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Solutions by

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PROBLEMS FOR JUNIORS

JP.391 In $\Delta ABC, P$ −inner point, $M, L \in [AB], D, E \in [BC], F, K \in [CA]$

$$AM = AF, BL = BE, CK = CD, |DE| = a_1, |FK| = b_1, |LM| = c_1$$

$(M, P, F), (C, P, L), (D, P, K)$ −are collinear.Prove that:

$$F = \frac{1}{2}(a_1 r_a + b_1 r_b + c_1 r_c)$$

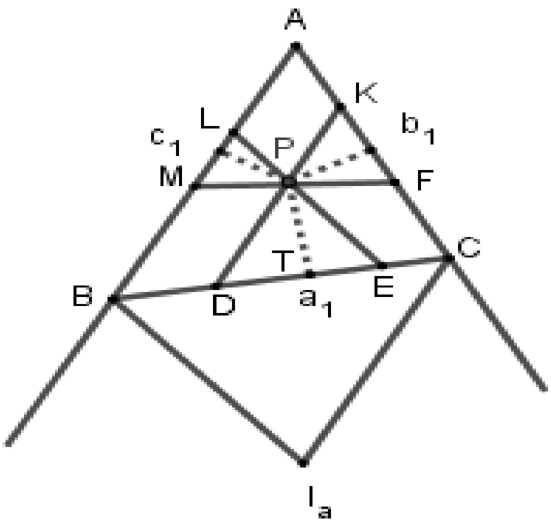
Proposed by Mehmet Şahin-Ankara-Turkyie

Solution by proposer

$\mu(\widehat{AMP}) = \mu(\widehat{AFM}) = 90° - \frac{\widehat{A}}{2}$

$$\mu(\widehat{BLE}) = \mu(\widehat{BEL}) = 90° - \frac{\widehat{B}}{2}$$

$$\mu(\widehat{CKD}) = 90° - \frac{\widehat{C}}{2}$$



Let $PT \perp BC, |LT| = h_1$.

$$\Delta PDE \sim \Delta I_a CB \Rightarrow \frac{h_1}{r_a} = \frac{a_1}{a}$$

$$\Rightarrow h_1 = \frac{a_1}{a} \cdot r_a$$

Similarly, $h_2 = \frac{b_1}{b} \cdot r_b$ and $h_3 = \frac{c_1}{c} \cdot r_c$.

$$F = [PBC] + [PCA] + [PAB] = \frac{1}{2} a \cdot h_1 + \frac{1}{2} b \cdot h_2 + \frac{1}{2} c \cdot h_3 =$$

$$= \frac{1}{2}\left(\frac{a_1}{a} \cdot r_a\right) a + \frac{1}{2}\left(\frac{b_1}{b} \cdot r_b\right) b + \frac{1}{2}\left(\frac{c_1}{c} \cdot r_c\right) c = \frac{1}{2}(a_1 r_a + b_1 r_b + c_1 r_c)$$

JP.392 $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that

$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$. Prove that:

$$\text{If } \sum_{cyc} \frac{z_2 z_3}{14 z_2 z_3 - z_2^2 - z_3^2} = \frac{1}{5} \text{ then } AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$A(z_1),B(z_2),C(z_3),\Delta ABC\subset C(0,1)$$
$$\sin^2 A=-\frac{(z_2-z_3)^2}{4z_2z_3}\Rightarrow\sum_{cyc}\frac{4z_2z_3}{12z_2z_3-(z_2-z_3)^2}=\frac{4}{5}\Leftrightarrow$$
$$\sum_{cyc}\frac{1}{3-\frac{(z_2-z_3)^2}{4z_2z_3}}=\frac{4}{5}\Leftrightarrow\sum_{cyc}\frac{1}{3+\sin^2 A}=\frac{4}{5};(1)$$
$$\text{But: }\sum_{cyc}\frac{1}{3+\sin^2 A}\ge\frac{4}{5};(2).\ \text{From }(1)\&(2)\Rightarrow AB=BC=CA.$$
$$\begin{cases}\sum_{cyc}\frac{1}{3+\sin^2 A}\overset{Bergstrom}{\ge}\frac{9}{9+\sum\sin^2 A}\\ \sum_{cyc}\sin^2 A=\frac{s^2-r^2-4Rr}{2R^2}\end{cases}\Rightarrow\frac{9}{9+\frac{s^2-r^2-4Rr}{2R^2}}\ge\frac{4}{5}\Leftrightarrow$$
$$\frac{18R^2}{18R^2+s^2-r^2-4Rr}\ge\frac{4}{5}\Leftrightarrow 45R^2\ge 36R^2+2s^2-2r^2-8Rr\Leftrightarrow$$
$$2s^2\le 9R^2+8Rr+2r^2$$
But: $s^2\le 4R^2+4Rr+3r^2\ (Gerretsen)\Rightarrow 2s^2\le 8R^2+8Rr+6r^2\Rightarrow$
$8R^2+8Rr+6r^2\le 4R^2+8Rr+2r^2\Leftrightarrow 4r^2\le R^2\Leftrightarrow R\ge 2r(Euler).$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$|z_1|=|z_2|=|z_3|=1\ \to\ \Delta ABC\in C(O,1)\ \to\ \sin^2 A=-\frac{(z_2-z_3)^2}{4z_2z_3}\ (And\ analogs)$$
$$(*)\leftrightarrow\sum_{cyc}\frac{1}{3-\frac{(z_2-z_3)^2}{4z_2z_3}}=\frac{4}{5}\leftrightarrow\sum_{cyc}\frac{1}{3+\sin^2 A}=\frac{4}{5}$$
$$But\ we\ have:\ \sum_{cyc}\frac{1}{3+\sin^2 A}\overset{CBS}{\ge}\frac{9}{(3+\sin^2 A)+(3+\sin^2 B)+(3+\sin^2 C)}$$
$$=\frac{9}{9+(\sin^2 A+\sin^2 B+\sin^2 C)}$$
$$Also, we\ have:\ \sum\sin^2 A=\sum\frac{a^2}{4R^2}=\frac{1}{4R^2}\sum a^2\overset{Leibniz}{\le}\frac{1}{4R^2}.9R^2=\frac{9}{4}$$
$$\to\sum_{cyc}\frac{1}{3+\sin^2 A}\ge\frac{4}{5}$$

With equality if ΔABC is equilateral.

$$Therefore,\quad \sum_{cyc}\frac{z_2z_3}{14z_2z_3-z_2{}^2-z_3{}^2}=\frac{1}{5}\ \to\ AB=BC=CA.$$

JP.393 In ΔABC the following relationship holds:

$$h_a^3+h_b^3+h_c^3\le\frac{81}{8}(9R^3-64r^3)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

Using inequality: $(x+y+z)^3\ge x^3+y^3+z^3+24xyz,\forall x,y,z>0$

$$(\Leftrightarrow (a+b)(b+c)(c+a)\ge 8abc)$$

$$h_a^3+h_b^3+h_c^3\le(h_a+h_b+h_c)^3-24h_ah_bh_c;(1)$$

$$h_a+h_b+h_c=\frac{s^2+r^2+4Rr}{2R}\text{ and }h_ah_bh_c=\frac{2s^2r^2}{R};(2)$$

From $(1),(2)$ it follows that:

$$h_a^3+h_b^3+h_c^3\le\frac{(s^2+r^2+4Rr)^3}{8R^3}-24\cdot\frac{2s^2r^2}{R};(3)$$

But $s^2\le 4R62+4Rr+3r^3$ $(Gerretsen)$ and $2s^2\ge 27Rr(Turtoiu);(4)$

From $(3),(4)$ it follows that:

$$h_a^3+h_b^3+h_c^3\le\frac{(4R^2+8Rr+4r^2)^3}{8R^3}-\frac{24\cdot 2+Rr^3}{R}=\frac{8(R+r)^6}{R^3}-24\cdot 2+R^3\overset{Euler}{\le}$$

$$\le 8\left(\frac{\left(\frac{3R}{2}\right)^6}{R^3}-81r^3\right)=8\left(\frac{3^6\cdot R^3}{2^6}-81r^3\right)=8\cdot 81\cdot\frac{9R^3-2^6r^3}{2^6}=\frac{81}{8}(9R^3-64r^3)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ know\ that:\ h_a\le w_a\le\sqrt{s(s-a)}=\sqrt{r_br_c}\ (And\ analogs)$$

$$\to\sum h_a{}^3\le\sum r_br_c\sqrt{r_br_c}\overset{AM-GM}{\le}\sum r_br_c.\frac{r_b+r_c}{2}=\frac{1}{2}\left[\prod(r_a+r_b)-2r_ar_br_c\right]$$

$$=\frac{1}{2}(4Rs^2-2rs^2)=$$

$$=2Rs^2-rs^2\overset{Mitrinovic}{\le}\frac{27}{2}R^3-27r^3\overset{?}{\le}\frac{81}{8}(9R^3-64r^3)$$

$$\leftrightarrow 4968r^3 \le 621R^3 \leftrightarrow (2r)^3 \le R^3 \leftrightarrow 2r \le R \ (Euler's\ inequality)$$

$$Therefore, \quad \sum {h_a}^3 \le \frac{81}{8}(9R^3 - 64r^3).$$

Solution 3 by Nguyen Van Canh-Ben Tre-Vietnam

$$\sum_{cyc} h_a^3 = \sum_{cyc}(h_a^2 \cdot h_a) \overset{h_a \le m_a}{\le} \sum_{cuc}(m_a^2 \cdot h_a) = \sum_{cyc} \frac{2b^2 + 2c^2 - a^2}{4} \cdot \frac{2F}{a} =$$

$$= \frac{F}{2}\sum_{cyc} \frac{2(b^2 + c^2) - a^2}{a} = \frac{F}{2} \cdot \frac{2\sum bc(b^2 + c^2) - abc\sum a}{abc} = \frac{2\sum bc(b^2 + c^2) - abc\sum a}{8R} \le$$

$$\overset{bc(b^2+c^2) \le b^4 + c^4}{\le} \frac{4(a^4 + b^4 + c^4) - abc\sum a}{8R} = \frac{8(s^4 - (9Rr + 12r^2)s^2 + (4Rr + r^2)^2)}{8R} \overset{(1)}{\le}$$

$$\le \frac{81(9R^3 - 64r^3)}{8}$$

$$(1) \Leftrightarrow 8(s^4 - (9Rr + 6r^2)s^2 + (4Rr + r^2)^2) \le 81(9R^4 - 64Rr^3)$$

Let $f(u) = u^2 - (9Rr + 6r^2)u + (4Rr + r^2)^2; (\because u = s^2)$

$$f'(u) = 2u - (9Rr + 6r^2) \overset{u \ge 16Rr - 5r^2}{\ge} 2(16Rr - 5r^2) - (9Rr + 6r^2)$$

$$= 23Rr - 16r^2 \overset{R \ge 2r}{\ge} 30r^2 \Rightarrow f'(u) \ge 0 \Rightarrow f(u) \nearrow [16Rr - 5r^2, 4R^2 + 4Rr + 3r^2]$$

$$\Rightarrow 8[(4R^2 + 4Rr + 3r^2)^2 - (9Rr + 6r^2)(4R^2 + 4Rr + 3r^2) + (4Rr + r^2)^2] \overset{(2)}{\le}$$

$$\le 81(9R^4 - 64Rr^3)$$

$$(2) \Leftrightarrow 8[(4t^2 + 4t + 3) - (9t + 6)(4t^2 + 4t + 3) + (4t + 1)^2]$$

$$\le 81(9t^4 - 64t), \left(\because t = \frac{R}{r} \ge 2\right)$$

$$\Leftrightarrow 8(16t^4 - 4t^3 - 4t^2 - 19t - 8) \le 729t^4 - 5184t$$

$$\Leftrightarrow 601t^4 + 32t^3 + 32t^2 - 5032t + 64 \ge 0$$

$$\Leftrightarrow (t - 2)(601t^3 + 1234t^2 + 2500t - 32) \ge 0$$

Which is true from $t \ge 2 \Rightarrow t - 2 \ge 0$ and

$$601t^3 + 1234t^2 + 2500t - 32 > 2500t - 32 > 16t - 32 \overset{t \ge 2}{>} 16 \cdot 2 - 32 = 0$$

$\Rightarrow (2) \Rightarrow (1)$ it's true.

JP.394 If $x, y, z, t > 0$ then:

$$\frac{x^4+1}{y\sqrt{x^4-x^2+1}}+\frac{y^4+1}{z\sqrt{y^4-y^2+1}}+\frac{z^4+1}{z\sqrt{z^4-z^2+1}}+\frac{t^4+1}{x\sqrt{t^4-t^2+1}}\geq 8$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{x^4+1}{y\sqrt{x^4-x^2+1}}=\sum_{cyc}\frac{x^4-x^2+1+x^2}{y\sqrt{x^4-x^2+1}}=$$

$$=\sum_{cyc}\frac{1}{y}\left(\frac{x^4-x^2+1}{\sqrt{x^4-x^2+1}}+\frac{x^2}{\sqrt{x^4-x^2+1}}\right)=\sum_{cyc}\frac{1}{y}\left(\sqrt{x^4-x^2+1}+\frac{x^2}{\sqrt{x^4-x^2+1}}\right)\overset{AM-GM}{\geq}$$

$$\geq\sum_{cyc}\frac{1}{y}\cdot 2\sqrt{\sqrt{x^4-x^2+1}\cdot\frac{x^2}{\sqrt{x^4-x^2+1}}}=2\sum_{cyc}\frac{x}{y}\overset{AM-GM}{\geq}2\cdot 4\cdot\sqrt[4]{\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{t}\cdot\frac{t}{x}}=8$$

Equality holds for $x=y=z=t=1$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\frac{x^4+1}{y\sqrt{x^4-x^2+1}}\geq$$

$$\geq 4\cdot\sqrt[4]{\frac{(x^4+1)(y^4+1)(z^4+1)(t^4+1)}{xyzt\sqrt{(x^4-x^2+1)(y^4-y^2+1)(z^4-z^2+1)(t^4-t^2+1)}}}\geq 8$$

$$[(x^4+1)(y^4+1)(z^4+1)(t^4+1)]^2\geq$$

$$\geq 4(xyzt)^2(x^4-x^2+1)(y^4-y^2+1)(z^4-z^2+1)(t^4-t^2+1)$$

Because $(x^4+1)^2\geq 4x^2(x^4-x^2+1)=4x^6-4x^4+4x^2$

$$x^8+2x^4+4x^4\geq 4x^6+4x^2$$

$$(x^8-x^6)+3(x^4-x^2)\geq 3(x^6-x^4)+(x^2-1)$$

$$x^6(x^2-1)+3x^2(x^2-1)\geq 3x^4(x^2-1)+(x^2-1)=(x^2-1)(3x^4+1)$$

$(x^2-1)\left(x^6+3x^2-(3x^4+1)\right)\geq 0$, which is true because

for $x\geq 1$: $x^6+3x^2\geq 3x^4+1$ and for $x<1$: $x^6+3x^2<3x^4+1$

Analogous,

$$(y^4+1)^2\geq 4y^2(y^4-y^2+1),(z^4+1)^2\geq 4z^2(z^4-z^2+1),$$

and $(t^4+1)^2 \ge 4t^2(t^4-t^2+1)$

Solution 3 by Ravi Prakash-New Delhi-India

We first show that for $x>0$:

$$\frac{x^4+1}{x\sqrt{x^4-x^2+1}} \ge 2 \Leftrightarrow \frac{x^2+\frac{1}{x^2}}{\sqrt{x^2+\frac{1}{x^2}-2}} \ge 2 \Leftrightarrow$$

$$\sqrt{x^2+\frac{1}{x^2}-1}+\frac{1}{\sqrt{x^2+\frac{1}{x^2}-1}} \ge 2, \text{true for } (AM-GM)$$

Now,

$$\sum_{cyc}\frac{x^4+1}{y\sqrt{x^4-x^2+1}} \ge 4\cdot\sqrt[4]{\prod_{cyc}\frac{x^4+1}{x\sqrt{x^4-x^2+1}}} \ge 4\cdot\sqrt[4]{2^4}=8$$

Solution 4 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have : }\frac{x^4+1}{\sqrt{x^4-x^2+1}}$$

$$=\sqrt{x^4-x^2+1}+\frac{x^2}{\sqrt{x^4-x^2+1}} \overset{AM-GM}{\ge} 2\sqrt{\sqrt{x^4-x^2+1}.\frac{x^2}{\sqrt{x^4-x^2+1}}}=2x$$

$$\rightarrow \frac{x^4+1}{y\sqrt{x^4-x^2+1}} \ge 2.\frac{x}{y}$$

Similarly, we have :

$$\frac{y^4+1}{z\sqrt{y^4-y^2+1}} \ge 2.\frac{y}{z}, \frac{z^4+1}{t\sqrt{z^4-z^2+1}} \ge 2.\frac{z}{t} \text{ and } \frac{t^4+1}{x\sqrt{t^4-t^2+1}} \ge 2.\frac{t}{x}$$

$$\rightarrow LHS_{(*)} \ge 2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x}\right) \overset{AM-GM}{\ge} 2.4\sqrt[4]{\frac{x}{y}.\frac{y}{z}.\frac{z}{t}.\frac{t}{x}}=8.$$

Solution 5 by Daniel Văcaru-Romania

We have:

$$\frac{x^4+1}{y\sqrt{x^4-x^2+1}}=\frac{x^5+x}{yx\sqrt{x^4-x^2+1}}=\frac{x^5+x}{y\sqrt{x^2(x^4-x^2+1)}} \overset{AM-GM}{\ge}$$

$$\geq \frac{2(x^5+x)}{y[x^2+(x^4-x^2+1)]}=\frac{2x}{y}$$

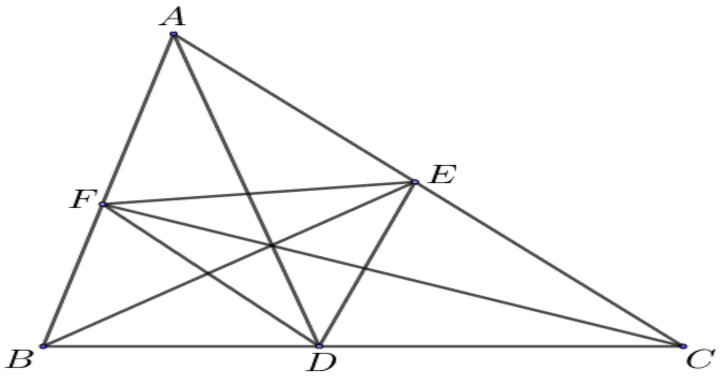
Therefore,

$$\sum_{cyc}\frac{x^4+1}{y\sqrt{x^4-x^2+1}}\geq 2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x}\right)\overset{AM-GM}{\geq} 2.4\sqrt[4]{\frac{x}{y}.\frac{y}{z}.\frac{z}{t}.\frac{t}{x}}=8.$$

JP.395 Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E and F be chosen on sides BC, CA and AB, so that AD, BE and CF bisect the angles of ABC. Prove $\frac{DE}{AB}+\frac{EF}{BC}+\frac{FD}{CA}\leq\frac{3}{4}\left(1+\frac{R}{2r}\right)$.

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer



Let $a=BC, b=CA, c=AB$ be the lengths of the sides. We know that $AF=\frac{bc}{a+b}$,

$$AE=\frac{bc}{a+c}.$$

Using the law of cosines in triangle AFE, we have $EF^2=AF^2+AE^2-2AF\cdot AE\cdot\cos A$.

Also, we know that: $\cos A=\frac{b^2+c^2-a^2}{2bc}$. So,

$$EF^2=\left(\frac{bc}{a+b}\right)^2+\left(\frac{bc}{a+c}\right)^2-\frac{2b^2c^2}{(a+b)(a+c)}\cdot\frac{b^2+c^2-a^2}{2bc}=$$

$$=b^2c^2\left(\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}-\frac{b^2+c^2-a^2}{bc(a+b)(a+c)}\right)=$$

$$=b^2c^2\cdot\frac{bc(a+c)^2+bc(a+b)^2-(b^2+c^2-a^2)(a+b)(a+c)}{bc(a+b)^2(a+c)^2}=$$

$$=\frac{b^2c^2}{(a+b)^2(a+c)^2}\cdot$$

$$\cdot\frac{bc(b^2+c^2)+2abc(b+c)+2a^2bc-a(a+b+c)(b^2+c^2-a^2)-bc(b^2+c^2)+a^2bc}{bc}$$

$$=\frac{b^2c^2}{(a+b)^2(a+c)^2}\cdot$$

$$\cdot\frac{a^2(bc+a^2+ab+ac)+2abc(a+b+c)-a(a+b+c)(b^2+c^2)}{bc}$$

$$=\left(\frac{bc}{(a+b)(a+c)}\right)^2\cdot\frac{a^2(a+b)(a+c)-a(a+b+c)(b^2+c^2-2bc)}{bc}=$$

$$=\left(\frac{bc}{(a+b)(a+c)}\right)^2\cdot\frac{a^2(a+b)(a+c)-a(a+b+c)(b-c)^2}{bc}\le$$

$$\le\left(\frac{bc}{(a+b)(a+c)}\right)^2\cdot\frac{a^2(a+b)(a+c)}{bc}=\frac{a^2bc}{(a+b)(a+c)}.$$

$$EF\le\sqrt{\frac{a^2bc}{(a+b)(a+c)}}\le\sqrt{\frac{a^2bc}{2\sqrt{ab}\cdot2\sqrt{ac}}}=\sqrt{\frac{a\sqrt{bc}}{4}}=\sqrt{\frac{\left(\frac{a+\sqrt{bc}}{2}\right)^2}{4}}=$$

$=\frac{a+\sqrt{bc}}{4}\le\frac{a+\frac{b+c}{2}}{4}=\frac{2a+b+c}{8}$. Namely, $EF\le\frac{2a+b+c}{8}$. Similarly,

$$FD\le\frac{a+2b+c}{8},DE\le\frac{a+b+2c}{8}.$$

Now, we have:

$$\frac{DE}{AB}+\frac{EF}{BC}+\frac{FD}{CA}\le\frac{1}{8}\left(\frac{a+b+2c}{c}+\frac{2a+b+c}{a}+\frac{a+2b+c}{b}\right)=$$

$$=\frac{1}{8}\left(6+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)\right);(*)$$

Now, we'll prove that $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}$. Consider the substitutions $a=y+z,b=z+x,c=x+y$, where x,y,z are positive real numbers. We know that $\frac{R}{r}=\frac{abc}{4(s-a)(s-b)(s-c)},s=\frac{a+b+c}{2}$.

We have:

$\frac{1}{(z+x)^2}+\frac{1}{(y+z)^2}\le\frac{1}{4zx}+\frac{1}{4xy}=\frac{x+y}{4xyz}$ and multiply by $(z+x)(y+z)$ both sides, we get

$$\frac{y+z}{z+x}+\frac{z+x}{y+z}\le\frac{(x+y)(y+z)(z+x)}{4xyz}$$

Namely, $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}$. Similarly, $\frac{b}{c}+\frac{c}{b}\le\frac{R}{r},\frac{c}{a}+\frac{a}{c}\le\frac{R}{r}$. Now, $(*)$ gives

$$\frac{DE}{AB}+\frac{EF}{BC}+\frac{FD}{CA}\le\frac{1}{8}\left(6+\frac{R}{r}+\frac{R}{r}+\frac{R}{r}\right)=\frac{3}{4}\left(1+\frac{R}{2r}\right)$$

So, $\frac{DE}{AB}+\frac{EF}{BC}+\frac{FD}{CA}\le\frac{3}{4}\left(1+\frac{R}{2r}\right)$.

Equality holds when the triangle ABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ a=BC,b=CA,c=AB.$$

$$We\ know\ that:\ AE=\frac{bc}{a+c}\ and\ AF=\frac{bc}{a+b}.$$

$$Using\ the\ Law\ of\ cosines\ in\ \Delta AEF:\ EF^2=AE^2+AF^2-2.AE.AF.\cos A=$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc(b^2+c^2-a^2)}{(a+b)(a+c)}$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=-\frac{bc(b-c)^2}{(a+b)^2(a+c)^2}[(a+b)(a+c)-bc]+\frac{a^2bc}{(a+b)(a+c)}$$

$$=-\frac{2sabc(b-c)^2}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}\le$$

$$\le\frac{a^2bc}{(a+b)(a+c)}\ \rightarrow\ \frac{EF}{BC}\le\sqrt{\frac{b}{a+b}\cdot\frac{c}{a+c}}\overset{AM-GM}{\le}\frac{1}{2}\left(\frac{b}{a+b}+\frac{c}{a+c}\right)$$

$$\rightarrow\sum\frac{EF}{BC}\le\sum\frac{1}{2}\left(\frac{b}{a+b}+\frac{c}{a+c}\right)=\frac{1}{2}\sum\left(\frac{b}{a+b}+\frac{a}{a+b}\right)=\frac{1}{2}\sum 1$$

$$=\frac{3}{4}(1+1)\overset{Euler}{\le}\frac{3}{4}\left(1+\frac{R}{2r}\right).$$

$$Therefore,\quad \frac{DE}{AB}+\frac{EF}{BC}+\frac{FD}{CA}\le\frac{3}{4}\left(1+\frac{R}{2r}\right).$$

JP.396 Let h_a,h_b,h_c be the altitudes from the vertices A,B,C respectively, R the circumradius and r the inradius of a triangle ABC. Let A_1,B_1 and C_1 be chose on the sides BC,CA and AB so that AA_1,BB_1 and CC_1 bisect the angles of ABC. Let h_A,h_b and h_C denote the altitudes of triangles AB_1C_1,BC_1A_1 and CA_1B_1 from the vertices A,B,C, respectively.

Prove that: $\sqrt[3]{\frac{h_a}{h_A}}+\sqrt[3]{\frac{h_b}{h_B}}+\sqrt[3]{\frac{h_c}{h_C}}\le 3\sqrt[3]{2}\cdot\frac{R}{2r}$.

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let $a=BC,b=CA,c=AB$ be the lengths of the sides. We know that $AC_1=\frac{bc}{a+b}$, $AB_1=\frac{bc}{a+c}$. Using the law of the cosines in triangle AB_1C_1, we have

$$B_1C_1^2 = AC_1^2 + AB_1^2 - 2AC_1 \cdot AB_1 \cdot \cos A =$$

$$= \left(\frac{bc}{a+b}\right)^2 + \left(\frac{bc}{a+c}\right)^2 - \frac{2b^2c^2}{(a+b)(a+c)}\cos A$$

We know that: $\cos A = \frac{b^2+c^2-a^2}{2bc}$; $(and\ analogs)$. So,

$$B_1C_1^2 = \left(\frac{bc}{a+b}\right)^2 + \left(\frac{bc}{a+c}\right)^2 - \frac{2b^2c^2}{(a+b)(a+c)} \cdot \frac{b^2+c^2-a^2}{2bc} =$$

$$= b^2c^2\left(\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} - \frac{b^2+c^2-a^2}{bc(a+b)(a+c)}\right) =$$

$$= b^2c^2 \cdot \frac{bc(a+c)^2 + bc(a+b)^2 - (b^2+c^2-a^2)(a+b)(a+c)}{bc(a+b)^2(a+c)^2} =$$

$$= \frac{b^2c^2}{(a+b)^2(a+c)^2} \cdot \frac{a^2(a+b)(a+c) - a(a+b+c)(b^2+c^2-2bc)}{bc} =$$

$$= \frac{b^2c^2}{(a+b)^2(a+c)^2} \cdot \frac{a^2(a+b)(a+c) - a(a+b+c)(b-c)^2}{bc} \le$$

$$\le \frac{b^2c^2}{(a+b)^2(b+c)^2} \cdot \frac{a^2(a+b)(a+c)}{bc} = \frac{a^2bc}{(a+b)(a+c)}$$

So,

$$B_1C_1 \le \sqrt{\frac{a^2bc}{(a+b)(a+c)}} \le \sqrt{\frac{a^2bc}{2\sqrt{ab} \cdot 2\sqrt{ac}}} = \sqrt{\frac{a\sqrt{bc}}{4}} \le \sqrt{\frac{\left(\frac{a+\sqrt{bc}}{2}\right)^2}{4}} = \frac{a+\sqrt{bc}}{4} \le$$

$$\le \frac{a + \frac{b+c}{2}}{4} = \frac{2a+b+c}{8}.$$

Namely, $B_1C_1 \le \frac{2a+b+c}{8}$. Similarly, $A_1C_1 \le \frac{a+2b+c}{8}$ and $A_1B_1 \le \frac{a+b+2c}{8}$.

Now, in triangle AB_1C_1, we have: $\frac{1}{2}B_1C_1 \cdot h_A = \frac{1}{2}AC_1 \cdot AB_1 \cdot \sin A$ or

$$h_A = \frac{\frac{bc}{a+b} \cdot \frac{bc}{a+c} \cdot \frac{a}{2R}}{B_1C_1} \Leftrightarrow \frac{1}{h_A} = \frac{2R(a+b)(b+c)B_1C_1}{ab^2c^2}$$

$$\Rightarrow \frac{1}{h_A} \le \frac{2R(a+b)(b+c)\frac{2a+b+c}{8}}{ab^2c^2}$$

Also, we have: $bc = 2R \cdot h_a$. So,

$$\frac{h_a}{h_A} \le \frac{2R(a+b)(a+c) \cdot \frac{2a+b+c}{8} \cdot h_a}{ab^2c^2} \le \frac{2R\left(\frac{a+b+a+c}{2}\right)^2 \cdot \frac{2a+b+c}{8} \cdot \frac{bc}{2R}}{ab^2c^2}$$

$$= \frac{(2a+b+c)^3}{32abc}$$

Namely, $\frac{h_a}{h_A} \le \frac{(2a+b+c)^3}{32abc} \Leftrightarrow \sqrt[3]{\frac{h_a}{h_A}} \le \frac{2a+b+c}{\sqrt[3]{32abc}}$. Similarly, $\sqrt[3]{\frac{h_b}{h_B}} \le \frac{a+2b+c}{\sqrt[3]{32abc}}$, $\sqrt[3]{\frac{h_c}{h_C}} \le \frac{a+b+2c}{\sqrt[3]{32abc}}$.

Adding up these inequalities, we have:

$$\sqrt[3]{\frac{h_a}{h_A}}+\sqrt[3]{\frac{h_b}{h_B}}+\sqrt[3]{\frac{h_c}{h_C}}\le\frac{2(a+b+c)}{\sqrt[3]{4}\cdot\sqrt{3}\cdot r}=\frac{4s}{\sqrt[3]{4}\cdot\sqrt[3]{abc}}$$

But $R\ge 2r, 2s\le 3\sqrt{3}R, s\ge 3\sqrt{3}r$ and $abc=4Rrs$, so

$$\sqrt[3]{\frac{h_a}{h_A}}+\sqrt[3]{\frac{h_b}{h_B}}+\sqrt[3]{\frac{h_c}{h_C}}\le\frac{6\sqrt{3}R}{\sqrt[3]{4}\cdot\sqrt[3]{4Rrs}}=3\sqrt[3]{2}\cdot\frac{R}{2r}$$

Equality holds if and only if triangle ABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=BC, b=CA, c=AB$ and F be the area of ΔABC.

We know that : $AB_1=\frac{bc}{a+c}$ *and* $AC_1=\frac{bc}{a+b}$.

Using the Law of cosines in ΔAB_1C_1 : $B_1C_1{}^2=AB_1{}^2+AC_1{}^2-2.AB_1.AC_1.\cos A=$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc(b^2+c^2-a^2)}{(a+b)(a+c)}$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc[(b-c)^2+2bc-a^2]}{(a+b)(a+c)}$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}$$

$$=-\frac{bc(b-c)^2[(a+b)(a+c)-bc]}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=-\frac{2sabc(b-c)^2}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}\le\frac{a^2bc}{(a+b)(a+c)}\rightarrow B_1C_1$$

$$\le a\sqrt{\frac{bc}{(a+b)(a+c)}}\quad(1)$$

Now, we have : $h_A=\frac{2[AB_1C_1]}{B_1C_1}$ *with* $[AB_1C_1]=\frac{1}{2}.AB_1.AC_1.\sin A=\frac{bc.F}{(a+b)(a+c)}$

$$\rightarrow h_A=\frac{2bc.F}{(a+b)(a+c).B_1C_1}\overset{(1)}{\ge}\frac{2bc.F}{(a+b)(a+c)}.\frac{\sqrt{(a+b)(a+c)}}{a\sqrt{bc}}$$

$$=h_a.\sqrt{\frac{bc}{(a+b)(a+c)}}$$

$$\to \frac{h_a}{h_A} \le \sqrt{\frac{(a+b)(a+c)}{bc}} \to \sqrt[3]{\frac{h_a}{h_A}} \le \sqrt[6]{\frac{(a+b)(a+c)}{bc}}$$

$$= \sqrt[3]{2}.\sqrt[6]{\frac{a+b}{2c}.\frac{a+c}{2b}.1.1.1.1} \overset{AM-GM}{\le} \frac{\sqrt[3]{2}}{6}\left(\frac{a+b}{2c}+\frac{a+c}{2b}+4.1\right)$$

$$\to \sum_{cyc}\sqrt[3]{\frac{h_a}{h_A}} \le \frac{\sqrt[3]{2}}{6}\sum_{cyc}\left(\frac{a+b}{2c}+\frac{a+c}{2b}+4\right)$$

$$= \frac{\sqrt[3]{2}}{6}\left(12+\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\right) \overset{\substack{Euler\\Bandila}}{\le} \frac{\sqrt[3]{2}}{6}\left(12.\frac{R}{2r}+\sum_{cyc}\frac{R}{r}\right) = 3\sqrt[3]{2}.\frac{R}{2r}$$

$$Therefore, \quad \sum_{cyc}\sqrt[3]{\frac{h_a}{h_A}} \le 3\sqrt[3]{2}.\frac{R}{2r}$$

JP.397 Solve for real numbers:

$$3^{\log_2(3^x-1)} = 2^{\log_3(2^x+1)} + 1$$

Proposed by Ionuț Florin Voinea-Romania

Solution 1 by proposer

Lemma. Let be $f, g: (0,\infty) \to \mathbb{R}$ have strictly monotonically and $\begin{cases} f(x) = g(y) \\ f(y) = g(z) \\ f(z) = g(x) \end{cases}$.

Then $x = y = z$ has unique solution for the system.

Proof. Let's suppose f, g – increasing functions.

$$\text{If } x < y \text{ then } f(x) < f(y) \Rightarrow g(y) < g(z), g \nearrow \Rightarrow y < z, f \nearrow \Rightarrow f(y) < f(z) \Rightarrow$$

$$g(z) < g(x), g \nearrow \Rightarrow z < x.$$

So, $x < y < z < x$ impossible. Analogous, for $x > y \Rightarrow x > y > z > x$ impossible.

Hence, $x = y \Rightarrow g(y) = g(z)$; $g \nearrow \Rightarrow g$ –injective function, then $y = z \Rightarrow x = y = z$.

$$3^x - 1 > 0 \Rightarrow 3^x > 1 \Rightarrow x \in (0,\infty)$$

$$3^{\log_2(3^x-1)} = 2^{\log_3(2^x+1)} + 1 \Leftrightarrow \log_2(3^x-1) = \log_3(2^{\log_3(2^x+1)}+1); (*)$$

Let $y = \log_2(3^x-1)$ and $z = \log_3(2^{\log_3(2^x+1)}+1) = \log_3(2^x+1)$ then $(*) \Leftrightarrow$

$$\log_2(2^x+1) = \log_3(2^y+1) = z \Leftrightarrow \log_2(3^x-1) = z \Leftrightarrow 3^x - 1 = 2^z$$

$$3^x = 2^z + 1 \Leftrightarrow x = \log_3(2^z+1)$$

So, $\begin{cases} x = \log_3(2^z+1) \\ y = \log_3(2^x+1) \\ z = \log_3(2^y+1) \end{cases} \Leftrightarrow \begin{cases} 3^x = 2^z+1 \\ 3^y = 2^x+1 \\ 3^z = 2^y+1 \end{cases}$

Let $f:(0,\infty)\to\mathbb{R}, f(x)=3^x, f\nearrow; g:(0,\infty)\to\mathbb{R}, g(z)=2^z+1, g\nearrow$, from Lemma it follows that $x=y=z$.

$$x=z\Rightarrow 3^x=2^x+1\Leftrightarrow\left(\frac{2}{3}\right)^x+\left(\frac{1}{3}\right)^x=1\Rightarrow x=1 \text{ solution.}$$

Solution 2 by Florentin Vişescu-Romania

$3^x-1>0\Rightarrow x>0$.Let $f:(0,\infty)\to\mathbb{R}, f(x)=\log_2(3^x-1)$ increasing, then f –injective.

Let $y\in\mathbb{R}\Rightarrow\exists x>0$ such that $f(x)=y\Leftrightarrow\log_2(3^x-1)=y\Leftrightarrow 3^x=2^y+1\Leftrightarrow$ $x=\log_3(2^y+1)$. So, f –surjective. Hence, f –invertible with $f^{-1}(x)=\log_3(2^x+1)$

So, we have:

$$3^{f(x)}=2^{f^{-1}(x)}+1\Rightarrow f(x)=\log_3\left(2^{f^{-1}(x)}+1\right)\Rightarrow x=\log_3\left(2^{f^{-1}\left(f^{-1}(x)\right)}+1\right)$$

$$\Rightarrow 3^x=2^{f^{-1}\left(f^{-1}(x)\right)}+1\Rightarrow 3^x-1=2^{f^{-1}\left(f^{-1}(x)\right)}$$

$$\Rightarrow\log_2(3^x-1)=f^{-1}\left(f^{-1}(x)\right)\Rightarrow f\left(f\left(f(x)\right)\right)=x$$

Lemma. If f is strictly increasing function with $f\left(f\left(f(x)\right)\right)=a$, then $f(a)=a$.

Proof. Let's suppose $f(a)=b>a$. Then $f\left(f(a)\right)=f(b)>f(a)=a$ and

$$f\left(f\left(f(a)\right)\right)>f(a)>a \text{ impossible!}$$

Let's suppose $f(a)=b<a$. Then $f\left(f(a)\right)=f(b)<f(a)<a$ and

$$f\left(f\left(f(a)\right)\right)<f(a)<a \text{ impossible!}$$

Return to the problem. $f\left(f\left(f(x)\right)\right)=x\Rightarrow f(x)=x\Rightarrow\log_2(3^x-1)=x\Rightarrow 3^x=2^x+1$

$$\left(\frac{3}{2}\right)^x-\left(\frac{1}{2}\right)^x=1;\left(x\to\left(\frac{3}{2}\right)^x\nearrow;x\to\left(\frac{1}{2}\right)^x\searrow\right)$$

So, $u(x)=\left(\frac{3}{2}\right)^x-\left(\frac{1}{2}\right)^x$ injective and $x=1$ unique solution.

Solution 3 by Agayev Seddredin-Azerbaijan

$$3^{\log_2(3^x-1)}=2^{\log_3(2^x+1)}+1$$

$$\begin{cases}\log_2(3^x-1)=a\\ \log_3(2^x+1)=b\end{cases}\Rightarrow\begin{cases}3^x=2^a+1\\ 3^b=2^x+1\end{cases}$$

We have: $3^a=2^b+1$

If $a\ge b\ge x;(*)\Rightarrow 2^a+1=3^x\le 3^b=2^x+1\Rightarrow a\le x;(**)$

From $(*),(**)$ we get $x=a$.

$$3^x=2^x+1\Rightarrow\left(\frac{2}{3}\right)^x+\left(\frac{1}{3}\right)^x=1$$

$$x>1\Rightarrow\left(\frac{2}{3}\right)^x<\left(\frac{2}{3}\right)^1,\left(\frac{1}{3}\right)^x<\left(\frac{1}{3}\right)^1\Rightarrow\left(\frac{2}{3}\right)^x+\left(\frac{1}{3}\right)^x<1;(\textit{no solution}!)$$

$$x<1\Rightarrow\left(\frac{2}{3}\right)^x+\left(\frac{1}{3}\right)^x>1;(\textit{no solution}!)$$

Therefore, $x=1$ is unique solution.

Solution 4 by Jamal Issah-Ghana

$$3^{\log_2(3^x-1)}=2^{\log_3(2^x+1)}+1$$

$$\begin{cases}\log_2(3^x-1)=m\\ \log_3(2^x+1)=n\end{cases}\Rightarrow\begin{cases}2^m=3^x-1\\ 3^n=2^x+1\end{cases}$$

We have: $3^{2^m}=2^{3^n}+1\Leftrightarrow 3^{2^m}-2^{3^n}=1$

From Catalan's conjecture: $m=n=1$. Hence, $3^x-1=2\Leftrightarrow x=1$.

Solution 5 by Hikmat Mammmadov-Azerbaijan

$$\because a^{\log_b c}=c^{\log_b a}\Rightarrow(3^x-1)^{\log_2 3}=(2^x+1)^{\log_3 2}+1$$

$$\begin{cases}3^x-1=v\\ 2^x+1=u\end{cases}\Rightarrow 2^{v\cdot\log_2 3}+3^{u\cdot\log_3 2}+1=0$$

$$3^v=2^u+1\Leftrightarrow 3^v-2^u=1\Rightarrow u=v=1\Rightarrow x=1$$

Solution 6 by Khaled Abd Imouti-Damascus-Syria

$$n+1=n+1\Rightarrow e^{\log(n+1)}=e^{\log n}+1$$

$$e^{\frac{\log(n+1)}{\log n}\cdot\log(n+1-1)}=e^{\frac{\log n}{\log(n+1)}\cdot\log(n+1)}$$

$$e^{\frac{\log(n+1)}{\log n}\cdot\log\left(e^{\log(n+1)}-1\right)}=e^{\frac{\log n}{\log(n+1)}\cdot\log\left(e^{\log n}+1\right)}+1$$

$$(n+1)^{\log_n((n+1)^x-1)}=e^{\log\left(e^{x\cdot\log(n+1)-1}\right)\cdot\frac{\log(n+1)}{\log n}}$$

$$n^{\log_{n+1}(n^x+1)} = e^{\log(e^{x\log n}+1)\cdot\frac{\log n}{\log(n+1)}} = 1 \Rightarrow x = 1$$

JP.398 If $a, b, c > 0;\ ab\sqrt{ab} + bc\sqrt{bc} + ca\sqrt{ca} = 3$ then:

$$\sum_{cyc}\left(a\sqrt{a} - b\sqrt{b}\right)^2 + 24 \le \sum_{cyc}(a+b)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

First, we prove that:

$$\left(a\sqrt{a} - b\sqrt{b}\right)^2 + 8ab\sqrt{ab} \le (a+b)^3;\ (1)$$

$$a^3 + b^3 - 2ab\sqrt{ab} + 8ab\sqrt{ab} \le a^3 + b^3 + 3ab(a+b) \Leftrightarrow$$

$$3ab(a+b) - 6ab\sqrt{ab} \ge 0 \Leftrightarrow 3ab\left(a + b - 2\sqrt{ab}\right) \ge 0$$

$$\Leftrightarrow 3ab\left(\sqrt{a} - \sqrt{b}\right)^2 \ge 0$$

By adding in (1), we get:

$$\sum_{cyc}\left(a\sqrt{a} - b\sqrt{b}\right)^2 + 8\sum_{cyc} ab\sqrt{ab} \le \sum_{cyc}(a+b)^3 \Leftrightarrow$$

$$\sum_{cyc}\left(a\sqrt{a} - b\sqrt{b}\right)^2 + 8\cdot 3 \le \sum_{cyc}(a+b)^3 \Leftrightarrow \sum_{cyc}\left(a\sqrt{a} - b\sqrt{b}\right)^2 + 24 \le \sum_{cyc}(a+b)^3$$

Equality holds for $a = b = c$.

Solution 2 by George Florin Şerban-Romania

$$\sum_{cyc}\left(a\sqrt{a} - b\sqrt{b}\right)^2 + 24 = \sum_{cyc}\left(a^3 + b^3 - 2ab\sqrt{ab}\right) + 24 =$$

$$= \sum_{cyc}(a^3 + b^3) - 2\sum_{cyc} ab\sqrt{ab} + 24 = 2\sum_{cyc} a^3 + 18 \overset{(1)}{\le} \sum_{cyc}(a+b)^3 =$$

$$= 2\sum_{cyc} a^3 + 3\sum_{cyc} ab(a+b)$$

$$(1) \Leftrightarrow \sum_{cyc} ab(a+b) \ge 6;$$

$$\sum_{cyc} ab(a+b) \overset{AM-GM}{\geq} 2\sum_{cyc} ab\sqrt{ab} = 6$$

Therefore,

$$\sum_{cyc}\left(a\sqrt{a}-b\sqrt{b}\right)^2 + 24 \leq \sum_{cyc}(a+b)^3$$

Equality holds for $a=b=c$.

Solution 3 by Fayssal Abdelli-Bejaia-Algerie

$$\sum_{cyc}(a+b)^3 = \sum_{cyc}(a^3+b^3+3a^2b+3ab^2) = 2\sum_{cyc} a^3 + 3\sum_{cyc} ab(a+b)\,;(1)$$

$$\sum_{cyc}\left(a\sqrt{a}-b\sqrt{b}\right)^2 = \sum_{cyc}\left(a^3+b^3-2ab\sqrt{ab}\right) = 2\sum_{cyc} a^3 - 2\sum_{cyc} ab\sqrt{ab} =$$

$$= 2(a^3+b^3+c^3) - 6$$

Hence,

$$24 + \sum_{cyc}\left(a\sqrt{a}-b\sqrt{b}\right)^2 = 18 + 2(a^3+b^3+c^3);(2)$$

We must to prove:

$$3\sum_{cyc} ab(a+b) \geq 18 \Leftrightarrow \sum_{cyc} ab(a+b) \geq 6;(3)$$

$$\sum_{cyc} ab(a+b) \overset{AM-GM}{\geq} 2\sum_{cyc} ab\sqrt{ab} \Rightarrow (3)\text{true} \Rightarrow (2)\ \text{true}.$$

Therefore,

$$\sum_{cyc}\left(a\sqrt{a}-b\sqrt{b}\right)^2 + 24 \leq \sum_{cyc}(a+b)^3$$

Equality holds for $a=b=c$.

Solution 4 by Daniel Văcaru-Romania

We have:

$$\sum_{cyc}\left(a\sqrt{a}-b\sqrt{b}\right)^2 + 24 = 2(a^3+b^3+c^3) + 18$$

$$2(a^3+b^3+c^3)+18\le\sum_{cyc}(a+b)^3\Leftrightarrow$$

$$3\sum_{cyc}ab(a+b)\ge 18\Leftrightarrow\sum_{cyc}ab(a+b)\ge 6,\sum_{cyc}ab(a+b)\overset{AM-GM}{\ge}2\sum_{cyc}ab\sqrt{ab}$$

Therefore,

$$\sum_{cyc}\left(a\sqrt{a}-b\sqrt{b}\right)^2+24\le\sum_{cyc}(a+b)^3$$

Equality holds for $a=b=c$.

JP.399 In ΔABC the following relationship holds:

$$a^4+b^4+c^4\ge 16F^2+\frac{1}{2}\left((a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\sum_{cyc}(a^2-b^2)^2=2(a^4+b^4+c^4)-2\sum_{cyc}a^2b^2$$

$$a^4+b^4+c^4=\sum_{cyc}a^2b^2+\frac{1}{2}\sum_{cyc}(a^2-b^2)^2=$$

$$=\frac{(ab)^2}{1}+\frac{(bc)^2}{1}+\frac{(ca)^2}{1}+\frac{1}{2}\sum_{cyc}(a^2-b^2)^2\ge$$

$$\overset{Bergstrom}{\ge}\frac{(ab+bc+ca)^2}{1+1+1}+\frac{1}{2}\sum_{cyc}(a^2-b^2)^2\ge$$

$$\overset{GORDON}{\ge}\frac{\left(4\sqrt{3}F\right)^2}{3}+\frac{1}{2}\sum_{cyc}(a^2-b^2)^2=16F^2+\frac{1}{2}\sum_{cyc}(a^2-b^2)^2$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$2(a^4+b^4+c^4)-(a^2-b^2)^2-(b^2-c^2)^2-(c^2-a^2)^2=$$

$$=a^4+b^4-(a^2-b^2)^2+b^4+c^4-(b^2-c^2)^2+c^4+a^4-(c^2-a^2)^2=$$

$$=a^4+a^2(2c^2-a^2)+b^4+b^2(2a^2-b^2)+c^4+c^2(2b^2-c^2)=$$

$$=2(ab)^2+2(bc)^2+2(ca)^2\ge\frac{2}{3}(ab+bc+ca)^2\ge\frac{2}{3}\left(4\sqrt{3}F\right)^2=32F^2$$

$$\Rightarrow 2(a^4+b^4+c^4)\ge 32F^2+(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2$$

$$\Rightarrow a^4+b^4+c^4\ge 16F^2+\frac{1}{2}[(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2]$$

Solution 3 by Marian Ursărescu-Romania

We must show that:

$$a^4+b^4+c^4\ge 16F^2+\frac{1}{2}(2a^4+2b^4+2c^4-2a^2b^2-2b^2c^2-2c^2a^2)\Leftrightarrow$$

$$a^2b^2+b^2c^2+c^2a^2\ge 16F^2;(1)$$

But $(ab)^2+(bc)^2+(ca)^2\ge abc(a+b+c);(2)$

From (1)&(2) we must show that: $abc(a+b+c)\ge 16F^2;(3)$

But $abc=4Rrs, a+b+c=2s, F=sr;(4)$

From (3)&(4) we must show that: $4Rrs\cdot 2s\ge 16r^2s^2\Leftrightarrow R\ge 2r\ (Euler)$.

Solution 4 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}a^4=16F^2+\frac{1}{2}\sum_{cyc}(a^2-b^2)^2\Leftrightarrow\sum_{cyc}a^4\ge 16F^2+\frac{1}{2}\left(2\sum_{cyc}a^4-2\sum_{cyc}a^2b^2\right)$$

$$\sum_{cyc}a^2b^2\ge 16F^2(need\ to\ prove)$$

$$\because (1^2+1^2+1^2)\left(\sum_{cyc}a^2b^2\right)\overset{CBS}{\ge}\left(\sum_{cyc}ab\right)^2\Rightarrow$$

$$\sum_{cyc}a^2b^2\ge\frac{(s^2+r^2+4Rr)^2}{3}\overset{Gerretsen}{\ge}\left(\frac{16Rr-5r^2+r^2+4Rr}{3}\right)^2=$$

$$=\left(\frac{20Rr-4r^2}{3}\right)^2\overset{Mitrinovic}{\ge}\left(\frac{20\cdot\frac{2s}{3\sqrt{3}}r+r^2-4r\cdot\frac{s}{3\sqrt{3}}}{3}\right)^2=\frac{\left(\frac{36F}{3\sqrt{3}}\right)^2}{3}=16F^2$$

JP.400 If $x,y,z>0;\ \sqrt[3]{xy}+\sqrt[3]{yz}+\sqrt[3]{zx}=3$ then:

$$\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}\ge 3\sqrt[3]{2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

First we prove that if $x,y>0$ then $\sqrt[3]{4(x+y)}\ge\sqrt[3]{x}+\sqrt[3]{y};(1)$

Denote $x=a^3, y=b^3$ we have: $\sqrt[3]{4(a^3+b^3)}\ge a+b\Leftrightarrow$

$$4(a^3+b^3)\geq(a+b)^3\Leftrightarrow 4a^3+4b^3\geq a^3+b^3+3ab(a+b)\Leftrightarrow$$

$$3(a^3-b^3)-3ab(a+b)\geq 0\Leftrightarrow(a+b)(a^2-ab+b^2)-ab(a+b)\geq 0$$

$\Leftrightarrow (a+b)(a^2-2ab+b^2)\geq 0\Leftrightarrow(a+b)(a-b)^2\geq 0$, which is true for all $a,b\geq 0$.

Equality holds for $a=b\Leftrightarrow x=y$.

Multiplying (1) with $\sqrt[3]{z}$, it follows that:

$$\sqrt[3]{4z(x+y)}\geq\sqrt[3]{xz}+\sqrt[3]{yz};(2)$$

Analogous,

$$\sqrt[3]{4x(y+z)}\geq\sqrt[3]{xy}+\sqrt[3]{xz};(3)$$

$$\sqrt[3]{4y(z+x)}\geq\sqrt[3]{zy}+\sqrt[3]{xy};(4)$$

By adding (2),(3),(4) we obtain:

$$\sqrt[3]{4}\left(\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}\right)\geq 2\left(\sqrt[3]{xy}+\sqrt[3]{yz}+\sqrt[3]{zx}\right)=2\cdot 3=6$$

Hence,

$$\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}\geq\frac{6}{\sqrt[3]{4}}=\frac{6\sqrt[3]{2}}{\sqrt[3]{8}}=3\sqrt[3]{2}$$

Equality holds for $x=y=z=1$.

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}\geq 3\sqrt[3]{\sqrt[3]{xyz(x+y)(y+z)(z+x)}}$$

But: $(x+y)(y+z)(z+x)\overset{AM-GM}{\geq}8xyz\ (Cesaro);(1)$

$$\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}\geq 3\sqrt[3]{\sqrt[3]{8(xyz)^2}}=3\sqrt[3]{2\sqrt[3]{(xyz)^2}}$$

But $\sqrt[3]{xy}+\sqrt[3]{yz}+\sqrt[3]{zx}\geq=3\sqrt[3]{\sqrt[3]{(xyz)^2}}\Rightarrow 3\geq 3\sqrt[6]{(xyz)^2}\Rightarrow xyz\leq 1;(2)$

$$\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}\geq 3\sqrt[3]{2\sqrt[3]{(xyz)^2}}\overset{?}{\geq}3\sqrt[3]{2}\Rightarrow$$

$$\sqrt[3]{2}\cdot\sqrt[6]{(xyz)^2}\geq\sqrt[3]{2}\Rightarrow xyz\geq 1;(3)$$

$$\Rightarrow\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}\geq 3\sqrt[3]{2\sqrt[3]{(xyz)^2}}\geq 3\sqrt[3]{2}$$

Equality holds for $x=y=z=1$.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sqrt[3]{x(y+z)}+\sqrt[3]{y(z+x)}+\sqrt[3]{z(x+y)}=$$

$$=\sqrt[3]{\frac{1}{4}(1+1)(x+x)(y+z)}+\sqrt[3]{\frac{1}{4}(1+1)(y+y)(z+x)}+\sqrt[3]{\frac{1}{4}(1+1)(z+z)(x+y)}$$

$$\geq\sqrt[3]{\frac{1}{4}}\left(\sqrt[3]{1\cdot xy}+\sqrt[3]{1\cdot yz}+\sqrt[3]{1\cdot yx}+\sqrt[3]{1\cdot zx}+\sqrt[3]{1\cdot zy}+\sqrt[3]{1\cdot xz}\right)=$$

$$\geq 2\cdot\sqrt[3]{\frac{1}{4}}\left(\sqrt[3]{xy}+\sqrt[3]{yz}+\sqrt[3]{zx}\right)=2\cdot\sqrt[3]{\frac{1}{4}}\cdot 3=3\sqrt[3]{2}$$

Equality holds for $x=y=z=1$.

Solution 4 by Angel Plaza-Spain

By doing $\sqrt[3]{xy}=a,\sqrt[3]{yz}=b,\sqrt[3]{zx}=c$, the problem becomes

If $a,b,c>0, a+b+c=3$ then:

$$\sqrt[3]{\frac{a^3+c^3}{2}}+\sqrt[3]{\frac{b^3+c^3}{2}}+\sqrt[3]{\frac{c^3+b^3}{2}}\geq 3$$

By power mean arithmetic mean inequality:

$$\sqrt[3]{\frac{a^3+c^3}{2}}+\sqrt[3]{\frac{b^3+c^3}{2}}+\sqrt[3]{\frac{c^3+b^3}{2}}\geq\frac{a+c}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$

Equality holds for $a=b=c=1\Leftrightarrow x=y=z=1$

JP.401 Find all sets x,y,z of positive integers such that

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=\frac{1}{2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

We have $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=\frac{1}{2}\Leftrightarrow xyz=2(x+y+z)$

Assume WLOG that $x\leq y\leq z$. Then it follows that (since x,y,z are positive integers).

$xyz \le 6z \Rightarrow xy \le 6$ or $x^2 \le 6$ namely $x = 1$ or $x = 2$.

For $x = 1 \Rightarrow yz = 2 + 2y + 2z \Leftrightarrow yz - 2y - 2z = 2 \Leftrightarrow (y-2)(z-2) = 6 = 2 \cdot 3$ with $y - 2$ being the smaller positive factor.

This leads to the two solutions $(x,y,z) = (1,3,8),(1,4,5)$.

For $x = 2$, we have $(y-1)(z-1) = 3 = 1 \cdot 3$ namely $y = 2$ and $z = 4$.

So, $(x,y,z) \in \{(1,3,8),(1,4,5),(2,2,4)\}$

Solution 2 by Ertan Yildirim-Turkiye

$$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = \frac{x+y+z}{xyz} = \frac{1}{2} \Rightarrow 2(x+y+z) = xyz; (*)$$

Suppose that: $x \ge y \ge z \Rightarrow 2(x+y+z) \le 2(x+x+x) = 6x$

$$\Rightarrow xyz \le 6x \Rightarrow yz \le 6$$

$$(i) yz = 6 \Rightarrow y = 6, z = 1.$$

$$(*): 2(x+7) = 6x \Rightarrow 14 = 4x \Rightarrow x = \frac{7}{2} \notin \mathbb{Z}$$

$$(ii) yz = 6 \Rightarrow y = 3, z = 2$$

$$(*): 2(x+5) = 6x \Rightarrow 10 = 4x \Rightarrow x = \frac{5}{2} \notin \mathbb{Z}$$

$$(iii) yz = 5 \Rightarrow y = 5, z = 1$$

$$(*): 2(x+6) = 5x \Rightarrow 12 = 3x \Rightarrow x = 4$$

$$\Rightarrow (x,y,z) \in \{(4,5,1),(4,1,5),(5,1,4),(5,4,1),(1,4,5),(1,5,4)\}$$

$$(iv) yz = 4, y = 2, z = 2$$

$$(*): 2(x+4) = 4x \Rightarrow 8 = 2x \Rightarrow x = 4$$

$$(x,y,z) \in \{(4,2,2),(2,4,2),(2,2,4)\}$$

$$(v) yz = 4, y = 4, z = 1$$

$(*): 2(x+5) = 4x \Rightarrow x = 5$ (same case in (iii))

$$(vi) yz = 3, y = 3, z = 1$$

$$(*): 2(x+4) = 3x \Rightarrow x = 8$$

$$(x,y,z) \in \{(8,3,1),(8,1,3),(1,3,8),(1,8,3),(3,1,8),(3,8,1)\}$$

$$(vii) yz = 2, y = 2, z = 1$$

$(*): 2(x+3) = 2x \Rightarrow 6 = 0$ no solution!

$$(viii) yz=1, y=z=1$$

$$(*): 2(x+2)=x \Rightarrow x=-4 \notin \mathbb{Z}_+$$

Solution 3 by Fayssal Abdelli-Bejaia-Algerie

$$x,y,z \in \mathbb{N}^*, \frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=\frac{1}{2} \Rightarrow \frac{x+y+z}{xyz}=\frac{1}{2} \Leftrightarrow 2(x+y+z)=xyz; (1)$$

Let suppose: $x \le y \le z \Rightarrow 2(x+y+z) \le 6z$

$x=1, y=1; (1) \Rightarrow z=2z+4$ impossible.

$x=1, y=2; (1) \Rightarrow 2z=2(3+z)$ impossible.

$x=1, y=3; (1) \Rightarrow z=8$

$x=1, y=4; (1) \Rightarrow z=5$

$x=1, y=4; (1) \Rightarrow z=4$

$x=1, y=5; (1) \Rightarrow$impossible.

$x=2, y=1$ impossible.

$x=2, y=2; (1) \Rightarrow z=4$

$x=2, y=3$ impossible.

$x=3, y=1; (1) \Rightarrow z=8$

$x=3, y=2$ impossible.

$x=4, y=1; (1) \Rightarrow z=5$

$x=5, y=1; (1) \Rightarrow z=4$

$x=6, y=1; (1) \Rightarrow$impossible.

Hence,

$$(x,y,z) \in \{(1,3,8),(1,4,5),(1,5,4),(2,2,4),(3,1,8),(4,1,5),(5,1,4)\}$$

By applying permutations $x \le y \le z$ and $x \le z \le y$, we get:

$$(x,y,z) \in \left\{\begin{matrix}(1,3,8),(1,4,5),(1,5,4),(2,2,4),(3,1,8),(4,1,5),(5,1,4), \\ (1,4,5),(1,5,4),(4,1,5),(4,5,1),(5,1,4),(5,4,1) \\ (2,2,4),(2,4,2),(4,2,2)\end{matrix}\right\}$$

JP.402 In acute ΔABC the following relationship holds:

$$R-2r \le 3(\max\{h_a,h_b,h_c\}-\min\{h_a,h_b,h_c\})$$

Proposed by Cristian Miu-Romania

Solution 1 by proposer

Let us recall two inequalities. The first one is Bankhoff inquality:

$\sum h_a \le 2R+5r$ and the second one is Erdos inequality. In any acute triangle,

$R+r \le \max\{h_a,h_b,h_c\}$. Now, we can write:

$$\min\{h_a,h_b,h_c\} \le \frac{2R+5r}{3} \le R+r \le \max\{h_a,h_b,h_c\}$$

So, we obtain: $\max\{h_a,h_b,h_c\}-\min\{h_a,h_b,h_c\}\ge R+r-\frac{2r+5r}{3}=\frac{R-2r}{3}$

Bankhoff inequality is easy to prove, because $\sum h_a=\frac{1}{2R}\sum ab=\frac{s^2+r^2+4Rr}{2R}$.

But $\frac{s^2+r^2+4Rr}{2R}\le 2R+5r$. This inequality can be proved using Gerretsen inequality

$$s^2\le 4R^2+4Rr+3r^2.$$

Let us prove Erdos inequality. It is easy to see that $\sum a\cot A=2(R+r)$ and

$\sum a^2\cot A=4F$, where F –is area of triangle ABC. So, we obtain:

$\min\{h_a,h_b,h_c\}\le R+r\le\max\{h_a,h_b,h_c\}$ because x,y,z are real numbers and u,v,w are positive, then $\min\left\{\frac{x}{u},\frac{y}{v},\frac{z}{w}\right\}\le\frac{x+y+z}{u+v+w}\le\max\left\{\frac{x}{u},\frac{y}{v},\frac{z}{w}\right\}$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$WLOG, we\ may\ assume\ that:a\ge b\ge c\to h_a\le h_b\le h_c\to(*)\leftrightarrow R-2r\le 3(h_c-h_a)=\frac{6sr(a-c)}{ca}$$

$$Since\ s>a\ \to\ It's\ suffices\ to\ prove\ that:\ R-2r\le\frac{6r(a-c)}{c}\leftrightarrow\frac{R}{r}\le\frac{6a-4c}{c}$$

$$We\ have:\frac{6a-4c}{c}=\frac{2a+4(a-c)}{c}\overset{a\ge c}{\ge}\frac{2a}{c}\overset{?}{\ge}\frac{R}{r}=\frac{2abc}{(-a+b+c)(a-b+c)(a+b-c)}$$

$$(-a+b+c)(a-b+c)(a+b-c)\ge bc^2\ \leftrightarrow\ (-a+b+c)(a^2-b^2-c^2+2bc)\ge bc^2$$
$$\leftrightarrow(-a+b)\big(-2bc.\cos A-c(a-b)+c(a+b)\big)+c(a^2-b^2-c^2+2bc)\ge bc^2$$
$$\leftrightarrow(a-b)[2bc.\cos A+c(a-b)]+c(b^2-a^2)+c(a^2-b^2-c^2+2bc)\ge bc^2$$

$$\leftrightarrow\ (a-b)[2bc.\cos A+c(a-b)]+c^2(b-c)\ge 0\ which\ is\ true\ (\because a\ge b\ge c,\cos A\ge 0)$$
$$Therefore,\quad R-2r\le 3(max\{h_a,h_b,h_c\}-min\{h_a,h_b,h_c\}).$$

JP.403 If $x,y,z\in(0,1)$ then in ΔABC the following relationship holds:

$$\frac{1}{(y+z)(1-x^2)h_a^4}+\frac{1}{(z+x)(1-y^2)h_b^4}+\frac{1}{(x+y)(1-z^2)h_c^4}\ge\frac{3\sqrt{3}}{4F^2}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$2x^2(1-x^2)^2 \overset{AM-GM}{\leq} \left(\frac{2x^2+1-x^2+1-x^2}{3}\right)^3 = \frac{8}{27}$$

$$x^2(1-x^2)^2 \leq \frac{4}{27} \Rightarrow x(1-x^2) \leq \frac{2}{3\sqrt{3}}; (1)$$

$$\sum_{cyc} \frac{1}{(y+z)(1-x^2)h_a^4} = \sum_{cyc} \frac{x}{x(y+z)(1-x^2)h_a^4} \overset{(1)}{\geq} \frac{3\sqrt{3}}{2} \sum_{cyc} \frac{x}{(y+z)h_a^4} =$$

$$= \frac{3\sqrt{3}}{2} \sum_{cyc} \frac{xa^4}{(y+z)(ah_a)^4} = \frac{3\sqrt{3}}{2} \cdot \frac{1}{16F^4} \sum_{cyc} \frac{x}{y+z} a^4 \overset{Tsintsifas}{\geq}$$

$$\geq \frac{3\sqrt{3}}{32F^4} \cdot 8F^2 = \frac{3\sqrt{3}}{4F^2}$$

Equality holds for $a=b=c; x=y=z=\frac{\sqrt{3}}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM-GM, we\ have: \ x^2 + \frac{\sqrt{3}}{9x} + \frac{\sqrt{3}}{9x} \geq 3\sqrt[3]{x^2 . \frac{\sqrt{3}}{9x} . \frac{\sqrt{3}}{9x}} = 1 \rightarrow$$

$$1 - x^2 \leq \frac{2\sqrt{3}}{9x} \ \ (And\ analogs)$$

$$LHS_{(*)} \geq \sum_{cyc} \frac{9}{2\sqrt{3}} . \frac{x}{y+z} . \left(\frac{a}{2F}\right)^4 = \frac{3\sqrt{3}}{32F^4} \sum_{cyc} \frac{(x+y+z)-(y+z)}{y+z} . a^4$$

$$= \frac{3\sqrt{3}}{32F^4}\left[\left(\sum_{cyc} x\right)\left(\sum_{cyc} \frac{a^4}{y+z}\right) - \sum_{cyc} a^4\right] \geq$$

$$\overset{Bergstrom}{\geq} \frac{3\sqrt{3}}{32F^4}\left[\left(\sum_{cyc} x\right) . \frac{(a^2+b^2+c^2)^2}{2(x+y+z)} - \sum_{cyc} a^4\right] = \frac{3\sqrt{3}}{64F^4}\left[\left(\sum_{cyc} a^2\right)^2 - 2\sum_{cyc} a^4\right]$$

$$= \frac{3\sqrt{3}}{64F^4}\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right)$$

$$Since\ 2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4 = 16F^2 \rightarrow \sum_{cyc} \frac{1}{(y+z)(1-x^2){h_a}^4} \geq \frac{3\sqrt{3}}{64F^4}.16F^2 = \frac{3\sqrt{3}}{4F^2}.$$

$$Therefore,$$

$$\frac{1}{(y+z)(1-x^2){h_a}^4} + \frac{1}{(z+x)(1-y^2){h_b}^4} + \frac{1}{(x+y)(1-z^2){h_c}^4} \geq \frac{3\sqrt{3}}{4F^2}.$$

Equality holds for $a = b = c; x = y = z = \frac{\sqrt{3}}{3}$.

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4 = 16F^2, h_a = \frac{2F}{a}, h_b = \frac{2F}{b}, h_c = \frac{2F}{c}$$

Let $f(x) = x - x^3, \forall x \geq 0, f'(x) = 1 - 3x^2, f''(x) = -6x$

$f'(x) = 0 \Rightarrow x = \frac{1}{\sqrt{3}} \Rightarrow f''\left(\frac{1}{\sqrt{3}}\right) = -\frac{6}{\sqrt{3}} < 0 \Rightarrow$ max $f(x)$ is attains at $x = \frac{1}{\sqrt{3}}$

$$f\left(\frac{1}{\sqrt{3}}\right) \geq f(x) \Rightarrow \frac{2}{3\sqrt{3}} \geq f(x)$$

$$\sum_{cyc} \frac{1}{(y+z)(1-x^2)h_a^4} = \sum_{cyc} \frac{x}{y+z} \cdot \frac{1}{x-x^3} \cdot \frac{1}{h_a^4} \geq \frac{3\sqrt{3}}{2} \sum_{cyc} \frac{x}{y+z} \cdot \frac{1}{h_a^4} =$$

$$= \frac{3\sqrt{3}}{2} \sum_{cyc} \left(\frac{x}{y+z} + 1 - 1\right) \cdot \frac{1}{h_a^4} =$$

$$= \frac{3\sqrt{3}}{2}(x+y+z) \sum_{cyc} \frac{\left(\frac{1}{h_a^2}\right)^2}{y+z} - \frac{3\sqrt{3}}{2} \sum_{cyc} \frac{1}{h_a^4} \overset{Bergstrom}{\geq}$$

$$\geq \frac{3\sqrt{3}}{4}\left(\sum_{cyc} \frac{1}{h_a^2}\right)^2 - \frac{3\sqrt{3}}{2} \sum_{cyc} \frac{1}{h_a^4} = \frac{3\sqrt{3}}{4}\left(\frac{a^2+b^2+c^2}{4F^2}\right)^2 - \frac{3\sqrt{3}}{2} \sum_{cyc} \frac{a^4}{16F^2} =$$

$$= \frac{3\sqrt{3}}{4F^2}\left\{\frac{(a^2+b^2+c^2)^2}{16F^2} - \frac{2(a^4+b^4+c^4)}{16F^2}\right\} =$$

$$= \frac{3\sqrt{3}}{4F^2} \cdot \frac{2(a^2b^2+b^2c^2+c^2a^2) - a^4 - b^4 - c^4}{16F^2} = \frac{3\sqrt{3}}{4F^2}$$

Equality holds for $a = b = c; x = y = z = \frac{\sqrt{3}}{3}$.

JP.404 If $x,y>0; z\in\left[0,\frac{\pi}{2}\right]; \frac{1}{x+\sin z}+\frac{1}{y+\sin z}\ge 1$ then:

$$\frac{1}{x}+\frac{1}{y}\ge 2\sin z$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposer

$$(x+\sin z)\left(\frac{1}{x}+\sin z\right)\overset{AM-GM}{\ge} 2\sqrt{x\cdot\sin z}\cdot 2\sqrt{\frac{\sin z}{x}}=4\sin z$$

$$\frac{1}{x}+\sin z\ge\frac{4\sin z}{x+\sin z};(1)$$

Analogous,

$$\frac{1}{y}+\sin z\ge\frac{4\sin z}{y+\sin z};(2)$$

By adding (1),(2) we get:

$$\frac{1}{x}+\frac{1}{y}+2\sin z\ge\frac{4\sin z}{x+\sin z}+\frac{4\sin z}{y+\sin z}\Leftrightarrow$$

$$\frac{1}{x}+\frac{1}{y}\ge\frac{4\sin z}{x+\sin z}+\frac{4\sin z}{y+\sin z}-2\sin z=2\sin z\left(2\left(\frac{1}{x+\sin z}+\frac{1}{y+\sin z}\right)-1\right)\ge$$

$$\ge 2\sin z(2\cdot 1-1)=2\sin z$$

Equality holds for $x=y=1; z=\frac{\pi}{2}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

If $z=0$ →we need to prove that: $\frac{1}{x}+\frac{1}{y}\ge 0$ which is true.

$$\text{If } z\neq 0\ \to\ \sin z>0\ \to\ 1\le\frac{1}{x+\sin z}+\frac{1}{y+\sin z}\overset{AM-GM}{\le}\frac{1}{2\sqrt{x.\sin z}}+\frac{1}{2\sqrt{y.\sin z}}\le$$

$$\overset{CBS}{\le}\frac{1}{2}\cdot\sqrt{\left(\frac{1}{\sin z}+\frac{1}{\sin z}\right)\left(\frac{1}{x}+\frac{1}{y}\right)}\ \to\ 1\le\frac{1}{2\sin z}\left(\frac{1}{x}+\frac{1}{y}\right).$$

Therefore,

$$\frac{1}{x}+\frac{1}{y}\ge 2\sin z.$$

Equality holds for $x = y = 1; z = \frac{\pi}{2}$.

Solution 3 by Ravi Prakash-New Delhi-India

If $z = 0$, then is nothing to prove.

Assume $0 < z \le \frac{\pi}{2} \Leftrightarrow \sin z > 0$

$$1 \le \frac{1}{x + \sin z} + \frac{1}{y + \sin z} \le \frac{1}{2\sqrt{x \cdot \sin z}} + \frac{1}{2\sqrt{y \cdot \sin z}}$$

$$\Rightarrow 2\sqrt{\sin z} \le \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} \Rightarrow 4\sin z \le \frac{1}{x} + \frac{1}{y} + \frac{1}{\sqrt{xy}} \le \frac{1}{x} + \frac{1}{y} + \frac{1}{x} + \frac{1}{y}$$

Hence,

$$2\sin z \le \frac{1}{x} + \frac{1}{y}$$

Equality holds for $x = y = 1; z = \frac{\pi}{2}$.

Solution 4 by Amrit Awasthi-India

$$0 \le z \le \frac{\pi}{2} \Rightarrow 0 \le \sin z \le 1$$

$$x + \sin z \le (1 + x) \Rightarrow \frac{1}{x + \sin z} \ge \frac{1}{1 + x}$$

Similarly,

$$\frac{1}{y + \sin z} \ge \frac{1}{1 + y}$$

Adding, we get:

$$\frac{1}{x + \sin z} + \frac{1}{y + \sin z} \ge \frac{1}{1 + x} + \frac{1}{1 + y} = \frac{x + y + 2}{(1 + x)(1 + y)}$$

Now it's given that

$$\frac{1}{x + \sin z} + \frac{1}{y + \sin z} \ge 1$$

Hence, we must have

$$\frac{(x + y + 2)}{(1 + x)(1 + y)} \ge 1 \Leftrightarrow x + y + 2 \ge 1 + x + y + xy \Rightarrow xy \le 1$$

Now, as $xy \le 1 \Rightarrow \sqrt{xy} \le 1 \Rightarrow \frac{1}{\sqrt{xy}} \ge 1; (*)$

Therefore,

$$\frac{1}{x}+\frac{1}{y}\overset{AM-GM}{\geq}\frac{2}{\sqrt{xy}}\overset{(*)}{\geq}2\overset{(**)}{\geq}2\sin z$$

$(**)\Leftrightarrow 1\geq \sin z$ true. So, $\frac{1}{x}+\frac{1}{y}\geq 2\sin z$

Equality holds for $x=y=1; z=\frac{\pi}{2}$.

JP.405 In ΔABC the following relationship holds:

$$\frac{a^3}{2b+2c-a}+\frac{b^3}{2c+2a-b}+\frac{c^3}{2a+2b-c}\geq\frac{4\sqrt{3}F}{3}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{a^3}{2(b+c)-a}=\sum_{cyc}\frac{a^4}{2ab+2ac-a^2}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a^2+b^2+c^2)^2}{4(ab+bc+ca)-(a^2+b^2+c^2)}\geq\frac{(a^2+b^2+c^2)^2}{4(a^2+b^2+c^2)-(a^2+b^2+c^2)}=$$

$$=\frac{a^2+b^2+c^2}{3}\overset{Ionescu-Weitzenbock}{\geq}\frac{4\sqrt{3}F}{3}$$

Equality holds for $a=b=c$.

Solution 2 by Daniel Văcaru-Romania

$$\sum_{cyc}\frac{a^3}{2(b+c)-a}=\sum_{cyc}\frac{a^4}{2a(b+c)-a^2}\overset{Bergstrom}{\geq}\frac{(a^2+b^2+c^2)^2}{4(ab+bc+ca)-a^2-b^2-c^2};(1)$$

We use $a^2+b^2+c^2\geq ab+bc+ca$ for both counter and denominator and we have:

$$\sum_{cyc}\frac{a^3}{2(b+c)-a}\geq\frac{(a^2+b^2+c^2)^2}{4(ab+bc+ca)-a^2-b^2-c^2}\geq$$

$$\geq\frac{(ab+bc+ca)^2}{3(ab+bc+ca)}=\frac{ab+bc+ca}{3}\geq\frac{4\sqrt{3}F}{3}$$

PROBLEMS FOR SENIORS

SP.391 Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\left(\sum_{i=1}^{k} i^2\right)^{-1}\right)^{\sum_{i=1}^{k} i^2} - ne\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} \overset{L'H}{=} \lim_{x\to 0}\left(1+\frac{1}{x}\right)^{\frac{1}{x}}\left(\frac{\log(1+x)}{x}\right)' = e\cdot\lim_{x\to 0}\frac{\frac{x}{x+1}-\log(x+1)}{x^2} =$$

$$= e\cdot\lim_{x\to 0}\frac{x-(x+1)\log(x+1)}{x^2(x+1)} \overset{L'H}{=} e\cdot\lim_{x\to 0}\frac{1-\log(x+1)-1}{3x^2+2x}$$

$$= e\cdot\lim_{x\to 0}\frac{-1}{(x+1)(6x+2)} = -\frac{e}{2}$$

Because: $\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} = -\frac{e}{2}$ then $\forall\varepsilon>0, \exists\delta(\varepsilon)>0$ such that $|x|<\delta(\varepsilon)$

$$\Rightarrow \left|\frac{(1+x)^{\frac{1}{x}}-e}{x}-\left(\frac{e}{2}\right)\right| < \varepsilon \Leftrightarrow -\varepsilon < \frac{(1+x)^{\frac{1}{x}}-e}{x}-\left(-\frac{e}{2}\right) < \varepsilon$$

$$\Leftrightarrow -\varepsilon-\frac{e}{2} < \frac{(1+x)^{\frac{1}{x}}-e}{x} < \varepsilon-\frac{e}{2}$$

$$0 < \frac{1}{1^2+2^2+\cdots+n^2} < \cdots < \frac{1}{1^2+2^2} < \frac{1}{1^2} < \delta(\varepsilon)$$

We have the following relationship:

$$-\varepsilon-\frac{e}{2} < \frac{\left(1+\frac{1}{1^2}\right)^{1^2}-e}{\frac{1}{1^2}} < \varepsilon-\frac{e}{2}, -\varepsilon-\frac{e}{2} < \frac{\left(1+\frac{1}{1^2+2^2}\right)^{1^2+2^2}-e}{\frac{1}{1^2+2^2}} < \varepsilon-\frac{e}{2}$$

$$\vdots$$

$$-\varepsilon-\frac{e}{2} < \frac{\left(1+\frac{1}{1^2+2^2+\cdots+n^2}\right)^{1^2+2^2+\cdots+n^2}-e}{\frac{1}{1^2+2^2+\cdots+n^2}} < \varepsilon-\frac{e}{2}$$

Summing these up relations, it follows that:

$$\left(-\varepsilon-\frac{e}{2}\right)\sum_{k=1}^{n}\frac{1}{1^2+2^2+\cdots+k^2}<\sum_{k=1}^{n}\left(1+\left(\sum_{i=1}^{k}i^2\right)^{-1}\right)^{\sum_{i=1}^{k}i^2}-ne<$$

$$<\left(\varepsilon-\frac{e}{2}\right)\sum_{k=1}^{n}\frac{1}{1^2+2^2+\cdots+k^2}$$

$$\Leftrightarrow\left(-\varepsilon-\frac{e}{2}\right)\cdot a_n<\sum_{k=1}^{n}\left(1+\left(\sum_{i=1}^{k}i^2\right)^{-1}\right)^{\sum_{i=1}^{k}i^2}-ne<\left(\varepsilon-\frac{e}{2}\right)\cdot a_n$$

Where, $a_n=\frac{1}{1^2}+\frac{1}{1^2+2^2}+\cdots+\frac{1}{1^2+2^2+\cdots+n^2};n\geq 1.$

Now, we get:

$$a_n=\sum_{k=1}^{n}\frac{6}{k(k+1)(2k+1)}=6\sum_{k=1}^{n}\left(\frac{1}{k}+\frac{1}{k+1}-\frac{4}{2k+1}\right)=$$

$$=6\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k}\right)-24\sum_{k=1}^{n}\left(\frac{1}{2k+1}-\frac{1}{2k}\right)=$$

$$=6\left(\frac{1}{n+1}-1\right)-24(H_{2n+1}-H_{2n}-1).$$

$$\lim_{n\to\infty}(H_n-\log n)=\gamma$$

$$H_{2n+1}-H_{2n}=(H_{2n+1}-\log(2n+1))-(H_{2n}-\log 2n)+\log\frac{2n+1}{2n}$$

$$\lim_{n\to\infty}(H_{2n+1}-H_{2n})=\log 2\ \text{ and }\lim_{n\to\infty}a_n=6(3-4\log 2)$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\left(\sum_{i=1}^{k}i^2\right)^{-1}\right)^{\sum_{i=1}^{k}i^2}-ne\right)=-\frac{e}{2}\cdot 6(3-4\log 2)=$$

$$=3e(4\log 2-3)$$

Solution 2 by Adrian Popa-Romania

$$\sum_{i=1}^{k}i^2=\frac{k(k+1)(2k+1)}{6}$$

$$\sum_{k=1}^{n}\left(1+\frac{6}{k(k+1)(2k+1)}\right)^{\frac{k(k+1)(2k+1)}{6}}=$$

$$= (1+1)^1 + \left(1+\frac{1}{5}\right)^5 + \left(1+\frac{1}{14}\right)^{14} + \dots + \left(1+\frac{6}{n(n+1)(2n+1)}\right)^{\frac{n(n+1)(2n+1)}{6}} \geq$$

$$\overset{Bernoulli}{\geq} 2n$$

$$e_n = \left(1+\frac{1}{n}\right)^n \text{ increasing, } e_n \to e$$

$$(1+1)^1 < e, \left(1+\frac{1}{5}\right)^5 < e, \left(1+\frac{1}{14}\right)^{14} < e, \dots,$$

$$\left(1+\frac{6}{n(n+1)(2n+1)}\right)^{\frac{n(n+1)(2n+1)}{6}} < e$$

$$\left(-\varepsilon-\frac{e}{2}\right)\sum_{k=1}^{n}\frac{1}{1^2+2^2+\dots+k^2} < \sum_{k=1}^{n}\left(1+\left(\sum_{i=1}^{k} i^2\right)^{-1}\right)^{\sum_{i=1}^{k} i^2} - ne <$$

$$< \left(\varepsilon-\frac{e}{2}\right)\sum_{k=1}^{n}\frac{1}{1^2+2^2+\dots+k^2}$$

$$\Leftrightarrow \left(-\varepsilon-\frac{e}{2}\right)\cdot a_n < \sum_{k=1}^{n}\left(1+\left(\sum_{i=1}^{k} i^2\right)^{-1}\right)^{\sum_{i=1}^{k} i^2} - ne < \left(\varepsilon-\frac{e}{2}\right)\cdot a_n$$

Where, $a_n = \frac{1}{1^2} + \frac{1}{1^2+2^2} + \dots + \frac{1}{1^2+2^2+\dots+n^2}; n \geq 1.$

Now, we get:

$$a_n = \sum_{k=1}^{n}\frac{6}{k(k+1)(2k+1)} = 6\sum_{k=1}^{n}\left(\frac{1}{k}+\frac{1}{k+1}-\frac{4}{2k+1}\right) =$$

$$= 6\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k}\right) - 24\sum_{k=1}^{n}\left(\frac{1}{2k+1}-\frac{1}{2k}\right) =$$

$$= 6\left(\frac{1}{n+1}-1\right) - 24(H_{2n+1}-H_{2n}-1).$$

$$\lim_{n\to\infty}(H_n - \log n) = \gamma$$

$$H_{2n+1} - H_{2n} = (H_{2n+1} - \log(2n+1)) - (H_{2n} - \log 2n) + \log\frac{2n+1}{2n}$$

$$\lim_{n\to\infty}(H_{2n+1} - H_{2n}) = \log 2 \text{ and } \lim_{n\to\infty} a_n = 6(3 - 4\log 2)$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\left(\sum_{i=1}^{k} i^2\right)^{-1}\right)^{\sum_{i=1}^{k} i^2} - ne\right) = -\frac{e}{2}\cdot 6(3-4\log 2) =$$

$$= 3e(4\log 2 - 3)$$

SP.392 Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ be sequences of real numbers such that

$$x_1 = 1, y_1 = a^2 + a + 1,$$

$$a > 1, \forall n \in \mathbb{N}^* \text{ and } x_{n+1} = \frac{a^2 + ax_n + x_ny_n}{y_n}, y_{n+1} = \frac{a^2 + ay_n + x_ny_n}{x_n}. \text{ Find:}$$

$$\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n.$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$x_{n+1} = \frac{a^2 + ax_n + x_ny_n}{y_n} = \frac{a(a + x_n) + (a + x_n)y_n - ay_n}{y_n} = \frac{(a + x_n)(a + y_n)}{y_n} - a$$

$$\Rightarrow a + x_{n+1} = \frac{(a + x_n)(a + y_n)}{y_n} \Rightarrow \frac{1}{a + x_{n+1}} = \frac{y_n}{(a + x_n)(a + y_n)}$$

$$\text{Similarly, } \frac{1}{a + y_{n+1}} = \frac{x_n}{(a + x_n)(a + y_n)} \text{ then:}$$

$$\frac{1}{a + x_{n+1}} - \frac{1}{a + y_{n+1}} = \frac{y_n - x_n}{(a + x_n)(a + y_n)} = \frac{(y_n + a) - (x_n + a)}{(a + x_n)(a + y_n)} = \frac{1}{a + x_n} - \frac{1}{a + y_n}$$

Namely,

$$\frac{1}{a + x_{n+1}} - \frac{1}{a + y_{n+1}} = \frac{1}{a + x_n} - \frac{1}{a + y_n} = \cdots = \frac{1}{a + x_1} - \frac{1}{a + y_1} =$$

$$= \frac{1}{a + 1} - \frac{1}{(a + 1)^2} = \frac{a}{(a + 1)^2}$$

$$\frac{1}{a + x_{n+1}} = \frac{a}{(a + 1)^2} + \frac{1}{a + y_{n+1}} > \frac{a}{(a + 1)^2}; (x_n > 0, y_n > 0, \forall n \in \mathbb{N}^* - \text{induction})$$

$$\Rightarrow a + x_n < \frac{(a + 1)^2}{a} \Rightarrow x_n < \frac{(a + 1)^2}{a} - a = 2 + \frac{1}{a} < 3, \forall a > 1$$

$$\Rightarrow x_n < 3, \forall n \in \mathbb{N}^*; \Rightarrow (x_n)_{n\geq 1} - \text{is bounded}; (1)$$

$$x_{n+1} - x_n = \frac{a^2 + ax_n}{y_n} > 0 \text{ and } y_{n+1} - y_n = \frac{a^2 + ay_n}{x_n} > 0 \Rightarrow (x_n)_{n\geq 1}, (y_n)_{n\geq 1} \nearrow; \ (2).$$

From $(1), (2)$ we obtain that $(x_n)_{n\geq 1}$ is convergent.

Let $l = \lim_{n\to\infty} x_n$, $l \in (1, 3)$ and suppose that $l' = \lim_{n\to\infty} y_n$; $l' \in \mathbb{R}$ hence,

$$l=\frac{a^2+al+ll'}{l'}\Leftrightarrow ll'=a^2+al+ll'\Leftrightarrow l=-a\text{ contradiction! }\Rightarrow \lim_{n\to\infty} y_n=\infty.$$

$$\frac{1}{a+x_n}-\frac{1}{a+y_n}=\frac{a}{(a+1)^2};(y_n\to+\infty)\Rightarrow\frac{1}{a+x_n}=\frac{a}{(a+1)^2}$$

$$\Leftrightarrow a+l=\frac{(a+1)^2}{a}\Leftrightarrow l=\frac{(a+1)^2}{a}-a=\frac{2a+1}{a}\Rightarrow \lim_{n\to\infty} x_n=2+\frac{1}{a}.$$

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$x_{n+1}=\frac{a^2+ax_n+y_nx_n}{y_n},y_{n+1}=\frac{a^2+ay_n+y_nx_n}{x_n}$$

$$x_2=\frac{2a62+2a+1}{a^2+a+1}<y_2$$

$$x_{n+1}-y_{n+1}=\frac{a^2+ax_n+y_nx_n}{y_n}-\frac{a^2+ay_n+y_nx_n}{x_n}=$$

$$=\frac{(x_n-y_n)(a^2+ay_n+ax_n+x_ny_n)}{x_ny_n}=\frac{(x_n-y_n)(a+x_n)(a+y_n)}{x_ny_n}$$

By mathematical induction $y_n>x_n,\forall n\in\mathbb{N}$

Now, $x_1,x_2>0$. Suppose $x_n>0,y_n>0$ then

$$x_{n+1}=\frac{a^2+ax_n+y_nx_n}{y_n}=\frac{a^2+ax_n}{y_n}+x_n>0$$

$$y_{n+1}=\frac{a^2+ay_n+y_nx_n}{x_n}=\frac{a^2+ay_n}{x_n}+y_n>0,\forall n\in\mathbb{N}$$

$$\Rightarrow a+x_n<\frac{(a+1)^2}{a}\Rightarrow x_n<\frac{(a+1)^2}{a}-a=2+\frac{1}{a}<3,\forall a>1$$

$$\Rightarrow x_n<3,\forall n\in\mathbb{N}^*;\Rightarrow (x_n)_{n\ge1}-\text{is bounded };(1)$$

$$x_{n+1}-x_n=\frac{a^2+ax_n}{y_n}>0\text{ and }y_{n+1}-y_n=\frac{a^2+ay_n}{x_n}>0\Rightarrow(x_n)_{n\ge1},(y_n)_{n\ge1}\nearrow;\quad(2).$$

From $(1),(2)$ we obtain that $(x_n)_{n\ge1}$ is convergent.

Let $x=\lim_{n\to\infty}x_n$, $x\in(1,3)$ and suppose that $y=\lim_{n\to\infty}y_n$; $y\in\mathbb{R}$ hence,

$$x=\frac{a^2+ax+xy}{y}\Leftrightarrow xy=a^2+ax+xy\Leftrightarrow x=-a\text{ contradiction! }\Rightarrow\lim_{n\to\infty}y_n=\infty.$$

$$\frac{1}{a+x_n}-\frac{1}{a+y_n}=\frac{a}{(a+1)^2};(y_n\to+\infty)\Rightarrow\frac{1}{a+x_n}=\frac{a}{(a+1)^2}$$

$$\Leftrightarrow a+x=\frac{(a+1)^2}{a}\Leftrightarrow x=\frac{(a+1)^2}{a}-a=\frac{2a+1}{a}\Rightarrow \lim_{n\to\infty}x_n=2+\frac{1}{a}.$$

Solution 3 by Ruxandra Daniela Tonilă-Romania

$$x_{n+1}=\frac{a^2+ax_n+x_ny_n}{y_n}=\frac{a(a+x_n)+(a+x_n)y_n-ay_n}{y_n}=\frac{(a+x_n)(a+y_n)}{y_n}-a$$

$$\Rightarrow a+x_{n+1}=\frac{(a+x_n)(a+y_n)}{y_n}\Rightarrow \frac{1}{a+x_{n+1}}=\frac{y_n}{(a+x_n)(a+y_n)}$$

Similarly, $\frac{1}{a+y_{n+1}}=\frac{x_n}{(a+x_n)(a+y_n)}$ **then:**

$$\frac{1}{a+x_{n+1}}-\frac{1}{a+y_{n+1}}=\frac{y_n-x_n}{(a+x_n)(a+y_n)}=\frac{(y_n+a)-(x_n+a)}{(a+x_n)(a+y_n)}=\frac{1}{a+x_n}-\frac{1}{a+y_n}$$

Namely,

$$\frac{1}{a+x_{n+1}}-\frac{1}{a+y_{n+1}}=\frac{1}{a+x_n}-\frac{1}{a+y_n}=\cdots=\frac{1}{a+x_1}-\frac{1}{a+y_1}=$$

$$=\frac{1}{a+1}-\frac{1}{(a+1)^2}=\frac{a}{(a+1)^2}$$

$$\frac{1}{a+x_{n+1}}=\frac{a}{(a+1)^2}+\frac{1}{a+y_{n+1}}>\frac{a}{(a+1)^2};(x_n>0,y_n>0,\forall n\in\mathbb{N}^*-\text{induction})$$

$$\Rightarrow a+x_n<\frac{(a+1)^2}{a}\Rightarrow x_n<\frac{(a+1)^2}{a}-a=2+\frac{1}{a}<3,\forall a>1$$

$$\Rightarrow x_n<3,\forall n\in\mathbb{N}^*;\Rightarrow (x_n)_{n\ge1}-\text{is bounded};(1)$$

$$x_{n+1}-x_n=\frac{a^2+ax_n}{y_n}>0 \text{ and } y_{n+1}-y_n=\frac{a^2+ay_n}{x_n}>0\Rightarrow (x_n)_{n\ge1},(y_n)_{n\ge1}\nearrow;\ \ (2).$$

From $(1),(2)$ we obtain that $(x_n)_{n\ge1}$ is convergent.

Let $x=\lim_{n\to\infty}x_n$, $x\in(1,3)$ and suppose that $y=\lim_{n\to\infty}y_n$; $y\in\mathbb{R}$ hence,

$$x=\frac{a^2+ax+xy}{y}\Leftrightarrow xy=a^2+ax+xy\Leftrightarrow x=-a \text{ contradiction!}\Rightarrow \lim_{n\to\infty}y_n=\infty.$$

$$\frac{1}{a+x_n}-\frac{1}{a+y_n}=\frac{a}{(a+1)^2};(y_n\to+\infty)\Rightarrow\frac{1}{a+x_n}=\frac{a}{(a+1)^2}$$

$$\Leftrightarrow a+x=\frac{(a+1)^2}{a}\Leftrightarrow x=\frac{(a+1)^2}{a}-a=\frac{2a+1}{a}\Rightarrow \lim_{n\to\infty}x_n=2+\frac{1}{a}.$$

SP.393 If $S_{n,k}=e^{\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n+k\cdot n}}, n,k\in\mathbb{N}, k\geq 1$ then find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\frac{1}{n}\right)^{\frac{k}{S_{n,k}}}-n\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

We want to prove the following inequality:

$$(*): \log(1+x)<x<-\log(1-x), \forall x\in(0,1)$$

Let $f:[0,1]\to\mathbb{R}, f(x)=x-\log(1+x)$ with $f'(x)=\frac{x}{1+x}>0, \forall x\in(0,1)$ then,

f –increasing function on $(0,1)$. How, $f(0)=0$ then, $x-\log(1+x)>0, \forall x\in(0,1)$.

Now, let be the function $g:[0,1)\to\mathbb{R}, g(x)=x+\log(1-x)$ with $g'(x)=-\frac{x}{1-x}<0$,

$\forall x\in(0,1)$ then, g –decreasing on $(0,1)$ then, $g(x)<g(0)=0, \forall x\in(0,1)$.

$$\text{Let: } x_n=\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n+k\cdot n}$$

Using the inequality $(*)$, it follows that:

$$\log\left(1+\frac{1}{n}\right)<\frac{1}{n}<-\log\left(1-\frac{1}{n}\right)$$

$$\log\left(1+\frac{1}{n+1}\right)<\frac{1}{n+1}<-\log\left(1-\frac{1}{n+1}\right)$$

$$\vdots$$

$$\log\left(1+\frac{1}{n+k\cdot n}\right)<\frac{1}{n+k\cdot n}<-\log\left(1-\frac{1}{n+kn}\right)$$

Adding these relations, it follows that:

$$\log\left(\prod_{i=0}^{kn}\left(1+\frac{1}{n+i}\right)\right)<x_n<-\log\left(\prod_{i=0}^{kn}\left(1-\frac{1}{n+i}\right)\right)$$

$$\text{But: }\prod_{i=0}^{kn}\left(1+\frac{1}{n+i}\right)=k+1+\frac{1}{n}\ \text{ and }\ \prod_{i=0}^{kn}\left(1-\frac{1}{n+i}\right)=\frac{n-1}{n+kn}\ \text{thus,}$$

$$\log\left(k+1+\frac{1}{n}\right)<x_n<\log\left(k+1+\frac{k+1}{n-1}\right)\ \text{then,}$$

$$\lim_{n\to\infty} x_n = \log(k+1) \Rightarrow \lim_{n\to\infty} S_n = e^{\lim_{n\to\infty} x_n} = k+1$$

$$\Rightarrow \Omega = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{\frac{1}{2}} + \left(1+\frac{1}{n}\right)^{\frac{2}{3}} + \cdots + \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} - n\right)$$

Let: $y_n = \sum_{k=1}^{n}\left(1+\frac{1}{n}\right)^{\frac{k}{S_n}} - n; n, k \geq 1 \Rightarrow y_n = \sum_{k=1}^{n}\left(1+\frac{1}{n}\right)^{\frac{k}{k+1}} - n; n \geq 1.$

We have:

$$y_n \leq n\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} - n = \frac{\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} - 1}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} - 1}{\frac{1}{n}} = \lim_{x\to 0} \frac{(1+x)^{\frac{1}{1+x}} - 1}{x} \overset{L'H}{=} 1; (1)$$

From Bernoulli inequality, $(1+t)^{\frac{1}{m}} \leq 1 + \frac{t}{m}$ for $t = \frac{1}{n}$ and $m = k+1$, we get:

$$\left(1+\frac{1}{n}\right)^{\frac{1}{k+1}} \leq 1 + \frac{1}{n(k+1)}$$

Namely,

$$\left(1+\frac{1}{n}\right)^{\frac{k}{k+1}} = \frac{1+\frac{1}{n}}{\left(1+\frac{1}{n}\right)^{\frac{1}{k+1}}} \geq \frac{1+\frac{1}{n}}{1+\frac{1}{n(k+1)}} = \frac{n(k+1)+k+1}{n(k+1)+1}$$

It follows that:

$$y_n \geq \sum_{k=1}^{n}\left(\frac{n(k+1)+k+1}{n(k+1)+1} - 1\right) = \sum_{k=1}^{n}\frac{k}{n(k+1)+1} \geq \sum_{k=1}^{n}\frac{k-1}{nk} =$$

$$\sum_{k=1}^{n}\left(\frac{1}{n} - \frac{1}{nk}\right) = 1 - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} = 1 - \frac{H_n}{n} \to 1; (2)$$

From $(1), (2)$ it follows that: $\lim_{n\to\infty} a_n = 1$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\frac{1}{n}\right)^{\frac{k}{S_{n,k}}}-n\right)=1$$

Solution 2 by Adrian Popa-Romania

$$S_{n,k}=e^{\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}\right)+\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\cdots+\frac{1}{3n}\right)+\cdots+\left(\frac{1}{kn}+\frac{1}{kn+1}+\cdots+\frac{1}{kn+n}\right)}$$

$$\lim_{n\to\infty}\left(\frac{1}{kn}+\frac{1}{kn+1}+\cdots+\frac{1}{kn+n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\frac{1}{k+\frac{i}{n}}=\int_0^1\frac{1}{k+x}dx=$$

$$=\log(k+x)|_0^1=\log\left(\frac{k+1}{k}\right)$$

Hence,

$$\lim_{n\to\infty}S_{n,k}=e^{\log 2+\log\frac{3}{2}+\cdots+\log\frac{k-1}{k}}=e^{\log\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\ldots\cdot\frac{k+1}{k}\right)}=k+1$$

$$\Rightarrow\Omega=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{\frac{1}{2}}+\left(1+\frac{1}{n}\right)^{\frac{2}{3}}+\cdots+\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}-n\right)$$

We have:

$$\sum_{k=1}^{n}\left(1+\frac{1}{n}\right)^{\frac{k}{S_n}}-n\le n\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}-n=\frac{\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}-1}{\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}-1}{\frac{1}{n}}=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{1+x}}-1}{x}\overset{L'H}{=}1;(1)$$

From Bernoulli inequality, $(1+t)^{\frac{1}{m}}\le 1+\frac{t}{m}$ for $t=\frac{1}{n}$ and $m=k+1$, we get:

$$\left(1+\frac{1}{n}\right)^{\frac{1}{k+1}}\le 1+\frac{1}{n(k+1)}$$

Namely,

$$\left(1+\frac{1}{n}\right)^{\frac{k}{k+1}}=\frac{1+\frac{1}{n}}{\left(1+\frac{1}{n}\right)^{\frac{1}{k+1}}}\ge\frac{1+\frac{1}{n}}{1+\frac{1}{n(k+1)}}=\frac{n(k+1)+k+1}{n(k+1)+1}$$

It follows that:

$$\sum_{k=1}^{n}\left(1+\frac{1}{n}\right)^{\frac{k}{S_n}}-n\ge\sum_{k=1}^{n}\left(\frac{n(k+1)+k+1}{n(k+1)+1}-1\right)=\sum_{k=1}^{n}\frac{k}{n(k+1)+1}\ge\sum_{k=1}^{n}\frac{k-1}{nk}=$$

$$\sum_{k=1}^{n}\left(\frac{1}{n}-\frac{1}{nk}\right)=1-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}=1-\frac{H_n}{n}\to 1;\ (2)$$

From $(1),(2)$ it follows that: $\lim_{n\to\infty}a_n=1$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\frac{1}{n}\right)^{\frac{k}{S_{n,k}}}-n\right)=1$$

SP.394 If $(F_n)_{n\ge0}, F_0=F_1=1, F_{n+2}=F_{n+1}+F_n, \forall n\in\mathbb{N}$ and $a,b,c\in\mathbb{R}_+^*$ with $a+b+c\le 24$, then prove:

$$\frac{F_n}{\sqrt{F_n^2+aF_{n+1}F_{n+2}}}+\frac{F_{n+1}}{\sqrt{F_{n+1}^2+bF_{n+2}F_n}}+\frac{F_{n+2}}{\sqrt{F_{n+2}^2+cF_nF_{n+1}}}\ge 1,\forall n\in\mathbb{N}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

Let: $U_n=\frac{F_n}{\sqrt{F_n^2+aF_{n+1}F_{n+2}}}+\frac{F_{n+1}}{\sqrt{F_{n+1}^2+bF_{n+2}F_n}}+\frac{F_{n+2}}{\sqrt{F_{n+2}^2+cF_nF_{n+1}}}$

$$U_n=\sum_{cyc}\frac{F_n}{\sqrt{F_n^2+aF_{n+1}F_{n+2}}}=\sum_{cyc}\frac{F_n^2}{\sqrt{F_n}\cdot\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}}=$$

$$=\sum_{cyc}\frac{F_n^2}{\sqrt{F_n}\cdot\sqrt{F_n^3+ap_n}},\forall n\in\mathbb{N},\text{where } p_n=F_nF_{n+1}F_{n+2},\forall n\in\mathbb{N};(1)$$

We denote: $v_n=\sqrt{F_n}\cdot\sqrt{F_n^3+ap_n}$ and $V_n=v_n+v_{n+1}+v_{n+2}=$

$$=\sqrt{F_n}\cdot\sqrt{F_n^3+ap_n}+\sqrt{F_{n+1}}\cdot\sqrt{F_{n+1}^3+ap_{n+1}}+\sqrt{F_{n+2}}\cdot\sqrt{F_{n+2}^3+ap_{n+2}},\forall n\in\mathbb{N}.$$

We have: $V_n^2=\left(\sum\sqrt{F_n}\cdot\sqrt{F_n^3+ap_n}\right)^2\overset{CBS}{\leq}s_n\left(F_n^3+F_{n+1}^3+F_{n+2}^3+(a+b+c)p_n\right),\forall n\in$ $\mathbb{N}$, where we denote $s_n=F_n+F_{n+1}+F_{n+2},\forall n\in\mathbb{N}$.

Since $(x+y+z)^3\geq x^3+y^3+z^3+24xyz,\forall x,y,z\in\mathbb{R}_+^*$ and

$$V_n^2\leq s_n\left(F_n^3+F_{n+1}^3+F_{n+2}^3+(a+b+c)p_n\right)$$
$$\leq s_n\left(F_n^3+F_{n+1}^3+F_{n+2}^3+24F_nF_{n+1}F_{n+2}\right),$$

then $V_n^2\leq s_n\cdot s_n^3=s_n^4,\forall n\in\mathbb{N}\Leftrightarrow V_n\leq s_n^2,\forall n\in\mathbb{N};(2)$

Applying Bergstrom's inequality, and from $(1),(2)$ it follows that:

$$U_n\geq\frac{s_n^2}{V_n}\geq\frac{s_n^2}{s_n^2}=1,\forall n\in\mathbb{N}.$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\frac{F_n}{\sqrt{F_n^2+aF_{n+1}F_{n+2}}}+\frac{F_{n+1}}{\sqrt{F_{n+1}^2+bF_nF_{n+2}}}+\frac{F_{n+2}}{\sqrt{F_{n+2}^2+cF_nF_{n+1}}}=$$
$$=\frac{F_n^{\frac{3}{2}}}{\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}}+\frac{F_{n+1}^{\frac{3}{2}}}{\sqrt{F_{n+1}^3+bF_nF_{n+1}F_{n+2}}}+\frac{F_{n+2}^{\frac{3}{2}}}{\sqrt{F_{n+2}^3+cF_nF_{n+1}F_{n+2}}}\overset{Radon}{\geq}$$
$$=\frac{\sqrt{(F_n+F_{n+1}+F_{n+2})^3}}{\sqrt{F_n^3+F_{n+1}^3+F_{n+2}^3+(a+b+c)F_nF_{n+1}F_{n+2}}}=$$
$$=\frac{\sqrt{F_n^3+F_{n+1}^3+F_{n+2}^3+3(F_n+F_{n+1})(F_{n+1}+F_{n+2})(F_{n+2}+F_n)}}{\sqrt{F_n^3+F_{n+1}^3+F_{n+2}^3+(a+b+c)F_nF_{n+1}F_{n+2}}}\overset{AM-GM}{\geq}$$
$$\geq\frac{\sqrt{F_n^3+F_{n+1}^3+F_{n+2}^3+24F_nF_{n+1}F_{n+2}}}{\sqrt{F_n^3+F_{n+1}^3+F_{n+2}^3+(a+b+c)F_nF_{n+1}F_{n+2}}}\geq 1$$

Since $a+b+c\leq 24$.

Solution 3 by Amrit Awasthi-India

$$\Omega=\sum_{cyc}\frac{F_n}{\sqrt{F_n^2+aF_{n+1}F_{n+2}}}=\sum_{cyc}\frac{F_n\sqrt{F_n}}{\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}}=$$

$$=\sum_{cyc}\frac{F_n^2}{\sqrt{F_n}\cdot\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}}\overset{Bergstrom}{\geq}\frac{(F_n+F_{n+1}+F_{n+2})^2}{\sum\sqrt{F_n}\cdot\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}}=$$

$$=\frac{4F_{n+2}^2}{\sqrt{F_n}\cdot\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}};(*)$$

Now apply Cauchy-Schwarz in the term in denominator, we get

$$\left(\sum\sqrt{F_n}\cdot\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}\right)^2\leq\left(\sum F_n\right)\left(\sum(F_n^3+aF_nF_{n+1}F_{n+2})\right)$$

$$\left(\sum\sqrt{F_n}\cdot\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}\right)^2$$

$$\leq(\sum F_n)(F_n^3+F_{n+1}^3+F_{n+2}^3+(a+b+c)F_nF_{n+1}F_{n+2}$$

$$\leq\left(\sum F_n\right)(F_n^3+F_{n+1}^3+F_{n+2}^3+24F_nF_{n+1}F_{n+2});(a+b+c=24)$$

$$\leq\left(\sum F_n\right)\left(\sum F_n\right)^3=\left(\sum F_n\right)^4$$

$$\because (x+y+z)^3\geq x^3+y^3+z^3+24xyz$$

$$\frac{1}{\sum\sqrt{F_n}\cdot\sqrt{F_n^3+aF_nF_{n+1}F_{n+2}}}\geq\frac{1}{(\sum F_n)^2}=\frac{1}{4F_{n+2}^2}$$

Substitute in $(*)$ we get $\Omega\geq\frac{4F_{n+2}^2}{4F_{n+2}^2}=1$.Therefore,

$$\sum_{cyc}\frac{F_n}{\sqrt{F_n^2+aF_{n+1}F_{n+2}}}\geq 1$$

SP.395 $A\in M_2(\mathbb{Q})$ **such that** $\det(A^2-2I_2)=0$. **Find:**

$$\Omega=\det(A^2-3A+3I_2)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

Let $f(x)=\det(A+xI_2)=\det A+a_1x+x^2, a_1\in\mathbb{Q};(1)$

$$\det(A^2-2I_2)=0\Leftrightarrow\det\left(A+\sqrt{2}I_2\right)\cdot\det\left(A-\sqrt{2}I_2\right)=0\Leftrightarrow$$

$$\det\left(A+\sqrt{2}I_2\right)=0\text{ or}\det\left(A-\sqrt{2}I_2\right)=0\Rightarrow f\left(\sqrt{2}\right)=0\text{ or }f\left(-\sqrt{2}\right)=0;(2)$$

From $(1),(2)$ it follows that:

$$\det A\pm a_1\sqrt{2}+2=0\Rightarrow a_1=0\Rightarrow\det A=-2\Rightarrow f(x)=\det(A+xI_2)=x^2-2;(3)$$

$$\Omega = \det(A^2 - 3A + 3I_2) = \det(A + x_1I_2)(A + x_2I_2) =$$

$= \det(A^2 + (x_1 + x_2)A + x_1x_2I_2) = \det(A^2 - 3A + 3I_2)$; (4),where$x_{1,2}$ – are roots of the equation $x^2 + 3x + 3 = 0$ and $x_1 + x_2 = -3, x_1x_2 = 3$.

From $(3),(4)$ we have:

$$\Omega = \det(A^2 - 3A + 3I_2) = f(x_1) \cdot f(x_2) = (x_1^2 - 2)(x_2^2 - 2) =$$
$$= (x_1x_2)^2 - 2(x_1^2 + x_2^2) + 4 = (x_1x_2)^2 - 2[(x_1 + x_2)^2 - 2x_1x_2] + 4 =$$
$$= 9 - 2(9 - 6) + 4 = 7.$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $a,b,c,d \in \mathbb{Q}, \det(A^2 - 2I_2) = 0$

$$\Rightarrow \det(A - \sqrt{2}I_2)(A + \sqrt{2}I_2) = 0 \Rightarrow \det(A - \sqrt{2}I_2) \cdot \det(A + \sqrt{2}I_2) = 0$$
$$\Rightarrow \det(A - \sqrt{2}I_2) = 0 \text{ or } \det(A + \sqrt{2}I_2) = 0$$

Assume that $\det(A - \sqrt{2}I_2) = 0 \Rightarrow \begin{vmatrix} a - \sqrt{2} & b \\ c & d - \sqrt{2} \end{vmatrix} = 0$

$$\Rightarrow (a - \sqrt{2})(d - \sqrt{2}) - bc = 0 \Rightarrow (ad - bc + 2) - \sqrt{2}(a + d) = 0$$

As $a,b,c,d \in \mathbb{Q}, ad - bc + 2 = 0$ and $a + d = 0$

$$\because ad - bc = -2$$

By the Cauchy-Hamilton Theorem, we have

$$A^2 - (a + d)A + (ad - bc)I_2 = O_2 \Rightarrow A^2 - 2I_2 = O_2 \text{ or } A^2 = 2I_2$$
$$\because A^2 - 3A + 3I_2 = 2I_2 - 3A + 3I_2 = \begin{pmatrix} 5 - 3a & -3b \\ -3c & 5 - 3d \end{pmatrix}$$
$$\det(A^2 - 3A + 3I_2) = (5 - 3a)(5 - 3d) - 9bc =$$
$$= 25 - 15(a + d) + 9(ad - bc) = 7$$

Solution 3 by Ruxandra Daniela Tonilă-Romania

$$\det(A^2 - 2I_2) = \det(A - \sqrt{2}I_2)(A + \sqrt{2}I_2) = \det(A - \sqrt{2}I_2) \cdot \det(A + \sqrt{2}I_2) = 0$$
$$\Rightarrow \det(A - \sqrt{2}I_2) = 0 \text{ or } \det(A + \sqrt{2}I_2) = 0$$

Let λ_1, λ_2 be the eigenvalues of $A \Rightarrow \lambda_1 = \sqrt{2}$ or $\lambda_2 = -\sqrt{2}$

$$A \in M_2(\mathbb{Q}) \Leftrightarrow \begin{cases} Tr(A) = \lambda_1 + \lambda_2 \in \mathbb{Q} \\ \det(A) = \lambda_1 \cdot \lambda_2 \in \mathbb{Q} \Leftrightarrow \lambda_2 = -\lambda_1. \\ \lambda_1 \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

So, $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2} \Rightarrow Tr(A) = 0, \det(A) = -2$

$$P_A(x) = x^2 - Tr(A) \cdot x + \det(A) = x^2 - 2$$

Let be the equation $x^2 - 3x + 3 = 0, \Delta = -3, x_{1,2} = \frac{3 \pm i\sqrt{3}}{2}$

$$\det(A^2 - 3A + 3I_2) = \det(A - x_1 I_2) \cdot \det(A - x_2 I_2) \Leftrightarrow$$

$$\det(A^2 - 3A + 3I_2) = P_A(x_1) \cdot P_A(x_2) = \left(\left(\frac{3 + i\sqrt{3}}{2}\right)^2 - 2\right)\left(\left(\frac{3 - i\sqrt{3}}{2}\right)^2 - 2\right) =$$

$$= \left(\frac{6 + 6i\sqrt{2}}{4} - 2\right)\left(\frac{6 - 6i\sqrt{2}}{4} - 2\right) = \frac{-2 + 6i\sqrt{3}}{4} \cdot \frac{-2 - 6i\sqrt{3}}{4} = 7$$

SP.396 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x) - f(ax) + f(a^2x) - f(a^3x) + f(a^4x) = x, \forall x \in \mathbb{R}, a \in (0,1)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$x \to ax \Rightarrow f(a) - f(a^2x) + f(a^3x) - f(a^4x) + f(a^5x) = ax$$

$$\Rightarrow f(x) + f(a^5x) = (1 + a)x$$

$$x \to a^5x \Rightarrow f(a^5x) + f(a^{10}x) = (1 + a)a^5x \overset{(-)}{\Rightarrow} f(x) - f(a^{10}x) = (1 + a)(1 - a^5)x$$

$$x \to a^{10}x \Rightarrow f(a^{10}x) - f(a^{20}x) = (1 + a)(1 - a^5)a^{10}x$$

$$\vdots$$

$$x \to a^{10}x \Rightarrow f(a^{10(n-1)}x) - f(a^{10n}x) = (1 + a)(1 - a^5)a^{10(n-1)}x \overset{(+)}{\Rightarrow}$$

$$f(x) - f(a^{10n}x) = (1 + a)(1 - a^5)x(1 + a^{10} + \dots + a^{10(n-1)})$$

$$\lim_{n\to\infty}\left(f(x) - f(a^{10n}x)\right) = \lim_{n\to\infty}(1 + a)(1 - a^5)x \cdot \frac{a^{10n} - 1}{a^{10} - 1} \Rightarrow$$

$$f(x) - f(\lim_{n\to\infty} a^{10n}x) = \frac{(1 + a)(1 - a^5)x}{1 - a^{10}}$$

$$\begin{cases} f(x) - f(0) = \dfrac{(1 + a)x}{1 + a^5} \\ f(0) = 0 \end{cases} \Rightarrow f(x) = \frac{x}{1 - a + a^2 - a^3 + a^4}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ f(x)-f(ax)+f(a^2x)-f(a^3x)+f(a^4x)=x\ (1)$$

$$x=0\ \rightarrow\ f(0)=0\ \ and\ x\sim ax\ \rightarrow$$

$$f(ax)-f(a^2x)+f(a^3x)-f(a^4x)+f\left(a^5x\right)=ax\ (2)$$

$$(1)+(2)\ \rightarrow\ f(x)+f\left(a^5x\right)=(a+1)x\ \ (3)$$

$$x\sim a^5x\ in\ (3)\rightarrow\ \ f\left(a^5x\right)+f(a^{10}x)=(a+1)a^5x\ \ (4)$$

$$(4)-(3)\ \rightarrow\ \ f(a^{10}x)-f(x)=(a+1)\left(a^5-1\right)x\ \ (5)$$

$$x\sim a^{10k}x,k\in N\ in\ (5)\ \rightarrow\ f\left(a^{10(k+1)}x\right)-f\left(a^{10k}x\right)=(a+1)\left(a^5-1\right)a^{10k}x$$

$$\rightarrow\sum_{k=0}^{n-1}\left[f\left(a^{10(k+1)}x\right)-f\left(a^{10k}x\right)\right]=(a+1)\left(a^5-1\right)x\sum_{k=0}^{n-1}a^{10k}\,,n\in N^*$$

$$\leftrightarrow\ f(a^{10n}x)-f(x)=\frac{(a+1)\left(a^5-1\right)(1-a^{10n})}{1-a^{10}}x$$

$$\overset{n\to\infty,a\in(0,1)}{\rightarrow}\quad f(0)-\ f(x)=-\frac{a+1}{a^5+1}x=-\frac{x}{a^4-a^3+a^2-a+1}$$

$$Therefore,\qquad f(x)=\frac{x}{a^4-a^3+a^2-a+1},\forall x\in R$$

Solution 3 by Ravi Prakash-New Delhi-India

$$f(x)-f(ax)+f(a^2x)-f(a^3x)+f(a^4x)=x;\ (1)$$

Put $x=0$ to obtain $f(0)=0$. In (1) replace x by ax so that

$$f(ax)-f(a^2x)+f(a^3x)-f(a^4x)+f\left(a^5x\right)=ax;\ (2)$$

Adding (1) and (2) we get:

$$f(x)+f\left(a^5x\right)=(a+1)x$$

$$f\left(a^5x\right)+f(a^{10}x)=(a+1)a^5x$$

$$f(a^{10}x)+f\left(a^{15}x\right)=(a+1)x$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f\left(a^{5n-5}x\right)+f\left(a^{5n}x\right)=(a+1)a^{5n-5}x$$

Multiply the 2nd equation by (-1), 4th equation by $(-1)^3$, 6th equation by $(-1)^5$ and so adding, we get

$$f(x)+(-1)^{n-1}f(a^{5n}x)=(a+1)[1-a^5+a^{10}+\cdots+(-1)^{n-1}a^{5n-5}]x;\ (3)$$

As $0<a<1, a^{5n}\to 0$ as $n\to\infty$. Since f –is continuous, $f(a^{5n}x)\xrightarrow{n\to\infty}f(0)=0$.

Taking limit as $n\to\infty$ in (3) we get:

$$f(x)=\frac{x(a+1)}{1+a^5};\forall x\in\mathbb{R}.$$

SP.397 Find:

$$\Omega=\lim_{p\to\infty}\frac{1}{p^a}\cdot\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

Let $a_k=\frac{k^2}{2k^2-2nk+n^2}$. Therefore,

$$a_k+a_{n-k}=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{2(n-k)^2-2n(n-k)+n^2}=$$

$$=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{(n-k)[2(n-k)-2n]+n^2}=$$

$$=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{-2k(n-k)+n^2}=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{n^2-2kn+2k^2}=$$

$$=\frac{k^2+(n-k)^2}{n^2-2kn+2k^2}=1$$

Hence,

$$2\sum_{k=1}^{n-1}a_k=\sum_{k=1}^{n-1}(a_k+a_{n-k})=n-1$$

and then

$$\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}=\frac{n+1}{2}$$

$$\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}=\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{n+1}{2}=\sum_{m=1}^{p}\left(\frac{m(m+1)}{4}+\frac{m}{2}\right)=$$

$$=\frac{1}{4}\sum_{m=1}^{p}m^2+\frac{1}{2}\sum_{m=1}^{p}m=\frac{1}{4}\cdot\frac{p(p+1)(2p+1)}{6}+\frac{1}{2}\cdot\frac{p(p+1)}{2}=$$

$$=\frac{p(p+1)}{4}\left(\frac{2p+1}{6}+1\right)=\frac{p(p+1)(2p+7)}{24}$$

Therefore,

$$\Omega=\lim_{p\to\infty}\frac{1}{p^a}\cdot\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}=\lim_{p\to\infty}\frac{1}{p^a}\cdot\frac{p(p+1)(2p+7)}{24}=\begin{cases}0, if\ a>3\\ \frac{1}{12}, if\ a=3\\ +\infty, if\ a<3\end{cases}$$

Solution 2 by Adrian Popa-Romania

$$a_k=\frac{k^2}{2k^2-2nk+n^2}\Rightarrow a_{n-k}=\frac{(n-k)^2}{2(n-k)^2-2n(n-k)+n^2}=\frac{n^2-2nk+k^2}{n^2-2nk+2k^2}$$

$$a_k+a_{n-k}=\frac{k^2}{2k^2-2nk+n^2}+\frac{n^2-2nk+k^2}{n^2-2nk+2k^2}=$$

$$=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{n^2-2kn+2k^2}$$

$$a_k+a_{n-k}=\frac{k^2+(n-k)^2}{n^2-2kn+2k^2}=1$$

$$\sum_{k=1}^{n}a_k=\frac{1}{2}\sum_{k=1}^{n}(a_k+a_{n-k})=\frac{n}{2}$$

Hnece,

$$\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}=\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{n+1}{2}=\sum_{m=1}^{p}\left(\frac{m(m+1)}{4}+\frac{m}{2}\right)=$$

$$=\frac{1}{4}\sum_{m=1}^{p}m^2+\frac{1}{2}\sum_{m=1}^{p}m=\frac{1}{4}\cdot\frac{p(p+1)(2p+1)}{6}+\frac{1}{2}\cdot\frac{p(p+1)}{2}=$$

$$=\frac{p(p+1)}{4}\left(\frac{2p+1}{6}+1\right)=\frac{p(p+1)(2p+7)}{24}$$

Therefore,

$$\Omega=\lim_{p\to\infty}\frac{1}{p^a}\cdot\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}=\lim_{p\to\infty}\frac{1}{p^a}\cdot\frac{p(p+1)(2p+7)}{24}=\begin{cases}0, if\ a>3\\ \frac{1}{12}, if\ a=3\\ +\infty, if\ a<3\end{cases}$$

Solution 3 by Soumitra Mandal-Chandar—Nagore-India

Let $a_k=\frac{k^2}{2k^2-2nk+n^2}$. Therefore,

$$a_k+a_{n-k}=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{2(n-k)^2-2n(n-k)+n^2}=$$

$$=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{(n-k)[2(n-k)-2n]+n^2}=$$

$$=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{-2k(n-k)+n^2}=\frac{k^2}{2k^2-2nk+n^2}+\frac{(n-k)^2}{n^2-2kn+2k^2}=$$

$$=\frac{k^2+(n-k)^2}{n^2-2kn+2k^2}=1$$

Hence,

$$2\sum_{k=1}^{n-1}a_k=\sum_{k=1}^{n-1}(a_k+a_{n-k})=n-1$$

and then

$$\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}=\frac{n+1}{2}$$

$$\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}=\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{n+1}{2}=\sum_{m=1}^{p}\left(\frac{m(m+1)}{4}+\frac{m}{2}\right)=$$

$$=\frac{1}{4}\sum_{m=1}^{p}m^2+\frac{1}{2}\sum_{m=1}^{p}m=\frac{1}{4}\cdot\frac{p(p+1)(2p+1)}{6}+\frac{1}{2}\cdot\frac{p(p+1)}{2}=$$

$$=\frac{p(p+1)}{4}\left(\frac{2p+1}{6}+1\right)=\frac{p(p+1)(2p+7)}{24}$$

Therefore,

$$\Omega=\lim_{p\to\infty}\frac{1}{p^a}\cdot\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}=\lim_{p\to\infty}\frac{1}{p^a}\cdot\frac{p(p+1)(2p+7)}{24}=\begin{cases}0, if\ a>3\\ \frac{1}{12}, if\ a=3\\ +\infty, if\ a<3\end{cases}$$

SP.398 If $a,b>0, b\neq 1$ solve for real numbers:

$$\sum_{k=1}^{n}\frac{x+\log_b(a+k)}{2x+\log_b(a+k)(a-k+n)}-\frac{n+1}{2}=0$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

Let $a_k=\frac{x+\log_b(a+k)}{2x+\log_b(a+k)(a-k+n)}$. Hence,

$$a_k+a_{n-k}=\frac{x+\log_b(a+k)}{2x+\log_b(a+k)(a-k+n)}+\frac{x+\log_b(c+n-k)}{2x+\log_b(a+n-k)(a+k)}=1$$

Hence,

$$2\sum_{k=1}^{n-1}a_k=\sum_{k=1}^{n-1}(a_k+a_{n-k})=n-1$$

and then

$$\sum_{k=1}^{n}a_k=\sum_{k=1}^{n-1}a_k+a_n=\frac{n-1}{2}+\frac{x+\log_b(a+n)}{2x+\log_b[a(a+n)]}$$

So, we get:

$$\frac{n-1}{2}+\frac{x+\log_b(a+n)}{2x+\log_b[a(a+n)]}=\frac{n+1}{2}\Leftrightarrow\frac{x+\log_b(a+n)}{2x+\log_b[a(a+n)]}=1\Leftrightarrow$$

$$2x+\log_b[a(a+n)]=x+\log_b(a+n)$$

$$x=\log_b\left(\frac{1}{a}\right)$$

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$S=\sum_{k=1}^{n}\frac{x+\log_b(a+k)}{2x+\log_b(a+k)(a-k+n)}$$

$$n-k=p\to k=n-p$$

$$S=\sum_{p=0}^{n-1}\frac{x+\log_b(a+n-p)}{2x+\log_b(a+n-p)(a+p)}=\sum_{p=1}^{n}\frac{x+\log_b(a+n-p)}{2x+\log_b(a+n-p)(a+p)}+$$

$$+\frac{x+\log_b(a+n)}{2x+\log_b a(a+n)}-\frac{x+\log_b a}{2x+\log_b a(a+n)}$$

$$\Rightarrow 2S = \sum_{p=0}^{n} \frac{x+\log_b(a+n-p)}{2x+\log_b(a+n-p)(a+p)} +$$

$$+\frac{x+\log_b(a+n)}{2x+\log_b a(a+n)} - \frac{x+\log_b a}{2x+\log_b a(a+n)} =$$

$$= n + \frac{x+\log_b(a+n)}{2x+\log_b a(a+n)} - \frac{x+\log_b a}{2x+\log_b a(a+n)}$$

$$S = \frac{n}{2} + \frac{1}{2}\left(\frac{x+\log_b(a+n)}{2x+\log_b a(a+n)} - \frac{x+\log_b a}{2x+\log_b a(a+n)}\right)$$

$$\frac{x+\log_b(a+n)}{2x+\log_b a(a+n)} - \frac{x+\log_b a}{2x+\log_b a(a+n)} = 1$$

So, we get:

$$\frac{n-1}{2} + \frac{x+\log_b(a+n)}{2x+\log_b[a(a+n)]} = \frac{n+1}{2} \Leftrightarrow \frac{x+\log_b(a+n)}{2x+\log_b[a(a+n)]} = 1 \Leftrightarrow$$

$$2x+\log_b[a(a+n)] = x+\log_b(a+n)$$

$$x = \log_b\left(\frac{1}{a}\right)$$

Solution 3 by Ravi Prakash-New Delhi-India

Let $a_k = \frac{x+\log_b(a+k)}{2x+\log_b(a+k)(a-k+n)}$. Hence,

$$a_k + a_{n-k} = \frac{x+\log_b(a+k)}{2x+\log_b(a+k)(a-k+n)} + \frac{x+\log_b(c+n-k)}{2x+\log_b(a+n-k)(a+k)} = 1$$

Hence,

$$S = a_1 + a_2 + \cdots + a_n$$

$$S + a_0 = a_0 + a_1 + \cdots + a_{n-1} + a_n$$

$$S + a_0 = a_n + a_{n-1} + \ldots + a_1 + a_0$$

So, we get: $2(S+a_0) = n+1 \Rightarrow S = \frac{n+1}{2} - a_0$

$$S = \frac{n+1}{2} - a_0 \Rightarrow S - \frac{n+1}{2} = 0 \Rightarrow a_0 = 0 \Rightarrow x + \log_b a = 0$$

$$x = \log_b\left(\frac{1}{a}\right)$$

SP.399 If $m,n>0;\ x,y,z\in(0,1)$ then:

$$\frac{1}{(1-x^2)(my+nz)}+\frac{1}{(1-y^2)(mz+nx)}+\frac{1}{(1-z^2)(mx+ny)}\ge\frac{9\sqrt{3}}{2}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution 1 by proposer

$$2t^2(1-t^2)\overset{AM-GM}{\le}\left(\frac{2t^2+1-t^2+1-t^2}{3}\right)^3=\frac{8}{27}$$

$$t(1-t^2)\le\frac{2}{3\sqrt{3}};t\in(0,1)$$

$$\sum_{cyc}\frac{1}{(1-x^2)(my+nz)}=\sum_{cyc}\frac{x}{x(1-x^2)(my+nz)}\ge\frac{3\sqrt{3}}{3}\cdot\sum_{cyc}\frac{x}{my+nz}=$$

$$=\frac{3\sqrt{3}}{2}\cdot\sum_{cyc}\frac{x^2}{mxy+nxz}\overset{Bergstrom}{\ge}\frac{3\sqrt{3}}{2}\cdot\frac{(x+y+z)^2}{(m+n)(x+y+z)}\ge$$

$$\ge\frac{3\sqrt{3}}{2}\cdot\frac{3(xy+yz+zx)}{(m+n)(xy+yz+zx)}=\frac{9\sqrt{3}}{2}$$

Equality holds for $m=n;\ x=y=z=\frac{1}{\sqrt{3}}$.

Solution 2 by George Florin Șerban-Romania

$$\sum_{cyc}\frac{1}{(1-x^2)(my+nz)}\ge\frac{9\sqrt{3}}{2(m+n)};(*)$$

$$x^2+\frac{2\sqrt{3}}{9x}=x^2+\frac{\sqrt{3}}{9x}+\frac{\sqrt{3}}{9x}\overset{AGM}{\ge}3\cdot\sqrt[3]{x^2\cdot\frac{\sqrt{3}}{9x}\cdot\frac{\sqrt{3}}{9x}}=1$$

$$\Rightarrow x^2+\frac{2\sqrt{3}}{9x}\ge1;\forall x\in(0,1)\Rightarrow 1-x^2\le\frac{2\sqrt{3}}{9x}\Rightarrow\frac{1}{1-x^2}\ge\frac{9x}{2\sqrt{3}}$$

$$\sum_{cyc}\frac{1}{(1-x^2)(my+nz)}\ge\sum_{cyc}\frac{9x}{2\sqrt{3}(my+nz)}=\frac{9}{2\sqrt{3}}\sum_{cyc}\frac{x}{my+nz}\overset{(1)}{\ge}\frac{9\sqrt{3}}{2(m+n)}$$

$$(1)\Leftrightarrow\sum_{cyc}\frac{x}{my+nz}\ge\frac{3}{m+n}$$

We have:

$$\sum_{cyc}\frac{x}{my+nz}=\sum_{cyc}\frac{x^2}{mxy+nxz}\overset{Bergstrom}{\geq}\frac{(x+y+z)^2}{m\sum xy+n\sum xz}=\frac{(x+y+z)^2}{(m+n)(xy+yz+zx)}\geq$$

$$\geq\frac{3}{m+n},\text{which is true from }(x+y+z)^2\geq 3(xy+yz+zx)$$

Equality holds for $m=n; x=y=z=\frac{1}{\sqrt{3}}$.

Solution 3 by Soumitra Mandal-Chandar-Nagore-India

$$\text{Let } f(x)=x-x^3, x\geq 0, f'(x)=1-3x^2, f''(x)=-6x$$

$$f(x)=0\Rightarrow x=\frac{1}{\sqrt{3}}.\text{Now } f''\left(\frac{1}{\sqrt{3}}\right)=-\frac{6}{\sqrt{3}};\forall x\geq 0$$

Hence, f –attains maximum value at $x=\frac{1}{\sqrt{3}}$ then $f\left(\frac{1}{\sqrt{3}}\right)\geq f(x);\forall x\geq 0$

$$\frac{2}{3\sqrt{3}}\geq f(x);\forall x\geq 0$$

Hence,

$$\sum_{cyc}\frac{1}{(1-x^2)(my+nz)}=\sum_{cyc}\frac{1}{x-x^3}\cdot\frac{x}{my+nz}\geq\frac{3\sqrt{3}}{2}\sum_{cyc}\frac{x}{my+nz}=$$

$$=\frac{3\sqrt{3}}{2}\sum_{cyc}\frac{x^2}{mxy+nxz}\overset{Bergstrom}{\geq}\frac{3\sqrt{3}}{2}\cdot\frac{1}{m+n}\cdot\frac{(x+y+z)^2}{xy+yz+zx}\geq\frac{9\sqrt{3}}{2(m+n)}$$

Equality holds for $m=n; x=y=z=\frac{1}{\sqrt{3}}$.

SP.400 In $\Delta ABC, \lambda>0$ the following relationship holds:

$$\prod_{cyc}\left(\frac{\sin A}{\sin B\sin C}+\lambda^2\right)\geq 9\lambda^2\cdot\sqrt{\frac{3}{4}}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

For any positive real numbers x,y,z,λ holds inequality:

$$(x^2+\lambda^2)(y^2+\lambda^2)(z^2+\lambda^2)\geq\frac{3}{4}\lambda^2(x+y+z)^2;(1)$$

Proof. We have:

$$(x^2+\lambda^2)(y^2+\lambda^2)\geq\lambda^2(x+y)^2\Leftrightarrow(xy-\lambda^2)^2\geq 0$$

and

$$(x^2+\lambda^2)(y^2+\lambda^2)\ge\frac{3}{4}\lambda^2((x+y)^2+\lambda^2)\Leftrightarrow\left(xy-\frac{\lambda^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\ge 0$$

Hence,

$$(x^2+\lambda^2)(y^2+\lambda^2)(z^2+\lambda^2)\ge\frac{3\lambda^2}{4}((x+y)^2+\lambda^2)(z^2+\lambda^2)\ge$$

$$\ge\frac{3}{4}\lambda^2\big((x+y)+z\big)^2=\frac{3}{4}\lambda^2(x+y+z)^2$$

Let $x=\sqrt{\frac{\sin A}{\sin B\sin C}},y=\sqrt{\frac{\sin B}{\sin C\sin A}},z=\sqrt{\frac{\sin C}{\sin A\sin B}}$ then

$$\left(\frac{\sin A}{\sin B\sin C}+\lambda^2\right)\left(\frac{\sin B}{\sin C\sin A}+\lambda^2\right)\left(\frac{\sin C}{\sin A\sin B}+\lambda^2\right)$$

$$\ge\frac{3}{4}\lambda^2\left(\sum_{cyc}\sqrt{\frac{\sin A}{\sin B\sin C}}\right)^2;(2)$$

We prove that:

$$\sum_{cyc}\sqrt{\frac{\sin A}{\sin B\sin C}}\ge 3\cdot\sqrt[4]{\frac{4}{3}};(3)\Leftrightarrow\sum_{cyc}\sin A\ge 3\cdot\sqrt{\prod_{cyc}\sin A}\cdot\sqrt[4]{\frac{4}{3}}$$

How

$$\sum_{cyc}\sin A\ge 3\cdot\sqrt[3]{\prod_{cyc}\sin A}$$

Remains to prove that:

$$3\cdot\sqrt[3]{\prod_{cyc}\sin A}\ge 3\cdot\sqrt{\prod_{cyc}\sin A}\cdot\sqrt[4]{\frac{4}{3}}\Leftrightarrow\prod_{cyc}\sin A\le\frac{3\sqrt{3}}{8}\ (true)$$

From (2),(3) it follows that:

$$\prod_{cyc}\left(\frac{\sin A}{\sin B\sin C}+\lambda^2\right)\ge\frac{3}{4}\lambda^2\cdot 9\cdot\sqrt{\frac{4}{3}}=9\lambda^2\cdot\sqrt{\frac{3}{4}}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\prod_{cyc}\left(\frac{\sin A}{\sin B\sin C}+\lambda^2\right)\overset{Hölder}{\ge}\left(\sqrt[3]{\prod_{cyc}\frac{\sin A}{\sin B\sin C}}+\sqrt[3]{(\lambda^2)^3}\right)^3$$

$$=\left(\frac{1}{\sqrt[3]{\sin A\sin B\sin C}}+\lambda^2\right)^3\ge$$

$$\overset{AM-GM}{\ge}\left(\frac{3}{\sin A+\sin B+\sin C}+\lambda^2\right)^3\overset{Jensen}{\ge}$$

$$\left(\frac{3}{3\sin\frac{\pi}{3}}+\lambda^2\right)^3=\left(\frac{2}{\sqrt{3}}+\lambda^2\right)^3=\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\lambda^2\right)^3\geq$$

$$\overset{AM-GM}{\geq}\ 27.\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}.\lambda^2=9\lambda^2\geq 9\lambda^2.\sqrt{\frac{3}{4}}$$

$$\rightarrow\ Therefore,\quad \prod_{cyc}\left(\frac{\sin A}{\sin B\sin C}+\lambda^2\right)\geq 9\lambda^2.\sqrt{\frac{3}{4}}$$

SP.401 Let I be the incentre of triangle ABC and let A', B' and C' be the intersections of the rays AI, BI, and CI with the circumcircle of the triangle. Prove that $[A'B'C']\geq[ABC]$, where $[*]$ −represent the area.

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

We have: $\angle C'A'B'=\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}-\frac{A}{2}$. Using the law of sines, we have

$$B'C'=2R\cdot\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)=2R\cdot\cos\frac{C}{2};A'B'=2R\cdot\cos\frac{C}{2};C'A'=2R\cdot\cos\frac{B}{2}$$

We know that: $[A'B'C']=\frac{A'B'\cdot B'C'\cdot C'A'}{4R}=\frac{8R^3\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4R}=2R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Now, we'll prove that $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\geq\sin A\sin B\sin C$. We have:

$$\sin A\sin B=\frac{1}{2}(\cos(A-B)-\cos(A+B))\leq\frac{1}{2}(1+\cos C)=\cos^2\frac{C}{2}$$

So, $\sin A\sin B\leq\cos^2\frac{C}{2}$ and similarly $\sin B\sin C\leq\cos^2\frac{A}{2}$, $\sin C\sin A\leq\cos^2\frac{B}{2}$

Multiplying up these inequalities, we get

$$\sin^2A\sin^2B\sin^2C\leq\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}\Leftrightarrow\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\geq\sin A\sin B\sin C$$

Now, $[A'B'C']=2R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\geq 2R^2\sin A\sin B\sin C=$

$=2R^2\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right)=\frac{abc}{4R}=[ABC]$, where $a=|BC|,b=|CA|,c=|AB|$.

So, $\{A'B'C']\geq[ABC]$

Equality holds if and only if triangle ABC is equilateral.

Solution 2 by Marian Ursărescu-Romania

$$\widehat{ABA'}=2\widehat{C}+\widehat{A};\widehat{B'AC'}=\widehat{B}+\widehat{C}$$

In $\Delta ACA'$ by law of sines: $AA'=2R\sin\left(\frac{2C+A}{2}\right)$

In $\Delta AB'C'$ by law of sines $B'C'=2R\sin\left(\frac{B+C}{2}\right)=2R\sin\frac{A}{2}$ and analogs.

$$[A'B'C']=\frac{A'B'\cdot A'C'\cdot\sin\widehat{B'A'C'}}{2}=\frac{1}{2}2R\sin\frac{C}{2}\cdot 2R\sin\frac{B}{2}\cdot\cos\frac{A}{2}\Rightarrow$$

$$[A'B'C']=2R^2\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2};(1)$$

But $[ABC]=2R^2\cdot\sin A\sin B\sin C;(2)$

From (1),(2) it follows that

$$\frac{[A'B'C']}{[ABC]}=\frac{2R^2\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2R^2\sin A\sin B\sin C}=\frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}=$$

$$=\frac{1}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}};(3)$$

But $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R};(4)$. From (3),(4) we have

$\frac{[A'B'C']}{[ABC]}=\frac{R}{2r};(5)$ and $R\ge 2r\ (Euler);(6)$

From (5),(6) we get $[A'B'C']\ge[ABC]$

SP.402 Let m_a,m_bm_c be the lengths of the medians of a triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{8r^2}{R^4}\le\sum_{cyc}\frac{\sin^2 B+\sin^2 C}{m_a^2}\le\frac{1}{2r^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let $a=|BC|,b=|CA|,c=|AB|$. We know that $s_a\ge h_a$, with $s_a=\frac{2bc}{b^2+c^2}m_a(symmedian)$ and $h_a=\frac{bc}{2R}$.So, $\frac{2m_a}{b^2+c^2}\ge\frac{1}{2R}\Leftrightarrow\frac{m_a^2}{b^2+c^2}\ge\frac{m_a}{4R}\Leftrightarrow\frac{b^2+c^2}{m_a^2}\le\frac{4R}{m_a}$.

Also, we know that $m_a \geq h_a(and\ analogs) \Leftrightarrow \frac{1}{m_a} \leq \frac{1}{h_a}$ and $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$.

$$\sum_{cyc}\frac{b^2+c^2}{m_a^2} \leq 4R\sum_{cyc}\frac{1}{m_a} \leq 4R\sum_{cyc}\frac{1}{h_a}=\frac{4R/1}{r}=\frac{4R}{r}$$

Using the law of the sines, we get:

$$\sum_{cyc}\frac{\sin^2 B+\sin^2 C}{m_a^2} \leq \frac{4}{Rr} \overset{Euler}{\leq} \frac{1}{2r^2}$$

For the left inequality, we have:

$$\sum_{cyc}\frac{b^2+c^2}{m_a^2}=\left(\frac{c^2}{m_a^2}+\frac{a^2}{m_b^2}+\frac{b^2}{m_c^2}\right)+\left(\frac{b^2}{m_a^2}+\frac{c^2}{m_b^2}+\frac{a^2}{m_c^2}\right) \overset{CBS}{\geq}$$

$$\geq \frac{(c+a+b)^2}{m_a^2+m_b^2+m_c^2}+\frac{(b+c+a)^2}{m_a^2+m_b^2+m_c^2}=\frac{8s^2}{m_a^2+m_b^2+m_c^2}; a+b+c=2s$$

We know that: $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$ and $a^2+b^2+c^2 \leq 9R^2$.

So, $m_a^2+m_b^2+m_c^2 \leq \frac{27}{4}R^2$. Hence,

$$\sum_{cyc}\frac{b^2+c^2}{m_a^2} \geq \frac{8s^2}{\frac{27}{4}R^2} \overset{Mitrinovic}{\geq} \frac{32r^2}{R^2}$$

Using Law of sines, it follows that

$$\sum_{cyc}\frac{\sin^2 B+\sin^2 C}{m_a^2} \geq \frac{8r^2}{R^4}$$

Therefore,

$$\frac{8r^2}{R^4} \leq \sum_{cyc}\frac{\sin^2 B+\sin^2 C}{m_a^2} \leq \frac{1}{2r^2}$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{\sin^2 B+\sin^2 C}{m_a^2}=\sum_{cyc}\frac{b^2+c^2}{4R^2}\cdot\frac{1}{m_a^2} \overset{\boxed{m_a\geq\frac{b^2+c^2}{4R}}}{\leq} \sum_{cyc}\frac{(b^2+c^2)}{4R^2}\cdot\frac{16R^2}{(b^2+c^2)^2}=$$

$$=4\sum_{cyc}\frac{1}{b^2+c^2}=4\sum_{cyc}\frac{1}{2bc}=\frac{2}{abc}\sum_{cyc}a=\frac{4s}{4Rrs}=\frac{2}{Rr} \overset{Euler}{\leq} \frac{1}{2r^2}$$

$$\sum_{cyc}\frac{\sin^2 B+\sin^2 C}{m_a^2}\overset{(Power-Mean)}{\geq}\sum_{cyc}\frac{\frac{(\sin B+\sin C)^2}{2}}{m_a^2}\overset{Leibniz}{\geq}$$

$$\geq\frac{1}{2}\cdot\frac{(\sum(\sin B+\sin C))^2}{\sum m_a^2}=\frac{1}{2}\cdot\frac{(2\sum\sin A)^2}{\frac{3}{4}\sum a^2}\overset{Leibniz}{\geq}$$

$$\geq\frac{1}{2}\cdot\frac{\left(2\sum\frac{a}{2R}\right)^2}{\frac{3}{4}\cdot 9R^2}=\frac{1}{2R^4}\cdot\frac{4}{27}\cdot 4s^2\overset{Mitrinovic}{\geq}\frac{8r^2}{R^4}$$

SP.403 In $\Delta ABC,\lambda>0,n\in\mathbb{N}$ the following relationship holds:

$$\prod_{cyc}\left(\frac{\sin^{2n}A}{\sin^{2n+4}B}+\lambda^2\right)\geq 12\lambda^2$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

For any positive real numbers x,y,z,λ holds inequality:

$$(x^2+\lambda^2)(y^2+\lambda^2)(z^2+\lambda^2)\geq\frac{3}{4}\lambda^2(x+y+z)^2;(1)$$

Proof. We have:

$$(x^2+\lambda^2)(y^2+\lambda^2)\geq\lambda^2(x+y)^2\Leftrightarrow(xy-\lambda^2)^2\geq 0$$

and

$$(x^2+\lambda^2)(y^2+\lambda^2)\geq\frac{3}{4}\lambda^2((x+y)^2+\lambda^2)\Leftrightarrow\left(xy-\frac{\lambda^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0$$

Hence,

$$(x^2+\lambda^2)(y^2+\lambda^2)(z^2+\lambda^2)\geq\frac{3\lambda^2}{4}((x+y)^2+\lambda^2)(z^2+\lambda^2)\geq$$

$$\geq\frac{3}{4}\lambda^2\big((x+y)+z\big)^2=\frac{3}{4}\lambda^2(x+y+z)^2$$

Let $x=\frac{\sin^n A}{\sin^{n+2}B},y=\frac{\sin^n B}{\sin^{n+2}C},z=\frac{\sin^{n+1}C}{\sin^{n+2}A}$ then

$$\prod_{cyc}\left(\frac{\sin^{2n}A}{\sin^{2n+4}B}+\lambda^2\right)\geq\frac{3}{4}\lambda^2\left(\sum_{cyc}\frac{\sin^n A}{\sin^{n+2}B}\right)^2;(2)$$

Remains to prove that:

$$\sum_{cyc}\frac{\sin^n A}{\sin^{n+2}B}\ge 4;(3)$$

From AM-GM inequality, we have:

$$\sum_{cyc}\frac{\sin^n A}{\sin^{n+2}B}\ge 3\cdot\sqrt[3]{\frac{1}{(\prod\sin A)^2}}\overset{(*)}{\ge}\frac{3}{\sqrt[3]{\left(\frac{3\sqrt{3}}{8}\right)^2}}=4,(*)\Leftrightarrow\prod_{cyc}\sin A=\frac{rs}{2R^2}\le\frac{3\sqrt{3}}{8}$$

From (2),(3) it follows that:

$$\prod_{cyc}\left(\frac{\sin^{2n}A}{\sin^{2n+4}B}+\lambda^2\right)\ge\frac{3}{4}\lambda^2\left(\sum_{cyc}\frac{\sin^n A}{\sin^{n+2}B}\right)^2\ge 12\lambda^2$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\prod_{cyc}\left(\frac{\sin^{2n}A}{\sin^{2n+4}B}+\lambda^2\right)\overset{Hölder}{\ge}\left(\sqrt[3]{\prod_{cyc}\frac{\sin^{2n}A}{\sin^{2n+4}B}}+\sqrt[3]{(\lambda^2)^3}\right)^3$$

$$=\left(\frac{1}{\sqrt[3]{\sin A\sin B\sin C}^4}+\lambda^2\right)^3\ge$$

$$\overset{AM-GM}{\ge}\left[\left(\frac{3}{\sin A+\sin B+\sin C}\right)^4+\lambda^2\right]^3\overset{Jensen}{\ge}\left[\left(\frac{3}{3\sin\frac{\pi}{3}}\right)^4+\lambda^2\right]^3=\left[\left(\frac{2}{\sqrt{3}}\right)^4+\lambda^2\right]^3$$

$$=\left(\frac{8}{9}+\frac{8}{9}+\lambda^2\right)^3\overset{AM-GM}{\ge}\left(3.\sqrt[3]{\frac{8}{9}.\frac{8}{9}.\lambda^2}\right)^3=\frac{64\lambda^2}{3}\ge 12\lambda^2$$

$$\textit{Therefore,}\quad\prod_{cyc}\left(\frac{\sin^{2n}A}{\sin^{2n+4}B}+\lambda^2\right)\ge 12\lambda^2$$

SP.404 In ΔABC the following relationship holds:

$$3(a^2+b^2+c^2)+4(h_a^2+h_b^2+h_c^2)\ge 24\sqrt{3}F$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$2h_a^2-2m_a^2=2h_a^2-2\cdot\frac{2(b^2+c^2)-a^2}{4}=2h_a^2-b^2-c^2+\frac{a^2}{2}$$

Hence,

$$a^2+b^2+c^2+2h_a^2-2m_a^2=2h_a^2+a^2+\frac{a^2}{2}=2h_a^2+\frac{3}{2}a^2\overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{2h_a^2\cdot\frac{3a^2}{2}}=2\sqrt{3a^2h_a^2}=2\sqrt{2(2F)^2}=4\sqrt{3}F$$

$$3(a^2+b^2+c^2)+2\left(h_a^2+h_b^2+h_c^2\right)-2\left(m_a^2+m_b^2+m_c^2\right)\geq 12\sqrt{3}F$$

Thus,

$$3(a^2+b^2+c^2)+2\left(h_a^2+h_b^2+h_c^2\right)-2\cdot\frac{3}{4}(a^2+b^2+c^2)\geq 12\sqrt{3}F$$

$$\frac{3}{2}(a^2+b^2+c^2)+2\left(h_a^2+h_b^2+h_c^2\right)\geq 12\sqrt{3}F$$

Therefore,

$$3(a^2+b^2+c^2)+4\left(h_a^2+h_b^2+h_c^2\right)\geq 24\sqrt{3}F$$

Equality holds for $a=b=c$.

Solution 2 by Daniel Văcaru-Romania

We have $3a^2+4h_a^2=3a^2+\frac{16F^2}{a^2}\overset{AGM}{\geq}8F\sqrt{3}$ (and analogs for b and c)

Summing these relationships, we find:

$$3(a^2+b^2+c^2)+4\left(h_a^2+h_b^2+h_c^2\right)\geq 24\sqrt{3}F.$$

SP.405 If $a,b,c>0$ then:

$$\sqrt{a^2+5ab+7b^2}+\sqrt{b^2+5bc+7c^2}+\sqrt{c^2+5ca+7a^2}\geq\sqrt{13}(a+b+c)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$3(a-b)^2+(7a+19b)^2=3(a^2-2ab+b^2)+49a^2+361b^2+266ab=$$

$$=52a^2+260ab+364b^2=52(a^2+5ab+7b^2)$$

Hence,

$$a^2+5ab+7b^2=\frac{1}{52}\left(3(a-b)^2+(7a+19b)^2\right)\geq\frac{1}{52}(7a+19b)^2$$

So, we have:

$$\sqrt{a^2+5ab+7b^2}\ge\frac{1}{\sqrt{52}}(7a+19b)$$

Therefore,

$$\sum_{cyc}\sqrt{a^2+5ab+7b^2}\ge\frac{1}{2\sqrt{13}}\left(7\sum_{cyc}a+19\sum_{cyc}b\right)=$$

$$=\frac{1}{2\sqrt{13}}\big(7(a+b+c)+19(a+b+c)\big)=\frac{26}{2\sqrt{13}}(a+b+c)=\sqrt{13}(a+b+c)$$

Equality holds for $a=b=c$.

Solution 2 by Adrian Popa-Romania

$$\sum_{cyc}\sqrt{a^2+5ab+7b^2}\overset{AGM}{\ge}\sum_{cyc}\sqrt{13\sqrt[13]{a^7\cdot b^{19}}}=\sqrt{13}\sum_{cyc}\sqrt[26]{a^7\cdot b^{19}}=\sqrt{13}\sum_{cyc}a^{\frac{7}{26}}\cdot b^{\frac{19}{26}}$$

$$\text{By Holder: }\sum_{i=1}^{n}a_ib_i\le\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\cdot\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}},\text{ where }\frac{1}{p}+\frac{1}{q}=1.$$

$$\text{For } n=3:\ \sum_{cyc}a^{\frac{7}{26}}\cdot b^{\frac{19}{26}}\ge\left(\sum_{cyc}\left(a^{\frac{7}{26}}\right)^{\frac{26}{7}}\right)^{\frac{7}{26}}\cdot\left(\sum_{cyc}\left(b^{\frac{19}{26}}\right)^{\frac{26}{19}}\right)^{\frac{19}{26}}=$$

$$=\left(\sum_{cyc}a\right)^{\frac{7}{26}}\cdot\left(\sum_{cyc}b\right)^{\frac{19}{26}}=a+b+c\Rightarrow\sum_{cyc}\sqrt{a^2+5ab+7b^2}\ge\sqrt{13}(a+b+c).$$

Equality holds for $a=b=c$.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2\sqrt{a^2+5ab+7b^2}=\sqrt{(2a+5b)^2+3b^2}$$

$$=\frac{1}{\sqrt{52}}\sqrt{\left[\underbrace{\left(\frac{2a+5b}{7}\right)^2+\cdots+\left(\frac{2a+5b}{7}\right)^2}_{49\ times}+b^2+b^2+b^2\right]\Big(\underbrace{1+\cdots+1}_{52\ times}\Big)}\ge$$

$$\overset{CBS}{\ge}\frac{\sqrt{13}}{26}\left(\underbrace{\left(\frac{2a+5b}{7}\right)+\cdots+\left(\frac{2a+5b}{7}\right)}_{49\ times}+b+b+b\right)=\frac{\sqrt{13}}{13}(7a+19b)$$

$$\to\sqrt{a^2+5ab+7b^2}\ge\frac{\sqrt{13}}{26}(7a+19b)\ \ (And\ analogs)$$

$$\to \sum_{cyc} \sqrt{a^2+5ab+7b^2} \geq \frac{\sqrt{13}}{26} \sum_{cyc} (7a+19b) = \frac{\sqrt{13}}{26}.26 \sum_{cyc} a = \sqrt{13}(a+b+c)$$

$$Therefore, \quad \sum_{cyc} \sqrt{a^2+5ab+7b^2} \geq \sqrt{13}(a+b+c)$$

Equality holds for $a=b=c$.

Solution 4 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc} \sqrt{a^2+5ab+7b^2} =$$

$$= \frac{1}{2} \sum_{cyc} \sqrt{(2a+5b)^2+3b^2} \overset{Minkowski}{\geq} \frac{1}{2} \sqrt{\left(\sum_{cyc} (2a+5b)\right)^2 + 3\left(\sum_{cyc} b\right)^2} =$$

$$= \frac{1}{2} \sqrt{49\left(\sum_{cyc} a\right)^2 + 3\left(\sum_{cyc} a\right)^2} = \sqrt{13} \sum_{cyc} a.$$

$$\text{Therefore,} \quad \sum_{cyc} \sqrt{a^2+5ab+7b^2} \geq \sqrt{13}(a+b+c)$$

Equality holds for $a=b=c$.

Solution 5 by Angel Plaza-Gran Canaria-Spain

The proposed inequality is equivalent to

$$\sqrt{b^2+5ba+7a^2} + \sqrt{c^2+5cb+7b^2} + \sqrt{a^2+5ac+7c^2} \geq \sqrt{13}(a+b+c)$$

Or, after summing up both of them, to

$$S[a,b,c] + S[b,a,c] \geq 2\sqrt{13}(a+b+c),$$

where $S[a,b,c]$ and $S[b,a,c]$ respectively denote the left-hand side of both inequalities.

We will prove that

$$\sqrt{a^2+5ab+7b^2} + \sqrt{b^2+5ba+7a^2} \geq \sqrt{13}(a+b)$$

Last inequality is homogeneous, so we may assume that $a+b=1$, and the inequality becomes $f(a)+f(1-a) \geq \sqrt{13}$, where $f(x) = \sqrt{x^2+5x(1-x)+7(1-x)^2}$.

Function $f(x)$ is convex because $f''(x) = \frac{3}{4\sqrt{(3x^2-9x+7)^3}} > 0, \forall x \in [0,1]$.

Therefore, by Jensen's inequality $f(a)+f(1-a)\geq 2f\left(\frac{1}{2}\right)=\sqrt{13}$, and the problem is done. Equality holds for $a=b=c$.

UNDERGRADUATE PROBLEMS

UP.391

$$x_n=\sum_{i=1}^{n}\sin\frac{(2i-1)x}{n^2}, x>0$$

Find:

$$\Omega=\lim_{x\to 0}\left(\lim_{n\to\infty}\left(\left(\sum_{i=1}^{n}a_i^{1+x_n}\right)\cdot\left(\sum_{i=1}^{n}a_i\right)^{-1}\right)^{\frac{1}{x_n}}\right), a_i>0, i=\overline{1,n}$$

Proposed by Florică Anastase-Romania

Solution by proposer

For $a>0$. We prove that:

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sin\frac{(2i-1)a}{n^2}\right)=a$$

Using the well-known inequality: $x-\frac{x^3}{6}<\sin x<x, \forall x>0\Rightarrow$

$$\frac{a}{n^2}-\frac{1}{6}\cdot\frac{a^3}{n^6}<\sin\frac{a}{n^2}<\frac{a}{n^2}$$

$$\frac{3a}{n^2}-\frac{1}{6}\cdot\frac{3^3a^3}{n^6}<\sin\frac{3a}{n^2}<\frac{3a}{n^2}$$

……………………………………

$$\frac{(2n-1)a}{n^2}-\frac{1}{6}\cdot\frac{(2n-1)^3a^3}{n^6}<\sin\frac{(2n-1)a}{n^2}<\frac{(2n-1)a}{n^2}$$

Summing, we get:

$$\frac{a}{n^2}\cdot\sum_{i=1}^{n}(2i-1)-\frac{a^3}{n^6}\cdot\sum_{i=1}^{n}(2i-1)^3<\sum_{i=1}^{n}\sin\frac{(2i-1)a}{n^2}<\frac{a}{n^2}\cdot\sum_{i=1}^{n}(2i-1)$$

Let us denote:

$$a_n=\frac{a}{n^2}\cdot\sum_{i=1}^{n}(2i-1),\ b_n=x_n-\frac{a^3}{n^6}\cdot\sum_{i=1}^{n}(2i-1)^3$$

$$\Rightarrow a_n=a\Rightarrow b_n=a-\frac{a^3}{n^6}\cdot\sum_{i=1}^{n}(2i-1)^3=a-\frac{a^3}{n^4(2n^2-1)}$$

So, $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=a$ **then,**

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sin\frac{(2i-1)x}{n^2}\right)=x$$

$$\Omega=\lim_{x\to 0}\left(\lim_{n\to\infty}\left(\left(\sum_{i=1}^{n}a_i^{1+x_n}\right)\cdot\left(\sum_{i=1}^{n}a_i\right)^{-1}\right)^{\frac{1}{x_n}}\right)=\lim_{x\to 0}\left(\left(\sum_{i=1}^{n}a_i^{1+x}\right)\cdot\left(\sum_{i=1}^{n}a_i\right)^{-1}\right)^{\frac{1}{x}}$$

Let be the function

$$f(x)=\left(\left(\sum_{i=1}^{n}a_i^{1+x}\right)\cdot\left(\sum_{i=1}^{n}a_i\right)^{-1}\right)^{\frac{1}{x}}$$

$$f(x)=e^{\log g(x)};\ g(x)=\frac{1}{x}\left(\log\left(\sum_{i=1}^{n}a_i^{x+1}\right)-\log\left(\sum_{i=1}^{n}a_i\right)\right)$$

$$\lim_{x\to 0}g(x)=\lim_{x\to 0}\left(\sum_{i=1}^{n}a_i^{x+1}\cdot\log a_i\right)\cdot\left(\sum_{i=1}^{n}a_i^{x+1}\right)^{-1}=\left(\sum_{i=1}^{n}a_i\cdot\log a_i\right)\cdot\left(\sum_{i=1}^{n}a_i\right)^{-1}=$$

$$=\left(\sum_{i=1}^{n}\log a_i^{a_i}\right)\cdot\left(\sum_{i=1}^{n}a_i\right)^{-1}=\log\left(\prod_{i=1}^{n}a_i^{a_i}\right)^{\left(\sum_{i=1}^{n}a_i\right)^{-1}}$$

Therefore,

$$\Omega=\lim_{x\to 0}\left(\lim_{n\to\infty}\left(\left(\sum_{i=1}^{n}a_i^{1+x_n}\right)\cdot\left(\sum_{i=1}^{n}a_i\right)^{-1}\right)^{\frac{1}{x_n}}\right)=\left(\prod_{i=1}^{n}a_i^{a_i}\right)^{\left(\sum_{i=1}^{n}a_i\right)^{-1}}$$

UP.392 Find

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{(n!)((2n-1)!!)}}\cdot\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]$$

where $[*]$ great integer function.

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{(n!)((2n-1)!!)}}\cdot\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]=$$

$$=\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{(n!)((2n-1)!!)}}\cdot\frac{1}{n^2}\cdot\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right];(1)$$

$$\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{(n!)((2n-1)!!)}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{n!\cdot((2n-1)!!)}}\overset{C-D'A}{=}$$

$$=\lim_{n\to\infty}\frac{(n+1)^{2n+2}}{(n+1)!\,((2n+1)!!)}\cdot\frac{n!\cdot((2n-1)!!)}{n^{2n}}=\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\left(\frac{n+1}{n}\right)^{2n}=\frac{e^2}{2}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\cdot\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]\overset{C-S}{=}\lim_{n\to\infty}\frac{\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]}{(n+1)^2-n^2}$$

$$=\lim_{n\to\infty}\frac{\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]}{2n+1};(2)$$

We have:

$$\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]\le\left(\sqrt{n+2}+\sqrt{n+1}\right)^2<\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]+1,\forall n\ge 1$$

Hence,

$$\frac{\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]}{2n+1}\le\frac{\left(\sqrt{n+2}+\sqrt{n+1}\right)^2}{2n+1}<\frac{\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]}{2n+1}+\frac{1}{2n+1},\forall n\ge 1$$

$$\lim_{n\to\infty}\frac{\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]}{2n+1}\le 2<\lim_{n\to\infty}\frac{\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]}{2n+1}+\lim_{n\to\infty}\frac{1}{2n+1}\Rightarrow$$

$$\lim_{n\to\infty}\frac{\left[\left(\sqrt{n+2}+\sqrt{n+1}\right)^2\right]}{2n+1}=2$$

Therefore,

$$=\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{(n!)\left((2n-1)!!\right)}}\cdot\frac{1}{n^2}\cdot\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]\overset{(1)}{=}\frac{e^2}{2}\cdot 2=e^2$$

Solution 2 by Syed Shahabudeen-Kerala-India

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[n]{(n!)(2n-1)!!}}\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]=\lim_{n\to\infty}\frac{2}{\sqrt[n]{2n!}}\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]=$$

$$\because (2n-1)!!=\frac{(2n)!}{2^n n!}$$

$$=\lim_{n\to\infty}\frac{2n^2}{\sqrt[n]{2n!}}\cdot\frac{1}{n^2}\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]=L\cdot S$$

$$L=2\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{2n!}}\overset{Stirling}{=}2\lim_{n\to\infty}\frac{n^2}{(4n\pi)^{\frac{1}{2n}}\left(\frac{2n}{e}\right)^2}=\frac{1}{2}\lim_{n\to\infty}\frac{e^2}{(4n\pi)^{\frac{1}{2n}}}=\frac{e^2}{2}$$

$$S=\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]\overset{Stolz}{=}\lim_{n\to\infty}\frac{\left[\left(\sqrt{n+1}+\sqrt{n}\right)^2\right]}{2n+1}\le$$

$$=\lim_{n\to\infty}\frac{2n+1+2\sqrt{n(n+1)}}{2n+1}=2$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[n]{(n!)(2n-1)!!}}\sum_{k=1}^{n}\left(\sqrt{k+1}+\sqrt{k}\right)^2=\frac{e^2}{2}\cdot 2=e^2$$

Solution 3 by Kaushik Mahanta-Assam-India

$$\sum_{k=1}^{n}\left[\left(\sqrt{k+1}+\sqrt{k}\right)^2\right]=\sum_{k=1}^{n}\left(2k+1+\left[\sqrt{(2k+1)^2-1}\right]\right)=\sum_{k=1}^{n}(4k+1)=2n^2+3n$$

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[n]{(n!)(2n-1)!!}}\sum_{k=1}^{n}\left(\sqrt{k+1}+\sqrt{k}\right)^2=\lim_{n\to\infty}\frac{1}{\sqrt[n]{(n!)(2n-1)!!}}(2n^2+3n)=$$

$$= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt[n]{(n!)\left(\frac{(2n)!}{2^n(n!)}\right)}} = \lim_{n\to\infty} \frac{2(2n^2+3n)}{\sqrt[n]{(2n)!}} =$$

$$\because (2n)! = \sqrt{2\pi\cdot 2n}\left(\frac{2n}{e}\right)^{2n}$$

$$= \lim_{n\to\infty} \frac{2(2n^2+3n)}{(2\pi\cdot 2n)^{\frac{1}{2n}}\left(\frac{2n}{e}\right)^{\frac{2n}{n}}} = \lim_{n\to\infty} \frac{2(2n^2+3n)}{(2n)^2}\cdot e^2 = e^2$$

Solution 4 by Ravi Prakash-New Delhi-India

$$\text{Let: } a_k = \left[\left(\sqrt{k}+\sqrt{k+1}\right)^2\right], b_n = \sum_{k=1}^{n} a_k \text{. We have:}$$

$$2k < 2\sqrt{k(k+1)} < 2k+1 \Leftrightarrow 4k^2 < 4k(k+1) < 4k^2+4k+1$$

$$4k+1 < \left(\sqrt{k}+\sqrt{k+1}\right)^2 < 4k+2 \Leftrightarrow \left[\left(\sqrt{k}+\sqrt{k+1}\right)^2\right] = 4k+1$$

$$\Rightarrow b_n = \sum_{k=1}^{n} a_k = 2n(n+1)+n = n(2n+3)$$

$$\text{Let: } c_n = \frac{b_n}{\sqrt[n]{n!\,(2n+1)!!}} = \sqrt[n]{\frac{n^n(2n+3)^n}{n!\,(2n+1)!!}} = \sqrt[n]{d_n}, \text{ where } d_n = \frac{n^n(2n+3)^n}{n!\,(2n+1)!!}; \forall n \in \mathbb{N}^*$$

$$\frac{d_{n+1}}{d_n} = \frac{(n+1)^{n+1}(2n+5)^{n+1}}{(n+1)!\,(2n+3)!!}\cdot\frac{n!\,(2n+1)!!}{n^n(2n+3)^n} = \left(1+\frac{1}{n}\right)^n\left(1+\frac{2}{2n+3}\right)^{n+1} =$$

$$= \left(1+\frac{1}{n}\right)^n\left[\left(1+\frac{2}{2n+3}\right)^{\frac{2n+3}{2}}\right]^{\frac{2n+1}{2n+3}} \Rightarrow \lim_{n\to\infty}\frac{d_{n+1}}{d_n} = e^2$$

We show that

$$\lim_{n\to\infty}\sqrt[n]{d_n} = \lim_{n\to\infty}\frac{d_{n+1}}{d_n} = e^2$$

Given $0 < \varepsilon < 1$ there exists a positive integer m such that:

$$\left|\frac{d_{n+1}}{d_n} - e^2\right| < \varepsilon, \forall n > m$$

$$\text{Let: } k_1 = \min\left\{d_1, \frac{d_2}{d_1}, \dots, \frac{d_m}{d_{m-1}}\right\} \text{ and } k_2 = \max\left\{d_1, \frac{d_2}{d_1}, \dots, \frac{d_m}{d_{m-1}}\right\}$$

Also, for $n > m: e^2 - \varepsilon < \frac{d_{n+1}}{d_n} > e^2 + \varepsilon$

For $n > m: k_1^m (e^2 - \varepsilon)^{n-m} \le d_1 \cdot \frac{d_2}{d_1} \cdot \ldots \cdot \frac{d_m}{d_{m-1}} \cdot \frac{d_{m+1}}{d_m} \cdot \ldots \cdot \frac{d_n}{d_{n-1}} \le k_2^m (e^2 + \varepsilon)^{n-m}$

$$\Rightarrow k_1^{\frac{m}{n}} (e^2 - \varepsilon)^{1-\frac{m}{n}} \le d_n^{\frac{1}{n}} \le k_2^{\frac{m}{n}} (e^2 + \varepsilon)^{1-\frac{m}{n}}$$

$$e^2 - \varepsilon \le \lim_{n\to\infty} \sqrt[n]{d_n} \le e^2 + \varepsilon \Rightarrow \lim_{n\to\infty} \sqrt[n]{d_n} = e^2$$

UP.393 Calculate the integral:

$$\Omega = \int_0^\infty \frac{\tan^{-1} x}{x^4 + 1} dx$$

It is required to express the integral value with the usual mathematical constants and $\psi_1\left(\frac{3}{8}\right)$, where $\psi_1(x)$ is the trigamma function.

Proposed by Vasile Mircea Popa-Romania

Solution 1 by proposer

Consider the integral with parameter:

$$\Omega(\mathrm{a}) = \int_0^\infty \frac{\tan^{-1}(ax)}{x^4 + 1} dx \Rightarrow \Omega'(\mathrm{a}) = \int_0^\infty \frac{x}{(x^4 + 1)(1 + a^2 x^2)} dx$$

We make notations: $a^2 = b$ and we have:

$$\Omega^1(a) = \int_0^\infty \frac{x}{(x^4 + 1)(1 + bx^2)} dx \overset{u=x^2;du=2xdx}{=} \frac{1}{2} \int_0^\infty \frac{1}{(u^2 + 1)(1 + bu)} du$$

We can write: $\frac{1}{(u^2+1)(1+bu)} = \frac{A}{1+bu} + \frac{Bu+C}{u^2+1}$, where A, B, C have the following values:

$$A = \frac{b^2}{b^2 + 1}, B = -\frac{b}{b^2 + 1}, C = \frac{1}{b^2 + 1}$$

We have:

$$\int \frac{A}{1 + bu} du = \frac{A}{b} \log(1 + bu) + K$$

$$\int \frac{Bu + C}{u^2 + 1} du = \frac{B}{2} \log(u^2 + 1) + C \tan^{-1} u + K$$

Hence,

$$\int_0^\infty \frac{1}{(u^2 + 1)(1 + bu)} du = \frac{A}{b} \log(1 + bu) + \frac{B}{2} \log(u^2 + 1) + C \tan^{-1} u + K$$

We make notation:

$$P(u) = \int \frac{1}{(u^2 + 1)(1 + bu)} du = Q(u) + R(u), \text{where}: Q(u) = C \tan^{-1} u$$

$$R(u) = \frac{A}{b} \log(1 + bu) + \frac{B}{2} \log(u^2 + 1)$$

We calculate:

$$\Delta_1 = \lim_{u\to\infty} Q(u) - Q(0)\text{, we obtain: } \Delta_1 = \frac{\pi}{2(b^2+1)} \Rightarrow C = \frac{1}{b^2+1}$$

We calculate:

$$\Delta_2 = \lim_{u\to\infty} R(u) - R(0)$$

We can write: $R(u) = \frac{b}{b^2+1}\log\left(\frac{bu+1}{\sqrt{u^2+1}}\right)$ because $\frac{A}{b} = \frac{b}{b^2+1}$ and $\frac{B}{2} = -\frac{b}{2(b^2+1)}$.

We obtain: $\Delta_2 = \frac{b}{b^2+1}\log b$.

Therefore, using Newton-Leibniz formula, we can write:

$$\int_0^\infty \frac{1}{(u^2+1)(1+bu)}du = \Delta_1 + \Delta_2 = \frac{\pi + 2b\log b}{2+2b^2}\text{, or:}$$

$$\int_0^\infty \frac{1}{(u^2+1)(1+a^2u)}du = \frac{\pi + 2a^2\log b}{2+2a^4}$$

$$\Omega^1(a) = \frac{1}{2}\cdot\frac{\pi + 2a^2\log b}{2(1+a^4)}\text{, because } \Omega(0) = 0 \text{ we have:}$$

$$\Omega = \Omega(1) = \int_0^1 \Omega^1(a)\,da$$

$$\Omega = \frac{\pi}{4}\int_0^1 \frac{1}{1+a^4}da + \int_0^1 \frac{a^2\log a}{1+a^4}da\,;(1)$$

The first integral in the above relation are integral of rational function and are calculated relatively easily with the Newton-Leibniz formula. The integral have the following value:

$$\int_0^1 \frac{1}{1+a^4}da = \frac{\sqrt{2}}{8}\left(\pi + 2\log\left(1+\sqrt{2}\right)\right);(2)$$

We calculate now the second integral: $\int_0^1 \frac{x^2\log x}{1+x^4}dx$

We make the notation: $f(x) = \frac{x^2}{1+x^4}$ and develop the function in power series.

We have for $x \in (0,1)$: $f(x) = x^2 - x^6 + x^{10} - x^{14} + x^{18} - x^{22} + x^{26} - x^{30} + x^{34} - x^{38}$ .... We obtain:

$$\int_0^1 f(x)\log x\,dx = -\frac{1}{3^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{15^2} - \frac{1}{19^2} + \frac{1}{23^2} - \frac{1}{27^2} + \frac{1}{31^2} - \frac{1}{35^2} + \frac{1}{39^2} - \cdots$$

$$= -\left(\frac{1}{3^2} + \frac{1}{11^2} + \frac{1}{19^2} + \frac{1}{27^2} + \frac{1}{35^2} + \cdots\right) + \left(\frac{1}{7^2} + \frac{1}{15^2} + \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{39^2} + \cdots\right)$$

Now we will use the trigamma function, which is defined by the relationship:

$$\psi_1(x) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

$$\text{We can write: } \psi_1\left(\frac{3}{8}\right) = \sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{8}+n\right)^2}; \psi_2\left(\frac{7}{8}\right) = \sum_{n=0}^{\infty}\frac{1}{\left(\frac{7}{8}+n\right)^2}$$

$$\text{So, } \int_0^1 f(x)\log x\,dx = \frac{1}{64}\left(-\psi_1\left(\frac{3}{8}\right)+\psi_1\left(\frac{7}{8}\right)\right)$$

But we have the following equality (the duplication formula):

$$4\psi_1(2x) = \psi_1(x)+\psi_1\left(x+\frac{1}{2}\right), \text{which for } x=\frac{3}{8} \text{ we gives us:}$$

$$4\psi_1\left(\frac{3}{4}\right) = \psi_1\left(\frac{3}{8}\right)+\psi_1\left(\frac{7}{8}\right)$$

The following special value is known:

$$\psi_1\left(\frac{3}{4}\right) = \pi^2 - 8G, \text{where G} - \textit{is Catalan's} \text{ constant.}$$

Therefore, we can write:

$$\int_0^1 \frac{x^2\log x}{x^4+1}dx = \frac{1}{64}\left(-2\psi_1\left(\frac{3}{8}\right)+4\pi^2-32G\right); (3)$$

Replacing the values of the integrals (2) and (3) in the relation (1), we obtain the value of the integral from the problem statement. This is:

$$\Omega = \frac{\sqrt{2}}{32}(1+\sqrt{2})\pi^2+\frac{\pi\sqrt{2}}{16}\log(1+\sqrt{2})-\frac{1}{32}\psi_1\left(\frac{3}{8}\right)-\frac{1}{2}G$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^\infty \frac{\tan^{-1}x}{x^4+1}dx = \int_0^\infty\int_0^\infty \frac{x}{(1+x^4)(1+x^2y^2)}dxdy =$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+y^4}\int_0^\infty\left(\frac{x}{1+x^4}-\frac{x^3y^2}{1+x^4}+\frac{xy^4}{1+x^2y^2}\right)dxdy =$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+y^4}\left[\tan^{-1}(x^2)+y^2\log(1+x^2y^2)-\frac{1}{2}y^2\log(1+x^4)\right]_0^\infty dy =$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+y^4}\left(\frac{\pi}{2}+2y^2\log y\right)dy$$

$$\Omega = \frac{\pi}{4}\underbrace{\int_0^1\frac{dy}{1+y^4}}_{A}+\underbrace{\int_0^1\frac{y^2\log y}{1+y^4}dy}_{B}$$

$$A = \frac{1}{2}\int_0^1\frac{y^2+1}{1+y^4}dy-\frac{1}{2}\int_0^1\frac{y^2-1}{1+y^4}dy$$

$$= \frac{1}{2}\int_0^1\frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy-\frac{1}{2}\int_0^1\frac{1-\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy \overset{\boxed{u=y-\frac{1}{y};\ v=y+\frac{1}{y}}}{=}$$

$$=\frac{1}{2}\int_{-\infty}^{0}\frac{du}{u^2+2}+\frac{1}{2}\int_{2}^{\infty}\frac{dv}{v^2-1}=\frac{\pi}{4\sqrt{2}}-\frac{\log(3-2\sqrt{2})}{4\sqrt{2}}$$

$$B=\int_0^1\frac{(y^2-y^6)\log y}{1-y^8}dy\overset{\boxed{y\to y^8}}{=}\frac{1}{64}\int_0^1\frac{\left(y^{\frac{3}{8}-1}-y^{\frac{7}{8}-1}\right)\log y}{1-y}dy=$$

$$=\frac{1}{64}\left\{\psi^{(1)}\left(\frac{7}{8}\right)-\psi^{(1)}\left(\frac{3}{8}\right)\right\}$$

Therefore,

$$\Omega=\int_0^\infty\frac{\tan^{-1}x}{x^4+1}dx=\frac{\pi^2}{16\sqrt{2}}-\frac{\pi\log(3-2\sqrt{2})}{16\sqrt{2}}+\frac{1}{64}\left\{\psi^{(1)}\left(\frac{7}{8}\right)-\psi^{(1)}\left(\frac{3}{8}\right)\right\}$$

UP.394

$$\Omega(a)=\int_1^a\frac{x}{\log(1+x^2)}dx,a>2$$

Prove that:

$$\frac{(a+b+c)^2}{3\sqrt{2}}+\frac{1}{\sqrt{2}}\log\left(\frac{a^2b^2c^2}{e^3}\right)\le\sum_{cyc}\Omega(a)\le(a+b+c)^2+\log\left(\frac{a^2b^2c^2}{e\sqrt{e}}\right)$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\because\frac{x-y}{\log x-\log y}\le\frac{x+y}{2};\forall x,y>0, let\ x>y\Rightarrow\frac{\frac{x}{y}-1}{\log\left(\frac{x}{y}\right)}\le\frac{\frac{x}{y}+1}{2};t=\frac{x}{y}>1\Rightarrow$$

$$\frac{t-1}{\log t}<\frac{t+1}{2},\forall t>1;(t\to t+1)\Leftrightarrow\frac{t}{\log(1+t)}<\frac{t}{2}+1;$$

$$(t=x^2)\Rightarrow\frac{x^2}{\log(1+x^2)}<\frac{1}{2}x^2+1\Leftrightarrow$$

$$\frac{x}{\log(1+x^2)}<\frac{1}{2}x+\frac{1}{x};\forall x>0;(1)$$

$$\because\log(1+t)\le\frac{t}{\sqrt{1+t}};\forall t>0\Leftrightarrow\sqrt{1+t}\le\frac{t}{\log(1+t)};\forall t>0\Rightarrow$$

$$\sqrt{1+x^2}\le\frac{x^2}{\log(1+x^2)}\text{ and }\sqrt{1+x^2}\ge\frac{1+x}{\sqrt{2}}\Rightarrow\frac{1}{\sqrt{2}}\left(x+\frac{1}{x}\right)\le\frac{x}{\log(1+x^2)};\forall x>0;(2)$$

From (1),(2) we get:

$$\frac{1}{\sqrt{2}}\left(x+\frac{1}{x}\right)<\frac{x}{\log(1+x^2)}<\frac{x}{2}+\frac{1}{x}$$

$$\frac{1}{\sqrt{2}}\int_1^a\left(x+\frac{1}{x}\right)dx\le\Omega(a)\le\int_1^a\left(\frac{x}{2}+\frac{1}{x}\right)dx\Leftrightarrow$$

$$\frac{1}{\sqrt{2}}\left(\frac{x^2}{2}+\log x\right)\Bigg|_1^a\le\Omega(a)\le\left(\frac{x^2}{4}+\log x\right)\Bigg|_1^a\Leftrightarrow$$

$$\frac{1}{\sqrt{2}}\left(\frac{a^2}{2}+\log a-\frac{1}{2}\right)\le\Omega(a)\le\frac{a^2}{4}+\log a-\frac{1}{4}$$

$$\frac{1}{\sqrt{2}}(a^2+\log a^2-1)\le\Omega(a)\le\frac{a^2}{2}+\log a^2-\frac{1}{2}$$

$$\frac{1}{\sqrt{2}}\left(a^2+\log\left(\frac{a^2}{e}\right)\right)\le\Omega(a)\le\frac{a^2}{2}+\log\left(\frac{a^2}{\sqrt{e}}\right)$$

Adding, we get:

$$\frac{1}{\sqrt{2}}\left(a^2+b^2+c^2+\log\left(\frac{a^2b^2c^2}{e^3}\right)\right)\le\sum_{cyc}\Omega(a)\le\frac{a^2+b^2+c^2}{2}+\log\left(\frac{a^2b^2c^2}{e\sqrt{e}}\right)\Leftrightarrow$$

$$\frac{(a+b+c)^2}{3\sqrt{2}}+\frac{1}{\sqrt{2}}\log\left(\frac{a^2b^2c^2}{e^3}\right)\le\sum_{cyc}\Omega(a)\le(a+b+c)^2+\log\left(\frac{a^2b^2c^2}{e\sqrt{e}}\right)$$

UP.395 For $a>0,b>1$ find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k+2a+b^{n-k}}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$\because\tan^{-1}\left(\frac{k}{n}\right)+\tan^{-1}\left(\frac{n-k}{n}\right)=\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)$$

Let us denote:

$$a_k=\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k+2a+b^{n-k}}=\frac{(a+b^k)\left(\tan^{-1}\left(\frac{k}{n}\right)+\tan^{-1}\left(\frac{n-k}{n}\right)\right)}{b^k+2a+b^{n-k}}$$

Hence,

$$a_k + a_{n-k} = \frac{(a+b^k)\left(\tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right)\right)}{b^k + 2a + b^{n-k}}$$

$$+\frac{(a+b^{n-k})\left(\tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right)\right)}{b^k + 2a + b^{n-k}} = \tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right)$$

and then,

$$2\sum_{k=1}^{n-1} a_k = \sum_{k=1}^{n}(a_k + a_{n-k}) = \sum_{k=1}^{n-1}\left(\tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right)\right) = 2\sum_{k=1}^{n-1}\tan^{-1}\left(\frac{k}{n}\right)$$

Therefore,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n-1} a_k + a_n = \sum_{k=1}^{n-1}\tan^{-1}\left(\frac{k}{n}\right) + \frac{(a+b^n)\tan^{-1}(1)}{b^n + 2a + 1} =$$

$$= \sum_{k=1}^{n-1}\tan^{-1}\left(\frac{k}{n}\right) + \frac{\pi}{4}\cdot\frac{a+b^n}{1+2a+b^n}$$

So, we have:

$$\sum_{k=1}^{n}\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k + 2a + b^{n-k}} = \sum_{k=1}^{n-1}\tan^{-1}\left(\frac{k}{n}\right) + \frac{\pi}{4}\cdot\frac{a+b^n}{1+2a+b^n}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k + 2a + b^{n-k}} = \frac{1}{n}\sum_{k=1}^{n-1}\tan^{-1}\left(\frac{k}{n}\right) + \frac{\pi}{4n}\cdot\frac{a+b^n}{1+2a+b^n} =$$

$$= \frac{1}{n}\sum_{k=1}^{n}\tan^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{4n} + \frac{\pi}{4n}\cdot\frac{a+b^n}{1+2a+b^n} =$$

$$= \frac{1}{n}\sum_{k=1}^{n}\tan^{-1}\left(\frac{k}{n}\right) + \frac{\pi}{4n}\left(\frac{a+b^n}{1+2a+b^n} - 1\right) = \frac{1}{n}\sum_{k=1}^{n}\tan^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{4n}\cdot\frac{a+1}{1+2a+b^n}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k + 2a + b^{n-k}} = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\tan^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{4n}\cdot\frac{a+1}{1+2a+b^n}\right)$$

$$= \int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \log\sqrt{2}$$

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$I(n)=\frac{1}{n}\sum_{k=1}^{n}\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k+2a+b^{n-k}}=$$

$$=\frac{1}{n}\sum_{k=0}^{n-1}\frac{(a+b^{n-k}\tan^{-1}\left(\frac{n^2}{n^2-nk+k^2}\right)}{b^k+2a+b^{n-k}}=$$

$$=\frac{1}{n}\sum_{k=1}^{n}\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k+2a+b^{n-k}}+\frac{1}{n}\cdot\frac{\pi}{4}\cdot\frac{a+b^n}{1+2a+b^n}+\frac{1}{n}\cdot\frac{\pi}{4}\cdot\frac{a+1}{1+2a+b^n}$$

$$\Rightarrow 2I(n)=\frac{1}{n}\sum_{k=1}^{n}\frac{(2a+b^{n-k}+b^k)\tan^{-1}\left(\frac{n^2}{n^2-nk+k^2}\right)}{b^k+2a+b^{n-k}}+\frac{\pi}{4}\cdot\frac{a+b^n}{1+2a+b^n}+$$

$$+\frac{a+1}{1+2a+b^n}\cdot\frac{\pi}{4}=\frac{1}{n}\sum_{k=1}^{n}\tan^{-1}\left(\frac{n^2}{n^2-nk+k^2}\right)+\frac{\pi}{2n}$$

$$I(n)=\frac{1}{2n}\sum_{k=1}^{n}\tan^{-1}\left(\frac{n^2}{n^2-nk+k^2}\right)+\frac{\pi}{8n}$$

$$\lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^{n}\tan^{-1}\left(\frac{n^2}{n^2-nk+k^2}\right)+\frac{\pi}{8n}=\int_0^1\tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx=$$

$$=\int_0^1(\tan^{-1}x+\tan^{-1}(1-x))dx=$$

$$=\left[x\tan^{-1}x-\frac{1}{2}\log(x^2+1)\right]_0^1+\left[\frac{1}{2}\log(x^2-2x+2)+(x-1)\tan^{-1}(1-x)\right]_0^1=$$

$$=\frac{1}{2}(\pi-2\log 2)$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k+2a+b^{n-k}}=\frac{1}{4}(\pi-2\log 2)$$

Solution 3 by Ravi Prakash-New Delhi-India

For $0\le k\le n$ let

$$a_k=\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k+2a+b^{n-k}}$$

$$\Rightarrow a_{n-k}=\frac{(a+b^{n-k})\tan^{-1}\left(\frac{n^2}{n^2-(n-k)n+(n-k)^2}\right)}{b^{n-k}+2a+b^n}$$

$$a_k+a_{n-k}=\tan^{-1}\left(\frac{1}{1-\frac{k}{n}\left(1-\frac{k}{n}\right)}\right)=\tan^{-1}\left(\frac{\left(1-\frac{k}{n}\right)+\frac{k}{n}}{1-\frac{k}{n}\left(1-\frac{k}{n}\right)}\right)=$$

$$=\tan^{-1}\left(1-\frac{k}{n}\right)+\tan^{-1}\left(\frac{k}{n}\right)=\tan^{-1}\left(\frac{n-k}{n}\right)+\tan^{-1}\left(\frac{k}{n}\right)$$

$$\text{Let: } S=\sum_{k=1}^{n}a_k\Rightarrow S+a_0=a_0+a_1+\cdots+a_n=a_n+\cdots+a_1+a_0$$

$$2S+2a_0=\sum_{k=1}^{n}(a_k+a_{n-k})=\sum_{k=0}^{n}\left(\tan^{-1}\left(\frac{n-k}{n}\right)+\tan^{-1}\left(\frac{k}{n}\right)\right)=$$

$$=2\sum_{k=0}^{n}\tan^{-1}\left(\frac{k}{n}\right)\Rightarrow S+a_0=\sum_{k=0}^{n}\tan^{-1}\left(\frac{k}{n}\right)$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(a+b^k)\tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)}{b^k+2a+b^{n-k}}=\int_0^1\tan^{-1}x\,dx=\frac{\pi}{4}-\log\sqrt{2}$$

UP.396. Find:

$$\Omega(a)=\int_{\frac{1}{a}}^{a}\frac{\sin^2 x\cdot\tan\left(\frac{1}{x^2}\right)}{(1+x^2)\left(\sin^2\left(\frac{1}{x}\right)\tan\left(\frac{1}{x^2}\right)+\sin^2 x\tan(x^2)\right)}dx$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$t=\frac{1}{x}\Rightarrow x=\frac{1}{t},dx=-\frac{1}{t^2}dt,x=\frac{1}{a}\Rightarrow t=a;x=a\Rightarrow t=\frac{1}{a}$$

Hence,

$$\Omega(a)=\int_a^{\frac{1}{a}}\frac{\sin^2\left(\frac{1}{t}\right)\tan t^2\cdot\left(-\frac{1}{t^2}\right)dt}{\left(1+\frac{1}{t^2}\right)\left(\sin^2 t\tan^2\left(\frac{1}{t^2}\right)+\sin^2\left(\frac{1}{t}\right)\tan t^2\right)}=$$

$$=\int_{\frac{1}{a}}^{a}\frac{\sin^2\left(\frac{1}{t}\right)\tan t^2}{(1+t^2)\left(\sin^2 t\tan^2\left(\frac{1}{t^2}\right)+\sin^2\left(\frac{1}{t}\right)\tan t^2\right)}dt$$

Therefore,

$$2\Omega(a)=\int_{\frac{1}{a}}^{a}\frac{\sin^2 x\tan\left(\frac{1}{x^2}\right)+\sin^2\left(\frac{1}{x}\right)\tan x^2}{(1+x^2)\left(\sin^2 x\tan\left(\frac{1}{x^2}\right)+\sin^2\left(\frac{1}{x}\right)\tan x^2\right)}dx=\int_{\frac{1}{a}}^{a}\frac{dx}{1+x^2}$$

$2\Omega(a)=\tan^{-1}a-\tan^{-1}\left(\frac{1}{a}\right)$. Hence,

$$\Omega(a)=\frac{1}{2}\left(\tan^{-1}a-\tan^{-1}\left(\frac{1}{a}\right)\right)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega(a)=\int_{\frac{1}{a}}^{a}\frac{\sin^2 x\cdot\tan\left(\frac{1}{x^2}\right)}{(1+x^2)(\sin^2 x\tan\left(\frac{1}{x^2}\right)+\sin^2\left(\frac{1}{x^2}\right)\tan(x^2)}dx\overset{x=\frac{1}{x}}{=}$$

$$=\int_{\frac{1}{a}}^{a}\frac{\sin^2\left(\frac{1}{x^2}\right)\tan(x^2)}{(1+x^2)\left(\sin^2 x\tan\left(\frac{1}{x^2}\right)+\sin^2\left(\frac{1}{x^2}\right)\tan(x^2)\right)}dx$$

$$2\Omega(a)=\int_{\frac{1}{a}}^{a}\frac{dx}{1+x^2}=\tan^{-1}a-\tan^{-1}\left(\frac{1}{a}\right)$$

$$\Omega(a)=\tan^{-1}(a)-\frac{\pi}{4}sgn(a)$$

UP.397 Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\right)^2\left(e^{H_{2n+2}-H_{n+1}}-e^{H_{2n}-H_n}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposer

Denote $x_n=H_{2n}-H_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$

$$x_{n+1}-x_n=\frac{1}{(2n+1)(2n+2)};n\in\mathbb{N}^*,\lim_{n\to\infty}(x_{n+1}-x_n)=0$$

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}=\int_0^1\frac{1}{1+x}dx=\log 2$$

$$\lim_{n\to\infty}\frac{e^{x_{n+1}}-e^{x_n}}{x_{n+1}-x_n}=\lim_{n\to\infty}\frac{e^{x_n}(e^{x_{n+1}-x_n}-1)}{x_{n+1}-x_n}=\lim_{n\to\infty}\frac{e^{x_{n+1}-x_n}-1}{x_{n+1}-x_n}\cdot\lim_{n\to\infty}e^{x_n}=1\cdot e^{\log 2}=2$$

Hence,

$$\Omega=\lim_{n\to\infty}(e^{x_{n+1}}-e^{x_n})\left(\sqrt[n]{(2n-1)!!}\right)^2=\lim_{n\to\infty}\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^2\cdot n^2\cdot(e^{x_{n+1}}-e^{x_n})=$$

$$=\frac{4}{e^2}\cdot\lim_{n\to\infty}\frac{e^{x_{n+1}}-e^{x_n}}{x_{n+1}-x_n}\cdot n^2(x_{n+1}-x_n)=\frac{4}{e^2}\cdot 2\cdot\lim_{n\to\infty}\frac{n^2}{(2n+1)(2n+2)}=\frac{8}{e^2}\cdot\frac{1}{4}=\frac{2}{e^2}$$

Observation:

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D}{=}\lim_{n\to\infty}\frac{(2n+!)!!}{(n+!)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\lim_{n\to\infty}\frac{(2n-1)!!\,(2n+1)}{(2n-1)!!\,(n+1)}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}=2\cdot\frac{1}{e}=\frac{2}{e}$$

Solution 2 by Asmat Qatea-Afghanistan

$$(2n-1)!!=\frac{(2n)!}{2^n n!},H_n\cong\gamma+\log n+\frac{1}{2n}+o\left(\frac{1}{n^2}\right);n!\cong\sqrt{2n\pi}\left(\frac{n}{e}\right)^n$$

$$\Omega=4\lim_{n\to\infty}\left(\sqrt[n]{\frac{(2n)!}{n!}}\right)^2\left(e^{\log(2n+2)+\gamma+\frac{1}{4n+4}-\left(\log(n+1)+\gamma+\frac{1}{2n+2}\right)}-e^{\log(2n)+\gamma+\frac{1}{4n}-\left(\log n+\gamma+\frac{1}{2n}\right)}\right)$$

$$=$$

$$=4\lim_{n\to\infty}\left(\sqrt[n]{\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}}\right)^2\left(e^{\log 2-\frac{1}{4(n+1)}}-e^{\log 2-\frac{1}{4n}}\right)=$$

$$=\frac{1}{e^2}\lim_{n\to\infty}(\sqrt[n]{2}\cdot n^2)\left(2\cdot e^{-\frac{1}{4(n+1)}}-2\cdot e^{-\frac{1}{4n}}\right)=$$

$$=\frac{8}{e^2}\lim_{n\to\infty}\left(n^2\cdot e^{-\frac{1}{4n}}\right)\left(\frac{e^{-\frac{1}{4(n+1)}+\frac{1}{4n}}-1}{-\frac{1}{4(n+1)}+\frac{1}{4n}}\right)\cdot\left(-\frac{1}{4(n+1)}+\frac{1}{4n}\right)=$$

$$=\frac{2}{e^2}\lim_{n\to\infty}n^2\cdot\frac{1}{n(n+1)}=\frac{2}{e^2}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\right)^2\left(e^{H_{2n+2}-H_{n+1}}-e^{H_{2n}-H_n}\right)=\frac{2}{e^2}$$

UP.398 If $0<a\le b$ then:

$$\int_0^{\sqrt{ab}}e^{-x^2}\,dx-\int_0^{\frac{a+b}{2}}e^{-x^2}\,dx\ge\tan^{-1}(\sqrt{ab})-\tan^{-1}\left(\frac{a+b}{2}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}, f(x)=\tan^{-1}x-\int_0^x e^{-t^2}\,dt$ then,

$$f'(x)=\frac{1}{1+x^2}-e^{-x^2}=\frac{e^{x^2}-(x^2+1)}{(1+x^2)e^{x^2}}\ge 0\Rightarrow$$

f –increasing. It is well-known that: $e^{x^2}\ge x^2+1, \forall x\ge 0$

$\sqrt{ab}\le\frac{a+b}{2}$, f –increasing, then $f(\sqrt{ab})\le f\left(\frac{a+b}{2}\right)$

$$\tan^{-1}(\sqrt{ab})-\int_0^{\sqrt{ab}}e^{-t^2}\,dt\le\tan^{-1}\left(\frac{a+b}{2}\right)-\int_0^{\frac{a+b}{2}}e^{-t^2}\,dt$$

Hence,

$$\int_0^{\sqrt{ab}}e^{-x^2}\,dx-\int_0^{\frac{a+b}{2}}e^{-x^2}\,dx\ge\tan^{-1}(\sqrt{ab})-\tan^{-1}\left(\frac{a+b}{2}\right)$$

Equality holds for $a=b$.

Solution 2 by Ravi Prakash-New Delhi-India

Let $g_1(x)=(1+x^2)e^{-x^2}-1, x\ge 0$ then

$$g_1'(x)=2xe^{-x^2}-2x(1+x^2)e^{-x^2}=-2x^3e^{-x^2}<0,\forall x>0$$

$$\Rightarrow g_1(x)<g_1(0)=1\Rightarrow e^{-x^2}<\frac{1}{1+x^2};\forall x>0$$

$$\text{Let: } g(x)=\int_0^x e^{-t^2}dt-\int_0^x \frac{dt}{1+t^2}, x\ge 0$$

$$g'(x)=e^{-x^2}-\frac{1}{1+x^2}<0; \forall x>0 \Rightarrow g(x) \text{ decreases on } [0,\infty)$$

$$\frac{a+b}{2}\ge\sqrt{ab}, \forall\, a,b>0 \Rightarrow g\left(\frac{a+b}{2}\right)\le g(\sqrt{ab}).$$

$$\Rightarrow \int_0^{\frac{a+b}{2}} e^{-x^2}dx-\int_0^{\frac{a+b}{2}}\frac{dx}{1+x^2}\le\int_0^{\sqrt{ab}}e^{-x^2}dx-\int_0^{\sqrt{ab}}\frac{dx}{1+x^2}$$

$$\Rightarrow \int_0^{\frac{a+b}{2}} e^{-x^2}dx-\int_0^{\sqrt{ab}}e^{-x^2}dx\le\tan^{-1}\left(\frac{a+b}{2}\right)-\tan^{-1}(\sqrt{ab})$$

$$\int_0^{\sqrt{ab}}e^{-x^2}dx-\int_0^{\frac{a+b}{2}}e^{-x^2}dx\ge\tan^{-1}(\sqrt{ab})-\tan^{-1}\left(\frac{a+b}{2}\right)$$

Equality holds for $a=b$.

Solution 3 by Adrian Popa-Romania

$$0<a\le b\Rightarrow\sqrt{ab}\le\frac{a+b}{2}$$

$$\int_0^{\frac{a+b}{2}}e^{-x^2}dx=\int_0^{\sqrt{ab}}e^{-x^2}dx+\int_{\sqrt{ab}}^{\frac{a+b}{2}}e^{-x^2}dx$$

$$\int_0^{\sqrt{ab}}e^{-x^2}dx-\int_0^{\frac{a+b}{2}}e^{-x^2}dx=-\int_{\sqrt{ab}}^{\frac{a+b}{2}}e^{-x^2}dx=\int_{\frac{a+b}{2}}^{\sqrt{ab}}e^{-x^2}dx$$

$$\tan^{-1}(\sqrt{ab})-\tan^{-1}\left(\frac{a+b}{2}\right)=\tan^{-1}x\Big|_{\frac{a+b}{2}}^{\sqrt{ab}}=\int_{\frac{a+b}{2}}^{\sqrt{ab}}\frac{1}{1+x^2}dx$$

So, we must prove that:

$$\int_{\frac{a+b}{2}}^{\sqrt{ab}}e^{-x^2}dx\ge\int_{\frac{a+b}{2}}^{\sqrt{ab}}\frac{1}{1+x^2}dx\Leftrightarrow$$

$$-e^{-x^2}\ge-\frac{1}{1+x^2}\Leftrightarrow e^{-x^2}\le\frac{1}{x^2+1}\Leftrightarrow\frac{1}{e^{x^2}}\le\frac{1}{1+x^2}\Leftrightarrow e^{x^2}\ge x^2+1\ (\text{true})$$

Equality holds for $a=b$.

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\because e^t\ge 1+t; \forall t\ge 0 \overset{t=x^2}{\Longrightarrow} e^{x^2}\ge 1+x^2\Rightarrow\frac{1}{1+x^2}-e^{x^2}\ge 0; \forall x\ge 0$$

$$\text{Let: } f(y)=\int_0^y\left(\frac{1}{1+x^2}-e^{-x^2}\right)dx \text{ then } f'(y)=\frac{1}{1+y^2}-e^{-y^2}\ge 0\Rightarrow f\nearrow$$

$$\text{For } \frac{a+b}{2}\ge\sqrt{ab}\Rightarrow f\left(\frac{a+b}{2}\right)\ge f\left(\sqrt{ab}\right)$$

$$\Rightarrow\int_0^{\frac{a+b}{2}}\left(\frac{1}{1+x^2}-e^{-x^2}\right)dx\ge\int_0^{\sqrt{ab}}\left(\frac{1}{1+x^2}-e^{-x^2}\right)dx$$

$$\Rightarrow\int_0^{\sqrt{ab}}e^{-x^2}dx-\int_0^{\frac{a+b}{2}}e^{-x^2}dx\ge\int_0^{\sqrt{ab}}\frac{1}{1+x^2}dx-\int_0^{\frac{a+b}{2}}\frac{1}{1+x^2}dx\ge$$

$$\ge\tan^{-1}\left(\sqrt{ab}\right)-\tan^{-1}\left(\frac{a+b}{2}\right)$$

Equality holds for $a=b$.

UP.399 Find:

$$\Omega(t)=\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{t}{x+1}}-\left(\Gamma(x+1)\right)^{\frac{t}{x}}\right)\cdot x^{1-t};\, t>0$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution by proposers

$$\text{Denote: } u(x)=\frac{\left(\Gamma(x+2)\right)^{\frac{t}{x+1}}}{\left(\Gamma(x+1)\right)^{\frac{t}{x}}};\ \lim_{x\to\infty}\frac{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}{x}=\frac{1}{e}$$

$$\lim_{x\to\infty}u(x)=\lim_{x\to\infty}\left(\frac{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}{x+1}\right)^t\cdot\left(\frac{x}{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}\right)^t\cdot\left(\frac{x+1}{x}\right)^t=\frac{1}{e^t}\cdot e^t\cdot 1=1$$

$$\lim_{x\to\infty}\frac{u(x)-1}{\log(u(x))}=1;$$

$$\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\right)^t\cdot\frac{1}{\left(\Gamma(x+2)\right)^{\frac{t}{x+1}}}=\lim_{x\to\infty}(x+1)^t\cdot\frac{1}{\left(\Gamma(x+2)\right)^{\frac{t}{x+1}}}=$$

$$=\lim_{x\to\infty}\left(\frac{x+1}{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}\right)^t=e^t$$

Hence,

$$\Omega(t)=\lim_{x\to\infty}\left(\left(\Gamma(x+1)\right)^{\frac{t}{x}}\right)(u(x)-1)x^{1-t}=$$

$$=\lim_{x\to\infty}\left(\frac{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}{x}\right)^t\cdot\frac{u(x)-1}{\log u(x)}\cdot\log\left(u(x)\right)^x=\frac{1}{e^t}\cdot 1\cdot\log e^t=\frac{t}{e^t}$$

UP.400 If $0<a\le b<\frac{\pi}{2}$ then:

$$\int_a^b\frac{(3-\tan x)^2}{\cos^2\left(\sqrt{\tan x}\right)}dx+\int_a^b\frac{(3-\cot x)^2}{\cos^2\left(\sqrt{\cot x}\right)}dx\ge 18(b-a)+3\log(\cot a\cdot\tan b)$$

Proposed by Florică Anastase-Romania

Solution by proposer

From MacLaurin series expansion for $f(x)=\tan x$, we have that:

$$\tan x\ge x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}$$

Hence,

$$(3-x^2)\tan x-3x\ge(3-x^2)\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}\right)-3x=$$

$$=\left(3x+x^3+\frac{6x^5}{15}+\frac{51x^7}{315}\right)-\left(x^3+\frac{x^5}{3}+\frac{2x^7}{15}+\frac{17x^9}{315}\right)-3x=$$

$$=\frac{x^5(21+9x^2-17x^4)}{315}\ge 0$$

Hence,

$$(3-x^2)\tan x\ge 3x,\forall x\in\left(0,\frac{\pi}{2}\right)\Leftrightarrow\tan x\ge\frac{3x}{3-x^2},\forall x\in\left(0,\frac{\pi}{2}\right)$$

Integrating (1) we have:

$$\int_0^x\tan t\,dt\ge\int_0^x\frac{3t}{3-t^2}dt\Rightarrow-\log(\cos x)\ge-\frac{3}{2}\left(\log(3-x^2)-\log 3\right)$$

$$\cos^2 x\le\left(\frac{3-x^2}{3}\right)^3,\forall x\in\left(0,\frac{\pi}{2}\right)\Leftrightarrow\frac{(3-x^2)^2}{\cos^2 x}\ge 9+3x^2,\forall x\in\left(0,\frac{\pi}{2}\right);(2)$$

Now, in (2) putting $x\to\sqrt{\tan x}$ and $x\to\sqrt{\cot x}$, we get:

$$\begin{cases} \dfrac{(3-\tan x)^2}{\cos^2(\sqrt{\tan x})} \geq 9+3\tan x \\ \dfrac{(3-\cot x)^2}{\cos^2(\sqrt{\cot x})} \geq 9+3\cot x \end{cases} \Rightarrow$$

$$\frac{(3-\tan x)^2}{\cos^2(\sqrt{\tan x})}+\frac{(3-\cot x)^2}{\cos^2(\sqrt{\cot x})} \geq 18+3(\tan x+\cot x)$$

Hence,

$$\int_a^b \frac{(3-\tan x)^2}{\cos^2(\sqrt{\tan x})}dx+\int_a^b \frac{(3-\cot x)^2}{\cos^2(\sqrt{\cot x})}dx \geq 18(b-a)+3\log(\cot a\cdot\tan b)$$

Equality holds for $a=b$.

UP.401 If $0<a<b<\frac{\pi}{2}$ then:

$$\int_a^b\left(\frac{\cos x}{\tan(\sin x)}+\frac{\sin x}{\tan(\cos x)}+\frac{\sin x\cdot\tan x}{\tan(\sin x)}+\frac{\cos x\cdot\cot x}{\tan(\cos x)}\right)dx \leq \frac{5}{3}\log(\cot a\cdot\tan b)$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\frac{\cos x}{\tan(\sin x)}+\frac{\sin x}{\tan(\cos x)}+\frac{\sin x\cdot\tan x}{\tan(\sin x)}+\frac{\cos x\cdot\cot x}{\tan(\cos x)}=$$

$$=\frac{\cos x}{\tan(\sin x)}+\frac{\sin x}{\tan(\cos x)}+\frac{\sin^2 x}{\cos x\cdot\tan(\sin x)}+\frac{\cos^2 x}{\sin x\cdot\tan(\cos x)}=$$

$$=\left(\frac{\sin x}{\tan(\sin x)}+\frac{\cos x}{\tan(\cos x)}\right)(\tan x+\cot x);\ (1)$$

Now, from Maclaurin series expansion for $f(x)=\tan x$, we have that:

$$\tan x \geq x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}$$

Hence,

$$(3-x^2)\tan x-3x \geq (3-x^2)\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}\right)-3x=$$

$$=\left(3x+x^3+\frac{6x^5}{15}+\frac{51x^7}{315}\right)-\left(x^3+\frac{x^5}{3}+\frac{2x^7}{15}+\frac{17x^9}{315}\right)-3x=$$

$$=\frac{x^5(21+9x^2-17x^4)}{315} \geq 0$$

Hence,

$$(3-x^2)\tan x \geq 3x, \forall x \in \left(0, \frac{\pi}{2}\right)$$

For $x \to \sin x$ and $x \to \cos x$ it follows that:

$$\frac{\sin x}{\tan(\sin x)} + \frac{\cos x}{\tan(\cos x)} \leq \frac{3-\sin^2 x}{3} + \frac{3-\cos^2 x}{3} = \frac{5}{3}, \forall x \in \left(0, \frac{\pi}{2}\right); (2)$$

From (1),(2) it follows that:

$$\int_a^b \frac{\cos x}{\tan(\sin x)}dx + \int_a^b \frac{\sin x}{\tan(\sin x)}dx + \int_a^b \frac{\sin x \cdot \tan x}{\tan(\sin x)}dx + \int_a^b \frac{\cos x \cdot \cot x}{\tan(\cos x)}dx =$$

$$= \int_a^b \left(\frac{\sin x}{\tan(\sin x)} + \frac{\cos x}{\tan(\cos x)}\right)(\tan x + \cot x)dx \leq$$

$$\leq \frac{5}{3}\int_a^b (\tan x + \cot x)\, dx = \frac{5}{3}\log(\cot a \cdot \tan b)$$

UP.402 If $\frac{1}{e} < a \leq b < e$ then:

$$\sin^{-1}(\log(\sqrt{ab})) - \sin^{-1}\left(\log\left(\frac{a+b}{2}\right)\right) \leq \sqrt{1-\log^2\left(\frac{a+b}{2}\right)} - \sqrt{1-\log^2(\sqrt{ab})}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f: \left[\frac{1}{e}, e\right] \to \mathbb{R}, f(x) = \sin^{-1}(\log x) - \sqrt{1-\log^2 x}$ then,

$$f'(x) = \frac{1}{x\sqrt{1-\log^2 x}} + \frac{-\frac{2}{x}\log x}{2\sqrt{1-\log^2 x}} = \frac{1-\log x}{x\sqrt{1-\log^2 x}} \geq 0, \forall x \in \left[\frac{1}{e}, e\right]$$

f –inreasing and from $\sqrt{ab} \leq \frac{a+b}{2} \Rightarrow f(\sqrt{ab}) \leq f\left(\frac{a+b}{2}\right)$.

Hence,

$$\sin^{-1}(\log(\sqrt{ab})) + \sqrt{1-\log^2(\sqrt{ab})} \leq \sin^{-1}\left(\log\left(\frac{a+b}{2}\right)\right) + \sqrt{1-\log^2\left(\frac{a+b}{2}\right)}$$

And thus,

$$\sin^{-1}(\log(\sqrt{ab}))-\sin^{-1}\left(\log\left(\frac{a+b}{2}\right)\right)\le\sqrt{1-\log^2\left(\frac{a+b}{2}\right)}-\sqrt{1-\log^2(\sqrt{ab})}$$

Equality holds for $a=b$.

Solution 2 by Daniel Văcaru-Romania

Consider $f:(-1,1)\to\mathbb{R}, f(x)=\sin^{-1}x+\sqrt{1-x^2}$, with derivatives

$$f'(x)=\frac{1-x}{\sqrt{1-x^2}}>0;\forall x\in(-1,1)$$

We have $\log\sqrt{ab}<\log\frac{a+b}{a}$. Using monotony for f, we obtain

$$\sin^{-1}(\log(\sqrt{ab}))+\sqrt{1-\log^2(\sqrt{ab})}\le\sin^{-1}\left(\log\left(\frac{a+b}{2}\right)\right)+\sqrt{1-\log^2\left(\frac{a+b}{2}\right)}$$

That is another form for

$$\sin^{-1}(\log(\sqrt{ab}))-\sin^{-1}\left(\log\left(\frac{a+b}{2}\right)\right)\le\sqrt{1-\log^2\left(\frac{a+b}{2}\right)}-\sqrt{1-\log^2(\sqrt{ab})}$$

Equality holds for $a=b$.

UP.403 If $0<a\le b$ then:

$$\int_a^b\int_a^b\sqrt{x^2+5xy+8y^2}\,dxdy\ge\frac{\sqrt{14}}{2}(b+a)(b-a)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1by proposer

$$7(x-y)^2+(7x+21y)^2=7(x^2-2xy+y^2)+49x^2+441y^2+294xy=$$

$$=56x^2-14xy+7y^2+441y^2+294xy=$$

$$=56x^2+280xy+448y^2=56(x^2+5xy+8y^2)$$

Hence,

$$x^2+5xy+8y^2=\frac{1}{56}(7(x-y)^2+(7x+21y)^2)\ge$$

$$\ge\frac{1}{56}(7x+21y)^2=\frac{49}{56}(x+3y)^2=\frac{7}{8}(x+3y)^2$$

So, we get:

$$\sqrt{x^2+5xy+8y^2} \ge \frac{\sqrt{7}}{2\sqrt{2}}(x+3y)$$

Therefore,

$$\int_a^b \int_a^b \sqrt{x^2+5xy+8y^2}\,dxdy \ge \frac{\sqrt{7}\cdot\sqrt{2}}{4}\int_a^b \int_a^b (x+3y)\,dxdy =$$

$$= \frac{\sqrt{14}}{4}(b-a)\left(\int_a^b x\,dx + \int_a^b 3y\,dy\right) = \frac{\sqrt{14}}{4}(b-a)\cdot 4\cdot \frac{x^2}{2}\Big|_a^b =$$

$$= \frac{\sqrt{14}}{2}(b-a)(b^2-a^2) = \frac{\sqrt{14}}{2}(b+a)(b-a)^2$$

Equality holds for $a=b$.

Solution 2 by George Florin Șerban-Romania

If $x \ge y, x^5+5xy+8y^2 \overset{AGM}{\ge} 14\sqrt[14]{x^2\cdot x^5y^5\cdot y^{16}} = 14y\sqrt{xy} \overset{y\ge x}{\ge} 14y^2$

Equality holds for $x=y$.

$$\Rightarrow \sqrt{x^5+5xy+8y^2} \ge \sqrt{14y^2} = y\sqrt{14}$$

$$\int_a^b \int_a^b \sqrt{x^2+5xy+8y^2}\,dxdy \ge \int_a^b \int_a^b \sqrt{14}y\,dxdy = (b-a)\sqrt{14}\cdot \frac{y^2}{2}\Big|_a^b =$$

$$= \frac{(b-a)^2(b+a)\sqrt{14}}{2}$$

If $x \le y, x^2+5xy+8y^2 \ge x\sqrt{14} \Rightarrow$

$$\int_a^b \int_a^b \sqrt{x^2+5xy+8y^2}\,dxdy \ge \int_a^b \int_a^b \sqrt{14}x\,dxdy = (b-a)\sqrt{14}\cdot \frac{x^2}{2}\Big|_a^b =$$

$$= \frac{(b-a)^2(b+a)\sqrt{14}}{2}$$

Therefore,

$$\int_a^b \int_a^b \sqrt{x^2+5xy+8y^2}\,dxdy \ge \frac{(b-a)^2(b+a)\sqrt{14}}{2}$$

Equality holds for $a=b$.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$2\sqrt{x^2+5xy+8y^2}=\sqrt{(2x+5y)^2+7y^2}=\sqrt{7}\cdot\sqrt{7\left(\frac{2x+5y}{7}\right)^2+y^2}=$$

$$=\sqrt{\frac{7}{8}}\sqrt{\Bigg(\underbrace{1+\dots+1}_{8-times}\Bigg)\left(\underbrace{\left(\frac{2x+5y}{7}\right)^2+\dots+\left(\frac{2x+5y}{7}\right)^2}_{7-times}+y^2\right)}\overset{CBS}{\geq}$$

$$\geq\frac{\sqrt{14}}{4}\left(\underbrace{\left(\frac{2x+5y}{7}\right)+\dots+\left(\frac{2x+5y}{7}\right)}_{7-times}+y\right)$$

$$\Rightarrow\sqrt{x^2+5xy+8y^2}\geq\frac{\sqrt{14}}{4}(x+3y);\forall x,y\geq 0$$

$$\int_a^b\int_a^b\sqrt{x^2+5xy+8y^2}\,dxdy\geq\frac{\sqrt{14}}{4}\int_a^b\int_a^b(x+3y)\,dxdy=$$

$$=\frac{\sqrt{14}}{4}(1+3)\left(\frac{b^2}{2}-\frac{a^2}{2}\right)(b-a)=\frac{(b-a)^2(b+a)\sqrt{14}}{2}$$

Therefore,

$$\int_a^b\int_a^b\sqrt{x^2+5xy+8y^2}\,dxdy\geq\frac{(b-a)^2(b+a)\sqrt{14}}{2}$$

Equality holds for $a=b$.

UP.404 If $0<a\leq b<1$ then:

$$\log\left(\frac{1-a+b-ab}{1+a-b-ab}\right)\geq\frac{3\sqrt{3}}{2}(b^2-a^2)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Inequality can be written:

$$\log\left(\frac{(1+b)(1-a)}{(1+a)(1-b)}\right)\geq\frac{3\sqrt{3}}{2}(b^2-a^2)$$

$$-\frac{1}{2}\log\left(\frac{1-b}{1+b}\cdot\frac{1+a}{1-a}\right)\geq\frac{3\sqrt{3}}{4}(b^2-a^2)$$

$$-\frac{1}{2}\left(\log\left(\frac{1-b}{1+b}\right)-\log\left(\frac{1+a}{1-a}\right)\right)\geq\frac{3\sqrt{3}}{4}(b^2-a^2)$$

$$-\frac{1}{2}\cdot\log\left(\frac{1-x}{1+x}\right)\Bigg|_a^b\geq\frac{3\sqrt{3}}{3}\cdot\frac{x^2}{2}\Bigg|_a^b$$

$$\int_a^b\frac{1}{1-x^2}dx\geq\frac{3\sqrt{3}}{2}\int_a^b x\,dx$$

It is enough to prove that $\forall x\in(0,1):\frac{1}{1-x^2}\geq\frac{3\sqrt{3}}{2}x\Leftrightarrow$

$$1-x^2\leq\frac{2}{3\sqrt{3}x}\Leftrightarrow 3\sqrt{3}x(1-x^2)\leq 2\Leftrightarrow 3\sqrt{3}x(x^2-1)+2\geq 0$$

$$3\sqrt{3}x^3-3\sqrt{3}x+2\geq 0$$

$$3\sqrt{3}x^3+6x^2-6x^2-4x\sqrt{3}+x\sqrt{3}+2\geq 0$$

$$(3x^2-2x\sqrt{3}+1)(x\sqrt{3}+2)\geq 0$$

$$(x\sqrt{3}-1)^2(x\sqrt{3}+2)\geq 0 \text{ true.}$$

Equality holds for $a=b$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*)\leftrightarrow\log\left(\frac{(1+b)(1-a)}{(1+a)(1-b)}\right)\geq\frac{3\sqrt{3}}{2}(b^2-a^2)\ \leftrightarrow$$

$$\log\left(\frac{1+b}{1+a}\right)-\log\left(\frac{1-b}{1-a}\right)\geq 3\sqrt{3}\left(\frac{b^2}{2}-\frac{a^2}{2}\right)$$

$$\leftrightarrow\int_a^b\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx\geq\int_a^b 3\sqrt{3}x\,dx$$

$$\rightarrow\ \textit{It's suffices to prove}\ :\ \frac{1}{1+x}+\frac{1}{1-x}\geq 3\sqrt{3}x,\forall x\in(0,1)$$

$$\frac{1}{1+x}+\frac{1}{1-x}\geq 3\sqrt{3}x\ \leftrightarrow\ \frac{2}{3\sqrt{3}}\geq x(1-x^2)\ \leftrightarrow\ \frac{8}{27}\geq 2x^2(1-x^2)^2$$

$$\textit{By AM}-\textit{GM, we have}:\ 2x^2(1-x^2)^2\leq\left(\frac{2x^2+(1-x^2)+(1-x^2)}{3}\right)^3=\frac{8}{27}$$

$$\text{Therefore,}\qquad\log\left(\frac{1-a+b-ab}{1+a-b-ab}\right)\geq\frac{3\sqrt{3}}{2}(b^2-a^2)$$

Equality holds for $a=b$.

Solution 3 by Ravi Prakash-New Delhi-India

$$\text{For } 0<a\le b<1, \log\left(\frac{1-a+b-ab}{1+a-b-ab}\right)\ge\frac{3\sqrt{3}}{2}(b^2-a^2);(1)$$

Is clearly holds for $a=b$.

Let us assume $0<a<b<1$, then we can rewrite (1) as

$$\frac{\log\left(\frac{1+b}{1-b}\right)-\log\left(\frac{1+a}{1-a}\right)}{b^2-a^2}\ge\frac{3\sqrt{3}}{2};(2)$$

Let $f(x)=\log\left(\frac{1+x}{1-x}\right), 0<x<1$ and $g(x)=x^2, 0<x<1$.

For $0<a<b<1$, by the Cauchy's mean value theorem,

$$\text{for some } c\in(a,b): \frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}=\frac{1}{2c}\left(\frac{1}{1+c}+\frac{1}{1-c}\right)=\frac{1}{c(1-c^2)};(3)$$

Let $h(x)=x(1-x^2), 0<x<1$, then $h'(x)=1-3x^2=(1-\sqrt{3}x)(1+\sqrt{3}x)$

$$h'(x)>0 \text{ if } 0<x<\tfrac{1}{\sqrt{3}}\Rightarrow h(x)\le\frac{1}{\sqrt{3}\left(1-\frac{1}{3}\right)}=\frac{2}{3\sqrt{3}}, \forall x\in(0,1)$$

$$\Rightarrow h(c)\le\frac{2}{3\sqrt{3}}\Rightarrow\frac{1}{c(1-c^2)}\ge\frac{2}{3\sqrt{3}}, \forall c\in(0,1);(4)$$

From (3),(4) we get (2).

Solution 4 by Hikmat Mammadov-Azerbaijan

$$a\le b\Rightarrow -a\ge -b-a\ge 1-b, b\ge a\Rightarrow 1+b\ge 1+a$$

$$\Rightarrow(1-a)(1+b)\ge(1-b)(1+a)>0$$

$$\Rightarrow\frac{(1-a)(1+b)}{(1+a)(1-b)}\ge 1\Rightarrow\frac{1-a+b-ab}{1+a-b-ab}\ge 1\Rightarrow\log\left(\frac{1-a+b-ab}{1+a-b-ab}\right)\ge 0$$

$$1\ge b^2-a^2\ge 0$$

$$\frac{3\sqrt{3}}{2}>\frac{3\sqrt{3}}{2}(b^2-a^2)\ge 0$$

$$\text{When: } \frac{3\sqrt{3}}{2}(b^2-a^2)\cong\frac{3\sqrt{3}}{2}\log\left(\frac{1-a+b-ab}{1+a-b-ab}\right)\cong\infty$$

$$\text{Therefore,}\quad \log\left(\frac{1-a+b-ab}{1+a-b-ab}\right)\ge\frac{3\sqrt{3}}{2}(b^2-a^2)$$

Equality holds for $a = b$.

UP.405 Find:

$$\Omega(m) = \int_0^{\frac{\pi}{m}} \sin(mx) \cdot \log\left(1 + \tan\left(\frac{\pi}{m}\right)\tan x\right) dx\,; m > 2$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution by proposers

Let $x = \frac{\pi}{m} - y, dx = -dy$ then,

$$\Omega(m) = \int_{\frac{\pi}{m}}^{0} \sin\left(m\left(\frac{\pi}{m} - y\right)\right)\log\left(1 + \tan\left(\frac{\pi}{m}\right) tan\,\left(\frac{\pi}{m} - y\right)\right)(-dy) =$$

$$= \int_0^{\frac{\pi}{m}} \sin(\pi - my)\log\left(1 + \tan\left(\frac{\pi}{m}\right) \cdot \frac{\tan\left(\frac{\pi}{m}\right) - \tan y}{1 + \tan\left(\frac{\pi}{m}\right)\tan y}\right) dy =$$

$$= \int_0^{\frac{\pi}{m}} \sin(my)\log\left(\frac{1 + \tan\left(\frac{\pi}{m}\right)\tan y + \tan^2\left(\frac{\pi}{m}\right) - \tan\left(\frac{\pi}{m}\right)\tan y}{1 + \tan\left(\frac{\pi}{m}\right)\tan y}\right) dy =$$

$$= \int_0^{\frac{\pi}{m}} \sin(mx)\log\left(\frac{1 + \tan^2\left(\frac{\pi}{m}\right)}{1 + \tan\left(\frac{\pi}{m}\right)\tan x}\right) dx =$$

$$= \int_0^{\frac{\pi}{m}} \sin(mx)\log\left(1 + \tan^2\left(\frac{\pi}{m}\right)\right) dx - \int_0^{\frac{\pi}{m}} \sin(mx)\log\left(1 + \tan\left(\frac{\pi}{m}\right)\tan x\right) dx =$$

$$= \log\left(1 + \tan^2\left(\frac{\pi}{m}\right)\right)\int_0^{\frac{\pi}{m}} \sin(mx)\, dx - \Omega(m)$$

Hence,

$$2\Omega(m) = \log\left(1 + \tan^2\left(\frac{\pi}{m}\right)\right) \cdot \left(-\frac{1}{m}\right)\cos(mx)\Big|_0^{\frac{\pi}{m}} =$$

$$= \log\left(\frac{1}{\cos^2\left(\frac{\pi}{m}\right)}\right)\left(-\frac{1}{m}\right)\left(\cos\left(m \cdot \frac{\pi}{m}\right) - \cos 0\right) =$$

$$-\frac{1}{2}\log\left(\cos\left(\frac{\pi}{m}\right)\right) \cdot \left(-\frac{1}{m}\right)(\cos\pi - \cos 0) = \frac{1}{2m}\log\left(\cos\left(\frac{\pi}{m}\right)\right) \cdot (-2)$$

Therefore,

$$\Omega(m)=-\frac{1}{2m}\log\left(\cos\left(\frac{\pi}{m}\right)\right)$$

Solution 2 by Marian Ursărescu-Romania

$$\text{Let } x=\frac{\pi}{m}-t\Rightarrow dx=-dt\Rightarrow\begin{cases}x=0\\x=\frac{\pi}{m}\end{cases}\Rightarrow\begin{cases}t=\frac{\pi}{m}\\t=0\end{cases}$$

$$\Omega(m)=\int_{\frac{\pi}{m}}^{0}\sin(\pi-mt)\cdot\log\left(1+\tan\left(\frac{\pi}{m}\right)\cdot\tan\left(\frac{\pi}{m}-t\right)\right)(-dt)=$$

$$=\int_0^{\frac{\pi}{m}}\sin(mt)\cdot\log\left(1+\tan\left(\frac{\pi}{m}\right)\cdot\frac{\tan\left(\frac{\pi}{m}\right)-\tan t}{1+\tan\left(\frac{\pi}{m}\right)\cdot\tan t}\right)dt=$$

$$=\int_0^{\frac{\pi}{m}}\sin(mt)\cdot\log\left(\frac{1+\tan\left(\frac{\pi}{m}\right)\cdot\tan\left(\frac{\pi}{t}\right)+\tan^2\left(\frac{\pi}{m}\right)-\tan\left(\frac{\pi}{m}\right)\cdot\tan t}{1+\tan\left(\frac{\pi}{m}\right)\cdot\tan t}\right)dt=$$

$$=\int_0^{\frac{\pi}{m}}\sin(mt)\cdot\log\left(\frac{1+\tan\left(\frac{2\pi}{m}\right)}{1+\tan\left(\frac{\pi}{m}\right)\cdot\tan t}\right)dt=$$

$$=\int_0^{\frac{\pi}{m}}\sin(mt)\cdot\log\left(1+\tan\left(\frac{2\pi}{m}\right)\right)dt-\int_0^{\frac{\pi}{m}}\sin(mt)\cdot\log\left(1+\tan\left(\frac{\pi}{m}\right)\cdot\tan t\right)$$

$$\Omega(m)=\log\left(1+\tan^2\left(\frac{\pi}{m}\right)\right)\int_0^{\frac{\pi}{m}}\sin(mx)\,dx-\Omega(m)$$

Hence,

$$2\Omega(m)=\log\left(1+\tan^2\left(\frac{\pi}{m}\right)\right)\cdot\left(-\frac{1}{m}\right)\cos(mx)\Big|_0^{\frac{\pi}{m}}=$$

$$=\log\left(\frac{1}{\cos^2\left(\frac{\pi}{m}\right)}\right)\left(-\frac{1}{m}\right)\left(\cos\left(m\cdot\frac{\pi}{m}\right)-\cos 0\right)=$$

$$-\frac{1}{2}\log\left(\cos\left(\frac{\pi}{m}\right)\right)\cdot\left(-\frac{1}{m}\right)(\cos\pi-\cos 0)=\frac{1}{2m}\log\left(\cos\left(\frac{\pi}{m}\right)\right)\cdot(-2)$$

Therefore,

$$\Omega(m)=-\frac{1}{2m}\log\left(\cos\left(\frac{\pi}{m}\right)\right)$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\text{Using: } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \text{ we get:}$$

$$\Omega(m) = \int_0^{\frac{\pi}{m}} \sin\left[m\left(\frac{\pi}{m}-x\right)\right]\cdot\log\left[1+\tan\left(\frac{\pi}{m}\right)\tan\left(\frac{\pi}{m}-x\right)\right]dx =$$

$$= \int_0^{\frac{\pi}{m}} \sin(mx)\cdot\log\left[1+\frac{\tan\left(\frac{\pi}{m}\right)\left[\tan\left(\frac{\pi}{m}\right)-\tan x\right]}{1+\tan\left(\frac{\pi}{m}\right)\cdot\tan x}\right]dx =$$

$$= \int_0^{\frac{\pi}{m}} \sin(mx)\cdot\log\left(\sec^2\left(\frac{\pi}{m}\right)\right)dx - \Omega(m)$$

Hence,

$$2\Omega(m) = \frac{2}{m}\log\left(\sec\left(\frac{\pi}{m}\right)\right)\{-\cos(mx)\}\Big|_0^{\frac{\pi}{m}} = \frac{2}{m}\log\left(\sec\left(\frac{\pi}{m}\right)\right)(1+1)$$

Therefore,

$$\Omega(m) = \frac{2}{m}\log\left(\sec\left(\frac{\pi}{m}\right)\right)$$

Solution 4 by Ankush Kumar Parcha-India

$$\Omega(m) = \int_0^{\frac{\pi}{m}} \sin(mx)\cdot\log\left[1+\tan\left(\frac{\pi}{m}\right)\tan x\right]dx =$$

$$= -\frac{1}{m}\cos(mx)\cdot\log\left(1+\tan\left(\frac{\pi}{m}\right)\tan x\right)\Big|_0^{\frac{\pi}{m}} - \frac{\tan\left(\frac{\pi}{m}\right)}{m}\int_0^{\frac{\pi}{m}} \frac{\sec^2 x\cdot\cos(mx)}{1+\tan\left(\frac{\pi}{4}\right)\tan x}dx =$$

$$= \frac{2}{m}\log\left(\sec\left(\frac{\pi}{m}\right)\right) - \frac{1}{m}\cos(mx)\cdot\log\left(1+\tan\left(\frac{\pi}{m}\right)\tan x\right)\Big|_0^{\frac{\pi}{4}} - \Omega(m)$$

Therefore,

$$2\Omega(m) = \frac{4}{m}\log\left(\sec\left(\frac{\pi}{m}\right)\right) \Rightarrow \Omega(m) = \frac{2}{m}\log\left(\sec\left(\frac{\pi}{m}\right)\right)$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru