$= 27-6abc; a+b+c=3$$

Equality holds for $a=b=c=1$

Solution 4 by Nikos Ntorvas-Greece

$$a^3+b^3+c^3 = (a+b+c)^3-3[(a+b+c)(ab+bc+ca)-abc]$$

$$a^2+b^2+c^2 = (a+b+c)^2-2(ab+bc+ca)$$

We denote: $p=a+b+c, q=ab+bc+ca, r=abc$

$$LHS = 9(p^2-2q)-2[p^3-3(pq-r)] \geq 21$$

$$9p^2-18q-2p^3+6pq-6r\ge 21$$

$$9\cdot 3^2-18q-2\cdot 3^3+18q-6r\ge 21$$

$-6r+27\ge 21\Leftrightarrow r-1\le 0$ which is true from:

$$a+b+c\ge 3\sqrt[3]{abc}\Leftrightarrow 3\ge 3\sqrt[3]{abc}\Leftrightarrow 1\ge\sqrt[3]{abc}\Leftrightarrow 1\ge abc=r.$$

Equality holds for $a=b=c=1$

Solution 5 by Angel Plaza-Spain

By changing variables by $a=3x, b=3y, c=3z$, the problem becomes to proving that $81(x^2+y^2+z^2)-54(x^3+y^3+z^3)\ge 21$, where x,y,z are positive numbers with $x+y+z=1$. The inequality becomes homogeneous by multiplying by the appropriate power of $(x+y+z)$:

$81(x^2+y^2+z^2)(x+y+z)-54(x^3+y^3+z^3)\ge 21(x+y+z)^3$, which is equivalent to $6x^3+6y^3+6z^3+18x^2(y+z)+18y^2(x+z)+18z^2(x+y)\ge 126xyz$ which follows by the weighted arithmetic mean geometric mean inequality.

SP.361 Let $(x_n)_{n\ge 1},\ x_1=1$ such that

$$n^2[2(x_{n+1}-x_n-1)-n^2]+2x_n=n[3(n^2+x_n)-x_{n+1}]$$

Find:

$$\lim_{n\to\infty}\left(1+\frac{1}{2^n}\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\right)^{\frac{x_n}{n^2}}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$n^2[2(x_{n+1}-x_n-1)-n^2]+2x_n=n[3(n^2+x_n)-x_{n+1}]$$

$$2n^2(x_{n+1}-x_n-1)+n(x_{n+1}-3x_n)+2x_n=n^4+3n^3\Leftrightarrow$$

$$\frac{2n+1}{(n+1)(n+2)}x_{n+1}-\frac{2n-1}{n(n+1)}x_n=n$$

$$\sum_{k=1}^{n}\left(\frac{2k+1}{(k+1)(k+2)}x_{k+1}-\frac{2k-1}{k(k+1)}x_k\right)=\frac{n(n+1)}{2}\Leftrightarrow$$

$$x_n=\frac{n(n+1)(n^2-n+1)}{2(2n-1)}$$

$$S_n=\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\Rightarrow$$

$$S_n-S_{n+1}=\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}-\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}}{2k+1}=$$

$$=\frac{1}{2n+1}+\sum_{k=1}^{n-1}\frac{\binom{n}{k}-\binom{n-1}{k}}{2k+1}=\frac{1}{2n+1}+\sum_{k=1}^{n-1}\frac{\binom{n-1}{k-1}}{2k+1}=\sum_{k=1}^{n}\frac{\binom{n-1}{k-1}}{2k+1}=$$

$$=\sum_{k=1}^{n}\frac{\frac{k}{n}\binom{n}{k}}{2k+1}=\frac{1}{2n}\sum_{k=0}^{n}\frac{2k}{2k+1}\binom{n}{k}=\frac{1}{2n}\sum_{k=0}^{n}\left(1-\frac{1}{2k+1}\right)\binom{n}{k}=\frac{1}{2n}(2^n-S_n)\Rightarrow$$

$$\left(1+\frac{1}{2n}\right)S_n=S_{n-1}+\frac{2^n}{2n}\Rightarrow S_n=\frac{2n}{2n+1}S_{n-1}+\frac{2^n}{2n+1}$$

$$P(n): S_n\ge\frac{2^n}{2n},\qquad Q(n): S_n\le\frac{2^n}{n-1},\forall n\ge 4$$

$$P(4):\ S_4\ge\frac{2^4}{4}\Leftrightarrow\sum_{k=0}^{4}\frac{\binom{4}{k}}{2k+1}\ge 4\ (true)$$

If $P(k-1), k\ge 5$ is true, thus

$$S_k=\frac{2k}{2k+1}S_{k-1}+\frac{2^k}{2k+1}>\frac{2k}{2k+1}\cdot\frac{2^{k-1}}{k-1}=\frac{2^k}{2k+1}\left(\frac{k}{k+1}+1\right)=$$

$$=\frac{2^k(2k-1)}{k(2k-1)-1}>\frac{2^k}{k}\Rightarrow P(n)\text{ is true for all } n\ge 4.$$

$$Q(4):\ S_4\le\frac{2^4}{3}\Leftrightarrow\sum_{k=0}^{4}\frac{\binom{4}{k}}{2k+1}\le\frac{2^4}{3}\ (true).$$

If $Q(k-1),\forall k\ge 5$ is true, thus

$$s_k\le\frac{2k}{2k+1}\cdot\frac{2^{k-1}}{k-2}+\frac{2^k}{2k+1}=\frac{2^k}{2k+1}\left(\frac{k}{k-2}+1\right)=\frac{2^k}{2k+1}\cdot\frac{2k-2}{k-2}=$$

$$=\frac{2^k(2k-2)}{(k-1)(2k-2)+k-4}\le\frac{2^k}{k-1}\Rightarrow Q(k)\text{ is true for all } n\ge 4.\text{ So,}$$

$$\frac{2^n}{n}\le\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\le\frac{2^n}{n-1}\Rightarrow\frac{1}{n}\le\frac{1}{2^n}\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\le\frac{1}{n-1}$$

$$\left(1+\frac{1}{n}\right)^{\frac{x_n}{n^2}} \le \left(1+\frac{1}{2^n}\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\right)^{\frac{x_n}{n^2}} \le \left(1+\frac{1}{n-1}\right)^{\frac{x_n}{n^2}}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{x_n}{n^2}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n(n+1)(n^2-n+1)}{2n^2(2n-1)}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{(n+1)(n^2-n+1)}{2n^2(2n-1)}} = \sqrt[4]{e}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{\frac{x_n}{n^2}} = \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{\frac{n(n+1)(n^2-n+1)}{2n^2(2n-1)}} =$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n-1}\right)^{n-1}\right]^{\frac{n(n+1)(n^2-n+1)}{2n^2(2n-1)(n-1)}} = \sqrt[4]{e}$$

$$\Rightarrow \Omega = \lim_{n\to\infty}\left(1+\frac{1}{2^n}\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\right)^{\frac{x_n}{n^2}} = \sqrt[4]{e}$$

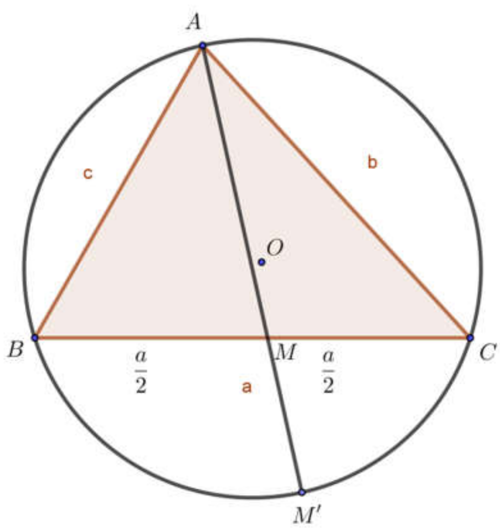
SP.362 Let m_a, m_b, m_c be the lengths of the medians of a triangle ΔABC. Prove

$$\frac{4\sqrt{3}}{3R} \le \frac{cscA}{m_a} + \frac{cscB}{m_b} + \frac{cscC}{m_c} \le \frac{\sqrt{3}R}{3r^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let a, b, c be the lengths of the sides BC, CA, AB, respectively. We have:



$$b^2+c^2 = 2AM^2 + \frac{BC^2}{2} = 2AM^2 + 2BM^2$$
$$= 2(AM^2 + BM\cdot MC) =$$
$$= 2(AM^2 + AM\cdot MM') = 2AM(AM+MM')$$
$$= 2AM\cdot AM'$$

So, $b^2+c^2 = 2m_a\cdot AM'$; $AM' \le 2R$. Namely, $b^2+c^2 \le 4m_aR$ or

$4m_aR \ge b^2+c^2 \ge 2bc$; $(by\ AM-GM)$. We know that: $F=\frac{1}{2}bc\cdot sinA$. So,

$4m_aR\ge\frac{4F}{sinA}$ or $\frac{1}{m_asinA}\le\frac{R}{F}$. Also, we have: $F=rs$ and $s\ge 3\sqrt{3}r$. So,

$\frac{1}{m_asinA}\le\frac{R}{rs}\le\frac{\sqrt{3}R}{9r^2}$ or $\frac{cscA}{m_a}\le\frac{\sqrt{3}R}{9r^2}$, and similarly: $\frac{cscB}{m_b}\le\frac{\sqrt{3}R}{9r^2}$, $\frac{cscC}{m_c}\le\frac{\sqrt{3}R}{9r^2}$

Now, using AM-GM inequality, we have:

$$\frac{1}{m_asinA}+\frac{1}{m_bsinB}+\frac{1}{m_csinC}\ge\frac{3}{\sqrt[3]{m_am_bm_c}\cdot\sqrt[3]{sinAsinBsinC}}\ge$$

$$\ge\frac{3}{\frac{m_a+m_b+m_c}{3}\cdot\frac{sinA+sinB+sinC}{3}}=\frac{27}{(m_a+m_b+m_c)(sinA+sinB+sinC)}$$

We know that $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)\le\frac{3}{4}\cdot 9R^2=\frac{27}{4}R^2$, so

$(m_a+m_b+m_c)^2\le 3(m_a^2+m_b^2+m_c^2)$, namely

$(m_a+m_b+m_c)^2\le\frac{81}{4}R^2$ or $m_a+m_b+m_c\le\frac{9}{2}R$. Also, it is well-known that:

$$sinA+sinB+sinC\le\frac{3\sqrt{3}}{2}$$

Hence, $\frac{cscA}{m_a}+\frac{cscB}{m_b}+\frac{cscC}{m_c}\ge\frac{27}{\frac{9}{2}R\cdot\frac{3\sqrt{3}}{2}}=\frac{4\sqrt{3}}{3R}$

Therefore,

$$\frac{4\sqrt{3}}{3R}\le\frac{cscA}{m_a}+\frac{cscB}{m_b}+\frac{cscC}{m_c}\le\frac{\sqrt{3}R}{3r^2}$$

Solution 2 by Aggeliki Papaspyropoulos-Greece

$$\frac{4\sqrt{3}}{3R}\le\frac{cscA}{m_a}+\frac{cscB}{m_b}+\frac{cscC}{m_c}\le\frac{\sqrt{3}R}{3r^2};(1)$$

$$\csc A=\frac{1}{\sin A};\csc B=\frac{1}{\sin B};\csc C=\frac{1}{\sin C}$$

$$(1)\to\frac{4\sqrt{3}}{3R}\le\sum_{cyc}\frac{1}{m_a\cdot\sin A}\le\frac{\sqrt{3}R}{3r^2};(2)$$

$$\frac{4\sqrt{3}}{3R}\le\sum_{cyc}\frac{2R}{a\cdot m_a}\le\frac{\sqrt{3}R}{3r^2};(3)$$

$$(I): \frac{4\sqrt{3}}{3R} \le \sum_{cyc} \frac{2R}{a \cdot m_a} \Leftrightarrow \sum_{cyc} \frac{1}{a \cdot m_a} \ge \frac{2\sqrt{3}}{3R^2}; (4)$$

$$a \cdot m_a \le \frac{R}{2r} \cdot ah_a; \left(\because m_a \le \frac{R}{2r} h_a (Panaitopol)\right)$$

$$\to a \cdot m_a \le \frac{R \cdot F}{r}, (\, F = area\ of\ \Delta ABC) \to \frac{1}{a \cdot m_a} \ge \frac{r}{R \cdot F} = \frac{r}{R \cdot rs} = \frac{1}{Rs}$$

$$\sum_{cyc} \frac{1}{a \cdot m_a} \ge \frac{3}{Rs}.$$

So it is enough to prove that: $\frac{3}{Rs} \ge \frac{2\sqrt{3}}{3R^2}$, from $(4) \to 9R \ge 2s\sqrt{3}; (5)$

$$\frac{R}{2} \ge \frac{s}{3\sqrt{3}}; (Mitrinovic)$$

$So,\quad R \ge \frac{2s}{3\sqrt{3}} \to 9R \ge \frac{18s}{3\sqrt{3}} = \frac{6s}{\sqrt{3}} = 2\sqrt{3}s \to 9R \ge 2s\sqrt{3} \to (I)$ is true.

$$(II): \sum_{cyc} \frac{2R}{a \cdot m_a} \le \frac{\sqrt{3}R}{3r^2} \Leftrightarrow \sum_{cyc} \frac{1}{a \cdot m_a} \le \frac{\sqrt{3}}{6r^2}$$

$\because m_a \ge \frac{b^2 + c^2}{4R}$, we have:

$$a \cdot m_a \ge \frac{a(b^2 + c^2)}{4R} \to \frac{1}{a \cdot m_a} \le \frac{4R}{a(b^2 + c^2)} \le \frac{4R}{2abc} = \frac{2R}{abc}$$

$$\to \sum_{cyc} \frac{1}{a \cdot m_a} \le \frac{6R}{abc} = \frac{6R}{4R \cdot rs} = \frac{3}{2 \cdot rs}$$

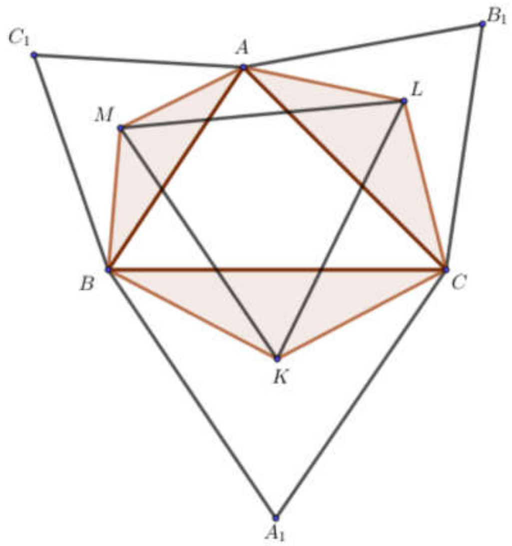
So, we have to prove: $\frac{3}{2rs} \le \frac{\sqrt{3}}{6r^2} \Leftrightarrow \frac{\sqrt{3}}{s} \le \frac{1}{3r} \Leftrightarrow r \le \frac{s}{3\sqrt{3}}$ $(Mitrinovic) \to (II)$ is true.

SP.363 Triangle ABC has $|BC| = a, |CA| = b, |AB| = c$, inradius r and circumradius R. Equilateral triangles A_1BC, B_1CA and C_1AB with centroids K, L and M respectively, are drawn externally to triangle ABC. Prove that:

$$3\sqrt{3} \le [ALM] + [BMK] + [CKL] \le \frac{3\sqrt{3}}{4} R^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer



We have:

$$AM = BM = \frac{2}{3}\cdot\frac{c\sqrt{3}}{2} = \frac{c\sqrt{3}}{3},$$

$$AM = CK = \frac{a\sqrt{3}}{2} \text{ and } CL = AL = \frac{b\sqrt{3}}{3}$$

$$\angle LAM = 30^\circ + A + 30^\circ = 60^\circ + A$$

$$\angle MBK = 60^\circ + B, \angle KCL = 60^\circ + C$$

Also, we have:

$$[ALM] = \frac{1}{2}AM\cdot AL\cdot sin(LAM) =$$

$$= \frac{1}{2}\cdot\frac{c\sqrt{3}}{3}\cdot\frac{b\sqrt{3}}{3}\cdot sin(60^\circ + A) =$$

$$= \frac{bc}{6}\left(\frac{\sqrt{3}}{2}cosA + \frac{1}{2}sinA\right) = \frac{bc}{6}\left(\frac{\sqrt{3}}{2}\cdot\frac{b^2+c^2-a^2}{2bc} + \frac{1}{2}\cdot\frac{a}{2R}\right) =$$

$$= \frac{\sqrt{3}}{24}(b^2 + c62 - a^2) + \frac{abc}{6\cdot 4R} = \frac{\sqrt{3}}{24}(b^2+c^2-a^2) + \frac{[ABC]}{6}$$

So, $[ALM] = \frac{\sqrt{3}}{24}(b^2+c^2-a^2) + \frac{[ABC]}{6}$**. Similarly, we have:**

$[BMK] = \frac{\sqrt{3}}{24}(a^2+c^2-b^2) + \frac{[ABC]}{6}$ **and** $[CKL] = \frac{\sqrt{3}}{24}(a^2+b^2-c^2) + \frac{[ABC]}{6}$**. So,**

$$[ALM] + [BMK] + [CKL] = \frac{\sqrt{3}}{24}(a^2+b^2+c^2) + \frac{[ABC]}{2}$$

We know that $a^2+b^2+c^2 \le 9R^2$**, then** $s = 3s - 2s = (s-a)+(s-b)+(s-c)$

By AM-GM we have: $s \ge 3\sqrt[3]{(s-a)(s-b)(s-c)} \Leftrightarrow s^3 \ge 27(s-a)(s-b)(s-c)$

$\Leftrightarrow s^4 \ge 27s(s-a)(s-b)(s-c)$**. We know that:** $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$

$s^4 \ge 27[ABC]^2 \Leftrightarrow s^2 \ge 3\sqrt{3}[ABC] \Leftrightarrow [ABC] \le \frac{s^2}{3\sqrt{3}}$**. Also, we know that:**

$s = \frac{3\sqrt{3}}{2}R$**, so** $[ABC] \le \frac{3\sqrt{3}}{4}R^2$**. Now, we have:**

$$[ALM + [BKM] + [CKL] = \frac{\sqrt{3}}{24}(a^2+b^2+c^2) + \frac{[ABC]}{2} \le$$

$$\leq \frac{\sqrt{3}}{24}\cdot 9R^2+\frac{3\sqrt{3}}{8}=\frac{3\sqrt{3}}{4}R^2$$

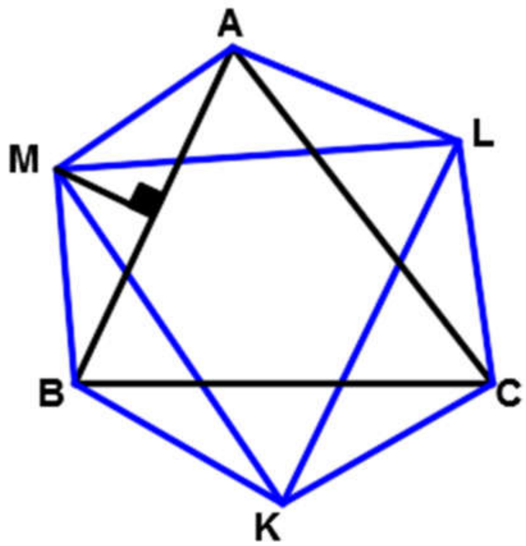
Also, we have: $[ALM+[BKM]+[CKL]=\frac{\sqrt{3}}{24}(a^2+b^2+c^2)+\frac{[ABC]}{2}\geq$

$$\geq \frac{\sqrt{3}}{24}\cdot 36r^2+\frac{rs}{2}\geq \frac{\sqrt{3}}{24}\cdot 36r^2+\frac{r(3\sqrt{3}r)}{2}=\frac{3\sqrt{3}}{2}r^2+\frac{3\sqrt{3}}{2}r^2=3\sqrt{3}r^2$$

Therefore,

$$3\sqrt{3}\leq [ALM]+[BMK]+[CKL]\leq \frac{3\sqrt{3}}{4}R^2$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ have: AM=\frac{\frac{AB}{2}}{\cos MAB}=\frac{c}{2\cos\frac{\pi}{6}}=\frac{\sqrt{3}}{3}c, similarly: AL=\frac{\sqrt{3}}{3}b, and\ \mu(MAL)=A+\frac{\pi}{3}.$$

$$\to [ALM]=\frac{1}{2}.AL.AM.\sin MAL=\frac{1}{2}.\frac{\sqrt{3}}{3}b.\frac{\sqrt{3}}{3}c.\sin\left(A+\frac{\pi}{3}\right)$$

$$=\frac{1}{6}bc\left(\frac{1}{2}\sin A+\frac{\sqrt{3}}{2}\cos A\right)=$$

$$=\frac{1}{12}.\frac{abc}{2R}+\frac{\sqrt{3}}{12}bc.\frac{b^2+c^2-a^2}{2bc}=\frac{1}{6}F+\frac{\sqrt{3}}{24}(b^2+c^2-a^2).\quad (\because F=[ABC])$$

$$\to [ALM]=\frac{1}{6}F+\frac{\sqrt{3}}{24}(b^2+c^2-a^2)\quad (and\ analogs)$$

$$\to \sum [ALM] = \sum \left[\frac{1}{6}F + \frac{\sqrt{3}}{24}(b^2 + c^2 - a^2)\right] = \frac{1}{2}F + \frac{\sqrt{3}}{24}\sum a^2.$$

$$\text{Now,} \quad \sum [ALM] \overset{\text{Ionescu Weitzenbock}}{\geq} \frac{1}{2}F + \frac{\sqrt{3}}{24}.4\sqrt{3}F = F = sr \overset{\text{Mitrinovic}}{\geq} 3\sqrt{3}r^2 \ (1).$$

$$\text{And}: \sum [ALM] \overset{\text{Leibniz}}{\leq} \frac{1}{2}sr + \frac{\sqrt{3}}{24}.9R^2 \overset{\substack{\text{Euler}\\ \text{Mitrinovic}}}{\leq} \frac{1}{2}.\frac{3\sqrt{3}}{2}R.\frac{1}{2}R + \frac{3\sqrt{3}}{8}R^2$$

$$= \frac{3\sqrt{3}}{4}R^2 \ (2).$$

$$(1),(2) \to 3\sqrt{3}r^2 \leq \sum [ALM] \leq \frac{3\sqrt{3}}{4}R^2.$$

SP.364 Let ABC be a non-right triangle with circumradius R. Squares with sides AB, BC, CA and centroids K, L, M respectively, are drawn externally to triangle ABC. Let α, β, γ be the distance from the vertices A, B, C to the segments $\overline{KM}, \overline{KL}, \overline{LM}$, respectively. Prove that:

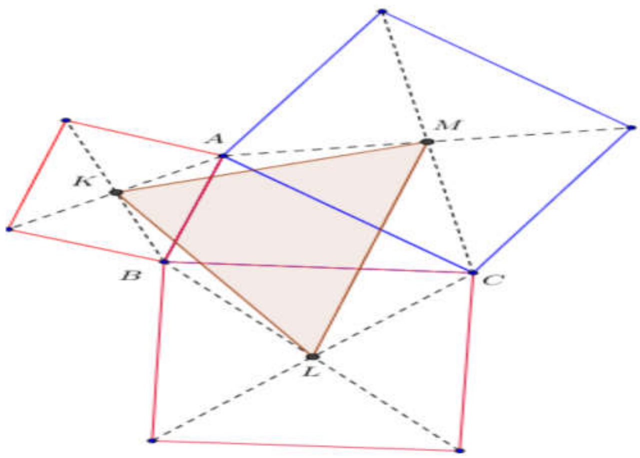
$$\left(\frac{\cot A}{\alpha}\right)^2 + \left(\frac{\cot B}{\beta}\right)^2 + \left(\frac{\cot C}{\gamma}\right)^2 \geq \frac{8 \cdot \tan 75^\circ}{3R^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let a, b, c be the lengths of the sides BC, CA, AB of ΔABC, respectively.

We have $AK = \frac{c\sqrt{2}}{2}, AM = \frac{b\sqrt{2}}{3}$ and using the law of the cosines in ΔAKH, we have:



$$KM^2 = AK^2 + AM^2 - 2AK \cdot AM\cos(90^\circ + A)$$

$$= \frac{b^2 + c^2}{2} + bc \cdot \frac{a}{2R} = \frac{b^2 + c^2}{2} + 2F,$$

Where $F = area(\Delta ABC) = \frac{abc}{4R}$.

So, $KM^2=\frac{b^2+c^2}{2}+2F, KL^2=\frac{c^2+a^2}{2}+2F, LM^2=\frac{a^2+b^2}{2}+2F$

Now, we have $area(\Delta AKM)=\frac{1}{2}KM\cdot\alpha=\frac{\alpha}{2}\sqrt{\frac{b^2+c^2}{2}+2F}=\frac{1}{2}AK\cdot AM\cdot\sin(KAM)$

But $\alpha\sqrt{\frac{b^2+c^2}{2}+2F}=\frac{c\sqrt{2}}{2}\cdot\frac{b\sqrt{2}}{2}\cdot cosA$, so $\alpha\sqrt{\frac{b^2+c^2}{2}+2F}=\frac{bc}{2}\cdot\frac{b^2+c^2-a^2}{2bc}$

$$\alpha\cdot\sqrt{\frac{b^2+c^2}{2}+2F}=\frac{b^2+c^2-a^2}{4}\Leftrightarrow\alpha=\frac{b^2+c^2-a^2}{4\sqrt{\frac{b^2+c^2}{2}+2F}}\le\frac{b^2+c^2-a^2}{4\sqrt{bc+2F}}$$

We have: $F=\frac{1}{2}bc\cdot sinA\Leftrightarrow bc=\frac{2F}{sinA}$ or $\alpha\le\frac{b^2+c^2-a^2}{4\sqrt{\frac{2F}{sinA}+2F}}\Leftrightarrow\left(\frac{b^2+c^2-a^2}{\alpha}\right)^2\ge 32F\left(\frac{1}{sinA}+1\right)$

Similarly, $\left(\frac{c^2+a^2-b^2}{\beta}\right)^2\ge 32F\left(\frac{1}{sinB}+1\right)$ and $\left(\frac{a^2+b^2-c^2}{\gamma}\right)^2\ge 32F\left(\frac{1}{sinC}+1\right)$.

Adding up these inequalities, we have:

$$\left(\frac{b^2+c^2-a^2}{\alpha}\right)^2+\left(\frac{c^2+a^2-b^2}{\beta}\right)^2+\left(\frac{a^2+b^2-c^2}{\gamma}\right)^2$$
$$\ge 32F\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}+3\right)$$

We know that: $\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\ge 2\sqrt{3}$, in any ΔABC.So,

$$\left(\frac{b^2+c^2-a^2}{\alpha}\right)^2+\left(\frac{c^2+a^2-b^2}{\beta}\right)^2+\left(\frac{a^2+b^2-c^2}{\gamma}\right)^2\ge 3F\left(2\sqrt{3}+3\right)$$

From the law of cosines in ΔABC, we have:

$$a^2=b^2+c^2-2bc\cdot cosA\Leftrightarrow b^2+c^2-a^2=2bc\cdot cosA$$

$b^2+c^2-a^2=2\cdot\frac{2F}{sinA}\cdot cosA=4F\cdot cotA$ and similarly,

$c^2+a^2-b^2=4F\cdot cotB, a^2+b^2-c^2=4FcotC$. So,

$$\left(\frac{4FcotA}{\alpha}\right)^2+\left(\frac{4FcotB}{\beta}\right)^2+\left(\frac{4FcotC}{\gamma}\right)^2\ge 32F\left(2\sqrt{3}+3\right)\Leftrightarrow$$

$$\left(\frac{cotA}{\alpha}\right)^2+\left(\frac{cotB}{\beta}\right)^2+\left(\frac{cotC}{\gamma}\right)^2\ge\frac{2\left(2\sqrt{3}+3\right)}{F}$$

Let $2s=a+b+c$ is the perimeter of the ΔABC, then

$$s = 3s - 2s = s + s + s - a - b - c = (s-a)(s-b)(s-c)$$

By AM-GM: $s \geq 3\sqrt[3]{(s-a)(s-b)(s-c)} \Leftrightarrow s^3 \geq 27(s-a)(s-b)(s-c)$

$\Leftrightarrow s^4 \geq 27s(s-a)(s-b)(s-c)$. We know that $F = \sqrt{s(s-a)(s-b)(s-c)}$, so

$s^4 \geq 27F^2$or$F \leq \frac{s^2}{27}$ and $s \leq \frac{3\sqrt{3}}{2}R$, namely $F \leq \frac{3\sqrt{3}}{4}R^2$. Now, we have:

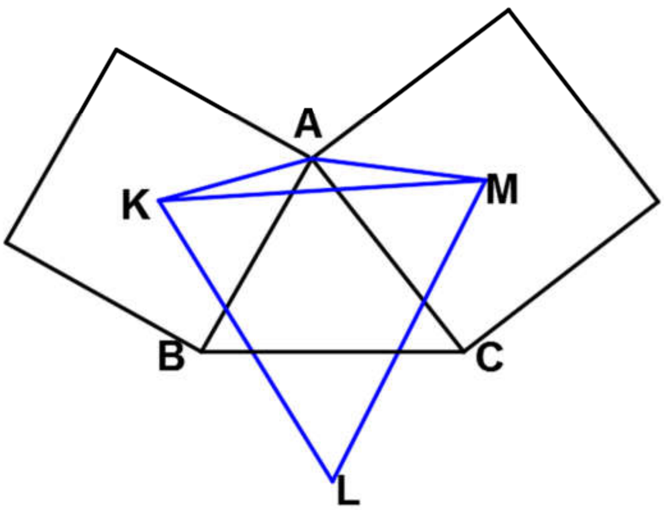
$$\left(\frac{cotA}{\alpha}\right)^2 + \left(\frac{cotB}{\beta}\right)^2 + \left(\frac{cotC}{\gamma}\right)^2 \geq \frac{2(2\sqrt{3}+3)}{F} \geq \frac{4\sqrt{3}+6}{\frac{3\sqrt{3}}{4}R^2} = \frac{24(2+\sqrt{3})}{9R^2} = \frac{8(2+\sqrt{3})}{3R^2}$$

Also, we know that $tan75^\circ = 2 + \sqrt{3}$, so

$$\left(\frac{cotA}{\alpha}\right)^2 + \left(\frac{cotB}{\beta}\right)^2 + \left(\frac{cotC}{\gamma}\right)^2 \geq \frac{8 \cdot tan75^\circ}{3R^2}$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ have: AK = \frac{\sqrt{2}}{2}c, AM = \frac{\sqrt{2}}{2}b \text{ and } \mu(KAM) = A + \frac{\pi}{2}.$$

$$\rightarrow [KAM] = \frac{1}{2}.AK.AM.\sin(KAM) = \frac{1}{4}bc.\sin\left(A + \frac{\pi}{2}\right) = \frac{1}{4}bc.\cos A.$$

$$And:\ KM^2 = AK^2 + AM^2 - 2.AK.AM.\cos(KAM) = \frac{1}{2}(b^2+c^2) - bc.\cos\left(A + \frac{\pi}{2}\right) =$$

$$= \frac{1}{2}(b^2+c^2) + bc.\sin A = \frac{1}{2}(b^2+c^2) + bc.\frac{a}{2R} = \frac{1}{2}(b^2+c^2) + 2F.\ (\because F = [ABC])$$

$$\to \alpha^2 = \left(\frac{2.[KAM]}{KM}\right)^2 = \frac{b^2c^2.\cos^2 A}{2(b^2+c^2+4F)} \to \left(\frac{\cot A}{\alpha}\right)^2 = \frac{2(b^2+c^2+4F)}{b^2c^2.\sin^2 A}$$

$$= \frac{8R^2(b^2+c^2+4F)}{(abc)^2} \to \left(\frac{\cot A}{\alpha}\right)^2 = \frac{b^2+c^2+4F}{2F^2} \quad (And\ analogs)$$

$$\to \sum\left(\frac{\cot A}{\alpha}\right)^2 = \sum\frac{b^2+c^2+4F}{2F^2} = \frac{1}{F^2}\left(\sum a^2 + 6F\right) \overset{Ionescu\,Weitzenbock}{\geq} \frac{4\sqrt{3}F+6F}{F^2} =$$

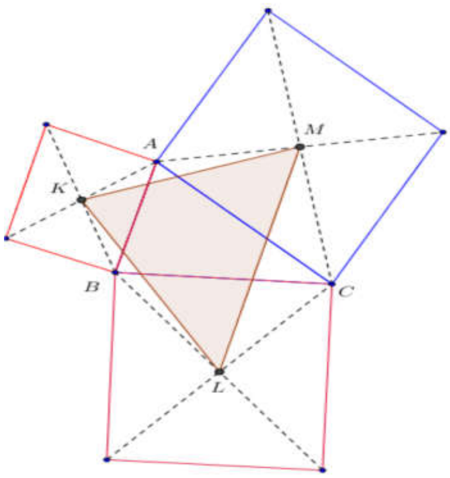
$$= \frac{4\sqrt{3}+6}{sr} \overset{\substack{Euler\\Mitrinovic}}{\geq} \frac{4\sqrt{3}+6}{\frac{3\sqrt{3}R}{2}.\frac{R}{2}} = \frac{8(2+\sqrt{3})}{3R^2}$$

$$We\ have:\ \tan^2 75^\circ = \frac{1-\cos(2.75)}{1+\cos(2.75)} = \frac{1-\cos(90+60)}{1+\cos(90+60)} = \frac{1+\sin 60^\circ}{1-\sin 60^\circ} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$$= (2+\sqrt{3})^2 \to \tan 75^\circ = 2+\sqrt{3} \quad \to \sum\left(\frac{\cot A}{\alpha}\right)^2 \geq \frac{8.\tan 75^\circ}{3R^2}.$$

Solution 3 by Aggeliki Papaspyropoulou-Greece

Working in triangle $AKM, \angle KAM = 90^\circ + A$



$$AK = \frac{c\sqrt{2}}{2}, AM = \frac{b\sqrt{2}}{2}$$

$$KM^2 = \left(\frac{c\sqrt{2}}{2}\right)^2 + \left(\frac{b\sqrt{2}}{2}\right)^2 - 2\cdot\frac{c\sqrt{2}}{2}\cdot\frac{b\sqrt{2}}{2}\cdot\cos\widehat{KAM}$$

$$KM^2 = \frac{1}{2}(c^2+b^2) - bc\cdot(-\sin A)$$

$$\to KM^2 = \frac{1}{2}(b^2+c^2) + bc\cdot\sin A$$

Similarly,

$$LM^2 = \frac{1}{2}(a^2+b^2) + ab\cdot\sin C$$

$$LK^2 = \frac{1}{2}(c^2+a^2) + ac\cdot\sin B$$

$$\alpha\cdot KM = \frac{bc}{2}\sin\left(\frac{\pi}{2}+A\right) = \frac{bc}{2}\cos A$$

$$\to \alpha = \frac{bc\cdot\cos A}{2KM} \to \frac{1}{\alpha} = \frac{2KM}{bc\cdot\cos A}$$

So, $\frac{\cot A}{\alpha}=\frac{\cos A}{\sin A}\cdot\frac{2KM}{bc\cdot\cos A}=\frac{2KM}{bc\cdot\sin A}=\frac{2KM}{2F}=\frac{KM}{F}$

Because $bc\cdot\sin A=2F, F=[ABC]$ −area of triangle ABC.

Similarly, $\frac{\cot B}{\beta}=\frac{KL}{F}$ and $\frac{\cot C}{\gamma}=\frac{ML}{F}$

$$\sum_{cyc}\left(\frac{\cot A}{\alpha}\right)^2=\frac{KM^2+KL^2+LM^2}{F^2};(2F=ab\cdot\sin C=ac\cdot\sin B)$$

$$\rightarrow\sum_{cyc}\left(\frac{\cot A}{\alpha}\right)^2=\frac{1}{F^2}\left(\frac{b^2+c^2}{2}+2F+\frac{c^2+a^2}{2}+2F+\frac{a^2+b^2}{2}+2F\right)$$

We have to prove:

$$a^2+b^2+c^2+6F\geq\frac{8F^2\cdot\tan 75^\circ}{3R^2};(1)$$

$$\tan 75^\circ=2+\sqrt{3};(2), 3R^2\geq\frac{a^2+b^2+c^2}{3};(3)(Leibniz)$$

So, by $(2),(3)$ it is enough to prove that:

$$\frac{(a^2+b^2+c^2)^2}{3}+6F\cdot\frac{a^2+b^2+c^2}{3}\geq 8F^2\cdot(2+\sqrt{3});(4)$$

$$\Leftrightarrow\frac{(a^2+b^2+c^2)^2}{F^2}+\frac{6(s^2+b^2+c^2)}{F}\geq 24(2+\sqrt{3});(5)$$

Let $x=\frac{a^2+b^2+c^2}{F}$, then $(5)\Leftrightarrow x^2+6x-24(2+\sqrt{3})\geq 0;(6)$

$$\Leftrightarrow(x-4\sqrt{3})(x+6+4\sqrt{3})\geq 0;(7)$$

$a^2+b^2+c^2\geq 4\sqrt{3}F\rightarrow\ x\geq 4\sqrt{3}\rightarrow(7)$ is true.

SP.365. Let $f,F:[a,b]\rightarrow\mathbb{R}$, such that $F(x)=-f(x)+cosf(x)$. If F −is Riemann integrable, prove that f −is Riemann integrable.

Proposed by Cristian Miu-Romania

Solution by proposer

Let us prove first two things. The first things is that the function $x\rightarrow -x+cosx$ is injective. The second thing is that if $(x_n)_n$ −is a sequence and $-x_n+cox_n$ is convergent, then $(x_n)_n$ −is convergent.

To ptove that $x \to -x + cosx$ is injective we shall prove that $x \to x + sinx$ is injective.

From $x_1 + sinx_1 = x_2 + sinx_2$ we have:

$$\left|sin\frac{x_1 - x_2}{2}cos\frac{x_1 + x_2}{2}\right| = \left|\frac{x_1 - x_2}{2}\right|$$

From here we obtain $x_1 = x_2$, because $|sinx| \le |x|$ with equality if and only if $x = 0$.

So $x \to x + sinx$ is injective and because $x \to \frac{\pi}{2} - x$ is also injective, we obtain that

$x \to -x + cosx$ is injective.

Now if $-x_n + cosx_n$ is convergent, we obtain that $(x_n)_n$ –is bounded, so there exists

$\lim_{n\to\infty} inf x_n$ and $\lim_{n\to\infty} sup x_n$ and they are real numbers.

Let $l = \lim_{n\to\infty} inf x_n$ and $L = \lim_{n\to\infty} sup x_n$. Then exists x_{p_n} and x_{q_n} such as

$\lim_{n\to\infty} x_{p_n} = l$ and $\lim_{n\to\infty} x_{q_n} = L$; $\lim_{n\to\infty}(-x_n + cosx_n) = x$, so

$\lim_{n\to\infty}\left(-x_{p_n} + cosx_{p_n}\right) = x = \lim_{n\to\infty}\left(-x_{q_n} + cosx_{q_n}\right)$. So $-l + cosl = -L + cosL$.

But the function $x \to -x + cosx$ is injective. We obtain that $l = L$ which prove that

$(x_n)_n$ –is convergent. Now, we shall use Lebesque's theorem for Riemann integrability.

Because $-f(x) + cosf(x)$ is Riemann integrable, we obtain that f –is bounded.

Let x_0 –be a continuity point of F. We shall prove that x_0 is also a continuity point for f.

Let $(x_n)_n$ –be sequences such as $\lim_{n\to\infty} x_n = x_0$. Then $\lim_{n\to\infty}\left(-f(x_n) + cosf(x_n)\right) = F(x_0)$,

so we obtain that $f(x_n)$ –is convergent.

So, $\lim_{n\to\infty} f(x_n) = A$. W obtain that $-A + cosA = -f(x_0) + cosf(x_0)$ and because

$x \to -x + cosx$ is injective we obtain that $A = f(x_0)$.

According to Lebesque theorem the function f –is Riemann integrable.

SP.366 Let m_a, m_b, m_c be the medians, r_a, r_b, r_c the exradii, r inradius and R circumradius of a triangle ABC. Prove that:

$$\frac{3}{2}\left(\frac{R}{2r}\right)^{-2} \le \frac{r_a^2}{m_b^2 + m_c^2} + \frac{r_b^2}{m_c^2 + m_a^2} + \frac{r_c^2}{m_a^2 + m_b^2} \le 2\left(\frac{R}{2r}\right)^2 - \frac{1}{2}\left(\frac{R}{2r}\right)$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

First we will prove that: $r_a \le \frac{a^2}{4r}$. We have: $r_a = s \cdot tan\frac{A}{2}$, where s —is the semiperimeterof ΔABC. Let $BC = a, CA = b, AB = c$ and F – the area of ΔABC. We have:

$$\frac{a^2}{4F} = \frac{(2RsinA)^2}{4\left(\frac{1}{2}bcsinA\right)} = \frac{2R^2sinA}{bc} = \frac{2R^2sinA}{(2RsinB)(2RsinC)} = \frac{sinA}{2sinBsinC} =$$

$$= \frac{2sin\frac{A}{2}cos\frac{A}{2}}{cos(B-C) - cos(B+C)} = \frac{2sin\frac{A}{2}cos\frac{A}{2}}{cos(B-C) + cosA} \ge \frac{2sin\frac{A}{2}cos\frac{A}{2}}{1 + cosA} =$$

$$= \frac{2sin\frac{A}{2}cos\frac{A}{2}}{1 + \left(2cos^2\frac{A}{2} - 1\right)} = \frac{sin\frac{A}{2}}{cos\frac{A}{2}} = tan\frac{A}{2}$$

So, $tan\frac{A}{2} \le \frac{a^2}{4F} = \frac{a^2}{4rs}$. Namely, $r_a \le \frac{a^2}{4r}$. Similarly, $r_b \le \frac{b^2}{4r}, r_c \le \frac{c^2}{4r}$. Now, $\frac{r_a^2}{m_b^2+m_c^2} \le \frac{r_a^2}{r_ar_c+r_ar_b} = \frac{r_a}{r_b+r_c}$, because $m_a^2 \ge r_br_c, m_b^2 \ge r_cr_a, m_c^2 \ge r_ar_b$. So,

$$\frac{r_a^2}{m_b^2+m_c^2} + \frac{r_b^2}{m_c^2+m_a^2} + \frac{r_c^2}{m_a^2+m_b^2} \le \frac{r_a}{r_b+r_c} + \frac{r_b}{r_c+r_a} + \frac{r_c}{r_a+r_b} \le$$

$$\le \frac{\frac{a^2}{4r}}{\frac{F}{s-a} + \frac{F}{s-b}} + \frac{\frac{b^2}{4r}}{\frac{F}{s-c} + \frac{F}{s-a}} + \frac{\frac{c^2}{4r}}{\frac{F}{s-a} + \frac{F}{s-b}} =$$

$$= \frac{\frac{a^2}{4r}}{F\left(\frac{1}{s-a} + \frac{1}{s-b}\right)} + \frac{\frac{b^2}{4r}}{F\left(\frac{1}{s-c} + \frac{1}{s-a}\right)} + \frac{\frac{c^2}{4r}}{F\left(\frac{1}{s-a} + \frac{1}{s-b}\right)} =$$

$$= \frac{\frac{a^2}{4r}}{F \cdot \frac{s}{(s-b)(s-c)}} + \frac{\frac{b^2}{4r}}{F \cdot \frac{s}{(s-c)(s-a)}} + \frac{\frac{c^2}{4r}}{F \cdot \frac{s}{(s-a)(s-b)}} =$$

$$= \frac{a(s-b)(s-c) + b(s-c)(s-a) + c(s-a)(s-b)}{4rF} =$$

$$= \frac{a\left(\frac{s-a+s-b}{2}\right)^2 + b\left(\frac{s-c+s-a}{2}\right)^2 + c\left(\frac{s-a+s-b}{2}\right)^2}{4rF} =$$

$$=\frac{a\cdot\frac{a^2}{4}+b\cdot\frac{b^2}{4}+c\cdot\frac{c^2}{4}}{4rF}=\frac{a^3+b^3+c^3}{16rF}$$

It is well-known that $a^3+b^3+c^3=2s(s^2-3r^2-6Rr)$ and

$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$. So,

$$\frac{r_a^2}{m_b^2+m_c^2}+\frac{r_b^2}{m_c^2+m_a^2}+\frac{r_c^2}{m_a^2+m_b^2}=\frac{a^3+b^3+c^3}{16rF}=\frac{2s(s^2-3r^2-6Rr)}{16r\cdot(rs)}=$$

$$=\frac{s^2-3r^2-6Rr}{8r^2}\le\frac{(4R^2+4Rr+3r^2)-3r^2-6Rr}{8r^2}=\frac{4R^2-2Rr}{8r^2}=$$

$$=2\left(\frac{R}{2r}\right)^2-\frac{1}{2}\left(\frac{R}{2r}\right)$$

Now, using Cauchy-Schwarz Inequality, we have:

$$\frac{r_a^2}{m_b^2+m_c^2}+\frac{r_b^2}{m_c^2+m_a^2}+\frac{r_c^2}{m_a^2+m_b^2}\ge\frac{(r_a+r_b+r_c)^2}{2(m_a^2+m_b^2+m_c^2)}$$

We know that: $r_a+r_b+r_c=4R+r; a^2+b^2+c^2\le 9R^2$ and

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)\le\frac{27}{4}R^2$$

So,

$$\frac{r_a^2}{m_b^2+m_c^2}+\frac{r_b^2}{m_c^2+m_a^2}+\frac{r_c^2}{m_a^2+m_b^2}\ge\frac{(4R+r)^2}{2\cdot\frac{27}{4}R^2}\ge\frac{2\cdot 81r^2}{27R^2}=\frac{3}{2}\left(\frac{R}{2r}\right)^{-2}$$

Namely,

$$\frac{3}{2}\left(\frac{R}{2r}\right)^{-2}\le\frac{r_a^2}{m_b^2+m_c^2}+\frac{r_b^2}{m_c^2+m_a^2}+\frac{r_c^2}{m_a^2+m_b^2}\le 2\left(\frac{R}{2r}\right)^2-\frac{1}{2}\left(\frac{R}{2r}\right)$$

Solution 2 by Marian Ursărescu-Romania

For LHS, using Bergstrom's inequality, we have:

$$\sum_{cyc}\frac{r_a^2}{m_b^2+m_c^2}\ge\frac{(r_a+r_b+r_c)^2}{2(m_a^2+m_b^2+m_c^2)}=\frac{(4R+r)^2}{2\cdot\frac{3}{4}(a^2+b^2+c^2)}=\frac{2(4R+r)^2}{3(a^2+b^2+c^2)}$$

We must show that:

$$\frac{2(4R+r)^2}{3(a^2+b^2+c^2)}\ge\frac{3}{2}\cdot\frac{4r^2}{R^2}\Leftrightarrow R^2(4R+r)^2\ge 9r^2(a^2+b^2+c^2)$$

But: $a^2+b^2+c^2\le 9R^2$; (2). From (1), (2) we must to prove:

$$R^2(4R+r)^2\ge 81R^2r^2\Leftrightarrow (4R+r)^2\ge 81r^2\Leftrightarrow 4R+r\ge 9r$$

$$4R\ge 8r\Leftrightarrow R\ge 2r\ (Euler).$$

For RHS, using $m_a\ge\sqrt{s(s-a)}$, we have:

$$m_b^2+m_c^2\ge s(s-b)+s(s-c)=sa$$

We must show that:

$$\frac{1}{s}\cdot\sum_{cyc}\frac{r_a^2}{a}\le\frac{2R^2}{4r^2}-\frac{R}{4r}=\frac{2R^2-Rr}{4r^2};(3)$$

But: $r_a\le\frac{a^2}{4r}$; (4). From (3), (4) we must show that:

$$\frac{1}{s\cdot 16r^2}\cdot\sum_{cyc}a^3\le\frac{2R^2-Rr}{4r^2}\Leftrightarrow\frac{1}{4s}\cdot\sum_{cyc}a^3\le 2R^2-Rr;(5)$$

But: $\sum a^3=2s(s^2-3r^2-6Rr)$; (6)

From (5), (6) we must show:

$$\frac{1}{4s}\cdot 2s(s^2-3r^2-6Rr)\le 2R^2-2Rr\Leftrightarrow$$

$$s^2-3r^2-6Rr\le 4R^2-2Rr\Leftrightarrow s^2\le 4R^2+4Rr+3r^2(Gerretsen)$$

SP.367. Let m_a,m_b,m_c be the medians, r_a,r_b,r_c the exradii, r the inradius and R the circumradius of a triangle ABC. Prove that:

$$\frac{8r}{R^2}<\frac{r_a+r_b}{m_am_b}+\frac{r_b+r_c}{m_bm_c}+\frac{r_c+r_a}{m_cm_a}\le\frac{1}{r}\left(3\left(\frac{R}{2r}\right)^4-1\right)$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

First we will prove that: $r_a\le\frac{a^2}{4r}$. We have: $r_a=s\cdot tan\frac{A}{2}$, where s –is the semiperimeter of ΔABC. Let $BC=a, CA=b, AB=c$ and $F-$ the area of ΔABC. We have:

$$\frac{a^2}{4F}=\frac{(2RsinA)^2}{4\left(\frac{1}{2}bcsinA\right)}=\frac{2R^2sinA}{bc}=\frac{2R^2sinA}{(2RsinB)(2RsinC)}=\frac{sinA}{2sinBsinC}=$$

$$=\frac{2sin\frac{A}{2}cos\frac{A}{2}}{cos(B-C)-cos(B+C)}=\frac{2sin\frac{A}{2}cos\frac{A}{2}}{cos(B-C)+cosA}\geq\frac{2sin\frac{A}{2}cos\frac{A}{2}}{1+cosA}=$$

$$=\frac{2sin\frac{A}{2}cos\frac{A}{2}}{1+\left(2cos^2\frac{A}{2}-1\right)}=\frac{sin\frac{A}{2}}{cos\frac{A}{2}}=tan\frac{A}{2}$$

So, $tan\frac{A}{2}\leq\frac{a^2}{4F}=\frac{a^2}{4rs}$**. Namely,** $r_a\leq\frac{a^2}{4r}$**. Similarly,** $r_b\leq\frac{b^2}{4r},r_c\leq\frac{c^2}{4r}$**.Also, we have:**

$$m_a^2=\frac{2(b^2+c^2)-a^2}{4}\geq\frac{(b+c)^2-a^2}{4}=\frac{(b+c+a)(b+c-a)}{4}=$$

$$=\frac{2s(2s-2a)}{4}=s(s-a)=\frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)}=\frac{F^2}{(s-b)(s-c)}=$$

$$=\frac{F}{s-b}\cdot\frac{F}{s-c}=r_ar_b$$

Now, we have:

$$\frac{r_a+r_b}{m_am_b}\leq\frac{r_a+r_b}{\sqrt{r_br_c}\cdot\sqrt{r_cr_a}}=\frac{r_a+r_b}{r_c\sqrt{r_ar_b}}\leq\frac{\frac{a^2}{4r}+\frac{b^2}{4r}}{\frac{F}{s-c}\sqrt{\frac{F}{s-a}\cdot\frac{F}{s-b}}}=\frac{\frac{a^2+b62}{4r}}{\frac{F^2}{(s-c)\sqrt{(s-a)(s-b)}}}$$

$$=\frac{(a^2+b^2)(s-c)\sqrt{(s-a)(s-b)}}{4rF^2}\leq\frac{(a^2+b^2)(s-c)\cdot\frac{s-a+s-b}{2}}{4rF^2}=$$

$=\frac{(a^2+b^2)(s-c)c}{8rF^2}$**. Similarly:** $\frac{r_b+r_c}{m_bm_c}\leq\frac{(b^2+c^2)(s-a)a}{8rF^2};\frac{r_c+r_a}{m_cm_a}\leq\frac{(c^2+a^2)(s-b)b}{8rF^2}$

$$\frac{r_a+r_b}{m_am_b}+\frac{r_b+r_c}{m_bm_c}+\frac{r_c+r_a}{m_cm_a}\leq$$

$$\leq\frac{1}{8rF^2}\Big((a^2+b^2)(s-c)c+(b^2+c^2)(s-a)a+(c^2+a^2)(s-b)b\Big)=$$

$$=\frac{1}{8rF^2}\cdot\frac{(a^2+b^2)(a+b-c)+(b^2+c^2)(-a+b+c)+(c^2+a^2)(a-b+c)b}{2}=$$

$$=\frac{1}{8rF}\cdot\frac{2(a^2bc+ab^2c+abc^2)+ab^3+a^3b+bc^3+b^3c+ac^3+a^3c-2(a^2b^2+b^2c^2+c^2a^2)}{2}=$$

$$=\frac{1}{8rF^2}\left(abc(a+b+c)+\frac{ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)}{2}-(a^2b^2+b^2c^2+c^2a^2)\right)\leq$$

$$\le \frac{1}{8rF^2}\left(abc(a+b+c)+\frac{\left(\frac{a+b}{2}\right)^2(a^2+b^2)+\left(\frac{b+c}{2}\right)^2(b^2+c^2)+\left(\frac{c+a}{2}\right)^2(c^2+a^2)}{2}-(a^2b^2+b^2c^2+c^2a^2)\right)\le$$

$$\le \frac{1}{8rF^2}\left(abc(a+b+c)+\frac{\frac{2(a^2+b^2)^2}{4}+\frac{2(b^2+c^2)^2}{4}+\frac{2(c^{22}+a^2)^2}{4}}{2}-(a^2b^2+b^2c^2+c^2a^2)\right)\le$$

$$\le \frac{1}{8rF^2}\left(abc(a+b+c)+\frac{a^4+b^4+c^4-(a^2b^2+b^2c^2+c^2a^2)}{2}\right)$$

Now, we will prove that $a^2b^2+b^2c^2+c^2a^2\le\frac{4R^2F^2}{r^2}$

We have: $a^2b^2+b^2c^2+c^2a^2=(abc)^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$

Also, we have $(b-c)^2\ge 0 \Leftrightarrow a^2-(b-c)^2\le a^2 \Leftrightarrow \frac{1}{a^2}\le\frac{1}{a^2-(b-c)^2}=$

$=\frac{1}{(a+b-c)(a-b+c)}=\frac{1}{(2s-2c)(2s-2b)}=\frac{1}{4(s-b)(s-c)}$. Similarly

$\frac{1}{b^2}\le\frac{1}{4(s-c)(s-a)}$ and $\frac{1}{c^2}\le\frac{1}{4(s-a)(s-b)}$. So,

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4}\left(\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}\right)=$$

$$=\frac{1}{4}\cdot\frac{s-a+s-b+s-c}{(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{s(3s-2s)}{s(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{s^2}{F^2}\overset{Heron}{=}$$

$=\frac{1}{4}\cdot\frac{s^2}{r^2s^2}=\frac{1}{4r^2}$. So, $a^2b^2+b^2c^2+c^2a^2\le(abc)^2\cdot\frac{1}{4r^2}$. Also we know that:

$abc=4R\cdot F=4Rrs;\ a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)-16F^2$. So,

$$\frac{r_a+r_b}{m_am_b}+\frac{r_b+r_c}{m_bm_c}+\frac{r_c+r_a}{m_cm_a}\le\frac{1}{8rF^2}\left(8RsF+\frac{2R^2F^2}{r^2}-8F^2\right)=$$

$$=\frac{8Rrs\cdot rs}{8r\cdot r^2s^2}+\frac{R^2}{4r^3}-\frac{1}{r}=\frac{R}{r^2}+\frac{R^2}{4r^3}-\frac{1}{r}=\frac{4Rr+R^2-4r^2}{4r^3}\overset{R\ge 2r}{\le}\frac{3R^2}{4r^3}-\frac{1}{r}=$$

$=\frac{1}{r}\left(3\left(\frac{R}{2r}\right)^2-1\right)$. For the left inequality, we have:

$$\frac{r_a+r_b}{m_am_b}+\frac{r_b+r_c}{m_bm_c}+\frac{r_c+r_a}{m_cm_a}\overset{AM-GM}{\ge}\frac{3\sqrt[3]{(r_a+r_b)(r_b+r_c)(r_c+r_a)}}{\left(\sqrt[3]{m_am_bm_c}\right)^2}\ge$$

$$\geq \frac{3\sqrt[3]{(r_a+r_b)(r_b+r_c)(r_c+r_a)}}{\left(\frac{m_a+m_b+m_c}{3}\right)^2}=\frac{27\sqrt[3]{(r_a+r_b)(r_b+r_c)(r_c+r_a)}}{(m_a+m_b+m_c)^2}$$

We know that: $(m_a+m_b+m_c)^2 \leq 3(m_a^2+m_b^2+m_c^2)=3\cdot\frac{3}{4}(a^2+b^2+c^2)\leq$

$\leq 3\cdot\frac{3}{4}\cdot 9R^2=\frac{81}{4}R^2$**and** $(r_a+r_b)(r_b+r_c)(r_c+r_a)=4s^2R,;s\geq 3\sqrt{3}r$. **So,**

$$\frac{r_a+r_b}{m_am_b}+\frac{r_b+r_c}{m_bm_c}+\frac{r_c+r_a}{m_cm_a}\geq\frac{27\sqrt[3]{4s^2R}}{\frac{81R^2}{4}}\geq\frac{27\sqrt[3]{4\left(3\sqrt{3}r\right)^2\cdot 2r}}{\frac{81R^2}{4}}=$$

$$=\frac{\sqrt[3]{(2\cdot 3\cdot r)^3}}{3R^2}=\frac{8r}{R^2}$$

Namely,

$$\frac{8r}{R^2}<\frac{r_a+r_b}{m_am_b}+\frac{r_b+r_c}{m_bm_c}+\frac{r_c+r_a}{m_cm_a}\leq\frac{1}{r}\left(3\left(\frac{R}{2r}\right)^4-1\right)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ know\ that\ m_a\leq\frac{Rh_a}{2r}\ (Panaitopol)\ and$$

$$\frac{2}{h_c}=\frac{1}{r_a}+\frac{1}{r_b}=\frac{r_a+r_b}{r_ar_b}\ (and\ analogs)$$

$$\to\sum\frac{r_a+r_b}{m_am_b}\geq\left(\frac{2r}{R}\right)^2\sum\frac{r_a+r_b}{h_ah_b}=\left(\frac{2r}{R}\right)^2\sum\frac{2r_ar_b}{h_ah_bh_c}=\left(\frac{2r}{R}\right)^2\cdot\frac{2}{h_ah_bh_c}\sum r_ar_b$$

$$=\left(\frac{2r}{R}\right)^2\cdot\frac{2R}{2F^2}.s^2=\frac{4}{R}\overset{Euler}{\geq}\frac{8r}{R^2}$$

$$Now, we\ know\ that\ m_a\geq h_a\ (and\ analogs).$$

$$\to\sum\frac{r_a+r_b}{m_am_b}\leq\sum\frac{r_a+r_b}{h_ah_b}=\sum\frac{2r_ar_b}{h_ah_bh_c}=\frac{2}{h_ah_bh_c}\sum r_ar_b=\frac{2R}{2F^2}.s^2$$

$$=\frac{R}{r^2}\overset{?}{\leq}\frac{1}{r}\left(3\left(\frac{R}{2r}\right)^4-1\right)$$

$$\leftrightarrow 3\left(\frac{R}{2r}\right)^4-2\left(\frac{R}{2r}\right)-1\geq 0\leftrightarrow(x-1)(3x^3+3x^2+3x+1)\geq 0\ \left(\because x=\frac{R}{2r}\right)$$

$$Which\ is\ true\ from\ Euler\ \left(x=\frac{R}{2r}\geq 1\right).$$

Therefore,

$$\frac{8r}{R^2} \le \sum \frac{r_a + r_b}{m_a m_b} \le \frac{1}{r}\left(3\left(\frac{R}{2r}\right)^4 - 1\right)$$

SP.368 If $0 < a < b < 1$, then prove:

$$\frac{a(2b-a)}{b\sqrt{a^2+b^2}} < \int_a^b \frac{dx}{x\sqrt{(x^2+a^2)^3}} + \frac{\sqrt{2}}{2} < \frac{a}{\sqrt{a^2+b^2}} + \frac{b-a}{a\sqrt{2}}$$

Proposed by Florică Anastase-Romania

Solution by proposer

Let be the function:

$$f, g: [a,b] \to R, f(x) = \frac{1}{x} \text{ și } g(x) = \frac{1}{\sqrt{(x^2+a^2)^3}}, h: [a,b] \to R,$$

$$h(x) = f(x) - f(b) \text{ decreasing.}$$

From second M.V.T. $\exists c \in (a,b)$ ***such that:***

$$\int_a^b g(x)h(x)dx = h(a)\int_a^c g(x)dx$$

$$\int_a^b g(x)\big(f(x) - f(b)\big)dx = \big(f(a) - f(b)\big)\int_a^c g(x)dx$$

$$\int_a^b f(x)g(x)dx =$$

$$= f(b)\int_a^b g(x)dx + \big(f(a) - f(b)\big)\int_a^c g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

$$G(x) = \int g(x)dx = \int \frac{dx}{\sqrt{(x^2+a^2)^3}} = (*) \begin{cases} a^2 + x^2 = t^2x^2 \\ x = \dfrac{a}{\sqrt{t^2-1}} \\ dx = -\dfrac{atdt}{\sqrt{(t^2-1)^3}} \end{cases}$$

$$(*)=-\int\frac{\sqrt{(t^2-1)^3}}{t^3}\cdot\frac{at}{\sqrt{(t^2-1)^3}}dt=-a\int\frac{dt}{t^2}=\frac{a}{t}=\frac{ax}{\sqrt{a^2+x^2}}+C$$

$$\int_a^b\frac{dx}{x\sqrt{(x^2+a^2)^3}}=f(a)(G(c)-G(a))+f(b)\big(G(b)-G(c)\big)$$

$$=\frac{a}{\sqrt{a^2+b^2}}+\frac{c(b-a)}{b\sqrt{a^2+c^2}}-\frac{\sqrt{2}}{2}$$

$$a<c<b\to\frac{c(b-a)}{b\sqrt{a^2+c^2}}>\frac{a(b-a)}{b\sqrt{a^2+b^2}}$$

Hence,

$$\int_a^b\frac{dx}{x\sqrt{(x^2+a^2)^3}}=\frac{a}{\sqrt{a^2+b^2}}+\frac{c(b-a)}{b\sqrt{a^2+c^2}}-\frac{\sqrt{2}}{2}>\frac{a}{\sqrt{a^2+b^2}}+\frac{a(b-a)}{b\sqrt{a^2+b^2}}-\frac{\sqrt{2}}{2}=$$

$$=\frac{a(2b-a)}{b\sqrt{a^2+b^2}}-\frac{\sqrt{2}}{2}$$

$$a<c<b\to\frac{1}{a^2+c^2}<\frac{1}{2a^2}\to\frac{c(b-a)}{b\sqrt{a^2+c^2}}<\frac{b(b-a)}{ab\sqrt{2}}=\frac{b-a}{a\sqrt{2}}$$

$$\int_a^b\frac{dx}{x\sqrt{(x^2+a^2)^3}}=\frac{a}{\sqrt{a^2+b^2}}+\frac{c(b-a)}{b\sqrt{a^2+c^2}}-\frac{\sqrt{2}}{2}<\frac{a}{\sqrt{a^2+b^2}}+\frac{b-a}{a\sqrt{2}}-\frac{\sqrt{2}}{2}$$

Therefore,

$$\frac{a(2b-a)}{b\sqrt{a^2+b^2}}<\int_a^b\frac{dx}{x\sqrt{(x^2+a^2)^3}}+\frac{\sqrt{2}}{2}<\frac{a}{\sqrt{a^2+b^2}}+\frac{b-a}{a\sqrt{2}}$$

SP.369 Let $(L_n)_{n\geq 0}, L_0=2, L_1=1, L_{n+2}=L_{n+1}+L_n, \forall n\in\mathbb{N}$, be the Lucas' sequences, and $a,b,c\in\mathbb{R}_+^*$ such that $abc=1$. Prove that:

$$\frac{1}{a^6(bL_n+cL_{n+1})^2}+\frac{1}{b^6(cL_n+aL_{n+1})^2}+\frac{1}{c^6(aL_n+bL_{n+1})^2}\geq\frac{3}{L_{n+2}^2},\forall n\in\mathbb{N}$$

D.M. Bătineţu – Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

By Bergström's inequality we have

$$A_n=\frac{1}{a^6(bL_n+cL_{n+1})^2}+\frac{1}{b^6(cL_n+aL_{n+1})^2}+\frac{1}{c^6(aL_n+bL_{n+1})^2}\geq$$

$$\geq\frac{1}{3}\left(\frac{1}{a^3(bL_n+cL_{n+1})}+\frac{1}{b^3(cL_n+aL_{n+1})}+\frac{1}{c^3(aL_n+bL_{n+1})}\right)^2,\forall n\in\mathbb{N};(1)$$

Also we have

$$B_n=\frac{1}{a^3(bL_n+cL_{n+1})}+\frac{1}{b^3(cL_n+aL_{n+1})}+\frac{1}{c^3(aL_n+bL_{n+1})}=$$

$$=\frac{\frac{1}{a^2}}{a(bL_n+cL_{n+1})}+\frac{\frac{1}{b^2}}{b(cL_n+aL_{n+1})}+\frac{\frac{1}{c^2}}{c(aL_n+bL_{n+1})},\forall n\in\mathbb{N}$$

Where we apply again the Bergström's inequality and than AM-GM inequality and we deduce that

$$B_n\geq\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{(ab+bc+ca)(L_n+L_{n+1})}=\frac{(ab+bc+ca)^2}{(abc)^2(ab+bc+ca)(abc)^2L_{n+2}}=$$

$$=\frac{ab+bc+ca}{L_{n+2}}\overset{AM-GM}{\geq}\frac{3\cdot\sqrt[3]{(abc)^2}}{L_{n+2}}=\frac{3}{L_{n+2}},\forall n\in\mathbb{N};(2)$$

From (1) and (2) follows that

$$A_n\geq\frac{1}{3}\left(\frac{3}{L_{n+2}}\right)^2=\frac{3}{L_{n+2}^2},\forall\, n\in\mathbb{N}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\frac{1}{a^6(bL_n+cL_{n+1})^2}=\sum\frac{\left(\frac{1}{a}\right)^4}{(abL_n+acL_{n+1})^2}\overset{abc=1}{=}\sum\frac{\left(\frac{1}{a}\right)^4}{\left(\frac{1}{c}.L_n+\frac{1}{b}.L_{n+1}\right)^2}\geq$$

$$\overset{Hölder}{\geq}\frac{\left(\sum\frac{1}{a}\right)^4}{3\left[\sum\left(\frac{1}{c}.L_n+\frac{1}{b}.L_{n+1}\right)\right]^2}=\frac{\left(\sum\frac{1}{a}\right)^4}{3\left(\sum\frac{1}{a}\right)^2(L_{n+1}+L_n)^2}=\frac{\left(\sum\frac{1}{a}\right)^2}{3{L_{n+2}}^2}\overset{AM-GM}{\geq}$$

$$\geq\frac{3^2}{3{L_{n+2}}^2}=\frac{3}{{L_{n+2}}^2}.$$

$$Therefore,$$

$$\sum \frac{1}{a^6(bL_n + cL_{n+1})^2} \geq \frac{3}{{L_{n+2}}^2}, \forall n \in N$$

SP.370 If ABC is a triangle with inradius r and circumradius R, then for any point M in the plane of triangle, $M \notin \{A, B, C\}$, holds the inequality

$$\frac{MA}{MB+MC} + \frac{MB}{MC+MA} + \frac{MC}{MA+MB} \geq \frac{R+r}{R} \geq \frac{3r}{R}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

In the inequality of Nesbitt $\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}, \forall x, y, z \in \mathbb{R}; (1)$ we put $x = MA, y = MB, z = MC$, and we deduce that:

$$U = \frac{MA}{MB+MC} + \frac{MB}{MC+MA} + \frac{MC}{MA+MB} \geq \frac{3}{2}; (2)$$

By Euler' inequality, we have in any triangle : $R \geq 2r \Leftrightarrow \frac{1}{2} \geq \frac{r}{R}; (3)$

From $(2), (3)$ it follows that:

$$U \geq 1 + \frac{1}{2} \geq 1 + \frac{r}{R} = \frac{R+r}{R} \geq \frac{2r+r}{R} = \frac{3r}{R}$$

Solution 2 by Daniel Văcaru-Romania

We have:

$$\frac{MA}{MB+MC} + \frac{MB}{MC+MA} + \frac{MC}{MA+MB} =$$

$$= \frac{MA^2}{MA \cdot MB + MA \cdot MC} + \frac{MB^2}{MB \cdot MC + MB \cdot MA} + \frac{MC^2}{MB \cdot MC + MB \cdot MA} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(MA+MB+MC)^2}{2(MA \cdot MB + MB \cdot MC + MC \cdot MA)} =$$

$$= 1 + \frac{MA^2 + MB^2 + MC^2}{2(MA \cdot MB + MB \cdot MC + MC \cdot MA)} \overset{S.O.S}{\geq} \frac{3}{2} = 1 + \frac{1}{2}; (1)$$

But we know Euler, namely $R \geq 2r; (2)$ we get: $\frac{1}{2} \geq \frac{r}{R}; (3)$. Then we have:

$$\frac{MA}{MB+MC} + \frac{MB}{MC+MA} + \frac{MC}{MA+MB} \overset{(1)}{\geq} \frac{3}{2} = 1 + \frac{1}{2} \overset{(3)}{\geq} 1 + \frac{r}{R} = \frac{R+r}{R} \overset{(2)}{\geq} \frac{3r}{R}$$

SP.371. Let $ABCD$ be a tetrahedron, and let M be a point in space, $M \notin \{A, B, C\}$. Prove that:

$$\frac{MA}{MB+MC+MD}+\frac{MB}{MC+MD+MA}+\frac{MC}{MD+MA+MB}+\frac{MD}{MA+MB+MC} \geq \frac{R+r}{R} \geq \frac{4r}{R}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

If $x, y, z, t \in \mathbb{R}_+^*, X = x+y+z+t$, then

$$U = \sum_{cyc} \frac{x}{X-x} \geq \frac{4}{3}; (1)$$

$$U = \sum_{cyc} \frac{x}{X-x} = \sum_{cyc} \frac{x-X+X}{X-x} = -4 + X \cdot \sum_{cyc} \frac{1}{X-x}$$

By Bergstrom' inequality, we get:

$$U \geq -4 + X \cdot \frac{16}{\sum(X-x)} = -4 + \frac{16}{3} = \frac{4}{3}$$

We put in (1): $x = MA, y = MB, z = MC, t = MD$ and we obtain

$$\sum_{cyc} \frac{MA}{MB+MC+MD} \geq \frac{4}{3} = 1 + \frac{1}{3}; (2)$$

By the inequality of Euler-Durrande for tetrahedron we have: $R \geq 3r \Leftrightarrow \frac{r}{R} \leq \frac{1}{3}$ and then by (2) we deduce that:

$$\sum_{cyc} \frac{MA}{MB+MC+MD} \geq 1 + \frac{1}{3} \geq 1 + \frac{r}{R} = \frac{R+r}{R} \geq \frac{4r}{R}$$

Solution 2 by Daniel Văcaru-Romania

$$\frac{MA}{MB+MC+MD}+\frac{MB}{MC+MD+MA}+\frac{MC}{MD+MA+MB}+\frac{MD}{MA+MB+MC} =$$

$$= \frac{MA^2}{MA \cdot MB + MA \cdot MC + MA \cdot MD} + \frac{MB^2}{MB \cdot MC + MB \cdot MD + MB \cdot MA} +$$

$$+ \frac{MC^2}{MC \cdot MD + MC \cdot MA + MC \cdot MB} + \frac{MD^2}{MD \cdot MA + MD \cdot MB + MD \cdot MC} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(MA+MB+MC+MD)^2}{2(MA\cdot MB+MA\cdot MC+MA\cdot MD+MB\cdot MC+MB\cdot MD+MC\cdot MD)}=$$

$$=1+\frac{MA^2+MB^2+MC^2+MD^2}{2(MA\cdot MB+MA\cdot MC+MA\cdot MD+MB\cdot MC+MB\cdot MD+MC\cdot MD)};(1)$$

But: $3(MA^2+MB^2+MC^2+MD^2)\geq$

$$\geq 2(MA\cdot MB+MA\cdot MC+MA\cdot MD+MB\cdot MC+MB\cdot MD+MC\cdot MD)$$

Hence,

$$\frac{MA^2+MB^2+MC^2+MD^2}{2(MA\cdot MB+MA\cdot MC+MA\cdot MD+MB\cdot MC+MB\cdot MD+MC\cdot MD)}\geq\frac{1}{3};(2)$$

On the other hand, we have $R\geq 3r;(3)\Leftrightarrow\frac{1}{3}\geq\frac{r}{R};(4)$. Then we have:

$$\frac{MA}{MB+MC+MD}+\frac{MB}{MC+MD+MA}+\frac{MC}{MD+MA+MB}+\frac{MD}{MA+MB+MC}\overset{(1)}{\cong}$$

$$=1+\frac{MA^2+MB^2+MC^2+MD^2}{2(MA\cdot MB+MA\cdot MC+MA\cdot MD+MB\cdot MC+MB\cdot MD+MC\cdot MD)}\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}1+\frac{1}{3}\overset{(4)}{\geq}1+\frac{r}{R}=\frac{R+r}{R}\overset{(2)}{\geq}\frac{4r}{R}$$

SP.372 If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ with $\lim_{n\to\infty}\frac{f(x)}{x}=a\in\mathbb{R}_+^*$, $(b_n)_{n\geq 1}$ is an arithmetic progression with $b_1,r\in\mathbb{R}_+^*$ and $u,v\in\mathbb{R}$ satisfy $u+v=1$, then compute:

$$\lim_{n\to\infty}\left((n+1)^{u}\sqrt[n+1]{\left(f(b_1)f(b_2)\dots f(b_n)f(b_{n+1})\right)^v}-n^{u}\sqrt[n]{\left(f(b_1)f(b_2)\dots f(b_n)\right)^v}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by proposers

We denote $f(b_1)f(b_2)\dots f(b_n)=f(b_n)!,\forall n\in\mathbb{N}^*$ and

$$B_n=(n+1)^{u}\sqrt[n+1]{(f(b_{n+1})!)^v}-n^{u}\sqrt[n]{\left(f(b_n)\right)^v}=n^{u}\sqrt[n]{\left(f(b_n)\right)^v}(w_n-1)=$$

$$=n^{u-1}\sqrt[n]{(f(b_n)!)^v}\cdot\frac{w_n-1}{\ln w_n}\cdot\ln w_n^n=\left(\frac{\sqrt[n]{f(b_n)!}}{n}\right)^v\cdot\frac{w_n-1}{\ln w_n}\cdot\ln w_n^n,\forall n\geq 2\text{, where}$$

$$w_n=\left(\frac{n+1}{n}\right)^u\left(\frac{\sqrt[n+1]{f(b_{n+1})!}}{\sqrt[n]{f(b_n)!}}\right)^v,\forall n\geq 2\text{. We have}$$

$\lim_{n\to\infty}\frac{f(b_n)}{n}=\lim_{n\to\infty}\frac{f(b_n)}{b_n}\cdot\frac{b_n}{n}=ar$ and $\lim_{n\to\infty}\frac{f(b_{n+1})!}{nf(b_n)!}=\lim_{n\to\infty}\frac{f(b_{n+1})}{n}=ar$. Also

we have that

$$\lim_{n\to\infty}\frac{\sqrt[n]{f(b_n)!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{f(b_n)!}{n^n}}=\lim_{n\to\infty}\left(\frac{f(b_{n+1})!}{(n+1)^{n+1}}\cdot\frac{n^n}{f(b_n)!}\right)=\lim_{n\to\infty}\left(\frac{1}{e_n}\cdot\frac{f(b_{n+1})}{n+1}\right)=\frac{ar}{e},$$

where $e_n=\left(1+\frac{1}{n}\right)^n\to e$. We get

$$\lim_{n\to\infty}w_n=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^u\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{f(b_{n+1})!}}{\sqrt[n]{f(b_n)!}}\right)^v=$$

$$=1\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{f(b_{n+1})!}}{n+1}\cdot\frac{n}{\sqrt[n]{f(b_n)!}}\cdot\frac{n+1}{n}\right)^v$$

$$=\left(\frac{ar}{e}\cdot\frac{e}{ar}\cdot1\right)^v=1\text{, so }\lim_{n\to\infty}\frac{w_n-1}{\ln w_n}=1\text{. Hence}$$

$$\lim_{n\to\infty}w_n^n=\lim_{n\to\infty}e_n^u\cdot\lim_{n\to\infty}\left(\frac{f(b_{n+1})!}{f(b_n)!}\cdot\frac{1}{\sqrt[n+1]{f(b_{n+1})!}}\right)^v=$$

$$=e^u\lim_{n\to\infty}\left(\frac{f(b_{n+1})}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{f(b_{n+1})!}}\right)^v=$$

$$=e^u\left(ar\cdot\frac{e}{ar}\right)^v=e^ue^v=e\text{. Then}$$

$$\lim_{n\to\infty}B_n=\lim_{n\to\infty}\left(\frac{\sqrt[n]{f(b_n)!}}{n}\right)^v\cdot1\cdot\ln e=\lim_{n\to\infty}\left(\frac{\sqrt[n]{f(b_n)!}}{n}\right)^v=\left(\frac{ar}{e}\right)^v$$

Solution 2 by Mikael Bernardo-Mozambique

$$u+v=1\to u-1=-v$$

$$\Omega=\lim_{n\to\infty}\left((n+1)^u\cdot\sqrt[n+1]{\left(f(b_1)f(b_2)\dots f(b_n)f(b_{n+1})\right)^v}-n^u\cdot\sqrt[n]{\left(f(b_1)f(b_2)\dots f(b_n)\right)^v}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)^u\cdot\sqrt[n+1]{\left(f(b_1)f(b_2)\dots f(b_n)f(b_{n+1})\right)^v}-n^u\cdot\sqrt[n]{\left(f(b_1)f(b_2)\dots f(b_n)\right)^v}}{n+1-n}\right)\overset{LC-S}{=}$$

$$=\lim_{n\to\infty}\frac{n^u\cdot\sqrt[n]{\left(f(b_1)f(b_2)\dots f(b_n)\right)^v}}{n}=\lim_{n\to\infty}n^{u-1}\cdot\sqrt[n]{\left(f(b_1)f(b_2)\dots f(b_n)\right)^v}=$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{\left(f(b_1)f(b_2)\ldots f(b_n)\right)^v}}{n^v}; (since\ u-1=v)$$

$$= \lim_{n\to\infty}\left(\sqrt[n]{\frac{f(b_1)\cdot f(b_2)\cdot\ldots\cdot f(b_n)}{n^n}}\right)^v \overset{C-D}{=} \left(\lim_{n\to\infty}\frac{f(b_{n+1})}{(n+1)^{n+1}}\cdot n^n\right)^v =$$

$$= \left(\lim_{n\to\infty}\frac{b_1+n\cdot r}{n+1}\cdot\frac{\alpha}{e}\right)^v = \left(\frac{r\cdot\alpha}{e}\right)^v$$

SP.373 If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a function such that $\lim_{x\to\infty}\frac{f(x)}{x}=c\in\mathbb{R}_+^*$ and $(a_n)_{n\ge1}$ is a positive sequence such that $\lim_{n\to\infty}(a_{n+1}-a_n)=a\in\mathbb{R}_+^*$, then compute:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{f(a_1)f(a_2)\ldots f(a_n)f(a_{n+1})}}-\frac{n^2}{\sqrt[n]{f(a_1)f(a_2)\ldots f(a_n)}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by proposers

Denoting $f(a_n)! = f(a_1)f(a_2)\ldots f(a_n), \forall n\in\mathbb{N}^*, e_n=\left(1+\frac{1}{n}\right)^n$ and

$$B_n=\frac{(n+1)^2}{\sqrt[n+1]{f(a_{n+1})!}}-\frac{n^2}{\sqrt[n]{f(a_n)!}}=\frac{n^2}{\sqrt[n]{f(a_n)!}}(u_n-1)=\frac{n}{\sqrt[n]{f(a_n)!}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n, \forall n\ge 2$$

where $u_n=\left(\frac{n+1}{n}\right)^2\frac{\sqrt[n]{f(a_n)!}}{\sqrt[n+1]{f(a_{n+1})!}}$. We have

$$\lim_{n\to\infty}\frac{a_n}{n}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)-n}=a; \lim_{n\to\infty}\frac{f(a_n)}{n}=\lim_{n\to\infty}\left(\frac{f(a_n)}{a_n}\cdot\frac{a_n}{n}\right)=ac$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{f(a_n)!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{f(a_n)!}}=\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{f(a_{n+1})!}\cdot\frac{f(a_n)!}{n^n}\right)=\lim_{n\to\infty}e_n\cdot\frac{n+1}{f(a_{n+1})}=$$

$$= e\lim_{n\to\infty}\left(\frac{n+1}{a_{n+1}}\cdot\frac{a_{n+1}}{f(a_n)}\right)=e\cdot\frac{1}{a}\cdot\frac{1}{c}=\frac{e}{ac}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{f(a_n)!}}{n}\cdot\frac{n+1}{\sqrt[n+1]{f(a_{n+1})!}}\cdot\frac{n}{n+1}\right)=1\cdot\frac{ac}{e}\cdot\frac{e}{ac}\cdot 1=1$$

So, $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$

$$\lim_{n\to\infty} u_n^n=\lim_{n\to\infty}\left(e_n^2\cdot\frac{f(a_n)!}{f(a_{n+1})!}\cdot\sqrt[n+1]{f(a_{n+1})!}\right)=e^2\cdot\frac{1}{ac}\cdot\frac{ac}{e}=e$$

Hence $\Omega=\lim_{n\to\infty}\frac{n}{\sqrt[n]{f(a_n)!}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=\frac{e}{ac}\cdot 1\cdot\ln e=\frac{e}{ac}$

SP.374 Let m_a, m_b, m_c be the lengths of the medians of a triangle with circumradius R and area F.Prove that:

$$\frac{4}{9R^2}\le\frac{1}{m_a(m_b+2m_c)}+\frac{1}{m_b(m_c+2m_a)}+\frac{1}{m_c(m_a+2m_b)}\le\frac{\sqrt{3}}{3F}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by proposer

For the right inequality we have:

First will prove that $\frac{1}{m_a(m_b+2m_c)}\le\frac{1}{9m_a}\cdot\left(\frac{2}{m_c}+\frac{1}{m_b}\right)$

We have $\frac{1}{9m_a}\cdot\left(\frac{2}{m_c}+\frac{1}{m_b}\right)-\frac{1}{m_a(m_b+2m_c)}=$

$$=\frac{1}{m_a}\left(\frac{1}{9}\left(\frac{2}{m_c}+\frac{1}{m_b}\right)-\frac{1}{m_b+2m_c}\right)=\frac{1}{m_a}\cdot\frac{(2m_b+m_c)(m_b+2m_c)-9m_bm_c}{9m_bm_c(m_a+2m_c)}=$$

$\frac{1}{m_a}\cdot\frac{2(m_b-m_c)^2}{9m_bm_c(m_b+2m_c)}\ge 0$. Similarly

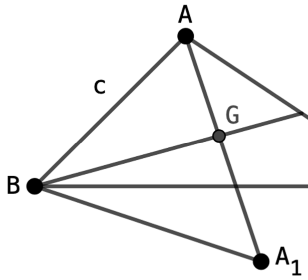
$\frac{1}{m_b(m_c+2m_a)}\le\frac{1}{9m_b}\left(\frac{2}{m_a}+\frac{1}{m_c}\right)$ and

$$\frac{1}{m_c(m_a+2m_b)}\le\frac{1}{9m_c}\left(\frac{2}{m_b}+\frac{1}{m_a}\right)$$

Adding up these inequalities, we get

$$\frac{1}{m_a(m_b+2m_c)}+\frac{1}{m_b(m_c+2m_a)}+\frac{1}{m_c(m_a+2m_b)}\le\frac{1}{3}\left(\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}\right)$$

Now, will prove that: $\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}\le\frac{\sqrt{3}}{F}$



We know that in triangle BGA, we have $GA_1=\frac{2}{3}m_a, BG=\frac{2}{3}m_b, A_1B=\frac{2}{3}m_c$ and area $\Delta BGA_1=\frac{1}{3}F$. Also we know that

$$a+b+c=2s\le 3\sqrt{3}R \text{ or } a+b+c\le 3\sqrt{3}\frac{abc}{4F}.(abc=4R\cdot F)$$

So for ΔBGA, we have

$$\frac{2}{3}m_a+\frac{2}{3}m_b+\frac{2}{3}m_c\le 3\sqrt{3}\cdot\frac{\frac{2}{3}m_a\cdot\frac{2}{3}m_b\cdot\frac{2}{3}m_c}{4\cdot\frac{1}{3}F} \text{ or } \frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}\le\frac{\sqrt{3}}{F}. \text{ So,}$$

$$\frac{1}{m_a(m_b+2m_c)}+\frac{1}{m_b(m_c+2m_a)}+\frac{1}{m_c(m_a+2m_b)}\le\frac{\sqrt{3}}{3F}$$

Now, for the left inequality, we have by Cauchy – Schwarz inequality,

$$\frac{1}{m_a(m_b+2m_c)}+\frac{1}{m_b(m_c+2m_a)}+\frac{1}{m_c(m_a+2m_b)}\ge\frac{(1+1+1)^2}{3(m_am_b+m_bm_c+m_cm_a)}$$

Also we have

$$m_am_b+m_bm_c+m_cm_a\le m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$$

We know that $a^2+b^2+c^2\le 9R^2$ in any triangle ABC. So

$$m_am_b+m_bm_c+m_cm_a\le\frac{27}{4}R^2. \text{ Namely}$$

$$\frac{1}{m_a(m_b+2m_c)}+\frac{1}{m_b(m_c+2m_a)}+\frac{1}{m_c(m_a+2m_b)}\ge\frac{9}{3\cdot\frac{27}{4}R^2}=\frac{4}{9R^2}$$

Equality holds if and only if the triangle ABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\frac{1}{m_a(m_b+2m_c)}\overset{CBS}{\ge}\frac{9}{\sum m_a(m_b+2m_c)}=\frac{3}{\sum m_am_b}\overset{\sum xy\le\sum x^2}{\ge}\frac{3}{\sum {m_a}^2}$$

$$=\frac{3}{\frac{3}{4}\sum a^2}\overset{Leibniz}{\ge}\frac{4}{9R^2}.$$

$Now, we\ know\ that\ m_a, m_b, m_c can\ be\ the\ sides-lengths\ of\ triangle.$

$Let\ F', s', R' be\ the\ area, semiperimeter\ and\ circumradii\ of\ \Delta m_am_bm_c.$

$$\sum\frac{1}{m_a(m_b+2m_c)}=\sum\frac{1}{m_am_b+2m_am_c}\overset{CBS}{\leq}\sum\frac{1}{9}\left(\frac{1}{m_am_b}+\frac{2}{m_am_c}\right)=\frac{1}{3}\sum\frac{1}{m_am_b}$$

$$=\frac{\sum m_a}{3m_am_bm_c}=$$

$$=\frac{2s'}{3.4R'F'}\overset{Mitrinovic}{\leq}\frac{3\sqrt{3}R'}{3.4R'F'}=\frac{\sqrt{3}}{4F'}=\frac{\sqrt{3}}{3F}.\ \left(\because F'=\frac{3}{4}F\right)$$

$$Therefore,\quad \frac{4}{9R^2}\leq\sum\frac{1}{m_a(m_b+2m_c)}\leq\frac{\sqrt{3}}{3F}$$

SP.375 If $x,y,z\in(0,1)$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{xa^4}{(y+z)^2(1-x^2)}+\frac{yb^4}{(z+x)^2(1-y^2)}+\frac{zc^4}{(x+y)^2(1-z)}\geq 6\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by proposers

Let be $f:(0,1)\to\mathbb{R}, f(t)=t(1-t^2)$ with $f'(t)=1-t^2-2t^2=$

$=1-3t^2, f'(t)=0\Rightarrow t=\frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right)=\frac{2}{3\sqrt{3}}$ and

t	0	$\frac{1}{\sqrt{3}}$	1
$f'(t)$	$+++++++$	0	$--------$
$f(t)$	$\nearrow$	$\frac{2}{3\sqrt{3}}$	$\searrow$

$$\Rightarrow f(t)\leq\frac{2}{3\sqrt{3}}, \forall t(0,1)$$

and then

$$\sum_{cyc}\frac{xa^4}{(y+z)^2(1-x^2)}=\sum_{cyc}\frac{x^2a^4}{(y+z)^2f(x)}\geq\frac{3\sqrt{3}}{2}\sum_{cyc}\frac{x^2a^4}{(y+z)^2}\geq$$

$$\overset{Bergstrom}{\geq}\frac{3\sqrt{3}}{2}\cdot\frac{1}{3}\left(\sum_{cyc}\frac{xa^2}{y+z}\right)^2\overset{Tsintsifas}{\geq}\frac{\sqrt{3}}{2}\left(2\sqrt{3}F\right)^2=$$

$$=\frac{3\sqrt{3}\cdot 4F^2}{2}=6\sqrt{3}F^2$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

From AM-GM, we have:

$$x^2+\frac{\sqrt{3}}{9x}+\frac{\sqrt{3}}{9x}\geq 3\cdot\sqrt[3]{x^2\cdot\frac{\sqrt{3}}{9x}\cdot\frac{\sqrt{3}}{9x}}=1$$

$$\rightarrow 1-x^2\leq\frac{2\sqrt{3}}{9x}\ (and\ analogs)$$

$$\sum_{cyc}\frac{xa^4}{(y+z)^2(1-x^2)}\geq\frac{9}{2\sqrt{3}}\sum_{cyc}\frac{x^2a^4}{(y+z)^2}=\frac{3\sqrt{3}}{2}\sum_{cyc}\left(\frac{xa^2}{y+z}\right)^2\overset{CBS}{\geq}\frac{\sqrt{3}}{2}\left(\sum_{cyc}\frac{xa^2}{y+z}\right)^2$$

We know that:

$$\sum_{cyc}\frac{x}{y+z}\cdot a^2\geq 2\sqrt{3}F\ (Tsintsifas)$$

Therefore,

$$\sum_{cyc}\frac{xa^4}{(y+z)^2(1-x^2)}\geq\frac{\sqrt{3}}{2}\left(2\sqrt{3}F\right)^2=6\sqrt{3}F^2$$

UP.361 Prove that:

$$\int_0^1\frac{tan^{-1}x}{x\sqrt{1-x^2}}dx=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log(sinx)\,dx$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$\because\prod_{k=1}^{n-1}sin\frac{k\pi}{n}=\frac{\sqrt{n}}{2^{n-1}}$$

Let: $x_k, k=1,2,\ldots,2n$ the roots of the unity.

$$x_k=cos\frac{k\pi}{2n}+isin\frac{k\pi}{2n},k=1,2,\ldots,2n$$

$$x^{2n}-1=\prod_{k=1}^{2n}(x-x_k)\overset{x_{1,2}=\pm 1-roots}{\triangleq}(x^2-1)\prod_{k=1}^{n-1}(x-x_k)(x-\overline{x_k})$$

$$=(x^2-1)\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right)$$

$$\Rightarrow x^{2n-2}+x^{2n-4}+\dots+x^2+1=\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right)\overset{x=1}{\Longrightarrow}$$

$$n=\prod_{k=1}^{n-1}\left(2-2\cos\frac{k\pi}{n}\right)=\prod_{k=1}^{n-1}\left(4\sin^2\frac{k\pi}{2n}\right)$$

$$n=2^{2(n-1)}\cdot\sin^2\frac{\pi}{2n}\cdot\sin^2\frac{2\pi}{2n}\cdot\ldots\cdot\sin^2\frac{(n-1)\pi}{2n}$$

$$2^{n-1}\cdot\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\ldots\cdot\sin\frac{(n-1)\pi}{2n}=\sqrt{n}\Rightarrow\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}=\frac{\sqrt{n}}{2^{n-1}}$$

$$\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx=\frac{1}{2}\int_0^{\pi}\log(\sin x)\,dx=\frac{\pi}{2}\int_0^1\log(\sin\pi x)\,dx=$$

$$=\lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=1}^{n-1}\log\left(\sin\frac{k\pi}{n}\right)=\lim_{n\to\infty}\frac{\pi}{2n}\log\left(\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}\right)=$$

$$=\lim_{n\to\infty}\frac{\pi}{2n}\log\left(\frac{\sqrt{n}}{2^{n-1}}\right)=\frac{\pi}{2}\lim_{n\to\infty}\frac{\log\sqrt{n}-(n-1)\log 2}{n}=-\frac{\pi}{2}\log 2;\ (1)$$

$$F(y)=\int_0^1\frac{\tan^{-1}xy}{x\sqrt{1-x^2}}dx\Rightarrow F'(x)=\int_0^1\frac{dx}{(1+x^2y^2)\sqrt{1-x^2}}=\int_0^{\frac{\pi}{2}}\frac{dx}{1+y^2\cos^2 t}$$

$$=\frac{1}{\sqrt{1+y^2}}\tan^{-1}\left(\frac{\tan t}{\sqrt{1+y^2}}\right)=\frac{\pi}{2\sqrt{1+y^2}}\Rightarrow$$

$$F(y)=\frac{\pi}{2}\log\left(y+\sqrt{1+y^2}\right)+C$$

$$\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\frac{\pi}{2}\log\left(1+\sqrt{2}\right);\ (2)$$

From (1), (2) we get:

$$\int_0^1 \frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx = \log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx$$

Solution 2 by Timson Azeez Folorunsho-Lagos-Nigeria

$$I = \log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx \overset{x\to\frac{\pi}{2}-x}{\cong} \log_2(\sqrt{2}-1)\int_{\frac{\pi}{2}}^{0}\log\left(\sin\left(\frac{\pi}{2}-x\right)\right)(-dx) =$$

$$= \log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx$$

$$2I = \log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log(\sin x\cdot\cos x)\,dx = \log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin 2x}{2}\right)dx =$$

$$= \log_2(\sqrt{2}-1)(\int_0^{\frac{\pi}{2}}\log(\sin 2x)\,dx - \int_0^{\frac{\pi}{2}}\log(2)\,dx \overset{2x=y}{\cong}$$

$$= \log_2(\sqrt{2}-1)(\frac{1}{2}\int_0^{\pi}\frac{1}{2}\log(\sin x)\,dx - \int_0^{\frac{\pi}{2}}\log(2)dx =$$

$$= \log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx - \frac{\pi}{2}\log(2)\log_2(\sqrt{2}-1) =$$

$$= I - \frac{\pi}{2}\log(2)\log_2(\sqrt{2}-1)$$

$$I = -\frac{\pi}{2}\log(2)\log_2(\sqrt{2}-1);\ (1)$$

$$J = \int_0^1 \frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx \overset{x=\sin y}{\cong} \int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\sin y)}{\sin y\cos y}\cos y\,dy = \int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\sin y)}{\sin y}dy$$

$$J(a) = \int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(a\sin y)}{\sin y}dy \to J'(a) = \int_0^{\frac{\pi}{2}}\frac{\sin y\,dy}{\sin y(1+a^2\sin^2 y)} = \int_0^{\frac{\pi}{2}}\frac{dy}{1+a^2\sin^2 y} =$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sec^2 y\,dy}{\sec^2 y + a^2\tan^2 y} = \int_0^{\frac{\pi}{2}}\frac{\sec^2 y\,dy}{1+\tan^2 y(1+a^2)} \overset{\substack{u=\tan y\\ du=\sec^2 y\,dy}}{\cong}$$

$$= \int_0^{\infty}\frac{du}{1+\left(u\sqrt{1+a^2}\right)^2} = \frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(u\sqrt{1+a^2}\right)\Big|_0^{\infty} = \frac{\pi}{2\sqrt{1+a^2}}$$

$$J'(a)=\frac{\pi}{2\sqrt{1+a^2}}\to J(a)=\frac{\pi}{2}\int\frac{da}{\sqrt{1+a^2}}\overset{a=\tan r}{\cong}\frac{\pi}{2}\int\frac{\sec^2 r\,dr}{\sec r}=$$

$$=\frac{\pi}{2}\int \sec r\,dr=\frac{\pi}{2}\log\left(a+\sqrt{1+a^2}\right)$$

$$J(1)=\frac{\pi}{2}\log(\sqrt{2}+1)=\frac{\pi}{2}\log\left(\frac{1}{\sqrt{2}-1}\right)=-\frac{\pi}{2}\log(\sqrt{2}-1);(2)$$

From $(1),(2)$ it follows that:

$$\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx$$

Solution 3 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\int_0^1\left(\int_0^1\frac{dx}{(1+x^2y^2)\sqrt{1-x^2}}\right)dy$$

$$\int_0^1\frac{dx}{(1+x^2y^2)\sqrt{1-x^2}}\overset{x=\sin\theta}{\cong}\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+y^2\sin^2\theta}\overset{t=\tan\theta}{\cong}\int_0^{\infty}\frac{dt}{1+t^2(y^2+1)}=\frac{\pi}{2\sqrt{y^2+1}}$$

$$\Omega=\frac{\pi}{2}\int_0^1\frac{dy}{\sqrt{y^2+1}}=\frac{\pi}{2}\sinh^{-1}(1)=\frac{\pi}{2}\log(\sqrt{2}+1)=-\frac{\pi}{2}\log(\sqrt{2}-1)=$$

$$=-\frac{\pi}{2}\log 2\cdot\frac{\log(\sqrt{2}-1)}{\log 2}$$

$$\text{Known }\int_0^{\frac{\pi}{2}}\log(\sin x)dx=-\frac{\pi}{2}\log 2$$

Therefore,

$$\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx$$

Solution 4 by Samar Das-India

$$I=\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx\overset{x=\sin y}{\cong}\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\sin y)}{\sin y\cos y}\cos y\,dy=\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\sin y)dy}{\sin y}$$

$$I(m)=\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(m\sin y)}{\sin y}dy$$

$$\frac{\partial I}{\partial m}=\int_0^{\frac{\pi}{2}}\frac{siny}{(1+m^2sin^2y)siny}dy=\int_0^{\frac{\pi}{2}}\frac{dy}{1+m^2sin^2y}=$$

$$=\int_0^{\frac{\pi}{2}}\frac{sec^2ydy}{1+(1+m^2)tan^2y}=\frac{1}{m^2+1}\int_0^{\frac{\pi}{2}}\frac{sec^2ydy}{\left(\frac{1}{\sqrt{m^2+1}}\right)^2+tan^2y}=$$

$$=\frac{1}{m^2+1}\cdot\frac{1}{\frac{1}{\sqrt{m^2+1}}}tan^{-1}\left(\frac{tany}{\frac{1}{\sqrt{m^2+1}}}\right)_0^{\frac{\pi}{2}}=\frac{\pi}{2\sqrt{m^2+1}};(1)$$

$$\to I=\frac{\pi}{2}\int\frac{dm}{\sqrt{m^2+1}}=\frac{\pi}{2}log\left(m+\sqrt{m^2+1}\right)+C;(2)$$

For $m=0\to I(0)=\int_0^{\frac{\pi}{2}}\frac{tan^{-1}(0\cdot siny)}{siny}dy=0$ from (2):

$$I(0)=\frac{\pi}{2}log(1)+C\to C=0$$

$$I(1)=\int_0^{\frac{\pi}{2}}\frac{tan^{-1}(1\cdot siny)}{siny}dy\overset{(2)}{\cong}\frac{\pi}{2}log(1+\sqrt{2});(I)$$

$$J=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log(sinx)\,dx\overset{x\to\frac{\pi}{2}-x}{\cong}log_2(\sqrt{2}-1)\int_{\frac{\pi}{2}}^0log\left(sin\left(\frac{\pi}{2}-x\right)\right)(-dx)=$$

$$=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log(cosx)\,dx$$

$$2J=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log(sinx\cdot cosx)\,dx=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log\left(\frac{sin2x}{2}\right)dx=$$

$$=log_2(\sqrt{2}-1)(\int_0^{\frac{\pi}{2}}log(sin2x)\,dx-\int_0^{\frac{\pi}{2}}log(2)\,dx\overset{2x=y}{\cong}$$

$$=log_2(\sqrt{2}-1)(\frac{1}{2}\int_0^{\pi}\frac{1}{2}log(sinx)\,dx-\int_0^{\frac{\pi}{2}}log(2)dx=$$

$$=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log(sinx)\,dx-\frac{\pi}{2}log(2)log_2(\sqrt{2}-1)=$$

$$=I-\frac{\pi}{2}log(2)log_2(\sqrt{2}-1)$$

$$J=-\frac{\pi}{2}log(2)log_2(\sqrt{2}-1);(II)$$

From $(I),(II)$ it follows that:

$$\int_0^1\frac{tan^{-1}x}{x\sqrt{1-x^2}}dx=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log(sinx)\,dx$$

Solution 5 by Akerele Segun-Lagos-Nigeria

$$tan^{-1}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1},|x|<1$$

$$\text{So, }\int_0^1\frac{tan^{-1}x}{x\sqrt{1-x^2}}dx=\int_0^1\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)\left(x\sqrt{1-x^2}\right)}dx=$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\int_0^1 x^{2k}(1-x^2)^{-\frac{1}{2}}dx\overset{x^2=u}{\cong}\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\int_0^1 u^{k-\frac{1}{2}}(1-u)^{-\frac{1}{2}}du=$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\beta\left(k+\frac{1}{2},\frac{1}{2}\right)=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k k\Gamma(2k)}{(2k+1)\Gamma(k+1)\Gamma(k+1)}=-\frac{\pi}{2}log(\sqrt{2}-1)=$$

$$=log_2(\sqrt{2}-1)\left(-\frac{\pi}{2}log(2)\right)$$

$$Recall:\int_0^{\frac{\pi}{2}}log(sinx)dx=-\frac{\pi}{2}log2$$

Therefore,

$$\int_0^1\frac{tan^{-1}x}{x\sqrt{1-x^2}}dx=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log(sinx)\,dx$$

$$\omega_n=\sum_{k=1}^{2n}cot\frac{k\pi}{2n+1}\cdot\left(sin\frac{2k\pi}{2n+1}+icos\frac{2k\pi}{2n+1}\right)$$

UP.362. Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{\omega_n+2-(k+n)}}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\because \sum_{k=1}^{2n}\left(cot\frac{k\pi}{2n+1}\right)\left(sin\frac{2k\pi}{2n+1}+icos\frac{2k\pi}{2n+1}\right)=2n-1, \forall n\in\mathbb{N}$$

$$\sum_{k=1}^{2n}\left(cot\frac{k\pi}{2n+1}\right)\left(sin\frac{2k\pi}{2n+1}+icos\frac{2k\pi}{2n+1}\right)=$$

$$=\sum_{k=1}^{n}\left(cot\frac{k\pi}{2n+1}\right)\left(sin\frac{2k\pi}{2n+1}+icos\frac{2k\pi}{2n+1}\right)+\sum_{k=n+1}^{2n}\left(cot\frac{k\pi}{2n+1}\right)\left(sin\frac{2k\pi}{2n+1}+icos\frac{2k\pi}{2n+1}\right)=$$

$$=S_1+S_2$$

Denote $2n+1-k=l$, **how** $k\in\{n+1,n+2,\dots,2n\}\Rightarrow l\in\{1,2,\dots,n-1,n\}$

$$S_2=\sum_{l=1}^{n}cot\frac{(2n+1-l)\pi}{2n+1}\left(sin\frac{(4n+2-2l)\pi}{2n+1}+icos\frac{(4n+2-2l)\pi}{2n+1}\right)=$$

$$=\sum_{k=1}^{n}cot\frac{l\pi}{2n+1}\left(sin\frac{2l\pi}{2n+1}-icos\frac{2l\pi}{2n+1}\right)\Rightarrow$$

$$S=S_1+S_2=2\sum_{k=1}^{n}cot\frac{2k\pi}{2n+1}sin\frac{2k\pi}{2n+1}=4\sum_{k=1}^{n}cos^2\frac{k\pi}{2n+1}=$$

$$=2\sum_{k=1}^{n}\left(1+cos\frac{2k\pi}{2n+1}\right)=2n+2\sum_{k=1}^{n}cos\frac{2k\pi}{2n+1}=2n-1\Rightarrow\omega_n=2n-1$$

$$\because cos\frac{[2n-(2k-1)]\pi}{2n+1}=-cos\frac{2k\pi}{2n+1}, \forall k=1,2\dots,n$$

$$\because \sum_{k=1}^{n}cos\frac{2k\pi}{2n+1}=1-2\left[cos\frac{(2n-1)\pi}{2n+1}+\cdots+cos\frac{\pi}{2n+1}\right]=-\frac{1}{2};$$

Hence,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{\omega_n+2-(k+n)}}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{2n-1+2-(k+n)}}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{n+1-k}};(1)$$

$$\frac{1}{k^{n+1-k}}=\frac{k^k}{k^{n+1}}=\frac{k^k}{\left(1+(k-1)\right)^{n+1}}\le\left(\frac{k}{k-1}\right)^k\cdot\frac{1}{\binom{n+1}{k}}=$$

$$=\left(1+\frac{1}{k-1}\right)\left(1+\frac{1}{k-1}\right)^{k-1}\cdot\frac{1}{\binom{n+1}{k}}\le\frac{2e}{\binom{n+1}{k}}\Rightarrow$$

$$1\le\sum_{k=1}^{n}\frac{1}{k^{\omega_n+2-(k+n)}}\le 1+2e\sum_{k=1}^{n-1}\frac{1}{\binom{n+1}{k}}+\frac{1}{n}\le 1+2e\cdot\frac{n-2}{\binom{n+1}{2}}+\frac{1}{n}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{\omega_n+2-(k+n)}}=1$$

UP.363 Let be $(a_n)_{n\ge1};(b_n)_{n\ge1}\subset(0,\infty)$ such that:

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a\in(0,\infty);b_n=\left(\prod_{k=1}^{n}a_{2k-1}\right)^{\frac{1}{n}}$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1}-\frac{a_n\cdot b_n}{n}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposer

By Cesaro-Stolz's theorem:

$$\lim_{n\to\infty}\frac{a_n}{n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}(a_{n+1}-a_n)=a$$

By Cauchy D'Alembert's theorem:

$$\lim_{n\to\infty}\frac{b_n}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{\prod_{k=1}^{n}a_{2k-1}}{n^n}}=\lim_{n\to\infty}\frac{\prod_{k=1}^{n+1}a_{2k-1}}{(n+1)^{n+1}}\cdot\frac{n^n}{\prod_{k=1}^{n}a_k}=$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{a_{2n+1}}{n+1}=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{a_{2n+1}}{2n+1}=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{a_{2n+1}}{2n+1}\cdot\frac{2n+1}{n+1}=\frac{2a}{e}$$

Hence,

$$\lim_{n\to\infty}\frac{a_n\cdot b_n}{n^2}=\lim_{n\to\infty}\frac{a_n}{n}\cdot\frac{b_n}{n}=a\cdot\frac{2a}{e}=\frac{2a^2}{e}$$

Denote $u_n=\frac{a_{n+1}\cdot b_{n+1}\cdot n}{a_n\cdot b_n\cdot(n+1)}$, hence $\lim_{n\to\infty}u_n=1$, $\lim_{n\to\infty}\frac{u_n-1}{\log u_n}=1$.

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \cdot \frac{\prod_{k=1}^{n+1} a_{2k-1}}{\prod_{k=1}^{n} a_{2k-1}} \cdot \frac{1}{\sqrt[n+1]{\prod_{k=1}^{n+1} a_{2k-1}}} \cdot \left(\frac{a_{2n+1}}{a_n}\right)^n =$$

$$= \frac{1}{e} \cdot \lim_{n\to\infty} \frac{a_{n+1}}{n+1} \cdot \frac{n+1}{b_{n+1}} \cdot \left(\left(1+\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}\cdot\frac{n}{a_n}\cdot(a_{n+1}-a_n)}\right) =$$

$$= \frac{1}{e} \cdot 2a \cdot \frac{e}{2a} \cdot e^{a\cdot\frac{1}{a}} = e$$

$$\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1} - \frac{a_n\cdot b_n}{n}\right) = \lim_{n\to\infty} \frac{a_n b_n}{n} \cdot \frac{u_n - 1}{log u_n} \cdot u_n =$$

$$= \lim_{n\to\infty} \frac{a_n b_n}{n^2} \cdot \frac{u_n - 1}{log u_n} \cdot log u_n^n = \frac{2a^2}{e} \cdot 1 \cdot e = \frac{2a^2}{e}$$

Solution 2 by Amrit Awasthi-India

We are given that: $\lim_{n\to\infty}(a_{n+1}-a_n) = a$. This can be rewritten as:

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{(n+1)-n} = a$$

Using Cesaro-Stolz Theorem, we have:

$$\lim_{n\to\infty}(a_{n+1}-a_n) = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{(n+1)-n} = \lim_{n\to\infty} \frac{a_n}{n} = a \to a_n = na$$

Now substituting this value in the given value of b_n we have

$$b_n = \left(\prod_{k=1}^{n}(2k-1)a\right)^{\frac{1}{n}} = a \cdot (1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1))^{\frac{1}{n}}$$

Now, the product can be rewritten as

$$(1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)) = \frac{(2n)!!}{n!\cdot 2^n} \to b_n = a \cdot \left(\frac{(2n)!}{n!\cdot 2^n}\right)^{\frac{1}{n}}$$

Now, we have:

$$\phi = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1} - \frac{a_n\cdot b_n}{n}\right) = \lim_{n\to\infty} \frac{\frac{a_{n+1}\cdot b_{n+1}}{n+1} - \frac{a_n\cdot b_n}{n}}{(n+1)-n}$$

Therefore, again using Cesaro-Stolz theorem, we have:

$$\phi=\lim_{n\to\infty}\frac{\frac{a_n\cdot b_n}{n}}{n}$$

Now, substituting previously obtained values, we have:

$$\phi\lim_{n\to\infty}\frac{a_n\cdot b_n}{n^2}=\lim_{n\to\infty}\frac{a^2\cdot\left(\frac{(2n)!}{n!\cdot 2^n}\right)^{\frac{1}{n}}}{n}$$

Therefore, using Stirling's approximation as n approaches infinity, we have:

$$\left((2n)!\right)^{\frac{1}{n}}\approx\left(\frac{2n}{e}\right)^2\Leftrightarrow(n!)^{\frac{1}{n}}\approx\frac{n}{e}$$

Hence, we have:

$$\phi=\lim_{n\to\infty}\frac{a^2}{2n}\cdot\frac{4n^2}{e^2}\cdot\frac{e}{n}=\frac{2a^2}{e}$$

Therefore,

$$\phi=\lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1}-\frac{a_n\cdot b_n}{n}\right)=\frac{2a^2}{e}$$

UP.364 In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{a^2}\right)\cdot\ldots\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)\geq\frac{3^n}{\sqrt{\left(R\sqrt{3}\right)^{n^2+n}}};\ n\in\mathbb{N}^*$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{1}{a^n}=\sum_{cyc}\frac{1^{n+1}}{a^n}\overset{Radon}{\geq}\frac{(1+1+\cdots+1)^{n+1}}{(a+b+c)^n}=\frac{3^{n+1}}{(2s)^n}=\frac{\left(\sqrt{3}\right)^{2n+2}}{(2s)^n}\overset{Mitrinovic}{\geq}$$

$$\geq\frac{\left(\sqrt{3}\right)^{2n+2}}{\left(3\sqrt{3}R\right)^n}=\frac{\left(\sqrt{3}\right)^{2n+2-3n}}{R^n}=\frac{\left(\sqrt{3}\right)^{2-n}}{R^n}$$

$$\prod_{i=1}^{n}\left(\sum_{cyc}\frac{1}{a_i}\right)\geq\prod_{cyc}\frac{\left(\sqrt{3}\right)^{2-i}}{R^i}=\frac{\left(\sqrt{3}\right)^{(2-1)+(2-2)+\cdots+(2-n)}}{R^{1+2+3+\cdots+n}}=\frac{\left(\sqrt{3}\right)^{2n-\frac{n(n+1)}{2}}}{R^{\frac{n(n+1)}{2}}}=$$

$$=\frac{(\sqrt{3})^{2n}}{(R\sqrt{3})^{\frac{n(n+1)}{2}}}=\frac{3^n}{\sqrt{(R\sqrt{3})^{n^2+n}}}.$$

Equality holds for an equilateral triangle.

Solution 2 by Adrian Popa-Romania

$$\sum_{cyc}\frac{1}{a^n}=\frac{1^{n+1}}{a^n}+\frac{1^{n+1}}{b^n}+\frac{1^{n+1}}{c^n}\overset{Radon}{\geq}\frac{(1+1+1)^{n+1}}{(a+b+c)^n}=\frac{3^{n+1}}{(2s)^n}\overset{s\leq\frac{3R\sqrt{3}}{2}}{\geq}$$

$$\geq\frac{3^{n+1}}{\left(2\cdot\frac{3R\sqrt{3}}{2}\right)^n}=\frac{3^{n+1}}{(3\sqrt{3}R)^n}=\frac{3^n\cdot 3}{3^{\frac{n}{2}}\cdot 3^n\cdot R^n}=\frac{3}{3^{\frac{n}{2}}\cdot R^n}$$

$$\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{a^2}\right)\cdot\ldots\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)\geq\frac{3}{3^{\frac{1}{2}}\cdot R^1}\cdot\frac{3}{3^{\frac{2}{2}}\cdot R^2}\cdot\ldots\cdot\frac{3}{3^{\frac{n}{2}}\cdot R^n}=$$

$$=\frac{3^n}{3^{\frac{1+2+3+\cdots+n}{2}}\cdot R^{1+2+3+\cdots+n}}=\frac{3^n}{\sqrt{3^{\frac{n(n+1)}{2}}\cdot R^{\frac{n(n+1)}{2}}}}\geq\frac{3^n}{\sqrt{(R\sqrt{3})^{n^2+n}}}$$

Solution 3 by George Florin Șerban-Romania

$$\sum_{cyc}\frac{1}{a}\geq\frac{\sqrt{3}}{R};(Petrovic`)$$

$$\sum_{cyc}\left(\frac{1}{a^2}\right)\overset{Holder}{\geq}\frac{1}{3}\left(\sum_{cyc}\frac{1}{a}\right)^2\geq\frac{1}{3}\left(\frac{\sqrt{3}}{R}\right)^2,\ldots,\sum_{cyc}\left(\frac{1}{a^n}\right)\overset{Holder}{\geq}\frac{1}{3^{n-1}}\left(\sum_{cyc}\frac{1}{a}\right)^n\geq\frac{1}{3^{n-1}}\left(\frac{\sqrt{3}}{R}\right)^n$$

$$\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{a^2}\right)\cdot\ldots\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)\geq\frac{\frac{\sqrt{3}}{R}\cdot\left(\frac{\sqrt{3}}{R}\right)^2\cdot\ldots\cdot\left(\frac{\sqrt{3}}{R}\right)^n}{3\cdot 3^2\cdot\ldots\cdot 3^{n-1}}=$$

$$=\frac{\sqrt{3}^{1+2+3+\cdots+n}}{R^{1+2+3+\cdots+n}\cdot 3^{\frac{n(n+1)}{2}}}=\frac{3^n\cdot 3^{\frac{n^2-3n}{4}}}{\sqrt{R^{n^2+n}}\cdot 3^{\frac{n^2-n}{2}}}=$$

$$=\frac{3^n\cdot 3^{\frac{n^2-3n}{4}-\frac{n^2-n}{2}}}{\sqrt{R^{n^2+n}}}=\frac{3^n}{\sqrt{R^{n^2+n}}\cdot 3^{\frac{n^2+n}{4}}}=\frac{3^n}{\sqrt{R^{n^2+n}}\cdot\sqrt{\sqrt{3^{n^2+n}}}}=\frac{3^n}{\sqrt{(R\sqrt{3})^{n^2+n}}}$$

UP.365 Let be $a_n=\sum_{k=1}^n tan^{-1}\left(\frac{1}{k^2+k+1}\right);n\geq 1$. Find:

$$\Omega=\lim_{n\to\infty}n^2(e^{a_{n+1}}-e^{a_n})$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

It's easy to prove that:

$$tan^{-1}\left(\frac{1}{k^2+k+1}\right)=tan^{-1}\left(\frac{1}{k}\right)-tan^{-1}\left(\frac{1}{k+1}\right),\forall k\in\mathbb{N}^*$$

It follows that:

$$a_n=tan^{-1}1-tan^{-1}\left(\frac{1}{n+1}\right)=\frac{\pi}{4}-tan^{-1}\left(\frac{1}{n+1}\right)\text{and }\lim_{n\to\infty}a_n=\frac{\pi}{4}$$

$$\Omega=\lim_{n\to\infty}n^2(e^{a_{n+1}}-e^{a_n})=\lim_{n\to\infty}n^2e^{a_n}(e^{a_{n+1}-a_n}-1)=$$

$$=\lim_{n\to\infty}n^2e^{a_n}\left(e^{\frac{1}{(n+1)^2+(n+1)+1}}-1\right)=\lim_{n\to\infty}n^2e^{a_n}\left(e^{\frac{1}{n^2+3n+3}}-1\right)=$$

$$=\lim_{n\to\infty}n^2e^{a_n}\left(\frac{e^{tan^{-1}\left(\frac{1}{n^2+3n+3}\right)}-1}{tan^{-1}\left(\frac{1}{n^2+3n+3}\right)}\right)\cdot tan^{-1}\left(\frac{1}{n^2+3n+3}\right)=$$

$$=\lim_{n\to\infty}e^{a_n}\cdot\frac{e^{\frac{1}{n^2+3n+3}}-1}{tan^{-1}\left(\frac{1}{n^2+3n+3}\right)}\cdot\frac{tan^{-1}\left(\frac{1}{n^2+3n+3}\right)}{\frac{1}{n^2+3n+3}}\cdot\frac{n^2}{n^2+3n+3}=$$

$$=e^{\frac{\pi}{4}}\cdot 1\cdot 1\cdot 1=e^{\frac{\pi}{4}}$$

Solution 2 by Asmat Qatea-Afghanistan

$$a_n=\sum_{k=1}^n\tan^{-1}\left(\frac{1}{k^2+k+1}\right)=\sum_{k=1}^n(\tan^{-1}(k+1)-\tan^{-1}k)=\tan^{-1}(n+1)-\tan^{-1}1$$

$$a_n=\tan^{-1}(n+1)-\frac{\pi}{4}$$

$$\Omega=\lim_{n\to\infty}n^2\left(e^{\tan^{-1}(n+2)-\frac{\pi}{4}}-e^{\tan^{-1}(n+1)-\frac{\pi}{4}}\right)=$$

$$=\frac{1}{\sqrt[4]{e^{\pi}}}\cdot\lim_{n\to\infty}n^2\,e^{\tan^{-1}(n+1)-\frac{\pi}{4}}\left(e^{\tan^{-1}(n+2)-\tan^{-1}(n+1)}-1\right)=$$

$$=\lim_{n\to\infty}\mathrm{e}^{\tan^{-1}(n+1)}\cdot\frac{e^{\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)}-1}{\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)}\cdot\frac{\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)}{\frac{1}{n^2}}=$$

$$=e^{\frac{\pi}{2}}\cdot\lim_{n\to\infty}\frac{\frac{-\frac{2n+3}{(n^2+3n+3)^2}}{1+\left(\frac{1}{n^2+3n+3}\right)^2}}{-\frac{2}{n^3}}=\frac{e^{\frac{\pi}{2}}}{2}\cdot\lim_{n\to\infty}\frac{2n^4+3n^3}{(n^2+3n+3)^2+9}=\frac{e^{\frac{\pi}{2}}}{2}\cdot\frac{2}{1}=e^{\frac{\pi}{2}}$$

Therefore,

$$\Omega=\lim_{n\to\infty}n^2(e^{a_{n+1}}-e^{a_n})=\frac{1}{\sqrt[4]{e^{\pi}}}\cdot e^{\frac{2\pi}{4}}=\sqrt[4]{e^{\pi}}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\textit{Let } b_k=\tan^{-1}\left(\frac{1}{k^2+k+1}\right)=\tan^{-1}\left(\frac{k+1-k}{1+k(k+1)}\right)=\tan^{-1}(k+1)-\tan^{-1}k$$

$$\Rightarrow\sum_{k=1}^{n}b_k=\tan^{-1}(n+1)-\frac{\pi}{4}=a_n$$

$$n^2(e^{a_{n+1}}-e^{a_n})=$$

$$=e^{\tan^{-1}(n+1)-\frac{\pi}{4}}\cdot\frac{e^{\tan^{-1}(n+2)}-e^{\tan^{-1}(n+1)}}{\tan^{-1}(n+2)-\tan^{-1}(n+1)}\cdot\frac{\tan^{-1}\left(\frac{1}{1+(n+1)(n+2)}\right)}{\frac{1}{1+(n+1)(n+2)}}$$

$$\cdot\frac{n^2}{(n+1)(n+2)+1}$$

$$\Omega=\lim_{n\to\infty}n^2(e^{a_{n+1}}-e^{a_n})=e^{\frac{\pi}{2}-\frac{\pi}{4}}=e^{\frac{\pi}{4}}$$

Solution 4 by Kaushik Mahanta-Assam-India

$$a_n=\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{k^2+k+1}\right)=\sum_{k=1}^{n}(\tan^{-1}(k+1)-\tan^{-1}k)=\tan^{-1}(n+1)-\tan^{-1}1$$

$$a_n=\tan^{-1}(n+1)-\frac{\pi}{4}$$

$$\Omega=\lim_{n\to\infty}n^2\left(e^{\tan^{-1}(n+2)-\frac{\pi}{4}}-e^{\tan^{-1}(n+1)-\frac{\pi}{4}}\right)=$$

$$=\lim_{n\to\infty}e^{-\frac{\pi}{4}}n^2\left(e^{\tan^{-1}(n+2)}-e^{\tan^{-1}(n+1)}\right)=$$

$$=\lim_{n\to\infty}e^{-\frac{\pi}{4}}n^2\left[\left(e^{\frac{\pi}{2}}-\frac{e^{\frac{\pi}{2}}}{n+2}+\cdots\right)-\left(e^{\frac{\pi}{2}}-\frac{e^{\frac{\pi}{2}}}{n+1}+\cdots\right)\right]=$$

$$=\lim_{n\to\infty}e^{-\frac{\pi}{4}}n^2\left(\frac{e^{\frac{\pi}{2}}}{n+1}-\frac{e^{\frac{\pi}{2}}}{n+2}\right)=\lim_{n\to\infty}e^{\frac{\pi}{2}-\frac{\pi}{4}}n^2\cdot\frac{1}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=e^{\frac{\pi}{4}}$$

UP.366. If $s_n=1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2\sqrt{n};n\geq 1$ find:

$$\Omega=\lim_{n\to\infty}\left(1+e^{s_{n+1}}-s^{s_n}\right)^{n\sqrt{n}}$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

It's known that: $\lim_{n\to\infty}s_n=s\in(-2,-1)$.

It follows that: $\lim_{n\to\infty}\left(e^{s_{n+1}}-e^{s_n}\right)=e^s-e^s=0$

$$\Omega=\lim_{n\to\infty}\left(\left(1+e^{s_{n+1}}-s^{s_n}\right)^{\frac{1}{e^{s_{n+1}}e^{s_n}}}\right)^{n\sqrt{n}\left(e^{s_{n+1}}-e^{s_n}\right)}=e^{\lim_{n\to\infty}n\sqrt{n}\left(e^{s_{n+1}}-e^{s_n}\right)};(1)$$

$$\lim_{n\to\infty}n\sqrt{n}\left(e^{s_{n+1}}-e^{s_n}\right)=\lim_{n\to\infty}e^{s_n}\left(e^{s_{n+1}-s_n}-1\right)n\sqrt{n}=$$

$$=e^s\cdot\lim_{n\to\infty}\frac{e^{s_{n+1}-s_n}-1}{s_{n+1}-s_n}\cdot n\sqrt{n}(s_{n+1}-s_n)=$$

$$=e^s\cdot 1\cdot\lim_{n\to\infty}n\sqrt{n}\left(\frac{1}{\sqrt{n+1}}-2\sqrt{n+1}+2\sqrt{n}\right)=$$

$$=e^s\cdot\lim_{n\to\infty}n\sqrt{n}\left(\frac{1}{\sqrt{n+1}}-2\left(\sqrt{n+1}-\sqrt{n}\right)\right)=e^s\lim_{n\to\infty}n\sqrt{n}\left(\frac{1}{\sqrt{n+1}}-\frac{2}{\sqrt{n+1}+\sqrt{n}}\right)=$$

$$=e^s\lim_{n\to\infty}n\sqrt{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}-2\sqrt{n+1}}{\sqrt{n+1}\left(\sqrt{n+1}+\sqrt{n}\right)}=$$

$$= -e^s \lim_{n\to\infty} \frac{n}{\sqrt{n+1}(\sqrt{n+1}-\sqrt{n})} \lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n}) =$$

$$= -\frac{1}{2} e^s \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = -\frac{1}{4} e^s; (2)$$

From (1),(2) it follows that:

$$\Omega = \lim_{n\to\infty} (1 + e^{s_{n+1}} - s^{s_n})^{n\sqrt{n}} = e^{-\frac{e^s}{4}}$$

Solution 2 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty} ((1 + e^{s_{n+1}} - s^{s_n})^{\frac{1}{e^{s_{n+1}}} e^{s_n}})^{n\sqrt{n}(e^{s_{n+1}} - e^{s_n})} = e^{\lim_{n\to\infty} n\sqrt{n}(e^{s_{n+1}} - e^{s_n})}; (1)$$

$$\lim_{n\to\infty} n\sqrt{n}(e^{s_{n+1}} - e^{s_n}) = \lim_{n\to\infty} e^{s_n}(e^{s_{n+1}-s_n} - 1)n\sqrt{n} =$$

$$= e^s \cdot \lim_{n\to\infty} \frac{e^{s_{n+1}-s_n} - 1}{s_{n+1} - s_n} \cdot n\sqrt{n}(s_{n+1} - s_n); (2)$$

$$\lim_{n\to\infty} e^{S_n} = e^s, where\ s - is\ Ioachimescu\ constant; (3)$$

$$\lim_{n\to\infty} \frac{e^{S_{n+1}-S_n} - 1}{S_{n+1} - S_n} = \log e = 1; (4)$$

$$\lim_{n\to\infty} n\sqrt{n}(S_{n+1} - S_n) = \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} + 2\sqrt{n}\right) =$$

$$= \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - 2(\sqrt{n+1} - \sqrt{n})\right) = \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - \frac{2}{\sqrt{n+1}+\sqrt{n}}\right) =$$

$$= \lim_{n\to\infty} n\sqrt{n} \cdot \frac{\sqrt{n+1} + \sqrt{n} - 2\sqrt{n+1}}{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})} =$$

$$= -\lim_{n\to\infty} \frac{n}{\sqrt{n+1}(\sqrt{n+1}-\sqrt{n})} \lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n}) =$$

$$= -\frac{1}{2} \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = -\frac{1}{4}; (5)$$

From $(1), (2), (3), (4), (5)$ it follows that: $\Omega = \lim_{n\to\infty} (1 + e^{s_{n+1}} - s^{s_n})^{n\sqrt{n}} = \frac{1}{\sqrt[4]{e^{e^s}}}$

Solution 3 by Felix Marin-Romania

$$s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}.$$

Indeed, s_n is related to the zeta function. Namely,

$$s_n = \zeta\left(\frac{1}{2}\right) + \underbrace{\frac{1}{2}\int_n^{\infty} \frac{\{x\}}{x^{\frac{3}{2}}}dx}_{<\frac{1}{\sqrt{n}}} \text{ such that } \lim_{n\to\infty} s_n = \zeta\left(\frac{1}{2}\right)$$

Moreover,

$$1 + e^{s_{n+1}} - e^{s_n} = 1 + e^{s_n}\left(e^{s_{n+1}-s_n} - 1\right) \overset{as\ n\to\infty}{\approx} 1 + e^{\zeta\left(\frac{1}{2}\right)}(s_{n+1} - s_n) =$$

$$= 1 + e^{\zeta\left(\frac{1}{2}\right)}\left(\frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} + 2\sqrt{n}\right)$$

$$\overset{as\ n\to\infty}{\approx} 1 + e^{\zeta\left(\frac{1}{2}\right)}\left[\frac{1}{n^{\frac{1}{2}}}\left(1 - \frac{1}{2n}\right) - 2n^{\frac{1}{2}}\left(1 + \frac{1}{2n} - \frac{1}{8n^2}\right) + 2n^{\frac{1}{2}}\right] = 1 - \frac{e^{\zeta\left(\frac{1}{2}\right)}}{4} - \frac{1}{n\sqrt{n}}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(1 + e^{s_{n+1}} - s^{s_n}\right)^{n\sqrt{n}} = \lim_{n\to\infty}\left\{1 - \left[exp\frac{\left(\zeta\left(\frac{1}{2}\right)\right)}{4}\right]\cdot\frac{1}{n\sqrt{n}}\right\}^{n\sqrt{n}} =$$

$$= exp\left(-\frac{\left(\zeta\left(\frac{1}{2}\right)\right)}{4}\right) \approx 0.9436$$

UP.367. Find: $\Omega = \lim_{n\to\infty}\sqrt[n]{\frac{((2n)!!)^n}{(2an)!!}}; a \in \mathbb{N}$

$$0!! = 1, (2k)!! = 2\cdot 4\cdot \ldots \cdot (2k), k \in \mathbb{N}^*$$

Proposed by D.M. Bătineţu-Giurgiu and Daniel Sitaru-Romania

Solution by proposers

Denote $x_n = \sqrt[n]{\frac{((2n)!!)^n}{(2an)!!}}; n \geq 2; e_n = \frac{(2n)!!}{n^n}$

For $a = 0; x_n = \sqrt[n]{\frac{(0!!)^n}{0!!}} = 1; n \geq 2;\ \lim_{n\to\infty} x_n = 1$

$$\text{If } a=1;\ x_n=\sqrt[n]{\frac{((2n)!!)^n}{(2an)!!}}=1;\ n\ge 2;\ \lim_{n\to\infty}x_n=1$$

$$\text{For } a\ge 1:\ x_n=\sqrt[n]{\frac{((2n)!!)^n}{(2an)!!}}=\left(\frac{\sqrt[n]{(2n)!!}}{n}\right)^a\cdot\frac{n^a}{\sqrt[n]{(2an)!!}}=\left(\sqrt[n]{\frac{(2n)!!}{n^n}}\right)^a\cdot\sqrt[n]{\frac{n^{na}}{(2an)!!}}$$

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\left(\left(\frac{(2n+2)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n)!!}\right)^a\cdot\frac{(n+1)^{a(n+1)}}{(2an+2a)!!}\cdot\frac{(2an)!!}{n^{na}}\right)=$$

$$=\lim_{n\to\infty}\left(\left(\frac{2n+2}{n+1}\cdot\frac{1}{e_n}\right)^a\cdot e_n^a\cdot\frac{(n+1)^a}{(2an+2)(2an+4)\cdot\ldots\cdot(2an+2a)}\right)=$$

$$=\frac{2^a}{e^a}\cdot e^a\cdot\lim_{n\to\infty}\frac{(n+1)^a}{2^a(an+1)(an+2)\cdot\ldots\cdot(an+a)}=$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{an+1}\cdot\frac{n+1}{an+2}\cdot\ldots\cdot\frac{n+1}{an+a}\right)=\frac{1}{a^a}$$

UP.368. Let be

$$\Omega_n=\int_n^{n+1}\frac{x^n}{e^x+1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}dx\,;n\in\mathbb{N}^*$$

Prove that: $\Omega_n<n!$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru-Romania

Solution 1 by proposers

$$\text{Let be } P_n(x)=1+\tfrac{x}{1!}+\tfrac{x^2}{2!}+\cdots+\tfrac{x^n}{n!}$$

$$P'_n(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}=P_{n-1}(x)$$

$$P_n(x)-P'_n(x)=\frac{x^n}{n!}\Rightarrow x^n=n!\left(P_n(x)-P'_n(x)\right)$$

$$x^n=n!\left(P_n(x)-P_{n-1}(x)\right)$$

$$\Omega_n(x)=\int_n^{n+1}\frac{n!\left(P_n(x)-P'_n(x)\right)}{e^x+P_n(x)}dx=n!\int_n^{n+1}\frac{n!\left(e^x+P_n(x)-e^x-P'_n(x)\right)}{e^x+P_n(x)}dx=$$

$$n!\int_n^{n+1}dx-n!\int_n^{n+1}\frac{e^x+P'_n(x)}{e^x+P_n(x)}dx=n!-n!\log\left(e^x+P_n(x)\right)_n^{n+1}=$$

$$= n!\left(1 - log\frac{e^{n+1} + P_n(n+1)}{e^n + P_n(n)}\right) < n!$$

Because $e^{n+1} > e^n; P_n(n+1) > P_n(n), P_n$ −increasing function, hence

$$\frac{e^{n+1}+P_n(n+1)}{e^n+P_n(n)} > 1 \text{and } log\frac{e^{n+1}+P_n(n+1)}{e^n+P_n(n)} > 0.$$

Solution 2 by Ravi Prakash-New Delhi-India

For $n \le x \le n+1$ we have:

$$e^x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} > e^x \to$$

$$f(x) = \frac{x^n}{e^x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}} < \frac{x^n}{e^x}$$

$$\to \int_n^{n+1} f(x)dx \le \int_n^{n+1} x^n e^{-x}\,dx < \int_0^{\infty} x^n e^{-x}\,dx$$

Therefore,

$$\int_n^{n+1} f(x)\,dx < \Gamma(n+1) = n!$$

Solution 3 by Jaihon Obaidullah-Afghanistan

For $n \le x \le n+1$ we have:

$$e^x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} > e^x \to$$

Also this integral smaller then

$$\int_0^{\infty} x^n e^{-x}dx = \int_0^{\infty} x^{(n+1)-1} e^{-x}\,dx = \Gamma(n+1) = n!$$

Solution 4 by Remus Florin Stanca-Romania

$$\int_n^{n+1} \frac{x^n}{e^x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}dx < n! \Leftrightarrow$$

$$\int_n^{n+1} \frac{\frac{x^n}{n!}}{e^x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}dx < 1$$

Let $f_n(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$ then,

$$\int_n^{n+1}\frac{f_n(x)-f_{n-1}(x)}{e^x+f_n(x)}dx<1\Leftrightarrow\int_n^{n+1}\frac{e^{-x}f_n(x)-e^{-x}f_{n-1}(x)}{1+e^{-x}f_n(x)}dx<1$$

$$f'_n(x)=1+\frac{x}{1!}+\cdots+\frac{x^{n-1}}{(n-1)!}=f_{n-1}(x)$$

$$\Leftrightarrow\int_n^{n+1}\frac{e^{-x}f_n(x)-e^{-x}f'_n(x)}{1+e^{-x}f_n(x)}dx<1\Leftrightarrow-\int_n^{n+1}\frac{-e^{-x}f_n(x)+e^{-x}f'_n(x)}{1+e^{-x}f_n(x)}dx<1$$

$$-\int_n^{n+1}\frac{(e^{-x})'f_n(x)+e^{-x}f'_n(x)}{1+e^{-x}f_n(x)}dx<1\Leftrightarrow-\int_n^{n+1}\frac{(e^{-x}f_n(x)+1)'}{e^{-x}f_n(x)+1}dx<1$$

$$\Leftrightarrow-\log(e^{-x}f_n(x)+1)|_n^{n+1}<1$$

$$\Leftrightarrow\log\left(\frac{e^{-n}f_n(n)+1}{e^{-n-1}f_n(n+1)+1}\right)<1\Leftrightarrow\frac{e^{-n}f_n(n)+1}{e^{-n-1}f_n(n+1)+1}<e$$

$$\Leftrightarrow e-1+e^{-n}\big(f_n(n+1)-f_n(n)\big)>0;(1)$$

$f'_n(x)=f_{n-1}(x),\forall x\geq0\to f_n(x)$ –increasing for $x\in\mathbb{R}_+\to f_n(n+1)>f_n(n)$

$\to e^{-n}\big(f_n(n+1)-f_n(n)\big)\geq0$ and $e-1>0$ then

$e-1+e^{-n}\big(f_n(n+1)-f_n(n)\big)>0$ and from (1) it follows that

$$\int_n^{n+1}\frac{x^n}{e^x+1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}dx<n!$$

UP.369. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function; $a,b>0;a<b;a+b=s;$ $f(s-x)+f(x)=c;\forall x\in\mathbb{R};c>0$. Find:

$$\Omega=\int_a^b(x^2-sx+s^2)f(x)dx$$

Proposed by D.M. Bătineţu-Giurgiu and Daniel Sitaru-Romania

Solution 1 by proposers

$$x=u(t)=s-x;u'(t)=-1;u(a)=b;u(b)=a$$

$$\Omega=\int_a^b(x^2-sx+s^2)f(x)dx=\int_a^b((s-t)^2-s(s-t)+s^2)f(s-t)dt=$$

$$=\int_a^b (t^2-st+s^2)\big(c-f(t)\big)dt=c\int_a^b (x^2-sx+s^2)dx-\int_a^b (x^2-sx+s^2)f(x)dx$$

$$\Omega=\frac{c}{2}\int_a^b (x^2-sx+s^2)dx=\frac{c}{2}\left(\frac{b^3-a^3}{3}-\frac{b^2-a^2}{3}+s^2(b-a)\right)=$$

$$=\frac{c(b-a)}{12}(2s^2-2sb+3s^2)=\frac{c(b-a)(5s^2-2sb)}{12}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Using: }\int_a^b g(x)dx=\int_a^b g(a+b-x)dx,\qquad \text{we get:}$$

$$\Omega=\int_a^b [(s-x)^2-s(s-x)+s^2]f(s-x)dx=\int_a^b (s^2-sx+x^2)\big(c-f(x)\big)dx$$

$$\Rightarrow 2\Omega=c\int_a^b (s^2-sx+x^2)dx=c\left[s^2(b-a)-\frac{s}{2}(b^2-a^2)+\frac{1}{3}(b^3-a^3)\right]=$$

$$=c(b-a)\left[s^2-\frac{s}{2}\cdot s+\frac{1}{3}(b^2+a^2+ab)\right]=\frac{c}{6}[3s^2-2(b^2+a^2+ab)](b-a)=$$

$$=\frac{c}{6}[3s^2-2(a+b)^2+2ab](b-a)=\frac{c}{6}(s^2+2ab)(b-a)$$

Solution 3 by Kamel Gandouli Rezgui-Tunisia

$$\int_a^b (x^2-sx+s^2)f(x)dx\overset{x=s-u}{=}\int_a^b ((s-u)^2-s(s-u)+s^2)f(s-u)du=$$

$$\int_a^b (-su+u^2+s^2)f(s-u)du=\int_a^b (-su+u^2+s^2)\big(c-f(u)\big)du$$

$$\Rightarrow 2\int_a^b (x^2-sx+s^2)f(x)dx=c\int_a^b (-su+u^2+s^2)du=$$

$$=c\left[-\frac{su^2}{2}+\frac{u^3}{3}+s^2u\right]_a^b=-\frac{scb^2}{2}+\frac{b^3c}{3}+s^2cb+\frac{csa^2}{2}+\frac{a^3c}{3}+s^2ca=$$

$$=-\frac{scb^2}{2}+\frac{b^3c}{3}+cs^3+\frac{sca^2}{2}+\frac{a^3c}{3}$$

Solution 4 by Kaushik Mahanta-Assam-India

$$\Omega=\int_a^b (x^2-(a+b)x+(a+b)^2)f(x)dx=$$

$$=\int_a^b \left((a+b)-x\right)^2-(a+b)\left((a+b-x)+(a+b)^2\right)f(a+b-x)dx=$$

$$=\int_a^b ((a+b)^2+x^2-2(a+b)x-(a+b)^2+(a+b)x+(a+b)^2)\left(c-f(x)\right)dx=$$

$$=\int_a^b (x^2-(a+b)x+(a+b)^2)c\,dx-\Omega$$

$$\Rightarrow 2\Omega=\int_a^b (x^2-(a+b)x+(a+b)^2)c\,dx=\left(\frac{x^3}{3}-(a+b)\frac{x^2}{2}+(a+b)^2x\right)\Bigg|_a^b\cdot c=$$

$$\Omega=\frac{c}{2}(\frac{(b^3-a^3)}{3}-(a+b)\cdot\frac{b^2-a^2}{2}+(a+b)^2(b-a)=\frac{c}{6}[(a+b)^2+2ab](b-a)$$

Solution 5 by Angel Plaza-Spain

By doing $x=s-t$, since $x^2-sx+s^2=(s-t)^2-s(s-t)+s^2=t^2-st+s^2$and $f(x)=f(s-t)=c-f(t)$, the integral becomes

$$\Omega=\int_a^b (t^2-st+s^2)\left(c-f(t)\right)dt=-\Omega+c\int_a^b (t^2-st+s^2)\,dt$$

$$\to\Omega=\frac{c}{2}\int_a^b (t^2-st+s^2)\,dt=\frac{c(b-a)}{12}(5a^2+8ab+5b^2)$$

UP. 370.If $a,b>0$, then

$$\int_0^{\frac{\pi}{4}}\frac{dx}{(x+1)(a^2cos^2x+b^2sin^2x)}<\frac{1}{ab(\pi+4)}\left(\pi\frac{b}{a}+4tan^{-1}\left(\frac{b}{a}\right)\right)$$

Proposed by Florică Anastase-Romania

Solution:

Theorem (Bonnet-Weierstrass):

If $f:[a,b]\to R$ decreasing function of C^1class and $g:[a,b]\to R$ continuous function, then $\exists c\in[a,b]$ such that:

$$\int_a^b f(x)g(x)dx=f(a)\int_a^c g(x)dx+f(b)\int_c^b g(x)dx$$

Demonstration:

$$\text{Let } h:[a,b]\to R, h(x)=f(x)-f(b) \text{ decreasing and } h(x)\ge 0, \forall x\in[a,b].$$

From second M.V.T. $\exists c\in[a,b]$ ***such that:***

$$\int_a^b g(x)h(x)dx = h(a)\int_a^c g(x)dx$$

$$\int_a^b g(x)\big(f(x)-f(b)\big)dx = (f(a)-f(b))\int_a^c g(x)dx$$

$$\int_a^b f(x)g(x)dx =$$

$$= f(b)\int_a^b g(x)dx + (f(b)-f(a))\int_a^c g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

q.e.d.

Let $f,g:\left[0,\frac{\pi}{4}\right]\to R, g(x)=\frac{1}{a^2cos^2x+b^2sin^2x}, f(x)=\frac{1}{x+1}, f'(x)=-\frac{1}{(x+1)^2}<0$ ***then f is decreasing***

$$G(x)=\int\frac{dx}{a^2cos^2x+b^2sin^2x}=$$

$$=\int\frac{1}{a^2+b^2tan^2x}\cdot\frac{dx}{cos^2x}=\frac{1}{b^2}\int\frac{dt}{\left(\frac{a}{b}\right)^2+t^2}=\frac{1}{ab}tan^{-1}\left(\frac{btanx}{a}\right)+C$$

Then $\exists c\in\left[0,\frac{\pi}{4}\right]$ ***for which:***

$$\int_0^{\frac{\pi}{4}}\frac{dx}{(x+1)(a^2cos^2x+b^2sin^2x)}=f(0)\big(G(c)-G(0)\big)+f\left(\frac{\pi}{4}\right)\big(G(b)-G(c)\big)=$$

$$=\frac{1}{ab}tan^{-1}\left(\frac{btanc}{a}\right)+\frac{1}{\frac{\pi}{4}+1}\cdot\frac{1}{ab}\left(tan^{-1}\frac{b}{a}-tan^{-1}\left(\frac{b}{a}tanc\right)\right)=$$

$$=\frac{1}{ab(\pi+4)}\left(\pi tan^{-1}\left(\frac{b}{a}tanc\right)+4tan^{-1}\frac{b}{a}\right)$$

$$\because tan^{-1}x < x, \forall x > 0 \rightarrow tan^{-1}\left(\frac{b}{a}tanc\right) < \frac{b}{a}\tan c < \frac{b}{a}tan\frac{\pi}{4} = \frac{b}{a}$$

Therefore,

$$\int_0^{\frac{\pi}{4}} \frac{dx}{(x+1)(a^2cos^2x + b^2sin^2x)} = \frac{1}{ab(\pi+4)}\left(\pi tan^{-1}\left(\frac{b}{a}tanc\right) + 4tan^{-1}\left(\frac{b}{a}\right)\right) <$$

$$< \frac{1}{ab(\pi+4)}\left(\pi\frac{b}{a} + 4tan^{-1}\left(\frac{b}{a}\right)\right)$$

UP.371. Let be $(a_n)_{n\geq1}; a_n = \prod_{k=1}^{n}((2k-1)!!)^{\frac{1}{k}}$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution by proposers

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{a_n}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!!}} =$$

$$= e\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}} = e\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}} = e\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n} =$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{n+1}{2n+1} = \frac{e^2}{2}.$$

If $u_n = \left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}; \forall n \geq 2$ then:

$$\lim_{n\to\infty}u_n = 1 \text{ and } \lim_{n\to\infty}\frac{u_n - 1}{logu_n} = 1$$

$$\lim_{n\to\infty}u_n^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{2n}{a_{n+1}}\cdot\sqrt[n+1]{a_{n+1}} = e^2\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{n+1} = e^2\cdot\frac{e}{2}\cdot\frac{2}{e^2}$$

$$= e$$

Hence,

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right) = \lim_{n\to\infty}\frac{n^2}{\sqrt[n]{a_n}}(u_n - 1) = \lim_{n\to\infty}\frac{n^2}{\sqrt[n]{a_n}}\cdot\frac{u_n - 1}{logu_n}\cdot u_n =$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{log u_n} \cdot u_n^n = \frac{e^2}{2} \cdot 1 \cdot loge = \frac{e^2}{2}$$

UP.372. $(x)_{n\geq 1}$ **−be a positive sequence of real numbers such that** $x_1 > 0$

and

$$x_{n+1} = \frac{x_n^2}{2x_n - log(1+x_n)}$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{(2n)!!}{(2n-1)!!} \cdot \frac{nx_n}{\sqrt{2n+1}}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

Let be the function: $f:(0,\infty) \to \mathbb{R}, f(x) = \frac{x^2}{2x - log(1+x)}$

$x_1 > 0$. Suppose $x_n > 0$ and how $x_n > log(1+x_n) \Rightarrow 2x_n > log(1+x_n) \Rightarrow$

$$\frac{x_n^2}{2x_n - log(1+x_n)} > 0 \Rightarrow x_{n+1} > 0$$

$$x_n > log(1+x_n) \Rightarrow x_n - log(1+x_n) > 0 \Rightarrow 2x_n - log(1+x_n) > x_n \Rightarrow \frac{x_{n+1}}{x_n} < 1$$

$\Rightarrow (x_n)_{n\geq 1} \searrow$. So, $(x_n)_{n\geq 1}$ −is convergent sequence.

$$\lim_{n\to\infty} \frac{1}{nx_n} = \lim_{n\to\infty} \left(\frac{1}{x_{n+1}} - \frac{1}{x_n}\right) = \lim_{n\to\infty} \frac{x_n - f(x_n)}{x_n f(x_n)} = \lim_{x\to 0} \frac{x - \frac{x^2}{2x - log(1+x)}}{\frac{x^3}{2x - log(1+x)}} =$$

$$= \lim_{x\to 0} \frac{x^2 - xlog(1+x)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{(1+x)^2}}{2} = \frac{1}{2}$$

Therefore, $\lim_{n\to\infty} nx_n = 2; (1)$

$$I_n(x) = \int_0^x \frac{t^n}{\sqrt{1-t^2}} dt; n \in \mathbb{N} \Rightarrow$$

$$I_n(x) = -t^{n-1} \cdot \sqrt{1-t^2}\Big|_0^x + (n-1)\int_0^x t^{n-2}\sqrt{1-t^2}dt =$$

$$= (n-1)I_{n-2} - (n-1)I_n - x^{n-1}\sqrt{1-x^2} \Rightarrow$$

$$I_n = \frac{n-1}{n} \cdot I_{n-2} = \dots = \frac{(n-1)!}{n!} I_0$$

$$I_{2n} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2};\ \ I_{2n+1} = \frac{(2n)!!}{(2n+1)!!}$$

How $\frac{I_{n+1}}{I_{n-1}} \to 1$, then $\frac{I_{2n+1}}{I_{2n}} \to 1$ and $\lim_{n\to\infty}\left(\frac{(2n)!!}{(2n+1)!!}\right)^2 \cdot \frac{1}{2n+1} = \frac{\pi}{2}; (2)$

From (1),(2) it follows that:

$$\Omega = \lim_{n\to\infty} \frac{(2n)!!}{(2n-1)!!} \cdot \frac{nx_n}{\sqrt{2n+1}} = 2\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}$$

Solution 2 by Adrian Popa-Romania

$$\frac{(2n)!!}{(2n-1)!!} = \frac{2\cdot 4\cdot 6\cdot \dots \cdot (2n)}{1\cdot 3\cdot 5\cdot \dots \cdot (2n-1)} = \frac{2^n \cdot n!}{\frac{(2n)!}{2^n \cdot n!}} = \frac{4^n \cdot (n!)^2}{(2n)!} = \frac{4^n}{\frac{(2n)!}{(n!)^2}} = \frac{4^n}{\binom{2n}{n}}$$

$$\lim_{n\to\infty} \frac{\binom{2n}{n}}{4^n}\sqrt{n\pi} = 1; (Wallis)$$

$$\frac{\binom{2n}{n}}{4^n} \cong \frac{1}{\sqrt{n\pi}} \to \frac{4^n}{\binom{2n}{n}} \cong \sqrt{n\pi}$$

$$\lim_{n\to\infty} \frac{(2n)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt{2n+1}} = \lim_{n\to\infty} \frac{\sqrt{n\pi}}{\sqrt{2n+1}} = \lim_{n\to\infty} \sqrt{\frac{n\pi}{2n+1}} = \frac{\pi}{2}$$

$$x_{n+1} = \frac{x_n^2}{2x_n - log(1+x_n)} \text{ and } \log(1+x_n) \le x_n \to$$

$$x_{n+1} < \frac{x_n^2}{2x_n - x_n} = \frac{x_n^2}{x_n} = x_n \to (x_n)_{n\ge 1} \searrow$$

Let be $l = \lim_{n\to\infty} x_n \to l = \frac{l^2}{2l - \log(1+l)} \to l^2 = l\log(1+l) \to l(l - \log(1+l)) = 0 \to l = 0$

So, $\lim_{n\to\infty} x_n = 0$.

Now, we have:

$$x_{n+1} = \frac{x_n^2}{2x_n - log(1+x_n)} = \frac{x_n^2}{2x_n - x_n + \frac{x_n^2}{2} - \frac{x_n^3}{3} + \frac{x_n^4}{4} - \cdots} =$$

$$=\frac{1}{\frac{2}{x_n}-\frac{1}{x_n}+\frac{1}{2}\underbrace{-\frac{x_n}{3}+\frac{x_n^2}{4}-\cdots}_{0}}=\frac{1}{\frac{1}{x_n}+\frac{1}{2}}=\frac{2x_n}{2+x_n}$$

$$x_2=\frac{2x_1}{2+x_1},x_3=\frac{2x_2}{2+x_2}=\frac{\frac{4x_1}{x_1+2}}{\frac{2x_1}{2+x_1}+2}=\frac{4x_1}{2+x_1}\cdot\frac{2+x_1}{4+4x_1}\to$$

$$x_3=\frac{2x_1}{2+2x_1}$$

$$x_4=\frac{2x_3}{2+x_3}=\frac{\frac{4x_1}{2+2x_1}}{\frac{2x_1}{2+2x_1}+2}=\frac{4x_1}{2+2x_1}\cdot\frac{2+2x_1}{4+6x_1}=\frac{2x_1}{2+3x_1}$$

Applying principle of mathematical induction, we get:

$$x_n=\frac{2x_1}{2+(n-1)x_1}\to\lim_{n\to\infty}\frac{2nx_1}{nx_1-x_1+2}=2$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!}\cdot\frac{nx_n}{\sqrt{2n+1}}=2\sqrt{\frac{\pi}{2}}=\sqrt{2\pi}$$

UP.373 $k\in\mathbb{N},k>0$ **and** $x_1>k,x_{n+1}=\frac{x_n^2}{x_n-k};n\in\mathbb{N}^*$

Find:

$$\Omega=\lim_{n\to\infty}\frac{n^2}{\log(\log n)}\cdot\frac{\sqrt[n]{H_n}-1}{x_n}$$

Proposed by Florică Anastase-Romania

Solution 1 by Ruxandra Daniela Tonilă-Romania

$$x_{n+1}=\frac{x_n^2}{x_n-k}\Leftrightarrow\frac{x_{n+1}}{x_n}=\frac{x_n}{x_n-k}>1\Rightarrow x_{n+1}>x_n,\text{so }(x_n)_{n\geq1}\nearrow$$

$$x_{n+1}=\frac{x_n^2}{x_n-k}|(-x_n)\Rightarrow x_{n+1}-x_n=\frac{kx_n}{x_n-k}$$

$$\begin{cases} x_n - x_{n-1} = \dfrac{kx_{n-1}}{x_{n-1}-k} \\ x_{n-1} - x_{n-2} = \dfrac{kx_{n-2}}{x_{n-2}-k} \\ \vdots \\ x_2 - x_1 = \dfrac{kx_1}{x_1-k} \end{cases}$$

Adding these relations, we get:

$$x_n = x_1 + \frac{kx_1}{x_1-k} + \frac{kx_2}{x_2-k} + \cdots + \frac{kx_{n-1}}{x_{n-1}-k} =$$

$$= x_1 + k\left(\frac{x_1}{x_1-k} + \frac{x_2}{x_2-k} + \cdots + \frac{x_{n-1}}{x_{n-1}-k}\right) > k \cdot n$$

Therefore,

$$\lim_{n\to\infty} x_n = \infty$$

$$\lim_{n\to\infty} \sqrt[n]{H_n} \overset{C-D}{=} \lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}}{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}} = 1$$

$$\Omega = \lim_{n\to\infty} \frac{n^2}{log(logn)} \cdot \frac{\sqrt[n]{H_n}-1}{x_n} = \lim_{n\to\infty} \frac{n}{x_n} \cdot \frac{e^{\log\sqrt[n]{H_n}}-1}{\log\sqrt[n]{H_n}} \cdot \log\sqrt[n]{H_n} \cdot \frac{n}{\log(\log n)} =$$

$$= \lim_{n\to\infty} \frac{n}{x_n} \cdot \lim_{n\to\infty} \frac{n}{\log(\log n)} \cdot \log\sqrt[n]{H_n}\,; (1)$$

$$\lim_{n\to\infty} \frac{n}{x_n} \overset{C-S}{=} \lim_{n\to\infty} \frac{n+1-n}{x_{n+1}-x_n} = \lim_{n\to\infty} \frac{1}{\frac{x_n^2}{x_n-k} - x_n} = \lim_{n\to\infty} \frac{x_n-k}{kx_n} = \frac{1}{k}; (2)$$

$$\lim_{n\to\infty} \frac{n}{\log(\log n)} \cdot \log\sqrt[n]{H_n} = \lim_{n\to\infty} \frac{\log H_n}{\log(\log n)} = \lim_{n\to\infty} \frac{\log H_n - \log(\log n) + \log(\log n)}{\log(\log n)} =$$

$$= \lim_{n\to\infty}\left(1 + \frac{\log\left(\frac{H_n}{\log n}\right)}{\log(\log n)}\right) = \lim_{n\to\infty}\left(1 + \frac{\log\left(\frac{H_n - \log n}{\log n} + 1\right)}{\log(\log n)}\right) = 1; (3)$$

$$\because \lim_{n\to\infty}(H_n - \log n) = \gamma \text{ and } \lim_{n\to\infty} \log\left(\frac{H_n - \log n}{\log n} + 1\right) = 0$$

From (1), (2), (3) it follows that:

$$\Omega = \lim_{n\to\infty} \frac{n^2}{log(logn)} \cdot \frac{\sqrt[n]{H_n} - 1}{x_n} = \frac{1}{k}$$

Solution 2 by proposer

$$\text{Let } y_n = \frac{1}{x_n};\ y_1 < \frac{1}{k} \Rightarrow y_1 \in \left(0, \frac{1}{k}\right), y_{n+1} = y_n(1 - ky_n)$$

Suppose $y_n \in \left(0, \frac{1}{k}\right) \Rightarrow 0 < ky_n < 1 \Rightarrow 0 < 1 - ky_n < 1 \Rightarrow 0 < y_n(1 - ky_n) < y_n \Rightarrow$

$y_{n+1} \in \left(0, \frac{1}{k}\right)$ and $(y_n)_{n\ge1} \searrow$. So, $(y_n)_{n\ge1}$ −convergent sequence, then $\exists y \in \mathbb{R}$ such that

$$\lim_{n\to\infty} y_n = y \Rightarrow y = y(1 - ky); \left(y < \frac{1}{k}\right) \Rightarrow y = 0$$

$$\lim_{n\to\infty} \frac{1}{ny_n} = \lim_{n\to\infty} \frac{\frac{1}{y_n}}{n} = \lim_{n\to\infty} \frac{y_n - y_{n+1}}{y_n \cdot y_{n+1}} = \lim_{n\to\infty} \frac{y_n - y_n(1 - ky_n)}{y_n^2(1 - ky_n)} =$$

$$= \lim_{n\to\infty} \frac{ky_n^2}{y_n^2(1 - ky_n)} = \lim_{y\to0} \frac{k}{1 - ky_n} = k$$

$$\text{Thus, } \lim_{n\to\infty} \frac{n}{x_n} = \frac{1}{k}$$

$$u_n = \sqrt[n]{H_n}; 1 < u_n < \sqrt[n]{n} \to 1 \Rightarrow u_n - 1 \to 0$$

$$\Omega = \lim_{n\to\infty} \frac{n^2}{\log(\log n)} \cdot \frac{\sqrt[n]{H_n} - 1}{x_n} = \lim_{n\to\infty} \frac{n}{x_n} \cdot \frac{n(\sqrt[n]{H_n} - 1)}{\log(\log n)} =$$

$$= \frac{1}{k} \cdot \lim_{n\to\infty} \frac{u_n - 1}{\frac{1}{n} log\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)} \cdot \frac{log\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)}{\log(\log n)} =$$

$$= \frac{1}{k} \cdot \lim_{n\to\infty} \frac{u_n - 1}{logu_n} \cdot \left(\frac{log\left(\frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{\log n}\right)}{\log(\log n)} + 1 \right) =$$

$$= \frac{1}{k} \cdot \lim_{n\to\infty} \frac{1}{log[1 + (u_n - 1)]^{\frac{1}{u_n - 1}}} \cdot \left(\frac{log\left(\frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{logn}\right)}{\log(\log n)} + 1 \right) = \frac{1}{k}$$

UP.374 Calculate the integral:

$$\int_0^1 \frac{x\ln x}{x^3 + x\sqrt{x} + 1}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by proposer

Let us denote: $I = \int_0^1 \frac{x\ln x}{x^3+x\sqrt{x}+1}dx$

In this integral we make the variable change: $x = z^{\frac{2}{3}}$

We obtain:

$$I = \frac{4}{9}\int_0^1 \frac{z^{\frac{1}{3}}\ln z}{z^2 + z + 1}dz$$

We have, successively:

$$I = \frac{4}{9}\int_0^1 \frac{(1-z)z^{\frac{1}{3}}\ln z}{1 - z^3}dz, \qquad I = \frac{4}{9}\left(\int_0^1 \frac{z^{\frac{1}{3}}\ln z}{1-z^3}dz - \int_0^1 \frac{z^{\frac{4}{3}}\ln z}{1-z^3}dz\right)$$

$$I = \frac{4}{9}\left(\int_0^1 \sum_{n=0}^{\infty} z^{3n+\frac{1}{3}}\ln z\,dz - \int_0^1 \sum_{n=0}^{\infty} z^{3n+\frac{4}{3}}\ln z\,dz\right)$$

$$I = \frac{4}{9}\sum_{n=0}^{\infty}\left(\int_0^1 z^{3n+\frac{1}{3}}\ln z\,dz - \int_0^1 z^{3n+\frac{4}{3}}\ln z\,dz\right)$$

We will use the following relationship:

$\int_0^1 x^3 \ln x\,dx = -\frac{1}{(a+1)^2}$, where $a \in \mathbb{R}, a \geq 0$

We obtain:

$$I = \frac{4}{9}\sum_{n=0}^{\infty}\left[\frac{1}{\left(3n+\frac{7}{3}\right)^2} - \frac{1}{\left(3n+\frac{4}{3}\right)^2}\right]$$

Or:

$$I=\frac{4}{9}\sum_{n=0}^{\infty}\left[\frac{\frac{1}{9}}{\left(n+\frac{7}{9}\right)^2}-\frac{\frac{1}{9}}{\left(n+\frac{4}{9}\right)^2}\right]$$

We now use the following relationship:

$$\Psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\Psi_1(x)$ is the trigamma function.

We have obtained the value of the integral required in the problem statement:

$$I=\frac{4}{81}\left[\Psi_1\left(\frac{7}{9}\right)-\Psi_1\left(\frac{4}{9}\right)\right]$$

Solution 2 by Kaushik Mahanta-Assam-India

$$\Omega=\int_0^1\frac{x\ln x}{x^3+x\sqrt{x}+1}dx\overset{\sqrt{x}=t}{\triangleq}\int_0^1\frac{t^2logt^2}{t^6+t^3+1}\cdot 2tdt=\int_0^1\frac{t^2\log t^2}{(t^3)^2+t^3+1}\cdot 2tdt=$$

$$=\int_0^1\frac{2t^2\log t\,(1-t^3)}{(1-t^3)(1+t^3+t^6)}\cdot 2tdt=2^2\int_0^1\frac{t^3(1-t^3)\log t}{1-t^9}dt\overset{t^9=m}{\triangleq}$$

$$=4\int_0^1\frac{m^{\frac{1}{3}}\left(1-m^{\frac{1}{3}}\right)\log m^{\frac{1}{9}}}{1-m}\cdot\frac{dm}{9m^{\frac{8}{9}}}=\frac{4}{81}\int_0^1\frac{m^{-\frac{5}{9}}\left(1-m^{\frac{1}{3}}\right)\log m\,dm}{1-m}=$$

$$=\frac{4}{81}\left(\int_0^1\frac{m^{-\frac{5}{9}}\log m}{1-m}dm-\int_0^1\frac{m^{-\frac{2}{9}}\log m}{1-m}dm\right)=$$

$$=\frac{4}{81}\left(\int_0^1\frac{m^{\frac{4}{9}-1}\log m}{1-m}dm-\int_0^1\frac{m^{\frac{7}{9}-1}\log m}{1-m}dm\right)=$$

$$=\frac{4}{81}\left(\psi^{(1)}\left(\frac{7}{9}\right)-\psi^{(1)}\left(\frac{4}{9}\right)\right)$$

Solution 3 by Serlea Kabay-Liberia

$$\Omega=\int_0^1\frac{x\ln x}{x^3+x\sqrt{x}+1}dx\overset{x=u^2}{\triangleq}4\int_0^1\frac{u^3\log u}{u^6+u^3+1}du=4\int_0^1\frac{(u^3-1)u^3\log u}{u^9-1}du$$

Let $\Omega(n)=\int_0^1 \frac{u^n}{u^9-1}du$ such that $\Omega(n)=4\left(\frac{\partial\Omega(n)}{\partial n}|_{n=6}-\frac{\partial\Omega(n)}{\partial n}|_{n=3}\right)$

$$\Omega(n)=-\sum_{i=0}^{\infty}\int_0^1 u^{9i+n}du=-\sum_{i=1}^{\infty}\frac{1}{n+9i+1}$$

$$\Rightarrow\frac{\partial\Omega(n)}{\partial n}=\sum_{i=1}^{\infty}\frac{1}{(n+9i+1)^2}$$

$$\text{Now, }\Omega=4(\sum_{i=0}^{infty}\frac{1}{(9i+7)^2}-\sum_{i=1}^{\infty}\frac{1}{(9i+4)^2}$$

$$\Omega=4\left(\frac{1}{81}\psi^{(1)}\left(\frac{7}{9}\right)-\frac{1}{81}\psi^{(1)}\left(\frac{4}{9}\right)\right)=\frac{4}{81}\left(\psi^{(1)}\left(\frac{7}{9}\right)-\psi^{(1)}\left(\frac{4}{9}\right)\right)$$

Therefore,

$$\Omega=\int_0^1\frac{x\ln x}{x^3+x\sqrt{x}+1}dx=\frac{4}{81}\left(\psi^{(1)}\left(\frac{7}{9}\right)-\psi^{(1)}\left(\frac{4}{9}\right)\right)$$

Solution 4 by Ajentunmobi Abdulquyyom-Nigeria

$$\Omega=\int_0^1\frac{x\ln x}{x^3+x\sqrt{x}+1}dx\overset{\sqrt{x}=u}{\cong}\int_0^1\frac{u^2\log u^2}{u^6+u^3+1}\cdot 2udu=4\int_0^1\frac{u^3\log u}{(u^3)^2+u^3+1}du=$$

$$=4\int_0^1\frac{u^3(1-u^3)\log u}{1-u^9}du=4\underbrace{\int_0^1\frac{u^3\log u}{1-u^9}du}_{A}-4\underbrace{\int_0^1\frac{u^6\log u}{1-u^9}du}_{B}$$

$$A=\int_0^1\sum_{n=0}^{\infty}u^{9n+3}\log u\,du\overset{IBP}{\cong}-4\int_0^1\frac{u^{9n+4-1}}{9n+4}du=-\left[\sum_{n=0}^{\infty}4\cdot\frac{u^{9n+4}}{(9n+4)^2}\right]_0^1=$$

$$=-\frac{4}{9^2}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{4}{9}\right)^2}=-\frac{4}{81}\psi^{(1)}\left(\frac{4}{9}\right)$$

$$B=4\int_0^1\frac{u^6\log u}{1-u^9}du=4\sum_{n=0}^{\infty}u^{9n+6}\log u\,du\overset{IBP}{\cong}-4\int_0^1\frac{u^{9n+7-1}}{9n+7}du$$

$$=-\left[\sum_{n=0}^{\infty}4\cdot\frac{u^{9n+7}}{(9n+7)^2}\right]_0^1=-\frac{4}{9^2}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{7}{9}\right)^2}=-\frac{4}{81}\psi^{(1)}\left(\frac{7}{9}\right)$$

Thus,

$$\Omega = A - B = \frac{4}{81}\left(\psi^{(1)}\left(\frac{7}{9}\right) - \psi^{(1)}\left(\frac{4}{9}\right)\right)$$

Solution 5 by Mohammad Rostami-Afghanistan

$$\Omega = \int_0^1 \frac{x\ln x}{x^3 + x\sqrt{x} + 1}dx \overset{\sqrt{x}=u}{=} \int_0^1 \frac{u^2 \log u^2}{u^6 + u^3 + 1}\cdot 2udu = 4\int_0^1 \frac{u^3 \log u}{u^6 + u^3 + 1}du \overset{u^3=t}{=}$$

$$= 4\int_0^1 \frac{t\log\sqrt[3]{t}}{t^2 + t + 1}\left(\frac{dt}{3\sqrt[3]{t^2}}\right) = \frac{4}{9}\int_0^1 \frac{t^{\frac{1}{3}}\log t}{t^2 + t + 1}dt = \frac{4}{9}\int_0^1 \frac{t^{\frac{1}{3}}(1-t)\log t}{1 - t^3}dt =$$

$$= \frac{4}{9}\left(\int_0^1 t^{\frac{1}{3}}\sum_{n=0}^{\infty}(t^3)^n \frac{\partial}{\partial a}|_{a=0}\, t^a dt - \int_0^1 t^{\frac{4}{3}}\sum_{k=0}^{\infty}(t^3)^k \frac{\partial}{\partial b}|_{b=0} t^b dt\right) =$$

$$= \frac{4}{9}\left(\sum_{n=0}^{\infty}\frac{\partial}{\partial a}|_{a=0}\int_0^1 t^{\frac{1}{3}+3n+a}dt - \sum_{k=0}^{\infty}\frac{\partial}{\partial b}|_{b=0}\int_0^1 t^{\frac{4}{3}+3k+b}dt\right) =$$

$$= \frac{4}{9}\left(\sum_{n=0}^{\infty}\left[\frac{3}{4+3a+9n}\right]'_{a=0} - \sum_{k=0}^{\infty}\left[\frac{3}{7+3b+9k}\right]'_{b=0}\right) =$$

$$= \frac{4}{9}\left(\sum_{n=0}^{\infty}\left[-\frac{9}{(4+3a+9n)^2}\right]_{a=0} - \sum_{k=0}^{\infty}\left[-\frac{9}{(7+3b+9k)^2}\right]_{b=0}\right) =$$

$$= \frac{4}{9}\left(-\frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{4}{9}\right)^2} + \frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{7}{9}\right)^2}\right) =$$

$$= \frac{4}{81}\left[\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{7}{9}\right)^2} - \sum_{k=0}^{\infty}\frac{1}{\left(n+\frac{4}{9}\right)^2}\right] = \frac{4}{81}\left(\psi^{(1)}\left(\frac{7}{9}\right) - \psi^{(1)}\left(\frac{4}{9}\right)\right)$$

UP.375 In any convex polygon $A_1A_2 \dots A_n, n \geq 3$ with the area F and the sides lengths $A_kA_{k+1} = a_k, k = \overline{1,n}, A_{n+1} = A_1$ the following inequality holds:

$$\sum_{k=1}^{n}\left(a_k - \sqrt{a_k a_{k+1}} + a_{k+1}\right)^2 \geq 4F \cdot \tan\frac{\pi}{n}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

Solution 1 by proposer

We have:

$$2\left(x-\sqrt{xy}+y\right)^2 \geq x^2+y^2, \forall x,y \in \mathbb{R}_+^* = (0,\infty) \quad (1)$$

Indeed, let be $u \in \mathbb{R}_+^*$, such that $x = u^2 y$ then the relationship (1) becomes:

$$2\left(u^2y-\sqrt{u^2y\cdot y}+y\right)^2 \geq u^4y^2+y^2 \Leftrightarrow 2y^2(u^2-u+1)^2 \geq (u^4+1)y^2 \Leftrightarrow$$

$$\Leftrightarrow 2(u^2-u+1)^2 \geq u^4+1 \Leftrightarrow 2u^4+2u^2+2-4u^3+4u^2-4u \geq u^4+1 \Leftrightarrow$$

$\Leftrightarrow u^4-4u^3+6u^2-4u+1 \geq 0 \Leftrightarrow (u-1)^4 \geq 0$ which is obvious with equality $\Leftrightarrow$

$$\Leftrightarrow u=1 \Leftrightarrow x=y.$$

Hence, according with (1) we have:

$$\sum_{k=1}^{n}\left(a_k-\sqrt{a_ka_{k+1}}+a_{k+1}\right)^2 \geq \frac{1}{2}\cdot\sum_{k=1}^{n}\left(a_k^2+a_{k+1}^2\right) =$$

E. Just-N. Schaunmberger

$$= \sum_{k=1}^{n} a_k^2 \geq 4\cdot F\cdot \tan\frac{\pi}{n}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let s be the semiperimeter of the polygon $A_1A_2\ldots A_n$.

We have : $a_k-\sqrt{a_ka_{k+1}}+a_{k+1} \overset{AM-GM}{\geq} a_k-\frac{a_k+a_{k+1}}{2}+a_{k+1} = \frac{a_k+a_{k+1}}{2}$,

$$\forall k=\overline{1,n},$$

$$\rightarrow \sum_{k=1}^{n}\left(a_k-\sqrt{a_ka_{k+1}}+a_{k+1}\right)^2 \geq \sum_{k=1}^{n}\left(\frac{a_k+a_{k+1}}{2}\right)^2 \overset{CBS}{\geq} \frac{1}{n}\left(\sum_{k=1}^{n}\frac{a_k+a_{k+1}}{2}\right)^2 = \frac{1}{n}\left(\sum_{k=1}^{n}a_k\right)^2$$

$$\rightarrow \sum_{k=1}^{n}\left(a_k-\sqrt{a_ka_{k+1}}+a_{k+1}\right)^2 \geq \frac{4s^2}{n}.$$

We know that, for any convex polygone with n sides, $s^2 \geq nF.\tan\frac{\pi}{n}$

Therefore,

$$\sum_{k=1}^{n}\left(a_k-\sqrt{a_ka_{k+1}}+a_{k+1}\right)^2 \geq 4F\cdot\tan\frac{\pi}{n}.$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru