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JP.376. Let ΔABC be an acute triangle. Prove that:

$$\sqrt{\frac{\sin A}{\sin B\cdot\sin C}}+\sqrt{\frac{\sin B}{\sin C\cdot\sin A}}+\sqrt{\frac{\sin C}{\sin A\cdot\sin B}}\geq\sqrt[4]{108}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

$$\sqrt{\frac{\sin A}{\sin B\cdot\sin C}}+\sqrt{\frac{\sin B}{\sin C\cdot\sin A}}+\sqrt{\frac{\sin C}{\sin A\cdot\sin B}}=$$

$$=\sqrt{\frac{\sin(B+C)}{\sin B\cdot\sin C}}+\sqrt{\frac{\sin(C+A)}{\sin C\cdot\sin A}}+\sqrt{\frac{\sin(A+B)}{\sin A\cdot\sin B}}=$$

$$=\sqrt{\frac{\sin B\cos C+\sin C\cos B}{\sin B\cdot\sin C}}+\sqrt{\frac{\sin C\cos A+\sin A\cos C}{\sin C\cdot\sin A}}$$

$$+\sqrt{\frac{\sin A\cos B+\sin B\cos A}{\sin A\cdot\sin B}}$$

$$=\sqrt{\cot C+\cot B}+\sqrt{\cot A+\cot C}+\sqrt{\cot B+\cot C}$$

$$\because (x+y+z)^3\geq 3(xy+yz+yz)$$

$$\left(\sqrt{\cot C+\cot B}+\sqrt{\cot A+\cot C}+\sqrt{\cot B+\cot C}\right)^2\geq$$

$$\geq 3\left(\sqrt{(\cot A+\cot B)(\cot B+\cot C)}+\sqrt{(\cot B+\cot C)(\cot C+\cot A)}\right.$$

$$\left.+\sqrt{(\cot C+\cot A)(\cot A+\cot B)}\right)=$$

$$=3\sqrt{\cot^2 A+(\cot A\cot B+\cot B\cot C+\cot C\cot A}+$$

$$+3\sqrt{\cot^2 B+(\cot A\cot B+\cot B\cot C+\cot C\cot A)}+$$

$$+3\sqrt{\cot^2 C+(\cot A\cot B+\cot B\cot C+\cot C\cot A)}$$

We know that: $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$. So,

$$\left(\sqrt{\cot C+\cot B}+\sqrt{\cot A+\cot C}+\sqrt{\cot B+\cot C}\right)^2\geq$$

$$\geq 3\left(\sqrt{\cot^2 A+1}+\sqrt{\cot^2 B+1}+\sqrt{\cot^2 C+1}\right)$$

Using Minkowski's inequality, we have:

$$3\left(\sqrt{\cot^2 A+1}+\sqrt{\cot^2 B+1}+\sqrt{\cot^2 C+1}\right)\geq$$

$$\geq 3\sqrt{(\cot A+\cot B+\cot C)+(1+1+1)^2}$$

$$\geq 3\sqrt{3(\cot A\cot B+\cot B\cot C+\cot C\cot A)+9}=3\sqrt{3\cdot 1+9}=6\sqrt{3}$$

So, $\sqrt{\cot C+\cot B}+\sqrt{\cot A+\cot C}+\sqrt{\cot B+\cot C}\geq\sqrt[4]{108}$

Therefore,

$$\sqrt{\frac{\sin A}{\sin B\cdot\sin C}}+\sqrt{\frac{\sin B}{\sin C\cdot\sin A}}+\sqrt{\frac{\sin C}{\sin A\cdot\sin B}}\geq\sqrt[4]{108}$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Henry Ricardo-New York-USA

We will use the known inequality

$$0<\sin A\sin B\sin C\leq\frac{3\sqrt{3}}{8};(*)$$

for acute triangles [Bottema *et al., Geometric Inequalities,* 2.8].

Now the AGM inequality yields

$$\sum_{cyc}\sqrt{\frac{\sin A}{\sin B\cdot\sin C}}\geq\frac{3}{\sqrt[6]{\sin A\sin B\sin C}}\overset{(*)}{\geq}\frac{3}{\sqrt[6]{\frac{3\sqrt{3}}{8}}}=\sqrt[4]{108}$$

Solution 3 by Daniel Văcaru-Romania

We have:

$$\sqrt{\frac{\sin A}{\sin B\cdot\sin C}}+\sqrt{\frac{\sin B}{\sin C\cdot\sin A}}+\sqrt{\frac{\sin C}{\sin A\cdot\sin B}}=\frac{\sin a+\sin B+\sin C}{\sqrt{\sin A\cdot\sin B\cdot\sin C}}\overset{AM-GM}{\geq}$$

$$\geq\frac{3\sqrt[3]{\sin A\cdot\sin B\cdot\sin C}}{\sqrt{\sin A\cdot\sin B\cdot\sin C}}=\frac{3}{\sqrt[6]{\sin A\cdot\sin B\cdot\sin C}}$$

But we know that:

$$\sin A\cdot\sin B\cdot\sin C\leq\left(\frac{\sqrt{3}}{2}\right)^2\Rightarrow\sqrt[6]{\sin A\cdot\sin B\cdot\sin C}\leq\frac{\sqrt[4]{3}}{\sqrt[4]{4}}\Rightarrow$$

$$\sqrt{\frac{\sin A}{\sin B\cdot\sin C}}+\sqrt{\frac{\sin B}{\sin C\cdot\sin A}}+\sqrt{\frac{\sin C}{\sin A\cdot\sin B}}\geq\frac{3\sqrt[4]{4}}{\sqrt[4]{3}}=\sqrt[4]{4\cdot27}=\sqrt[4]{108}$$

JP.377. Let a, b, c be positive real numbers such that

$$ab+bc+ca\leq a+b+c.\text{ Prove that:}$$

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}=\frac{a^3+b^3+c^3}{(abc)^2}$$

By Power Mean Inequality, we have: $\frac{a^3+b^3+c^3}{3}\geq\left(\frac{a+b+c}{3}\right)^3$. From the condition

$ab+bc+ca\leq a+b+c$, we have $a+b+c\geq abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$, so

$$\left(\frac{a+b+c}{3}\right)^3\geq\left(abc\cdot\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right)^3\rightarrow\frac{a^3+b^3+c^3}{3}\geq\left(abc\cdot\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right)^3$$

By the GM-HM Inequality, we have: $\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\geq\frac{1}{\sqrt[3]{abc}}$, so $\frac{a^3+b^3+c^3}{3}\geq(abc)^3\left(\frac{1}{\sqrt[3]{abc}}\right)^3=(abc)^2$

Namely,

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq 3$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$a+b+c\geq ab+bc+ca$$

$$a^3+b^3+c^3\geq\frac{(a+b+c)^3}{9}\geq\frac{(ab+bc+ca)^3}{9}=$$

$$=\frac{(ab)^3+(bc)^3+(ca)^3+3a^2b^3c++c^2a^3b+a^3c^2b+c^3b^2a+b^3a^2c+6(abc)^2}{9}\geq 3(abc)^2$$

$$(ab)^3+(bc)^3+(ca)^3+3a^2b^3c++c^2a^3b+a^3c^2b+c^3b^2a+b^3a^2c+6(abc)^2\geq 27(abc)^2$$

$$a^3+b^3+c^3\geq 3(abc)^2$$

$$\frac{a}{(bc)^2}+\frac{b}{(ca)^2}+\frac{c}{(ab)^2}\geq 3$$

Solution 3 by Fayssal Abdelli-Bejaia-Algerie

$$\frac{\frac{a}{(bc)^2}+\frac{b}{(ca)^2}+\frac{c}{(ab)^2}}{3}\geq\sqrt{\frac{abc}{(abc)^4}}\Rightarrow\frac{a}{(bc)^2}+\frac{b}{(ca)^2}+\frac{c}{(ab)^2}\geq 3\sqrt[3]{(abc)^3}$$

$$\Rightarrow\frac{a}{(bc)^2}+\frac{b}{(ca)^2}+\frac{c}{(ab)^2}\geq 3abc\overset{(?)}{\geq}3\Leftrightarrow abc\overset{(?)}{\geq}1;(A)$$

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc}\Rightarrow a+b+c\geq 3\sqrt[3]{abc}$$

$$\frac{ab+bc+ca}{3}\geq\sqrt[3]{(abc)^2}\Rightarrow ab+bc+ca\geq 3\sqrt[3]{(abc)^2}$$

$$\frac{a+b+c}{ab+bc+ca}\geq 1\Rightarrow\frac{3\sqrt[3]{abc}}{3\sqrt[3]{(abc)^2}}\geq 1\Rightarrow\frac{1}{\sqrt[3]{abc}}\geq 1\Rightarrow abc\leq 1;(B)$$

From $(A)\&(B)$ it follows that:

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq 3abc\geq 3$$

Solution 4 by Abdallah El Farissi-Bechar-Algerie

$$a+b+c\geq ab+bc+ca\Rightarrow$$

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}=\frac{1}{a^2b^2c^2}(a^3+b^3+c^3)\overset{x\to x^2-convex}{\geq}\frac{3}{a^2b^2c^2}\left(\frac{a+b+c}{3}\right)^3\geq$$

$$\geq\frac{3}{a^2b^2c^2}\left(\frac{ab+bc+ca}{3}\right)^3\overset{AGM}{\geq}3$$

Solution 5 by Michael Sterghiou-Greece

$$a+b+c\geq ab+bc+ca;(c)$$

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq 3;(1)$$

Let $(p,q,r)=(\sum a,\sum ab,\prod a)$ with $p\geq q$.

It holds that: $q^2\geq 3pr\geq 3qr$ or $q\geq 3r$. Now,

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}=\frac{1}{r}\cdot\sum_{cyc}\frac{a^2}{bc}\overset{CBS}{\geq}\frac{1}{r}\cdot\frac{p^2}{q}\overset{(c)}{\geq}\frac{q}{r}\overset{(2)}{\geq}3$$

Solution 6 by Henry Ricardo-New York-USA

We note that the AM-GM inequality gives us $ab+bc+ca \geq 3\sqrt[3]{(abc)^2}$, so that

$a+b+c \geq 3\sqrt[3]{(abc)^2}$. Furthermore, the power mean inequality yields

$$\left(\frac{a^3+b^3+c^3}{3}\right)^{\frac{1}{3}} \geq \frac{a+b+c}{3}, \text{or } a^3+b^3+c^3 \geq \frac{(a+b+c)^3}{9}$$

Now we have:

$$\sum_{cyc} \frac{a}{b^2c^2} = \sum_{cyc} \frac{a^3}{a^2b^2c^2} \geq \frac{(a+b+c)^3}{9(abc)^2} \geq \frac{27(abc)^2}{9(abc)^2} = 3$$

Equality holds if and only if $a=b=c=1$.

Comment. We could have used consequence of Holder's inequality:

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}$$

JP.378. Determine all triplets (x,y,z) of positive integers which satisfy the following two equations: $xy+z^2=31,\ \ x+yz^2=53$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Taking the sum and the difference of the corresponding sides of the given two equations, we obtain, respectively $(y+1)\cdot(x+z^2)=84$ and $(y-1)\cdot(-x+z^2)=22$.

Since both x and z are positive integers, we see that $y+1, y-1$ are factors of $84, 22$, respectively.

From this fact, we can conclude that only possible values of y are $2, 3$.

If $y=2$, we obtain by solving the simultaneous equations $x+z^2=28, -x+z^2=22$ that $(x,y,z)=(3,2,5)$.If $y=3$, analogous $(x,y,z)=(5,3,4)$.

Since it is easy to check that all of these triples satisfy the requirement, the desired answer is $(3,2,5), (5,3,4)$.

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

Let $xy+z^2=31; (1)$ and $x+yz^2=53; (2)$

$$(1)-(2)\Rightarrow x+yz^2-xy-z^2=22\Rightarrow x(1-y)+z^2(y-1)=22$$

$$\Rightarrow (z^2-x)(y-1)=22;(A)$$

$$(A)\Rightarrow (z^2-x=1)\&(y-1=22);(a)$$

$$\Rightarrow (z^2-x=22)\&(y-1=1);(b)$$

$$\Rightarrow (z^2-x=11)\&(y-1=2);(c)$$

$$\Rightarrow (z^2-x=2)\&(y-1=11);(a)$$

$$(a): y=23\ \&\ z^2-x=1\Rightarrow z^2=x+1\Rightarrow z^2+23x=31$$

From $(a)\&(1)\Rightarrow z^2=31-23x=x+1\Rightarrow 24x=31\Rightarrow x\notin\mathbb{N}$

$$(b):(y-2)\&(z^2-x=22)z^2=22+x,(1)\Rightarrow z^2=31-2x\Rightarrow$$

$$31-2x=22+x\Rightarrow 3x+9\Rightarrow x=3$$

$y=2\Rightarrow x=3\Rightarrow z^2=25\Rightarrow z=5$. Hence,$(x,y,z)=(3,2,5)$.

$$(c):(y=3)\&\ (z^20x=11)\Rightarrow z^2=11+x,(1)\Rightarrow z^2=31-3x\Rightarrow 31-3x=11+x$$

$\Rightarrow x=5, z^2=11+x=11+5=16\Rightarrow z=4$. Hence,

$$(x,y,z)=(5,3,4).$$

$$(d)(y=12)\&\ (z^2=2+x),(1)\Rightarrow z^2=31-12x\Rightarrow 31-12x=2+x\Rightarrow 13x=29$$

$\Rightarrow x\notin\mathbb{N}$. Finally $S=\{(5,3,4),(3,2,5)\}$.

JP.379. If $ABCD$ tetrahedron $AB=a, AD=b, AC=c, BD=d,$ $DC=e, CB=f, F$ –total area, then

$$a^4+b^4+c^4+d^4+e^4+f^4\geq 2F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

By Goldner's inequality:

In ΔABC: $a^4+c^4+f^4\geq 16[ABC]^2;(1)$

In ΔACD: $c^4+b^4+e^4\geq 16[ACD]^2;(2)$

In ΔABD: $a^4+b^4+d^4\geq 16[ABD]^2;(3)$

In ΔBCD: $e^4+f^4+d^4\geq 16[BCD]^2;(4)$

By adding $(1),(2),(3),(4)$ it follows that:

$$2(a^4+b^4+c^4+d^4+e^4+f^4)\ge 16\left(\frac{[ABC]^2}{1}+\frac{[ACD]^2}{1}+\frac{[ABD]^2}{1}+\frac{[BCD]^2}{1}\right)\ge$$

$$\overset{Bergstrom}{\ge} 16\cdot\frac{([ABC]+[ACD]+[ABD]+[BCD])^2}{1+1+1+1}=\frac{16}{4}\cdot F^2=4F^2$$

$$2(a^4+b^4+c^4+d^4+e^4+f^4)\ge 4F^2$$

Therefore,

$$a^4+b^4+c^4+d^4+e^4+f^4\ge 2F^2$$

Solution 2 by Ravi Prakash-New Delhi-India

First, we show that for a ΔABC, $a^4+b^4+c^4\ge 16F_1^2$, F_1 – area of ΔABC.

$$a^2+b^2+c^2-4\sqrt{3}=b^2+c^2-2bc\cdot\cos A+b^2+c^2-4\sqrt{3}\left(\frac{1}{2}bc\cdot\sin A\right)=$$

$$=2(b^2+c^2)-2bc(\cos A+\sqrt{3}\sin A)=$$

$$=2(b^2+c^2)-4bc\cdot\cos\left(\frac{\pi}{3}-A\right)=2(b^2+c^2)-4bc=2(b-c)^2\ge 0$$

$$\Rightarrow a^2+b^2+c^2\ge 4\sqrt{3}F_1$$

Now, $\dfrac{a^4+b^4+c^4}{3}\ge\left(\dfrac{a^2+b^2+c^2}{3}\right)^2\Rightarrow a^4+b^4+c^4\ge\dfrac{1}{3}\left(4\sqrt{3}F_1\right)^2=16F_1^2.$

In tetrahedron, let:

$$F_1=area(\Delta ABC); F_2=area(\Delta ACD); F_3=area(\Delta ABD); F_4=area(\Delta BCD)$$

$$AB=a, AD=b, CD=e, CB=f, AC=c, BD=d$$

$$F=F_1+F_2+F_3+F_4$$

Now,

In ΔABC: $a^4+c^4+f^4\ge 16\Delta_1^2; (1)$

In ΔACD: $c^4+b^4+e^4\ge 16\Delta_2^2; (2)$

In ΔABD: $a^4+b^4+d^4\ge 16\Delta_3^2; (3)$

In ΔBCD: $e^4+f^4+d^4\ge 16\Delta_4^2; (4)$

By adding $(1),(2),(3),(4)$ it follows that:

$$a^4+b^4+c^4+d^4+e^4+f^4\ge 8\left(\frac{\Delta_1^2}{1}+\frac{\Delta_2^2}{1}+\frac{\Delta_3^2}{1}+\frac{\Delta_4^2}{1}\right)\ge$$

$$\geq 32\cdot\frac{\Delta_1^2+\Delta_2^2+\Delta_3^2+\Delta_4^2}{4}\geq 32\left(\frac{\Delta_1+\Delta_2+\Delta_3+\Delta_4}{4}\right)^2=4F^2$$

JP.380. If $a,b,c,d>0,\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=4$ then:

$$\sum_{cyc}\frac{1}{a^4+b^4+c^4+5}\leq\frac{1}{2\sqrt{abcd}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$a^4+b^4+c^4+5=a^4+b^4+c^4+1+4\geq 4\sqrt[4]{a^4b^4c^4\cdot 1^4}+4=4abc+4=$$

$$=4(abc+1)=4\cdot 2\sqrt{abc}=8\sqrt{abc}$$

Hence,

$$\frac{1}{a^4+b^4+c^4+5}\leq\frac{1}{8\sqrt{abc}};(1)$$

$$\frac{1}{b^4+c^4+d^4+5}\leq\frac{1}{8\sqrt{bcd}};(2)$$

$$\frac{1}{c^4+d^4+a^4+5}\leq\frac{1}{8\sqrt{cda}};(3)$$

$$\frac{1}{a^4+b^4+d^4+5}\leq\frac{1}{8\sqrt{abd}};(4)$$

By adding $(1),(2),(3),(4)$ it follows that:

$$\sum_{cyc}\frac{1}{a^4+b^4+c^4+5}\leq\frac{1}{8}\sum_{cyc}\frac{1}{\sqrt{abc}}=\frac{1}{8}\cdot\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}}{\sqrt{abcd}}=$$

$$=\frac{4}{8\sqrt{abcd}}=\frac{1}{2\sqrt{abcd}}$$

Equality holds if and only if $a=b=c=d=1$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c,d>0$ and $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=4$ we give $a=x^2,b=y^2,c=z^2,d=t^2$

$$\Rightarrow x+y+z+t=4$$

$$\sum_{cyc}\frac{1}{a^4+b^4+c^4+5}=\sum_{cyc}\frac{1}{x^8+y^8+z^8+5}\leq\frac{1}{8}\sum_{cyc}\frac{1}{xyz}\leq\frac{1}{2xyzt}$$

which is true, because $x+y+z+x=4$.

Solution 3 by Asmat Qatea-Afghanistan

$$a^4+b^4+c^4+5\overset{AM-GM}{\geq}8\sqrt[8]{a^4b^4c^4}\Leftrightarrow a^4+b^4+c^4+5\geq 8\sqrt{abc}$$

$$\frac{1}{a^4+b^4+c^4+5}\leq\frac{1}{8\sqrt{abc}}$$

Hence,

$$\sum_{cyc}\frac{1}{a^4+b^4+c^4+5}\leq\sum_{cyc}\frac{1}{8\sqrt{abc}}\Leftrightarrow\sum_{cyc}\frac{1}{a^4+b^4+c^4+5}\leq\frac{1}{8\sqrt{abcd}}\sum_{cyc}\sqrt{a}\Leftrightarrow$$

$$\sum_{cyc}\frac{1}{a^4+b^4+c^4+5}\leq\frac{1}{2\sqrt{abcd}}$$

JP.381. If ABC and UVW are two triangles. Prove that:

$$\sum_{cyc}\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\left(1+\sin\frac{U}{2}\right)\geq\sum_{cyc}\cos\frac{U}{2}$$

Proposed by Cristian Miu-Romania

Solution 1 by proposer

Let us write Pedoe-Neuberg inequality for ABC and UVW. We obtain:

$$\sum_{cyc}u^2(b^2+c^2-a^2)\geq 16F_1F_2,$$

where F_1,F_2 are the areas of UVW and ABC.

This inequality can be written as:

$$\sum_{cyc}\cot A\,(\cot V+\cot W)\geq 2$$

We used the fact that:

$$\cot A=\frac{b^2+c^2-a^2}{4F_1}\ (and\ analogs)$$

Now, let us make the changes $A\to\frac{\pi+A}{4}, U\to\frac{\pi+U}{4}$ and the similarly. We obtain:

$$\sum_{cyc}\frac{\cos\frac{\pi+A}{4}}{\sin\frac{\pi+A}{4}}\left(\frac{\cos\frac{\pi+V}{4}}{\sin\frac{\pi+V}{4}}+\frac{\cos\frac{\pi+W}{4}}{\sin\frac{\pi+W}{4}}\right)\geq 2$$

So, we obtain:

$$\sum_{cyc}\frac{\cos\frac{A}{4}-\sin\frac{A}{4}}{\cos\frac{A}{4}+\sin\frac{A}{4}}\left(\frac{\cos\frac{V}{4}-\sin\frac{V}{4}}{\cos\frac{V}{4}+\sin\frac{V}{4}}+\frac{\cos\frac{W}{4}-\sin\frac{W}{4}}{\cos\frac{W}{4}+\sin\frac{W}{4}}\right)\geq 2$$

Which can be written as:

$$\sum_{cyc}\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\left(\frac{\cos\frac{V}{2}}{1+\sin\frac{V}{2}}+\frac{\cos\frac{W}{2}}{1+\sin\frac{W}{2}}\right)\geq 2$$

$$\sum_{cyc}\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\left(\frac{\cos\frac{U}{2}+\cos\frac{V}{2}+\cos\frac{W}{2}}{\left(1+\sin\frac{V}{2}\right)\left(1+\sin\frac{W}{2}\right)}\right)\geq 2$$

$$\left(\sum_{cyc}\cos\frac{U}{2}\right)\sum_{cyc}\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\left(1+\sin\frac{U}{2}\right)\geq 2\prod_{cyc}\left(1+\sin\frac{U}{2}\right)$$

But:

$$\left(\sum_{cyc}\cos\frac{U}{2}\right)^2=2\prod_{cyc}\left(1+\sin\frac{U}{2}\right)\to\sum_{cyc}\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\left(1+\sin\frac{U}{2}\right)\geq\sum_{cyc}\cos\frac{U}{2}$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if XYZ is an acute triangle then there is a triangle with angles $:\ \pi-2X,\pi-2Y,\pi-2Z.$

$$\to Let:\ \{A,B,C\}\to\{\pi-2A,\pi-2B,\pi-2C\}\ and\ \{U,V,W\}$$
$$\to\{\pi-2U,\pi-2V,\pi-2W\}$$

$$\to We\ need\ to\ prove:\ \sum_{cyc}\frac{\sin A}{1+\cos A}(1+\cos U)\geq\sum_{cyc}\sin U,\ \forall acute\ \Delta ABC,\Delta UVW$$

$$\leftrightarrow\sum_{cyc}\frac{2\sin\frac{A}{2}.\cos\frac{A}{2}}{2\cos^2\frac{A}{2}}.2\cos^2\frac{U}{2}\geq\sum_{cyc}\sin U\leftrightarrow 2\sum_{cyc}\tan\frac{A}{2}.\cos^2\frac{U}{2}\geq\sum_{cyc}\sin U\ \ (*)$$

Let again :

$$\{U,V,W\}\to\{\pi-2U,\pi-2V,\pi-2W\}$$

and F, R be the area and circumradius of ΔUVW.

$$\rightarrow (*) \leftrightarrow 2\sum_{cyc}\tan\frac{A}{2}.\sin^2 U \ge \sum_{cyc}\sin 2U = 4\prod_{cyc}\sin U = \frac{2F}{R^2}$$

$$\leftrightarrow \sum_{cyc}\tan\frac{A}{2}.\frac{u^2}{4R^2} \ge \frac{F}{R^2} \quad \leftrightarrow \quad \sum_{cyc}\tan\frac{A}{2}.u^2 \ge 4F.$$

Which is true from Oppenheim's inequality :

$$\sum_{cyc}\tan\frac{A}{2}.u^2 \ge 4F.\sqrt{\sum_{cyc}\tan\frac{A}{2}.\tan\frac{B}{2}} = 4F.$$

$$\textit{Therefore,} \qquad \sum_{cyc}\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\left(1+\sin\frac{U}{2}\right) \ge \sum_{cyc}\cos\frac{U}{2}.$$

JP.382 In acute $\Delta ABC, D \in (BC), E \in (AC), F \in (AB)$. Prove that:

$$\sqrt{\frac{AD^3+BE^3}{AD^5+BE^5}}+\sqrt{\frac{BE^3+CF^3}{BE^5+CF^5}}+\sqrt{\frac{CF^3+AD^3}{CF^5+AD^5}} \le \frac{1}{r}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

We have: $2(x^5+y^5) \le (x^3+y^3)(x^2+y^2)$

$$\begin{cases}\dfrac{x^3+y^3}{x^5+y^5} \le \dfrac{2}{x^2+y^2} \\ 2(x^2+y^2) \le (x+y)^2\end{cases} \rightarrow \frac{x^3+y^3}{x^5+y^5} \le \frac{4}{(x+y)^2} \rightarrow \sqrt{\frac{x^3+y^3}{x^5+y^5}} \le \frac{2}{x+y}$$

Hence,

$$\sum_{cyc}\sqrt{\frac{x^3+y^3}{x^5+y^5}} \le \sum_{cyc}\frac{2}{(x+y)^2}; (1)$$

$$\text{But: } \frac{1}{x+y} \le \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right) \rightarrow \frac{2}{x+y} \le \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right); (2)$$

From $(1), (2)$ it follows that:

$$\sum_{cyc}\sqrt{\frac{x^3+y^3}{x^5+y^5}} \le \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

Thus,

$$\sqrt{\frac{AD^3+BE^3}{AD^5+BE^5}}+\sqrt{\frac{BE^3+CF^3}{BE^5+CF^5}}+\sqrt{\frac{CF^3+AD^3}{CF^5+AF^5}}\le\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF};(3)$$

But: $AD\ge h_a, BE\ge h_b, CF\ge h_c\to\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}\le\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$

Therefore,

$$\sqrt{\frac{AD^3+BE^3}{AD^5+BE^5}}+\sqrt{\frac{BE^3+CF^3}{BE^5+CF^5}}+\sqrt{\frac{CF^3+AD^3}{CF^5+AF^5}}\le\frac{1}{r}$$

Solution 2 by Adrian Popa-Romania

$$\frac{x^3+y^3}{x^5+y^5}\le\frac{x^3+y^3}{x^4y+xy^4}=\frac{x^3+y^3}{xy(x^3+y^3)}=\frac{1}{xy}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge\frac{1}{\sqrt{xy}}+\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}};(\because\sum a^2\ge\sum ab)$$

Let $x=AD, y=BE, z=CF$ then,

$$\sqrt{\frac{AD^3+BE^3}{AD^5+BE^5}}+\sqrt{\frac{BE^3+CF^3}{BE^5+CF^5}}+\sqrt{\frac{CF^3+AD^3}{CF^5+AF^5}}\le\frac{1}{\sqrt{AD\cdot BE}}+\frac{1}{\sqrt{BE\cdot CF}}+\frac{1}{\sqrt{CF\cdot AD}}\le$$

$$\le\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}\le\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$$

JP.383 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{sinAsinB}}\ge 6$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$(a-b)^3\ge 0\Leftrightarrow 4a^2+4ab+4b^2\ge 3a^2+6ab+3b^2$$

$$\to\sqrt{a^2+ab+b^2}\ge\frac{\sqrt{3}}{2}(a+b)\to\frac{\sqrt{a^2+ab+b^2}}{ab}\ge\frac{\sqrt{3}}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$\sum_{cyc}\frac{\sqrt{a^2+ab+b^2}}{ab}\ge\sqrt{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\frac{3}{R}$$

$$\text{But: }\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\frac{\sqrt{3}}{R}(Leuenberger)\to$$

$$\sum_{cyc}\frac{\sqrt{4R^2\sin^2 A+4R^2\sin^2 B+4R^2 sinAsinB}}{4R^2\sin A\sin B}\ge\frac{3}{R}\Leftrightarrow$$

$$\sum_{cyc}\frac{\sqrt{\sin^2 A+\sin^2 B+sinAsinB}}{\sin A\sin B}\ge 6\Leftrightarrow\sum_{cyc}\sqrt{\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{sinAsinB}}\ge 6$$

JP.384. Solve for real numbers:

$$2^x+9^{\frac{1}{x}}+2^x\cdot 9^{\frac{1}{x}}=19$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Ionuț Florin Voinea - Romania

$$x\neq 0\Rightarrow\mathbb{D}=\mathbb{R}^*$$

1) If $x<0\Rightarrow 2^x+9^{\frac{1}{x}}+2^x\cdot 9^{\frac{1}{x}}<3$ **no has solutions.**

2) If $x>0$: $2^x+9^{\frac{1}{x}}+2^x\cdot 9^{\frac{1}{x}}=19\Leftrightarrow 2^x+9^{\frac{1}{x}}+2^x\cdot 9^{\frac{1}{x}}+1=20\Leftrightarrow$

$$(2^x+1)\left(9^{\frac{1}{x}}+1\right)=20\Leftrightarrow(2^x+1)\left(2^{\frac{1}{x}\log_2 9}+1\right)=20$$

Let $f:(0,\infty)\to\mathbb{R}, f(x)=(a^x+1)\left(a^{\frac{b}{x}}+1\right), a>1, b>0$, **which is decreasing on** $(0,\sqrt{b})$ **and increasing on** $[\sqrt{b},\infty)$. **We observe that** $f(x)=f\left(\frac{b}{x}\right),\forall x>0$.

For $a=2, b=\log_2 9\Rightarrow f(x)=(2^x+1)\left(2^{\frac{1}{x}\log_2 9}+1\right)$ **which is decreasing on** $(0,\sqrt{\log_2 9})$ **and increasing on** $(\sqrt{\log_2 9},\infty)$. **So, the equation** $f(x)=20$ **has two solutions or no has solutions.**

For $x=2\Rightarrow f(2)=(2^2+1)\left(2^{\frac{1}{2}\log_2 9}+1\right)=20\Rightarrow x=2$ **has solution.**

How $f(2)=f(\log_2 9)\Rightarrow x=\log_2 9$ **has solution.**

So, $x\in\{\log_2 3, 2\}$

Solution 2 by Jamal Issah-Teshie-Ghana

Let $2^x = a, 9^{\frac{1}{x}} = b \Rightarrow a + b + ab = 19 \Leftrightarrow a + (1 + a)b = 19$

Let's assume that a, b are consecutive positive integer numbers, hence $b = 1 + a \Rightarrow$

$a + (1 + a)(1 + a) = 19 \Leftrightarrow a^2 + 3a - 18 = 0 \Leftrightarrow a_1 = -6 < 0$ (impossible), $a_2 = 3$

$a = 3 \Rightarrow b = 4 \Rightarrow 2^x = 3 \Rightarrow x = \log_2 3$ and $a = 4 \Rightarrow x = 2$

So, $x \in \{\log_2 3, 2\}$

Solution 3 by proposer

If $x < 0$: $2^x + 9^{\frac{1}{x}} + 2^x \cdot 9^{\frac{1}{x}} < 3$ then, the equation has not solutions.

So, $x > 0$. Let $g: (0, \infty) \to R, f(x) = 2^x + 9^{\frac{1}{x}} + 2^x \cdot 9^{\frac{1}{x}}$ strictly increasing on $\left(\sqrt{\log_2 9}, \infty\right)$

and strictly decreasing on $\left(0, \sqrt{\log_2 9}\right)$. Thus, the equation admit unique solution

$x = 2$ on $\left(\sqrt{\log_2 9}, \infty\right)$ and $x = \log_2 3$ unique solution on $\left(0, \sqrt{\log_2 9}\right)$.

So, $x \in \{\log_2 3, 2\}$

JP.385 If $a, b, c > 0$ are such that $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{3}{4}$ then:

$$\sum_{cyc} \frac{a + 2b}{a^2 + 2b^2} + \sum_{cyc} \frac{b + 2a}{b^2 + 2a^2} \leq 3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

First we prove that:

$$\frac{a + 2b}{a^2 + 2b^2} + \frac{b + 2a}{b^2 + 2a^2} \leq \frac{4}{a + b}; (1)$$

$$\Leftrightarrow (a + b)(a + 2b)(b^2 + 2a^2) + (a + b)(b + 2a)(a^2 + 2b^2) \leq$$

$$\leq 4(a^2 + 2b^2)(b^2 + 2a^2)$$

$$\Leftrightarrow (a + b)(ab^2 + 2a^3 + 2b^3 + 4a^2b) + (a + b)(a^2b + 2b^3 + 2a^3 + 4ab^2) \leq$$

$$\leq 4(a^2b^2 + 2a^4 + 2b^4 + 4a^2b^2)$$

$$\Leftrightarrow a^2b^2 + 2a^4 + 2ab^3 + 4a^3b + ab^3 + 2a^3b + 2b^4 + 4a^2b^2 + a^3b + 2ab^3 + 2a^4 +$$

$$+4a^2b^2 + a^2b^2 + 2b^4 + 2a^3b + 4ab^3 \leq$$

$$\leq 4a^2b^2+8a^4+8b^4+16a^2b^2 \Leftrightarrow a^4-4ab^3+6a^2b^2-4ab^3+b^4\geq 0$$

$\Leftrightarrow (a-b)^4\geq 0$.Equality holds for $a=b$.Analogous:

$$\frac{b+2c}{b^2+2c^2}+\frac{c+2b}{c^2+2b^2}\leq\frac{4}{b+c};(2)$$

$$\frac{c+2a}{c^2+2a^2}+\frac{a+2c}{a^2+2c^2}\leq\frac{4}{c+a};(3)$$

By Adding $(1),(2),(3)$ it follows that:

$$\sum_{cyc}\frac{a+2b}{a^2+2b^2}+\sum_{cyc}\frac{b+2a}{b^2+2a^2}\leq 4\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)=4\cdot\frac{3}{4}=3.$$

Equality holds for $a=b=c=2$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\frac{a+2b}{a^2+2b^2}=\frac{a^2+2b^2+3ab}{(a+b)(a^2+2b^2)}=\frac{1}{a+b}+\frac{3ab}{(a+b)(a^2+2b^2)}$$

$$Similarly,\quad \frac{b+2a}{b^2+2a^2}=\frac{1}{a+b}+\frac{3ab}{(a+b)(b^2+2a^2)}$$

$$\rightarrow\frac{a+2b}{a^2+2b^2}+\frac{b+2a}{b^2+2a^2}$$

$$=\frac{2}{a+b}+\frac{3}{2(a+b)}\left(\frac{2ab}{a^2+2b^2}+\frac{2ab}{b^2+2a^2}\right)\overset{AM-GM}{\leq}\frac{2}{a+b}$$

$$+\frac{3}{2(a+b)}\left(\frac{2ab}{2ab+b^2}+\frac{2ab}{2ab+a^2}\right)=$$

$$=\frac{2}{a+b}+\frac{3}{2(a+b)}\left[2-\left(\frac{b^2}{2ab+b^2}+\frac{a^2}{2ab+a^2}\right)\right]\overset{Bergstrom}{\leq}$$

$$\leq\frac{2}{a+b}+\frac{3}{2(a+b)}\left(2-\frac{(a+b)^2}{(a+b)^2+2ab}\right)\leq$$

$$\overset{AM-GM}{\leq}\frac{2}{a+b}+\frac{3}{2(a+b)}\left(2-\frac{(a+b)^2}{(a+b)^2+\frac{(a+b)^2}{2}}\right)=\frac{2}{a+b}+\frac{3}{2(a+b)}\left(2-\frac{2}{3}\right)=\frac{4}{a+b}$$

$$\rightarrow\frac{a+2b}{a^2+2b^2}+\frac{b+2a}{b^2+2a^2}\leq\frac{4}{a+b}\ (And\ analogs)$$

$$Therefore,\sum_{cyc}\frac{a+2b}{a^2+2b^2}+\sum_{cyc}\frac{b+2a}{b^2+2a^2}\leq\sum_{cyc}\frac{4}{a+b}=3.$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{a+2b}{a^2+2b^2}+\frac{b+2a}{b^2+2a^2}=\frac{(a+2b)(b^2+2a^2)+(b+2a)(a^2+2b^2)}{(a^2+2b^2)(b^2+2a^2)}=$$

$$=\frac{ab^2+2a^3+2b^3+4a^2b+a^2b+2b^3+2a^3+4ab^2}{a^2b^2+2a^4+2b^4+4a^2b^2}\le\frac{4}{a+b}$$

$$(a+b)\left(ab^2+a^2b+4(a^2b+ab^2)+4(a^3+b^3)\right)\le$$

$$\le 4((ab)^2+2b^4+2c^4+4(ab)^2)$$

$$a^2b^2+a^3b+4(a^3b+a^2b^2)+4(a^4+ab^3)+ab^3+a^2b^2+4(a^2b^2+ab^3)$$
$$+4(a^3b+b^4)\le 2a(ab)^2+8(a^4+b^4)$$

$$a^3b+4a^3b+4ab^3+ab^3+4ab^3+4a^3b+4(a^4+b^4)\le 10(ab)^2+8(a^4+b^4)$$

$$9(a^3b+b^3a)\le 10(ab)^2+4(a^4+b^4)$$

$$5(a^3b^2-(ab)^2)+5(ab^4-(ab)^2)\le 4(a^4-a^3b)+4(b^4-b^2a)$$

$$5a^2b(a-b)+5ab^2(b-a)\le 4a^3(a-b)+4b^3(b-a)$$

$$(a-b)(5a^2b-5ab^2)\le 4(a-b)(a^3-b^3)$$

$$5ab(a-b)(a-b)\le 4(a-b)(a-b)(a^2+ab+b^2)$$

$5ab\le 4(a^2+ab+b^2)$. Hence,

$$\frac{a+2b}{a^2+2b^2}+\frac{b+2a}{b^2+2a^2}\le\frac{4}{b+c}\text{ and }\frac{c+2a}{c^2+2a^2}+\frac{a+2c}{a^2+2c^2}\le\frac{4}{c+a}$$

Therefore,

$$\sum_{cyc}\frac{a+2b}{a^2+2b^2}+\sum_{cyc}\frac{b+2a}{b^2+2a^2}\le\sum_{cyc}\frac{4}{a+b}=3$$

JP.386. In $\Delta ABC, n\in\mathbb{N}$ the following relationship holds:

$$\sum_{cyc}\frac{|a-b||a-c|^n a^{3n}}{b^{3n-3}|b-c|^{n-1}}\ge 16sr^2(4R^2+5Rr+r^2-s^2)$$

Proposed by George Florin Serban-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{x^{2n}}{y^{2n-2}}\ge\sum_{cyc}x^2,\forall x,y>0;(To\ prove)$$

$$\frac{x^{2n}}{y^{2n-2}}+\frac{x^{2n}}{y^{2n-2}}+y^2+y^2+\cdots+y^2\overset{AM-GM}{\geq}2n\cdot\sqrt[2n]{\frac{x^{4n}}{y^{4n-4}}\cdot y^{4n-4}}=2nx^2$$

$$\frac{2x^{2n}}{y^{2n-2}}+(2n-2)y^2\geq 2nx^2$$

$$2\sum_{cyc}\frac{x^{2n}}{y^{2n-2}}+(2n-2)\sum_{cyc}x^2\geq 2n\sum_{cyc}x^2\Rightarrow\sum_{cyc}\frac{x^{2n}}{y^{2n-2}}\geq\sum_{cyc}x^2$$

Let $x=a\sqrt{a|a-b|\cdot|a-c|}, y=b\sqrt{b|b-c|\cdot|b-a|}, z=c\sqrt{c|c-a|\cdot|c-b|}$

$$\sum_{cyc}\frac{x^{2n}}{y^{2n-2}}=\sum_{cyc}\frac{|a-b||a-c|^na^{3n}}{b^{3n-3}|b-c|^{n-1}}\geq\sum_{cyc}a^3|a-b|\cdot|a-c|\geq$$

$$\geq\sum_{cyc}a^3(a-b)(a-c)=16sr^2(4R^2+5Rr+r^2-s^2)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\sum_{cyc}\frac{|a-b||a-c|^na^{3n}}{b^{3n-3}|b-c|^{n-1}}$$

$$=\sum_{cyc}\frac{(|a-b||a-c|a^3)^n}{(|b-a||b-c|b^3)^{n-1}}\overset{Holder}{\geq}\frac{\left(\sum\ |a-b||a-c|a^3\right)^n}{\left(\sum\ |b-a||b-c|b^3\right)^{n-1}}$$

$$=\sum_{cyc}|a-b||a-c|a^3\geq$$

$$\geq\sum_{cyc}(a-b)(a-c)a^3=\sum_{cyc}a^5-\sum_{sym}a^4b+abc\sum_{cyc}a^2$$

$$=\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^4\right)-2\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}ab\right)+3abc\sum_{cyc}a^2=$$

$$=2s.2(s^4-6s^2r^2-8s^2Rr+8Rr^3+16R^2r^2+r^4)$$

$$-2.2s(s^2-3r^2-6Rr)(s^2+r^2+4Rr)+3.4Rrs.2(s^2-r^2-4Rr)$$

$$=16sr^2(4R^2+5Rr+r^2-s^2).$$

Therefore, $\sum_{cyc} \frac{|a-b||a-c|^n a^{3n}}{b^{3n-3}|b-c|^{n-1}} \geq 16sr^2(4R^2+5Rr+r^2-s^2), \forall n \in N_{\geq 2}$

JP.387. Let ΔABC be any triangle. Prove that for any $n \in \mathbb{N}-\{0,1\}$ the following relationship holds:

$$\frac{|a-b|\cdot|a-c|^n\cdot a^{2n}}{b^{2n-2}|b-c|^{n-1}}+\frac{|b-c|\cdot|b-a|^n\cdot b^{2n}}{c^{2n-2}|c-a|^{n-1}}+\frac{|a-c|\cdot|c-b|^n\cdot c^{2n}}{a^{2n-2}|a-b|^{n-1}} \geq 4r^2[(4R+r)^2-3s^2]$$

Proposed by George Florin Șerban-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{x^{2n}}{y^{2n-2}} \geq \sum_{cyc} x^2, \forall x,y>0; (To\ prove)$$

$$\frac{x^{2n}}{y^{2n-2}}+\frac{x^{2n}}{y^{2n-2}}+y^2+y^2+\cdots+y^2 \overset{AM-GM}{\geq} 2n\cdot\sqrt[2n]{\frac{x^{4n}}{y^{4n-4}}\cdot y^{4n-4}} = 2nx^2$$

$$\frac{2x^{2n}}{y^{2n-2}}+(2n-2)y^2 \geq 2nx^2$$

$$2\sum_{cyc}\frac{x^{2n}}{y^{2n-2}}+(2n-2)\sum_{cyc}x^2 \geq 2n\sum_{cyc}x^2 \Rightarrow \sum_{cyc}\frac{x^{2n}}{y^{2n-2}} \geq \sum_{cyc}x^2$$

Let $x=a\sqrt{|a-b|\cdot|a-c|}, y=b\sqrt{|b-c|\cdot|b-a|}, z=c\sqrt{|c-a|\cdot|c-b|}$

$$\sum_{cyc}\frac{x^{2n}}{y^{2n-2}}=\sum_{cyc}\frac{|a-b|\cdot|a-c|^n\cdot a^{2n}}{b^{2n-2}|b-c|^{n-1}} \geq \sum_{cyc}a^2|a-b|\cdot|a-c| \geq$$

$$\geq \sum_{cyc}a^2(a-b)(a-c)=4r^2[(4R+r)^2-3s^2]$$

Solution 2 by Adrian Popa-Romania

$$\sum_{cyc}\frac{|a-b|\cdot|a-c|^n\cdot a^{2n}}{b^{2n-2}|b-c|^{n-1}}=\sum_{cyc}\frac{|a-b|\cdot(a^2\cdot|a-c|)^n}{\left(b^2(b-c)\right)^{n-1}}=$$

$$=\sum_{cyc}\frac{(a^2|a-b|\cdot|a-c|)^n}{(|a-b|\cdot|b-c|\cdot b^2)^{n-1}}\overset{Radon}{\geq}\frac{(\sum(a^2|a-b|\cdot|a-c|))^n}{(\sum b^2\cdot|b-a|\cdot|b-c|)^{n-1}}=$$

$$=\sum_{cyc}a^2|a-b|\cdot|a-c|\geq\sum_{cyc}a^2(a-b)(a-c)=\sum_{cyc}a^2(a^2-ac-ab+bc)=$$

$$=\sum a^4-\sum a^3b-\sum a^3c+abc\cdot\sum a=\sum a\cdot\sum a^3-2\sum a^2\cdot\sum ab+3abc\cdot\sum a=$$

$$=2s\cdot 2s(s^2-3r^2-6Rr)-2(2s^2-8Rr-2r^2)(s^2+4Rr+r^2)+3\cdot 4Rrs\cdot 2s=$$

$$=-12r^2s^2+4r^2(16R^2+8Rr+r^2)=4r^2[(4R+r)^2-3s^2]$$

JP.388 Solve for complex numbers:

$$\begin{cases}|x-y|\geq\sqrt{3}|z|\\|y-z|\geq\sqrt{3}|x|\\|z-x|\geq\sqrt{3}|y|\end{cases}$$

Proposed by Ionuţ Florin Voinea-Romania

Solution 1 by proposer

$$\begin{cases}|x-y|\geq\sqrt{3}|z|\\|y-z|\geq\sqrt{3}|x|\\|z-x|\geq\sqrt{3}|y|\end{cases}\Leftrightarrow\begin{cases}|x-y|^2\geq 3|z|^2\\|y-z|^2\geq 3|x|^2\\|z-x|^2\geq 3|y|^2\end{cases}$$

$$\rightarrow|x-y|^2+|y-z|^2+|z-x|^2\geq 3(|x|^2+|y|^2+|z|^2)$$

How, $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2),\forall z_1,z_2\in\mathbb{C}$ **then,**

$$2(|x|^2+|y|^2)+2(|y|^2+|z|^2)+2(|z|^2+|x|^2)\geq$$

$$\geq 3(|x|^2+|y|^2+|z|^2)+|x+y|^2+|y+z|^2+|z+x|^2$$

$$\rightarrow|x|^2+|y|^2+|z|^2\geq|x+y|^2+|y+z|^2+|z+x|^2$$

How, $|z_1+z_2|^2+|z_2+z_3|^2+|z_3+z_1|^2=|z_1|^2+|z_2|^2+|z_3|^2,\forall z_1,z_2,z_3\in\mathbb{C}$ **then,**

$$|x|^2+|y|^2+|z|^2\geq|x|^2+|y|^2+|z|^2+|x+y+z|^2$$

Hence, $|x+y+z|\leq 0$ **namely,** $x+y+z=0$.

We distinguish the following cases:

(I) **If** $x=y$, **then** $|x-y|=0$ **and how** $|x-y|\geq\sqrt{3}|z|$, **we get** $|z|\leq 0$ **then,** $z=0$.

It follows that, $|y|\geq\sqrt{3}|x|$ **and** $|x|=|y|$ **then** $|x|\geq\sqrt{3}|x|\Leftrightarrow|x|\leq 0\Leftrightarrow x=0$.

So, $(x,y,z)=(0,0,0)$ **has solution for the system.**

(II) If $x, y, z \in \mathbb{C}$ different in pairs, let's consider $A(x), B(y), C(z)$.

$x+y+z=0, u=\frac{x+y+z}{3}=0$, and let $G(u)$ −centroid then $G=O$.

$$\begin{cases}|x-y| \geq \sqrt{3}|z| \\ |y-z| \geq \sqrt{3}|x| \\ |z-x| \geq \sqrt{3}|y|\end{cases} \Leftrightarrow \begin{cases}AB \geq \sqrt{3}GC \\ BC \geq \sqrt{3}GA \\ CA \geq \sqrt{3}GB\end{cases} \Leftrightarrow \begin{cases}c \geq \frac{2\sqrt{3}}{3}m_c \\ a \geq \frac{2\sqrt{3}}{3}m_a \\ b \geq \frac{2\sqrt{3}}{3}m_b\end{cases} \Leftrightarrow \begin{cases}c^2 \geq \frac{4}{3}m_c^2 \\ a^2 \geq \frac{4}{3}m_a^2 ; (*) \\ b^2 \geq \frac{4}{3}m_b^2\end{cases}$$

Thus, $a^2+b^2+c^2 \geq \frac{4}{3}(m_a^2+m_b^2+m_c^2)$. We know that:

$m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$ namely, in $(*)$ we have just equality. Therefore,

$$\begin{cases}c^2=\frac{4}{3}m_c^2 \\ a^2=\frac{4}{3}m_a^2 \\ b^2=\frac{4}{3}m_b^2\end{cases}$$

$$a^2=\frac{4}{3}m_a^2 \Leftrightarrow 3a^2=2(b^2+c^2)-a^2 \Leftrightarrow 2a^2=b^2+c^2.$$

Similarly, we get: $2b^2=c^2+a^2$ and $2c^2=a^2+b^2$.

We have, $\begin{cases}2a^2=b^2+c^2 \\ 2b^2=c^2+a^2\end{cases} \Rightarrow 2(a^2-b^2)=b^2-a^2 \Rightarrow a^2=b^2 \Rightarrow a=b$.

Similarly, $b=c$. Thus, $a=b=c \Rightarrow |x-y|=|y-z|=|z-x|$.

$$\begin{cases}|x-y| \geq \sqrt{3}|z| \\ |y-z| \geq \sqrt{3}|x| \\ |z-x| \geq \sqrt{3}|y|\end{cases} \Leftrightarrow \begin{cases}|x-y| = \sqrt{3}|z| \\ |y-z| = \sqrt{3}|x| \\ |z-x| = \sqrt{3}|y|\end{cases}$$

So, it follows that, $|x|=|y|=|z|$.

$$S=\{(0,0,0)\} \cup \{(u,v,w) \mid |u|=|v|=|w|; u,v,w \in \mathbb{C}^*, u+v+w=0, u \neq v \neq w\}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$|x-y| \geq \sqrt{3}|z| \Rightarrow |x-y|^2 \geq 3|z|^2; (1)$$

$|x|^2+|y|^2-x\bar{y}-\bar{x}y \geq 2|z|^2; (2)$. Similarly,

$|y|^2+|z|^2-y\bar{z}-\bar{y}z \geq 3|x|^2; (3)$ and $|x|^2+|z|^2-x\bar{z}-\bar{x}z \geq 3|y|^2; (4)$

From (2),(3) and (4) it follows,

$$2|x|^2+2|y|^2+2|z|^2-x\overline{y}-\overline{x}y-y\overline{z}-\overline{y}z-x\overline{z}-\overline{x}z\geq 3(|x|^2+|y|^2+|z|^2)\Rightarrow$$

$$|x|^2+|y|^2+|z|^2+x\overline{y}+\overline{x}y+y\overline{z}+\overline{y}z+x\overline{z}+\overline{x}z\leq 0\Leftrightarrow$$

$$|x+y+z|^2\leq 0\Leftrightarrow |x+y+z|=0\Leftrightarrow x+y+z=0.$$

Also, $|x-y|^2\geq 3|z|^2$ and $|y-z|^2\geq 3|x|^2\Rightarrow$

$|x|^2+|y|^2-x\overline{y}-\overline{x}y\geq 3|z|^2$ and $|y|^2+|z|^2-y\overline{z}-\overline{y}z\geq 3|x|^2$

$$|x|^2+2|y|^2+|z|^2-\overline{y}(x+z)-y(\overline{x}+\overline{z})\geq 3|x|^2+3|z|^2\Rightarrow$$

$$2|y|^2-\overline{y}(-y)-y(-\overline{y})\geq 2|x|^2+2|z|^2\Rightarrow |x|+|z|^2\leq 2|y|^2;(5)$$

Now, $|x-z|^2+|y|^2\geq 4|y|^2\Rightarrow |x-z|^2+|-x-z|^2\geq 4|y|^2\Rightarrow$

$$2(|x|^2+|z|^2)\geq 4|y|^2\Rightarrow |x|^2+|z|^2\geq 2|y|^2;(6)$$

From (5),(6), we get: $|x|^2+|z|^2=2|y|^2;(7)$.

Similarly, $|y|^2+|z|^2=2|x|^2\Rightarrow |x|^2-|y|^2=2(|y|^2-|x|^2)\Rightarrow 3(|x|^2-|y|^2)=0$

$\Rightarrow |x|=|y|$. From (7) we get: $|x|=|y|=|z|$.

Now, $x+y+z=0$ and $|x|=|y|=|z|$

If $x=0$, then $y=z=0$.

If $x\neq 0$, $|x-y|^2+|z|^2=|x-y|^2+|-x-y|^2=2|x|^2+2|y|^2=4|z|^2\Rightarrow$

$|x-y|^2=3|z|^2$. Similarly, $|y-z|^2=|z-x|^2=3|z|^2$.

Thus, if $x\neq 0$, then x,y,z are vertices of an equilateral triangle lying on a circle with center at the origin.

JP.389 A right parallelipiped $ABCDA'B'C'D'$ has the basis $ABCD$ rhombus, and areas of the two diagonals sections of the parallelipiped are F_1 and F_2 respectively. Let R be the circumradius of ΔABC, R_2circumradius of ΔABD and V volume of the right parallelipiped.

Prove that: $R_1R_2F_1F_2\geq V^2$.

Proposed by Radu Diaconu-Romania

Solution 1 by proposer

Let h be the side edge and altitude, $d_1=AC$, $d_2=BD$ diagonals of the bases.

Sections diagonals are rectangle, so $F_1 = h \cdot d_1, F_2 = h \cdot d_2$.

$F = \frac{d_1 \cdot d_2}{2}$ −area of the rhombus. We have: $F_1 \cdot F_2 = h^2 d_1 d_2 \Leftrightarrow h^2 = \frac{F_1 F_2}{2F}$ then,

$$h = \sqrt{\frac{F_1 F_2}{2F}}.$$

So, $V = h \cdot F = \frac{1}{2}\sqrt{2F \cdot F_1 F_2}$.

$$F_{\Delta ABC} = \frac{AB \cdot BC \cdot CA}{4R_1} = \frac{l^2 \cdot d_1}{4R_1} = \frac{d_1 \cdot d_2}{4} \Rightarrow d_2 = \frac{l^2}{R_1}$$

$$F_{\Delta ABD} = \frac{AB \cdot BD \cdot DA}{4R_2} = \frac{l^2 \cdot d_2}{4R_2} = \frac{d_1 \cdot d_2}{4} \Rightarrow d_1 = \frac{l^2}{R_2}$$

$$F = \frac{d_1 \cdot d_2}{2} = \frac{l^4}{2R_1 R_2};\ d_1^2 + d_2^2 = 4l^2 \Leftrightarrow \frac{l^4}{R_1^2} + \frac{l^4}{R_2^2} = 4l^2 \Rightarrow l^2 = \frac{4R_1^2 R_2^2}{R_1^2 + R_2^2}$$

$$F = \frac{16 R_1^4 R_2^4}{\left(R_1^2 + R_2^2\right)^2} \cdot \frac{1}{2R_1 R_2} = \frac{8 R_1^3 R_2^3}{\left(R_1^2 + R_2^2\right)^2}$$

Hence,

$$V = \frac{1}{\sqrt{2}} \cdot \frac{2\sqrt{2} R_1 R_2 \sqrt{R_1 R_2}}{R_1^2 + R_2^2} \cdot \sqrt{F_1 F_2} \le \sqrt{R_1 R_2 F_1 F_2}, \text{because} \frac{2R_1 R_2}{R_1^2 + R_2^2} \le 1.$$

Squaring, we get $R_1 R_2 F_1 F_2 \ge V^2$.

Solution 2 by Adrian Popa-Romania

$$F_1 = AC \cdot h \Rightarrow \frac{AC}{\sin B} = 2R_1 \Rightarrow AC = 2R \cdot \sin B$$

$$F_2 = BD \cdot h \Rightarrow \frac{BD}{\sin A} = 2R_2 \Rightarrow BD = 2R_2 \cdot \sin A$$

$$A = \pi - B \Rightarrow \sin A = \sin B$$

$$V = A_{ABCD} \cdot h = \frac{AC \cdot BD}{2} \cdot h = \frac{2R_1 \cdot \sin A \cdot 2R_2 \cdot \sin A}{2} \cdot h =$$

$$= 2R_1 R_2 \sin^2 A \cdot h^2$$

$$F_1 F_2 R_1 R_2 = AC \cdot h \cdot BD \cdot h R_1 R_2 \overset{(*)}{>} 4R_1^2 R_2^2 \sin^4 A$$

$$2R_1 \sin A \cdot h \cdot 2R_2 \sin A \cdot h \cdot R_1 R_2 \overset{(*)}{>} 4R_1^2 R_2^2 \sin^4 A \cdot h^2 \Leftrightarrow$$

$\sin^2 A \cdot h^2 \ge \sin^4 A \cdot h^2 \Leftrightarrow \sin^2 A \ge \sin^4 A$, which is clearly true.

JP.390 Let $x \in \mathbb{R}$ and ABC a triangle with F area. Prove that:

$$\frac{a^3}{\sqrt{b^2\sin^2 x + c^2\cos^2 x}} + \frac{b^3}{\sqrt{c^2\sin^2 x + a^2\cos^2 x}} + \frac{c^3}{\sqrt{a^2\sin^2 x + b^2\cos^2 x}} \geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution by proposer

We have:

$$\sum_{cyc}\frac{a^3}{\sqrt{b^2\sin^2 x + c^2\cos^2 x}} = \sum_{cyc}\frac{a^4}{a\sqrt{b^2\sin^2 x + c^2\cos^2 x}} \overset{AM-GM}{\geq}$$

$$\geq \sum_{cyc}\frac{a^4}{\frac{a^2 + b^2\sin^2 x + c^2\cos^2 x}{2}} = 2\cdot\sum_{cyc}\frac{a^4}{a^2 + b^2\sin^2 x + c^2\cos^2 x} \overset{Bergstrom}{\geq}$$

$$\geq 2\cdot\frac{(a^2+b^2+c^2)^2}{\sum(a^2 + b^2\sin^2 x + c^2\cos^2 x)} = 2\cdot\frac{(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)(1+\sin^2 x + \cos^2 x)} =$$

$$= 2\cdot\frac{(a^2+b^2+c^2)^2}{2(a^2+b^2+c^2)} = a^2+b^2+c^2 \overset{Ionescu-W}{\geq} 4\sqrt{3}F$$

SP.376 Let r, r_a, r_b, r_c and R be, respectively, the inradius, the exradii, and the circumradius of triangle ABC with side lengths a, b, c. Prove that:

$$36\sqrt{3}\frac{r^3}{R^2} \leq \frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + 4r\left(\frac{r_a}{b+c} + \frac{r_b}{c+a} + \frac{r_c}{a+b}\right) \leq \frac{9\sqrt{3}R^2}{4r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

We will prove that:

$$36\sqrt{3}\frac{r^3}{R^2} \leq \frac{a^2+b^2+4rr_c}{a+b} + \frac{b^2+c^2+4rr_a}{b+c} + \frac{c^2+a^2+4rr_b}{c+a} \leq \frac{9\sqrt{3}R^2}{4r}$$

We know that: $r_a = s\cdot tan\frac{A}{2}, r_b = s\cdot tan\frac{B}{2}, r_c = s\cdot tan\frac{C}{2}, F = rs$

$$\rightarrow b^2 + c^2 + 4rr_a = b^{2\frac{2}{c}} + 4r\cdot s\cdot tan\frac{A}{2} = b^2 + c^2 + 4F\cdot tan\frac{A}{2} =$$

$$= b^2+c^2+4F\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = b^2+c^2+\frac{8F}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cdot\sin^2\frac{A}{2} = b^2+c^2+\frac{8F}{\sin A}\cdot\sin^2\frac{A}{2}$$

We know that: $F=\frac{1}{2}bc\cdot\sin A$ **and** $\cos A = 1-2\sin^2\frac{A}{2}$, **so**

$$b^2+c^2+4rr_a = b^2+c^2+4bc\cdot\sin^2\frac{A}{2} = b^2+c^2+4bc\cdot\frac{1-\cos A}{2} =$$

$$= b^2+c^2+2bc-2bc\cdot\cos A = (b^2+c^2-2bc\cdot\cos A)+2bc = a^2+2bc$$

Similarly, $c^2+a^2+4rr_b = b^2+2ca, a^2+b^2+4rr_c = c^2+2ab.$

Namely, we'll prove that:

$$36\sqrt{3}\frac{r^3}{R^2} \le \frac{a^2+bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b} \le \frac{9\sqrt{3}R^2}{4r}$$

We have:

$$\frac{a^2+bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b} =$$

$$= \left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)+2\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right) \overset{CBS}{\ge}$$

$$\frac{(a+b+c)^2}{2(a+b+c)}+2abc\left(\frac{1}{c(a+b)}+\frac{1}{a(b+c)}+\frac{1}{b(c+a)}\right) \ge$$

$$\ge s+2abc\cdot\frac{(1+1+1)^2}{2(ab+bc+ca)} = s+\frac{9abc}{ab+bc+ca}$$

We know that: $ab+bc+ca \le a^2+b^2+c^2 \le 9R^2, abc = 4Rrs, R \ge 2r, s \ge 3\sqrt{3}r$. **So,**

$$\frac{a^2+bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b} \ge 3\sqrt{3}r+\frac{9(4Rrs)}{9R^2} = \frac{3\sqrt{3}R^2r+4Rrs}{R^2} \ge$$

$$\ge \frac{3\sqrt{3}r(2r)^2+4(2r)\cdot r\cdot 3\sqrt{3}r}{R^2} = 36\sqrt{3}\frac{r^3}{R^2}$$

$$\frac{a^2+bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b} \le \frac{a^2+b^2+c^2}{b+c}+\frac{b^2+c^2+a^2}{c+a}+\frac{c^2+a^2+b^2}{a+b} =$$

$$= (a^2+b^2+c^2)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \le (a^2+b^2+c^2)\left(\frac{1}{2\sqrt{ab}}+\frac{1}{2\sqrt{bc}}+\frac{1}{2\sqrt{ca}}\right) =$$

$$= \frac{(a^2+b^2+c^2)}{2}\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right) \overset{CBS}{\le} \frac{9R^2}{2}\sqrt{3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)} \le$$

$$\le \frac{9\sqrt{3}}{2}R^2\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

Now, we'll prove that: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}$**. We have:** $(b-c)^2\ge 0\to a^2-(b-c)^2\le a^2$

$$\frac{1}{a^2}\le\frac{1}{a^2-(b-c)^2}=\frac{1}{(a+b-c)(a-b+c)}=\frac{1}{2(s-c)\cdot 2(s-b)}=\frac{1}{4(s-b)(s-c)}$$

Similarly,

$$\frac{1}{b^2}\le\frac{1}{4(s-c)(s-a)},\frac{1}{c^2}\le\frac{1}{4(s-a)(s-b)}$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4}\left(\frac{1}{(s-a)(s-b)}+\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}\right)=$$

$$=\frac{1}{4}\cdot\frac{s-c+s-b+s-a}{(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{s^2}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{1}{4}\cdot\frac{s^2}{F^2}=\frac{1}{4}\cdot\frac{s^2}{r^2s^2}=\frac{1}{4r^2}$$

Now,

$$\frac{a^2+bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b}\le\frac{9\sqrt{3}R^2}{2}\cdot\sqrt{\frac{1}{4r^2}}=\frac{9\sqrt{3}R^2}{4r}$$

Namely,

$$36\sqrt{3}\frac{r^3}{R^2}\le\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+4r\left(\frac{r_a}{b+c}+\frac{r_b}{c+a}+\frac{r_c}{a+b}\right)\le\frac{9\sqrt{3}R^2}{4r}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:4rr_a=\frac{4sr^2}{s-a}=\frac{4sr^2(s-b)(s-c)}{(s-a)(s-b)(s-c)}$$

$$=4(s-b)(s-c)\overset{AM-GM}{\le}[(s-b)+(s-c)]^2=a^2$$

$$\to\sum\frac{b^2+c^2}{b+c}+4r\sum\frac{r_a}{b+c}\le\sum\frac{b^2+c^2}{b+c}+\sum\frac{a^2}{b+c}$$

$$=\sum\frac{a^2+b^2+c^2}{b+c}\overset{CBS}{\le}\left(\sum a^2\right)\sum\frac{1}{4}\left(\frac{1}{b}+\frac{1}{c}\right)\le$$

$$\overset{Leibniz}{\leq} 9R^2.\frac{1}{2}\sum\frac{1}{a}\overset{CBS}{\leq}\frac{9R^2}{2}\sqrt{3\sum\frac{1}{a^2}}\overset{Goldstone}{\leq}\frac{9R^2}{2}\sqrt{3.\frac{1}{4r^2}}=\frac{9\sqrt{3}}{4}.\frac{R^2}{r}$$

$$Also, we\ have:\sum\frac{b^2+c^2}{b+c}\overset{CBS}{\geq}\sum\frac{(b+c)^2}{2(b+c)}=\frac{1}{2}\sum(b+c)$$

$$=2s\overset{Mitrinovic}{\geq}2.3\sqrt{3}r\overset{Euler}{\geq}6\sqrt{3}r\left(\frac{2r}{R}\right)^2=24\sqrt{3}.\frac{r^3}{R^2}$$

$$And:\ 4r\sum\frac{r_a}{b+c}=4sr^2\sum\frac{1}{(s-a)(b+c)}\overset{CBS}{\geq}4sr^2.\frac{9}{\sum(s-a)(b+c)}$$

$$=\frac{36sr^2}{4s^2-2\sum ab}\geq$$

$$\overset{Mitrinovic}{\geq}\frac{36.3\sqrt{3}r.r^2}{(\sum a)^2-2\sum ab}=\frac{108\sqrt{3}r^3}{\sum a^2}\overset{Leibniz}{\geq}\frac{108\sqrt{3}r^3}{9R^2}=12\sqrt{3}.\frac{r^3}{R^2}$$

$$\to\sum\frac{b^2+c^2}{b+c}+4r\sum\frac{r_a}{b+c}\geq 24\sqrt{3}.\frac{r^3}{R^2}+12\sqrt{3}.\frac{r^3}{R^2}=36\sqrt{3}.\frac{r^3}{R^2}$$

$$Therefore,\quad 36\sqrt{3}.\frac{r^3}{R^2}\leq\sum\frac{b^2+c^2}{b+c}+4r\sum\frac{r_a}{b+c}\leq\frac{9\sqrt{3}}{4}.\frac{R^2}{r}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{b^2+c^2}{b+c}+4r\sum\frac{r_a}{b+c}\overset{\text{Bandila and A-G}}{\leq}\frac{R}{r}\sum\frac{abc}{a(b+c)}$$

$$+2r\sum\frac{r_a}{\sqrt{bc}}\overset{A-G}{\leq}\frac{R(4Rrs)}{2r}\sum\frac{1}{a\sqrt{bc}}+\frac{2r}{\sqrt{abc}}\sum(r_a\sqrt{a})$$

$$=2R^2s\sum\left(\sqrt{\frac{1}{ab}}\sqrt{\frac{1}{ac}}\right)+\frac{2r}{\sqrt{4Rrs}}\sum\left(\sqrt{ar_a}\sqrt{r_a}\right)$$

$$\overset{CBS}{\leq}2R^2s\sum\frac{1}{ab}+\frac{2r}{\sqrt{4Rrs}}\sqrt{\sum ar_a}\sqrt{\sum r_a}$$

$$=\frac{2R^2s(2s)}{4Rrs}+\frac{(2r)\sqrt{4R+r}}{\sqrt{4Rrs}}\sqrt{4Rs\sum\left(\sin\frac{A}{2}\cos\frac{A}{2}\tan\frac{A}{2}\right)}$$

$$=\frac{Rs}{r}+2\sqrt{\frac{4Rr+r^2}{2}}\sqrt{\sum(1-\cos A)}=\frac{Rs}{r}+\sqrt{\frac{2(2R-r)(4Rr+r^2)}{R}}$$

$$\overset{?}{\leq}\frac{3Rs}{2r}\Leftrightarrow\frac{Rs}{2r}\overset{?}{\geq}\sqrt{\frac{2(2R-r)(4Rr+r^2)}{R}}\Leftrightarrow R^3s^2\underset{(i)}{\overset{?}{\geq}}8(4R+r)(2R-r)r^3\ and$$

$$\because R^3s^2\overset{\text{Mitrinovic}}{\geq}27R^3r^2\ \therefore\text{ in order to prove (i), it suffices to prove}$$

$$:\ 27R^3\geq 8r(4R+r)(2R-r)$$

$$\Leftrightarrow 27t^3-64t^2+16t+8\geq 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left(21t^2+(t-2)(t+2)+5t(t-2)\right)$$

$$\geq 0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow\text{(i) is true}\Rightarrow\sum\frac{b^2+c^2}{b+c}+4r\sum\frac{r_a}{b+c}$$

$$\leq\frac{3Rs}{2r}\overset{\text{Mitrinovic}}{\leq}\frac{9\sqrt{3}R^2}{4r}$$

$$\therefore\boxed{\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+4r\left(\frac{r_a}{b+c}+\frac{r_b}{c+a}+\frac{r_c}{a+b}\right)\leq\frac{9\sqrt{3}R^2}{4r}}$$

$$\text{Again,}\sum\frac{b^2+c^2}{b+c}+4r\sum\frac{r_a}{b+c}$$

$$\geq\sum\frac{(b+c)^2}{2(b+c)}+4r\sum\frac{s\tan^2\frac{A}{2}}{4R\tan\frac{A}{2}\cos\frac{A}{2}\cos\frac{B-C}{2}}\overset{\because 0\leq\cos\frac{B-C}{2}\leq 1\text{ and analogs}}{\geq}\sum\frac{b+c}{2}$$

$$+\frac{rs}{R}\sum\frac{\tan^2\frac{A}{2}}{\sin\frac{A}{2}}\overset{\text{Bergstrom}}{\geq}=2s+\left(\frac{rs}{Rs^2}\right)\frac{(4R+r)^2}{\sum\sin\frac{A}{2}}\overset{\sum\sin\frac{A}{2}\leq\frac{3}{2}}{\geq}$$

$$2s+\frac{2r(4R+r)^2}{3Rs}=\frac{6Rs^2+2r(4R+r)^2}{3Rs}\overset{\text{Mitrinovic}}{\geq}\frac{2(6Rs^2+2r(4R+r)^2)}{3R(3\sqrt{3}R)}\overset{?}{\geq}36\sqrt{3}\frac{r^3}{R^2}$$

$$\Leftrightarrow 3Rs^2+r(4R+r)^2\underset{(ii)}{\overset{?}{\geq}}243r^3\ and$$

$$\because 3Rs^2+r(4R+r)^2\overset{\text{Mitrinovic}}{\geq}81Rr^2+r(4R+r)^2$$

$$\therefore\text{ in order to prove (ii), it suffices to prove}:\ (4R+r)^2+81Rr-243r^2\geq 0$$

$$\Leftrightarrow 16R^2+89Rr-242r^2\geq 0\Leftrightarrow(R-2r)(16R+121r)\geq 0\to\text{true}$$

$$\because R\overset{\text{Euler}}{\geq}2r\Rightarrow\text{(ii) is true}$$

$$\Rightarrow \sum \frac{b^2+c^2}{b+c} + 4r\sum \frac{r_a}{b+c} \geq 36\sqrt{3}\frac{r^3}{R^2}$$

$$\Rightarrow \boxed{36\sqrt{3}\frac{r^3}{R^2} \leq \frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a}} \text{ (QED)}$$

SP.377 Let w_a, w_b, w_c be the internal bisectors, r_a, r_b, r_c the exradii, r the inradius and R the circumradius of a triangle ABC. Prove that:

$$\left(\frac{w_a}{r_a}\right)^2 + \left(\frac{w_b}{r_b}\right)^2 + \left(\frac{w_c}{r_c}\right)^2 \leq 27\left(\frac{R}{2r}\right)^4 - 24$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let $BC = a, CA = b, AB = c$ and $2s = a+b+c$ is the semiperimeter of ΔABC. We know that: $w_a = \frac{2bc}{b+c}\cos\frac{A}{2}$ and $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$. Then:

$$w_a = \frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}} = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}$$

Also, we know that: $F = rs = \frac{abc}{4R}$, so $2s = \frac{abc}{2Rr}$. Now,

$$(b+c-a)^2 \geq 0 \leftrightarrow (b+c)^2 - 4a(b+c) + 4a^2 \geq 0 \leftrightarrow$$

$$4a(-a+b+c) \leq (b+c)^2 \leftrightarrow 4a(2s-2a) \leq (b+c)^2 \leftrightarrow 8a(s-a) \leq (b+c)^2 \leftrightarrow$$

$16a(2s-2a) \leq (b+c)^2$, so $b+c \geq 2\sqrt{2}\sqrt{a(s-a)}$. So,

$$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \leq \frac{2\sqrt{bc}\cdot\sqrt{s(s-a)}}{2\sqrt{2}\cdot\sqrt{a}\cdot\sqrt{s-a}} = \frac{1}{2}\sqrt{\frac{bc}{a}}\cdot\sqrt{2s} =$$

$$= \frac{1}{2}\sqrt{\frac{bc}{a}}\cdot\sqrt{\frac{abc}{2Rr}} = \frac{1}{2}\cdot\frac{bc}{\sqrt{2Rr}}$$

We know that: $r_a = \frac{F}{s-a}, r_b = \frac{F}{s-b}, r_c = \frac{F}{s-c}$. Now, we have:

$$\left(\frac{w_a}{r_a}\right)^2 + \left(\frac{w_b}{r_b}\right)^2 + \left(\frac{w_c}{r_c}\right)^2 \leq \frac{\left(\frac{bc}{2\sqrt{2Rr}}\right)^2}{\left(\frac{F}{s-a}\right)^2} + \frac{\left(\frac{ca}{2\sqrt{2Rr}}\right)^2}{\left(\frac{F}{s-b}\right)^2} + \frac{\left(\frac{ab}{2\sqrt{2Rr}}\right)^2}{\left(\frac{F}{s-c}\right)^2} =$$

$$=\frac{b^2c^2(s^2-2as+a^2)+c^2a^2(s^2-2bs+b^2)+a^2b^2(s^2-2cs+c^2)}{8Rr\cdot F^2}=$$

$$=\frac{a^2b^2+b^2c^2+c^2a^2-8Rr(ab+bc+ca)+48R^2r^2}{8Rr^3}$$

Now, we will prove that: $a^2b^2+b^2c^2+c^2a^2\le 4R^2r^2$. We have:

$$a^2b^2+b^2c^2+c^2a^2=a^2b^2c^2\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)$$

Also, we have: $(b-c)^2\ge 0\leftrightarrow a^2-(b-c)^2\le a^2\leftrightarrow$

$$\frac{1}{a^2}\le\frac{1}{a^2-(b-c)^2}=\frac{1}{(a+b-c)(a-b+c)}=\frac{1}{2(s-c)\cdot 2(s-b)}=\frac{1}{4(s-b)(s-c)}$$

Similarly: $\frac{1}{b^2}\le\frac{1}{4(s-c)(s-a)},\frac{1}{c^2}\le\frac{1}{4(s-b)(s-a)}$

$$\text{So, }\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4}\cdot\frac{(s-a)+(s-b)+(s-c)}{(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{s(3s-2s)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{1}{4}\cdot\frac{s^2}{F^2}=\frac{s^2}{4(rs)^2}=\frac{1}{4r^2}$$

$$\because F=sr=\sqrt{s(s-a)(s-b)(s-c)},R\ge 2r,s\le\frac{3\sqrt{3}}{2}R$$

Namely:$a^2b^2+b^2c^2+c^2a^2\le(abc)^2\cdot\frac{1}{4r^2}=(4Rrs)^2\cdot\frac{1}{4r^2}=4R^2s^2$

Also, we know that $ab+bc+ca\ge 18Rr$. So,

$$\left(\frac{w_a}{r_a}\right)^2+\left(\frac{w_b}{r_b}\right)^2+\left(\frac{w_c}{r_c}\right)^2\le\frac{4R^2s^2-8Rr\cdot 18Rr+48R^2r^2}{8Rr^3}\le$$

$$=\frac{4R^2\left(\frac{3\sqrt{3}}{2}R\right)^2-144R^2r^2+48R^2r^2}{8Rr^3}=\frac{27R^4-96R^2r^2}{8Rr^3}\le$$

$$\le\frac{27R^4-96\cdot 4r^4}{16r^4}=27\left(\frac{R}{2r}\right)^4-24$$

Equality holds if and only if triangle ABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that:

$$w_a\le\sqrt{s(s-a)};(and\ analogs);(1)\text{ and }\prod(r_b+r_c)=4Rs^2;(2)$$

$$\sum_{cyc}\left(\frac{w_a}{r_a}\right)^2 \overset{(1)}{\leq} \sum_{cyc}\frac{r_br_c}{r_a^2} = r_ar_br_c\sum_{cyc}\left(\frac{1}{r_a}\right)^3 = r_ar_br_c\left[\left(\sum_{cyc}\frac{1}{r_a}\right)^3 - 3\prod_{cyc}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\right] =$$

$$= r_ar_br_c\left[\left(\sum_{cyc}\frac{1}{r_a}\right)^3 - 3\prod_{cyc}\left(\frac{r_b+r_c}{r_br_c}\right)\right] \overset{(2)}{=}$$

$$= s^2r\left[\left(\frac{1}{r}\right)^3 - 3\cdot\frac{4Rs^2}{(s^2r)^2}\right] = \left(\frac{s}{r}\right)^2 - 12\cdot\frac{R}{r} \overset{\substack{Mitrinovic\\ Euler}}{\leq}$$

$$\leq \left(\frac{3\sqrt{3}R}{2r}\right)^2 - 12\cdot 2 = 27\left(\frac{R}{2r}\right)^2 - 24 \overset{Euler}{\leq} 27\left(\frac{R}{2r}\right)^2\cdot\left(\frac{R}{2r}\right)^2 - 24 = 27\left(\frac{R}{2r}\right)^4 - 24$$

Therefore,

$$\sum_{cyc}\left(\frac{w_a}{r_a}\right)^2 \leq 27\left(\frac{R}{2r}\right)^4 - 24$$

SP.378. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with area F.Prove that:

$$m_a^n + m_b^n + m_c^n \geq 3^{\frac{n}{4}+1}\cdot F^{\frac{n}{2}} \text{ for each integer } n \geq 1.$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let $a = BC, b = CA, c = AB$. It is well-known that:

$$a^2+b^2+c^2 \geq ab+bc+ca \geq 4\sqrt{3}F\ (Finsler - Hadwiger)$$

Proof. We have:

$$\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC} \geq 3\sqrt[3]{\frac{1}{sinAsinBsinC}}$$

We know that:

$$\sqrt[3]{sinAsinBsinC} \leq \frac{sinA+sinB+sinC}{3} \leq \frac{\frac{3\sqrt{3}}{2}}{3} = \frac{\sqrt{3}}{3}$$

So, $sinAsinBsinC \leq \frac{3\sqrt{3}}{8}$, namely $\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC} \geq 2\sqrt{3}$. Now,

$$a^2+b^2+c^2 \ge ab+bc+ac = 2F\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\right) \ge 2F\cdot 2\sqrt{3} = 4\sqrt{3}$$

Now, $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \ge 12\sqrt{3}F$. Namely,

$$F \le \frac{\sqrt{3}}{4}\left(\frac{a+b+c}{3}\right)^2$$

Also, we know that: $\left(\frac{a+b+c}{3}\right)^2 \le \left(\frac{a^n+b^n+c^n}{3}\right)^{\frac{2}{n}}$. So, $F \le \frac{\sqrt{3}}{4}\left(\frac{a^n+b^n+c^n}{3}\right)^{\frac{2}{n}}$; (1)

We know that in $\Delta BGA'$; $(AG = GA')$, we have:

$$BG = \frac{2}{3}m_b, GA' = \frac{2}{3}m_a, A'B = \frac{2}{3}m_c; (G-centroid), F_{\Delta BGA'} = \frac{1}{3}F.$$

The inequality (1) gives:

$$\frac{1}{3}F \le \frac{\sqrt{3}}{4}\left(\frac{\left(\frac{2}{3}m_a\right)^n+\left(\frac{2}{3}m_b\right)^n+\left(\frac{2}{3}m_c\right)^n}{3}\right)^{\frac{2}{n}} \text{ or } m_a^n+m_b^n+m_c^n \ge 3^{\frac{n}{4}+1}\cdot F^{\frac{n}{2}}$$

Equality holds if and only if triangle ABC is equilateral.

Solution 2 by Marian Ursărescu-Romania

$$m_a^n+m_b^n+m_c^n \ge 3\sqrt[3]{(m_am_bm_c)^n}$$

We must show that:

$$3\sqrt[3]{(m_am_bm_c)^n} \ge 3\sqrt[4]{3^n}\cdot\sqrt{F^n} \Leftrightarrow \sqrt[3]{m_am_bm_c} \ge \sqrt[4]{3}\cdot\sqrt{F} \Leftrightarrow$$

$$(m_am_bm_c)^2 \ge \sqrt{27}\cdot F^3; (1). \text{ But: } m_a \ge \sqrt{s(s-a)}; (2).$$

From (1),(2) remains to prove that:

$$s^2F^2 \ge \sqrt{2r}F \Leftrightarrow s^2 \ge 3\sqrt{3}rs \Leftrightarrow s \ge 3\sqrt{3}r\ (Mitrinovic).$$

Solution 3 by Adrian Popa-Romania

$$\frac{(m_a)^{\frac{n}{2}}}{1^{\frac{n}{2}-1}}+\frac{(m_b)^{\frac{n}{2}}}{1^{\frac{n}{2}-1}}+\frac{(m_c)^{\frac{n}{2}}}{1^{\frac{n}{2}-1}} \ge \frac{(m_a^2+m_b^2+m_c^2)^{\frac{n}{2}}}{3^{\frac{n}{2}-1}} = \frac{\left(\frac{3}{4}(a^2+b^2+c^2)\right)^{\frac{n}{2}}}{3^{\frac{n}{2}-1}} \ge$$

$$\ge \frac{\left(\frac{3}{4}\cdot 4\sqrt{3}F\right)^{\frac{n}{2}}}{3^{\frac{n}{2}-1}} = 3^{\frac{3n}{4}-\frac{n}{2}+1}\cdot F^{\frac{n}{2}} = 3^{\frac{n}{4}+1}\cdot F^{\frac{n}{2}}$$

SP.379 If $x, y, z \in (0,1)$ then:

$$\frac{x}{(y+z)^2(1-x^2)}+\frac{y}{(z+x)^2(1-y^2)}+\frac{z}{(x+y)^2(1-z^2)} \geq \frac{9\sqrt{3}}{8}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

Let be $f:(0,1) \to \mathbb{R}, f(x) = x - x^3, f'(x) = 1 - 3x^2$

$$f'(x) = 0 \to x^2 = \frac{1}{3} \to x = \frac{\sqrt{3}}{3}$$

$$maxf(x) = f\left(\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3} - \frac{1}{3}\cdot\frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{9} = \frac{2}{3\sqrt{3}}$$

$$f(x) \leq \frac{2}{3\sqrt{3}} \to \frac{1}{f(x)} \geq \frac{3\sqrt{3}}{2}; (1)$$

$$\sum_{cyc}\frac{x}{(y+z)^2(1-x^2)} = \sum_{cyc}\frac{\frac{x}{(y+z)^2}}{1-x^2} = \sum_{cyc}\frac{\left(\frac{x}{y+z}\right)^2}{x-x^3} = \sum_{cyc}\frac{1}{f(x)}\cdot\left(\frac{x}{y+z}\right)^2 \overset{(1)}{\geq}$$

$$\geq \frac{3\sqrt{3}}{2}\sum_{cyc}\frac{\left(\frac{x}{y+z}\right)^2}{1} \overset{Bergstrom}{\geq} \frac{3\sqrt{3}}{2}\cdot\frac{\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2}{1+1+1} \overset{Nesbitt}{\geq}$$

$$\geq \frac{3\sqrt{3}}{2}\cdot\frac{\left(\frac{3}{2}\right)^2}{3} = \frac{\sqrt{3}}{2}\cdot\frac{9}{4} = \frac{9\sqrt{3}}{8}$$

Equality holds for $x = y = z = \frac{\sqrt{3}}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ AM-GM, we\ have:\ x^2 + \frac{\sqrt{3}}{9x} + \frac{\sqrt{3}}{9x} \geq 3\sqrt[3]{x^2.\frac{\sqrt{3}}{9x}.\frac{\sqrt{3}}{9x}} = 1 \to$$

$$1 - x^2 \leq \frac{2\sqrt{3}}{9x}\ (And\ analogs)$$

$$\to \sum_{cyc}\frac{x}{(y+z)^2(1-x^2)} \geq \sum_{cyc}\frac{x}{(y+z)^2.\frac{2\sqrt{3}}{9x}}$$

$$=\frac{3\sqrt{3}}{2}\sum_{cyc}\left(\frac{x}{y+z}\right)^2 \overset{CBS}{\geq} \frac{3\sqrt{3}}{2}.\frac{1}{3}\left(\sum_{cyc}\frac{x}{y+z}\right)^2 \geq$$

$$\overset{Nesbitt}{\geq} \frac{\sqrt{3}}{2}.\left(\frac{3}{2}\right)^2 = \frac{9\sqrt{3}}{8}.$$

$$Therefore, \quad \sum_{cyc}\frac{x}{(y+z)^2(1-x^2)} \geq \frac{9\sqrt{3}}{8}, with\ equality\ if: x=y=z=\frac{\sqrt{3}}{3}.$$

Solution 3 Sanong Huayrerai-Nakon Pathom-Thailand

For $0 < x, y, z < 1$, we have:

$$\sum_{cyc}\frac{x}{(y+z)^2(1-x^2)} = \sum_{cyc}\frac{\left(\frac{x}{y+z}\right)^2}{x(1-x^2)} \geq \frac{\left(\sum\frac{x}{y+z}\right)^2}{x+y+x-(x^3+y^3+z^3)} \geq \frac{9\sqrt{3}}{8}$$

$$\left(\frac{3}{2}\right)^2 \geq \frac{9\sqrt{3}}{8}[(x+y+z)-(x^3+y^3+z^3)]$$

$$\frac{9}{4}+\frac{9\sqrt{3}}{8}(x^3+y^3+z^3) \geq \frac{9\sqrt{3}}{8}(x+y+z)$$

$2+\sqrt{3}(x^3+y^3+z^3) \geq \sqrt{3}(x+y+z)$, which is true because

$$\frac{2}{3}+\sqrt{3}x^3 \geq \sqrt{3}x \Leftrightarrow \frac{2}{3x}+\sqrt{3}x^2 \geq \sqrt{3}$$

$$\frac{1}{3x}+\frac{1}{3x}+\sqrt{3}x^2 \geq 3\sqrt[3]{\frac{\sqrt{2}}{9}} \geq \sqrt{3}$$

Similarly, $\frac{2}{3}+\sqrt{3}y^3 \geq \sqrt{3}y$ and $\frac{2}{3}+\sqrt{3}z^3 \geq \sqrt{3}z$.

SP.380 Let a, b, c be the sides of an arbitrary triangle. Denote by m_a, w_a, h_a the lengths of the median, the internal bisector and the altitude corresponding to the side a and ω its Brocard angle . Prove that:

$$\frac{1}{\sin\omega} \geq 2\sqrt[4]{\frac{m_a^2 m_b^2 m_c^2}{w_a w_b w_c h_a h_b h_c}}$$

Proposed by Vasile Jiglău-Romania

Solution 1 by proposer

The sums and the products used below are cyclic

Lemma 1:The sides of an arbitrary triangle verify:

$$8(a^2b^2 + b^2c^2 + c^2a^2) \geq (a+b+c)(a+b)(b+c)(c+a); (1)$$

Proof.

With the substitutions $x = s - a, y = s - b, z = s - c$ **after computations , the inequality becomes equivalent to:**

$$2\sum x^2 + 2\sum x^2yz \geq \sum x^3y + \sum x^3z + 2\sum x^2y^2 \Leftrightarrow$$

$$2\left(\sum x^2 + \sum x^2yz - \sum x^3y - \sum x^3z\right) + \sum xy(x-y)^2 \geq 0$$

The first paranthesis is positive ,since it is the inequality of Schur

$\sum x^r(x-y)(x-z) \geq 0$ **for** $r = 4$**, therefore the inequality is true.**

Lemma 2: The sides of an arbitrary triangle verify

$$\left(\sum a\right)^2\left(\sum a^2b^2\right) \geq \prod(2a^2 + 2b^2 - c^2); (2)$$

With the substitutions $x = s - a, y = s - b, z = s - c$**, after computations , the inequality becomes equivalent to :**

$$4\sum x^5y + 4\sum x^5z + 35\sum x^4y^2 + 35\sum x^4z^2 + 66\sum x^3y^3 \geq$$

$$\geq 54\sum x^4yz + 42\sum x^3y^2z + 42\sum x^3yz^2 + 18x^2y^2z^2$$

which immediately results from the Muirhead's lemma , then the inequality is true .

Let's go to the solution of the problem .With the formulas

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4}, w_a = \frac{2\sqrt{bc \cdot s(s-a)}}{b+c}, h_a = \frac{2F}{a}, \frac{1}{\sin\omega} = \frac{\sqrt{\sum a^2b^2}}{2F}$$

the enunciated inequality becomes:

$$s(a^2b^2 + b^2c^2 + c^2a^2)^2 \geq 4(a+b)(b+c)(c+a)m_a^2m_b^2m_c^2,$$

which immediately result if we multiply (1) and (2) .

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_a^2m_b^2m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2) \overset{(1)}{\equiv} \frac{1}{64}\left\{-4\sum a^6 + 6\left(\sum a^4b^2+\sum a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now, }\sum a^6 = \left(\sum a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum a^2\right)^3 - 3\left(2a^2b^2c^2+\sum a^2b^2\left(\sum a^2 - c^2\right)\right)$$

$$= \left(\sum a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum a^2b^2\right)\sum a^2 \overset{(2)}{\equiv} \sum a^6$$

$$\text{Again, }\sum a^4b^2+\sum a^2b^4 = \left(\sum a^2b^2\right)\left(\sum a^2 - c^2\right) \overset{(3)}{\equiv} \left(\sum a^2b^2\right)\sum a^2 - 3a^2b^2c^2$$

$\therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$

$$= \frac{1}{64}\left\{-4\left(\sum a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum a^2b^2\right)\sum a^2 + 6\left(\sum a^2b^2\right)\sum a^2 - 18a^2b^2c^2 + 3a^2b^2c^2\right\}$$

$$= \frac{1}{64}\left\{-4\left(\sum a^2\right)^3 + 18\left(\sum a^2b^2\right)\sum a^2 - 27a^2b^2c^2\right\}$$

$$= \frac{1}{64}\left\{-4\left(\sum a^2\right)^3 + 18\left(\left(\sum ab\right)^2 - 2abc(2s)\right)\left(\sum a^2\right) - 27a^2b^2c^2\right\}$$

$$= \frac{1}{64}\left\{-32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 - 576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2\right\}$$

$$= \frac{1}{16}\{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3\}$$

$$\therefore 16m_a^2m_b^2m_c^2 \overset{(*)}{\equiv} s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3$$

$$\text{Now, }\frac{1}{\sin\omega} \geq 2\sqrt[4]{\frac{m_a^2m_b^2m_c^2}{w_aw_bw_ch_ah_bh_c}} \Leftrightarrow \frac{1}{16r^4s^4}\left(\sum a^2b^2\right)^2(w_aw_bw_c)(h_ah_bh_c) \geq 16m_a^2m_b^2m_c^2$$

$$\Leftrightarrow \frac{1}{16r^4s^4}\left(\sum a^2b^2\right)^2\left(\frac{16Rr^2s^2}{s^2+2Rr+r^2}\right)\left(\frac{2r^2s^2}{R}\right)$$
$$\geq s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3$$
$$\Leftrightarrow 2\left(\left(s^2+4Rr+r^2\right)^2 - 16Rrs^2\right)^2$$
$$\geq \left(s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3\right)(s^2$$
$$+ 2Rr + r^2)$$
$$\Leftrightarrow s^8 - s^6(22Rr+26r^2) + r^2s^4(276R^2+34Rr+12r^2)$$
$$- r^3s^2(328R^3-220R^2r-230Rr^2-42r^3)$$
$$+ r^4(640R^4+672R^3r+264R^2r^2+46Rr^3+3r^4) \overset{(m)}{\gtreqqless} 0$$

$\because$ via Gerretsen, $\left(s^2-16Rr+5r^2\right)^4 + (42Rr-46r^2)(s^2-16Rr+5r^2)^3 \geq 0$

$\therefore$ in order to prove (m), it suffices to prove :

$$s^8 - s^6(22Rr+26r^2) + r^2s^4(276R^2+34Rr+12r^2)$$
$$- r^3s^2(328R^3-220R^2r-230Rr^2-42r^3)$$
$$+ r^4(640R^4+672R^3r+264R^2r^2+46Rr^3+3r^4)$$
$$\geq \left(s^2-16Rr+5r^2\right)^4 + (42Rr-46r^2)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (756R^2-1844Rr+552r^2)s^4 - (16200R^3-40348R^2r+20200Rr^2-2992r^3)rs^2$$
$$+ r^2(107136R^4-267104R^3r+188904R^2r^2-52404Rr^3+5128r^4) \geq 0$$
$$\Leftrightarrow (R-2r)(756R-332r)s^4 - 112r^2s^4$$
$$- (16200R^3-40348R^2r+20200Rr^2-2992r^3)rs^2$$
$$+ r^2(107136R^4-267104R^3r+188904R^2r^2-52404Rr^3+5128r^4) \overset{(i)}{\gtreqqless} 0$$

and $\because -112r^2s^4 \overset{\text{Gerretsen}}{\gtreqqless} -112r^2s^2(4R^2+4Rr+3r^2)$ $\therefore$ LHS of (i) $\geq$

$$(R-2r)(756R-332r)s^4$$
$$- rs^2\left(112r(4R^2+4Rr+3r^2) + 16200R^3 - 40348R^2r + 20200Rr^2\right.$$
$$\left.- 2992r^3\right)$$
$$+ r^2(107136R^4-267104R^3r+188904R^2r^2-52404Rr^3+5128r^4)$$

$\therefore$ in order to prove (i), it suffices to prove :

$$(R-2r)(756R-332r)s^4 - rs^2\left(112r\left(4R^2+4Rr+3r^2\right)+16200R^3-40348R^2r+20200Rr^2-2992r^3\right) + r^2\left(107136R^4-267104R^3r+188904R^2r^2-52404Rr^3+5128r^4\right) \overset{(ii)}{\overset{?}{\geq}} 0$$

$\because$ via Gerretsen, $(R-2r)(756R-332r)\left(s^2-16Rr+5r^2\right)^2 \geq 0 \therefore$

in order to prove (ii), it suffices to prove :

$$(R-2r)(756R-332r)s^4 - rs^2\left(112r\left(4R^2+4Rr+3r^2\right)+16200R^3-40348R^2r+20200Rr^2-2992r^3\right) + r^2\left(107136R^4-267104R^3r+188904R^2r^2-52404Rr^3+5128r^4\right)$$

$$\geq (R-2r)(756R-332r)\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow \left(7992R^3-26668R^2r+19040Rr^2-3984r^3\right)s^2 \geq r\left(86400R^4-325920R^3r+295020R^2r^2-99936Rr^3+11472r^4\right)$$

$$\Leftrightarrow (R-2r)\left((R-2r)(7992R+5300r)+8272r^2\right)s^2-8640r^3s^2 - r\left(86400R^4-325920R^3r+295020R^2r^2-99936Rr^3+11472r^4\right) \overset{(iii)}{\overset{?}{\geq}} 0$$

Again, LHS of (iii) $\overset{\text{Gerretsen}}{\geq}$

$$(R-2r)\left((R-2r)(7992R+5300r)+8272r^2\right)\left(16Rr-5r^2\right)-8640r^3\left(4R^2+4Rr+3r^2\right) - r\left(86400R^4-325920R^3r+295020R^2r^2-99936Rr^3+11472r^4\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow 41472t^4-140728t^3+108400t^2+44672t-60672 \overset{?}{\geq} 0 \quad \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(41472t^2+25160t+43152\right)+11664\right) \overset{?}{\geq} 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (m) is true $\therefore \dfrac{1}{\sin\omega} \geq 2\sqrt[4]{\dfrac{m_a^2m_b^2m_c^2}{w_aw_bw_ch_ah_bh_c}}$ (QED)

SP.381 Find all continuous functions $f:(0,\infty)\to\mathbb{R}$ such that

$$f(a^x)+f(a^{2x})+f(a^{4x})=x,\forall x\in\mathbb{R}, a>0, a\neq 1 \text{ −fixed.}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\text{Let } a^x=t\to x=\log_a t\to f(t)+f(t^2)+f(t^4)=\log_a t$$

$$t\sim t^2\to f(t^2)+f(t^4)+f(t^8)=\log_a t^2$$

$$\to f(t)-f(t^8)=-\log_a t\to f(t^8)-f(t)=\log_a t$$

$$t\sim t^{\frac{1}{8}}\to f(t)-f\left(t^{\frac{1}{8}}\right)=\log_a t^{\frac{1}{8}}$$

$$t\sim t^{\frac{1}{8}}\to f\left(t^{\frac{1}{8}}\right)-f\left(t^{\frac{1}{64}}\right)=\log_a t^{\frac{1}{64}}$$

… … … … … … … … … … … … … …

$$t\sim t^{\frac{1}{8}}\to f\left(t^{\frac{1}{8^{n-1}}}\right)-f\left(t^{\frac{1}{8^n}}\right)=\log_a t^{\frac{1}{8^n}}$$

Adding up these relations, it follows that:

$$f(t)-f\left(t^{\frac{1}{8^n}}\right)=\left(\frac{1}{8}+\frac{1}{8^2}+\cdots+\frac{1}{8^n}\right)\log_a t$$

$$\lim_{t\to\infty}\left(f(t)-f\left(t^{\frac{1}{8^n}}\right)\right)=\lim_{t\to\infty}\frac{\frac{1}{8}\left(\frac{1}{8^n}-1\right)}{\frac{1}{8}-1}\log_a t\to f(t)-f(1)=\frac{1}{7}\log_a t$$

$f(1)=0$ **(from hypothesis** $x=0\to 3f(1)=0$**). So,** $f(t)=\frac{1}{7}\log_a t$**.**

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\text{Let } t=a^x\Leftrightarrow x=\log_a t\Rightarrow f(t)+f(t^2)+f(t^4)=\log_a t;(1)$$

$$(1): t\to\sqrt{t}\Rightarrow f(\sqrt{t})+f(t)+f(t^2)=\frac{1}{2}\log_a t$$

$$(1): t\to\sqrt[4]{t}\Rightarrow f(\sqrt[4]{t})+f(\sqrt{t})+f(t)=\frac{1}{4}\log_a t$$

$$\Rightarrow f(t^2)-f(\sqrt[4]{t})=\frac{1}{4}\log_a t$$

$$t\to\sqrt{t}\Rightarrow f(t)-f(\sqrt[8]{t})=\frac{1}{2^3}\log_a t$$

… … … … . … … … … …

$$f\left(\sqrt[2^n]{t}\right)-f\left(\sqrt[2^{n+3}]{t}\right)=\frac{1}{2^{n+3}}\log_a t$$

Hence,

$$f(t)-f\left(\sqrt[2^{n+3}]{t}\right)=\left(\frac{1}{2^3}+\frac{1}{2^6}+\cdots+\frac{1}{2^{n+3}}\right)\log_a t$$

$$\lim_{n\to\infty}\left(f(t)-f\left(\sqrt[2^{n+3}]{t}\right)\right)=\lim_{n\to\infty}\left(\frac{1}{2^3}+\frac{1}{2^6}+\cdots+\frac{1}{2^{n+3}}\right)\log_a t=$$

$$=\log_a t\lim_{n\to\infty}\left(\frac{1}{2^3}+\frac{1}{2^6}+\cdots+\frac{1}{2^{n+3}}\right)=\frac{1}{2^3}\log_a t\lim_{n\to\infty}\frac{\left(\frac{1}{2^3}\right)^{\frac{n}{3}}-1}{\frac{1}{2^3}-1}$$

$$f(t)-\lim_{n\to\infty}f\left(\sqrt[2^{n+3}]{t}\right)=\frac{1}{7}\log_a t$$

f –continuous and $\lim_{n\to\infty}f\left(t^{\frac{1}{2^{n+3}}}\right)=f\left(\lim_{n\to\infty}t^{\frac{1}{2^{n+3}}}\right)=f(1)\Rightarrow f(t)=f(1)=\frac{1}{7}\log_a t$

From $f(t)+f(t^2)+f(t^4)=\log_a t$ and for $t=1$, we get: $3f(1)=\log_a 1\Leftrightarrow f(1)=0$.

Therefore,

$$f(x)=\frac{1}{7}\log_a x,\forall x\in(0,\infty)$$

SP.382. $z_1,z_2,z_3\in\mathbb{C}^*$ –different in pairs such that

$|z_1|=|z_2|=|z_3|=1, A(z_1),B(z_2),C(z_3)$. Prove that:

$$\sum_{cyc}\frac{z_2z_3}{(z_2-z_3)^2[z_2(z_1-z_3)^2-z_3(z_1+z_2)^2]}=\frac{1}{4z_1z_2z_3}\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$A(z_1),B(z_2),C(z_3)\to\Delta ABC\subset C(0,1)$$

$$\sum_{cyc}\frac{z_2z_3}{(z_2-z_3)^2[z_2(z_1-z_3)^2-z_3(z_1+z_2)^2]}=\frac{1}{4z_1z_2z_3}\Leftrightarrow$$

$$\sum_{cyc}\frac{z_1z_2^2z_3^2}{(z_2-z_3)^2[z_2(z_1-z_3)^2-z_3(z_1+z_2)^2]}=\frac{1}{4}\Leftrightarrow$$

$$\sum_{cyc}\frac{1}{\frac{(z_2-z_3)^2}{z_2z_3}\left[\frac{z_2(z_1-z_3)^2}{z_1z_2z_3}-\frac{z_3(z_1+z_2)^2}{z_1z_2z_3}\right]}=4;(1)$$

But: $\Delta ABC\subset C(0,1)\to\cos^2 A=\frac{(z_2+z_3)^2}{4z_2z_3},\sin^2 A=-\frac{(z_2-z_3)^2}{4z_2z_3};(2)$

From $(1),(2)$ it follows that:

$$\sum_{cyc}\frac{\csc^2 A}{\sin^2 B+\cos^2 C}=4;(3)$$

But: $\sum_{cyc}\frac{\csc^2 A}{\sin^2 B+\cos^2 C}\overset{Bergstrom}{\geq}\frac{(\sum\csc A)^2}{1+1+1}\geq\frac{(2\sqrt{3})^2}{3}=4;(4)$

From $(3),(4)$ equality holds if and only if $AB=BC=CA$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$|z_1|=|z_2|=|z_3|=1\ \to\ \Delta ABC\in C(O,1)\ \to\ \sin^2 A=-\frac{(z_2-z_3)^2}{4z_2z_3}\ and\ \cos^2 A=\frac{(z_2+z_3)^2}{4z_2z_3}\ (And\ analogs)$$

$$(*)\leftrightarrow\sum_{cyc}\frac{1}{-\frac{(z_2-z_3)^2}{4z_2z_3}\left[-\frac{(z_1-z_3)^2}{4z_1z_3}+\frac{(z_1+z_2)^2}{4z_1z_2}\right]}=4\leftrightarrow\sum_{cyc}\frac{1}{\sin^2 A\left[\sin^2 B+\cos^2 C\right]}=4$$

$$But\ from\ Hölder,we\ have:\sum_{cyc}\frac{1}{\sin^2 A(\sin^2 B+\cos^2 C)}\geq\frac{3^3}{(\sum\sin^2 A)[\sum(\sin^2 B+\cos^2 C)]}$$

$$Also,we\ have:\sum(\sin^2 B+\cos^2 C)=\sum(\sin^2 A+\cos^2 A)=\sum 1=3.$$

$$And:\sum\sin^2 A=\frac{1}{4R^2}\sum a^2\overset{Leibniz}{\leq}\frac{1}{4R^2}\cdot 9R^2=\frac{9}{4}$$

$$\to\sum_{cyc}\frac{1}{\sin^2 A(\sin^2 B+\cos^2 C)}\geq\frac{27}{\frac{9}{4}\cdot 3}=4,with\ equality\ if\ \Delta ABC\ is\ equilateral.$$

$$Therefore,\quad \sum_{cyc}\frac{z_2z_3}{(z_2-z_3)^2[z_2(z_1-z_3)^2-z_3(z_1+z_2)^2]}=\frac{1}{4z_1z_2z_3}\rightarrow$$

$$AB=BC=CA$$

SP.383 If $a,b,c>0$ then:

$$\frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{a^2b+b^2c+c^2a}\geq a^4b^4c^4$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let $f:(0,\infty)\rightarrow(0,\infty), f(x)=\left(\frac{a}{b}\right)^x+\left(\frac{b}{c}\right)^x+\left(\frac{c}{a}\right)^x$

$$f'(x)=\left(\frac{a}{b}\right)^x\log\left(\frac{a}{b}\right)+\left(\frac{b}{c}\right)^x\log\left(\frac{b}{c}\right)+\left(\frac{c}{a}\right)^x\log\left(\frac{c}{a}\right)$$

$$f''(x)=\left(\frac{a}{b}\right)^x\log^2\left(\frac{a}{b}\right)+\left(\frac{b}{c}\right)^x\log^2\left(\frac{b}{c}\right)+\left(\frac{c}{a}\right)^x\log^2\left(\frac{c}{a}\right)\geq 0$$

$\Rightarrow f''(x)\geq 0\Rightarrow f'$ –increasing, then $f'(x)\geq f'(0), \forall x\geq 0$

$$f'(x)\geq\log\left(\frac{a}{b}\right)+\log\left(\frac{b}{c}\right)+\log\left(\frac{c}{a}\right)=\log\left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}\right)=\log 1=0$$

$f'(x)\geq 0$, then f –increasing, so $f(5)\geq f(2)$.

Hence,

$$\left(\frac{a}{b}\right)^5+\left(\frac{b}{c}\right)^5+\left(\frac{c}{a}\right)^5\geq\left(\frac{a}{b}\right)^2+\left(\frac{b}{c}\right)^2+\left(\frac{c}{a}\right)^2\geq\frac{a}{b}\cdot\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}=$$

$$=\frac{a}{c}+\frac{b}{c}+\frac{c}{b}$$

Therefore,

$$\frac{a^5}{b^5}+\frac{b^5}{c^5}+\frac{c^5}{a^5}\geq\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$$

$$\frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{a^4b^4c^4}\geq a^2b+b^2c+c^2a$$

Finally, it follows that:

$$\frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{a^2b+b^2c+c^2a}\geq a^4b^4c^4$$

Equality holds for $a = b = c$.

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$\frac{a^{10}c^5 + b^{10}a^5 + c^{10}b^5}{3} \ge \sqrt[3]{a^{15}b^{15}c^{15}} \Rightarrow a^{10}c^5 + b^{10}a^5 + c^{10}b^5 \ge 3a^5b^5c^5$$

$$\frac{3a^5b^5c^5}{a^2b + b^2c + c^2a} \overset{(1)}{\ge} a^4b^4c^4 \Rightarrow abc \ge a^2b + b^2c + c^2a; (A)$$

$$\frac{a^2b + b^2c + c^2a}{3} \ge \sqrt[3]{a^3b^3c^3} \Rightarrow a^2b + b^2c + c^2a \ge 3abc; (B)$$

From $(A), (B)$, we have $a^2b + b^2c + c^2a = 3abc \Rightarrow \frac{3a^5b^5c^5}{3abc} \ge a^4b^4c^4$ true.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a^{10}c^5 + b^{10}a^5 + c^{10}b^5 = (a^2c)^5 + (b^2a)^5 + (c^2b)^5 \overset{Hölder}{\ge} \frac{(a^2c + b^2a + c^2b)^5}{3^4} =$$

$$= \frac{(a^2c + b^2a + c^2b)^3}{3^4}.(a^2c + b^2a + c^2b)^2 \overset{\substack{AM-GM \\ (\sum x)^2 \ge 3\sum xy}}{\ge}$$

$$\frac{\left(3\sqrt[3]{a^2c.b^2a.c^2b}\right)^3}{3^4}.3(a^2c.b^2a + b^2a.c^2b + c^2b.a^2c) =$$

$$= (abc)^3.abc(a^2b + b^2c + c^2a) = a^4b^4c^4(a^2b + b^2c + c^2a).$$

$$Therefore, \quad \frac{a^{10}c^5 + b^{10}a^5 + c^{10}b^5}{a^2b + b^2c + c^2a} \ge a^4b^4c^4.$$

Solution 4 by Adrian Popa-Romania

We must prove that:

$$a^{10}c^5 + b^{10}a^5 + c^{10}b^5 \ge a^5b^4c^6 + b^5c^4a^6 + c^5a^4b^6$$

$(10,0,5) > (5,4,6)$ because $10 > 5, 10 + 0 > 5 + 4, 10 + 0 + 5 = 5 + 4 + 6$.

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

We have:

$$\frac{a^6c}{b^4} + \frac{c^6b}{a^4} + \frac{b^6a}{c^4} \ge \frac{a^3 + b^3 + c^3}{3}\left(\frac{a^3c}{b^4} + \frac{c^3b}{a^4} + \frac{b^3a}{c^4}\right) \ge a^3 + b^3 + c^3$$

$$\ge a^2b + b^2c + c^2a$$

Hence,

$$\frac{a^{10}c^5 + c^{10}b^5 + b^{10}a^5}{(abc)^4} \ge a^2b + b^2c + c^2a$$

Finally, it follows that:

$$\frac{a^{10}c^5 + b^{10}a^5 + c^{10}b^5}{a^2b + b^2c + c^2a} \ge a^4b^4c^4$$

Equality holds for $a = b = c$.

SP.384 Let $(a_n)_{n\ge1}$ be sequence of real numbers with $a_n > 0, \forall n \in \mathbb{N}$ and $a_0 = 1,\ a_n^2 + a_n e^{a_n} = (n+1)(n+1+e^{a_n})$.Find:

$$\Omega = \lim_{n\to\infty} a_n \cdot \sqrt[n]{a_n \cdot \sin 1 \cdot \sin\frac{1}{2} \cdot \ldots \cdot \sin\frac{1}{n}}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$a_n^2 + a_n e^{a_n} = (n+1)(n+1+e^{a_n}) \Leftrightarrow$$

$$(n+1)e^{a_n} - a_n e^{a_n} + (n+1)^2 - a_n^2 = 0 \Leftrightarrow$$

$$e^{a_n}\big((n+1) - a_n\big) + (n+1)^2 - a_n^2 = 0 \Leftrightarrow$$

$$\big((n+1) - a_n\big)(e^{a_n} + a_n + n + 1) = 0 \Leftrightarrow a_n = n+1, (a_n > 0, \forall n \in \mathbb{N})$$

Now, we have:

$$\Omega = \lim_{n\to\infty} a_n \cdot \sqrt[n]{a_n \cdot \sin 1 \cdot \sin\frac{1}{2} \cdot \ldots \cdot \sin\frac{1}{n}} =$$

$$= \lim_{n\to\infty} \sqrt[n]{(n+1)^{n+1} \cdot \sin 1 \sin\frac{1}{2} \ldots \sin\frac{1}{n}} \overset{C-D}{=}$$

$$= \lim_{n\to\infty} \frac{(n+2)^{n+2} \cdot \sin 1 \sin\frac{1}{2} \ldots \sin\frac{1}{n} \sin\frac{1}{n+1}}{(n+1)^{n+1} \cdot \sin 1 \sin\frac{1}{2} \ldots \sin\frac{1}{n}} =$$

$$= \lim_{n\to\infty} \left[\left(\frac{n+2}{n+1}\right)^{n+1} \cdot (n+2)\sin\frac{1}{n+1}\right] =$$

$$=\left[\left(1+\frac{1}{n+1}\right)^{n+1}\cdot\frac{\sin\frac{1}{n+1}}{\frac{1}{n+1}}\cdot\frac{n+2}{n+1}\right]=e$$

Solution 2 by Mikael Bernardo-Mozambique

$$a_0=1, a_n^2+a_ne^{a_n}=(n+1)(n+1+e^{a_n})\Leftrightarrow$$

$$a_n^2+a_ne^n=(n+1)^2+(n+1)e^{a_n}\Leftrightarrow a_n=n+1;(a_1=1)$$

$$\Omega=\lim_{n\to\infty}a_n\cdot\sqrt[n]{a_n\cdot\sin 1\cdot\sin\frac{1}{2}\cdot\ldots\cdot\sin\frac{1}{n}}=\lim_{n\to\infty}\sqrt[n]{a_n^{n+1}\cdot\sin 1\cdot\sin\frac{1}{2}\cdot\ldots\cdot\sin\frac{1}{n}}=$$

$$\overset{C-D}{=}\lim_{n\to\infty}\frac{a_{n+1}^{n+2}\sin 1\cdot\sin\frac{1}{2}\cdot\ldots\cdot\sin\frac{1}{n}\cdot\sin\frac{1}{n+1}}{a_n^{n+1}\sin 1\cdot\sin\frac{1}{2}\cdot\ldots\cdot\sin\frac{1}{n}}=$$

$$=\lim_{n\to\infty}\frac{(n+2)^n\cdot(n+2)^2\cdot\sin\frac{1}{n+1}}{(n+1)^n\cdot(n+1)}=$$

$$=\lim_{n\to\infty}\left(\left(1+\frac{1}{n+1}\right)^{n+1}\right)^{\frac{n}{n+1}}\cdot\frac{(n+2)^2}{(n+1)^2}\cdot\frac{\sin\frac{1}{n+1}}{\frac{1}{n+1}}=e\cdot 1\cdot 1=e$$

SP.385 Let $(a_n)_{n\geq 1}$ be sequence of positive real numbers such that:

$$a_{n+1}^3-(a_n+a_1)a_{n+1}^2+(a_{n+1}-a_1)a_n^2+a_1a_na_{n+1}=0,\forall n\in\mathbb{N}^*,$$

$n>1$. Prove that:

$$\sum_{k=1}^{n}\log_3\left(\left(\frac{a_k}{a_{k+1}}\right)^2+\left(\frac{a_k}{a_{k+1}}\right)+1\right)\geq n$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$a_{n+1}^3-(a_n+a_1)a_{n+1}^2+(a_{n+1}-a_1)a_n^2+a_1a_na_{n+1}=0\Leftrightarrow$$

$$a_{n+1}^3+a_{n+1}a_n^2-a_na_{n+1}^2-a_1a_{n+1}^2-a_1a_n^2+a_1a_na_{n+1}=0\Leftrightarrow$$

$$(a_{n+1}-a_1)\cdot(a_{n+1}^2+a_n^2-a_na_{n+1})=0\Leftrightarrow$$

$$(a_{n+1}-a_1)\left(\frac{a_{n+1}}{a_n}+\frac{a_n}{a_{n+1}}-1\right)=0\Leftrightarrow a_{n+1}=a_1,\forall n\in\mathbb{N}^*,n>1$$

$$\text{because } \frac{a_{n+1}}{a_n}+\frac{a_n}{a_{n+1}}-1\ge 1,\forall n\in\mathbb{N}^*,n>1 \text{ and } a_n>0$$

Now, applying AM-GM inequality, we have:

$$\left(\frac{a_k}{a_{k+1}}\right)^2+\left(\frac{a_k}{a_{k+1}}\right)+1\ge 3\cdot\sqrt[3]{\frac{a_k^3}{a_{k+1}^3}}=3\cdot\frac{a_k}{a_{k+1}}$$

$$\prod_{k=1}^{n}\left(\left(\frac{a_k}{a_{k+1}}\right)^2+\left(\frac{a_k}{a_{k+1}}\right)+1\right)\ge 3^n\cdot\prod_{k=1}^{n}\frac{a_k}{a_{k+1}}=3^n,$$

$$\text{because } a_{n+1}=a_1,\forall n\in\mathbb{N}^*,n>1$$

Therefore,

$$\sum_{k=1}^{n}\log_3\left(\left(\frac{a_k}{a_{k+1}}\right)^2+\left(\frac{a_k}{a_{k+1}}\right)+1\right)\ge n$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*):\ {a_{n+1}}^3-(a_n+a_1){a_{n+1}}^2+(a_{n+1}-a_1){a_n}^2+a_1a_na_{n+1}=0,\forall n\in N^*,n>1.$$

$$Let\ n\in N^*,n>1.$$

$$(*)\ \leftrightarrow\ ({a_{n+1}}^3-a_1{a_{n+1}}^2)-(a_n{a_{n+1}}^2-a_1a_na_{n+1})+(a_{n+1}-a_1){a_n}^2=0$$

$$\leftrightarrow\ {a_{n+1}}^2(a_{n+1}-a_1)-a_na_{n+1}(a_{n+1}-a_1)+(a_{n+1}-a_1){a_n}^2=0$$

$$\leftrightarrow\ (a_{n+1}-a_1)({a_{n+1}}^2-a_na_{n+1}+{a_n}^2)=0$$

$$\leftrightarrow\ a_{n+1}=a_1,\forall n\in N^*,n>1\quad (\because {a_{n+1}}^2-a_na_{n+1}+{a_n}^2>0)$$

$$Now, from\ AM-GM, we\ have:$$

$$\left(\frac{a_k}{a_{k+1}}\right)^2+\left(\frac{a_k}{a_{k+1}}\right)+1\ge 3\sqrt[3]{\left(\frac{a_k}{a_{k+1}}\right)^2.\left(\frac{a_k}{a_{k+1}}\right).1}=3\left(\frac{a_k}{a_{k+1}}\right)$$

$$\to\sum_{k=1}^{n}\log_3\left(\left(\frac{a_k}{a_{k+1}}\right)^2+\left(\frac{a_k}{a_{k+1}}\right)+1\right)\ge\sum_{k=1}^{n}\log_3\left(3\left(\frac{a_k}{a_{k+1}}\right)\right)=\sum_{k=1}^{n}\left[1+\log_3\left(\frac{a_k}{a_{k+1}}\right)\right]$$

$$=n+\log_3\left(\frac{a_1}{a_{n+1}}\right)=n.$$

$$Therefore,\quad \sum_{k=1}^{n}\log_3\left(\left(\frac{a_k}{a_{k+1}}\right)^2+\left(\frac{a_k}{a_{k+1}}\right)+1\right)\geq n.$$

Solution 3 by Kamel Gandouli Rezgui-Tunisia

$$a_{n+1}^3-(a_1+a_n)a_{n+1}^2+(a_{n+1}-a_1)a_n^2-a_1a_na_{n+1}=0$$

$$(1+a_{n+1})(a_{n+1}^2-a_{n+1}a_n+a_n^2)=0\Leftrightarrow a_{n+1}=a_1,\forall n\geq 1 \text{ because}$$

$$a_{n+1}^2-a_{n+1}a_n+a_n^2>0.$$

$$v_n=\sum_{k=1}^{n}\log_3\left(\left(\frac{a_{k+1}}{a_k}\right)^2+\frac{a_{k+1}}{a_k}+1\right)$$

$$v_2=\log_3\left(\left(\frac{a_2}{a_1}\right)^2+\frac{a_2}{a_1}+1\right)+\log_3\left(\left(\frac{a_1}{a_2}\right)^2+\frac{a_2}{a_1}+1\right)=$$

$$=\frac{1}{\log 3}\cdot\log\left\{\left[\left(\frac{a_2}{a_1}\right)^2+\frac{a_2}{a_1}+1\right]\left[\left(\frac{a_1}{a_2}\right)^2+\frac{a_1}{a_2}+1\right]\right\}\geq\frac{\log 9}{\log 3}=2 \text{ because}$$

$$\left(x^2+1+\frac{1}{x}\right)\left(\frac{1}{x^2}+\frac{1}{x}+1\right)\geq 9$$

$$\text{Suppose: } v_n=\sum_{k=1}^{n}\log_3\left(\left(\frac{a_{k+1}}{a_k}\right)^2+\frac{a_{k+1}}{a_k}+1\right),\forall n>1$$

$$\Rightarrow\sum_{k=1}^{n+1}\log_3\left(\left(\frac{a_{k+1}}{a_k}\right)^2+\frac{a_{k+1}}{a_k}+1\right)=v_n+\log_3\left(\left(\frac{a_{n+2}}{a_{n+1}}\right)^2+\frac{a_{n+2}}{a_{n+1}}+1\right)\geq n+1$$

Solution 4 by Hyun Binh Yoo-South Korea

$$a_{n+1}^3-(a_n+a_1)a_{n+1}^2+(a_{n+1}-a_1)a_n^2+a_1a_na_{n+1}=0\Leftrightarrow$$

$$a_{n+1}^3+a_{n+1}a_n^2-a_na_{n+1}^2-a_1a_{n+1}^2-a_1a_n^2+a_1a_na_{n+1}=0\Leftrightarrow$$

$$(a_{n+1}-a_1)\cdot(a_{n+1}^2+a_n^2-a_na_{n+1})=0\Leftrightarrow a_{n+1}=a_1 \text{ or } a_{n+1}^2+a_n^2-a_na_{n+1}=0.$$

$$i)\ a_{n+1}=a_1\Rightarrow\sum_{k=1}^{n}\log_3\left(\left(\frac{a_k}{a_{k+1}}\right)^2+\left(\frac{a_k}{a_{k+1}}\right)+1\right)=$$

$$=\sum_{k=1}^{n}\log_3\left(\left(\frac{a_1}{a_1}\right)^2+\left(\frac{a_1}{a_1}\right)+1\right)=\sum_{k=1}^{n}\log_3 3=n$$

$$ii)\ a_{n+1}^2+a_n^2-a_na_{n+1}=0\Leftrightarrow\left(a_{n+1}-\frac{a_n}{2}\right)^2+\frac{3}{4}a_n^2=0\ impossible.$$

SP.386 Solve for real numbers:

$$\log_2(5^x-3)=\log_7(3^x+4)$$

Proposed by Ionuț Florin Voinea-Romania

Solution 1 by proposer

$$\text{C.E. }\begin{cases}5^x-3>0\\3^x+4>0\end{cases}\Rightarrow\begin{cases}x>\log_5 3\\x\in\mathbb{R}\end{cases}\Rightarrow x\in(\log_5 3,\infty)$$

$$\text{Let us denote: } y=\log_2(5^x-3)=\log_7(3^3+4)\Leftrightarrow\begin{cases}5^x-3=2^y\\3^x+4=7^y\end{cases}$$

I) If $x\ge 1$.

$$\text{i) For } y>x\Rightarrow\begin{cases}7^y>7^x\\7^y=3^x+4\end{cases}\Rightarrow 3^x+4>7^x|:7^x>0\Rightarrow\left(\frac{3}{7}\right)^x+4\left(\frac{1}{7}\right)^x>1;(*)$$

Let be the function $f:(0,\infty)\to\mathbb{R}, f(t)=\left(\frac{3}{7}\right)^t+4\left(\frac{1}{7}\right)^t$ decreasing and for $x\ge 1\Rightarrow$

$f(x)\le f(1)=1$ but from $(*)$: $f(x)>1$ then, $1<f(x)\le 1$ contradiction!

$$\text{ii) For } y<x\Rightarrow\begin{cases}2^y<2^x\\2^y=5^x-3\end{cases}\Rightarrow 5^x-3<2^x\Rightarrow 2^x+3>5^x|:5^x>0\Rightarrow$$

$$\left(\frac{2}{5}\right)^x+3\left(\frac{2}{5}\right)^x>1;(**)$$

Let be the function $g:(0,\infty)\to\mathbb{R}, g(t)=\left(\frac{2}{5}\right)^t+3\left(\frac{2}{5}\right)^t$ decreasing and for $x\ge 1\Rightarrow$

$g(x)\le g(1)=1$ but from $(**)$: $g(x)>1$ then, $1<g(x)\le 1$ cotradiction!

$$\text{iii) For } y=x\Rightarrow\begin{cases}5^x-3=2^x\\3^x+4=7^x\end{cases}$$

$$5^x-3=2^x\Leftrightarrow 2^x+3=5^x|:5^x\Leftrightarrow\left(\frac{2}{5}\right)^x+3\left(\frac{1}{5}\right)^x=1$$

We observe that: $x=1$ has unique solution, because function $g(x)=\left(\frac{2}{5}\right)^x+3\left(\frac{1}{5}\right)^x$ is decreasing.

II) If $x\in(\log_5 3,1)$

$$\text{i) For } y>x\Rightarrow\begin{cases}2^y>2^x\\2^y=5^x-3\end{cases}\Rightarrow 5^x-3>2^x\Rightarrow 2^x+3<5^x|:5^x>0$$

$$\Rightarrow\left(\frac{2}{5}\right)^x+3\left(\frac{1}{5}\right)^x<1;(1)$$

$x<1$ and $g(t)=\left(\frac{2}{5}\right)^t+3\left(\frac{1}{5}\right)^t$ decreasing then, $g(x)\ge g(1)=1$ but from (1): $g(x)<1$

thus, $1>g(x)>1$ contradiction!

ii) For $y<x\Rightarrow\begin{cases}7^y<7^x\\7^y<3^x+4\end{cases}\Rightarrow 3^x+4<7^x|:7^x>0$

$$\Rightarrow\left(\frac{3}{7}\right)^x+4\left(\frac{1}{7}\right)^x<1;(2)$$

$x<1$ and $f(t)=\left(\frac{3}{7}\right)^t+4\left(\frac{1}{7}\right)^t$ decreasing then, $f(x)>f(1)=1$ but from (2): $f(1)<1$

thus, $1>f(1)>1$ contradiction.

iii) For $y=x\Rightarrow\begin{cases}5^x-3=2^x\\3^x+4=7^x\end{cases}$ has solution $x=1$ but $1\notin(\log_5 3,1)$. No has solution.

So, $x=1$ has unique solution of the equation.

Solution 2 by Hyun Bin Yoo-South Korea

Let $f(x)=\log_2(5^x-3)$, $D=\{x|x>\log_5 3, x\in\mathbb{R}\}$ and $g(x)=\log_7(3^x+4)$ such that $f(x)=g(x)$. First $x=1$ is a trivial solution. Now, we will prove that there are no other solutions of $f(x)=g(x)$.

$$f'(x)=\frac{1}{\log 2}\cdot\frac{5^x\log 5}{5^x-3}=\log_2 5\cdot\frac{5^x}{5^x-3}$$

Because $\log_2 5>\log_2 2>1$ and $\frac{5^x}{5^x-3}>$ then $f'(x)>1,\forall x>\log_5 3$.

$$g'(x)=\frac{1}{\log 7}\cdot\frac{3^x\log 3}{3^x+4}=\log_7 3\cdot\frac{3^x}{3^x+4}$$

Because $0<\log_7 3<\log_7 7=1$ and $0<\frac{3^x}{3^x+4}<1$ then $0<g'(x)<1$.

So, $f'(x)>g'(x),\forall x>\log_5 3$.

$$i)\ x>1, f(x)=f(1)+\int_1^x f'(t)\,dt=1+\int_1^x f'(t)\,dt$$

$$g(x)=g(1)+\int_1^x g'(t)\,dt=1+\int_1^x g'(t)\,dt$$

Since $f'(x)>g'(x)\Rightarrow f(x)>g(x)\Rightarrow-\int_1^x f'(t)\,dt>\int_1^x g'(t)\,dt\Rightarrow$

$f(x)=g(x)$ no has solution.

$ii)\ \log_5 3<x<1$,

$$,f(x)=f(1)-\int_x^1 f'(t)\,dt=1-\int_x^1 f'(t)\,dt$$

$$g(x)=g(1)-\int_x^1 g'(t)\,dt=1-\int_x^1 g'(t)\,dt\Rightarrow$$

$f(x)=g(x)$ no has solution.

So, $x=1$ is unique solution of the equation.

SP387. Given $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=-2x^2+6x-3$, then find

$$\Omega=\lim_{n\to\infty}\int_1^2 f^n(x)\,dx$$

$$\text{(where } f^n(x)=\underbrace{f\left(f\left(f(\dots f(x))\right)\right)}_{n-times}$$

Proposed by Rajeev Rastogi-India

Solution by proposer

$$\because f(x)=-2(x^2-3x)-3=-2\left(x^2-3x+\frac{9}{4}\right)+\left(\frac{9}{2}-3\right)=\frac{3}{2}-2\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow f(x)=\frac{3-(2x-3)^2}{2}\Rightarrow(2f(x)-3)^2=(2x-3)^4$$

$$\therefore f(f(x))=\frac{3-2(f(x)-3)^2}{2}=\frac{3-(2x-3)^{2^2}}{2}$$

Now, by induction we will show that

$$f^n(x)=\frac{3-(2x-3)^{2^n}}{2}\quad(1)$$

Suppose (1) is true for some $n=k$ (where $k>1$)

$$\Rightarrow f^k(x)=\frac{3-(2x-3)^{2^k}}{2}$$

$$\therefore f^{k+1}(x)=\frac{3-(2f(x)-3)^{2^k}}{2}=\frac{3-(2x-3)^{2^{k+1}}}{2}$$

Hence by induction (1) is true for all $n\in\mathbb{N}$

Now

$$\Omega = \lim_{n\to\infty}\int_1^2\left(\frac{3-(2x-3)^{2^n}}{2}\right)dx = \lim_{x\to\infty}\frac{1}{2}\left(3x - \frac{(2x-3)^{2^n+1}}{(2^n+1)\cdot 2}\right)^2$$

$$= \lim_{n\to\infty}\frac{1}{2}\left[\left(6-\frac{1}{(2^n+1)\cdot 2}\right)-\left(3+\frac{1}{(2^n+1)\cdot 2}\right)\right] = \lim_{n\to\infty}\frac{1}{2}\left[3-\frac{1}{2^n+1}\right] = \frac{3}{2}$$

$$\therefore \Omega = \frac{3}{2}$$

SP.388 Given $\{a_n\}$ is a sequence of real numbers satisfying $a_1 = 0$ and

$$\frac{4}{4-a_{n+1}} - \frac{4}{4-a_n} = 2n+1; \forall n \geq 1$$

Define $b_n = \frac{4-a_n}{4}$ for $n \geq 1$, then find

$$\Omega = \lim_{n\to\infty}\left[4\cdot\sin^{-1}\frac{b_{n+1}}{b_n} - \left(\pi - \frac{1}{2} + \tan h^{-1}\sqrt{b_n}\right)\left(\prod_{k=2}^{n} a_k\right)\right]$$

Proposed by Rajeev Rastogi-India

Solution 1 by proposer

$$\frac{4}{4-a_{n+1}} - \frac{4}{4-a_n} = (n+1)^2 - n^2$$

$\Rightarrow \frac{4}{4-a_{n+1}} - (n+1)^2 = \frac{4}{4-a_n} - n^2$ **(1). Let** $c_n = \frac{4}{4-a_n} - n^2$**, then**

(1)$\Rightarrow c_n = c_{n-1} = c_{n-2} = \cdots = c_1 = \frac{4}{4-a_1} - 1^2 \Rightarrow c_n = \frac{4}{4} - 1 = 0$

$$\Rightarrow \frac{4}{4-a_n} = n^2 \Rightarrow a_n = 4 - \frac{4}{n^2}$$

$$\because b_n = \frac{4-a_n}{4} = \frac{1}{n^2}$$

$$\because \tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \tanh^{-1}\sqrt{b_n} = \frac{1}{2}\log\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right) = \frac{1}{2}\log\left(\frac{n+1}{n-1}\right)$$

Also,

$$\prod_{k=2}^{n} a_k = 4\prod_{k=2}^{n}\left(\frac{k-1}{k}\right)\cdot\left(\frac{k+1}{k}\right)$$

$$= 4\left[\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\dots\frac{n-1}{n}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\dots\frac{n+1}{n}\right)\right]$$

$$= 4\left[\frac{1}{n}\cdot\frac{n+1}{2}\right] \Rightarrow \prod_{k=2}^{n} a_k = \frac{2(n+1)}{n}$$

$$\therefore \Omega = \lim_{n\to\infty}\left[4\cdot\sin^{-1}\frac{n^2}{(n+1)^2} - \left(\pi - \frac{1}{2} + \frac{1}{2}\log\left(\frac{n+1}{n-1}\right)\right)\cdot\frac{2(n+1)}{n}\right]$$

$$= 4\sin^{-1}1 - \left(\pi - \frac{1}{2} + \frac{1}{2}\log 1\right)\cdot 2$$

$$= 4\cdot\frac{\pi}{2} - \left(\pi - \frac{1}{2}\right)\cdot 2 = 2\pi - 2\pi + 1$$

$$\Rightarrow \Omega = 1$$

Solution 2 by Adrian Popa-Romania

Denote: $\frac{4}{4-a_n} = x_n \Rightarrow \frac{4}{4-a_{n+1}} = x_{n+1}; x_1 = 1$

$$\left.\begin{array}{r} n=1 \Rightarrow x_2 - x_1 = 3 \\ n=2 \Rightarrow x_3 - x_2 = 5 \\ n=3 \Rightarrow x_4 - x_3 = 7 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ n=n-1 \Rightarrow x_n - x_{n-1} = 2n-1 \end{array}\right\} \Rightarrow x_n - x_1 = 3+5+7+\dots+(2n-1)$$

$$x_n = 1+3+5+\dots+(2n-1) = n^2 \Rightarrow \frac{4}{4-a_n} = n^2 \Rightarrow$$

$$a_n = \frac{4n^2-4}{n^2} = \frac{4(n-1)(n+1)}{n^2} = 4\left(1-\frac{1}{n^2}\right) \Rightarrow b_n = \frac{4-a_n}{4} = 1 - \frac{1}{4}a_n = \frac{1}{n^2}$$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} \to 1$$

$$\lim_{n\to\infty} 4\sin^{-1}\left(\frac{b_{n+1}}{b_n}\right) = 4\sin^{-1}1 = 2\pi$$

$$\tanh^{-1}\sqrt{b_n}=\tanh^{-1}\left(\frac{1}{n}\right)=\frac{1}{2}\log\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)=\frac{1}{2}\log\left(\frac{n+1}{n-1}\right)\to\frac{1}{2}\log 1=0$$

$$\prod_{k=2}^{n}a_k=\frac{2(n+1)}{n}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left[4\cdot\sin^{-1}\frac{n^2}{(n+1)^2}-\left(\pi-\frac{1}{2}+\frac{1}{2}\log\left(\frac{n+1}{n-1}\right)\right)\cdot\frac{2(n+1)}{n}\right]=$$

$$=4\sin^{-1}1-\left(\pi-\frac{1}{2}+\frac{1}{2}\log 1\right)\cdot 2=1$$

Solution 3 by Hyun Bin Yoo-South Korea

Let $c_n=\frac{4}{4-a_n}$ for $n\geq 1$ then, $c_1=1$ and $\frac{4}{4-a_{n+1}}-\frac{4}{4-a_n}=c_{n+1}-c_n=2n+1$

Hence,

$$\sum_{i=1}^{n}(c_{i+1}-c_i)=c_{n+1}-c_1=c_{n+1}-1,n\geq 1,\sum_{i=1}^{n}(2i+1)=2\cdot\frac{n(n+1)}{2}+n$$
$$=n^2+2n$$

Since $c_{n+1}-c_n=2n+1$ and

$$\sum_{i=1}^{n}(c_{i+1}-c_i)=\sum_{i=1}^{n}(2i+1)\Rightarrow c_{n+1}=(n+1)^2;n\geq 1\Rightarrow c_n=n^2;n\geq 1$$

Since $c_1=1,c_n=n^2;n\geq 1\Rightarrow n^2=\frac{4}{4-a_n}\Leftrightarrow n^2-1=\frac{a_n}{4-a_n}\Leftrightarrow a_n=4\left(1-\frac{1}{n^2}\right);n\geq 1$

Since $a_1=0,a_n=4\left(1-\frac{1}{n^2}\right);n\geq 1$ and $b_n=\frac{1}{c_n}\Rightarrow b_n=\frac{1}{n^2};n\geq 1$

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2=1,\lim_{n\to\infty}\sin^{-1}\left(\frac{b_{n+1}}{b_n}\right)=\lim_{t\to 1^-}\sin^{-1}t=\frac{\pi}{2}$$

$$\lim_{n\to\infty}\sqrt{b_n}=\lim_{n\to\infty}\sqrt{\frac{1}{n}}=0_+,\lim_{n\to\infty}\tanh^{-1}\sqrt{b_n}=\lim_{t\to 0^+}\tanh^{-1}t=0$$

$$\prod_{k=2}^{n} a_k = \prod_{k=2}^{n} 4\left(1-\frac{1}{k^2}\right) = 4\prod_{k=2}^{n}\left(\frac{k-1}{k}\right)\left(\frac{k+1}{k}\right) =$$

$$= 4\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\ldots\cdot\frac{n-1}{n}\cdot\frac{n+1}{n}\right) = \frac{2(n+1)}{n} \Rightarrow \lim_{n\to\infty}\prod_{k=2}^{n} a_k = \lim_{n\to\infty}\frac{2(n+1)}{n} = 2$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left[4\cdot\sin^{-1}\frac{n^2}{(n+1)^2} - \left(\pi - \frac{1}{2} + \frac{1}{2}\log\left(\frac{n+1}{n-1}\right)\right)\cdot\frac{2(n+1)}{n}\right] =$$

$$= 4\sin^{-1}1 - \left(\pi - \frac{1}{2} + \frac{1}{2}\log 1\right)\cdot 2 = 2\pi - 2\pi + 1 = 1$$

SP.389 Given $f:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the functional equation $f(x+y) - 3^y f(x) = 3^x f(y); \forall x,y\in\mathbb{R}, n\in\mathbb{N}^*$ then find

$$\Omega(\mathrm{n}) = \lim_{x\to\infty}\left[\frac{f(x)}{f'(x)} + \frac{f'(x)}{f''(x)} + \cdots + \frac{f^{n-1}(x)}{f^n(x)}\right]$$

(where $f^n(x)$ denotes the n^{th} derivative of $f(x)$ with respect to x)

Proposed by Rajeev Rastogi-India

Solution by proposer

$\because f(x+y) - 3^y f(x) = 3^x f(y); \forall x,y\in\mathbb{R}$ **(1)**

Define new function $g(x) = 3^{-x}\cdot f(x)$ then (1) $\Rightarrow$

$$3^{x+y}\cdot g(x+y) - 3^y\cdot 3^x g(x) = 3^x\cdot 3^y g(-1) \Rightarrow g(x+y) = g(x) + g(y)$$

Hence $g(x)$ satisfies Cauchy functional equation

$\therefore g(x) = ax$ where $a\in\mathbb{R} \Rightarrow ax = 3^{-x}\cdot f(x) \Rightarrow f(x) = ax\cdot 3^x$

$$f'(x) = a[x\cdot 3^x\ln 3 + 3^x\cdot 1] = a\cdot 3^x(x\cdot\ln 3 + 1)$$

$$f''(x) = a[3^x\cdot\ln 3 + (x\ln 3 + 1)\cdot 3^x\cdot\ln 3]$$

$$f''(x) = a\cdot 3^x\cdot\ln 3\,(x\ln 3 + 2)$$

$$f'''(x) = a(\ln 3)^2\cdot 3^x(x\ln 3 + 3)$$

$$\vdots$$

$$f^n(x) = a\cdot(\ln 3)^{n-1}\cdot 3^x(x\ln 3 + n)$$

$$\therefore\ \Omega=\lim_{x\to\infty}\left[\frac{f(x)}{f'(x)}+\frac{f'(x)}{f''(x)}+\cdots+\frac{f^{n-1}(x)}{f^{n}(x)}\right]$$

$$=\lim_{x\to\infty}\left[\frac{ax\cdot 3^x}{a\cdot 3^x(x\ln 3+1)}+\frac{a\cdot 3^x(x\ln 3+1)}{a\cdot 3^x\cdot\ln 3\,(x\ln 3+2)}+\cdots \atop \ldots+\frac{a\cdot(\ln 3)^{n-2}\cdot 3^x\big(x\ln 3+(n-1)\big)}{a(\ln 3)^{n-1}\cdot 3^x(x\ln 3+n)}\right]$$

$$=\lim_{x\to\infty}\left[\frac{x}{x\ln 3+1}+\frac{x\ln 3+1}{\ln 3\,(x\ln 3+2)}+\cdots+\frac{\big(x\ln 3+(n-1)\big)}{\ln 3\,(x\ln 3+n)}\right]$$

$$=\frac{1}{\ln 3}+\frac{1}{\ln 3}+\cdots+\frac{1}{\ln 3}\ \ (n-\text{times})\Rightarrow\Omega=\frac{n}{\ln 3}$$

SP.390 Given $f(x)$ be a non-constant function satisfying the integral equation

$$f(x)=2x^2-\int_0^2(f(t)-x)^2\,dx\ \text{ then find:}$$

$$\Omega=\lim_{x\to\infty}\left[\lim_{n\to\infty}\left(\frac{\sum_{r=0}^{n}f\left(\frac{x}{2^r}\right)}{x}\right)\right]$$

Proposed by Rajeev Rastogi-India

Solution 1 by proposer

Given

$$f(x)=2x^2-\int_0^2(f^2(t)-2xf(t)+x^2)dt$$

$$=2x^2-\int_0^2 f^2(t)\,dt+2x\int_0^2 f(t)dt-x^2\int_0^2 dt\Rightarrow f(x)=2x\int_0^2 f(t)\,dt-\int_0^2 f^2(t)\,dt\quad (1)$$

Let

$$\int_0^2 f(t)dt=a,\ \int_0^2 f^2(t)\,dt=b$$

then (1) $\Rightarrow f(x)=2ax-b$ (2)

Using this in the given equation, we have

$$2ax-b=2x^2-\int_0^2(2at-b-x)^2\,dt=2x^2-\left(\frac{(2at-b-x)^3}{6a}\right)_0^2$$

$$=2x^2-\frac{1}{6a}\left[\big(4a-(b+x)\big)^3+(b+x)^3\right]$$

$$= 2x^2 - \frac{1}{6a}[(4a)^3 - 3(4a)^2(b+x) + 3\cdot 4a\cdot(b+x)^2]$$

$$= 2x^2 - \frac{32}{3}a^2 + 8a(b+x) - 2(b+x)^2$$

$$\Rightarrow 2ax - b = (8a-4b) + \left(8ab - 32\frac{a^2}{3} - 2b^2\right); \forall x\in\mathbb{R}$$

Comparing coefficients

$$2a = 8a - 4b \ \ (3)$$

$$-b = 8ab - 32\frac{a^2}{3} - 2b^2 \ \ (4)$$

$$(3) \Rightarrow 6a = 4b \Rightarrow b = \frac{3}{2}a$$

$$\therefore (4) \Rightarrow -\frac{3}{2}a = 8a\cdot\frac{3}{2}a - 32\frac{a^2}{3} - 2\cdot\frac{9}{4}a^2$$

$$\Rightarrow a^2\left(12 - \frac{32}{3} - \frac{9}{2}\right) = -\frac{3}{2}a(\because a\neq 0)$$

$$\therefore a\left(\frac{72-64-27}{6}\right) = -\frac{3}{2}$$

$$\Rightarrow -\frac{19}{6}a = -\frac{3}{2} \Rightarrow a = \frac{9}{19} \therefore b = \frac{27}{38}$$

$$\therefore (2) \Rightarrow f(x) = \frac{18}{19}x - \frac{27}{38}$$

$$\therefore \Omega = \lim_{x\to\infty}\left[\lim_{n\to\infty}\left(\frac{f(x) + f\left(\frac{x}{2}\right) + f\left(\frac{x}{2^2}\right) + \cdots + f\left(\frac{x}{2^n}\right)}{x}\right)\right]$$

$$= \frac{18}{19}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots up\ to\ \infty\right) = \frac{18}{19}\times\frac{1}{1-\frac{1}{2}} = \frac{18}{19}\times 2$$

$$\Rightarrow \Omega = \frac{36}{19}$$

Solution 2 b Kamel Gandouli Rezgui-Tunisia

$$f(x) = 2x^2 - \int_0^2 (f(t)-x)^2 dx =$$

$$= 2x^2 - \left(\int_0^2 (f(t))^2 dt - 2x\int_0^2 f(t)dt + 2x^2\right) =$$

$$=-\left(\int_0^2 (f(t))^2 dt - 2x\int_0^2 f(t)dt\right) = 2x\alpha - \beta$$

$$\alpha = \int_0^2 (2t\alpha - \beta)dt = 2\alpha\cdot\frac{t^2}{2}\Big|_0^2 - 2\beta = 4\alpha - 2\beta$$

$$\int_0^2 (f(t))^2 dt = \int_0^2 (2t\alpha-\beta)^2 dt = \frac{1}{6\alpha}(2t\alpha-\beta)^3\Big|_0^2 =$$

$$=\frac{1}{6\alpha}(\alpha+\beta)^3 + \frac{1}{6\alpha}\cdot\beta^3 = \frac{1}{6\alpha}\left(\alpha+\frac{3\alpha}{2}\right)^3 + \frac{1}{6\alpha}\left(\frac{3\alpha}{2}\right)^3 = \frac{3\alpha}{2} = \beta$$

$$\frac{125}{48}\alpha^2 + \frac{27}{48}\alpha^2 = \frac{3}{2}\alpha; \alpha \neq 0 \Rightarrow \alpha = \frac{9}{19} \Rightarrow \beta = \frac{27}{38} \Rightarrow f(x) = \frac{18}{19}x - \frac{27}{38}$$

$$\int_0^2 (f(t)-x)^2 dt = \int_0^2 \left(\frac{18}{19}t - \frac{27}{38} - x\right)^2 dt \Rightarrow$$

$$f\left(\frac{x}{2^r}\right) = \frac{18}{19}\left(\frac{x}{2^r}\right) - \frac{27}{38}$$

$$\frac{1}{x}\sum_{r=0}^{n} f\left(\frac{x}{2^r}\right) = \frac{18}{19}x\cdot\frac{\frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}}}{x} - \underbrace{\frac{\frac{27}{38}(n+1)}{x}}_{\to 0 \text{ as } x\to\infty} = \frac{36}{19}$$

UP.376 If $a,b>0$ then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}}\cdot\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bn)^2}+\frac{1}{(a+b(n+1))^2}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposer

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}} \overset{(C-D)}{=} \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n} =$$

$$=\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\left(\frac{n}{n+1}\right)^n = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{e}{2}; (1)$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}}\cdot\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bn)^2}+\frac{1}{\left(a+b(n+1)\right)^2}}\right)=$$

$$=\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bn)^2}+\frac{1}{\left(a+b(n+1)\right)^2}}\right)\overset{(LCS)}{=}$$

$$=\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\lim_{n\to\infty}\frac{\sqrt{\frac{1}{b^2}+\frac{1}{(a+bn)^2}+\frac{1}{\left(a+b(n+1)\right)^2}}}{n+1-n}=$$

$$=\frac{e}{2}\cdot\lim_{n\to\infty}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bn)^2}+\frac{1}{\left(a+b(n+1)\right)^2}}=\frac{e}{2}\cdot\sqrt{\frac{1}{b^2}+0+0}=\frac{e}{2b}$$

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}}<\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bk)^2}+\frac{1}{\left(a+b(k+1)\right)^2}}\overset{AM-GM}{\le}$$

$$\le n\cdot\sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{b^2}+\frac{1}{(a+bk)^2}+\frac{1}{\left(a+b(k+1)\right)^2}\right)}\Leftrightarrow$$

$$\frac{n}{b}<\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bk)^2}+\frac{1}{\left(a+b(k+1)\right)^2}}\le\frac{n}{b}\sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{1}{\left(\frac{a}{b}+k\right)^2}+\frac{1}{\left(\frac{a}{b}+k+1\right)^2}\right)}$$

$$\Leftrightarrow\frac{n}{b\sqrt[n]{(2n-1)!!}}<\frac{1}{\sqrt[n]{(2n-1)!!}}\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bk)^2}+\frac{1}{\left(a+b(k+1)\right)^2}}\le$$

$$\le\frac{n}{b\sqrt[n]{(2n-1)!!}}\sqrt{1+\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{\left(\frac{a}{b}+k\right)^2}+\frac{1}{\left(\frac{a}{b}+(k+1)\right)^2}\right)}$$

Let: $x_n=\frac{n}{\sqrt[n]{(2n-1)!!}}$, it follows

$$\frac{1}{b}\lim_{n\to\infty} x_n < \Omega \le \frac{1}{b}\lim_{n\to\infty} x_n \sqrt{1+\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{\left(\frac{a}{b}+k\right)^2}+\frac{1}{\left(\frac{a}{b}+(k+1)\right)^2}\right)}; (1)$$

From Stirling approximation, we have:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{n\cdot\sqrt[n]{2^n\cdot n!}}{\sqrt[n]{(2n)!}} = \lim_{n\to\infty}\frac{2n\cdot\frac{n}{e}}{\left(\frac{2n}{e}\right)^2} = \frac{e}{2}; (2)$$

$$\sum_{k=1}^{n}\left(\frac{1}{\left(\frac{a}{b}+k\right)^2}+\frac{1}{\left(\frac{a}{b}+(k+1)\right)^2}\right) = 2\sum_{k=1}^{n}\left(\frac{1}{\left(\frac{a}{b}+k\right)^2}\right)+\frac{1}{\left(\frac{a}{b}+(n+1)\right)^2}-\frac{1}{\left(\frac{a}{b}+1\right)^2}$$

$$< 2\sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{\left(\frac{a}{b}+(n+1)\right)^2}-\frac{1}{\left(\frac{a}{b}+1\right)^2}$$

Hence,

$$1 < \sqrt{1+\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{\left(\frac{a}{b}+k\right)^2}+\frac{1}{\left(\frac{a}{b}+(k+1)\right)^2}\right)} \le$$

$$\le \sqrt{1+\frac{1}{n}\left(2\sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{\left(\frac{a}{b}+(n+1)\right)^2}-\frac{1}{\left(\frac{a}{b}+1\right)^2}\right)}$$

$$1 < \lim_{n\to\infty}\sqrt{1+\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{\left(\frac{a}{b}+k\right)^2}+\frac{1}{\left(\frac{a}{b}+(k+1)\right)^2}\right)} \le$$

$$\le \lim_{n\to\infty}\sqrt{1+\frac{1}{n}\left(2\sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{\left(\frac{a}{b}+(n+1)\right)^2}-\frac{1}{\left(\frac{a}{b}+1\right)^2}\right)} =$$

$$=\lim_{n\to\infty}\sqrt{1+\frac{1}{n}\left(2\cdot\frac{\pi^2}{2}+\frac{1}{\left(\frac{a}{b}+(n+1)\right)^2}-\frac{1}{\left(\frac{a}{b}+1\right)^2}\right)}=1$$

Hence,

$$\lim_{n\to\infty}\sqrt{1+\frac{1}{n}\left(2\sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{\left(\frac{a}{b}+(n+1)\right)^2}-\frac{1}{\left(\frac{a}{b}+1\right)^2}\right)}=1;(3)$$

From (1),(2),(3) it follows that:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}}\cdot\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bn)^2}+\frac{1}{\left(a+b(n+1)\right)^2}}\right)=\frac{e}{2b}$$

UP.377 Let $(x_n)_{n\ge0}$ sequence of positive real numbers such that

$$nx_n^2=ax_{n+1}^2+(an-1)x_{n+1}x_n;\ a>0,x_0>0-\text{fixed. Find:}$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^{n}x_k^{\frac{1}{\sqrt{m}}}}{n}\right)^{\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}}\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$ax_{n+1}^2+(an-1)x_{n+1}x_n-nx_n^2=0\Leftrightarrow anx_nx_{n+1}+ax_{n+1}^2-nx_n^2-x_nx_{n+1}=0\Leftrightarrow$$

$$(ax_{n+1}-x_n)(nx_n+x_{n+1})=0\Leftrightarrow\left(x_{n+1}-\frac{1}{a}x_n\right)(nx_n+x_{n+1})=0;(x_n>0,\forall n\in\mathbb{N})$$

$$\Leftrightarrow$$

$$x_{n+1}-\frac{1}{a}x_n=0\Leftrightarrow x_{n+1}=\frac{1}{a}x_n$$

So, $(x_n)_{n\ge0}$ −geometric progression with ratio $q=\frac{1}{a}$.

Now, we have:

$$\lim_{m\to\infty}\left(1+\frac{\sum_{k=1}^{n}\left(x_k^{\frac{1}{\sqrt{m}}}-1\right)}{n}\right)^{\frac{n}{\sum_{k=1}^{n}\left(x_k^{\frac{1}{\sqrt{m}}}-1\right)}}=e$$

$$\lim_{m\to\infty}\frac{\sum_{k=1}^{n}\left(x_k^{\frac{1}{\sqrt{m}}}-1\right)}{n\tan\left(\frac{1}{\sqrt{m}}\right)}=\frac{1}{n}\lim_{m\to\infty}\left[\frac{\frac{1}{\sqrt{m}}}{\tan\frac{1}{\sqrt{m}}}\cdot\sum_{k=1}^{n}\left(\frac{x_k^{\frac{1}{\sqrt{m}}}-1}{\frac{1}{\sqrt{m}}}\right)\right]=$$

$$=\frac{1}{n}\sum_{k=1}^{n}\log x_k=\frac{1}{n}\log\left(\prod_{k=1}^{n}x_k\right)=\frac{1}{n}\log\left(x_1^n\left(\frac{1}{a}\right)^{1+2+3+\dots+(n-1)}\right)=$$

$$=\frac{1}{n}\log\left(a_1^n\left(\frac{1}{a}\right)^{\frac{n(n-1)}{2}}\right)=\log\left(x_1\left(\frac{1}{a}\right)^{\frac{n-1}{2}}\right)$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^{n}x_k^{\frac{1}{\sqrt{m}}}}{n}\right)^{\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left[\left(1+\frac{\sum_{k=1}^{n}\left(x_k^{\frac{1}{\sqrt{m}}}-1\right)}{n}\right)^{\frac{n}{\sum_{k=1}^{n}\left(x_k^{\frac{1}{\sqrt{m}}}-1\right)}}\right]^{\frac{\sum_{k=1}^{n}\left(x_k^{\frac{1}{\sqrt{m}}}-1\right)}{n\tan\left(\frac{1}{\sqrt{m}}\right)}}\right)=$$

$$=\lim_{n\to\infty}\left(x_1\left(\frac{1}{a}\right)^{\frac{n-1}{2}}\right)=\begin{cases}0; & a>1\\ \infty; & a\in(0,1)\\ x_1; & a=1\end{cases}$$

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$ax_{n+1}^2+(an-1)x_nx_{n-1}-nx_n^2=0,\Delta=(an+1)^2x_n^2$$

$$x_{n+1}=\frac{1-an+(an+1)}{2a}x_n=\frac{x_n}{a}\Rightarrow x_n=\left(\frac{1}{a}\right)^n x_0$$

$$x_k^{\frac{1}{\sqrt{m}}}=\left(\frac{1}{a}\right)^{\frac{k}{\sqrt{m}}}x_0\Rightarrow\sum_{k=1}^{n}x_k^{\frac{1}{\sqrt{m}}}=\sum_{k=1}^{n}\left(\frac{1}{a}\right)^{\frac{k}{\sqrt{m}}}x_0=x_0x_1\frac{1-\left(\frac{1}{a}\right)^{\frac{n}{\sqrt{m}}}}{1-\left(\frac{1}{a}\right)^{\frac{1}{\sqrt{m}}}}\Rightarrow$$

$$\left(\frac{1}{n}\sum_{k=1}^{n}x_k^{\frac{1}{\sqrt{m}}}\right)^{\frac{1}{\tan\frac{1}{\sqrt{m}}}}=\left(\frac{x_0x_1}{n}\frac{1-\left(\frac{1}{a}\right)^{\frac{n}{\sqrt{m}}}}{1-\left(\frac{1}{a}\right)^{\frac{1}{\sqrt{m}}}}\right)^{\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}}$$

$$=\exp\left\{\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}\log\left(\frac{x_0x_1}{n}\frac{1-\left(\frac{1}{a}\right)^{\frac{n}{\sqrt{m}}}}{1-\left(\frac{1}{a}\right)^{\frac{1}{\sqrt{m}}}}\right)\right\}$$

$$\lim_{n\to\infty}\frac{x_0x_1}{n}\frac{1-\left(\frac{1}{a}\right)^{\frac{n}{\sqrt{m}}}}{1-\left(\frac{1}{a}\right)^{\frac{1}{\sqrt{m}}}}=\frac{\frac{n\log a}{\log a}}{n}x_0x_1=x_0x_1$$

$$\lim_{m\to\infty}\left(\frac{\sum_{k=1}^{n}x_k^{\frac{1}{\sqrt{m}}}}{n}\right)^{\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}}=e^{\lim_{m\to\infty}\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}\log x_0x_1}=\begin{cases}0, if\ x_0x_1=ax_1^2<1\\+\infty, if\ x_0x_1=ax_1^2>1\\1, if\ x_n\ constante\end{cases}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^{n}x_k^{\frac{1}{\sqrt{m}}}}{n}\right)^{\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}}\right)=\begin{cases}0, if\ x_0x_1=ax_1^2<1\\+\infty, if\ x_0x_1=ax_1^2>1\\1, if\ x_n\ constante\end{cases}$$

Solution 3 by Ruxandra Daniela Tonilă-Romania

$$nx_n^2=ax_{n+1}^2+(an-1)x_{n+1}\cdot x_n$$
$$n=a\left(\frac{x_{n+1}}{x_n}\right)^2+(an-1)\cdot\frac{x_{n+1}}{x_n}$$
$$1=\frac{a}{n}\cdot\left(\frac{x_{n+1}}{x_n}\right)^2+\left(a-\frac{1}{n}\right)\frac{x_{n+1}}{x_n}\Rightarrow\lim_{n\to\infty}\frac{a}{n}\cdot\left(\frac{x_{n+1}}{x_n}\right)^2+\left(a-\frac{1}{n}\right)\frac{x_{n+1}}{x_n}=1$$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\frac{1}{a}\Leftrightarrow\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\frac{1}{a}\lim_{n\to\infty}\frac{P(n+1)}{P(n)},$$

where $P(x)=a_kx^k+a_{k-1}x^{k-1}+\cdots+a_1x+a_0\Leftrightarrow x_{n+1}=\frac{P(n+1)}{P(n)}\cdot\frac{1}{a}\cdot x_n$

Let's suppose that $a_k=a_{k-1}=\cdots=a_1=0$ and $a_0\neq 0$, then $x_{n+1}=\frac{x_n}{a}$.

$\Rightarrow n\frac{x_0^2}{a^{2n}}=a\frac{x_0^2}{a^{2n+2}}+\frac{(an-1)x_0^2}{a^{2n+1}}\Rightarrow na=1+an-1\Rightarrow an=an$, true $\forall a>0$ and $n\in\mathbb{N}$.

$$\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^n x_k^{\frac{1}{\sqrt{m}}}}{n}\right)^{\frac{1}{\tan\left(\frac{1}{\sqrt{m}}\right)}}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^n x_k^{\frac{1}{\sqrt{m}}}}{n}\right)^{\sqrt{m}\cdot\frac{\frac{1}{\sqrt{m}}}{\tan\left(\frac{1}{\sqrt{m}}\right)}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^n\left(\frac{x_0}{a^k}\right)^{\frac{1}{\sqrt{m}}}}{n}\right)^{\sqrt{m}}\right)=x_0\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\sum_{k=1}^n\left(\frac{1}{a^{\frac{1}{\sqrt{m}}}}\right)^k}{n}\right)^{\sqrt{m}}\right)$$

Case 1. If $a\in(0,1)$, we have:

$$\Omega\overset{C-S}{=}x_0\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\frac{\sum_{k=1}^{n+1}\left(\frac{1}{a^{\frac{1}{\sqrt{m}}}}\right)^k-\sum_{k=1}^n\left(\frac{1}{a^{\frac{1}{\sqrt{m}}}}\right)^k}{n+1-n}\right)^{\sqrt{m}}\right)=$$

$$=x_0\lim_{m\to\infty}\left(\frac{1}{a^{\frac{1}{\sqrt{m}}}}\right)^{\sqrt{m}(n+1)}=x_0\lim_{n\to\infty}\frac{1}{a^{n+1}}=+\infty$$

Case 2. If $a=1$, we have:

$$\Omega=x_0\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^n 1^{\frac{1}{\sqrt{m}}}\right)^{\sqrt{m}}=x_0$$

Case 3. If $a>1$, we have:

$$\Omega = x_0 \lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{\frac{1}{a^{\frac{1}{\sqrt{m}}}}\cdot\frac{\left(\frac{1}{a^{\frac{1}{\sqrt{m}}}}\right)^n-1}{\frac{1}{a^{\frac{1}{\sqrt{m}}}}-1}}{n}\right)^{\sqrt{m}}\right) = x_0\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\frac{1-\left(\frac{1}{a^{\frac{1}{\sqrt{m}}}}\right)^n}{\left(a^{\frac{1}{\sqrt{m}}}-1\right)n}\right)^{\sqrt{m}}\right)$$

We have:

$$0\le\Omega\le x_0\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{1}{n^{\sqrt{m}}\left(a^{\frac{1}{\sqrt{m}}}-1\right)^{\sqrt{m}}}\right)=0$$

Therefore,

$$\Omega=\begin{cases}0; & a>1\\ \infty; & a\in(0,1)\\ x_1; & a=1\end{cases}$$

UP.378 If $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ are positive real sequences such that

$$b_n = a_1\cdot\sqrt{a_2!}\cdot\sqrt[3]{a_3!}\cdot\ldots\cdot\sqrt[n]{a_n!} \text{ and } \lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=\pi. \text{ Find:}$$

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by proposers

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+1}\cdot\lim_{n\to\infty}\frac{n\cdot a_n}{a_{n+1}}=\frac{e}{\pi}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{b}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{b_{n+1}}\cdot\frac{b}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+1}\cdot\frac{(n+1)b_n}{b_{n+1}}=$$

$$=e\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{a_{n+1}}}=e\cdot\frac{e}{\pi}=\frac{e^2}{\pi}$$

We have: $\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}=\frac{n^2}{\sqrt[n]{b_n}}\cdot(u_n-1)=\frac{n^2}{\sqrt[n]{b_n}}\cdot\frac{(u_n-1)}{\log u_n}\cdot\log u_n$

$$=\frac{n}{\sqrt[n]{b_n}}\cdot\frac{(u_n-1)}{\log u_n}\cdot\log u_n^n;(1)$$

where, $u_n=\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}$. We have: $\lim_{n\to\infty}u_n=1$, then $\lim_{n\to\infty}\frac{u_n-1}{\log u_n}$ $=1$. Namely,

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{b_n}{b_{n+1}}\cdot\sqrt[n+1]{b_{n+1}}=e^2\cdot\lim_{n\to\infty}\left(\frac{(n+1)\cdot b_n}{b_{n+1}}\cdot\sqrt[n+1]{b_{n+1}}\right)=$$

$$=e^2\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{n+1}=e^2\cdot\frac{e}{\pi}\cdot\frac{\pi}{e^2}=e$$

From (1) it follows that:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)=\frac{e^2}{\pi}\cdot 1\cdot\log e=\frac{e^2}{\pi}$$

Solution 2 by Marian Ursărescu-Romania

$$\Omega=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)=\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{b_n}}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{\sqrt[n]{b_n}}{n^2}-1\right)=$$

$$=\lim_{n\to\infty}\frac{n}{\sqrt[n]{b_n}}\cdot n\cdot\left(\frac{(n+1)^2}{n^2}\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}-1\right);(1)$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{b_n}}\overset{C-D}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{b_{n+1}}\cdot\frac{b_n}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{(n+1)b_n}{b_{n+1}}=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}!}}=e\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{a_n}}=e\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}}\overset{C-D}{=}$$

$$=e\cdot\lim_{n\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n}=e\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{(n+1)a_n}{a_{n+1}}=$$

$$=e\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{na_n}{a_{n+1}}\cdot\frac{n+1}{n}=e\cdot e\cdot\frac{1}{\pi}\cdot 1=\frac{e^2}{\pi};(2)$$

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{n^2}\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}-1\right)=\lim_{n\to\infty} n\left(e^{\log\left(\frac{(n+1)^2}{n^2}\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)}-1\right)=$$

$$=\lim_{n\to\infty} n\left(\frac{e^{\log\left(\frac{(n+1)^2}{n^2}\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)}-1}{\log\left(\frac{(n+1)^2}{n^2}\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)}\right)\cdot\log\left(\left(1+\frac{1}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)=$$

$$=\lim_{n\to\infty} n\cdot\log\left(\left(1+\frac{1}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)=\lim_{n\to\infty}\log\left(\left(1+\frac{1}{n}\right)^{2n}\cdot\frac{\sqrt[n]{b_n}}{\left(\sqrt[n+1]{b_{n+1}}\right)^n}\right)=$$

$$=\log(\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{2n}\cdot\frac{b_n}{b_{n+1}}\cdot\sqrt[n+1]{b_{n+1}}\right)=\log\left(e^2\cdot\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{a_{n+1}}}\cdot\sqrt[n+1]{b_{n+1}}\right)=$$

$$=\log\left(e^2\cdot\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{\sqrt[n]{a_n}}\right)=\log\left(e^2\cdot\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}\cdot\frac{n}{\sqrt[n]{a_n}}\right)\overset{(2)}{=}\log\left(e^2\cdot\frac{\pi}{e^2}\cdot\frac{e}{\pi}\right)=\log e=1;(3)$$

From (1),(2),(3) it follows:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)=\frac{e^2}{\pi}$$

UP.379. If $S_n=-2\sqrt{n}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$ is the Ioachimescu's sequence with $\lim_{n\to\infty} s_n=s$, then compute $\lim_{n\to\infty}(s_n-s)\sqrt[2n]{(2n-1)!!}$.

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

$$\text{Let: } x_n=(s_n-s)\sqrt[2n]{(2n-1)!!}=(s_n-s)\cdot\sqrt{n}\cdot\sqrt{\frac{\sqrt[n]{(2n-1)!!}}{n}};(1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}=\lim_{n\to\infty}\left(\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}\right)=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{2}{e};(2)$$

$$\lim_{n\to\infty}(s_n - s)\sqrt{n} = \lim_{n\to\infty}\frac{s_n - s}{\frac{1}{\sqrt{n}}} \overset{L.C-S}{\cong} \lim_{n\to\infty}\frac{s_{n+1} - s_n}{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}} =$$

$$= \lim_{n\to\infty}\frac{\sqrt{n(n+1)}}{\sqrt{n+1} - \sqrt{n}}(s_n - s_{n+1}) =$$

$$= \lim_{n\to\infty}\sqrt{n(n+1)}\left(\sqrt{n+1} + \sqrt{n}\right)\left(-2\sqrt{n} - \frac{1}{\sqrt{n+1}} + 2\sqrt{n+1}\right) =$$

$$= \lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1} + \sqrt{n}\right)\left(2n + 1 - 2\sqrt{n(n+1)}\right) =$$

$$= \lim_{n\to\infty}\frac{\sqrt{n}\left(\sqrt{n+1} + \sqrt{n}\right)}{2n + 1 + 2\sqrt{n(n+1)}} = \frac{1}{2}; (3)$$

From $(1), (2)$ and (3) we get:

$$\lim_{n\to\infty}(s_n - s)\sqrt[2n]{(2n-1)!!} = \lim_{n\to\infty}x_n = \frac{1}{2}\sqrt{\frac{2}{e}} = \frac{1}{\sqrt{2e}}$$

Solution 2 by Shivam Sharma-New Delhi-India

$$(2n-1)!! = \frac{(2n)!}{2^n n!} \Rightarrow \lim_{n\to\infty}(s_n - s)\left(\frac{(2n)!}{2^n n!}\right)^{\frac{1}{2n}} \overset{Stirling}{=}$$

$$= \lim_{n\to\infty}(s_n - s)\left(\frac{\left(\frac{2n}{e}\right)\sqrt{2}\sqrt{2\pi n}}{\sqrt{2}\left(\frac{n}{e}\right)^{\frac{1}{2}}\left(\sqrt{2\pi n}\right)^{\frac{1}{2n}}} = \sqrt{2}\lim_{n\to\infty}(s_n - s)\left(\sqrt{\frac{1}{e}}\right) = \sqrt{\frac{2}{e}}\lim_{n\to\infty}(s_n - s)\sqrt{n} \overset{C-S}{=}$$

$$= \sqrt{\frac{2}{e}}\lim_{n\to\infty}\frac{s_{n+1} - s_n}{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}} = \sqrt{\frac{2}{e}}\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}} - 2\left(\sqrt{n} - \sqrt{n+1}\right)}{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}} =$$

$$= \sqrt{\frac{2}{e}}\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}} - \frac{2}{\sqrt{n} + \sqrt{n+1}}}{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}} = \sqrt{\frac{2}{e}}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \sqrt{\frac{2}{e}}\lim_{n\to\infty}\frac{1}{1 + \sqrt{1 + \frac{1}{n}}} = \frac{1}{\sqrt{2e}}$$

UP.380 Let be $E(n) = \Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{2}{n}\right) \cdot \ldots \cdot \Gamma\left(\frac{n-1}{n}\right), n \geq 2, n \in \mathbb{N}^*$.

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{E(n)}{\sin\frac{1}{\sqrt{n}}}\cdot\sin\frac{1}{\sqrt{(2\pi)^{n-1}}}\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$E^2(n) = \left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\right]\dots\left[\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)\right] =$$

$$= \frac{\pi}{\sin\frac{1}{n}}\cdot\frac{\pi}{\sin\frac{2}{n}}\cdot\ldots\cdot\frac{\pi}{\sin\left(\frac{n-1}{n}\right)} = \prod_{k=1}^{n-1}\frac{\pi}{\sin\left(\frac{k\pi}{n}\right)};(1)$$

$$\sum_{k=0}^{n-1}x^k = \prod_{k=1}^{n-1}\left[x-\cos\left(\frac{2k\pi}{n}\right)-i\sin\left(\frac{2k\pi}{n}\right)\right]$$

For $x = 1$, we get:

$$n = \prod_{k=1}^{n-1}\left[1-\cos\left(\frac{2k\pi}{n}\right)-i\sin\left(\frac{2k\pi}{n}\right)\right] = \prod_{k=1}^{n-1}2\sqrt{\sin^2\left(\frac{k\pi}{n}\right)}$$

$$= 2^{n-1}\cdot\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)$$

Thus,

$$\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \to E(n) = \frac{1}{\sqrt{n}}\cdot(2\pi)^{\frac{n-1}{2}}$$

Hence, we have:

$$\Omega = \lim_{n\to\infty}\sqrt{\frac{(2\pi)^{n-1}}{n}}\cdot\frac{1}{\sin\frac{1}{\sqrt{n}}}\cdot\sin\frac{1}{\sqrt{(2\pi)^{n-1}}} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\cdot\frac{(2\pi)^{\frac{n-1}{2}}}{\sin\frac{1}{\sqrt{n}}}\cdot\sin\frac{1}{(2\pi)^{\frac{n-1}{2}}}$$

$$\lim_{n\to\infty}(2\pi)^{\frac{n-1}{2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\sin\frac{1}{\sqrt{n}}}\cdot\sin\frac{1}{(2\pi)^{\frac{n-1}{2}}} = \lim_{n\to\infty}(2\pi)^{\frac{n-1}{2}}\cdot\sin\frac{1}{(2\pi)^{\frac{n-1}{2}}} =$$

$$= \lim_{n\to\infty} \frac{\sin \frac{1}{(2\pi)^{\frac{n-1}{2}}}}{\frac{1}{(2\pi)^{\frac{n-1}{2}}}} = 1$$

Solution 2 by Kaushik Mahanta-Assam-India

$$\text{Recall, } E(n) = \Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{2}{n}\right) \cdot \ldots \cdot \Gamma\left(\frac{n-1}{n}\right) = \frac{1}{\sqrt{n}} \cdot (2\pi)^{\frac{n-1}{2}}$$

$$\Omega = \lim_{n\to\infty} \sqrt{\frac{(2\pi)^{n-1}}{n}} \cdot \frac{1}{\sin \frac{1}{\sqrt{n}}} \cdot \sin \frac{1}{\sqrt{(2\pi)^{n-1}}} =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{(2\pi)^{\frac{n-1}{2}}}{\sin \frac{1}{\sqrt{n}}} \cdot \sin \frac{1}{(2\pi)^{\frac{n-1}{2}}} = 1$$

Solution 3 by Syed Shahabudeen-Kerala-India

$$E(n) = \Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{2}{n}\right) \cdot \ldots \cdot \Gamma\left(\frac{n-1}{n}\right) = \frac{1}{\sqrt{n}} = \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)$$

From Gauss multiplication, we have:

$$\prod_{k=1}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-nz} \Gamma(nz)$$

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = (2\pi)^{\frac{n-1}{2}} \lim_{z\to 0} n^{\frac{1}{2}-nz} \frac{\Gamma(nz)}{\Gamma(z)} = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-1} = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{\pi}}$$

$$\Omega = \lim_{n\to\infty} \sqrt{\frac{(2\pi)^{n-1}}{n}} \cdot \frac{1}{\sin \frac{1}{\sqrt{n}}} \cdot \sin \frac{1}{\sqrt{(2\pi)^{n-1}}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{(2\pi)^{\frac{n-1}{2}}}{\sin \frac{1}{\sqrt{n}}} \cdot \sin \frac{1}{(2\pi)^{\frac{n-1}{2}}}$$

$$= \lim_{n\to\infty} \left(\frac{\sin\left(\frac{1}{\sqrt{(2\pi)^{n-1}}}\right)}{\frac{1}{\sqrt{(2\pi)^{n-1}}}} \cdot \frac{\frac{1}{\sqrt{n}}}{\sin\left(\frac{1}{\sqrt{n}}\right)} \right) = 1$$

UP.381 If $a,b\in\mathbb{R}_+,\gamma_n(a,b)=-\log(n+a)+\sum_{k=1}^{n}\frac{1}{k+b}$,

$\lim\limits_{n\to\infty}\gamma_n(a,b)=\gamma(a,)\in\mathbb{R}$, then find:

$$\Omega=\lim_{n\to\infty}\left(\log\left(\frac{e}{n+a}\right)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by proposers

$$\Omega=\lim_{n\to\infty}\left(\log\left(\frac{e}{n+a}\right)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n=\lim_{n\to\infty}\left(1+\gamma_n(a,b)-\gamma(a,b)\right)^n=$$

$$=\lim_{n\to\infty}\left(1+\gamma_n(a,b)-\gamma(a,b)\right)^{\frac{1}{\gamma_n(a,b)-\gamma(a,b)}\cdot n(\gamma_n(a,b)-\gamma(a,b))}=$$

$$=e^{\lim\limits_{n\to\infty}n(\gamma_n(a,b)-\gamma(a,b))};(1)$$

$$\lim_{n\to\infty}n\left(\gamma_n(a,b)-\gamma(a,b)\right)=\frac{\lim\limits_{n\to\infty}\left(\gamma_n(a,b)-\gamma(a,b)\right)}{\frac{1}{n}}\overset{C-S}{=}$$

$$=\lim_{n\to\infty}\frac{\gamma_{n+1}(a,b)-\gamma_n(a,b)}{\frac{1}{n+1}-\frac{1}{n}}=$$

$$=\lim_{n\to\infty}(n^2+n)\left(-\log(n+a)+\log(n+1+a)-\frac{1}{n+1+b}\right)=$$

$$=\lim_{n\to\infty}(n^2+n)\left(\log\left(\frac{n+1+a}{n+a}\right)-\frac{1}{n+1+b}\right)\overset{\substack{L'H\\ x=\frac{1}{a},x\to0,x>0}}{=}b-a+\frac{1}{2};(2)$$

From $(1),(2)$ it follows that:

$$\Omega=\lim_{n\to\infty}\left(\log\left(\frac{e}{n+a}\right)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n=e^{b-a+\frac{1}{2}}=e^{b-a}\cdot\sqrt{e}$$

UP.382 For $t>0$ find:

$$\Omega(\mathrm{t})=\lim_{n\to\infty}n^{1-t}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t}-\frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

$$\text{Let } x_n(t) = n^{1-t}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t} - \frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}\right). \text{We have:}$$

$$(*): x_n(t) = n^{1-t}\cdot\frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}\cdot(u_n - 1) = \left(\frac{\sqrt[n]{n!}}{n}\right)^{2t}\cdot n\cdot\frac{u_n - 1}{\log u_n}\cdot\log u_n =$$

$$= \left(\frac{\sqrt[n]{n!}}{n}\right)^{2t}\cdot\frac{u_n - 1}{\log u_n}\cdot\log u_n^n, \forall n \geq 2$$

$$\text{We denote: } u_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{2t}\cdot\left(\frac{n}{n+1}\right)^t, \forall n \geq 2.$$

$$\lim_{n\to\infty} u_n = 1, \lim_{n\to\infty}\frac{u_n - 1}{\log u_n} = 1$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(e_n^{-t}\cdot\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{2nt}\right) = e^{-t}\cdot\lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right)^{2t} =$$

$$= e^{-t}\cdot\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{2t} = e^{-t}\cdot e^{2t} = e^t$$

By $(*)$ and above, we obtain,

$$\lim_{n\to\infty} x_n(t) = e^{-2t}\cdot 1\cdot\log e^t = t\cdot e^{-2t}$$

Solution 2 by Asmat Qatea-Afghanistan

$$\Omega = \lim_{n\to\infty} n^{1-t}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t} - \frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}\right) =$$

$$= \lim_{n\to\infty} n^{1-t}\left(\frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}\right)\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t}\cdot\frac{n^t}{\left(\sqrt[n]{n!}\right)^{2t}} - 1\right) =$$

$$= \lim_{n\to\infty}\left(\frac{n\left(\sqrt[n]{n!}\right)^{2t}}{n^{2t}}\right)\left(\frac{n^t\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t\left(\sqrt[n]{n!}\right)^{2t}} - 1\right) =$$

$$\left(\lim_{n\to\infty} n\left(a^{\frac{1}{n}}-1\right)=\log a\right)$$

$$=\lim_{n\to\infty}\left(\frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^{2t}}\right)\cdot n\cdot\log\left(\frac{n^t\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t\left(\sqrt[n]{n!}\right)^{2t}}\right)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}\Rightarrow\lim_{n\to\infty}\sqrt[n]{n!}=\frac{n}{e}\Rightarrow\lim_{n\to\infty}\sqrt[n+1]{(n+1)!}=\frac{n+1}{e}$$

$$\Omega=\lim_{n\to\infty}\left(\frac{\left(\frac{n}{e}\right)^{2t}}{n^{2t}}\right)\cdot n\cdot\log\left(\frac{n^t\left(\frac{n+1}{e}\right)^{2t}}{(n+1)^t\left(\frac{n}{e}\right)^{2t}}\right)=\frac{1}{e^{2t}}\lim_{n\to\infty}n\cdot\log\left(\frac{n^t(n+1)^{2t}}{(n+1)^t n^{2t}}\right)=$$

$$=\frac{t}{e^{2t}}\lim_{n\to\infty}n\log\left(1+\frac{1}{n}\right)=\frac{t}{e^{2t}}\lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)^n=\frac{t}{e^{2t}}\log e=\frac{t}{e^{2t}}$$

Solution 3 by Syed Shahabudeen-India

$$\Omega=\lim_{n\to\infty}n^{1-t}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t}-\frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}\right)=\lim_{n\to\infty}n^{1-t}\frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}(a_n-1)=$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)^{2t}\cdot\frac{a_n-1}{\log a_n}\cdot\log a_n^n,\text{ where } a_n=\frac{n^t\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t\left(\sqrt[n]{n!}\right)^{2t}}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\overset{C-S}{=}\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{n+1-n}=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\overset{Lalescu}{=}\frac{1}{e}$$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n^t\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t\left(\sqrt[n]{n!}\right)^{2t}}=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\right)^{2t}\left(\frac{n}{\sqrt[n]{n!}}\right)^{2t}\left(\frac{n+1}{n}\right)^t=1$$

$$\lim_{n\to\infty}\frac{a_n-1}{\log a_n}=1,\ \lim_{n\to\infty}a_n^n=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{tn}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{2nt}=\frac{1}{e^t}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2nt}=e^t$$

Therefore,

$$\Omega=\lim_{n\to\infty}n^{1-t}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{2t}}{(n+1)^t}-\frac{\left(\sqrt[n]{n!}\right)^{2t}}{n^t}\right)=\frac{t}{e^{2t}}$$

UP. 383 Let R be the circumradius of ΔABC having the length of the sides a, b, c. Prove that:

$$\Delta = \begin{vmatrix} 3\sqrt{3}R & a & b & c \\ a & 3\sqrt{3}R & c & b \\ b & c & 3\sqrt{3}R & a \\ c & b & a & 3\sqrt{3}R \end{vmatrix} > 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Lemma 1. In any ΔABC: $s \le \frac{3\sqrt{3}R}{2}$ $(Mitrinovic\ inequality)$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \le 2\sin\frac{A+B}{2}$$

$$\sin C + \sin\frac{A+B+C}{3} \le 2\sin\frac{C+\frac{(A+B+C)}{3}}{2} = 2\sin\frac{A+B+4C}{6}$$

$$\sin A + \sin B + \sin C + \sin\frac{A+B+C}{3} \le 2\left(\sin\frac{A+B}{2} + \sin\frac{A+B+4C}{6}\right) \le$$

$$\le 2\cdot 2\sin\frac{\frac{A+B}{2}+\frac{A+B+4C}{6}}{2} = 4\sin\frac{3A+3B+A+B+4C}{6} =$$

$$= 3\sin\frac{4A+4B+4C}{6} = 4\sin\frac{2\pi}{3} = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$\sin A + \sin B + \sin C + \frac{\sqrt{3}}{2} \le 2\sqrt{3}$$

$$a+b+c \le 3\sqrt{3}R; (1)$$

Lemma 2. If $x, y, z, t \in \mathbb{R}$ then:

$$\Delta' = \begin{vmatrix} x & y & z & t \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{vmatrix} = (x+y+z+t)(x-y+z-t)(x+y-z-t)(x-y-z+t)$$

Proof.

$$\begin{vmatrix} x & y & z & t \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{vmatrix} \overset{L_1+L_2+L_3+L_4}{\cong} (x+y+z+t)\begin{vmatrix} 1 & 1 & 1 & 1 \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{vmatrix} \overset{C_4-C_3+C_2-C_1}{\cong}$$

$$= (x+y+z+t)\begin{vmatrix} 1 & 1 & 1 & 0 \\ y & x & t & z-x+x-y \\ z & t & x & -(z-t+x-y) \\ t & z & y & z-t+x-y \end{vmatrix} =$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} 1 & 1 & 1 & 0 \\ y & x & t & 1 \\ z & t & x & -1 \\ t & z & y & 1 \end{vmatrix} \overset{\substack{C_2-C_1 \\ C_3-C_1}}{\cong}$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} 1 & 0 & 0 & 0 \\ y & x-y & t-y & 1 \\ z & t-z & x-z & -1 \\ t & z-t & y-t & 1 \end{vmatrix} =$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} x-y & t-y & 1 \\ t-z & x-z & -1 \\ z-t & y-t & 1 \end{vmatrix} \overset{\substack{L_1+L_2 \\ L_3+L_2}}{\cong}$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} x-y+t-z & x-y-z+t \\ 0 & x-z+y-t \end{vmatrix} =$$

$$= (x+y+z+t)(x-y+z-t)(x+y-z-t)(x-y-z+t); (2)$$

If $x, y, z, t > 0; x > y+z+t$ **then** $\Delta' > 0$

$$x-y+z-t > y+z+t-y+z-t = 2z > 0$$

$$x+y-z-t > y+z+t+y-z-t = 2y > 0$$

$$x-y-z+t > y+z+t-y-z+t = 2t > 0$$

Let be $x = 3\sqrt{3}R; y = a; z = b; t = c$. **It follows** $\Delta = \Delta'; 3\sqrt{3}R \geq a+b+c$ **and from** $(1), (2)$:

$$\Delta = \begin{vmatrix} 3\sqrt{3}R & a & b & c \\ a & 3\sqrt{3}R & c & b \\ b & c & 3\sqrt{3}R & a \\ c & b & a & 3\sqrt{3}R \end{vmatrix} > 0$$

Solution 2 by George Florin Şerban-Romania

$$\Delta=\begin{vmatrix}3\sqrt{3}R & a & b & c\\ a & 3\sqrt{3}R & c & b\\ b & c & 3\sqrt{3}R & a\\ c & b & a & 3\sqrt{3}R\end{vmatrix}\overset{Laplace}{=}(-1)^6\begin{vmatrix}3\sqrt{3} & a\\ a & 3\sqrt{3}\end{vmatrix}\cdot\begin{vmatrix}3\sqrt{3}R & a\\ a & 3\sqrt{3}R\end{vmatrix}+$$

$$+(-1)^7\begin{vmatrix}3\sqrt{3}R & b\\ a & c\end{vmatrix}\cdot\begin{vmatrix}c & a\\ b & 3\sqrt{3}R\end{vmatrix}+(-1)^8\begin{vmatrix}3\sqrt{3}R & c\\ a & b\end{vmatrix}\cdot\begin{vmatrix}c & 3\sqrt{3}R\\ b & a\end{vmatrix}+$$

$$+(-1)^8\begin{vmatrix}a & b\\ 3\sqrt{3}R & c\end{vmatrix}\cdot\begin{vmatrix}b & a\\ c & 3\sqrt{3}R\end{vmatrix}+(-1)^{10}\begin{vmatrix}b & c\\ c & b\end{vmatrix}\cdot\begin{vmatrix}b & c\\ c & b\end{vmatrix}=$$

$$=729R^4+24\sqrt{3}Rabc+\sum a^4-2\sum a^2b^2-54R^2\sum a^2\overset{Heron}{=}$$

$$=729R^4+24\sqrt{3}R\cdot 4Rrs-54R^2(2s^2-2r^2-8Rr)-16F^2=$$

$$=729R^4+96\sqrt{3}R^2rs-108R^2s^2+108R^2r^2+432R^3r-16r^2s^2\ge$$

$$\ge 297R^4+584R^2r^2-64Rr^3-48r^4\overset{(?)}{>}0$$

Let $x=\frac{R}{r}\ge 2\Rightarrow 297x^4+584x^2-64x-48>0\Leftrightarrow$

$(x-2)(297x^3+594x^2+1772x+3480)+6912>0$ which is clearly true for all $x\ge 2$.

Therefore,

$$\Delta=\begin{vmatrix}3\sqrt{3}R & a & b & c\\ a & 3\sqrt{3}R & c & b\\ b & c & 3\sqrt{3}R & a\\ c & b & a & 3\sqrt{3}R\end{vmatrix}>0$$

Solution 3 by Ruxandra Daniela Tonilă-Romania

$$\Delta=\begin{vmatrix}3\sqrt{3}R & a & b & c\\ a & 3\sqrt{3}R & c & b\\ b & c & 3\sqrt{3}R & a\\ c & b & a & 3\sqrt{3}R\end{vmatrix}\overset{R_1+R_2+R_3+R_4}{=}$$

$$=(a+b+c+3\sqrt{3}R)\begin{vmatrix}1 & 1 & 1 & 1\\ a & 3\sqrt{3}R & c & b\\ b & c & 3\sqrt{3}R & a\\ c & b & a & 3\sqrt{3}R\end{vmatrix}\overset{C_2-C_4}{=}$$

$$=(a+b+c+3\sqrt{3}R)\begin{vmatrix}1 & 0 & 1 & 1\\ a & 3\sqrt{3}R-b & c & b\\ b & c-a & 3\sqrt{3}R & a\\ c & b-3\sqrt{3}R & a & 3\sqrt{3}R\end{vmatrix}\overset{C_1-C_3}{=}$$

$$=\left(a+b+c+3\sqrt{3}R\right)\begin{vmatrix}0 & 0 & 1 & 1\\ a-c & 3\sqrt{3}R-b & c & b\\ b-3\sqrt{3}R & c-a & 3\sqrt{3}R & a\\ c-a & b-3\sqrt{3}R & a & 3\sqrt{3}R\end{vmatrix}=$$

$$=\left(a+b+c+3\sqrt{3}R\right)\left(\begin{vmatrix}a-c & 3\sqrt{3}R-b & b\\ b-3\sqrt{3}R & c-a & a\\ c-a & b-3\sqrt{3}R & 3\sqrt{3}R\end{vmatrix}-\begin{vmatrix}a-c & 3\sqrt{3}R-b & c\\ b-3\sqrt{3}R & c-a & 3\sqrt{3}R\\ c-a & b-3\sqrt{3}R & a\end{vmatrix}\right)$$

$=$

$$=\left(a+b+c+3\sqrt{3}R\right)(\Delta_1-\Delta_2)$$

$$\Delta_1=\begin{vmatrix}a-c & 3\sqrt{3}R-b & b\\ b-3\sqrt{3}R & c-a & a\\ c-a & b-3\sqrt{3}R & 3\sqrt{3}R\end{vmatrix}\overset{\substack{R_1+R_2\\ C_3+C_2}}{=}\left(b-3\sqrt{3}R\right)^2\left(b+3\sqrt{3}R\right)$$

$$-(c-a)^2\left(b+3\sqrt{3}R\right)$$

$$\Delta_1=\left(b+3\sqrt{3}R\right)\left[\left(b-3\sqrt{3}R\right)^2-(c-a)^2\right]$$

$$\Delta_2=\begin{vmatrix}a-c & 3\sqrt{3}R-b & c\\ b-3\sqrt{3}R & c-a & 3\sqrt{3}R\\ c-a & b-3\sqrt{3}R & a\end{vmatrix}\overset{\substack{R_1+R_2\\ C_3+C_1}}{=}\left(b-3\sqrt{3}R\right)^2(a+c)-(a-c)^2(a+c)=$$

$$=(a+c)\left[\left(b-3\sqrt{3}R\right)^2-(c-a)^2\right]$$

$$\Delta=\left(a+b+c+3\sqrt{3}R\right)(\Delta_1-\Delta_2)=$$

$$=\left(a+b+c+3\sqrt{3}R\right)\left\{\left(b+3\sqrt{3}R\right)\left[\left(b-3\sqrt{3}R\right)^2-(c-a)^2\right]\right.$$

$$\left.-(a+c)\left[\left(b-3\sqrt{3}R\right)^2-(c-a)^2\right]\right\}=$$

$$=\left(a+b+c+3\sqrt{3}R\right)\left(b+3\sqrt{3}R-a-c\right)\left[\left(b-3\sqrt{3}R\right)^2-(c-a)^2\right];(1)$$

$$s=\frac{a+b+c}{2}\le\frac{3\sqrt{3}R}{2};(Mitrinovic)\Rightarrow a+b+c\le 3\sqrt{3}R\Rightarrow$$

$$0<2b<3\sqrt{3}R+b-c-a\Leftrightarrow b+3\sqrt{3}R-a-c>0;(2)$$

$\left(b-3\sqrt{3}R\right)^2-(c-a)^2=\left(b-3\sqrt{3}R-c+a\right)\left(b-3\sqrt{3}R+c-a\right)>0$ **because** $a+b+c\le 3\sqrt{3}R;(Mitrinovic)\Rightarrow b-3\sqrt{3}R-c+a<0$ **and** $b-3\sqrt{3}R+c-a<0\Rightarrow$

$$\left(b-3\sqrt{3}R\right)^2-(c-a)^2>0;(3)$$

From (1),(2),(3) it follows that

$$\Delta=\begin{vmatrix}3\sqrt{3}R & a & b & c\\ a & 3\sqrt{3}R & c & b\\ b & c & 3\sqrt{3}R & a\\ c & b & a & 3\sqrt{3}R\end{vmatrix}>0$$

Solution 4 by Mohammad Rostami-Afghanistan

$$3\sqrt{3}R=3\left(\frac{\sqrt{3}}{2}\right)(2R)=3\sin 60^\circ\cdot 2R$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R \text{ and } 3\sqrt{3}R=6R\sin 60^\circ\Rightarrow\begin{cases}3\sqrt{3}R=3a \text{ or } 3b \text{ or } 3c\\ \widehat{A}=60^\circ \text{ or } \widehat{B}=60^\circ \text{ or } \widehat{C}=60^\circ\end{cases}$$

If $3\sqrt{3}R=3a\Rightarrow\widehat{A}=60^\circ\Rightarrow\widehat{B}+\widehat{C}=120^\circ$

$$\Delta=\begin{vmatrix}3a & a & b & c\\ a & 3a & c & b\\ b & c & 3a & c\\ c & b & a & 3a\end{vmatrix}\overset{\substack{C_2+C_1\to C_1\\ C_3+C_4\to C_4}}{=}\begin{vmatrix}4a & a & b & b+c\\ 4a & 3a & c & b+c\\ a+b & c & 3a & 4a\\ b+C & b & a & 4a\end{vmatrix}$$

$$b+c=a\cdot\frac{\cos\left(\frac{B+C}{2}\right)}{\sin\frac{A}{2}}=a\cdot\frac{\cos 60^\circ}{\sin 30^\circ}=a\Rightarrow b+c=a$$

$$\Delta=\begin{vmatrix}4a & a & b & a\\ 4a & 3a & c & a\\ a & c & 3a & 4a\\ a & b & a & 4a\end{vmatrix}=a^2\begin{vmatrix}4 & a & b & 1\\ 4 & 3a & c & 1\\ 1 & c & 3a & 4\\ 1 & b & a & 4\end{vmatrix}=$$

$$=5a^2(-3)(-1)^5\begin{vmatrix}a & b & 1\\ 3a & c & 1\\ c-b & 2a & 0\end{vmatrix}=15a^2\begin{vmatrix}a & b & 1\\ 2a & c-b & 0\\ c-b & 2a & 0\end{vmatrix}=$$

$$=15a^2\cdot 1\cdot(-1)^4\begin{vmatrix}2a & c-b\\ c-b & 2a\end{vmatrix}=15a^2[4a^2-(c-b)^2]=$$

$$=15a^2[4a^2-(b^2+c^2-2bc)]$$

$$\widehat{A}=60^\circ\Rightarrow a^2=b^2+c62-2bc\cos 60^\circ\Rightarrow a^2=b^2+c^2-bc\Rightarrow$$

$$\Delta=15a^2(3a^2+bc)>0.$$

Solution 5 by Ravi Prakash-New Delhi-India

Let $d=3\sqrt{3}R$ so that:

$$\Delta = \begin{vmatrix} 3\sqrt{3}R & a & b & c \\ a & 3\sqrt{3}R & c & b \\ b & c & 3\sqrt{3}R & a \\ c & b & a & 3\sqrt{3}R \end{vmatrix} > 0$$

Using $C_1 \to C_1 + C_2 + C_3 + C_4$, we get: $(a+b+c+d)$ is factor of Δ.

Using $C_1 \to C_1 + C_2 - C_3 - C_4$ gives $(a+d-b-c)$ is factor of Δ.

Similarly, $(d-a-b-c)$ and $(d-b-a+c)$ are factor of Δ.

Thus, $\Delta = k(d+a+b+c)(d+a-b-c)(d-a+b-c)(d-b-a+c)$, where k is a constant. Comparing coefficients of d^4 on both the sides, we get k=1.

Hence, $\Delta = (d+a+b+c)(d+a-b-c)(d-a+b-c)(d-b-a+c)$.

To show $\Delta > 0$ it is sufficient to show that each of the three expressions with negative sign is positive. $d+a-b-c > 0 \Leftrightarrow b+c-a < d \Leftrightarrow$

$2R(\sin A + \sin B - \sin C) < 3\sqrt{3}R \Leftrightarrow \sin A + \sin B - \sin C < \frac{3\sqrt{3}}{2}$, which is true as $LHS < 2$ and $RHS > 2$.

UP. 384 Find:

$$\Omega = \lim_{n\to\infty}\left(1+\left(1+\frac{1}{n}\right)^{n+1} - e\right)^{n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

Its known that:

$$\frac{e}{2n+1} < \left(1+\frac{1}{n}\right)^{n+1} - e < \frac{e}{2n}; n \in \mathbb{N}^*$$

It follows that:

$$\frac{ne}{2n+1} < n\left(\left(1+\frac{1}{n}\right)^{n+1} - e\right) < \frac{e}{2}; n \in \mathbb{N}^*$$

Pass to the limit:

$$\lim_{n\to\infty}\frac{ne}{2n+1} \le \lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+1} - e\right) \le \lim_{n\to\infty}\frac{e}{2}$$

Hence,

$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right)=\frac{e}{2};(1)$$

Let be $u_n=\left(1+\frac{1}{n}\right)^{n+1}-e$; $\lim_{n\to\infty} u_n=0$ and by (1) $\lim_{n\to\infty} nu_n=\frac{e}{2}$

$$\Omega=\lim_{n\to\infty}\left(1+\left(1+\frac{1}{n}\right)^{n+1}-e\right)^n=\lim_{n\to\infty}(1+u_n)^n=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{u_n}\right)^{n\cdot u_n}=e^{\lim\limits_{n\to\infty} nu_n}=e^{\frac{e}{2}}$$

Solution 2 by Asmat Qatea-Afghanistan

$$\Omega=\lim_{n\to\infty}(1+p)^n=\lim_{n\to\infty}\left((1+p)^{\frac{1}{p}}\right)^{np}=\lim_{n\to\infty} e^{np}=\lim_{n\to\infty} e^{S}.$$

$$S=\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+1}-e\right)=e\lim_{n\to\infty} n\left(\left(\left(\frac{1}{e}\left(1+\frac{1}{n}\right)^{n+1}\right)^n\right)^{\frac{1}{n}}-1\right)=$$

$$=e\lim_{n\to\infty} n\log\left(\frac{\left(1+\frac{1}{n}\right)^{n+1}}{e}\right)=e\lim_{n\to\infty} n\left((n+1)\log\left(1+\frac{1}{n}\right)-1\right)=$$

$$=e\lim_{n\to\infty}\left((n^2+n)\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots\right)-n\right)=$$

$$=e\lim_{n\to\infty}\left(n+1-\frac{n^2+n}{2n^2}+\frac{n^2+n}{3n^3}-\frac{n^2+n}{4n^4}+\cdots-n\right)=e\left(1-\frac{1}{2}\right)=\frac{e}{2}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(1+\left(1+\frac{1}{n}\right)^{n+1}-e\right)^n=e^{\frac{e}{2}}$$

Solution 3 by Amrit Awasthi-Punjab-India

We start with expanding $y=\left(1+\frac{1}{x}\right)^{x+1}$.

Taking log both sides, we get: $\log y=(x+1)\log\left(1+\frac{1}{x}\right)=((x+1)(\frac{1}{x}-\frac{1}{2x^2}+\cdots)$

$$\log y = 1 + \frac{1}{x} - \frac{1}{2x} - \frac{1}{2x^2} + \cdots$$

$$y = e^{1+t} = e \cdot e^t = e\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right), \text{ where } t = \frac{1}{x} - \frac{1}{2x} - \frac{1}{2x^2} + \cdots$$

Substitute for t and after arranging, we get :

$$\left(1 + \frac{1}{x}\right)^{x+1} = e + \frac{e}{2x} + \frac{e}{24x^2} + \frac{e}{48x^3} + \cdots$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(1 + \left(1 + \frac{1}{n}\right)^{n+1} - e\right)^n = \lim_{n\to\infty}\left(1 + \frac{e}{2n}\left(1 + \frac{1}{12n} + \frac{1}{24n^2} + \cdots\right)\right)^n =$$

$$= \lim_{n\to\infty}\left[\left(1 + \frac{e}{2n}\left(1 + \frac{1}{12n} + \frac{1}{24n^2} + \cdots\right)\right)^{\frac{1}{\frac{e}{2n}\left(1+\frac{1}{12n}+\frac{1}{24n^2}+\cdots\right)}}\right]^{\frac{e}{2n}\left(1+\frac{1}{12n}+\frac{1}{24n^2}+\cdots\right)\cdot n} =$$

$$= \lim_{n\to\infty} e^{\frac{e}{2n}\left(1+\frac{1}{12n}+\frac{1}{24n^2}+\cdots\right)} = e^{\frac{e}{2}}$$

Solution 4 by Serlea Kabay-Liberia

$$\text{Let } \omega = \left(1 + \left(1 + \frac{1}{n}\right)^{n+1} - e\right)^n$$

$$\lim_{n\to\infty}\omega = \lim_{n\to\infty}\exp\left(n\log\left(1 + \exp\left((n+1)\log\left(1 + \frac{1}{n}\right)\right) - e\right)\right) \overset{n=\frac{1}{x}}{=}$$

$$\omega = \exp\left(\frac{1}{x}\log\left(1 + \exp\left(\left(\frac{1}{x} + 1\right)\log(1+x)\right) - e\right)\right) =$$

$$= \exp\left(\frac{1}{x}\log\left(1 + \exp(\left(\frac{1}{x} + 1\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon(x)\right) - e\right)\right); (\epsilon(s) \to 0)$$

$$= \exp\left(\left(\frac{1}{x}\log(1 + \exp(1 + \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{3} + \epsilon_1(x))\right) - e\right); ((\epsilon_1(s) \to 0)) =$$

$$= \exp\left(\frac{1}{x}\log(1 + ex^3 + \epsilon_2(x))\right) = \exp\left(\left(\frac{1}{x}\right)\left(\frac{ex}{2} - \frac{ex^2}{6} + \frac{ex^3}{3} + x^3\epsilon_2(x)\right)\right) =$$

$$= \exp\left(\frac{e}{2} - \frac{ex}{2} + \frac{ex^2}{3} + \epsilon_3(x)\right)$$

$$\lim_{x\to 0}\omega=\sqrt{e^e};\lim_{x\to 0}\left(-\frac{ex}{6}+\frac{ex^2}{3}+\epsilon_3(x)\right)=0$$

Therefore,

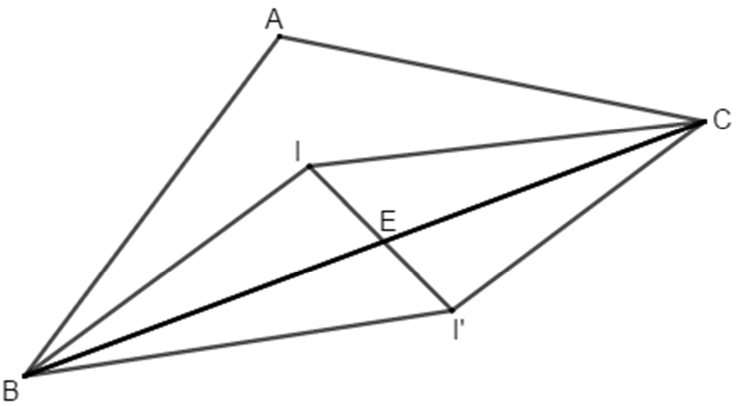
$$\Omega=\lim_{n\to\infty}\left(1+\left(1+\frac{1}{n}\right)^{n+1}-e\right)^n=e^{\frac{e}{2}}$$

UP.385 In any ΔABC let x,y,z be the distances from the incentre to the sides of triangle and $u\ge 1$ −fixed. Prove that:

$$u^x+u^y+u^z\le u^{\sqrt{\frac{bc(s-a)}{s}}}+u^{\sqrt{\frac{ca(s-b)}{s}}}u^{\sqrt{\frac{ab(s-c)}{s}}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by proposers



$E\in(BC);BE=EC$ and I' the symmetric of $I;IE=EI';I,E,I'$ −collinears.

$$IB+IC=I'C+IC>II'=2IE\ge 2x$$

$$IB+IC>2x\Rightarrow x<\frac{IB+IC}{2}$$

Analogous: $y<\frac{IC+IA}{2};z<\frac{IA+IB}{2}$

$$u^x\le u^{\frac{IB+IC}{2}};u^y\le u^{\frac{IC+IA}{2}};u^z\le u^{\frac{IA+IB}{2}}$$

$$u^x+u^y+u^z\le u^{\frac{IB}{2}}\cdot u^{\frac{IC}{2}}+u^{\frac{IC}{2}}\cdot u^{\frac{IA}{2}}+u^{\frac{IA}{2}}\cdot u^{\frac{IB}{2}}\le u^{IA}+u^{IB}+u^{IC}$$

$$\because ab+bc+ca\le a^2+b^2+c^2;a,b,c>0$$

$$u^x+u^y+u^z\le u^{\frac{r}{\sin\frac{A}{2}}}+u^{\frac{r}{\sin\frac{B}{2}}}+u^{\frac{r}{\sin\frac{C}{2}}}$$

$$\frac{r}{\sin\frac{A}{2}}=\frac{F}{s\sin\frac{A}{2}}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s\sqrt{\frac{(s-b)(s-c)}{bc}}}=\sqrt{\frac{bc(s-a)}{s}}$$

Therefore,

$$u^x+u^y+u^z\le u^{\sqrt{\frac{bc(s-a)}{s}}}+u^{\sqrt{\frac{ca(s-b)}{s}}}u^{\sqrt{\frac{ab(s-c)}{s}}}$$

UP.386 If $0<a\le b$ then:

$$\int_a^b\int_a^b\frac{dxdy}{xy+1}\le\frac{2(b-a)^2}{(a+1)(b+1)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $x,y>0$.

$$(xy+1)\left(\frac{x}{y}+1\right)\overset{CBS}{\geq}\left(\sqrt{xy}\cdot\sqrt{\frac{x}{y}}+\sqrt{1}\cdot\sqrt{1}\right)^2=(x+1)^2$$

$$\frac{1}{(x+1)^2}\ge\frac{1}{(xy+1)\left(\frac{x}{y}+1\right)};(1)$$

$$(xy+1)\left(\frac{y}{x}+1\right)\overset{CBS}{\geq}\left(\sqrt{xy}\cdot\sqrt{\frac{y}{x}}+\sqrt{1}\cdot\sqrt{1}\right)^2=(y+1)^2$$

$$\frac{1}{(y+1)^2}\ge\frac{1}{(xy+1)\left(\frac{y}{x}+1\right)};(2)$$

By adding $(1),(2)$ it follows that:

$$\frac{1}{(x+1)^2}+\frac{1}{(y+1)^2}\ge\frac{1}{(xy+1)\left(\frac{x}{y}+1\right)}+\frac{1}{(xy+1)\left(\frac{y}{x}+1\right)}=$$

$$=\frac{1}{xy+1}\left(\frac{1}{\frac{x}{y}+1}+\frac{1}{\frac{y}{x}+1}\right)=\frac{1}{xy+1}\left(\frac{y}{x+y}+\frac{x}{x+y}\right)=\frac{1}{xy+1}$$

$$\int_a^b\int_a^b \frac{dxdy}{xy+1} \le \int_a^b\int_a^b \frac{1}{(x+1)^2}dx\,dy + \int_a^b\int_a^b \frac{1}{(y+1)^2}dxdy =$$

$$= 2(b-a)\int_a^b \frac{1}{(x+1)^2}dx = 2(b-a)\left(-\frac{1}{b+1} - \frac{-1}{a+1}\right) =$$

$$= 2(b-a)\left(\frac{1}{a+1} - \frac{1}{b+1}\right) = \frac{2(b-a)^2}{(a+1)(b+1)}$$

Equality holds for $a = b$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let's\ prove\ that: \frac{1}{xy+1} \le \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2}, \forall x,y > 0$$

$$We\ have: (x+1)^2 \overset{CBS}{\le} (xy+1)\left(\frac{x}{y}+1\right) = \frac{(xy+1)(x+y)}{y}.$$

$$Similarly, (y+1)^2 \le \frac{(xy+1)(x+y)}{x}$$

$$\to \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} \ge \frac{y+x}{(xy+1)(x+y)} = \frac{1}{xy+1}, \forall x,y > 0$$

$$\to \int_a^b\int_a^b \frac{dxdy}{xy+1} \le \int_a^b\int_a^b \left(\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2}\right)dxdy = 2(b-a)\left(\frac{1}{a+1} - \frac{1}{b+1}\right)$$

$$= \frac{2(b-a)^2}{(a+1)(b+1)}.$$

$$Therefore, \quad \int_a^b\int_a^b \frac{dxdy}{xy+1} \le \frac{2(b-a)^2}{(a+1)(b+1)}.$$

UP.387 If in $\Delta ABC, 2s = 3$ then:

$$\frac{m_a+m_b}{m_c} + \frac{a^2b(m_b+m_c)}{m_a} + \frac{bc^2(m_c+m_a)}{m_b} \ge 8\sqrt{3}F$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\frac{m_a+m_b}{m_c} + \frac{a^2b(m_b+m_c)}{m_a} + \frac{bc^2(m_c+m_a)}{m_b} \overset{AM-GM}{\ge}$$

$$\geq \frac{2\sqrt{m_a m_b}}{m_c} + a^2b\cdot\frac{2\sqrt{m_b m_c}}{m_a} + bc^2\cdot\frac{2\sqrt{m_c m_a}}{m_b} \overset{AM-GM}{\geq}$$

$$\geq 2\cdot 3\cdot\sqrt[3]{\frac{2\sqrt{m_a m_b}}{m_c}\cdot a^2b\cdot\frac{2\sqrt{m_b m_c}}{m_a}\cdot bc^2\cdot\frac{2\sqrt{m_c m_a}}{m_b}} =$$

$$= 6\sqrt[3]{a^2b^2c^2} \overset{Carlitz}{\geq} 6\cdot\sqrt[3]{\left(\frac{4F}{\sqrt{3}}\right)^3} = 6\cdot\frac{4F}{\sqrt{3}} = \frac{24\sqrt{3}F}{3} = 8\sqrt{3}F$$

Equality holds for $a = b = c = 1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution :

$$LHS_{(*)} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{m_a+m_b}{m_c}.\frac{a^2b(m_b+m_c)}{m_a}.\frac{bc^2(m_c+m_a)}{m_b}}$$

$$= 3\sqrt[3]{(abc)^2.\frac{(m_a+m_b)(m_b+m_c)(m_c+m_a)}{m_am_bm_c}} \geq$$

$$\overset{Cesaro}{\geq} 3\sqrt[3]{(4RF)^2.8} = 3.4\sqrt[3]{2R.R.F^2} \overset{\substack{Mitrinovic\\Euler}}{\geq} 12\sqrt[3]{2.\frac{2s}{3\sqrt{3}}.2r.F^2} = 12\sqrt[3]{\left(\frac{2F}{\sqrt{3}}\right)^3}$$

$$= 12.\frac{2F}{\sqrt{3}} = 8\sqrt{3}F$$

$$Therefore,\quad \frac{m_a+m_b}{m_c} + \frac{a^2b(m_b+m_c)}{m_a} + \frac{bc^2(m_c+m_a)}{m_b} \geq 8\sqrt{3}F.$$

UP.388 If $x,y,z,p,q,r > 0; x+y+z = p+q+r = 3$ then:

$$x^p + x^q + x^r + y^p + y^q + y^r + z^p + z^q + z^r \geq 9$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$x^p + x^q + x^r + y^p + y^q + y^r + z^p + z^q + z^r =$$

$$= \frac{x^p}{1^{p-1}} + \frac{y^p}{1^{p-1}} + \frac{z^p}{1^{p-1}} + \frac{x^q}{1^{q-1}} + \frac{y^q}{1^{q-1}} + \frac{z^q}{1^{q-1}} + \frac{x^r}{1^{r-1}} + \frac{y^r}{1^{r-1}} + \frac{z^r}{1^{r-1}} \overset{Radon}{\geq}$$

$$\ge \frac{(x+y+z)^p}{(1+1+1)^{p-1}} + \frac{(x+y+z)^q}{(1+1+1)^{q-1}} + \frac{(x+y+z)^r}{(1+1+1)^{r-1}} \overset{AM-GM}{\ge}$$

$$\ge 3 \cdot \sqrt[3]{\frac{(x+y+z)^p}{3^{p-1}} \cdot \frac{(x+y+z)^q}{3^{q-1}} \cdot \frac{(x+y+z)^r}{3^{r-1}}} =$$

$$= 3 \cdot \sqrt[3]{\frac{(x+y+z)^{p+q+r}}{3^{p+q+r-3}}} = 3 \cdot \sqrt[3]{\frac{(x+y+z)^3}{3^0}} = 3 \cdot \sqrt[3]{3^3} = 9$$

Equality holds for $x = y = z = p = q = r = 1$.

Solution 2 by George Florin Șerban-Romania

$$x^p + x^q + x^r + y^p + y^q + y^r + z^p + z^q + z^r \overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{x^p x^q x^r} + 3\sqrt[3]{y^p y^q y^r} + 3\sqrt[3]{z^p z^q z^r} =$$

$$= 3\sqrt[3]{x^{p+q+r}} + 3\sqrt[3]{y^{p+q+r}} + 3\sqrt[3]{z^{p+q+r}} =$$

$$= 3(x+y+z) = 9$$

Solution 3 by Ravi Prakash-New Delhi-India

$$x^p + x^q + x^r \ge 3\sqrt[3]{x^p x^q x^r} \Rightarrow x^p + x^q + x^r \ge 3\sqrt[3]{x^3} = 3x; (1)$$

Similarly,

$y^p + y^q + y^r \ge 3y; (2)$ and $z^p + z^q + z^r \ge 3z; (3)$

From (1),(2),(3) it follows that:

$$x^p + x^q + x^r + y^p + y^q + y^r + z^p + z^q + z^r \ge 3(x+y+z) = 9$$

UP.389 If $0 < a \le b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{(xy+1)^2} \le \frac{4(b-a)^2(a^2+b^2+ab+3a+3b+3)}{3(a+1)^3(b+1)^3}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(xy+1)\left(\frac{x}{y}+1\right) = \left(\left(\sqrt{xy}\right)^2 + 1^1\right)\left(\left(\sqrt{\frac{x}{y}}\right)^2 + 1^2\right) \overset{CBS}{\ge}$$

$$\geq\left(\sqrt{xy}\cdot\sqrt{\frac{x}{y}}+1\cdot 1\right)^2=(x+1)^2$$

$$\frac{1}{(x+1)^2}\geq\frac{1}{(xy+1)\left(\frac{x}{y}+1\right)}\Rightarrow\frac{1}{(x+1)^2}\geq\frac{y}{(x+y)(xy+1)};(1)$$

$$(xy+1)\left(\frac{y}{x}+1\right)=\left(\left(\sqrt{xy}\right)^2+1^1\right)\left(\left(\sqrt{\frac{y}{x}}\right)^2+1^2\right)\overset{CBS}{\geq}$$

$$\geq\left(\sqrt{xy}\cdot\sqrt{\frac{y}{x}}+1\cdot 1\right)^2=(y+1)^2$$

$$\frac{1}{(y+1)^2}\geq\frac{1}{(xy+1)\left(\frac{y}{x}+1\right)}\Rightarrow\frac{1}{(y+1)^2}\geq\frac{x}{(x+y)(xy+1)};(2)$$

$$\frac{1}{(x+1)^4}+\frac{1}{(y+1)^4}=\frac{\frac{1}{(x+1)^2}}{1}+\frac{\frac{1}{(y+1)^2}}{1}\overset{Bergstrom}{\geq}\frac{\left(\frac{1}{(x+1)^2}+\frac{1}{(y+1)^2}\right)^2}{1+1}\overset{(1),(2)}{\geq}$$

$$\geq\frac{1}{2}\left(\frac{y}{(x+y)(xy+1)}+\frac{x}{(x+y)(xy+1)}\right)^2=$$

$$=\frac{1}{2}\left(\frac{x+y}{(x+y)(xy+1)}\right)^2=\frac{1}{2(xy+1)^2}$$

Hence,

$$\frac{1}{2(xy+1)^2}\leq\frac{1}{(x+1)^4}+\frac{1}{(y+1)^4}$$

$$\frac{1}{2}\int_a^b\int_a^b\frac{dxdy}{(xy+1)^2}\leq\int_a^b\int_a^b\left(\frac{1}{(x+1)^4}+\frac{1}{(y+1)^4}\right)dx\,dy$$

$$\int_a^b\int_a^b\frac{dxdy}{(xy+1)^2}\leq 4(b-a)\int_a^b\frac{1}{(x+1)^4}dx=4(b-a)\cdot\frac{(x+1)^{-4+1}}{-4+1}\Big|_a^b=$$

$$=-\frac{2}{3}(b-a)\left(\frac{1}{(b+1)^3}-\frac{1}{(a+1)^3}\right)=\frac{4}{3}(b-a)\cdot\frac{(b+1)^3-(a+1)^3}{(a+1)^3(b+1)^3}=$$

$$=\frac{4(b-a)(b^3+3b^2+3b+1-a^3-3a^2-3a-1)}{3(a+1)^3(b+1)^3}=$$

$$=\frac{4(b-a)^2(a^2+b^2+ab+3a+3b+3)}{3(a+1)^3(b+1)^3}$$

Equality holds for $a = b$.

Solution 2 by Ravi Prakash-New Delhi-India

We have, for $x, y > 0$, $(xy+1)\left(\frac{x}{y}+1\right) = x^2 + xy + \frac{x}{y} + 1 \ge x^2 + 2\sqrt{(xy)\left(\frac{x}{y}\right)} + 1$

$$\Rightarrow (xy+1)\left(\frac{x}{y}+1\right) \ge (x+y)^2$$

Similarly, $(xy+1)\left(\frac{y}{x}+1\right) \ge (y+1)^2$

$$\Rightarrow \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} \ge \frac{x}{(xy+1)(x+y)} + \frac{y}{(xy+1)(x+y)} =$$

$$= \frac{x+y}{(1+xy)(x+y)} = \frac{1}{1+xy}$$

$$\Rightarrow \frac{1}{(1+xy)^2} \le \frac{1}{(x+1)^4} + \frac{1}{(y+1)^4} + \frac{2}{(x+1)^2(y+1)^2}$$

$$\int_a^b \int_a^b \frac{dxdy}{(1+xy)^2} \le \int_a^b \int_a^b \left[\frac{1}{(x+1)^4} + \frac{1}{(y+1)^4} + \frac{2}{(x+1)^2(y+1)^2}\right] dxdy =$$

$$= \frac{2}{3}(b-a)\left[\frac{1}{(a+1)^3} - \frac{1}{(b+1)^3}\right] + 2\left[\frac{1}{a+1} - \frac{1}{b+1}\right]^2 =$$

$$= \frac{2}{3}(b-a) \cdot \frac{b^3 - a^3 + 3(b^2 - a^2) + 3(b-a)}{(a+1)^3(b+1)^3} + \frac{2}{3} \cdot \frac{(b-a)^2 A}{(a+1)^3(b+1)^3}$$

Where $A = b^2 + a^2 + ba + 3b + 3a + 3 + 3(b+1)(a+1) =$

$$= b^2 + a^2 + 4ba + 6b + 6a + 6 \le 2(b^2 + a^2 + ba + 3a + 3b + 3)$$

Therefore,

$$\int_a^b \int_a^b \frac{dxdy}{(xy+1)^2} \le \frac{4(b-a)^2(a^2 + b^2 + ab + 3a + 3b + 3)}{3(a+1)^3(b+1)^3}$$

UP.390 If $x, y, z \ge 1; x + y + z = 6$ then in ΔABC the following relationship holds:

$$(x^x + y^x + z^x)a^4 + (x^y + y^y + z^y)b^4 + (x^z + y^z + z^z)c^4 \ge 5184r^4$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$(x^x+y^x+z^x)a^4+(x^y+y^y+z^y)b^4+(x^z+y^z+z^z)c^4=$$

$$=\left(\frac{x^x}{1^{x-1}}+\frac{y^x}{1^{x-1}}+\frac{z^x}{1^{x-1}}\right)a^4+\left(\frac{x^y}{1^{y-1}}+\frac{y^y}{1^{y-1}}+\frac{z^y}{1^{y-1}}\right)b^4+\left(\frac{x^z}{1^{z-1}}+\frac{y^z}{1^{z-1}}+\frac{z^z}{1^{z-1}}\right)c^4\overset{Radon}{\geq}$$

$$\geq\frac{(x+y+z)^x}{(1+1+1)^{x-1}}a^4+\frac{(x+y+z)^y}{(1+1+1)^{y-1}}b^4+\frac{(x+y+z)^z}{(1+1+1)^{z-1}}c^4=$$

$$=\frac{6^x}{3^{x-1}}a^4+\frac{6^y}{3^{y-1}}b^4+\frac{6^z}{3^{z-1}}c^4\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{6^{x+y+z}}{3^{x+y+z-3}}\cdot a^4b^4c^4}=$$

$$=3\cdot\sqrt[3]{\frac{6^6}{3^3}\cdot(a^2b^2c^2)^2}=3\cdot\frac{36}{3}\cdot\sqrt[3]{(a^2b^2c^2)^2}\overset{Carlitz}{\geq}\frac{108}{3}\cdot\sqrt[3]{\left(\frac{4F}{\sqrt{3}}\right)^6}=\frac{108}{3}\cdot\frac{16F^2}{3}=$$

$$=\frac{36}{3}\cdot 16r^2s^2\overset{Mitrinovic}{\geq}12\cdot 16\cdot 27R^4=5184r^4$$

Equality holds for $a=b=c; x=y=z=2$.

Solution 2 by George Florin Şerban-Romania

$$(x^x+y^x+z^x)a^4+(x^y+y^y+z^y)b^4+(x^z+y^z+z^z)c^4=$$

$$=(a^4x^x+b^4x^y+c^4x^z)+(a^4y^x+b^4y^y+c^4y^z)+(a^4z^x+b^4z^y+c^4z^z)\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{(abc)^4x^xx^yx^z}+3\sqrt[3]{(abc)^4y^xy^yy^z}+3\sqrt[3]{(abc)^4z^xz^yz^z}=$$

$$=3\sqrt[3]{(abc)^4}\left(\sqrt[3]{x^{x+y+z}}+\sqrt[3]{y^{x+y+z}}+\sqrt[3]{z^{x+y+z}}\right)=$$

$$=3\sqrt[3]{(abc)^4}\sum x^2\overset{CBS}{\geq}3\sqrt[3]{(abc)^4}\cdot\frac{(\sum x)^2}{3}=36\sqrt[3]{(abc)^4}\overset{(*)}{\geq}5124r^4$$

$$(*)\Leftrightarrow\sqrt[3]{(abc)^4}\geq 144r^4=12^2r^4\Leftrightarrow(abc)^4\geq 12^6r^{12}$$

$$(abc)^2=16R^2r^2s^2\overset{Mitrinovic}{\geq}16R^2\,r^2\left(3\sqrt{3}r\right)^2=16R^2r^2\cdot 3^3r^2\overset{Euler}{\geq}12^6r^6$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum(x^x+y^x+z^x)a^4\overset{AM-GM}{\geq}3\sqrt[3]{\prod(x^x+y^x+z^x)a^4}=3\sqrt[3]{(abc)^4\prod(x^x+y^x+z^x)}\geq$$

$$\overset{Hölder}{\geq} 3\sqrt[3]{(4Rsr)^4.\left(x^{\frac{x+y+z}{3}}+y^{\frac{x+y+z}{3}}+z^{\frac{x+y+z}{3}}\right)^3} \overset{\substack{Euler\\Mitrinovic}}{\geq} 3(x^2+y^2$$

$$+z^2).\sqrt[3]{\left(4.2r.3\sqrt{3}r.r\right)^4} \geq$$

$$\overset{CBS}{\geq} (x+y+z)^2.\sqrt[3]{\left(2\sqrt{3}r\right)^{12}} = 6^2.\left(2\sqrt{3}r\right)^4 = 5184r^4.$$

$$Therefore, \quad \sum(x^x+y^x+z^x)a^4 \geq 5184r^4.$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru