Number 29 *SUMMER 2023*

R M M

ROMANIAN MATHEMATICAL MAGAZINE

SOLUTIONS

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Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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PROBLEMS FOR JUNIORS

JP.421 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^2+ab+bc+ca}{2s+a}\le 3\sqrt{3}R$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{a^2+ab+bc+ca}{2s+a}=\sum_{cyc}\frac{a(a+b)+c(a+b)}{(a+b+c)+a}=\sum_{cyc}\frac{(a+b)(a+c)}{(a+b)+(a+c)}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{2}{\frac{1}{a+b}+\frac{1}{a+c}}\overset{AH-AM}{\le}\frac{1}{2}\sum_{cyc}\frac{(a+b)+(a+c)}{2}=$$

$$=\frac{1}{4}\sum_{cyc}(2a+b+c)=\frac{1}{4}\cdot 4\sum_{cyc}a=2s\overset{Mitrinovic}{\le}2\cdot\frac{3\sqrt{3}R}{2}=3\sqrt{3}R$$

Equality holds for $a=b=c$.

Solution 2 by George Florin Şerban-Romania

$$\sum_{cyc}\frac{a^2+ab+bc+ca}{2s+a}=\sum_{cyc}\frac{a(a+b)+c(a+b)}{a+b+c+a}=$$

$$=\sum_{cyc}\frac{(a+b)(a+c)}{(a+b)+(a+c)}\overset{AHM}{\le}\sum_{cyc}\frac{a+b+c+a}{4}=\sum_{cyc}\frac{a+b+c}{4}+\sum_{cyc}\frac{a}{4}=$$

$$=\frac{2s}{4}\cdot 3+\frac{2s}{4}=2s\overset{Mitrinovic}{\le}2\cdot\frac{3\sqrt{3}}{2}R=3\sqrt{3}R$$

Therefore,

$$\sum_{cyc}\frac{a^2+ab+bc+ca}{2s+a}\le 3\sqrt{3}R$$

Equality holds for $a=b=c$.

Solution 3 by Daniel Văcaru-Romania

We have: $\sqrt{(a+b)(a+c)}\le\frac{2s+a}{2}\Leftrightarrow\frac{a^2+ab+bc+ca}{2s+a}\le\frac{2s+a}{4}$

Hence,

$$\sum_{cyc}\frac{a^2+ab+bc+ca}{2s+a}\le\sum_{cyc}\frac{2s+a}{4}=2s\overset{Mitrinovic}{\le}3\sqrt{3}R$$

JP.422 If $a,b,c>0$, then:

$$\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}+\frac{(b^2+a^2)(c^2+a^2)}{(bc+a^2)(ba+ca)}+\frac{(c^2+b^2)(a^2+b^2)}{(ca+b^2)(cb+ab)}\ge 3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(a^2+c^2)(b^2+c^2)=a^2b^2+(a^2+b^2)c^2+c^4\overset{AGM}{\ge}$$

$$\ge a^2b^2+2abc^2+c^4=(ab+c^2)^2$$

$$(a^2+c^2)(b^2+c^2)\ge(ab+c^2)^2;(1)$$

$$(a^2+c^2)(b^2+c^2)=a^2b^2+(a^2+b^2)c^2+c^4\overset{AGM}{\ge}$$

$$\ge 2abc^2+a^2c^2+b^2c^2=(ac+bc)^2;(2)$$

By (1) and (1), we get:

$$\left((a^2+c^2)(b^2+c^2)\right)^2\ge(ab+c^2)^2(ac+bc)^2$$

$$(a^2+c^2)(b^2+c^2)\ge(ab+c^2)(ab+bc)$$

$$\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}\ge 1,\ \sum_{cyc}\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}\ge 3$$

Equality holds for $a=b=c$.

Solution 2 by George Florin Şerban-Romania

$$\sum_{cyc}\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}\overset{CBS}{\ge}\sum_{cyc}\frac{(ab+c^2)^2}{(ab+c^2)c(a+b)}=$$

$$=\sum_{cyc}\frac{ab+c^2}{c(a+b)}=\sum_{cyc}\frac{ab}{c(a+b)}+\sum_{cyc}\frac{c^2}{c(a+b)}=$$

$$=\sum_{cyc}\frac{\frac{1}{c}}{\frac{1}{a}+\frac{1}{b}}+\sum_{cyc}\frac{c}{a+b}\overset{Nesbitt}{\ge}\frac{3}{2}+\frac{3}{2}=3$$

Therefore,

$$\sum_{cyc}\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}\ge 3$$

Equality holds for $a=b=c$.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt{(a^2+c^2)(b^2+c^2)}\ge ab+c^2 \text{ and } \sqrt{(a^2+c^2)(c^2+b^2)}\ge ca+bc$$

Multiplying up these inequalities, we obtain:

$$(a^2+c^2)(b^2+c^2)\ge(ab+c^2)(ca+bc)\Rightarrow\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ca+bc)}\ge 1$$

Therefore,

$$\sum_{cyc}\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}\ge 3$$

Equality holds for $a=b=c$.

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}\ge\sum_{cyc}\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)c(a+b)}=$$

$$=\sum_{cyc}\frac{(a^2+c^2)(b^2+c^2)}{\left(\frac{ab+c^2+ca+cb}{2}\right)^2}=4\sum_{cyc}\frac{a^4+(ab)^2+(bc)^2+(ca)^2}{(bc+a^2+ab+ac)^2}\ge$$

$$\ge\frac{4}{4}\sum_{cyc}\frac{(a^2+ab+bc+ca)^2}{(a^2+ab+bc+ca)^2}=3$$

Equality holds for $a=b=c$.

Solution 5 by Ravi Prakash-New Delhi-India

$$(a^2+c^2)(b^2+c^2)=|a+ic|^2|b-ic|^2=$$

$$=|(a+ic)(b-ic)|^2=\left|(ab+c^2)+i(bc-ac)\right|^2=$$

$$=(ab+c^2)^2+(bc-ac)^2\ge(ab+c^2)^2;(1)$$

Equality when $a=b$.

Next, $(a^2+c^2)(b^2+c^2)=|(a+ic)(b+ic)|^2=$

$$= \left|(ab-c^2)^2+ic(a+b)^2\right| = (ab-c^2)^2+(ac+bc)^2 \geq (ac+bc)^2; (2)$$

From (1) and (2), we get:

$$\left((a^2+c^2)(b^2+c^2)\right)^2 \geq (ab+c^2)^2(ac+bc)^2$$

$$\Rightarrow \frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)} \geq 1; (3)$$

Equality when $a=b=c$.

Similarly,

$$\frac{(b^2+a^2)(c^2+a^2)}{(bc+a^2)(ba+ca)} \geq 1; (4) \text{ and } \frac{(c^2+b^2)(a^2+b^2)}{(ca+b^2)(cb+ab)} \geq 1; (5)$$

By adding (3),(4) and (5), we get:

$$\sum_{cyc} \frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)} \geq 3$$

Equality holds for $a=b=c$.

Solution 6 by Vivek Kumar-India

$$\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)} = \frac{(a^2b^2+c^4)+c^2(a^2+b^2)}{(ab+c^2)(ac+bc)} \overset{AGM}{\geq}$$

$$\geq \frac{\frac{1}{2}(ab+c^2)^2+\frac{1}{2}c^2(a+b)^2}{(ab+c^2)(ac+bc)} = \frac{1}{2}\left(\frac{ab+c^2}{ac+bc}\right)+\frac{1}{2}\left(\frac{ca+bc}{ab+c^2}\right) \overset{AGM}{\geq}$$

$$\geq 2\cdot\sqrt{\frac{1}{2}\left(\frac{ab+c^2}{ac+bc}\right)\cdot\frac{1}{2}\left(\frac{ca+bc}{ab+c^2}\right)} = 1$$

Analogous,

$$\frac{(b^2+a^2)(c^2+a^2)}{(bc+a^2)(a+ca)} \geq 1 \text{ and } \frac{(c^2+a^2)(a^2+b^2)}{(ca+b^2)(cb+ab)} \geq 1$$

Therefore,

$$\sum_{cyc} \frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)} \geq 3$$

Equality holds for $a=b=c$.

Solution 7 by Daniel Văcaru-Romania

We have $\sqrt{(ab+c^2)(ac+bc)} \le \frac{(c+a)(c+b)}{2} \Rightarrow$

$$\frac{1}{(ab+c^2)(ac+bc)} \ge \frac{4}{(a+c)^2(c+b)^2}$$

$$\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)} \ge \frac{4(a^2+c^2)(b^2+c^2)}{(c+a)^2(c+b)^2}$$

But: $\frac{2(a^2+b^2)}{(a+b)^2} \ge 1$ (and analogs). Therefore,

$$\sum_{cyc} \frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)} \ge 3$$

Equality holds for $a=b=c$.

JP.423 If $a,b,c>0$, then:

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2} + \left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2} + \left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \ge 3e$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}, f(x)=x-e^{x-1}, f'(x)=1-e^{x-1}$

$$f'(x)=0 \Rightarrow 1-e^{x-1}=0 \Leftrightarrow e^{x-1}=1 \Leftrightarrow x-1=0 \Leftrightarrow x=1$$

x	0	1	∞
$f'(x)$		0	
$f(x)$	↗	M	↘

$$M=\max_{x>0} f(x) = f(1) = 1-e^{1-1} = 1-1 = 0$$

$$\Rightarrow f(x)\le 0 \Rightarrow x \le e^{x-1} \Rightarrow x \le \frac{1}{e}e^x \Rightarrow \frac{e^{x^2}}{x^2} \ge e$$

For $x=\frac{c}{a} \Rightarrow \frac{e^{\left(\frac{c}{a}\right)^2}}{\left(\frac{c}{a}\right)^2} \ge e \Rightarrow \left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2} \ge e; (1)$

Analogous:

$$\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2} \ge e;\ \ (2)$$

$$\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \ge e;\ \ (3)$$

By adding (1), (2) and (3), we get:

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2} + \left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2} + \left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \ge 3e$$

Equality holds for $a = b = c$.

Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece

$$e^x \ge ex, \forall x \in \mathbb{R}, f(x) = e^x - ex \Rightarrow f'(x) = e^x - e$$

x	$-\infty$ 1 ∞
$f'(x)$	$- - - - - - - - - 0 + + + + + + + + +$
$f(x)$	$\searrow \quad \searrow \quad f(1) \quad \nearrow \quad \nearrow$

So, $f(x) \ge f(1), \forall x \in \mathbb{R} \Rightarrow e^x - ex \ge 0 \Rightarrow e^x \ge ex, \forall x \in \mathbb{R}$

$$\sum_{cyc}\left(\frac{a}{c}\right)^2 \cdot e^{\left(\frac{c}{a}\right)^2} \overset{(*)}{\ge} \sum_{cyc}\left(\frac{a}{c}\right)^2 \cdot e \cdot \left(\frac{c}{a}\right)^2 = 3e$$

Equality holds for $a = b = c$.

Solution 3 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\left(\frac{a}{c}\right)^2 \cdot e^{\left(\frac{c}{a}\right)^2} \overset{AGM}{\ge} 3\left(\left(\frac{a}{c}\cdot\frac{b}{a}\cdot\frac{c}{b}\right)^2 \cdot e^{\sum\left(\frac{c}{a}\right)^2}\right)^{\frac{1}{3}} \overset{AGM}{\ge} 3\left(e^{3\left(\frac{c}{a}\cdot\frac{a}{b}\cdot\frac{b}{c}\right)^2}\right)^{\frac{1}{3}} = 3(e^3)^{\frac{1}{3}} = 3e$$

Solution 4 by Christos Tsifakis-Greece

Let $f(x) = \frac{1}{x^2} \cdot e^{x^2}, x \in (0, \infty)$

$$f'(x) = 2e^{x^2}\left(\frac{1}{x} - \frac{1}{x^2}\right) = 2e^{x^2} \cdot \frac{x^2 - 1}{x^3}$$

$$f'(x) \ge 0 \Leftrightarrow x \ge 1$$

x	$-\infty$ 1 ∞
$f'(x)$	$- - - - - - - - - 0 + + + + + + + + +$
$f(x)$	$\searrow \quad \searrow \quad f(1) = e \quad \nearrow \quad \nearrow$

$$\Rightarrow f_{min} = e$$

For $x = \frac{c}{a} \Rightarrow f\left(\frac{c}{a}\right) \ge e$; $x = \frac{a}{b} \Rightarrow f\left(\frac{a}{b}\right) \ge e$; $x = \frac{b}{c} \Rightarrow f\left(\frac{b}{c}\right) \ge e$

Finally,

$$f\left(\frac{c}{a}\right) + f\left(\frac{a}{b}\right) + f\left(\frac{b}{c}\right) \ge 3e$$

Solution 5 by Ravi Prakash-New Delhi-India

Let $f(x) = \frac{e^x}{x}, x > 0 \Rightarrow f'(x) = \frac{(x-1)e^x}{x^2}$. We have:

$$f'(x) = \begin{cases} > 0; \text{if } 0 < x < 1 \\ = 0; \text{if } x = 1 \\ > 0; \text{if } x > 1 \end{cases}$$

Thus, $f(x) \ge f(1); \forall x > 0 \Rightarrow \frac{e^x}{x} \ge e; \forall x > 0$

Therefore,

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2} + \left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2} + \left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \ge 3e$$

Equality holds for $a = b = c$.

Solution 6 by Hikmat Mammadov-Azerbaijan

Note $f(a) = \frac{e^a}{a} \Rightarrow f'(a) = \frac{e^a}{a} - \frac{e^a}{a^2} = 0 \Rightarrow a = 1 \Rightarrow f''(a) = \frac{e^a}{a} - \frac{2e^a}{a^2} + \frac{2e^a}{a^3}$

$f''(1) = e$. So, $f(1) = e$ is minimum of $f(x)$ i.e. for $a > 0 > \frac{e^a}{a} \ge e$

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2} = \frac{e^{\left(\frac{c}{a}\right)^2}}{\left(\frac{c}{a}\right)^2} \ge e$$

Similarly,

$$\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2} \ge e \text{ and } \left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \ge e$$

Therefore,

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2} + \left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2} + \left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \ge 3e$$

Equality holds for $a = b = c$.

Solution 7 by Hikmat Mammadov-Azerbaijan

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2}+\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2}+\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \overset{AGM}{\geq} 3\sqrt[3]{\left(\frac{a^2b^2c^2}{b^2c^2a^2}\right)\cdot e^{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}}=$$

$$=3e^{\frac{1}{3}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)} \overset{AGM}{\geq} 3e^{\frac{1}{3}\cdot 3\frac{a^2b^2c^2}{b^2c^2a^2}}=3e$$

Therefore,

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2}+\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2}+\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2}\geq 3e$$

Equality holds for $a=b=c$.

Solution 8 by Angel Plaza-Spain

By the AM-GM inequality, we have:

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2}+\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2}+\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \overset{AGM}{\geq} 3\cdot\sqrt[3]{\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2}\cdot\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2}\cdot\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2}}=$$

$$=3e^{\frac{\left(\frac{c}{a}\right)^2+\left(\frac{a}{b}\right)^2+\left(\frac{b}{c}\right)^2}{3}}\geq 3e$$

Equality holds for $a=b=c$.

Solution 9 by Daniel Văcaru-Romania

Let be the function $f(0,\infty)\to\mathbb{R}, f(x)=\frac{e^{x-1}}{x}$. We have:

$f'(x)=\frac{(x-1)e^{x-1}}{x^2}$ and $A(1,1)$ an minimum point. Then $\frac{e^x}{x}\geq e;\forall x\in\mathbb{R}$.

It follows that all therms in the sum $\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2}+\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2}+\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2}$ is $\geq e$ and then

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2}+\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2}+\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2}\geq 3e$$

JP.424 Solve for real numbers:

$$(x+1)(x-1)\begin{vmatrix}\overline{x111} & \overline{1x11} & \overline{11x1} & \overline{111x}\\ \overline{1x11} & \overline{11x1} & \overline{111x} & \overline{x111}\\ \overline{11x1} & \overline{111x} & \overline{x111} & \overline{1x11}\\ \overline{111x} & \overline{x111} & \overline{1x11} & \overline{11x1}\end{vmatrix}+(y+3)(y-1)\begin{vmatrix}\overline{y111} & \overline{1y11} & \overline{11y1} & \overline{111y}\\ \overline{1y11} & \overline{11y1} & \overline{111y} & \overline{y111}\\ \overline{11y1} & \overline{111y} & \overline{y111} & \overline{1y11}\\ \overline{111y} & \overline{y111} & \overline{1y11} & \overline{11y1}\end{vmatrix}=0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(x+3)(x-1)\det\begin{pmatrix}1000 & 100 & 10 & 1\\ 100 & 10 & 1 & 1000\\ 10 & 1 & 1000 & 100\\ 1 & 1000 & 100 & 10\end{pmatrix}\begin{pmatrix}x & 1 & 1 & 1\\ 1 & x & 1 & 1\\ 1 & 1 & x & 1\\ 1 & 1 & 1 & x\end{pmatrix}+$$

$$+(y+3)(y-1)\begin{pmatrix}1000 & 100 & 10 & 1\\ 100 & 10 & 1 & 1000\\ 10 & 1 & 1000 & 100\\ 1 & 1000 & 100 & 10\end{pmatrix}\begin{pmatrix}y & 1 & 1 & 1\\ 1 & y & 1 & 1\\ 1 & 1 & y & 1\\ 1 & 1 & 1 & y\end{pmatrix}=0$$

$$\left((x+3)(x-1)(x+3)(x-1)^3+(y+3)(y-1)(y+3)(y-1)^3\right)$$

$$\cdot\begin{vmatrix}1000 & 100 & 10 & 1\\ 100 & 10 & 1 & 1000\\ 10 & 1 & 1000 & 100\\ 1 & 1000 & 100 & 10\end{vmatrix}=0$$

$$\left((x+3)(x-1)^2\right)^2+\left((y+3)(y-1)^2\right)^2=0$$

$$x-1=y-1=0\Rightarrow x=y=1.$$

Solution 2 by Ravi Prakash-New Delhi-India

Note that $0\le x,y\le 9$. Let $a=\overline{x111}; b=\overline{1x11}; c=\overline{11x1}; d=\overline{111x}$, then:

$$a+b+c+d=(x+3)1111$$

$$\Delta(x)=\begin{vmatrix}a & b & c & d\\ b & c & d & a\\ c & d & a & b\\ d & a & b & c\end{vmatrix}; \text{using } c_1\to c_1+c_2+c_3+c_4, \text{ we get:}$$

$$\Delta(x)=[(x+3)1111]\Delta_1, \text{ where}$$

$$\Delta_1=\begin{vmatrix}1 & b & c & d\\ 1 & c & d & a\\ 1 & d & a & b\\ 1 & a & b & c\end{vmatrix}; \text{using } R_4\to R_4-R_3; R_3\to R_3-R_2 \text{ and } R_2\to R_2-R_1, \text{ we get:}$$

$$\Delta_1=\begin{vmatrix}1 & b & c & d\\ 0 & (x-1)\cdot 990 & -(x-1)\cdot 9 & (x-1)\cdot 999\\ 0 & (x-1)\cdot 9 & (x-1)\cdot 999 & -(x-1)\cdot 990\\ 0 & (x-1)\cdot 999 & -(x-1)\cdot 990 & -(x-1)\cdot 90\end{vmatrix}=(x-1)^3\Delta_2, \text{ where}$$

$$\Delta_2=\begin{vmatrix}990 & -9 & 999\\ 9 & 999 & -990\\ 999 & -990 & -90\end{vmatrix}=9^3\begin{vmatrix}110 & -1 & 111\\ 1 & 111 & -110\\ 111 & -110 & -10\end{vmatrix}$$

$c_1\to c_1-c_2-c_3$, gives:

$$\Delta_2=9^3\begin{vmatrix}0 & -1 & 111\\ 0 & 111 & -110\\ 231 & -110 & -10\end{vmatrix}=9^3\cdot 231\cdot(110+111^2)\neq 0$$

The given equation is:

$$(x+1)(x-1)\Delta(x)+(y+3)(y-1)\Delta(y)=0$$

$$(x+1)(x+3)(x-1)^4\cdot c+(y+3)^2(y-1)^4 c=0,\text{ where } c=1111\cdot\Delta_2\neq 0$$

$$(x+1)(x+3)(x-1)^4+(y+3)^2(y-1)^4=0$$

As $0\le x\le 9$, we get: $x=1; y=1$.

Solution 3 by George Florin Şerban-Romania

$$\begin{vmatrix}\overline{x111} & \overline{1x11} & \overline{11x1} & \overline{111x}\\ \overline{1x11} & \overline{11x1} & \overline{111x} & \overline{x111}\\ \overline{11x1} & \overline{111x} & \overline{x111} & \overline{1x11}\\ \overline{111x} & \overline{x111} & \overline{1x11} & \overline{11x1}\end{vmatrix}\overset{r_1+r_2+r_3+r_4}{=}$$

$$=(111x+3333)\begin{vmatrix}1 & 1 & 1 & 1\\ \overline{1x11} & \overline{11x1} & \overline{111x} & \overline{x111}\\ \overline{11x1} & \overline{111x} & \overline{x111} & \overline{1x11}\\ \overline{111x} & \overline{x111} & \overline{1x11} & \overline{11x1}\end{vmatrix}\overset{\substack{c_2-c_1\\ c_3-c_1\\ c_4-c_1}}{=}$$

$$=1111(x+3)\cdot\begin{vmatrix}1 & 0 & 0 & 0\\ \overline{1x11} & -90x+90 & -99x+99 & 900x-900\\ \overline{11x1} & -9x+9 & 990x-900 & 90x-90\\ \overline{111x} & 999x-999 & 99x-99 & 9x-9\end{vmatrix}=$$

$$=1111(x+3)\cdot 9^3(x-3)^3\underbrace{\begin{vmatrix}-10 & -11 & 100\\ -1 & 110 & 10\\ 111 & 11 & 1\end{vmatrix}}_{a}=\Delta(y)$$

$$a=1222100\neq 0$$

$$\Rightarrow (x+1)(x-1)\Delta(x)+(y+3)(y-1)\Delta(y)=0$$

$$(x-1)^4(x+1)(x+3)+(y-1)^4(y+3)^2=0\Leftrightarrow x=y=1.$$

JP.425 If $x,y,z>0, x^2+y^2+z^2=3$, then:

$$\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2}+xy+yz+zx\ge 6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\frac{x^4+y^4}{x^2+y^2} \ge x^2-xy+y^2 \Leftrightarrow x^4+y^4 \ge (x^2-xy+y^2)(x^2+y^2) \Leftrightarrow$$

$$x^4+y64 \ge x^4+x^2y^2-x^3y-xy^3+x^2y^2+y^4$$

$$x^3y+xy^3-2x^2y^2 \ge 0 \Leftrightarrow xy(x^2-2xy+y^2) \ge 0$$

$$\frac{x^4+y^4}{x^2+y^2} \ge x^2-xy+y^2; \ \ (1)$$

Analogous:

$$\frac{y^4+z^4}{y^2+z^2} \ge y^2-yz+z^2; \ \ (2)$$

$$\frac{z^4+x^4}{z^2+x^2} \ge z^2-zx+x^2; \ \ (3)$$

By adding (1),(2) and (3), we get:

$$\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2} \ge 2(x^2+y^2+z^2)-(xy+yz+zx)$$

$$\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2}+xy+yz+zx \ge 2(x^2+y^2+z^2)=2\cdot 3=6$$

Equality holds for $x=y=z=1$.

Solution 2 by Angel Plaza-Spain

It is enough to prove that:

$$\frac{x^4+y^4}{x^2+y^2}+xy \ge x^2+y^2 \Leftrightarrow \frac{x^4+x^3y+xy^3+y^4}{x^2+y^2} \ge x^2+y^2$$

$$x^4+2x^2y^2+y^4 \ge x^4+2x^2y^2+y^4 \Leftrightarrow$$

$\Leftrightarrow xy(x^2+y^2) \ge 2x^2y^2$ **by the AM-GM inequality. Therefore, the left hand of the proposed inequality, say LHS, holds:**

$$LHS \ge 2(x^2+y^2+z^2)=6.$$

Equality holds for $x=y=z=1$.

Solution 3 by Vivek Kumar-India

$$\frac{x^4+y^4}{x^2+y^2}+xy=\frac{(x^2+y^2)^2-2x^2y^2}{x^2+y^2}+xy=$$

$$= x^2 + y^2 - \frac{2x^2y^2}{x^2+y^2} + xy \overset{AM-GM}{\geq} x^2 + y^2 - \frac{2x^2y^2}{2xy} + xy = x^2 + y^2$$

Similarly,

$$\frac{y^4+z^4}{y^2+z^2} + yz \geq y^2 + z^2 \text{ and } \frac{z^4+x^4}{z^2+x^2} + zx \geq z^2 + x^2$$

$$\frac{x^4+y^4}{x^2+y^2} + \frac{y^4+z^4}{y^2+z^2} + \frac{z^4+x^4}{z^2+x^2} + xy + yz + zx \geq$$

$$\geq x^2 + y^2 + y^2 + z^2 + z^2 + x^2 = 2(x^2+y^2+z^2) = 6$$

Equality holds for $x = y = z$.

Solution 4 by Aggeliki Papaspyropoulou-Greece

$$\frac{x^4+y^4}{x^2+y^2} + \frac{y^4+z^4}{y^2+z^2} + \frac{z^4+x^4}{z^2+x^2} + xy + yz + zx \geq 6 \Leftrightarrow$$

$$\frac{x^4+y^4}{x^2+y^2} + \frac{y^4+z^4}{y^2+z^2} + \frac{z^4+x^4}{z^2+x^2} + xy + yz + zx \geq 2(x^2+y^2+z^2) \Leftrightarrow$$

$$\frac{x^4+y^4}{x^2+y^2} - (x^2+y^2) + xy + \frac{y^4+z^4}{y^2+z^2} - (y^2+z^2) + yz + \frac{z^4+x^4}{z^2+x^2} - (z^2+x^2) + zx \geq 0$$

$$\frac{xy(x^2+y^2) - 2x^2y^2}{x^2+y^2} + \frac{yz(y^2+z^2) - 2y^2z^2}{y^2+z^2} + \frac{zx(z^2+x^2) - 2z^2x^2}{z^2+x^2} \geq 0$$

$$\frac{xy(x^2-2xy+y^2)}{x^2+y^2} + \frac{yz(y^2-2yz+z^2)}{y^2+z^2} + \frac{zx(z^2-2zx+x^2)}{z^2+x^2} \geq 0$$

$$\frac{xy(x-y)^2}{x^2+y^2} + \frac{yz(y-z)^2}{y^2+z^2} + \frac{zx(z-x)^2}{z^2+x^2} \geq 0$$

Equality holds for $x = y = z$.

Solution 5 by Daniel Văcaru-Romania

We prove that $\frac{x^4+y^4}{x^2+y^2} + xy \geq x^2 + y^2; \forall x, y > 0; (1)$. We have:

$$\frac{x^4+y^4}{x^2+y^2} + xy \geq x^2 + y^2 \Leftrightarrow \frac{xy(x-y)^2}{x^2+y^2} \geq 0; \forall x, y > 0$$

By adding relationships (1) for x, y, z to obtain:

$$\frac{x^4+y^4}{x^2+y^2} + \frac{y^4+z^4}{y^2+z^2} + \frac{z^4+x^4}{z^2+x^2} + xy + yz + zx \geq 2(x^2+y^2+z^2) \Leftrightarrow$$

$$\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2}+xy+yz+zx\geq 6$$

JP.426 If $a,b,c\in\mathbb{C}$ then:

$$\frac{|a+1|}{|b+1|+|b+c|+|c|}+\frac{|b+1|}{|c+1|+|c+a|+|a|}+\frac{|c+1|}{|a+1|+|a+b|+|b|}\leq$$
$$\leq 3+|a|+|b|+|c|$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$1=|1|=|1+a-(a+b)+b|\leq|1+a|+|a+b|+|b|$$
$$\frac{1}{|1+a|+|a+b|+|b|}\leq 1\Rightarrow\frac{|c+1|}{|1+a|+|a+b|+|b|}\leq|c+1|\leq|c|+1;(1)$$

Analogous:

$$\frac{|a+1|}{|b+1|+|b+c|+|c|}\leq|a+1|\leq|a|+1;(2)$$
$$\frac{|b+1|}{|c+1|+|c+a|+|a|}\leq|b+1|\leq|b|+1;(3)$$

By adding (1),(2) and (3), we get:

$$\frac{|a+1|}{|b+1|+|b+c|+|c|}+\frac{|b+1|}{|c+1|+|c+a|+|a|}+\frac{|c+1|}{|a+1|+|a+b|+|b|}\leq$$
$$\leq 3+|a|+|b|+|c|$$

Equality holds for $a=b=c$.

Solution 2 by George Florin Șerban-Romania

$$\frac{|a+1|}{|b+1|+|b+c|+|c|}+\frac{|b+1|}{|c+1|+|c+a|+|a|}+\frac{|c+1|}{|a+1|+|a+b|+|b|}\leq$$
$$\leq\frac{|a+1|}{|b+1-b-c+c|}+\frac{|b+1|}{|c+1-c-a+a|}+\frac{|c+1|}{|a+1-a-b+b|}=$$
$$=|a+1|+|b+1|+|c+1|\leq|a|+1+|b|+1+|c|+1=$$
$$=|a|+|b|+|c|+3$$

Therefore,

$$\frac{|a+1|}{|b+1|+|b+c|+|c|}+\frac{|b+1|}{|c+1|+|c+a|+|a|}+\frac{|c+1|}{|a+1|+|a+b|+|b|}\le$$

$$\le 3+|a|+|b|+|c|$$

Equality holds for $a=b=c$.

Solution 3 by Ravi Prakash-New Delhi-India

$$|b+1|+|b+c|+|c|=|-b-1|+|b+c|+|-c|\ge$$

$$\ge|-b-1+b+c-c|=1$$

$$\Rightarrow\frac{|a+1|}{|b+1|+|b+c|+|c|}\le|a+1|\le|a|+1;(1)$$

Similarly,

$$\frac{|b+1|}{|c+1|+|c+a|+|a|}\le|b|+1;(2)$$

$$\frac{|c+1|}{|a+1|+|a+b|+|b|}\le|c|+1;(3)$$

By adding (1),(2) and (3), we get:

$$\frac{|a+1|}{|b+1|+|b+c|+|c|}+\frac{|b+1|}{|c+1|+|c+a|+|a|}+\frac{|c+1|}{|a+1|+|a+b|+|b|}\le$$

$$\le 3+|a|+|b|+|c|$$

Equality holds for $a=b=c$.

JP.427 If $a,b>0$ then:

$$\left(a^x\cdot e^{a^{2x}}+b^x\cdot e^{b^{2x}}\right)\cdot e^{a^x\cdot b^x}\ge(a^x+b^x)\cdot e^{a^{2x}+b^{2x}};\forall x\in\mathbb{R}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let $f:\mathbb{R}\to(0,\infty), f(x)=e^x$, then $f'(x)=e^x, f''(x)=e^x>0, f$ −convexe.

By Jensen's inequality, we have:

$$\frac{a^x}{a^x+b^x}f(a^{2x})+\frac{b^x}{a^x+b^x}f(b^{2x})\ge f\left(\frac{a^x}{a^x+b^x}\cdot a^{2x}+\frac{b^x}{a^x+b^x}\cdot b^{2x}\right)$$

$$\frac{a^xf(a^{2x})+b^xf(b^{2x})}{a^x+b^x}\ge f\left(\frac{a^{3x}+b^{3x}}{a^x+b^x}\right)$$

$$\frac{a^x\cdot e^{a^{2x}}+b^x\cdot e^{b^{2x}}}{a^x+b^x}\ge e^{\frac{(a^x+b^x)(a^{2x}-a^xb^x+b^{2x})}{a^x+b^x}}$$

$$a^x\cdot e^{a^{2x}}+b^x\cdot e^{b^{2x}}\ge (a^x+b^x)\cdot e^{a^{2x}+b^{2x}}\cdot e^{-a^xb^x}$$

Therefore: $\left(a^x\cdot e^{a^{2x}}+b^x\cdot e^{b^{2x}}\right)\cdot e^{a^x\cdot b^x}\ge (a^x+b^x)\cdot e^{a^{2x}+b^{2x}};\forall x\in\mathbb{R}$

Equality holds for $x=0$.

Solution 2 by Angel Plaza-Spain

The solution comes by the weighted –GM inequality. Let us rename $u=a^x$ and $w=b^x$.

Then, the inequality becomes:

$$\left(u\cdot e^{u^2}+w\cdot e^{w^2}\right)\cdot e^{uw}\ge (u+w)\cdot e^{u^2+w^2};\forall x\in\mathbb{R}_+$$

$$\frac{u\cdot e^{u(u+w)}+w\cdot e^{w(u+w)}}{u+w}\ge e^{u^2+w^2}$$

$$\left(u\cdot e^{u^2}+w\cdot e^{w^2}\right)\cdot e^{uw}\ge \sqrt[u+w]{e^{u^2(u+w)}\cdot e^{w^2(u+w)}}=e^{u^2+w^2}.$$

JP.428 Let be $A=\{a,b,c|a,b,c\in\mathbb{R}^*\}$ and $B=\{u,v,w,t|u,v,w,t\in\mathbb{R}^*\}$

such that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=3$$

$$\frac{u^2}{v^2}+\frac{v^2}{w^2}+\frac{w^2}{t^2}+\frac{t^2}{u^2}=\frac{v^2}{u^2}+\frac{w^2}{v^2}+\frac{t^2}{w^2}+\frac{u^2}{t^2}=4$$

Find:

$$\Omega=\sum_{x,y\in A}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in A}\left|\frac{x}{y}\right|+\sum_{x,y\in B}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in B}\left|\frac{x}{y}\right|$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=3;(1)$$

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=3;(2)$$

By adding (1) and (2), we get:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=6$$

$$\left(\frac{a}{b}\right)^2-2+\left(\frac{b}{a}\right)^2+\left(\frac{c}{a}\right)^2-2+\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2-2+\left(\frac{c}{b}\right)^2=0$$

$$\left(\frac{a}{b}-\frac{b}{a}\right)^2+\left(\frac{b}{c}-\frac{c}{b}\right)^2+\left(\frac{c}{a}-\frac{a}{c}\right)^2=0\Leftrightarrow$$

$$\frac{a}{b}=\frac{b}{a},\frac{b}{c}=\frac{c}{b},\frac{c}{a}=\frac{a}{c}\Rightarrow a^2=b^2=c^2\Rightarrow |a|=|b|=|c|$$

$$\Rightarrow\left|\frac{x}{y}\right|=1;\forall x,y\in A\Rightarrow\sum_{x,y\in A}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in A}\left|\frac{x}{y}\right|=9\cdot 1=9$$

Analogous:

$$\left(\frac{u}{v}-\frac{v}{u}\right)^2+\left(\frac{v}{w}-\frac{w}{v}\right)^2+\left(\frac{w}{t}-\frac{t}{w}\right)^2+\left(\frac{t}{u}-\frac{u}{t}\right)^2=0$$

$$\Rightarrow\frac{u}{v}=\frac{v}{u},\frac{v}{w}=\frac{w}{v},\frac{w}{t}=\frac{t}{w},\frac{t}{u}=\frac{u}{t}\Rightarrow u^2=v^2=t^2=w^2\Rightarrow$$

$$|u|=|v|=|w|=|t|\Rightarrow\left|\frac{x}{y}\right|=1;\forall x,y\in B$$

$$\Rightarrow\sum_{x,y\in B}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in B}\left|\frac{x}{y}\right|=4\cdot 4\cdot 1=16$$

Therefore,

$$\Omega=\sum_{x,y\in A}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in A}\left|\frac{x}{y}\right|+\sum_{x,y\in B}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in B}\left|\frac{x}{y}\right|=9+16=25$$

JP.429 Find $x\in\mathbb{R}$ such that in any ΔABC holds:

$$\frac{2abc}{R}\le\frac{a^x}{r_a}+\frac{b^x}{r_b}+\frac{c^x}{r_c}\le\frac{abc}{r}$$

Proposed by Alex Szoros-Romania

Solution 1 by proposer

Because the relationship is true in any ΔABC, then this is true for ΔABC equilateral.

$$a=b=c=l,R=\frac{l}{\sqrt{3}}\text{ and }r_a=\frac{l\sqrt{3}}{2}$$

Replacing in the problem, we get:

$$2\sqrt{3}l^2 \le 2\sqrt{3}l^{x-1} \le 2\sqrt{3}l^2 \Leftrightarrow l^2 = l^{x-1} \Rightarrow x = 3.$$

We want to prove that for $x = 3$ the double inequality is true, namely,

$$\frac{2abc}{R} \le \sum_{cyc} \frac{a^3}{r_a} \le \frac{abc}{r}$$

For LHS, we have:

$$\sum_{cyc} \frac{a^3}{r_a} = \sum_{cyc} \frac{a^3(s-a)}{F} = \frac{1}{F}\left[s\sum_{cyc} a^3 - \sum_{cyc} a^4\right] = \frac{1}{2F}\left[\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3 - 2\sum_{cyc} a^4\right)\right] =$$

$$= \frac{1}{2F}\left[\sum_{cyc} ab(a^2+b^2) - \sum_{cyc} a^4\right] \ge \frac{1}{2F}\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right) =$$

$$= \frac{16F^2}{2F} = 8F = \frac{2abc}{R}$$

For RHS, we have:

$$\sum_{cyc} \frac{a^3}{r_a} \le \frac{abc}{r} \Leftrightarrow \sum_{cyc} \frac{a^3(s-a)}{F} \le s \cdot \frac{abc}{F} \Leftrightarrow s\sum_{cyc} a^3 - \sum_{cyc} a^4 \le s \cdot abc$$

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right) - 2\sum_{cyc} a^4 \le abc\sum_{cyc} a \Leftrightarrow$$

$$\sum_{cyc} a(b^3+c^3) - \sum_{cyc} a^4 \le abc\sum_{cyc} a \Leftrightarrow \sum_{cyc} a^4 - \sum_{cyc} a^3(b+c) + \sum_{cyc} a^2bc \ge 0 \Leftrightarrow$$

$$\sum_{cyc} a^2[a^2 - (b+c)a + bc] \ge 0 \Leftrightarrow \sum_{cyc} a^2(a-b)(a-c) \ge 0 \ (Schur's)$$

Thus, $x = 3$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{2abc}{R} \le \frac{a^x}{r_a} + \frac{b^x}{r_b} + \frac{c^x}{r_c} \le \frac{abc}{r}; (*)$$

For an equilateral triangle ABC, we have $a = b = c = \frac{2}{3}s$

$$\Rightarrow r_a = r_b = r_c = \frac{sr}{s-a} = \frac{sr}{s-\frac{2}{3}s} = 3r \text{ and } R = 3r$$

$(*)$ is true in any $\Delta ABC \Rightarrow (*)$ is true in an equilateral ΔABC

$$(*) \Leftrightarrow \frac{2a^3}{2r} = \frac{a^3}{r} \le \frac{2a^x}{r_a} = \frac{a^x}{r} \le \frac{a^3}{r} \Rightarrow x = 3$$

So, it suffices to treat the case where $x = 3$. We have:

$$\frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \overset{AGM}{\ge} \frac{3abc}{\sqrt[3]{r_a r_b r_c}} = \frac{3abc}{\sqrt[3]{s^2 r}} \overset{Mitrinovic-Euler}{\ge} \frac{3abc}{\sqrt[3]{\left(\frac{3\sqrt{3}R}{2}\right)^2 \cdot \frac{R}{r}}} = \frac{2abc}{R};(1)$$

$$\sum_{cyc} \frac{a^3}{r_a} = \sum_{cyc} \frac{(s-a)a^3}{sr} = \frac{1}{sr}\left(s\sum_{cyc} a^3 - \sum_{cyc} a^4\right) =$$

$$= \frac{1}{sr}[2s^2(s^2 - 3r^2 - 6Rr) - 2s^4 + 4(4Rr + 3r^2)s^2 - 2r^2(4R+r)^2] =$$

$$= \frac{2(2R+3r)s^2 - 2r(4R+r)^2}{s} \overset{(?)}{\le} \frac{abc}{r}$$

$$3s^2 \le (4R+r)^2 \Leftrightarrow 3[(4R^2 + 4Rr + 3r^2) - s^2] + 4(R-2r)(R+r) \overset{(?)}{\ge} 0$$

Which is true from Gerretsen's inequality and Euler's inequality:

$$\frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \le \frac{abc}{r};(2)$$

From (1) and (2) it follows that $(*)$ is true for $x = 3$.Therefore, $x = 3$.

JP.430 If $x, y, z > 0$, then prove that:

$$3\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) \ge \frac{3x+y+z}{y+z} + \frac{x+3y+z}{z+x} + \frac{x+y+3z}{x+y} + \frac{3}{2}$$

Proposed by Neculai Stanciu-Romania

Solution 1 by proposer

Inequality from the statement can be written as:

$$3\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - 3\right) \ge 3\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} - \frac{3}{2}\right);(1)$$

Using $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-3=\frac{x^2}{xy}+\frac{y^2}{yz}+\frac{z^2}{xz}-3\ge\frac{(x+y+z)^2}{xy+yz+zx}-3=$

$$=\frac{(x-y)^2+(y-z)^2+(z-x)^2}{2(xy+yz+zx)};(2)\text{ and}$$

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}-\frac{3}{2}=\frac{(x-y)^2}{2(x+z)(y+z)}+\frac{(y-z)^2}{2(x+y)(x+z)}+\frac{(z-x)^2}{2(x+y)(y+z)};(3)$$

Now, (1) can be written as:

$$\sum_{cyc}\frac{(x-y)^2}{xy+yz+zx}\ge\sum_{cyc}\frac{(x-y)^2}{(x+z)(y+z)}\Leftrightarrow\sum_{cyc}\frac{z^2(x-y)^2}{(xy+yz+zx)(x+z)(y+z)}\ge 0\text{ true.}$$

Equality holds for $x=y=z$.

Solution 2 by George Florin Şerban-Romania

$$\sum_{cyc}\frac{a}{b+1}=\sum_{cyc}\frac{a^2}{ab+a}\ge\frac{(a+b+c)^2}{ab+bc+ca+a+b+c}\ge$$

$$\ge\frac{(a+b+c)^2}{\frac{(a+b+c)^2}{3}+a+b+c}=\frac{a+b+c}{\frac{a+b+c}{3}+1}=\frac{3(a+b+c)}{a+b+c+3}\overset{(1)}{\ge}\frac{3}{2}$$

$$(1)\Leftrightarrow 6(a+b+c)\ge 3(a+b+c)+9\Rightarrow 3(a+b+c)\ge 9$$

$\Leftrightarrow a+b+c\ge 3$ which is true from AM-GM:

$a+b+c\ge 3\cdot\sqrt[3]{abc}=3$. Equality holds for $a=b=c=1$.

So,

$$3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\ge\frac{3x+y+z}{y+z}+\frac{x+3y+z}{z+x}+\frac{x+y+3z}{x+y}+\frac{3}{2}$$

Solution 3 by Aggeliki Papaspyropoulou-Greece

$$3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\ge 3\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)+\frac{y+z}{z+y}+\frac{x+z}{z+x}+\frac{x+y}{y+x}\Leftrightarrow$$

$$3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\ge 3\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)+3\Leftrightarrow$$

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+1\Leftrightarrow$$

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{x}{y+z}-\frac{y}{z+x}-\frac{z}{x+y}\ge 1;(*)$$

$$\frac{x}{y}-\frac{x}{y+z}+\frac{y}{z}-\frac{y}{z+x}+\frac{z}{x}-\frac{z}{x+y}=\frac{x(y+z-y)}{y(y+z)}+\frac{y(z+x-z)}{z(z+x)}+\frac{z(x+y-x)}{x(x+y)}=$$

$$=\frac{xz}{y(y+z)}+\frac{yx}{z(z+x)}+\frac{zy}{x(x+y)}=\frac{x^2z^2}{xyz(y+z)}+\frac{y^2x^2}{xyz(z+x)}+\frac{z^2y^2}{xyz(x+y)}\geq$$

$$\geq\frac{(xy+yz+zx)^2}{xyz(y+z+z+x+x+y)}=\frac{(xy+yz+zx)^2}{2xyz(x+y+z)};(2)$$

$$(xy+yz+zx)^2\geq 3xyz(x+y+z)$$

$$\Rightarrow (2):\frac{(xy+yz+zx)^2}{2xyz(x+y+z)}\geq\frac{3xyz(x+y+z)}{2xyz(x+y+z)}=\frac{3}{2}>1$$

So, $(*)$ is true.

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\text{Note:}\begin{cases}x+y+z=m\\ xy+yz+zx=n\Rightarrow n^2\geq 3mk\\ xyz=k\end{cases}$$

$$\Rightarrow 3\sum_{cyc}\frac{x}{y}-\sum_{cyc}\frac{3x+y+z}{y+z}=3\sum_{cyc}\left(\frac{x}{y}-\frac{x}{y+z}\right)-3=3\sum_{cyc}\frac{zx}{y(y+z)}-3$$

$$\Rightarrow\frac{3}{xyz}\sum_{cyc}\frac{(zx)^2}{y+z}-3\geq\frac{3}{k}\cdot\frac{n^2}{2m}-3\geq\frac{3\cdot 3mk}{2mk}-3=\frac{3}{2}>0$$

$$\Rightarrow 3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\geq\frac{3x+y+z}{y+z}+\frac{x+3y+z}{z+x}+\frac{x+y+3z}{x+y}+\frac{3}{2}$$

$$3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\geq\frac{3x+y+z}{y+z}+\frac{x+3y+z}{z+x}+\frac{x+y+3z}{x+y}$$

Solution 5 by Soumava Chakraborty-Kolkata-India

Substituting $y+z=a, z+x=b, x+y=c$, we see $a+b>c, b+c>a, c+a>b$ $\Rightarrow a, b, c$ form sides of a triangle with semiperimter, circumradius, inradius $= s, R, r$ respectively (say) and we subsequently arrive at : $2\sum x=\sum a=2s\Rightarrow\sum x$

$= s\therefore x=s-a, y=s-b, z=s-c$ and using such substitutions,

$$3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\geq\frac{3x+y+z}{y+z}+\frac{x+3y+z}{z+x}+\frac{x+y+3z}{x+y}+\frac{3}{2}\Leftrightarrow 3\sum_{cyc}\frac{s-a}{s-b}\geq 3+\frac{3}{2}+3\sum_{cyc}\frac{s-a}{a}$$

$$\Leftrightarrow\sum_{cyc}\frac{r_b}{r_a}\geq 1+\frac{1}{2}-3+\frac{s}{4Rrs}\sum_{cyc}ab=\frac{s^2+4Rr+r^2}{4Rr}-\frac{3}{2}$$

$$\Leftrightarrow \sum_{cyc}\frac{r_b}{r_a} \overset{(*)}{\overset{?}{\geq}} \frac{s^2-2Rr+r^2}{4Rr} \text{ and } \sum_{cyc}\frac{r_b}{r_a} = \frac{1}{r_ar_br_c}\sum_{cyc} r_b^2r_c$$

$$= \frac{1}{rs^2}\sum_{cyc}\frac{r_b^2r_c^2}{r_c} \overset{\text{Bergstrom}}{\geq} \frac{1}{rs^2}\cdot\frac{(r_ar_b+r_br_c+r_cr_a)^2}{r_a+r_b+r_c} = \frac{s^4}{rs^2(4R+r)}$$

$$= \frac{s^2}{r(4R+r)} \overset{?}{\geq} \frac{s^2-2Rr+r^2}{4Rr}$$

$$\Leftrightarrow 4Rs^2 \overset{?}{\geq} (4R+r)s^2 - (4R+r)(2Rr-r^2)$$

$$\Leftrightarrow (4R+r)(2R-r) \underset{(**)}{\overset{?}{\geq}} s^2 \text{ and via Gerretsen, RHS of } (**)$$

$$\leq 4R^2+4Rr+3r^2 \overset{?}{\leq} (4R+r)(2R-r) \Leftrightarrow 2R^2-3Rr-2r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (2R+r)(R-2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore 3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)$$

$$\geq \frac{3x+y+z}{y+z}+\frac{x+3y+z}{z+x}+\frac{x+y+3z}{x+y}+\frac{3}{2}\ \forall\, x,y,z>0\ (QED)$$

JP.431 If $x,y,z \in \left(0,\frac{\pi}{2}\right)$ then:

$$3+\sqrt[3]{\prod_{cyc}(2+\tan^6 x)} \geq \sec^2 x+\sec^2 y+\sec^2 z$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$3+\sqrt[3]{\prod_{cyc}(2+\tan^6 x)} = 3+\sqrt[3]{\prod_{cyc}(1^3+1^3+(\tan^2 x)^3)} \overset{\text{Holder}}{\geq}$$

$$\geq 3+\sqrt[3]{(1\cdot 1\cdot \tan^2 x+1\cdot 1\cdot \tan^2 y+1\cdot 1\cdot \tan^2 z)^3} =$$

$$= 3+\tan^2 x+\tan^2 y+\tan^2 z =$$

$$= (1+\tan^2 x)+(1+\tan^2 y)+(1+\tan^2 z) = \sec^2 x+\sec^2 y+\sec^2 z$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt[3]{\prod_{cyc}(2+\tan^6 x)} \geq \sum_{cyc}\tan^2 x$$

$$3+\sqrt[3]{\prod_{cyc}(2+\tan^6 x)}\geq\sum_{cyc}\sec^2 x$$

Solution 3 by Ravi Prakash-New Delhi-India

Let $a=\tan^2 x, b=\tan^2 y, c=\tan^2 z; a,b,c>0$ as $0<x,y,z<\frac{\pi}{2}$.

We have:

$$1+a^3+a^3b^3\geq 3\sqrt[3]{a^6b^3}=3a^2b$$

$$1+b^3+a^3b^3\geq 3ab^2$$

$$1+a^3+a^3c^3\geq 3ac^2$$

$$1+c^3+a^3c^3\geq 3ac^2$$

$$1+b^3+b^3c^3\geq 3bc^2$$

$$1+c^3+b^3c^3\geq 3bc^2$$

$$a^3+b^3+c^3\geq 3abc$$

$$1+1+a^3b^3c^3\geq 3abc$$

By adding the above inequalities, we get:

$$8+3(a^3+b^3+c^3)+2(a^3b^3+b^3c^3+c^3a^3)+a^3b^3c^3\geq 3\sum a^2b+3\sum ab^2+6abc$$

$$8+4(a^3+b^3+c^3)+2(a^3b^3+b^3c^3+c^3a^3)+a^3b^3c^3$$

$$\geq\sum a^3+3\sum a^2b+3\sum ab^2+6abc$$

$$(2+a^3)(2+b^3)(2+c^3)\geq(a+b+c)^3$$

Therefore,

$$\sqrt[3]{\prod_{cyc}(2+\tan^6 x)}\geq\sum_{cyc}\tan^2 x$$

$$3+\sqrt[3]{\prod_{cyc}(2+\tan^6 x)}\geq\sum_{cyc}\sec^2 x$$

JP.432 In ΔABC the following relationship holds:

$$\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)\geq\frac{4R}{r}+1$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

Using $3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)\ge xyz(x+y+z)^3;\forall x,y,z>0$

$$3x^2y^2z^2\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)\left(\frac{y}{z}+\frac{z}{x}+\frac{x}{y}\right)\ge xyz(x+y+z)^3\Leftrightarrow$$

$$\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)\left(\frac{y}{z}+\frac{z}{x}+\frac{x}{y}\right)\ge\frac{(x+y+z)^3}{3xyz}\Rightarrow$$

$$\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)\ge\frac{(r_a+r_b+r_c)^3}{3r_ar_br_c};(1)$$

$$\because r_a+r_b+r_c=4R+r,r_ar_br_c=s^2r;(2)$$

From (1),(2), it follows that:

$$\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)\ge\frac{(4R+r)^3}{3s^2r};(3)$$

$$3s^2\le(4R+r)^2\ (Doucet)\Rightarrow\frac{(4R+r)^2}{3s^2}\ge 1;(4)$$

From (3),(4), it follows that:

$$\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)\ge\frac{4R}{r}+1$$

Solution 2 by Alex Szoros-Romania

$$\sum_{cyc}\frac{r_a}{r_b}=\sum_{cyc}\frac{r_a^2}{r_ar_b}\ge\frac{(\sum r_a)^2}{\sum r_ar_b}=\frac{(4R+r)^2}{s^2};(1)$$

$$\sum_{cyc}\frac{r_b}{r_a}=\sum_{cyc}\frac{s-a}{s-b}=\sum_{cyc}\frac{(s-a)^2}{(s-a)(s-b)}\le\frac{\left(\sum(s-a)\right)^2}{\sum(s-a)(s-b)}$$

$$\sum_{cyc}\frac{r_b}{r_a}\ge\frac{s^2}{\sum[s^2-(a+b)s+ab]}=\frac{s^2}{3s^2-4s^2+s^2+r^2+4Rr}$$

$$\Rightarrow\sum_{cyc}\frac{r_b}{r_a}\ge\frac{s^2}{r(4R+r)};(2)$$

From (1) and (2), it follows that:

$$\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)\ge\frac{4R}{r}+1$$

Solution 3 by Alex Szoros-Romania

Lemma. If $x, y, z > 0$ then:

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right) \geq (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\Leftrightarrow \sum_{cyc}\left(1+\frac{xz}{y^2}+\frac{x^2}{yz}\right) \geq \sum_{cyc}\left(1+\frac{x}{y}+\frac{x}{z}\right); (1)$$

From Schur's inequality, we have:

$$\sum x(x-y)(x-z) \geq 0 \Rightarrow \sum x[x^2-(y+z)x+yz] \geq 0$$

$$\sum x^3 - \sum x^2(y+z) + \sum xyz \geq 0$$

$$\sum x^3 + 3xyz \geq \sum xy(x+y); (2)$$

So, we have:

$$\sum_{cyc}\frac{x^2}{yz} = \sum_{cyc}\frac{x^3}{xyz} = \frac{\sum x^3}{xyz} \geq \frac{\sum xy(x+y) - 3xyz}{xyz}$$

$$\Rightarrow \sum_{cyc}\frac{x^2}{yz} \geq \sum_{cyc}\frac{x+y}{z} - 3; (3)$$

On the other hand, from AM-GM inequality, we have:

$$\sum_{cyc}\frac{xz}{y^2} \geq 3\cdot\sqrt[3]{\frac{xy\cdot yz\cdot zx}{x^2y^2z^2}} = 3 \Rightarrow \sum_{cyc}\frac{xz}{y^2} \geq 3; (4)$$

From (3) and (4), it follows:

$$\sum_{cyc}\frac{x^2}{yz} + \sum_{cyc}\frac{xz}{y^2} \geq \sum_{cyc}\frac{x+y}{z}$$

Using $\sum r_a = 4R + r$ and $\sum \frac{1}{r_a} = \frac{1}{r}$, for $x = r_a, y = r_b, z = r_c$ from Lemma, we obtain the proposed problem.

Solution 4 by Adrian Popa-Romania

$$r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\ (and\ analogs) \Rightarrow \frac{r_a}{r_b} = \frac{\cot\frac{B}{2}}{\cot\frac{A}{2}} \text{ and } \frac{r_b}{r_a} = \frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}$$

$$\because \sum_{cyc} \cot\frac{A}{2}\cot\frac{B}{2} = \frac{4R}{r} + 1$$

We must show that:

$$\left(\sum_{cyc} \frac{\cot\frac{B}{2}}{\cot\frac{A}{2}}\right)\left(\sum_{cyc} \frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}\right) \ge \sum_{cyc} \cot\frac{A}{2}\cot\frac{B}{2}$$

Let us denote: $x = \cot\frac{A}{2}; y = \cot\frac{B}{2}; z = \cot\frac{C}{2}; x, y, z > 0$

$$x + y + z = xyz$$

$$\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right)\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) \overset{?}{\ge} xy + yz + zx$$

$$(y^2z + z^2x + x^2y)(x^2z + y^2x + z^2y) \overset{?}{\ge} x^3y^3z^2 + y^3z^3x^2 + x^3z^3y^2; (*)$$

$$x^2y^2z^2 + y^4xz + y^3z^3 + x^3z^3 + z^4xy + x^4yz + x^3y^3$$

$$+ x^2y^2z^2 \overset{?}{\ge} x^2y^2z^2(xy + yz + zx)$$

$$x^3y^3 + y^3z^3 + z^3x^3 \ge 3x^2y^2z^2$$

We must show:

$$6x^2y^2z^2 + x^4z(x^3 + y^3 + z^3) \ge (x + y + z)^2(xy + yz + zx)$$

$$6xyz(x + y + z) + (x + y + z)(x^3 + y^3 + z^3) \ge (x + y + z)^2(xy + yz + zx)$$

$$6xyz(x + y + z) + (x + y + z)(x^3 + y^3 + z^3) \ge$$

$$\ge (x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)(xy + yz + zx)$$

From Schur's inequality, we have:

$$x^2(x - y)(x - z) + y^2(y - x)(y - z) + z^2(z - x)(z - y) \ge 0$$

$\Rightarrow (*)$ is true.

Solution 5 by George Florin Șerban-Romania

$$\left(\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c}\right) = 3 + \sum_{cyc} \frac{r_a^2}{r_b r_c} + \sum_{cyc} \frac{r_b r_c}{r_a^2} \overset{?}{\ge} \frac{4R}{r} + 1$$

$$\sum_{cyc} \frac{r_a^2}{r_b r_c} + \sum_{cyc} \frac{r_b r_c}{r_a^2} \overset{?}{\ge} \frac{4R}{r} - 2$$

$$\frac{\sum r_a^3}{\prod r_a}+\frac{s^2-12Rr}{r^2}\ge\frac{4R}{r}-2$$

$$\frac{(4R+r)^3-12Rs^2}{s^2r}+\frac{s^2}{r^2}-\frac{12R}{r}\ge\frac{4R}{r}-2$$

$$\frac{(4R+r)^3}{s^2r}-\frac{12R}{r}+\frac{s^2}{r^2}\ge\frac{16R}{r}-2$$

$$\frac{(4R+r)^3}{s^2r}+\frac{s^2}{r^2}\overset{?}{\ge}\frac{28R}{r}-2$$

$$\frac{(4R+r)^3}{s^2r}+\frac{s^2}{r^2}\overset{Blundon-G}{\ge}\frac{(4R+r)^3}{r\cdot\frac{R(4R+r)^2}{2(2R-r)}}+\frac{16Rr-5r^2}{r^2}\ge\frac{28R}{r}-2$$

$$\frac{(4R-2r)(4R+r)}{Rr}+\frac{16R}{r}-5\overset{?}{\ge}\frac{28R}{r}-2$$

$$\frac{16R^2-4Rr-2r^2}{Rr}-\frac{12R}{r}-3\ge 0$$

$$\frac{4R}{r}-\frac{2}{\frac{R}{r}}-7\ge 0.\text{ Let } x=\frac{R}{r}\ge 2\Rightarrow\frac{(x-2)(4x+1)}{x}\ge 0\text{ true }\forall x\ge 2.$$

Therefore,

$$\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)\ge\frac{4R}{r}+1$$

Solution 6 by Nguyen Van Canh-Ben Tre-Vietnam

Using $r_a=\frac{S}{p-a}$ **(analogs). We have:**

$$\frac{r_b+r_c}{r_a}+\frac{r_a+r_b}{r_c}+\frac{r_c+r_a}{r_b}=\frac{4R}{r}-2;$$

$$\leftrightarrow\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)+\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)+3=\frac{4R}{r}+1;$$

Let $x=\frac{r_a}{r_b}>0; y=\frac{r_b}{r_c}>0; z=\frac{r_c}{r_a}>0\to xyz=1$**. We need to prove that:**

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3;$$

$$\leftrightarrow(x+y+z)(xy+yz+zx)\ge x+y+z+xy+yz+zx+3;$$

$$\leftrightarrow xy(x+y)+yz(y+z)+zx(z+x)+3xyz\ge x+y+z+xy+yz+zx+3;$$

$$\leftrightarrow xy(x+y)+yz(y+z)+zx(z+x)\ge x+y+z+xy+yz+zx;$$

$$\leftrightarrow\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}\ge x+y+z+xy+yz+zx;$$

$$\leftrightarrow \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z} \ge x+y+z+xy+yz+zx; (*)$$

By AM-GM we have:

$$\frac{y}{x}+\frac{z}{y}+\frac{x}{z} \ge 3;$$

Thus, we need to prove that:

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+3 \ge x+y+z+xy+yz+zx; (**)$$

Since: $xyz=1 \to \exists \alpha,\beta,\gamma>0$ **such that:** $x=\frac{\alpha}{\beta}, y=\frac{\beta}{\gamma},\ z=\frac{\gamma}{\alpha}$

$$(**) \leftrightarrow \frac{\alpha\beta}{\gamma^2}+\frac{\beta\gamma}{\alpha^2}+\frac{\alpha\gamma}{\beta^2}+3 \ge \frac{\alpha}{\beta}+\frac{\beta}{\gamma}+\frac{\gamma}{\alpha}+\frac{\beta}{\alpha}+\frac{\gamma}{\beta}+\frac{\alpha}{\gamma};$$

$$\leftrightarrow \alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3+3\alpha^3\beta^3\gamma^3 \ge \alpha\beta\gamma\big(\alpha\beta(\alpha+\beta)+\beta\gamma(\beta+\gamma)+\gamma\alpha(\gamma+\alpha)\big);$$

Which is true by Schur's Inequality:

$$m^3+n^3+p^3+3mnp \ge mn(m+n)+np(n+p)+pm(p+m)$$

(Where: $m=\alpha\beta; n=\beta\gamma; p=\gamma\alpha$**)**

Therefore, $(**)$ **true** $\to$ $(*)$ **true. Proved.**

JP.433 In $\Delta ABC, AA', BB', CC'$ —internal bisectors and A'', B'', C'' —contact points with circumcircle of ΔABC. Prove that:

$$\frac{1}{3}\left(7-\frac{2r}{R}\right)^2 \le \frac{AA''}{A'A''}+\frac{BB''}{B'B''}+\frac{CC''}{C'C''} \le 6\left(\left(\frac{R}{r}\right)^2-2\right)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\frac{AA''}{A'A''}=\frac{AA'+A'A''}{A'A''}=\frac{AA'}{A'A''}+1=\frac{AA'^2}{AA'\cdot A'A''}+1=\frac{w_a^2}{BA'\cdot A'C}+1; (1)$$

From theorem of bisector:

$$BA'\cdot A'C=\frac{a^2bc}{(b+c)^2}; (2)$$

From (1) and (2), we get:

$$\frac{AA''}{A'A''}=\frac{4b^2c^2}{(b+c)^2}\cos^2\frac{A}{2}\cdot\frac{(b+c)^2}{a^2bc}+1=\frac{4bc}{a^2}\cos^2\frac{A}{2}=\frac{4bc}{ac}\cdot\frac{s(s-a)}{bc}=$$

$$=\frac{4s(s-a)}{a^2}+1=\frac{(b+c+a)(b+c-a)}{a^2}+1=\frac{(b+c)^2}{a^2}\ (and\ analogs)$$

We must to prove:

$$6\left(\left(\frac{R}{r}\right)^2-2\right)\le\sum_{cyc}\frac{(b+c)^2}{a^2}\le\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$$

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\ge\frac{1}{3}\left(\sum_{cyc}\frac{b+c}{a}\right)^2=\frac{1}{3}\left(\frac{s^2+r^2-2Rr}{2Rr}\right)^2\overset{Gerretsen}{\ge}$$

$$\ge\frac{1}{3}\left(\frac{14Rr-4r^2}{2Rr}\right)^2=\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$$

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\overset{CBS}{\le}\sum_{cyc}\frac{2(b^2+c^2)}{a^2}=2\sum_{cyc}\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)$$

But $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}\Rightarrow\frac{b^2}{a^2}+\frac{a^2}{b^2}\le\frac{R^2}{r^2}-2$, then

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\le 2\left(\frac{3R^2}{r^2}-6\right)=6\left(\frac{R^2}{r^2}-2\right)$$

Solution 2 by George Florin Șerban-Romania

From bisector theorem we have:

$$\frac{A'B}{A'C}=\frac{AB}{AC}=\frac{c}{b}\Rightarrow\frac{A'B}{A'B+A'C}=\frac{c}{b+c}=\frac{A'B}{BC}=\frac{A'B}{a}$$

$$\Rightarrow A'B=\frac{ac}{b+c}\Rightarrow A'C=BC-A'B=a-\frac{ac}{b+c}=\frac{ab}{b+c}$$

$$\rho(A')=A'B\cdot A'C=AA'\cdot A'A''=\frac{a^2bc}{(b+c)^2}\Rightarrow AA'=\frac{2bc\cos\frac{A}{2}}{b+c}$$

$$\frac{AA'}{A'A''}=\frac{AA'}{\frac{a^2bc}{(b+c)^2AA'}}=\frac{A'A^2(b+c)^2}{a^2bc}=\frac{\frac{4b^2c^2}{(b+c)^2}\cos^2\frac{A}{2}(b+c)^2}{a^2bc}=$$

$$=\frac{4bc\cdot\frac{s(s-a)}{bc}}{a^2}=\frac{4s(s-a)}{a^2}$$

$$\sum_{cyc}\frac{AA''}{A'A''}=\sum_{cyc}\frac{AA'+A'A''}{A'A''}=\sum_{cyc}\left(1+\frac{AA'}{A'A''}\right)=3+\sum_{cyc}\frac{AA'}{A'A''}=$$

$$= 3 + 4s\sum_{cyc}\frac{s-a}{a^2} = 3 + 4s\cdot\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s} =$$

$$= 3 + \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4R^2r^2}$$

$$s^2 + 2r62 - 12Rr \overset{Gerretsen}{\geq} 16Rr - 5r62 + 2r^2 - 12Rr =$$

$$= r(4R-3r) \overset{Euler}{\geq} r(8r-4r) = 5r^2 > 0$$

$$\sum_{cyc}\frac{AA''}{A'A''} \geq 3 + \frac{(16Rr-5r^2)(4Rr-3r^2)+r^3(4R+r)}{4R^2r^2} =$$

$$= 3 + \frac{(16R-5r)(4R-3r)+r(4R+r)}{4R^2} \overset{?}{\geq} \frac{1}{3}\left(7-\frac{2r}{R}\right)^2$$

Let: $\frac{R}{r} = x \geq 2 \Rightarrow 3 + \frac{(16x-5)(4x-3)+4x+1}{4x^2} \geq \frac{1}{3}\left(7-\frac{2}{x}\right)^2$

$$\Leftrightarrow (x-2)(32x-16) \geq 0 \text{ true } \forall x \geq 2.$$

$$\sum_{cyc}\frac{AA'}{A'A''} = 3 + \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4R^2r^2} \overset{Gerretsen}{\leq}$$

$$\leq 3 + \frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^3(4R+r)}{4R^2r^2} \leq$$

$$\leq 6\left(\left(\frac{R}{r}\right)^2 - 2\right)$$

Let: $\frac{R}{r} = x \geq 2 \Rightarrow (x-2)(2x^3+8x^2+x+2) \geq 0$ true $\forall x \geq 2$

Therefore,

$$\frac{1}{3}\left(7-\frac{2r}{R}\right)^2 \leq \frac{AA''}{A'A''} + \frac{BB''}{B'B''} + \frac{CC''}{C'C''} \leq 6\left(\left(\frac{R}{r}\right)^2 - 2\right)$$

JP.434 If $x, y, z \in (0,1); 4(x^2+y^2+z^2) = 3$ then:

$$x^2y^2(1-x^2)^3 + y^2z^2(1-y^2)^3 + z^2x^2(1-z^2)^3 \leq \frac{243}{1024}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$3x^2(1-x^2)^3 \le \left(\frac{3x^2+1-x^2+1-x^2+1-x^2}{4}\right)^4 = \left(\frac{3}{4}\right)^4 = \frac{243}{256}$$

$$x^2(1-x^2)^3 \le \frac{81}{256} \quad (1)$$

By multiplying (1) with y^2:

$$x^2y^2(1-x^2)^3 \le \frac{81}{256}y^2 \quad (2)$$

Analogous:

$$y^2z^2(1-y^2)^3 \le \frac{81}{256}z^2 \quad (3)$$

$$z^2x^2(1-z^2)^3 \le \frac{81}{256}x^2 \quad (4)$$

By adding (2); (3); (4):

$$x^2y^2(1-x^2)^3 + y^2z^2(1-y^2)^3 + z^2x^2(1-z^2)^3 \le$$

$$\le \frac{81}{256}(y^2+z^2+x^2) = \frac{81}{256}\cdot\frac{3}{4} = \frac{243}{1024}$$

Equality holds for:

$$x = y = z = \frac{1}{2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that:

$$1 = x^2 + \frac{1-x^2}{3} + \frac{1-x^2}{3} + \frac{1-x^2}{3} \overset{AGM}{\ge} 4\cdot\sqrt[4]{x^2\left(\frac{1-x^2}{3}\right)^3}$$

$$x^2(1-x^2)^3 \le \frac{3^3}{4^4} = \frac{27}{256} \Rightarrow x^2y^2(1-x^2)^3 \le \frac{27}{256}y^2$$

Similalry, we have:

$$y^2z^2(1-y^2)^3 \le \frac{27}{256}z^2 \text{ and } z^2x^2(1-z^2)^3 \le \frac{27}{256}x^2$$

By adding these inequalities, we get:

$$\sum_{cyc} x^2y^2(1-x^2)^3 \le \frac{27}{256}\sum_{cyc} x^2 = \frac{27}{256}\cdot\frac{3}{4} = \frac{81}{1024}$$

Therefore,

$$\sum_{cyc} x^2y^2(1-x^2)^3 \le \frac{81}{1024}$$

Equality holds for $x=y=z=\frac{1}{2}$.

Solution 3 by Ravi Prakash-New Delhi-India

Let $f(t)=t^2(1-t^2)^3, t\in(0,1)$, then $f'(t)=2t(1-t^2)^2(1-2t)(1+2t)$

$$f'(t)=\begin{cases} >0;\text{ if } t\in\left(0;\frac{1}{2}\right) \\ =0;\text{ if } t=\frac{1}{2} \\ <0;\text{ if } t>\frac{1}{2}\end{cases}$$

$f(t)$ is maximum when $t=\frac{1}{2}\Rightarrow f(t)\le f\left(\frac{1}{2}\right)=\frac{27}{256}$. Equality holds for $t=\frac{1}{2}$.

Thus, for $0<x,y,z<1$ we have: $x^2+y^2+z^2=\frac{3}{4}$

$$x^2y^2(1-x^2)^3 \le \frac{27}{256}y^2$$

$$y^2z^2(1-y^2)^3 \le \frac{27}{256}z^2$$

$$x^2z^2(1-z^2)^3 \le \frac{27}{256}x^2$$

Hence, we get:

$$x^2y^2(1-x^2)^3+y^2z^2(1-y^2)^3+x^2z^2(1-z^2)^3 \le \frac{27}{256}(x^2+y^2+z^2)=\frac{81}{1024}$$

Equality holds for $x=y=z=\frac{1}{2}$.

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$x^2y^2(1-x^2)^3+y^2z^2(1-y^2)^3+x^2z^2(1-z^2)^3=$$

$$=(xy)^2[(1-x)(1+x)]^3+(yz)^2[(1-y)(1+y)]^3+(zx)^2[(1-z)(1+z)]^3\le$$

$$\le (xy)^2\left[\left(\frac{1-x+1+x}{2}\right)^2\right]^3+(yz)^2\left[\left(\frac{1-y+1+y}{2}\right)^2\right]^3+(zx)^2\left[\left(\frac{1-z+1+z}{2}\right)^2\right]^3$$

$$=(xy)^2+(yz)^2+(zx)^2\le\frac{(x^2+y^2+z^2)^2}{3}\le\frac{3}{16}\le\frac{243}{1024}$$

Solution 5 by Nguyen Van Canh-Ben Tre-Vietnam

Let $a = x^2, b = y^2, c = z^2 \to 4(a+b+c) = 3 \to \frac{4(a+b+c)}{3} = 1$

Then, we have:

$$x^2y^2(1-x^2)^3 + y^2z^2(1-y^2)^3 + z^2x^2(1-z^2)^3 \le \frac{243}{1024};$$

$$\leftrightarrow ab\left(\frac{4(a+b+c)}{3} - a\right)^3 + bc\left(\frac{4(a+b+c)}{3} - b\right)^3 + ca\left(\frac{4(a+b+c)}{3} - c\right)^3 \le \frac{81}{1024}.\left(\frac{4(a+b+c)}{3}\right)^5;$$

$$\leftrightarrow ab(a+4b+4c)^3 + bc(4a+b+4c)^3 + ca(4a+4b+c)^3 \le 9(a+b+c)^5;$$

By AM-GM Inequality we have:

- $ab(a+4b+4c)^3 = \frac{b}{9}.9a.(a+4b+4c)(a+4b+4c)(a+4b+4c) \le \frac{b(12a+12b+12c)^4}{9.4^4};$

Similary:

- $bc(4a+b+4c)^3 \le \frac{c(12a+12b+12c)^4}{9.4^4};$
- $ca(4a+4b+c)^3 \le \frac{c(12a+12b+12c)^4}{9.4^4};$

Therefore,

$$ab(a+4b+4c)^3 + bc(4a+b+4c)^3 + ca(4a+4b+c)^3 \le \frac{b(12a+12b+12c)^4}{9.4^4} + \frac{c(12a+12b+12c)^4}{9.4^4} + \frac{c(12a+12b+12c)^4}{9.4^4} = 9(a+b+c)^5;$$

Proved. Equality if and only if $a = b = c = \frac{1}{4} \leftrightarrow x = y = z = \frac{1}{2}$

JP.435 If $x, y, z > 0; x^2 + y^2 + z^2 = 1$ then:

$$(x^6 + y^6 + z^6)^3 \ge \left(x^5 + y^5 + z^5\right)^4$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Denote $s = x^6 + y^6 + z^6$

$$4 = 3 + 1 = \frac{3(x^6 + y^6 + z^6)}{x^6 + y^6 + z^6} + x^2 + y^2 + z^2 =$$

$$=\frac{3x^6}{s}+\frac{3y^6}{s}+\frac{3z^6}{s}+x^2+y^2+z^2=$$

$$=\left(\frac{x^6}{s}+\frac{x^6}{s}+\frac{x^6}{s}+x^2\right)+\left(\frac{y^6}{s}+\frac{y^6}{s}+\frac{y^6}{s}+y^2\right)+\left(\frac{z^6}{s}+\frac{z^6}{s}+\frac{z^6}{s}+z^2\right)\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq} 4\cdot\sqrt[4]{\left(\frac{x^6}{s}\right)^3\cdot x^2}+4\cdot\sqrt[4]{\left(\frac{y^6}{s}\right)^3\cdot y^2}+4\cdot\sqrt[4]{\left(\frac{z^6}{s}\right)^3\cdot z^2}$$

$$4\geq 4\left(\frac{x^5}{\sqrt[4]{s^3}}+\frac{y^5}{\sqrt[4]{s^3}}+\frac{z^5}{\sqrt[4]{s^3}}\right)$$

$$\sqrt[4]{s^3}\geq x^5+y^5+z^5 \Leftrightarrow s^3\geq\left(x^5+y^5+z^5\right)^4$$

Therefore,

$$(x^6+y^6+z^6)^3\geq\left(x^5+y^5+z^5\right)^4$$

Equality holds for $x=y=z=\frac{\sqrt{3}}{3}$.

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$x^2+y^2+z^2=\frac{x^{20}}{x^{18}}+\frac{y^{20}}{y^{18}}+\frac{z^{20}}{z^{18}}=\frac{\left(x^5\right)^4}{(x^6)^3}+\frac{\left(y^5\right)^4}{(y^6)^3}+\frac{(z^3)^4}{(z^6)^3}\overset{Radon}{\geq}$$

$$\geq\frac{\left(x^5+y^5+z^5\right)^4}{(x^6+y^6+z^6)^3}\Leftrightarrow 1\geq\frac{\left(x^5+y^5+z^5\right)^4}{(x^6+y^6+z^6)^3}\Leftrightarrow(x^6+y^6+z^6)^3\leq\left(x^5+y^5+z^5\right)^4$$

Equality holds for $x=y=z=\frac{\sqrt{3}}{3}$.

Solution 3 by Aggeliki Papaspyropoulou-Greece

$$(x^6+y^6+z^6)^3(x^2+y^2+z^2)\geq\left(\sqrt[4]{x^{20}}+\sqrt[4]{y^{20}}+\sqrt[4]{z^{20}}\right)^4$$

$$(x^6+y^6+z^6)^3(x^2+y^2+z^2)\geq\left(x^5+y^5+z^5\right)^4$$

$$\Leftrightarrow(x^6+y^6+z^6)^3\leq\left(x^5+y^5+z^5\right)^4$$

Equality holds for $x=y=z=\frac{\sqrt{3}}{3}$.

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$(x^6+y^6+z^6)^3\leq\left(x^5+y^5+z^5\right)^4\Leftrightarrow$$

$$\left(\frac{x^6+y^6+z^6}{x^5+y^5+z^5}\right)^3 \geq x^5+y^5+z^5 \Leftrightarrow$$

$$\left(\frac{x^5+y^5+z^5}{x^4+y^4+z^4}\right)^2 \geq x^4+y^4+z^4 \Leftrightarrow$$

$$\frac{x^4+y^4+z^4}{x^3+y^3+z^3} \geq x^3+y^3+z^3 \Leftrightarrow$$

$$(x^2+y^2+z^2)(x^4+y^4+z^4) \geq (x^3+y^3+z^3)(x^3+y^3+z^3) \Leftrightarrow$$

$$x^6+y^6+z^6+x^2y^4+x^2z^4+y^2z^4+y^2x^4+z^2x^4+z^2y^4 \geq$$

$$\geq x^6+y^6+z^6+2((xy)^3+(yz)^3+(zx)^3) \Leftrightarrow$$

$$x^2z^4+y^2z^4+y^2x^4+z^2x^4+z^2y^4 \geq 2((xy)^3+(yz)^3+(zx)^3) \text{ true, because}$$

$$x^4y^2+y^4x^2 \geq 2(xy)^3, y^4z^2+y^2z^4 \geq 2(yz)^3, z^4x^2+z^2x^4 \geq 2(zx)^3$$

Equality holds for $x=y=z=\frac{\sqrt{3}}{3}$.

Solution 5 by Vivek Kumar-India

$$x^2+y^2+z^2=1$$

$$\left(\sum_{cyc} x^6\right)\left(\sum_{cyc} x^2\right) \geq \left(\sum_{cyc} x^4\right)^2$$

$$\left(\sum_{cyc} x^4\right)^2 \leq \sum_{cyc} x^6$$

Also,

$$\left(\sum_{cyc} x^6\right)\left(\sum_{cyc} x^4\right) \geq \left(\sum_{cyc} x^5\right)^2$$

$$\sum_{cyc} x^6 \geq \frac{\left(\sum x^5\right)^2}{\sum x^4} \Rightarrow \left(\sum_{cyc} x^6\right)^2 \geq \frac{\left(\sum x^5\right)^4}{(\sum x^4)^2} \geq \frac{\left(\sum x^5\right)^4}{\sum x^6}$$

$$\left(\sum_{cyc} x^6\right)^3 \geq \left(\sum_{cyc} x^5\right)^4$$

Equality holds for $x = y = z = \frac{\sqrt{3}}{3}$.

Solution 6 by Marin Chirciu-Romania

Using Holder's inequality, we get:

$$LHS = (x^{4n-2} + y^{4n-2} + z^{4n-2})^3 = (x^{4n-2} + y^{4n-2} + z^{4n-2})^3(x^2 + y^2 + z^2) =$$

$$= (x^{4n-2} + y^{4n-2} + z^{4n-2})(x^{4n-2} + y^{4n-2} + z^{4n-2})(x^{4n-2} + y^{4n-2} + z^{4n-2})(x^2 + y^2 + z^2) \geq$$

$$\overset{Holder}{\geq} \left(\sqrt[4]{x^{4n-2}x^{4n-2}x^{4n-2}x^2} + \sqrt[4]{y^{4n-2}y^{4n-2}y^{4n-2}y^2} + \sqrt[4]{z^{4n-2}z^{4n-2}z^{4n-2}z^2}\right)^4 =$$

$$= \left(\sqrt[4]{x^{12n-4}} + \sqrt[4]{y^{12n-4}} + \sqrt[4]{z^{12n-4}}\right)^4 = (x^{3n-1} + y^{3n-1} + z^{3n-1})^4 = RHS$$

Equality holds for $x = y = z = \frac{1}{\sqrt{3}}$ and $n = 1$.

For $n = 2$ we get the Proposed Problem JP.435 by Daniel Sitaru-Romania.

PROBLEMS FOR SENIORS

SP.421 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}}\ge\frac{8}{3}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{x+y}{x^2+xy+y^2}\ge\frac{4(x+y+z)}{\sum x^2+\sum xy}\Rightarrow$$

$$\sum_{cyc}\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}}\ge\frac{4\left(\sum\cos^2\frac{A}{2}\right)}{\sum\cos^4\frac{A}{2}+\sum\cos^2\frac{A}{2}\cos^2\frac{B}{2}};(1)$$

$$\sum_{cyc}\cos^2\frac{A}{2}=\frac{4R+r}{2R};(2)$$

$$\sum_{cyc}\cos^4\frac{A}{2}+\sum_{cyc}\cos^2\frac{A}{2}\cos^2\frac{B}{2}=\frac{(4R+r)^2-s^2}{8R^2}+\frac{s^2+(4R+r)^2}{16R^2}=$$

$$=\frac{3(4R+r)^2-s^2}{16R^2}\overset{Doucet}{\le}\frac{3(4R+r)^2-3r(4R+r)}{16R^2}=$$

$$=\frac{3(4R+r)(4R+r-r)}{16R^2}=\frac{3(4R+r)}{4R};(3)$$

From (1),(2) and (3), it follows that:

$$\sum_{cyc}\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}}\ge\frac{4(4R+r)}{2R}\cdot\frac{4R}{3(4R+r)}=\frac{8}{3}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let's\ prove\ that\ (*):\sum_{cyc}\frac{x+y}{x^2+xy+y^2}\ge\frac{4(x+y+z)}{x^2+y^2+z^2+xy+yz+zx},\forall x,y,z>0.$$

$$\left(\sum_{cyc} x^2 + \sum_{cyc} xy\right)\left(\sum_{cyc} \frac{x+y}{x^2+xy+y^2}\right) = \sum_{cyc} \frac{[(x^2+xy+y^2)+z(x+y+z)](x+y)}{x^2+xy+y^2}$$

$$= \sum_{cyc}\left(x+y+\frac{z[(x^2+xy+y^2)+xy+yz+zx]}{x^2+xy+y^2}\right)$$

$$= 2\sum_{cyc} x + \sum_{cyc} z + (xy+yz+zx)\sum_{cyc}\frac{z}{x^2+xy+y^2} \overset{CBS}{\geq} 3\sum_{cyc} x$$

$$+\left(\sum_{cyc} xy\right)\cdot\frac{(x+y+z)^2}{\sum z(x^2+xy+y^2)} = 3\sum_{cyc} x + \frac{(xy+yz+zx)(x+y+z)^2}{(xy+yz+zx)(x+y+z)}$$

$$\rightarrow \left(\sum_{cyc} x^2 + \sum_{cyc} xy\right)\left(\sum_{cyc}\frac{x+y}{x^2+xy+y^2}\right) \geq 4\sum_{cyc} x \leftrightarrow \sum_{cyc}\frac{x+y}{x^2+xy+y^2}$$

$$\geq \frac{4(x+y+z)}{x^2+y^2+z^2+xy+yz+zx}, \forall x,y,z > 0.$$

$$For\ x = \cos^2\frac{A}{2}, y = \cos^2\frac{B}{2}, z = \cos^2\frac{C}{2}.\ Using: \sum_{cyc}\cos^2\frac{A}{2} = \frac{4R+r}{2R}, \sum_{cyc}\cos^4\frac{A}{2}$$

$$= \frac{(4R+r)^2-s^2}{8R^2}, \sum_{cyc}\cos^2\frac{A}{2}\cos^2\frac{B}{2} = \frac{(4R+r)^2+s^2}{16R^2}$$

$$(*) \leftrightarrow \sum_{cyc}\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}} \geq \frac{4.\frac{4R+r}{2R}}{\frac{(4R+r)^2-s^2}{8R^2}+\frac{(4R+r)^2+s^2}{16R^2}}$$

$$= \frac{32R(4R+r)}{3(4R+r)^2-s^2} \overset{Doucet}{\geq} \frac{32R(4R+r)}{3(4R+r)^2-3r(4R+r)} = \frac{8}{3}.$$

$$Therefore,\quad \sum_{cyc}\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}} \geq \frac{8}{3}.$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2} \therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\sum\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}}\overset{\text{via (i) and analogs}}{\cong}4R\sum\frac{r_b+r_c+r_c+r_a}{(r_b+r_c)^2+(r_c+r_a)^2+(r_b+r_c)(r_c+r_a)}$$

$$=4R\sum\frac{(4R+r)+r_c}{r_b^2+r_c^2+2r_br_c+r_c^2+r_a^2+2r_cr_a+(r_ar_b+r_br_c+r_cr_a)+r_c^2}$$

$$=4R.\frac{1}{4R+r}\sum\frac{(4R+r)^2+2r_c(4R+r)-r_c(4R+r)-s^2+s^2}{(4R+r)^2-2s^2+s^2+2r_c(4R+r)}$$

$$=4R.\frac{1}{4R+r}\sum\frac{(4R+r)^2+2r_c(4R+r)-s^2}{(4R+r)^2+2r_c(4R+r)-s^2}$$

$$+4R.\frac{1}{4R+r}\sum\frac{s^2-r_c(4R+r)}{(4R+r)^2-s^2+2r_c(4R+r)}$$

$$=4R.\frac{3}{4R+r}+4R.\frac{1}{4R+r}.\sum\frac{r_ar_b+r_br_c+r_cr_a-r_c(r_a+r_b+r_c)}{s^2+(4R+r)^2-2(r_ar_b+r_br_c+r_cr_a)+2r_c(r_a+r_b+r_c)}$$

$$=4R.\frac{3}{4R+r}$$

$$+4R.\frac{1}{2(4R+r)}.\sum\frac{2\left(r_ar_b-r_c^2\right)-s^2-(4R+r)^2+s^2+(4R+r)^2}{s^2+(4R+r)^2+2\left(r_c^2-r_ar_b\right)}$$

$$=4R.\frac{3}{4R+r}+4R.\frac{1}{2(4R+r)}.\sum\frac{2\left(r_ar_b-r_c^2\right)-s^2-(4R+r)^2}{s^2+(4R+r)^2+2\left(r_c^2-r_ar_b\right)}$$

$$+4R.\frac{s^2+(4R+r)^2}{2(4R+r)}\sum\frac{1}{(4R+r)^2-s^2+2r_c(4R+r)}$$

$$=4R.\frac{3}{2(4R+r)}$$

$$+4R.\frac{s^2+(4R+r)^2}{2(4R+r)}.\frac{(\alpha+\beta r_a)(\alpha+\beta r_b)+(\alpha+\beta r_b)(\alpha+\beta r_c)+(\alpha+\beta r_c)(\alpha+\beta r_a)}{(\alpha+\beta r_a)(\alpha+\beta r_b)(\alpha+\beta r_c)}\Big(\alpha$$
$$=(4R+r)^2-s^2\ \textit{and}\ \beta=2(4R+r)\Big)$$

$$=4R.\frac{3}{2(4R+r)}$$

$$+4R.\frac{s^2+(4R+r)^2}{2(4R+r)}.\frac{7(4R+r)^4+3s^4-6s^2(4R+r)^2}{3(4R+r)^6-s^6+8rs^2(4R+r)^3-3s^2(4R+r)^4+s^4(4R+r)^2}\geq\frac{8}{3}$$

$$\Leftrightarrow\frac{s^2+(4R+r)^2}{2(4R+r)}.\frac{7(4R+r)^4+3s^4-6s^2(4R+r)^2}{3(4R+r)^6-s^6+8rs^2(4R+r)^3-3s^2(4R+r)^4+s^4(4R+r)^2}$$
$$\geq\frac{7R+4r}{6R(4R+r)}$$

$$\Leftrightarrow(16R+4r)s^6+\left(96R^2+16Rr-20r^2\right)(4R+r)^3s^2-(16R+4r)(4R+r)^2s^4$$
$$-12r(4R+r)^6\overset{(*)}{\gtrsim}0$$

Now, LHS of $(*)$ $\overset{\text{Gerretsen}}{\gtrsim}$ $-\left((16R+4r)(4R+r)^2-(16R+4r)\left(16Rr-5r^2\right)\right)s^4$

$$+\left(96R^2+16Rr-20r^2\right)(4R+r)^3s^2-12r(4R+r)^6$$
$$=-(16R+4r)\left(16R^2-8Rr+6r^2\right)s^4+\left(96R^2+16Rr-20r^2\right)(4R+r)^3s^2$$
$$-12r(4R+r)^6\overset{\text{Gerretsen}}{\gtrsim}$$

$$\left(-(16R+4r)(16R^2-8Rr+6r^2)(4R^2+4Rr+3r^2)+(96R^2+16Rr-20r^2)(4R+r)^3\right)s^2 - 12r(4R+r)^6$$

$$=\left(5120R^5+4864R^4r-128R^3r^2-832R^2r^3-512Rr^4-92r^5\right)s^2-12r(4R+r)^6 \overset{\text{Gerretsen}}{\gtrless} \left(5120R^5+4864R^4r-128R^3r^2-832R^2r^3-512Rr^4-92r^5\right)(16Rr-5r^2)-12r(4R+r)^6$$

$$\left(\because 5120R^5+4864R^4r-128R^3r^2-832R^2r^3-512Rr^4-92r^5 = (R-2r)(5120R^4+15104R^3r+30080R^2r^2+59328Rr^3+118144r^4) + 236196r^5 \overset{\text{Euler}}{>} 0\right)$$

$$\overset{?}{\geq} 0 \Leftrightarrow 1024t^6-672t^5-2264t^4-876t^3-216t^2+25t+14 \overset{?}{\geq} 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(1024t^5+1376t^4+488t^3+8t(t-2)+2(t^2-4)+90t^2+1\right) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (*) \text{ is true } \therefore \sum \frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}} \geq \frac{8}{3} \text{ (QED)}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$A+B+C=\pi, \cos A+\cos B+\cos C \leq \frac{3}{2} \Rightarrow \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$$

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2} \text{ and } \frac{A}{2}=\frac{\pi}{2}-\left(\frac{B}{2}+\frac{C}{2}\right)$$

$$\cos\frac{A}{2}=\cos\left(\frac{\pi}{2}-\left(\frac{B}{2}+\frac{C}{2}\right)\right)=\sin\left(\frac{B}{2}+\frac{C}{2}\right)=\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}$$

Hence,

$$\sum_{cyc}\cos^2\frac{A}{2}=\left(\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}\right)^2+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=$$

$$=\sin^2\frac{B}{2}\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\cos^2\frac{B}{2}+2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\cos^2\frac{B}{2}\cos^2\frac{C}{2} \leq \frac{9}{4}$$

Let $x=\cos\frac{A}{2}, y=\cos\frac{B}{2}, z=\cos\frac{C}{2}$, then

$$\sum_{cyc}\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}}=\sum_{cyc}\frac{x^2+y^2}{x^4+x^2y^2+y^4}=$$

$$=\sum_{cyc}\frac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{x^2}{y^2}+1+\frac{y^2}{x^2}} \geq \frac{4\left(\sum\frac{1}{x^2}\right)}{\sum\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left(\frac{x^2}{y^2}+1+\frac{y^2}{x^2}\right)} \geq \frac{8}{3}$$

$$3\sum_{cyc}\frac{1}{x^4}+6\sum_{cyc}\frac{1}{x^2y^2}\ge 8\sum_{cyc}\frac{1}{x^2}+2\sum_{cyc}\left(\frac{x^2}{y^4}+\frac{y^2}{x^4}\right)\text{ true because}$$

$$6\sum_{cyc}\frac{1}{x^2y^2}\ge 8\sum_{cyc}\frac{1}{x^2}\Leftrightarrow 3\sum_{cyc}\frac{1}{x^2y^2}\ge 4\sum_{cyc}\frac{1}{x^2}$$

$$3\sum_{cyc}x^2\ge 4\sum_{cyc}x^2y^2\Leftrightarrow 3\sum_{cyc}x^2\ge\frac{4}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^2\right)$$

$$\frac{9}{4}\ge x^2+y^2+z^2\text{ true.}$$

$$\text{and }3\sum_{cyc}\frac{1}{x^4}\ge 2\sum_{cyc}\left(\frac{x^2}{y^4}+\frac{y^2}{x^4}\right)$$

$$3\sum_{cyc}x^4y^4\ge 12\sum_{cyc}(x^6y^4+x^4y^6)$$

$$3\sum_{cyc}x^4y^4\ge\frac{4}{3}\left(\sum_{cyc}(xy)^4\right)\left(\sum_{cyc}x^2\right)\Leftrightarrow\frac{9}{4}\ge x^2+y^2+z^2\text{ true.}$$

SP.422 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_b+r_c}{r_b^2+r_br_c+r_c^2}\ge\frac{2}{2R-r}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{x+y}{x^2+xy+y^2}\ge\frac{4(x+y+z)}{\sum x^2+\sum xy}$$

$$\Leftrightarrow\sum_{cyc}\frac{(x+y)(x+y+z)}{x^2+xy+y^2}=3+\sum_{cyc}\frac{xy+yz+zx}{x^2+y^2+z^2}$$

We must to prove:

$$\sum_{cyc}\frac{xy+yz+zx}{x^2+xy+y^2}\ge 1+\frac{4\sum xy}{\sum x^2+\sum xy}\Leftrightarrow\frac{4\sum xy}{\sum x^2+\sum xy}-2\ge\sum_{cyc}\left(1-\frac{xy+yz+zx}{x^2+xy+y^2}\right)\Leftrightarrow$$

$$\sum_{cyc}\frac{(x-z)^2}{\sum x^2+\sum xy}\ge\sum_{cyc}\frac{(x-z)^2(y^2-xz)}{(x^2+xy+y^2)(y^2+yz+z^2)}\Leftrightarrow$$

$(x^2+xy+y^2)(y^2+yz+z^2)\geq(y^2-xz)(x^2+y^2+z^2+xy+yz+zx)$ true.

$$\sum_{cyc}\frac{r_a+r_b}{r_b^2+r_br_c+r_c^2}\geq\frac{4(r_a+r_b+r_c)}{r_a^2+r_b^2+r_c^2+r_ar_b+r_br_c+r_cr_a};(1)$$

$$\because r_a+r_b+r_c=4R+r;(2)$$

$$r_a^2+r_b^2+r_c^2+r_ar_b+r_br_c+r_cr_a=(4R+r)^2-2s^2+s^2=(4R+r)^2-s^2\overset{Doucet}{\leq}$$

$$\leq(4R+r)^2-3r(4R+r)=(4R+r)(4R+r-3r)=2(4R+r)(2R-r);(3)$$

From (1),(2) and (3) it follows that:

$$\sum_{cyc}\frac{r_a+r_b}{r_b^2+r_br_c+r_c^2}\geq\frac{2}{2R-r}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let's\ prove\ that\ (*):\ \sum_{cyc}\frac{x+y}{x^2+xy+y^2}\geq\frac{4(x+y+z)}{x^2+y^2+z^2+xy+yz+zx},\forall x,y,z>0.$$

$$We\ have:\ (x^2+y^2+z^2+xy+yz+zx)(x+y)$$

$$=(x^2+xy+y^2)(x+y)+z(x+y+z)(x+y)=$$

$$=(x^2+xy+y^2)(x+y)+z[(x^2+xy+y^2)+xy+yz+zx]$$

$$=(x+y+z)(x^2+xy+y^2)+z(xy+yz+zx)$$

$$\rightarrow(x^2+y^2+z^2+xy+yz+zx)\sum_{cyc}\frac{x+y}{x^2+xy+y^2}$$

$$=\sum_{cyc}\left((x+y+z)+\frac{(xy+yz+zx).z}{x^2+xy+y^2}\right)$$

$$=3\sum_{cyc}x+\sum_{cyc}xy.\sum_{cyc}\frac{z^2}{z(x^2+xy+y^2)}\geq$$

$$\overset{Bergstrom}{\geq}3\sum_{cyc}x+\sum_{cyc}xy.\frac{(x+y+z)^2}{\sum z(x^2+xy+y^2)}=3\sum_{cyc}x+\frac{(xy+yz+zx)(x+y+z)^2}{(xy+yz+zx)(x+y+z)}$$

$$=4(x+y+z)\ \rightarrow\ (*)\ is\ true.$$

$$For\ x=r_a,y=r_b,z=r_c,we\ obtain:$$

$$(*)\leftrightarrow \sum_{cyc}\frac{r_b+r_c}{{r_b}^2+r_br_c+{r_c}^2}\ge\frac{4(r_a+r_b+r_c)}{{r_a}^2+{r_b}^2+{r_c}^2+r_ar_b+r_br_c+r_cr_a}$$

$$=\frac{4(4R+r)}{(4R+r)^2-s^2}\overset{Doucet}{\overbrace{\ge}}\frac{4(4R+r)}{(4R+r)^2-3r(4R+r)}=\frac{2}{2R-r}.$$

$$Therefore,\quad \sum_{cyc}\frac{r_b+r_c}{{r_b}^2+r_br_c+{r_c}^2}\ge\frac{2}{2R-r}.$$

Solution 3 by Nguyen Van Canh-BenTre-Vietnam

First, for all $m,n,p>0$, we have:

$$(n^2+np+p^2)(mn+np+mp)\overset{\text{AM-GM}}{\overbrace{\le}}\frac{1}{4}(n^2+np+p^2+mn+np+mp)^2$$

$$=\frac{1}{4}(n+p)^2(m+n+p)^2;$$

$$\rightarrow\frac{1}{n^2+np+p^2}\ge\frac{4(mn+np+mp)}{(n+p)^2(m+n+p)^2};(\text{analogs})$$

$$\rightarrow\frac{n+p}{n^2+np+p^2}\ge\frac{4(mn+np+mp)}{(n+p)(m+n+p)^2};(\text{analogs})$$

Hence,

$$\sum\frac{n+p}{n^2+np+p^2}\ge\frac{4\sum np}{(m+n+p)^2}\sum\frac{1}{n+p}=\frac{4\sum np}{(m+n+p)^2}\cdot\frac{\sum m^2+3\sum np}{(m+n)(n+p)(p+n)}$$

$$\overset{m=r_a;n=r_b;p=r_c}{\overbrace{=}}\frac{4p^2}{(4R+r)^2}\cdot\frac{(4R+r)^2+p^2}{4Rp^2}=\frac{(4R+r)^2+p^2}{R(4R+r)^2}\overset{(*)}{\overbrace{\ge}}\frac{2}{2R-r};$$

$$(*)\leftrightarrow(2R-r)((4R+r)^2+p^2)\ge 2R(4R+r)^2;$$

$$\leftrightarrow(2R-r)p^2\ge r(4R+r)^2;$$

But: $p^2\ge 16Rr-5r^2$ (Gerretsen's Inequality). We need to prove that:

$$(2R-r)(16Rr-5r^2)\ge r(4R+r)^2;$$

$$\leftrightarrow(2R-r)(16R-5r)\ge 16R^2+8Rr+r^2;;$$

$$\leftrightarrow 16R^2-34Rr+4r^2\ge 0;;$$

$$\leftrightarrow 2(R-2r)(8R-r)\ge 0;$$

Which is true by $R\ge 2r$ (Euler).

Thus, $(*)$ is true. Proved.

Solution 4 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\sum\frac{r_b+r_c}{r_b^2+r_br_c+r_c^2}=\sum\frac{r_b+r_c}{(r_b+r_c)^2-r_br_c}\overset{\text{via (i) and analogs}}{=}\sum\frac{4R\cos^2\frac{A}{2}}{16R^2\cos^4\frac{A}{2}-bc\cos^2\frac{A}{2}}$$

$$=4R\frac{1}{16R^2\cos^2\frac{A}{2}-bc}\overset{\text{Bergstrom}}{\geq}\frac{36R}{8R^2\sum(1+\cos A)-\sum ab}$$

$$=\frac{36R}{8R^2\frac{4R+r}{R}-s^2-4Rr-r^2}$$

$$=\frac{36R}{32R^2+4Rr-r^2-s^2}\overset{?}{\geq}\frac{2}{2R-r}$$

$$\Leftrightarrow s^2+18R(2R-r)\overset{?}{\geq}32R^2+4Rr-r^2\overset{?}{\Leftrightarrow}s^2+4R^2-22Rr+r^2\underset{(*)}{\overset{?}{\geq}}0$$

Now, via Gerretsen, LHS of $(*)\geq 16Rr-5r^2+4R^2-22Rr+r^2=4R^2-6Rr-4r^2$

$$=2(R-2r)(2R+r)\overset{\text{Euler}}{\geq}0\Rightarrow(*)\text{ is true}\therefore\sum\frac{r_b+r_c}{r_b^2+r_br_c+r_c^2}$$

SP.423 If $z_1,z_2,z_3\in\mathbb{C}^*$ different in pairs such that $|z_1|=|z_2|=|z_3|=1$, $A(z_1),B(z_2),C(z_3)$. Prove that:

$$\sum_{cyc}\frac{z_2z_3}{3z_2z_3-z_2^2-z_3^2}=\frac{3}{4}\Leftrightarrow AB=BC=CA.$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$|z_1|=|z_2|=|z_3|=1,A(z_1),B(z_2),C(z_3)\Rightarrow\Delta ABC\in C(0,1)$$

$$\sum_{cyc}\frac{z_2z_3}{3z_2z_3-z_2^2-z_3^2}=\frac{3}{4}\Leftrightarrow\sum_{cyc}\frac{z_2z_3}{z_2z_3-(z_2-z_3)^2}=\frac{3}{4}\Leftrightarrow$$

$$\sum_{cyc}\frac{1}{1-\frac{(z_2-z_3)^2}{z_2z_3}}=\frac{3}{4};(1)$$

$$\text{But: } \sin^2 A = -\frac{(z_2 - z_3)^2}{4z_2z_3}; (2)$$

From (1) and (2), we have:

$$\sum_{cyc} \frac{1}{1 + 4\sin^2 A} = \frac{3}{4}; (3)$$

$$\text{But: } \sum_{cyc} \frac{1}{1 + 4\sin^2 A} \overset{Bergstrom}{\geq} \frac{9}{3 + 4\sum \sin^2 A} \geq \frac{3}{4} \Leftrightarrow$$

$12 \geq 3 + 4\sum \sin^2 A \Leftrightarrow \sum \sin^2 A \leq \frac{9}{4}$, true because $\sum \sin^2 A = \frac{a^2+b^2+c^2}{4R^2} \Rightarrow$

$a^2 + b^2 + c^2 \leq 9R^2 (Leibniz)$. So, we get:

$$\sum_{cyc} \frac{1}{1 + 4\sin^2 A} \geq \frac{3}{4}; (4)$$

From (3),(4) equality holds if and only if triangle ABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$|z_1| = |z_2| = |z_3| = 1 \rightarrow \Delta ABC \in C(0,1), then: \sin^2 A = -\frac{(z_2 - z_3)^2}{4z_2z_3}$$

$$\rightarrow \frac{z_2z_3}{3z_2z_3 - z_2^2 - z_3^2} = \frac{1}{1 + 4\sin^2 A} \ (And\ analogs)$$

$$\rightarrow \sum_{cyc} \frac{z_2z_3}{3z_2z_3 - z_2^2 - z_3^2} = \frac{3}{4} \leftrightarrow \sum_{cyc} \frac{1}{1 + 4\sin^2 A} = \frac{3}{4}.$$

$$But\ we\ have: \sum_{cyc} \frac{1}{1 + 4\sin^2 A} \overset{CBS}{\geq} \frac{3^2}{3 + 4(\sin^2 A + \sin^2 B + \sin^2 C)}$$

$$= \frac{9}{3 + \frac{a^2 + b^2 + c^2}{R^2}} \overset{Leibniz}{\geq} \frac{9}{3 + 9} = \frac{3}{4}.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

$$Therefore, \quad \sum_{cyc} \frac{z_2z_3}{3z_2z_3 - z_2^2 - z_3^2} = \frac{3}{4} \leftrightarrow AB = BC = CA.$$

SP.424 If $x,y,z>0,27(x^3y+y^3z+z^3x)=1$ then

$$45(x^2y+y^2z+z^2x)+6(xy+yz+zx)\le 4+3(x+y+z)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(3x-1)^2(4x+1)\ge 0;\forall x>0$$

$$(9x^2-6x+1)(4x+1)\ge 0$$

$$36x^3+9x^2-24x^2-6x+4x+1\ge 0$$

$$36x^3+1\ge 15x^2+2x;(1)$$

By multiplying (1) with y:

$$36x^3y+y\ge 15x^2y+2xy$$

$$36\sum_{cyc}x^3y+x+y+z\ge 15\sum_{cyc}x^2y+2(xy+yz+zx)$$

$$36\cdot\frac{1}{27}+x+y+z\ge 15\sum_{cyc}x^2y+2(xy+yz+zx)$$

$$15\sum_{cyc}x^2y+2(xy+yz+zx)\le x+y+z+\frac{4}{3}$$

$$45(x^2y+y^2z+z^2x)+6(xy+yz+zx)\le 4+3(x+y+z)$$

Equality holds for $x=y=z=\frac{1}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have that}:\ 45(x^2y+y^2z+z^2x)+6(xy+yz+zx)\overset{?}{\le}4+3(x+y+z)$$

$$\leftrightarrow 45(x^2y+y^2z+z^2x)+6(xy+yz+zx)\le 4.27(x^3y+y^3z+z^3x)+3(x+y+z)$$

$$\leftrightarrow 3y(36x^3-15x^2-2x+1)+3z(36y^3-15y^2-2y+1)$$

$$+3x(36z^3-15z^2-2z+1)\ge 0$$

$$\leftrightarrow y(3x-1)^2(4x+1)+z(3y-1)^2(4y+1)+x(3z-1)^2(4z+1)$$

$$\ge 0\ \ (\because (3x-1)^2(4x+1)=36x^3-15x^2-2x+1)$$

$$\textit{Which is true for all } x,y,z>0.$$

$$\textit{Therefore},\quad 45(x^2y+y^2z+z^2x)+6(xy+yz+zx)\le 4+3(x+y+z).$$

$$Equality\ holds\ iff\ x = y = z = \frac{1}{3}.$$

SP.425 If $x \in [0,1]$, then

$$1 + x^2 \le \int_0^1 e^{t^2}\,dt + \int_0^x 2t^2 e^{t^2}\,dt + \int_x^1 (2t^2 - 2t)e^{t^2}\,dt \le e^x$$

Proposed by Alex Szoros-Romania

Solution 1 by proposer

$$I = \int_0^x 2t^2 e^{t^2}\,dt = \int_0^x t\left(e^{t^2}\right)'\,dt = te^{t^2}\Big|_0^x - \int_0^x e^{t^2}\,dt = xe^{x^2} - \int_0^x e^{t^2}\,dt;\ (1)$$

$$J = \int_x^1 (2t^2 - 2t)e^{t^2}\,dt = \int_x^1 (t-1)2te^{t^2}\,dt = \int_x^1 (t-1)\left(e^{t^2}\right)'\,dt =$$

$$= (t-1)e^{t^2}\Big|_1^x - \int_x^1 e^{t^2}dt = -(x-1)e^{x^2} - \int_x^1 e^{t^2}\,dt =$$

$$= -xe^{x^2} + e^{x^2} - \int_x^1 e^{t^2}\,dt;\ (2)$$

From (1) and (2), we get:

$$I + J = e^{x^2} - \left(\int_0^x e^{t^2}\,dt + \int_x^1 e^{t^2}\,dt\right) = e^{x^2} - \int_0^1 e^{t^2}\,dt$$

$$I + J + \int_0^1 e^{t^2}\,dt = e^{x^2}$$

$$\int_0^1 e^{t^2}\,dt + \int_0^x 2t^2 e^{t^2}\,dt + \int_x^1 (2t^2 - 2t)e^{t^2}\,dt = e^{x^2};\ (3)$$

On the other hand, we have: $e^u \ge u + 1;\ \forall\, u \ge 0$, so

$$1 + x^2 \le e^{x^2}$$

How, $x \in [0,1] \Rightarrow x^2 \le x \Rightarrow e^{x^2} \le e^x$. We get:

$1 + x^2 \le e^{x^2} \le e^x\ ;\forall x \in [0,1]$ and using (3) we get the problem.

Solution 2 by Ravi Prakash-New Delhi-India

$$I = \int_0^1 e^{t^2}\,dt + \int_0^x 2t^2 e^{t^2}\,dt + \int_x^1 2t^2 e^{t^2}\,dt - \int_x^1 2te^{t^2}\,dt =$$

$$= te^{t^2}\Big|_0^1 - \int_0^1 t\cdot 2te^{t^2}\,dt + \int_0^1 2t^2e^{t^2}\,dt - e^{t^2}\Big|_x^1 = e^{x^2}$$

As $0\le x\le 1 \Rightarrow 0\le x^2\le x\le 1 \Rightarrow e^{x^2}\le e^x$. Also, $e^{x^2}\ge 1+x^2$.

Thus, for $0\le x\le 1$,

$$1+x^2\le \int_0^1 e^{t^2}\,dt + \int_0^x 2t^2e^{t^2}\,dt + \int_x^1 (2t^2-2t)e^{t^2}\,dt \le e^x$$

SP.426 Let R_1, R_2, R_3 be circumradii of $\Delta A_1B_1C_1, \Delta A_2B_2C_2, \Delta A_3B_3C_3$ with sides a_1, a_2, a_3 respectively b_1, b_2, b_3 and c_1, c_2, c_3. Prove that:

$$\frac{1}{a_1a_2a_3}+\frac{1}{b_1b_2b_3}+\frac{1}{c_1c_2c_3}\ge\frac{9\sqrt{3}}{(R_1+R_2+R_3)^3}$$

Proposed by D.M. Bătineţu-Giurgiu –Romania

Solution 1 by proposer

$$\frac{1}{a_1a_2a_3}+\frac{1}{b_1b_2b_3}+\frac{1}{c_1c_2c_3}\overset{AGM}{\ge}\frac{1}{\left(\frac{a_1+a_2+a_3}{3}\right)^3}+\frac{1}{\left(\frac{b_1+b_2+b_3}{3}\right)^3}+\frac{1}{\left(\frac{c_1+c_2+c_3}{3}\right)^3}$$

$=$

$$=27\left(\frac{1^4}{(2s_1)^3}+\frac{1^4}{(2s_2)^3}+\frac{1^4}{(2s_3)^2}\right)\overset{Radon}{\ge}27\cdot\frac{(1+1+1)^4}{(2s_1+2s_2+2s_3)^3}\overset{Mitrinovic}{\ge}$$

$$\ge\frac{27\cdot 81}{\left(2\cdot\frac{3\sqrt{3}}{2}R_1+2\cdot\frac{3\sqrt{3}}{2}R_2+2\cdot\frac{3\sqrt{3}}{2}R_3\right)^3}=\frac{27\cdot 81}{(3\sqrt{3})(R_1+R_2+R_3)^3}=$$

$$=\frac{27}{\sqrt{3}(R_1+R_2+R_3)^3}=\frac{9\sqrt{3}}{(R_1+R_2+R_3)^3}$$

Equality holds for $a_1=a_2=a_3; b_1=b_2=b_3; c_1=c_2=c_3$.

Solution 2 by Marian Ursărescu-Romania

In any ΔABC we have:

$$abc=4Rrs\le 4\cdot\frac{3\sqrt{3}}{2}R\cdot R\cdot R=3\sqrt{3}R^3$$

$$\frac{1}{abc}\ge\frac{1}{3\sqrt{3}R^3}$$

We must show that:

$$\frac{1}{R_1^3}+\frac{1}{R_2^3}+\frac{1}{R_3^3}\geq\frac{81}{(R_1+R_2+R_3)^3};(1)$$

But: $R_1+R_2+R_3\geq 3\sqrt[3]{R_1R_2R_3}\Rightarrow\frac{1}{(R_1+R_2+R_3)^3}\leq\frac{1}{27R_1R_2R_3};(2)$

From (1) and (2), we must show that:

$$\frac{1}{R_1^3}+\frac{1}{R_2^3}+\frac{1}{R_3^3}\geq\frac{3}{R_1R_2R_3}\text{ true because}$$

$$\frac{1}{R_1^3}+\frac{1}{R_2^3}+\frac{1}{R_3^3}\geq 3\sqrt[3]{\frac{1}{R_1^3R_2^3R_3^3}}=\frac{3}{R_1R_2R_3}$$

Solution 3 by George Florin Șerban-Romania

$$\frac{1}{a_1a_2a_3}+\frac{1}{b_1b_2b_3}+\frac{1}{c_1c_2c_3}\overset{AGM}{\geq}3\sqrt[3]{\frac{1}{a_1b_1c_1\cdot a_2b_2c_2\cdot a_3b_3c_3}}=$$

$$=\frac{3}{\sqrt[3]{4R_1F_1\cdot 4R_2F_2\cdot 4R_3F_3}}=\frac{3}{4}\cdot\frac{1}{\sqrt[3]{R_1R_2R_3}}\cdot\frac{1}{\sqrt[3]{s_1s_2s_3\cdot r_1r_2r_3}}\overset{AGM}{\geq}$$

$$\geq\frac{3}{4}\cdot\frac{3}{R_1+R_2+R_3}\cdot\frac{1}{\sqrt[3]{s_1s_2s_3}}\cdot\frac{1}{\sqrt[3]{r_1r_2r_3}}\overset{Mitrinovic}{\geq}$$

$$\geq\frac{27}{4(R_1+R_2+R_3)(r_1+r_2+r_3)}\cdot\frac{1}{\sqrt[3]{\frac{3\sqrt{3}R_1}{2}\cdot\frac{3\sqrt{3}R_2}{2}\cdot\frac{3\sqrt{3}R_3}{2}}}=$$

$$=\frac{27}{4(R_1+R_2+R_3)(r_1+r_2+r_3)}\cdot\frac{2}{3\sqrt{3}}\cdot\frac{1}{\sqrt[3]{R_1R_2R_3}}\overset{AGM}{\geq}$$

$$\geq\frac{54}{12\sqrt{3}(R_1+R_2+R_3)(r_1+r_2+r_3)}\cdot\frac{3}{R_1+R_2+R_3}=$$

$$=\frac{54\sqrt{3}}{12(R_1+R_2+R_3)^2(r_1+r_2+r_3)}\overset{Euler}{\geq}$$

$$\geq\frac{9\sqrt{3}}{2(R_1+R_2+R_3)^2\left(\frac{R_1}{2}+\frac{R_2}{2}+\frac{R_3}{3}\right)}=\frac{9\sqrt{3}}{(R_1+R_2+R_3)^3}$$

Solution 4 by Alex Szoros-Romania

In any ΔABC: $\sqrt[3]{abc}\leq\frac{a+b+c}{3}\overset{Mitrinovic}{\leq}\frac{3\sqrt{3}R}{2}\Rightarrow\sqrt[3]{abc}\leq R\sqrt{3}$

$$abc \leq 3\sqrt{3}R^3 \Rightarrow \frac{1}{abc} \geq \frac{1}{3\sqrt{3}R^3}; (1)$$

Using (1), we can write as:

$$\frac{1}{a_1a_2a_3} + \frac{1}{b_1b_2b_3} + \frac{1}{c_1c_2c_3} \geq \frac{1}{3\sqrt{3}}\left(\frac{1^4}{R_1^3} + \frac{1^4}{R_2^3} + \frac{1^4}{R_3^4}\right) \overset{Radon}{\geq}$$

$$\geq \frac{(1+1+1)^4}{3\sqrt{3}(R_1+R_2+R_3)^3} = \frac{81}{3\sqrt{3}(R_1+R_2+R_3)} = \frac{9\sqrt{3}}{(R_1+R_2+R_3)^3}$$

SP.427 If $f: [0, n] \to \left[0, \frac{1}{n-1}\right]$ continuous function, $n \in \mathbb{N}, n \geq 3$ then:

$$\int_0^n x^\alpha f(x) \cdot \sqrt[n-1]{1-(n-1)f(x)}\, dx \leq \frac{1}{\alpha+1} \cdot \sqrt[n-1]{n^{\alpha(n-1)-1}}; \alpha > 0$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$\sqrt[n]{f^{n-1}(x)\big(1-(n-1)f(x)\big)} \leq \frac{(n-1)f(x) + 1 - (n-1)f(x)}{n} = \frac{1}{n}$$

$$f^{n-1}(x)\big(1-(n-1)f(x)\big) \leq \frac{1}{n^n}$$

$$f(x)\sqrt[n-1]{1-(n-1)f(x)} \leq \frac{1}{n \cdot \sqrt[n-1]{n}}$$

$$\int_0^n x^\alpha f(x) \cdot \sqrt[n-1]{1-(n-1)f(x)}\, dx \leq \frac{1}{n \cdot \sqrt[n-1]{n}} \cdot \int_0^n x^\alpha\, dx$$

$$\int_0^n x^\alpha f(x) \cdot \sqrt[n-1]{1-(n-1)f(x)}\, dx \leq \frac{1}{n \cdot \sqrt[n-1]{n}} \cdot \frac{n^{\alpha+1}}{\alpha+1}$$

$$\int_0^n x^\alpha f(x) \cdot \sqrt[n-1]{1-(n-1)f(x)}\, dx \leq \frac{1}{\alpha+1} \cdot \frac{n^\alpha}{\sqrt[n-1]{n}}$$

$$\int_0^n x^\alpha f(x) \cdot \sqrt[n-1]{1-(n-1)f(x)}\, dx \leq \frac{1}{\alpha+1} \cdot \frac{n^\alpha}{n^{\frac{1}{n-1}}}$$

$$\int_0^n x^\alpha f(x) \cdot \sqrt[n-1]{1-(n-1)f(x)}\, dx \leq \frac{1}{\alpha+1} \cdot n^{\alpha - \frac{1}{n-1}}$$

Therefore,

$$\int_0^n x^\alpha f(x)\cdot\sqrt[n-1]{1-(n-1)f(x)}\,dx \le \frac{1}{\alpha+1}\cdot\sqrt[n-1]{n^{\alpha(n-1)-1}}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ a\ge 0. From\ AM-GM\ Inequality, we\ have:\ \underbrace{a^n+\cdots+a^n}_{n-1\ times}+\frac{1}{n^n}$$

$$\ge n\sqrt[n]{(a^n)^{n-1}.\frac{1}{n^n}}=a^{n-1}\ \leftrightarrow\ (n-1)a^n+\frac{1}{n^n}\ge a^{n-1}$$

$$\leftrightarrow\ a^{n-1}[1-(n-1)a]\le\frac{1}{n^n}\ \leftrightarrow\ a.\sqrt[n-1]{1-(n-1)a}\le\sqrt[n-1]{n^{-n}}, \forall a\ge 0.$$

$$Let\ x\in[0,n], for\ a=f(x), we\ have\ that:\ f(x).\sqrt[n-1]{1-(n-1)f(x)}\le\sqrt[n-1]{n^{-n}}$$

$$\to\ x^\alpha f(x).\sqrt[n-1]{1-(n-1)f(x)}\le\sqrt[n-1]{n^{-n}}.x^\alpha, \forall x\in[0,n], \alpha>0.$$

$$Therefore,\quad \int_0^n x^\alpha f(x).\sqrt[n-1]{1-(n-1)f(x)}dx\le\sqrt[n-1]{n^{-n}}.\int_0^n x^\alpha dx=\sqrt[n-1]{n^{-n}}.\frac{n^{\alpha+1}}{\alpha+1}$$

$$=\frac{1}{\alpha+1}.\sqrt[n-1]{n^{\alpha(n-1)-1}}, \alpha>0.$$

SP.428 Solve for real numbers:

$$\sum_{k=1}^n\frac{1}{\cos x-\cos(2k+1)x}=\frac{\sin nx}{\sin(n+1)x}\cdot\cot x$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$\sum_{k=1}^n\frac{1}{\cos x-\cos(2k+1)x}=\sum_{k=1}^n\frac{1}{2\sin kx\cdot\sin(k+1)x}=$$

$$=\frac{1}{2}\sum_{k=1}^n\frac{\sin x}{\sin kx\cdot\sin(k+1)x}\cdot\frac{1}{\sin x}=$$

$$=\frac{1}{2}\sum_{k=1}^n\frac{\sin(k+1)x\cdot\cos kx-\sin kx\cdot\cos(k+1)\,x}{\sin kx\cdot\sin(k+1)x}\cdot\frac{1}{\sin x}=$$

$$=\frac{1}{2\sin x}\sum_{k=1}^n(\cot x-\cot(k+1)x)=\frac{1}{2\sin x}(\cot x-\cot(n+1)x)=$$

$$=\frac{1}{2\sin x}\left(\frac{\cos x}{\sin x}-\frac{\cos(n+1)x}{\sin(n+1)x}\right)=$$

$$=\frac{1}{2\sin x}\cdot\frac{\cos x\cdot\sin(n+1)x-\cos(n+1)x\cdot\sin x}{\sin x\cdot\sin(n+1)x}=\frac{\sin nx}{2\sin^2 x\cdot\sin(n+1)x}$$

So, we have:

$$\sum_{k=1}^{n}\frac{1}{\cos x-\cos(2k+1)x}=\frac{\sin nx}{\sin(n+1)x}\cdot\cot x\Leftrightarrow$$

$$\frac{\sin nx}{2\sin^2 x\cdot\sin(n+1)x}=\frac{\sin nx}{\sin(n+1)x}\cdot\cot x\Leftrightarrow\frac{1}{2\sin^2 x}=\frac{\cos x}{\sin x}\Leftrightarrow$$

$$2\sin x\cos x=1\Leftrightarrow\sin 2x=1\Leftrightarrow 2x=\frac{\pi}{2}+k\pi$$

$$x\in\left\{\frac{k\pi}{2}+(-1)^k\frac{\pi}{12}\middle|k\in\mathbb{Z}\right\}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\cos x-\cos(2k+1)x=2\sin(kx)\sin\big((k+1)x\big)$$

$$\because\sum_{k=1}^{n}\frac{1}{\cos x-\cos(2k+1)x}=\frac{1}{2\sin x}\sum_{k=1}^{n}\frac{\sin[(k+1)x-kx]}{\sin(kx)\sin(k+1)\,x}=$$

$$=\frac{1}{2\sin x}\sum_{k=1}^{n}\frac{\sin(k+1)x\cos(kx)-\cos(k+1)x\sin(kx)}{\sin(kx)\sin(k+1)x}=$$

$$=\frac{1}{2\sin x}\sum_{k=1}^{n}[\cot(kx)-\cot)k+1)x]=\frac{1}{2\sin x}[\cot x-\cot(n+1)x]=$$

$$=\frac{1}{2\sin^2 x}\cdot\frac{\sin(nx)}{\sin(n+1)x}$$

Thus,

$$\frac{1}{2\sin^2 x}\cdot\frac{\sin(nx)}{\sin(n+1)x}=\frac{\sin(nx)\cot x}{\sin(n+1)x}$$

Assuming $\sin(nx)\neq 0$, we get:

$$\frac{1}{2\sin^2 x}=\cot x=\frac{\cos x}{\sin x}\Rightarrow\sin 2x=\frac{1}{2}=\sin\frac{\pi}{6}$$

$$x=\frac{1}{2}m\pi+(-1)^m\frac{\pi}{12},m\in\mathbb{Z}$$

SP.429 Let $(x_n)_{n\geq 1}$ is a sequence of real numbers such that

$$x_n = \int_0^1 x^n \cdot \log(1+x)\,dx.\ \text{Find:}$$

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} (-1)^{k-1} x_k$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$x \in [0,1] \Rightarrow \begin{cases} 0 \leq \log(x+1) \leq \log 2 \\ \frac{1}{2} \leq \frac{1}{x+1} \leq 1 \end{cases} \Rightarrow 0 \leq \frac{\log(1+x)}{1+x} \leq \log 2$$

$$\left|\frac{(-x)^{n+1}\log(1+x)}{1+x}\right| \leq x^{n+1} \cdot \log 2$$

$$\left|\int_0^1 \frac{(-x)^{n+1}\log(1+x)}{1+x}dx\right| \leq \int_0^1 \left|\frac{(-x)^{n+1}\log(1+x)}{1+x}\right| dx \leq$$

$$\leq \log 2 \cdot \int_0^1 x^{n+1}\,dx = \frac{\log 2}{n+1}$$

$$\lim_{n\to\infty} \int_0^1 \frac{(-x)^{n+1}\log(1+x)}{1+x}dx = 0 \Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k-1} x_k = \log\frac{4}{e} - \frac{1}{2}\log^2 2$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} (-1)^{k-1} x_k = 0$$

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$x_n \geq 0, \forall n \in \mathbb{N}$$

$$0 \leq |\Omega| \leq \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \left|(-1)^{k-1} \cdot x_k\right|$$

$$0 \leq |\Omega| \leq \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} x_k$$

$$0 \leq |\Omega| \leq \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \int_0^1 x^k \cdot \log(1+x)\,dx \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_0^1 x^{(k+1)}\,dx$$

$$0 \leq |\Omega| \leq \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{1}{k+2}$$

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{1}{k+2} = \lim_{n\to\infty} \frac{H_{n+2} - \frac{3}{2}}{n} = 0$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} (-1)^{k-1} x_k = 0$$

Solution 3 by Naren Bhandari-Bajura-Nepal

For all $n \geq 1$, the following equality holds:

$$\int_0^1 x^{2n-1} \cdot \log(1+x)\,dx = \frac{H_{2n} - H_n}{2n}; (1)$$

where H_n is n^{th} harmonic number defined by $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Proof. Since $f(x) = \log(1+x)$ possesses Maclaurin series for all $x \in [0,1]$, we exploit it here

$$\int_0^1 x^{2n-1} f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^1 x^{2n-1+k}\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(2n+k)} =$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{(2k-1)(2k+2n-1)} - \frac{1}{2k(2k+2n)} \right) =$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+2n-1} - \frac{1}{2k} + \frac{1}{2k+2n} \right) \cdot \frac{1}{2n} =$$

$$= \left(\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k} \right) \cdot \frac{1}{2n} = \frac{H_{2n} - H_n}{2n}$$

After partial fraction, one can use definition of digamma function and by multiplication formula of harmonic number, $H_{n-\frac{1}{2}}=2H_{2n}-H_n-\log 4$ easily leads us to proposed result.

Now, replace n by $\frac{n}{2}$ and enforcing $n\to n+1$ gives

$$x_n=\int_0^1 x^n\cdot\log(x+1)\,dx=\frac{H_{2n}-H_{\frac{n+1}{2}}}{n+1};(2)$$

So, we find x_n explicitly, now we check if $\sum_{k=1}^{\infty}(-1)^{k+1}x_k$ convergent or divergent.

It is easy to see that

$$\lim_{k\to\infty}x_k=\lim_{k\to\infty}\frac{H_{k+1}-H_{\frac{k+1}{2}}}{k+1}\cong\lim_{k\to\infty}\frac{1}{k(k+1)}=0$$

By asymptotic expansion of harmonic number and $x_k>x_{k+1}$ and by Leibniz test we conclude that $\sum_{k=1}^{n}(-1)^{k+1}x_k$ is convergent. Therefore we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k+1}x_k=L,L\in\mathbb{R}$$

Further, $\lim_{n\to\infty}n\neq 0$. So, by quotient rule of limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}x_k=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty}(-1)^{k+1}x_k=\lim_{n\to\infty}\frac{L}{n}=0$$

Solution 4 by Naren Bhandari-Bajura-Nepal

Using the finite geometric series formula,

$$\sum_{k=1}^{n}(-1)^{k+1}x^k=\frac{x}{1+x}-\frac{(-x)^n}{1+x}$$

and by linearity of integral

$$\Phi(n)=\sum_{k=1}^{n}(-1)^{k+1}\int_0^1 x^k f(x)\,dx=\int_0^1 f(x)\sum_{k=1}^{n}(-1)^{k+1}x^k\,dx=$$

$$=\int_0^1\frac{x\log(1+x)}{1+x}dx-\int_0^1(-x)^n\cdot\frac{\log(1+x)}{1+x}dx\overset{IBP}{=}$$

$$= 2\log 2 - \frac{\log^2 2}{2} - 1 - \int_0^1 g_n(x)\,dx$$

Now, the required limit is $\lim_{n\to\infty}\frac{\Phi(n)}{n} = 0 - \lim_{n\to\infty}\frac{1}{n}\int_0^1 g_n(x)\,dx$, We notice that

$$\frac{1}{n}|g_n(x)| \le \frac{1}{n}x^n = h_n(x); \forall x \in [0,1]$$

and is clearly integrable function $\int_0^1 h_n(x)\,dx = \frac{1}{n(n+1)}$ and by DCT, it follows

$$\lim_{n\to\infty}\frac{1}{n}\int_0^1 g_n(x)\,dx = \int_0^1 \lim_{n\to\infty} n^{-1}g_n(x)\,dx = 0, \text{since}$$

$$\lim_{n\to\infty}\int_0^1 h_n(x)\,dx = 0$$

Solution 5 by Naren Bhandari-Bajura-Nepal

We have a classical inequality $1 - \frac{1}{x} \le \log x \le x - 1; \forall x > 0$

$$\text{or}\ \frac{x}{1+x} \le \log(x+1) \le x; \forall x > -1$$

$$0 < \int_0^1 \frac{x^{k+1}}{x+1}dx \le \int_0^1 x^k\log(1+x)\,dx \le \int_0^1 x^{k+1}\,dx = \frac{1}{k+2}; (3)$$

The right most quantity

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k+2} = \sum_{k=3}^{n}\frac{(-1)^{k-3}}{k} = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} - \frac{1}{2} = \frac{2\log 2 - 1}{2} - \sum_{k=n+1}^{\infty}\frac{(-1)^{k+1}}{k}$$

Show that if $n \to \infty$, then $A(n) = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k+2} \to \frac{2\log 2 - 1}{2}$ and hence $\frac{A(n)}{n} \to 0$ if $n \to \infty$. By squeeze theorem in (3) tells us that the required limit is 0

SP.430 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{\cos\frac{A}{2}} \le \frac{3}{2s}\sum_{cyc}\frac{a}{\cos\frac{A}{2}} \le \frac{6R}{s}\sqrt{2+\frac{r}{2R}} \le \sqrt{2+\frac{5R}{r}}$$

Proposed by Alex Szoros-Romania

Solution 1 by proposer

The triplets (a,b,c) and $\left(\frac{1}{\cos\frac{A}{2}},\frac{1}{\cos\frac{B}{2}},\frac{1}{\cos\frac{C}{2}}\right)$ are same ordered, using Chebyshev's inequality:

$$\sum_{cyc}\frac{a}{\cos\frac{A}{2}}\geq\frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\right)=\frac{2s}{3}\sum_{cyc}\frac{1}{\cos\frac{A}{2}}$$

$$\Rightarrow\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\leq\frac{3}{2s}\sum_{cyc}\frac{a}{\cos\frac{A}{2}};(1)$$

On the other hand, using CBS we have:

$$\sum_{cyc}\sqrt{bc(s-a)}\leq\sqrt{\sum_{cyc}bc}\cdot\sqrt{\sum_{cyc}(s-a)}=\sqrt{s}\cdot\sqrt{s^2+r^2+4Rr}$$

$$\Rightarrow\sum_{cyc}\sqrt{\frac{bc(s-a)}{s}}\leq\sqrt{4R^2+4Rr+3r^2+r^2+4Rr}=2(R+r);(2)$$

Using (2) we can write:

$$\sum_{cyc}\frac{1}{\sin\frac{A}{2}}=\sum_{cyc}\sqrt{\frac{bc}{(s-b)(s-c)}}=\frac{1}{F}\sum_{cyc}\sqrt{bcs(s-a)}=\frac{s}{F}\sum_{cyc}\sqrt{\frac{bc(s-a)}{s}}$$

$$\Rightarrow\sum_{cyc}\frac{1}{\sin\frac{A}{2}}\leq\frac{1}{r}\cdot 2(R+r)=2\left(1+\frac{R}{r}\right);(3)$$

$$(3)\Rightarrow\sum_{cyc}\frac{1}{\sin\frac{A}{2}}=\sum_{cyc}\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\prod\sin\frac{A}{2}}=\frac{4R}{r}\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\leq 2\left(1+\frac{R}{r}\right)$$

$$\Rightarrow\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{2}\left(1+\frac{r}{R}\right);(4)\ (V.Nicula)$$

$$\sum_{cyc}\sin^2\frac{A}{2}=\sum_{cyc}\frac{1-\cos A}{2}=\frac{1}{2}\left(3-\sum_{cyc}\cos A\right)=\frac{1}{2}\left(3-1-\frac{r}{R}\right)=1-\frac{r}{2R}$$

Hence,

$$\left(\sum_{cyc}\sin\frac{A}{2}\right)^2=\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\le 1-\frac{r}{2R}+1+\frac{r}{2R}$$

$$\Rightarrow\sum_{cyc}\sin\frac{A}{2}\le\sqrt{2+\frac{r}{2R}};(5)(M.Bencze)$$

So, we have:

$$\frac{3}{2s}\sum_{cyc}\frac{a}{\cos\frac{A}{2}}=\frac{3}{2s}\sum_{cyc}\frac{2a\cdot\sin\frac{A}{2}}{\sin A}=\frac{6R}{s}\sum_{cyc}\sin\frac{A}{2}\le\frac{6R}{s}\sqrt{2+\frac{r}{2R}};(6)$$

$$(6)\Leftrightarrow\frac{6R}{s}\sqrt{2+\frac{r}{2R}}\le\sqrt{2+\frac{5R}{r}}\Leftrightarrow$$

$$\frac{36R^2}{s^2}\left(2+\frac{r}{2R}\right)\le 2+\frac{5R}{r}\Leftrightarrow s^2\ge\frac{18Rr(4R+r)}{5R+2r};(7)$$

Using Gerretsen inequality: $s^2\ge 16Rr-5r^2$, it is enough to prove:

$$16Rr-5r^2\ge\frac{18Rr(4R+r)}{5R+2r}\Leftrightarrow(16Rr-5r^2)(5R+2r)\ge 18Rr(4R+r)$$

$$\Leftrightarrow(16R-5r)(5R+2r)\ge 18R(4R+r)\Leftrightarrow 8R^2-11Rr-10r^2\ge 0$$

$$\Leftrightarrow(R-2r)(8R+5r)\ge 0 \text{ true from } R\ge 2r\ (Euler)\Rightarrow(7) \text{ its true.}$$

From (1),(6) and (7) we get the conclusion.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a\ge b\ge c\ \rightarrow\ \mu(A)\ge\mu(B)\ge\mu(C)\ \rightarrow$$

$$\frac{1}{\cos\frac{A}{2}}\ge\frac{1}{\cos\frac{B}{2}}\ge\frac{1}{\cos\frac{C}{2}},\textit{from Chebyshev's inequality, we have that}:$$

$$3\sum_{cyc}\frac{a}{\cos\frac{A}{2}}\ge\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\right)=2s\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\leftrightarrow\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\le\frac{3}{2s}\sum_{cyc}\frac{a}{\cos\frac{A}{2}}\quad(1)$$

$$\textit{Now, we have that}:\sum_{cyc}\frac{a}{\cos\frac{A}{2}}=\sum_{cyc}\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\cos\frac{A}{2}}=4R\sum_{cyc}\sin\frac{A}{2}$$

$$=4R\sum_{cyc}\sqrt{\frac{(s-b)(s-c)}{bc}}\overset{CBS}{\le}4R\sqrt{\left(\sum_{cyc}(s-b)(s-c)\right)\left(\sum_{cyc}\frac{1}{bc}\right)}=$$

$$= 4R\sqrt{r(4R+r).\frac{1}{2Rr}} = 4R\sqrt{2+\frac{r}{2R}} \to \frac{3}{2s}\sum_{cyc}\frac{a}{\cos\frac{A}{2}} \le \frac{6R}{s}\sqrt{2+\frac{r}{2R}}\ (2)$$

$$Also, we\ have: \frac{6R}{s}\sqrt{2+\frac{r}{2R}} \overset{?}{\leq} \sqrt{2+\frac{5R}{r}} \leftrightarrow \frac{18R(4R+r)}{s^2} \overset{?}{\leq} \frac{5R+2r}{r}$$

$$\leftrightarrow\ 18Rr(4R+r) \overset{?}{\leq} (5R+2r)s^2$$

$$\leftrightarrow\ (5R+2r)[s^2-(16Rr-5r^2)]$$

$$+r(R-2r)(8R$$

$$+5r) \overset{?}{\geq} 0\ which\ is\ true\ from\ Gerretsen\ and\ Euler's\ inequalities.$$

$$\to \frac{6R}{s}\sqrt{2+\frac{r}{2R}} \le \sqrt{2+\frac{5R}{r}}\ (3).\ \ From\ (1),(2)\ and\ (3), we\ get: \sum_{cyc}\frac{1}{\cos\frac{A}{2}}$$

$$\le \frac{3}{2s}\sum_{cyc}\frac{a}{\cos\frac{A}{2}} \le \frac{6R}{s}\sqrt{2+\frac{r}{2R}} \le \sqrt{2+\frac{5R}{r}}$$

SP.431 In ΔABC the following relationship holds:

$$1 \ge \frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{2s^2(s^2+r^2+2Rr)} \ge \frac{2r}{R}$$

Proposed by Alex Szoros-Romania

Solution 1 by proposer

Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{bc}{b+c} \ge \frac{2F}{R}\ (Tsintsifas)$$

Proof. Using $\frac{R}{r} \ge \frac{b}{c}+\frac{c}{b}$ (Băndilă), **we get:** $\frac{R}{r} \ge \frac{b^2+c^2}{bc} \Rightarrow \frac{bc}{b+c} \ge \frac{r}{R}\cdot\frac{b^2+c^2}{b+c} \Rightarrow$

$$\sum_{cyc}\frac{bc}{b+c} \ge \frac{r}{R}\sum_{cyc}\frac{b^2+c^2}{b+c}; (1)$$

$$\text{But} \sum_{cyc}\frac{b^2+c^2}{b+c} = \sum_{cyc}\frac{b^2}{b+c}+\sum_{cyc}\frac{c^2}{b+c} \overset{Bergstrom}{\geq} \frac{(\sum b)^2}{\sum(b+c)}+\frac{(\sum c)^2}{\sum(b+c)} =$$

$$=\sum_{cyc} a = 2s;(2)$$

From (1) and (2) it follows:

$$\sum_{cyc}\frac{bc}{b+c}\ge\frac{2sr}{R}=\frac{2F}{R}$$

From Lemma, we get:

$$\frac{1}{s}\sum_{cyc}\frac{bc}{b+c}\ge\frac{2r}{R};(3)$$

On the other hand, we have:

$$\frac{b+c}{4}\ge\frac{bc}{b+c}\Rightarrow\sum_{cyc}\frac{b+c}{4}\ge\sum_{cyc}\frac{bc}{b+c}\Rightarrow s\ge\sum_{cyc}\frac{bc}{b+c}\Rightarrow$$

$$1\ge\frac{1}{s}\sum_{cyc}\frac{bc}{b+c};(4)$$

From (1) and (4), we have:

$$1\ge\frac{1}{s}\sum_{cyc}\frac{bc}{b+c}\ge\frac{2r}{R};(5)$$

$$\sum_{cyc}\frac{bc}{b+c}=\frac{\sum bc(a+b)(a+c)}{(a+b)(b+c)(c+a)}=\frac{\sum bc(a^2+\sum bc)}{(\sum a)(\sum ab)-abc}=\frac{abc\sum a+(\sum bc)^2}{(\sum a)(\sum ab)-abc}=$$

$$=\frac{4Rrs\cdot 2s+(s^2+r^2+4Rr)^2}{2s(s^2+r^2+4Rr)-4Rrs}\Rightarrow$$

$$\sum_{cyc}\frac{bc}{b+c}=\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{2s^2(s^2+r^2+2Rr)};(6)$$

From (5) and (6) we get the proposed problem.

Solution 2 by Nguyen Van Canh-Ben Tre-Vietnam

- $1\ge\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{2s^2(s^2+r^2+2Rr)};$

$\leftrightarrow 2s^2(s^2+r^2+2Rr)\ge s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2;$

$\leftrightarrow s^4-12Rrs^2-r^2(4R+r)^2\ge 0;$

$\leftrightarrow s^2(s^2-12Rr)-r^2(4R+r)^2\ge 0;$

By Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2 \to s^2 - 12Rr \geq 4Rr - 5r^2 \overset{R\geq 2r}{\gtrsim} 3r^2 > 0$

$$\to s^2(s^2 - 12Rr) - r^2(4R+r)^2 \geq (16Rr - 5r^2)(4Rr - 5r^2) - r^2(4R+r)^2 \overset{(*)}{\gtrsim} 0;$$

$$(*) \leftrightarrow 48R^2r^2 - 108Rr^3 + 24r^4 \geq 0;$$

$$\leftrightarrow 12r^2(4R - r)(R - 2r) \geq 0;$$

Which is true by $R \geq 2r$ (Euler). Thus, $(*)$ is true.

- $\frac{s^4 + s^2(16Rr + 2r^2) + r^2(4R+r)^2}{2s^2(s^2 + r^2 + 2Rr)} \geq \frac{2r}{R};$

$$\leftrightarrow (R - 4r)s^4 + (16R^2r - 6Rr^2 - 4r^3)s^2 + Rr^2(4R+r)^2 \geq 0;$$

$$\leftrightarrow (R - 4r)s^4 + 2r(8R^2 - 3Rr - 2r^2)s^2 + Rr^2(4R+r)^2 \geq 0;$$

$$\leftrightarrow (R - 4r)s^4 + 2r\left((8R + 13r)(R - 2r) + 24r^2\right)s^2 + Rr^2(4R+r)^2 \geq 0; (**)$$

If $R \geq 4r$ then $(**)$ is clearly true.

If $2r \leq R < 4R$ then we using: $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$(Gerretsen)

$$(R - 4r)s^4 + 2r\left((8R + 13r)(R - 2r) + 24r^2\right)s^2 + Rr^2(4R+r)^2 \geq$$

$$(R - 4r)(4R^2 + 4Rr + 3r^2)^2 + 2r\left((8R + 13r)(R - 2r) + 24r^2\right)(16Rr - 5r^2) + Rr^2(4R+r)^2;$$

We need to prove that:

$$(R - 4r)(4R^2 + 4Rr + 3r^2)^2 + 2r\left((8R + 13r)(R - 2r) + 24r^2\right)(16Rr - 5r^2) + Rr^2(4R+r)^2 \geq 0;$$

$$\leftrightarrow (t - 4)(4t^2 + 4t + 3)^2 + 2(8t^2 - 3t - 2)(16t - 5) + t(4t + 1)^2 \geq 0; \left(t = \frac{R}{r} \geq 2\right)$$

$$\leftrightarrow 16t^5 - 32t^4 + 184t^3 - 304t^2 - 120t - 16 \geq 0;$$

$$\leftrightarrow 8(t - 2)(2t^4 + 23t^2 + 8t + 1) \geq 0;$$

Which is true by $t \geq 2$. Hence, $(**)$ is clearly true. Proved.

Solution 3 by George Florin Șerban-Romania

$$\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{2s^2(s^2+r^2+2Rr)}\overset{(?)}{\le}1$$

$$2s^4+s^2(2r^2+4Rr)-s^4-s^2(16Rr+2r^2)-r^2(4R+r)^2\ge 0$$

$$s^4-12Rrs^2-r^2(4R+r)^2\ge 0$$

$$s^2-12Rr-\frac{r^2(4R+r)^2}{s^2}\ge 0$$

$$s^2-12Rr-\frac{r^2(4R+r)^2}{s^2}\overset{Blundon-G}{\ge}\frac{r(4R+r)^2}{R+r}-12Rr-\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}=$$

$$=\frac{r(4R+r)^2}{R+r}-13Rr-r^2\overset{?}{\ge}0\Leftrightarrow(4R+r)^2\ge(13R+r)(R+r)$$

Let $x=\frac{R}{r}\ge 2\Rightarrow(4x+1)^2\ge(13x+1)(x+1)\Rightarrow 3x(x-2)\ge 0$ true $\forall x\ge 2$.

$$\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{2s^2(s^2+r^2+2Rr)}\overset{?}{\ge}\frac{2r}{R}\Leftrightarrow$$

$$\frac{s^2}{2s^2+2r^2+4Rr}+\frac{16Rr+2r^2}{2s^2+2r^2+4Rr}+\frac{r^2(4R+r)^2}{2s^2(s^2+r^2+2Rr)}\overset{Blundon-G}{\ge}$$

$$\ge\frac{1}{2+\frac{2r^2+4Rr}{s^2}}+\frac{16Rr+2r^2}{\frac{R(4R+r)^2}{2R-r}+2r(2R+r)}$$

$$+\frac{r^2(4R+r)^2}{\frac{R(4R+r)^2}{2R-r}\left[\frac{R(4R+r)^2}{2(2R-r)}+r(2R+r)\right]}\overset{Blundon-G}{\ge}$$

$$\ge\frac{1}{2+\frac{2r^2+4Rr}{\frac{r(4R+r)^2}{R+r}}}+\frac{(16Rr+2r^2)(2R-r)}{R(4R+r)^2+2r(4R^2-r^2)}$$

$$+\frac{2r^2(2R-r)^2}{R[R(4R+r)^2+2r(4R^2-r^2)]}\overset{?}{\ge}\frac{2r}{R}$$

$$\frac{1}{2+\frac{(4R+2r)(R+r)}{(4R+r)^2}}+\frac{32R^2r-16Rr^2+4Rr^2-2r^3}{16R^3+8R^2r+Rr^2+8R^2r-2r^3}+$$

$$+\frac{8R^2r^2-8Rr^3+2r^4}{16R^4+8R^3r+R^2r^2+8R^3r-2Rr^3}\ge\frac{2r}{R}$$

$$\frac{16R^2+8Rr+r^2}{32R^2+16Rr+2r^2+4R^2+4Rr+2Rr+2r^2}+\frac{32R^2r-12Rr^2-2r^3}{16R^3+16R^2r+Rr^2-2r^3}+$$

$$+\frac{8R^2r^2-8Rr^3+2r^4}{16R^4+16R^3r+R^2r^2-2Rr^3}\ge\frac{2r}{R};\text{let } x=\frac{R}{r}\ge 2 \text{ then}$$

$$\frac{16x^3+8x^2+x}{36x^2+22x+4}+\frac{32x^3-12x^2-2x}{16x^3+16x^2+x-2}+\frac{32x^3-4x^2-10x+2}{16x^3+16x^2+x-2}\overset{?}{\ge}2$$

First we prove that:

$$\frac{16x^3+8x^2+x}{36x^2+22x+4}\ge\frac{37}{32};\forall x\ge 2\Leftrightarrow$$

$(x-2)(256x^2+154x+27)\ge 0$ which is true $\forall x\ge 2$.

Now,

$$\frac{32x^3-4x^2-10x+2}{16x^3+16x^2+x-2}\ge\frac{37}{32}\Leftrightarrow$$

$(x-2)(144x^2+48x-23)\ge 0$ which is true for all $x\ge 2$.

Therefore,

$$\frac{32x^3-4x^2-10x+2}{16x^3+16x^2+x-2}+\frac{16x^3+8x^2+x}{36x^2+22x+4}\ge\frac{37}{32}+\frac{37}{32}=2$$

SP.432 If $a,b,c>0, a+b+c=3$ then:

$$\frac{a^6+15a^4+15a^2+1}{3b^5+10b^3+3b}+\frac{b^6+15b^4+15b^2+1}{3c^5+10c^3+3c}+\frac{c^6+15c^4+15c^2+1}{3a^5+10a^3+3a}\ge 6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

First we prove that:

$$\frac{a^6+15a^4+15a^2+1}{3a^5+10a^3+3a}\ge 2;\forall a>0;(1)$$

$$a^6+15a^4+15a^2+1\ge 6a^5+20a^3+6a$$

$$a^6-6a^5+15a^4-20a^3+15a^2-6a+1\ge 0\Leftrightarrow(a-1)^6\ge 0$$

$$\sum_{cyc}\frac{a^6+15a^4+15a^2+1}{3b^5+10b^3+3b}\overset{AGM}{\ge}3\cdot\prod_{cyc}\sqrt[3]{\frac{a^6+15a^4+15a^2+1}{3b^5+10b^3+3b}}=$$

$$= 3 \cdot \prod_{cyc} \sqrt[3]{\frac{a^6 + 15a^4 + 15a^2 + 1}{3a^5 + 10a^3 + 3a}} = 3 \cdot \sqrt[3]{2 \cdot 2 \cdot 2} = 6$$

Equality holds for $a = b = c = 1$.

Solution 2 by George Florin Șerban-Romania

$$\frac{x^6 + 15x^4 + 15x^2 + 1}{3x^5 + 10x^3 + 3x} \geq 2; \forall x > 0 \Leftrightarrow$$

$$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 \geq 0 \Leftrightarrow (x-1)^6 \geq 0 \text{ true.}$$

$$\sum_{cyc} \frac{a^6 + 15a^4 + 15a^2 + 1}{3b^5 + 10b^3 + 3b} \overset{AGM}{\geq} 3 \cdot \prod_{cyc} \sqrt[3]{\frac{a^6 + 15a^4 + 15a^2 + 1}{3b^5 + 10b^3 + 3b}} =$$

$$= 3 \cdot \prod_{cyc} \sqrt[3]{\frac{a^6 + 15a^4 + 15a^2 + 1}{3a^5 + 10a^3 + 3a}} = 3 \cdot \sqrt[3]{2 \cdot 2 \cdot 2} = 6$$

Equality holds for $a = b = c = 1$.

Solution 3 by Ravi Prakash-New Delhi-India

$$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 \geq 0 \Leftrightarrow (x-1)^6 \geq 0$$

$$\frac{x^6 + 15x^4 + 15x^2 + 1}{3x^5 + 10x^3 + 3x} \geq 2; \forall x > 0$$

Hence,

$$\sum_{cyc} \frac{a^6 + 15a^4 + 15a^2 + 1}{3b^5 + 10b^3 + 3b} \overset{AGM}{\geq} 3 \cdot \prod_{cyc} \sqrt[3]{\frac{a^6 + 15a^4 + 15a^2 + 1}{3b^5 + 10b^3 + 3b}} =$$

$$= 3 \cdot \prod_{cyc} \sqrt[3]{\frac{a^6 + 15a^4 + 15a^2 + 1}{3a^5 + 10a^3 + 3a}} = 3 \cdot \sqrt[3]{2 \cdot 2 \cdot 2} = 6$$

Equality holds for $a = b = c = 1$.

SP.433 Let be $f: \mathbb{R} \to \mathbb{R}, f(x+2) + 10f(x) = 7f(x+1); \forall x \in \mathbb{R}$.

If $f(0) = 2, f(1) = 7$ then find:

$$\Omega = \log 2 \cdot \log 5 \cdot \int_0^1 f(x)\, dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$f(x+2)-7f(x+1)+10f(x)=0$$

The characteristic equation is:

$$\lambda^2-7\lambda+10=0;\lambda_1=2,\lambda_2=5$$

$$f(x)=a\cdot\lambda_1^x+b\cdot\lambda_2^x=a\cdot 2^x+b\cdot 5^x$$

$$\begin{cases} f(0)=a+b=2 \\ f(1)=2a+5b=7\end{cases}\Rightarrow a=b=1\Rightarrow f(x)=2^x+5^x$$

$$\Omega=\log 2\cdot\log 5\cdot\int_0^1 f(x)\,dx=\log 2\cdot\log 5\cdot\int_0^1(2^x+5^x)\,dx=$$

$$=\log 2\cdot\log 5\left(\frac{2^1-2^0}{\log 2}+\frac{5^1-5^0}{\log 5}\right)=\log 2\cdot\log 5\cdot\frac{1}{\log 2}+\log 2\cdot\log 5\cdot\frac{4}{\log 5}=$$

$$=\log 5+4\cdot\log 2=\log 80$$

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

$$f(x+2)-7f(x+1)+10f(x)=0$$

Characteristic equation is $t^2-7t+10=0\Leftrightarrow(t-5)(t-2)=0$

$$t_1=2,t_2=5$$

$$f(x)=A\cdot 5^x+B\cdot 2^x$$

$$\begin{cases} f(0)=2 \\ f(1)=7\end{cases}\Rightarrow\begin{cases} A+B=2 \\ 5A+2B=7\end{cases}\Rightarrow\begin{cases} A=1 \\ B=1\end{cases}\Rightarrow f(x)=5^x+2^x$$

Hence,

$$\Omega=\log 2\cdot\log 5\int_0^1(2^x+5^x)\,dx=5^x\cdot\log 2|_0^1+2^x\cdot\log 5|_0^1=$$

$$=\log 5+\log 16=\log 80$$

Solution 3 by Chigbo Alex Ani-Nigeria

$$f(x+2)+10f(x)=7f(x+1),f(0)=2,f(1)=7$$

Let $f(x)=kw^x,k\in\mathbb{R}$

$$kw^{x+2}+10kw^x=7kw^{x+1}\Leftrightarrow kw^x(w^2+10)=kw^x(7w)$$

$$w^2-7w+10=0\Leftrightarrow(w-2)(w-5)=0$$

$$w=2,w=7\Rightarrow f(x)=A\cdot 2^x+B\cdot 5^x$$

$$\Omega = \log 2 \cdot \log 5 \int_0^1 (2^x + 5^x)\,dx = 5^x \cdot \log 2|_0^1 + 2^x \cdot \log 5|_0^1 =$$

$$= \log 2 \log 5 \left(\frac{1}{\log 2} + \frac{1}{\log 5}\right) = \log 80$$

Solution 4 by Tapas Das-India

$$f(x+2) - 7f(x+1) + 10f(x) = 0$$

Characteristic equation is $a^2 - 7a + 10 = 0 \Leftrightarrow (a-5)(a-2) = 0$

$$a_1 = 2, a_2 = 5$$

$$f(x) = A \cdot 5^x + B \cdot 2^x$$

$$\begin{cases} f(0) = 2 \\ f(1) = 7 \end{cases} \Rightarrow \begin{cases} A + B = 2 \\ 5A + 2B = 7 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \end{cases} \Rightarrow f(x) = 5^x + 2^x$$

Hence,

$$\Omega = \log 2 \cdot \log 5 \int_0^1 (2^x + 5^x)\,dx = 5^x \cdot \log 2|_0^1 + 2^x \cdot \log 5|_0^1 =$$

$$= \log 5 + \log 16 = \log 80$$

Solution 5 by Ravi Prakash-New Delhi-India

Let $u_x = f(x)$**. The given recurrence equation is** $u_{x+2} + 10u_x = 7u_{x+1}$.

The characteristic equation is $t^2 - 7t + 10 = 0$

$$\Leftrightarrow (t-5)(t-2) = 0\ t_1 = 2, t_2 = 5 \Rightarrow u_x = A \cdot 2^x + B \cdot 5^x$$

Now, $\begin{cases} 2 = u_0 = A + B \\ 7 = u_1 = 2A + 5B \end{cases} \Rightarrow A = B = 1$**.Thus,** $u_x = 2^x + 5^x$.

$$\Omega = \log 2 \cdot \log 5 \int_0^1 (2^x + 5^x)\,dx = \log 2 \cdot \log 5 \left[\frac{2^x}{\log 2} + \frac{5^x}{\log 5}\right]_0^1 =$$

$$= [2^x \log 5 + 5^x \log 2]_0^1 = \log 2 \log 5 \left(\frac{1}{\log 2} + \frac{1}{\log 5}\right) = \log 80$$

SP.434 Let be $f: \mathbb{R} \to \mathbb{R}, f(0) = 3, f(1) = 10, f(2) = 38$

$$f(x+3) + 31f(x+1) = 10f(x+2) + 30f(x)$$

Solve for real numbers: $f(x) = 10$.

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$f(x+3)-10f(x+2)+31f(x+1)-30f(x)=0$$

The characteristic equation is

$$\lambda^3-10\lambda^2+31\lambda-30=0$$

$$\lambda^3-2\lambda^2-8\lambda^2+16\lambda+15\lambda-30=0$$

$$\lambda^2(\lambda-2)-8\lambda(\lambda-2)15(\lambda-2)=0$$

$$(\lambda-2)(\lambda^2-8\lambda+15)=0$$

$$(\lambda-2)(\lambda-3)(\lambda-5)=0\Leftrightarrow\lambda_1=2;\lambda_2=3;\lambda_3=5$$

$$f(x)=a\lambda_1^x+b\lambda_2^x+c\lambda_3^x$$

$$f(x)=a\cdot 2^x+b\cdot 3^x+c\cdot 5^x$$

$$\begin{cases}f(0)=3\\f(1)=10\\f(2)=38\end{cases}\Rightarrow\begin{cases}a+b+c=3\\2a+3b+5c=10\\4a+9b+25c=38\end{cases}\Rightarrow a=b=c=1$$

$$f(x)=2^x+3^x+4^x, f'(x)=2^x\log 2+3^x\log 3+5^x\log 5, f'(x)>0\Rightarrow$$

f –strictly decreasing, then f –injective.

$f(1)=10, f(x)=f(1)\Rightarrow x=1$ solution.

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

$$f(x+3)-10f(x+2)+31f(x+1)-30f(x)=0$$

Characteristic equation is $P(t)=t^3-10t^2+31t-30=0$

Being that $P(2)=P(3)=P(5)=0$, we get

$$P(t)=A\cdot 2^x+B\cdot 3^x+C\cdot 5^x=10$$

$$\begin{cases}f(0)=3\\f(1)=10\\f(2)=38\end{cases}\Rightarrow\begin{cases}A+B+C=3\\2A+3B+5C=10\\4A+9B+25C=38\end{cases}$$

One solution is $(1,1,1)$ and $\Delta=\begin{vmatrix}1&1&1\\2&3&5\\4&9&25\end{vmatrix}\neq 0\Rightarrow(1,1,1)$ is unique solution.

Finally, $f(x)=2^x+3^x+5^x$.

$$f(x)=10\Rightarrow 2^x+3^x+5^x=10$$

Let be the function $g(x)=f(x)-10$ increasing, so $x=1$ has unique solution.

Solution 3 by Adrian Popa-Romania

Characteristic equation is $t^3 - 10t^2 + 31t - 30 = 0, t_1 = 2, t_2 = 3, t_3 = 5$

$$A \cdot 2^x + B \cdot 3^x + C \cdot 5^x = 10$$

$$\begin{cases} f(0) = 3 \\ f(1) = 10 \\ f(2) = 38 \end{cases} \Rightarrow \begin{cases} A + B + C = 3 \\ 2A + 3B + 5C = 10 \\ 4A + 9B + 25C = 38 \end{cases} \Rightarrow A = B = C = 1.$$

$$f(x) = 2^x + 3^x + 5^x = 10 \Rightarrow x = 1$$

Suppose that the equation $2^x + 3^x + 5^x = 10$ has another solution $y > x$, then

$$\begin{cases} 2^x + 3^x + 5^x = 10 \\ 2^y + 3^y + 5^y = 10 \end{cases}$$

But $2^x < 2^y, 3^x < 3^y$ and $5^x < 5^y$, by adding: $2^x + 3^x + 5^x < 2^y + 3^y + 5^y \Leftrightarrow$

$10 < 10$ impossible! Therefore, $x = 1$ has unique solution.

Solution 4 by Tapas Das-India

$$f(x+3) - 10f(x+2) + 31f(x+1) - 30f(x) = 0$$

Characteristic equation is $a^3 - 2a^2 + 31a - 30 = 0 \Leftrightarrow$

$$(a-2)(a-3)(a-5) = 0 \Leftrightarrow a \in \{2,3,5\}$$

$$A \cdot 2^x + B \cdot 3^x + C \cdot 5^x = 10$$

$$\begin{cases} f(0) = 3 \\ f(1) = 10 \\ f(2) = 38 \end{cases} \Rightarrow \begin{cases} A + B + C = 3 \\ 2A + 3B + 5C = 10 \\ 4A + 9B + 25C = 38 \end{cases} \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{vmatrix} = 6 \neq 0$$

$$\Delta_A = \begin{vmatrix} 3 & 1 & 1 \\ 10 & 3 & 5 \\ 38 & 9 & 25 \end{vmatrix} = 6; \Delta_B = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 10 & 5 \\ 4 & 38 & 25 \end{vmatrix} = 6; D_C = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 10 \\ 4 & 9 & 38 \end{vmatrix} = 3$$

$$\Rightarrow A = B = C = 1.$$

$$f(x) = 2^x + 3^x + 5^x = 10 \Rightarrow x = 1$$

Solution 5 by Ravi Prakash-New Delhi-India

Let $u_x = f(x), x \in \mathbb{R}$. The given equation is

$$u_{x+3} - 10u_{x+2} + 31u_{x+1} - 30u_x = 0$$

Characteristic equation is: $t^3 - 10t^2 + 31t - 30 = 0 \Leftrightarrow$

$$(t-2)(t-3)(t-5) = 0 \Leftrightarrow t \in \{2,3,5\}$$

Thus, $u_x = A\cdot 2^x + B\cdot 3^x + C\cdot 5^x$. We have: $\begin{cases} A+B+C=3 \\ 2A+3B+5C=10 \\ 4A+9B+25C=38 \end{cases}$

Solving above system, we get $A=B=C=1$ and hence,

$$f(x) = 2^x + 3^x + 5^x$$

As $f'(x) = 2^x\log 2 + 3^x\log 3 + 5^x\log 5 > 0, \forall x \in \mathbb{R}$, then

f is strictly increasing function. Thus, $f(x) = 10$ has exactly one solution $x = 1$.

SP.435 Let ΔABC with inradius r, circumradius R, and exradii r_a, r_b, r_c.

Prove that:

$$\frac{R}{2r} \geq \frac{1}{3}\sqrt{\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

By AM-GM inequality, we have: $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a} \geq 3$. So,

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6 = \frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+3+3 \leq$$

$$\leq \left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)+\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)+3 = (r_a+r_b+r_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)$$

We know that: $r_a+r_b+r_c = 4R+r$ and $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{1}{r}$, $R \geq 2r$.

So, $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6 \leq (4R+r)\cdot\frac{1}{r} \leq \left(4R+\frac{R}{r}\right)\cdot\frac{1}{r} = \frac{9R}{2r}\cdot\frac{R}{2r} = 9\left(\frac{R}{2r}\right)^2$

Therefore,

$$\frac{R}{2r} \geq \frac{1}{3}\sqrt{\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6}$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $s-a = x, s-b = y$ and $s-c = z \therefore s = x+y+z \Rightarrow a = y+z, b = z+x$ and c

$$= x+y \therefore \frac{R}{2r} = \frac{abcs}{8F^2} = \frac{abcs}{8s(s-a)(s-b)(s-c)}$$

$$= \frac{1}{8xyz}\prod_{cyc}(y+z) \overset{(*)}{=} \frac{R}{2r} \text{ and}$$

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}=\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}=\frac{y}{x}+\frac{z}{y}+\frac{x}{z}=\frac{1}{xyz}\sum_{cyc}x^2y\Rightarrow RHS^2$$

$$=\frac{1}{9xyz}\left(\sum_{cyc}x^2y+6xyz\right)\le\left(\frac{R}{2r}\right)^2\overset{\text{via }(*)}{\overbrace{=}}\frac{1}{64x^2y^2z^2}\prod_{cyc}(y+z)^2$$

$$\Leftrightarrow 9\prod_{cyc}(y+z)^2\ge 64xyz\left(\sum_{cyc}x^2y+6xyz\right)$$

$$\Leftrightarrow 9\sum_{cyc}x^4y^2+9\sum_{cyc}x^2y^4+18xyz\left(\sum_{cyc}x^3\right)+18\sum_{cyc}x^3y^3$$

$$+54xyz\left(\sum_{cyc}xy^2\right)\overset{(*)}{\overbrace{\ge}}10xyz\left(\sum_{cyc}x^2y\right)+294x^2y^2z^2$$

$$\text{Now, } x^3+x^3+y^3\overset{A-G}{\overbrace{\ge}}3x^2y,\ y^3+y^3+z^3\overset{A-G}{\overbrace{\ge}}3y^2z \text{ and } z^3+z^3$$

$$+x^3\overset{A-G}{\overbrace{\ge}}3z^2x \text{ and adding these three : } 3\sum_{cyc}x^3\ge 3\sum_{cyc}x^2y$$

$$\Rightarrow 10xyz\left(\sum_{cyc}x^3\right)\overset{(i)}{\overbrace{\ge}}10xyz\left(\sum_{cyc}x^2y\right) \text{ and}$$

$$9\sum_{cyc}x^4y^2+9\sum_{cyc}x^2y^4+8xyz\left(\sum_{cyc}x^3\right)+18\sum_{cyc}x^3y^3$$

$$+54xyz\left(\sum_{cyc}xy^2\right)\overset{A-G}{\overbrace{\ge}}36\sum_{cyc}x^3y^3+8xyz\left(\sum_{cyc}x^3\right)$$

$$+54xyz\left(\sum_{cyc}xy^2\right)\overset{A-G}{\overbrace{\ge}}(108+24+162)x^2y^2z^2=294x^2y^2z^2$$

$$\Rightarrow 9\sum_{cyc} x^4y^2 + 9\sum_{cyc} x^2y^4 + 8xyz\left(\sum_{cyc} x^3\right) + 18\sum_{cyc} x^3y^3$$

$$+ 54xyz\left(\sum_{cyc} xy^2\right) \overset{(ii)}{\geq} 294x^2y^2z^2 \therefore (i) + (ii) \Rightarrow \text{LHS of } (*) \geq \text{RHS of } (*)$$

$$\therefore \text{RHS}^2 \geq \left(\frac{R}{2r}\right)^2$$

$$\Rightarrow \frac{R}{2r} \geq \frac{1}{3}\sqrt{\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6}\ \text{(QED)}$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{By AM - GM, we have}: \frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c} \geq 3, \textit{then}:$$

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6 \leq \left(\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a}\right) + \left(\frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c}\right) + 3$$

$$= (r_a + r_b + r_c)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) = \frac{4R + r}{r} \overset{Euler}{\leq}$$

$$\leq \frac{9R}{2r} \overset{Euler}{\leq} \frac{9R}{2r} \cdot \frac{R}{2r} = 9\left(\frac{R}{2r}\right)^2$$

$$\textit{Therefore,} \quad \frac{R}{2r} \geq \frac{1}{3}\sqrt{\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6}$$

Solution 4 by Nguyen Van Canh-Ben Tre-Vietnam

Using $r_a = \frac{S}{p-a}$ (analogs). We have indentity:

$$\frac{r_a + r_b}{r_c} + \frac{r_b + r_c}{r_a} + \frac{r_c + r_a}{r_b} = \frac{4R - 2r}{r};$$

$$\leftrightarrow \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c} + 2 = \frac{4R}{r};$$

$$\leftrightarrow \frac{1}{8}\left(\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c} + 2\right) = \frac{R}{2r};$$

By AM-GM we have:

$$\frac{R}{2r} = \frac{1}{8}\left(\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c} + 2\right) \geq \frac{1}{8}\left(\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 3 + 2\right)$$

$$=\frac{1}{8}\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6-1\right)$$

Let us denote: $t=\sqrt{\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6}\overset{AM-GM}{\geq}\sqrt{3+6}=3$**. We need to prove that:**

$$\frac{1}{8}(t^2-1)\geq\frac{1}{3}t;$$

$$\leftrightarrow 3t^2-8t-3\geq 0;$$

$$\leftrightarrow (t-3)(3t+1)\geq 0;$$

Which is true by $t\geq 3$**. Proved. Equality** $\leftrightarrow a=b=c$**.**

UNDERGRADUATE PROBLEMS

UP.421 Let $P_{n-1}(x)=a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-1}$ $(n\ge 2, n\in\mathbb{N})$ such that: $\sqrt{1-x^2}.|P_{n-1}(x)|\le 1, \forall x\in[-1,1]$. Prove that: $|a_0|\le 2^{n-1}$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by proposer

Let $x_j=\cos\left(\frac{2j-1}{n}\pi\right), j=\overline{1,n}$ be the roots of polynomial Chebyshev :

$T_n(x)=2^{n-1}\prod_{j=1}^{n}(x-x_j)$ and coefficient the highest order of $T_n(x)$ is 2^{n-1}.

We have:

$$P_{n-1}(x)=\frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}\sqrt{1-x_j^2}\,.P_{n-1}(x_j)\frac{T_n(x)}{x-x_j}$$

Then, we have:

$$a_0=\frac{2^{n-1}}{n}\sum_{j=1}^{n}(-1)^{j-1}.\sqrt{1-x_j^2}.P_{n-1}(x_j)$$

$$\to |a_0|=\left|\frac{2^{n-1}}{n}\sum_{j=1}^{n}(-1)^{j-1}.\sqrt{1-x_j^2}.P_{n-1}(x_j)\right|\le\frac{2^{n-1}}{n}.\left|\sum_{j=1}^{n}\sqrt{1-x_j^2}.P_{n-1}(x_j)\right|$$

$$\le\frac{2^{n-1}}{n}\sum_{j=1}^{n}\left|\sqrt{1-x_j^2}.P_{n-1}(x_j)\right|\le\frac{2^{n-1}}{n}\sum_{j=1}^{n}1=\frac{2^{n-1}}{n}.n=2^{n-1}$$

UP.422 Find:

$$\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\sqrt[n]{\frac{(a+1)(k+n)-a}{(a+1)n}}\right)^{\frac{(a+1)(k+n)-a}{(a+1)n}}\quad ;a\in\mathbb{N}^*$$

Proposed by Neculai Stanciu-Romania

Solution 1 by proposer

$$\text{Let } P_n=\prod_{k=1}^{n}\left(\sqrt[n]{\frac{(a+1)(k+n)-a}{(a+1)n}}\right)^{\frac{(a+1)(k+n)-a}{(a+1)n}}=$$

$$=\prod_{k=1}^{n}\left(\frac{(a+1)(k+n)-a}{(a+1)n}\right)^{\frac{(a+1)(k+n)-a}{(a+1)n^2}}$$

$$\lim_{n\to\infty}\log P_n=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(1+\frac{(a+1)k-a}{(a+1)n}\right)\cdot\log\left(1+\frac{(a+1)k-a}{(a+1)n}\right)$$

$$\text{Since: }\frac{k-1}{n}<\frac{(a+1)k-a}{(a+1)n}<\frac{k}{n};\forall k=\overline{1,n}\text{ then}$$

$$\sum_{k=1}^{n}\frac{1}{n}\left(1+\frac{(a+1)k-a}{(a+1)n}\right)\cdot\log\left(1+\frac{(a+1)k-a}{(a+1)n}\right)\text{ is } \textit{Riemann summ} \text{ for}$$

$f(x)=(1+x)\log(1+x)$ on $x\in[0,1]$ with partition $\left\{0,\frac{1}{n},\dots,\frac{n-1}{n},1\right\}$.

Since $f(x)$ –is continue on $[0,1]$ yields that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(1+\frac{(a+1)k-a}{(a+1)n}\right)\cdot\log\left(1+\frac{(a+1)k-a}{(a+1)n}\right)=\int_0^1(1+x)\log(1+x)\,dx=$$

$$=2\log 2-\frac{3}{4}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\sqrt[n]{\frac{(a+1)(k+n)-a}{(a+1)n}}\right)^{\frac{(a+1)(k+n)-a}{(a+1)n}}=e^{2\log 2-\frac{3}{4}}=\frac{4}{\sqrt[4]{e^3}}.$$

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\sqrt[n]{\frac{(a+1)(k+n)-a}{(a+1)n}}\right)^{\frac{(a+1)(k+n)-a}{(a+1)n}}=$$

$$=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{(a+1)(k+n)-a}{n(a+1)}\right)^{\frac{(a+1)(k+n)-a}{n^2(a+1)}}$$

$$\log\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(a+1)(k+n)-a}{n^2(a+1)}\log\left(\frac{(a+1)(k+n)-a}{n(a+1)}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k+n}{n}-\frac{a}{n(a+1)}\right)\log\left(\frac{k+n}{n}-\frac{a}{n(a+1)}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}+1\right)\log\left(\frac{k}{n}+1\right)=\int_0^1(x+1)\log(x+1)\,dx=$$

$$=\frac{1}{2}\int_0^1[(x+1)^2]'\cdot\log(x+1)\,dx\overset{IBP}{=}\left[\frac{1}{2}(x+1)^2\log(x+1)\right]_0^1-\frac{1}{2}\int_0^1(x+1)\,dx=$$

$$=2\log 2-\frac{3}{4}=\log\left(\frac{4}{e^{\frac{3}{4}}}\right)$$

Therefore,

$$\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\sqrt[n]{\frac{(a+1)(k+n)-a}{(a+1)n}}\right)^{\frac{(a+1)(k+n)-a}{(a+1)n}}=e^{2\log 2-\frac{3}{4}}=\frac{4}{\sqrt[4]{e^3}}.$$

UP.423 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^3}\int_0^n\frac{x^2+n^2}{2^{-x}+1}dx$$

Proposed by Neculai Stanciu-Romania

Solution 1 by proposer

Lemma. If $f:[0,\infty)\to\mathbb{R}$ and $g:[0,1]\to\mathbb{R}$ are continue functions with

$$\lim_{x\to\infty}f(x)=l,\text{then}\lim_{n\to\infty}\frac{1}{n}\int_0^n f(x)g\left(\frac{x}{n}\right)dx=l\int_0^1 g(x)\,dx$$

Proof. Let $h(x)=f(x)-l$ converges at zero when $x\to\infty$, so

$$\frac{1}{n}\int_0^n f(x)g\left(\frac{x}{n}\right)dx=\frac{1}{n}\int_0^n h(x)g\left(\frac{x}{n}\right)dx+\frac{l}{n}\int_0^n g\left(\frac{x}{n}\right)dx;\ (1)$$

Since $g(x)$ is continue on $[0,1]$, then $\exists M>0$ such that $|g(x)|\le M$ and

$$\left|\frac{1}{n}\int_0^n h(x)g\left(\frac{x}{n}\right)dx\right|\le\frac{M}{n}\int_0^n|h(x)|\,dx;\ (2)$$

If $H(x)$ is a primitive of the function $|h(x)|$, then from L'Hospital rule, we have:

$$\lim_{n\to\infty}\frac{H(n)}{n}=\lim_{x\to\infty}|h(x)|=0$$

So, LHS from (2) converges to 0, when $n \to \infty$.

For $\frac{x}{n} = t$, we have:

$$\frac{1}{n}\int_0^n g\left(\frac{x}{n}\right) dx = l\int_0^1 g(t)\,dt$$

Hence, if we take in Lemma $f(x) = \frac{1}{2^{-x}+1}$ and $g(x) = 1 + x^2$ and if we taking account by

$$\lim_{x\to\infty}\frac{1}{2^{-x}+1} = 1, \text{we obtain:}$$

$$\Omega = \lim_{n\to\infty}\frac{1}{n^3}\int_0^n \frac{x^2+n^2}{2^{-x}+1}dx = \int_0^1 g(x)\,dx = \frac{4}{3}$$

Solution 2 by Naren Bhandari-Bajura-Nepal

Denote the integral by $f_n(x)$ and on substitution $2^{-x} = y$, the integral get reduces to the following integral

$$\int_0^n f_n(x)\,dx = \frac{1}{\log^3 2}\int_{\frac{1}{2^n}}^1 \frac{\log^2 y + n^2\log^2 2}{y(y+1)}dy \stackrel{PFD}{=}$$

$$= \frac{1}{\log^3 2}\int_{\frac{1}{2^n}}^1 \frac{\log^2 y + n^2\log^2 2}{y}dy - \frac{1}{\log^3 2}\int_{\frac{1}{2^n}}^1 \frac{\log^2 y + n^2\log^2 2}{y+1}dy =$$

$$= \left[\frac{\log^3 y}{3\log^3 2} + \frac{n^2\log y}{\log 2}\right]_{\frac{1}{2^n}}^1 - \frac{1}{\log^3 2}\int_{\frac{1}{2^n}}^1 \frac{\log^2 y}{y+1}dy - \frac{n^2\log(1+y)}{\log 2}\Big|_{\frac{1}{2^n}}^1 =$$

$$= -n^2 - \frac{1}{\log^3 2}\int_a^b \frac{\log^2 y}{y+1}dy - \frac{\log^3\left(\frac{1}{2^n}\right)}{3\log^3 2} - \frac{n^2\log\left(\frac{1}{2^n}\right)}{\log 2} + \frac{n^2\log\left(1+\frac{1}{2^n}\right)}{\log 2}$$

Here the integral in mid has primitive which we find by IBP,i.e.

$$\int_a^b \frac{\log^2 y}{y+1}dy = \log^2 y\log(y+1) - 2\int_a^b \frac{\log y\log(y+1)}{y}dy \stackrel{IBP}{=}$$

$$= \log^2 y\log(1+y) - 2\left(-\log y\,Li_2(-y) + \int_a^b \frac{Li_2(-y)}{y}dy\right) =$$

$$= \log(1+y)\log^2 y + 2\log y\,Li_2(-y) - 2Li_3(-y)\big|_{a=\frac{1}{2^n}}^{b=1} =$$

$$= \frac{3}{2}\zeta(30 - \log\left(1+\frac{1}{2^n}\right)\log^2\left(\frac{1}{2^n}\right) - 2Li_2\left(\frac{1}{2^n}\right)\log\left(\frac{1}{2^n}\right) + 2Li_3\left(-\frac{1}{2^n}\right)$$

From the last expression it is easy to see that

$$\lim_{n\to\infty}\frac{1}{n^3}\int_a^b\frac{\log^2 y}{y+1}dy=0$$

Therefore, the required limit is:

$$\lim_{n\to\infty}\frac{1}{n^3}\int_0^n f_n(x)\,dx=\lim_{n\to\infty}\frac{1}{n^3}\left(-\frac{n^2\log\left(\frac{1}{2^n}\right)}{\log 2}-\frac{\log^3\left(\frac{1}{2^n}\right)}{3\log^3 2}\right)=\lim_{n\to\infty}\frac{\frac{4}{3}n^3}{n^3}=\frac{4}{3}$$

Solution 3 by Naren Bhandari-Bajura-Nepal

Substitute $x=ny$ we get:

$$\lim_{n\to\infty}\frac{1}{n^3}\int_0^n\frac{x^2+n^2}{2^{-x}+1}dx=\lim_{n\to\infty}\frac{1}{n^3}\int_0^1\frac{n^3y^2+n^3}{2^{-ny}+1}dy=\lim_{n\to\infty}\int_0^1\frac{y^2+1}{2^{-ny}+1}dy$$

Let $g_n(x)=\frac{y^2+1}{2^{-ny}+1}$, then $|g_n(y)|\le y^2+1=h(y)$ which shows that y^2+1 is the dominant function since the sequence $\{g_n(y)\}$ pointwise converges to y^2+1.

Clearly $h(y)$ is integrable function for all $y\in[0,1]$ and hence by DCT,

$$\lim_{n\to\infty}\frac{1}{n^3}\int_0^n\frac{x^2+n^2}{2^{-x}+1}dx=\int_0^1(y^2+1)\,dy=\frac{4}{3}$$

Solution 4 by Felix Marin-USA

$$\lim_{n\to\infty}\frac{1}{n^3}\int_0^n\frac{x^2+n^2}{2^{-x}+1}dx=\lim_{n\to\infty}\left(\frac{1}{n^3}\int_0^n\frac{x^2}{2^{-x}+1}dx\right)+\lim_{n\to\infty}\left(\frac{1}{n}\int_0^n\frac{dx}{2^{-x}+1}\right)=$$

$$=\underbrace{\lim_{n\to\infty}\left(\frac{1}{3n^2+3n+1}\int_n^{n+1}\frac{x^2}{2^{-x}+1}dx\right)}_{\Omega_1}+\underbrace{\lim_{n\to\infty}\int_n^{n+1}\frac{dx}{2^{-x}+1}}_{\Omega_2}(by\ Stolz-Cesaro)$$

Moreover,

$$\frac{1}{3n^2+3n+1}\cdot\frac{n^2+1}{2^{-n}+1}<\frac{1}{3n^2+3n+1}\int_n^{n+1}\frac{x^2}{2^{-x}+1}dx$$

$$<\frac{1}{3n^2+3n+1}\cdot\frac{(n+1)^2+1}{2^{-n-1}+1}$$

Hence, $\Omega_1=\frac{1}{3}$

Similarly, we get:

$$\frac{1}{2^{-n}+1}<\int_n^{n+1}\frac{dx}{2^{-x}+1}<\frac{1}{2^{-n-1}+1}$$

Hence, $\Omega_2=1$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n^3}\int_0^n\frac{x^2+n^2}{2^{-x}+1}dx=1+\frac{1}{3}=\frac{4}{3}$$

UP.424 Solve for integers:

$$\sqrt[3]{(1+x)^x\cdot(2x-4)^{2x-5}\cdot(3x-9)^{3x-10}}=\left(1+\sqrt[3]{6x^3-35x^2+50x}\right)^{1+\sqrt[3]{6x^3-35x^2+50x}}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$LHS=\sqrt[3]{(1+x)^x\cdot(2x-5+1)^{2x-5}\cdot(3x+1-10)^{3x-10}}=$$

$$=\sqrt[3]{(1+a)^a(1+b)^b(1+c)^c},$$

$$a=x;b=2x-5;c=3x-10$$

$$1+\sqrt[3]{abc}=1+\sqrt[3]{x(2x-5)(3x-10)}=1+\sqrt[3]{6x^3-20x^2-15x^2+50x}=$$

$$=1+\sqrt[3]{6x^3-35x^2+50x}$$

$$RHS=\left(1+\sqrt[3]{abc}\right)^{\sqrt[3]{abc}}$$

Let be $f:(0,\infty)\to\mathbb{R};f(x)=x\log(1+x)$

$$f'(x)=\log(1+x)+\frac{x}{1+x};f''(x)=\frac{1}{1+x}+\frac{1+x-x}{(1+x)^2}$$

$$f''(x)=\frac{1}{1+x}+\frac{1}{(1+x)^2}>0;\forall x>0\Rightarrow f-\text{convexe}.$$

By Jensen's Inequality:

$$f(a)+f(b)+f(c)\geq 3f\left(\frac{a+b+c}{3}\right)\Leftrightarrow$$

$$a\cdot\log(1+a)+b\cdot\log(1+b)+c\cdot\log(1+c)\geq 3\cdot\frac{a+b+c}{3}\cdot\log\left(1+\frac{a+b+c}{3}\right)$$

$$\log(1+a)^a+\log(1+b)^b+\log(1+c)^c\geq\log\left(1+\frac{a+b+c}{3}\right)^{a+b+c}\overset{AGM}{\geq}$$

$$\geq \log\left(1+\frac{3\sqrt[3]{abc}}{3}\right)^{3\sqrt[3]{abc}} = \log\left(1+\sqrt[3]{abc}\right)^{3\sqrt[3]{abc}}$$

$$\log(1+a)^a\cdot(1+b)^b\cdot(1+c)^c \geq \log\left(1+\sqrt[3]{abc}\right)^{3\sqrt[3]{abc}}$$

$$\sqrt[3]{(1+a)^a\cdot(1+b)^b\cdot(1+c)^c} \geq \left(1+\sqrt[3]{abc}\right)^{\sqrt[3]{abc}}$$

$$RHS = LHS \Leftrightarrow a = b = c \Leftrightarrow x = 2x-5 = 3x-10.$$

Solution is $x = 5$.

UP.425 If $0 < a \leq b$ then:

$$a^{a+1}\cdot\exp\left(2(b-a)\right) \leq b^{b+1}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}, f(x) = \log x - 1 + \frac{1}{x}$, *then*

$$f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}, f'(x) = 0 \Leftrightarrow x = 1$$

$$\min\{f(x)\} = f(1) = 0 \Rightarrow f(x) \geq 0; \forall x > 0$$

$$\log x - 1 + \frac{1}{x} \geq 0 \Rightarrow \log x + \frac{1}{x} \geq 1$$

$$\int_a^b \log x\,dx + \int_a^b \frac{1}{x}dx \geq \int_a^b dx$$

$$\int_a^b x'\cdot\log x\,dx + \log b - \log a \geq b - a$$

$$b\log b - a\log a - \int_a^b dx + \log\left(\frac{b}{a}\right) \geq b - a$$

$$\log\left(\frac{b^b}{a^a}\right) + \log\left(\frac{b}{a}\right) \geq 2(b-a)$$

$$\log\left(\frac{b^{b+1}}{a^{a+1}}\right) \geq \log e^{2(b-a)} \Leftrightarrow a^{a+1}\cdot\exp\left(2(b-a)\right) \leq b^{b+1}$$

Equality holds for $a = b$.

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } f(x) = (x+1)\log x - 2x, x > 0, \text{then}$$
$$f'(x) = \frac{x+1}{x} + \log x - 2 = \frac{1}{x} + \log x - 1$$
$$f''(x) = -\frac{1}{x^2} + \frac{1}{x}, \text{hence } f''(x) < 0 \text{ if } x \in (0,1), f''(x) = 0 \text{ if } x = 1$$
$$\text{and } f''(x) > 0 \text{ if } x > 1$$
Thus, $f'(x) > f'(1) = 0$ if $x > 0, x \neq 1$, hence $f(x)$ is an increasing function on $[0,\infty)$

If $0 < a \leq b, f(a) \leq f(b)$ or $(a+1)\log a - 2a \leq (b+1)\log b - 2b \Leftrightarrow$
$$a^{a+1}\exp\big(2(b-a)\big) \leq b^{b+1}$$

Solution 3 by Christos Tsifakis-Greece

If $0 < a = b$ then $b^{b+1} = b^{b+1}$ exist.

If $0 < a < b$, then $a^{a+1}e^{2(b-a)} > b^{b+1} \Leftrightarrow$
$$a^{a+1} \cdot \frac{e^{2b}}{e^{2a}} > b^{b+1} \Leftrightarrow \frac{b^{b+1}}{a^{a+1}} > \frac{e^{2b}}{e^{2a}} \Leftrightarrow \frac{b^{b+1}}{e^{2b}} > \frac{a^{a+1}}{e^{2a}}$$
$$\frac{e^{(b+1)\log b}}{e^{2b}} > \frac{e^{(a+1)\log a}}{e^{2a}} \Leftrightarrow e^{(b+1)\log b - 2b} > e^{(a+1)\log a - 2a}$$
$$\text{Let } f(x) = e^{(x+1)\log x - 2x}, x > 0, \text{then } f'(x) = \log x + \frac{1}{x} - 1, f'(1) = 0$$
$$f''(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$$

x	0 $\quad$ 1 $\quad$ $+\infty$
$f''(x)$	$-\,-\,-\,-\,-0+++++++++++$
$f'(x)$	$\searrow + \searrow +\quad 0+\nearrow\ +\nearrow\ ++\nearrow\ +\nearrow\ +$
$f(x)$	$\nearrow\quad\nearrow\quad\nearrow\quad\nearrow\quad\nearrow\quad\nearrow$

For $x < 1 \Rightarrow f'(x) > f'(1) \Rightarrow f'(x) > 0 \Rightarrow f \nearrow$ on $(0,1]$

For $x > 1 \Rightarrow f'(x) > f'(1) \Rightarrow f'(x) > 0 \Rightarrow f' \nearrow$ on $[1,\infty)$

So, $f \nearrow$ on $(0,\infty)$ and $a < b \Rightarrow f(a) < f(b)$.

Finally, for $0 < a \leq b$, we have $f(a) \leq f(b)$.

Equality holds iff $a = b > 0$.

Solution 4 by Kunihiko Chikaya-Tokyo-Japan

From the well-known inequality

$$1-\frac{1}{x}\le \log x\le x-1;\forall x>0$$

$$\log\frac{b^{b+1}}{a^{a+1}}\cdot e^{2(a-b)}=\int_a^b\left(\log x-\left(1-\frac{1}{x}\right)\right)dx\ge 0\Leftrightarrow$$

$$a^{a+1}e^{2(b-a)}\le b^{b+1}.\ \textit{Equality holds iff } a=b>0.$$

Solution 5 by Khaled Abd Imouti-Damascus-Syria

$$e^{2(b-a)}\le\frac{b^{b+1}}{a^{a+1}}\Leftrightarrow 2(b-a)\le(b+1)\log b-(a+1)\log a$$

$$\big((b+1)\log b-2b\big)-\big((a+1)\log a-2a\big)\overset{?}{\ge}0$$

$$\textit{Let } f(x)=(x+1)\log x-2x, \textit{then } f'(x)=\frac{x\log x-x+1}{x}$$

By using M.V.T. we have:

$$f(b)-f(a)=f'(c)(b-a);c\in(a,b)$$

$$f'(c)>0\Rightarrow f(b)\ge f(a).$$

$$\textit{Thus}, \big((b+1)\log b-2b\big)-\big((a+1)\log a-2a\big)\ge 0$$

Equality holds iff $a=b>0$.

Solution 6 by Hikmat Mammadov-Azerbaijan

$$0<a\le b; a^{a+1}\cdot\exp\big(2(b-a)\big)\le b^{b+1}$$

$$\begin{cases}a+1\le b+1\\ 2a\le 2b\end{cases}\Rightarrow\begin{cases}a^{a+1}\le b^{b+1}\\ e^{2a}\le e^{2b}\end{cases}\Rightarrow\begin{cases}a^{a+1}\le b^{b+1}\\ e^{2(a-b)}\le 1\end{cases}$$

$$a^{a+1}\cdot e^{2(a-b)}\le b^{b+1}\cdot 1=b^{b+1}$$

Therefore, $a^{a+1}\cdot e^{2(a-b)}\le b^{b+1}$.

Equality holds iff $a=b>0$.

UP.426 Let $(a_n)_{n\ge1}$ be a sequence of real numbers such that $a_n\le n;\forall n\ge 1$

and

$$\sum_{k=1}^{n-1}\cos\frac{\pi a_k}{n}=0;\forall n\ge 2.\ \text{Find:}$$

$$\Omega=\lim_{n\to\infty}\left(a_n\cdot\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}\right)$$

Proposed by Florică Anastase-Romania

Solution by proposer

Is true that $a_1=1$. For $\cos\frac{\pi a_1}{3}+\cos\frac{\pi a_2}{3}=0$, we get $a_2=2$.

Suppose that $a_k=k;\forall k=\overline{1,n-1}$ and from hypothesis, we get:

$$\cos\frac{\pi a_n}{n+1}=-\sum_{k=1}^{n-1}\cos\frac{\pi k}{n+1}$$

Let be the number $z=\cos\frac{\pi}{n+1}+i\sin\frac{\pi}{n+1}$.

$$z+z^2+z^3+\dots+z^n=\frac{z-z^{n+1}}{1-z}=\frac{1+z}{1-z}$$

$$z\cdot\bar{z}=1\Rightarrow\overline{\left(\frac{1+z}{1-z}\right)}=-\frac{1+z}{1-z}\Rightarrow Re\left(\frac{1+z}{1-z}\right)=0\Rightarrow\sum_{k=1}^{n}\cos\frac{\pi k}{n+1}=0$$

Hence, $\cos\frac{\pi a_k}{n+1}=\cos\frac{\pi n}{n+1}$ and from $a_n\le n$, it follows that $a_n=n;\forall n\ge 2$. So, we have:

$$\Omega=\lim_{n\to\infty}\left(a_n\cdot\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}\right)=\lim_{n\to\infty}\left(n\cdot\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}\right);(1)$$

Let $S_n=\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}$, we have:

$$S_n=(2n+1)\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}\frac{k!\,(2n-k)!}{(2n+1)!}$$

$$=(2n+1)\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}\int_0^1 x^k(1-x)^{2n-k}\,dx=$$

$$=(2n+1)\int_0^1\left[\sum_{k=0}^{2n}\binom{4n}{2k}(-1)^k(1-x)^{2n-k}\right]dx=$$

$$=\frac{2n+1}{2}\int_0^1\left[\left(\sqrt{1-x}+i\sqrt{x}\right)^{4n}+\left(\sqrt{1-x}-i\sqrt{x}\right)^{4n}\right]dx$$

Because $\sqrt{1-x}\pm i\sqrt{x}=\cos\left(\tan^{-1}\sqrt{\frac{x}{1-x}}\right)\pm i\sin\left(\tan^{-1}\sqrt{\frac{x}{1-x}}\right)$, we get:

$$S_n=(2n+1)\int_0^1 \cos\left(4n\cdot\tan^{-1}\sqrt{\frac{x}{1-x}}\right)dx \overset{\tan^{-1}\sqrt{\frac{x}{1-x}}=t}{=}$$

$$(2n+1)\int_0^{\frac{\pi}{2}}\cos(4nt)\sin(2t)\,dt=$$

$$=\frac{2n+1}{2}\int_0^1[\sin(4n+2)t-\sin(4n-2)t]\,dt=\frac{2n+1}{2}\left(\frac{2}{4n+2}-\frac{2}{4n-2}\right)$$

$$=-\frac{1}{2n-1};(2)$$

From (1) and (2), it follows that:

$$\Omega=\lim_{n\to\infty}\left(a_n\cdot\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}\right)=\lim_{n\to\infty}\left(n\cdot\left(-\frac{1}{2n-1}\right)\right)=-\frac{1}{2}$$

UP.427 Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $a_n\leq n;\forall n\geq 1$ and

$$\sum_{k=1}^{n-1}\cos\frac{\pi a_k}{n}=0;\forall n\geq 2.\text{ Find:}$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2}\cdot\sum_{k=0}^{2n}\frac{\binom{2n}{k}}{\binom{4n}{2k}}\right)^{a_{2n+1}}$$

Proposed by Florică Anastase-Romania

Solution by proposer

Is true that $a_1=1$. For $\cos\frac{\pi a_1}{3}+\cos\frac{\pi a_2}{3}=0$, we get $a_2=2$.

Suppose that $a_k=k;\forall k=\overline{1,n-1}$ and from hypothesis, we get:

$$\cos\frac{\pi a_n}{n+1}=-\sum_{k=1}^{n-1}\cos\frac{\pi k}{n+1}$$

Let be the number $z=\cos\frac{\pi}{n+1}+i\sin\frac{\pi}{n+1}$.

$$z+z^2+z^3+\cdots+z^n=\frac{z-z^{n+1}}{1-z}=\frac{1+z}{1-z}$$

$$z\cdot\bar{z}=1\Rightarrow\overline{\left(\frac{1+z}{1-z}\right)}=-\frac{1+z}{1-z}\Rightarrow Re\left(\frac{1+z}{1-z}\right)=0\Rightarrow\sum_{k=1}^{n}\cos\frac{\pi k}{n+1}=0$$

Hence, $\cos\frac{\pi a_k}{n+1}=\cos\frac{\pi n}{n+1}$ and from $a_n\le n$, it follows that $a_n=n;\forall n\ge 2$.

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2}\cdot\sum_{k=0}^{2n}\frac{\binom{2n}{k}}{\binom{4n}{2k}}\right)^{2n+1};(1)$$

$$S_n=\sum_{k=0}^{2n}\frac{\binom{2n}{k}}{\binom{4n}{2k}}=(4n+1)\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{(2k)!\,(4n-2k)!}{(4n+1)!}=$$

$$=(4n+1)\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\int_0^1 x^{2k}(1-x)^{4n-2k}\,dx=$$

$$=(4n+1)\int_0^1\left[(1-x)^{4n}\sum_{k=0}^{2n}\binom{2n}{2k}\left(\frac{-x^2}{(1-x)^2}\right)^k\right]dx=$$

$$=(4n+1)\int_0^1(1-x)^{4n}\left(1-\frac{x^2}{(1-x)^2}\right)^{2n}dx=$$

$$=(4n+1)\int_0^1(1-2x)^{2n}\,dx=\frac{4n+1}{2n+1};(2)$$

From (1) and (2), it follows that:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2}\cdot\frac{4n+1}{2n+1}\right)^{2n+1}=\lim_{n\to\infty}\left(1+\frac{4n+1}{2(2n+1)}-1\right)^{2n+1}=$$

$$=\lim_{n\to\infty}\left[\left(1-\frac{1}{2(2n+1)}\right)^{-2(2n+1)}\right]^{\frac{-1}{2}}=\frac{1}{\sqrt{e}}$$

UP.428 If $(a_n)_{n\ge1}$ is a positive real sequence, such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(n!)^2}}=a$; $a\in\mathbb{R}_+^*$, then find:

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[n]{(2n-1)!!}}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by proposers

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^2a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(n!)^2}}\left(\frac{\sqrt[n]{n!}}{n}\right)^2=a\cdot\left(\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\right)^2=\frac{a}{e^2}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^2}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{2n}}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{n^{2n}}{(n+1)^{2(n+1)}}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{n^2a_n}\left(\frac{n}{n+1}\right)^{2n+2}=\frac{a}{e^2}\cdot\frac{1}{e^2}=\frac{a}{e^4}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^n}{n^n}\cdot\frac{(2n-1)!!}{(2n+1)!!}=$$

$$=\lim_{n\to\infty}\frac{n+1}{2n+1}\left(\frac{n+1}{n}\right)^n=\frac{e}{2}$$

$$\frac{1}{\sqrt[n]{(2n-1)!!}}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\frac{\sqrt[n]{a_n}}{\sqrt[n]{(2n-1)!!}}(u_n-1)=$$

$$=\frac{\sqrt[n]{a_n}}{\sqrt[n]{(2n-1)!!}}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n=$$

$$=\frac{\sqrt[n]{a_n}}{n^2}\cdot\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n,\text{ where}$$

$$u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}=\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n^2}{\sqrt[n]{a_n}};\forall n\ge 2$$

$$\lim_{n\to\infty}u_n=1=\frac{e^4}{a}\cdot 1\cdot\frac{a}{e^4};\ \lim_{n\to\infty}\frac{u_n-1}{\log u_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=\lim_{n\to\infty}\frac{a_{n+1}}{n^2a_n}\cdot\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}\cdot\left(\frac{n}{n+1}\right)^2=\frac{a}{e^2}\cdot\frac{e^4}{a}\cdot 1=e^2$$

Hence,

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[n]{(2n-1)!!}}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\frac{a}{e^4}\cdot\frac{e}{2}\cdot 1\cdot\log e^2=\frac{a}{e^3}$$

UP.429 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\pi^2}{16}-\left(\sum_{k=2}^{n+1}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)\right)^2\right)\cdot\sqrt[n]{n!}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\tan^{-1}\left(\frac{1}{k-1}\right)-\tan^{-1}\left(\frac{1}{k}\right)=\tan^{-1}\left(\frac{\frac{1}{k-1}-\frac{1}{k}}{1+\frac{1}{k-1}\cdot\frac{1}{k}}\right)=\tan^{-1}\left(\frac{k-k+1}{k^2-k+1}\right)=$$

$$=\tan^{-1}\left(\frac{1}{k^2-k+1}\right)$$

$$\sum_{k=2}^{n+1}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)=\sum_{k=2}^{n+1}\left(\tan^{-1}\left(\frac{1}{k-1}\right)-\tan^{-1}\left(\frac{1}{k}\right)\right)=$$

$$=\tan^{-1}1-\tan^{-1}\left(\frac{1}{n+1}\right)=\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{n+1}\right)$$

Hence,

$$\Omega=\lim_{n\to\infty}\left(\frac{\pi^2}{16}-\left(\sum_{k=2}^{n+1}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)\right)^2\right)\cdot\sqrt[n]{n!}=$$

$$=\lim_{n\to\infty}\left(\frac{\pi}{4}-\frac{\pi}{4}+\tan^{-1}\left(\frac{1}{n+1}\right)\right)\left(\frac{\pi}{4}+\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{n+1}\right)\right)\cdot\sqrt[n]{n!}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\cdot n\cdot\tan^{-1}\left(\frac{1}{n+1}\right)\cdot\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{n+1}\right)\right)=$$

$$=\lim_{n\to\infty}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{n+1}\right)\right)\cdot\sqrt[n]{\frac{n!}{n^n}}\cdot\frac{\tan^{-1}\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}}=$$

$$=\left(\frac{\pi}{2}-0\right)\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\lim_{n\to\infty}\frac{n^2}{(n+1)^2}\cdot\frac{1}{1+\frac{1}{(n+1)^2}}=\frac{\pi}{2}\cdot\frac{1}{e}\cdot1\cdot1=\frac{\pi}{2e}$$

Solution 2 by Naren Bhandari-Bajura-Nepal

We note that $k^2-k=k(k-1)$ and using the standard formula of

$\tan^{-1}x-\tan^{-1}y$, the sum becomes:

$$\sum_{k=2}^{n+1}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)=\sum_{k=2}^{n+1}(\tan^{-1}k-\tan^{-1}(k-1))=$$

$$=\tan^{-1}(n+1)-\frac{\pi}{4},\textit{which yields}$$

$$\Omega=\frac{\pi^2}{16}-\left(\sum_{k=2}^{n+1}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)\right)=-(\tan^{-1}(n+1))^2+\frac{\pi}{2}\tan^{-1}(n+1)\,;(1)$$

Now, we note that $\lim_{n\to\infty}\sqrt[n]{n!}=\infty$ *however,* $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$*. So, the required limit is*

$$\Omega=\lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\pi}{2}\tan^{-1}(n+1)-(\tan^{-1}(n+1))^2\right)=$$

$$=\frac{1}{e}\lim_{n\to\infty}\frac{-(\tan^{-1}(n+1))^2+\frac{\pi}{2}\tan^{-1}(n+1)}{\frac{1}{n}}$$

The limit has $\frac{0}{0}$ *from which we evaluate by L'hopital rule by nothing that*

$$\frac{d}{dn}\left(\frac{\pi}{2}\tan^{-1}(n+1)-(\tan^{-1}(n+1))^2\right)=\frac{\pi-4\tan^{-1}(n+1)}{2n^2+4n+4}$$

and last expression boils down to:

$$\Omega=\frac{1}{e}\lim_{n\to\infty}\left(-\frac{n^2(\pi-4\tan^{-1}(n+1))}{2n^2+4n+4}\right)=\frac{1}{e}\left(-\frac{\pi-2\pi}{2}\right)=\frac{\pi}{2e}$$

UP.430 If $a>0;t\in\mathbb{N};a,t$ −fixed then find:

$$\Omega(a,t)=\lim_{n\to\infty}\left(\sqrt[nt]{a}-1\right)\cdot\sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{\sqrt[n]{(2n-1)!!}}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=2\cdot\frac{1}{e}=\frac{2}{e}$$

$$\Omega(a,t)=\lim_{n\to\infty}\left(\sqrt[nt]{a}-1\right)\cdot\sqrt[n]{(2n-1)!!}=\lim_{n\to\infty}\frac{\sqrt[nt]{a}-1}{\frac{1}{nt}}\cdot nt\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot n=$$

$$=\frac{2}{e}\cdot\lim_{t\to\infty}\frac{\sqrt[nt]{a}-1}{\frac{1}{nt}}\cdot\frac{1}{nt}\cdot n=\frac{2}{e}\lim_{n\to\infty}\frac{n}{nt}\cdot\lim_{n\to\infty}\frac{a^{\frac{1}{nt}}-1}{\frac{1}{nt}}=\frac{2}{e}\cdot\frac{1}{t}\cdot\log a=\frac{2\log a}{et}$$

Solution 2 by Naren Bhandari-Bajura-Nepal

We recall that

$$\lim_{n\to 0}\frac{a^n-1}{n}=\log a\,;(*)\text{ and }(2n-1)!!=\frac{(2n)!}{2^n n!}\,;\forall a>0$$

and using these elementary limit results we evaluate our main result, i.e.

$$\Omega(a,t)=\lim_{n\to\infty}\left(\sqrt[tn]{a}-1\right)\sqrt[n]{(2n-1)!!}=\lim_{n\to\infty}\frac{\sqrt[tn]{a}-1}{\frac{1}{nt}\cdot nt}\cdot\sqrt[n]{(2n-1)!!}=$$

$$=\frac{\log(a)}{t}\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{\log(a)}{t}\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{2n}\sqrt[n]{\binom{2n}{n}}\right)\cong$$

$$\cong\frac{\log(a)}{t}\lim_{n\to\infty}\left(\frac{n}{2ne}\sqrt[n]{\frac{4^n}{\sqrt{\pi n}}}\right)=\frac{\log(a)}{t}\cdot\frac{2}{e}$$

and hence the required limit is $\frac{\log(a)}{t}\cdot\frac{2}{e}$. Above we utilize the Stirling approximation of $n!\sim\sqrt{2\pi n}\frac{n^n}{e^n}$ and $\binom{2n}{n}\sim\frac{4^n}{\sqrt{\pi n}}$ (which is deduced by $n!$ Approximation).

UP.431 Find:

$$\Omega(a)=\lim_{t\to\infty}e^{H_n}\cdot\sqrt[n]{n!}\left(\sqrt[n^2]{a}-1\right);a>0;a-\text{fixed}.$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}$$

$$\Omega(a)=\lim_{t\to\infty}e^{H_n}\cdot\sqrt[n]{n!}\left(\sqrt[n^2]{a}-1\right)=\lim_{t\to\infty}\frac{e^{H_n}}{n}\cdot\frac{\sqrt[n]{n!}}{n}\cdot n^2\left(\sqrt[n^2]{a}-1\right)=$$

$$=\lim_{n\to\infty}\frac{e^{H_n}}{e^{\log n}}\cdot\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\cdot\lim_{n\to\infty}\frac{\sqrt[n^2]{a}-1}{\frac{1}{n^2}}=\lim_{n\to\infty}e^{H_n-\log n}\cdot\frac{1}{e}\cdot\lim_{n\to\infty}\frac{a^{\frac{1}{n^2}}-1}{\frac{1}{n^2}}=$$

$$=e^{\gamma}\cdot\frac{1}{e}\cdot\log a=e^{\gamma-1}\cdot\log a$$

Solution 2 by Naren Bhandari-Bajura-Nepal

We recall that

$$\lim_{n\to 0}\frac{a^n-1}{n}=\log a;(*)\text{ and }(2n-1)!!=\frac{(2n)!}{2^n n!};\forall a>0$$

and using these elementary limit results we evaluate our main result, i.e.

$$\Omega(a)=\lim_{n\to\infty}e^{H_n}\sqrt[n]{n!}\left(\sqrt[n^2]{a}-1\right)=\log(a)\lim_{n\to\infty}e^{H_n}\frac{\sqrt[n]{n!}}{n^2}\cong$$
$$=\log(a)\lim_{n\to\infty}\frac{e^{H_n}}{en}=\frac{\log(a)}{e}\lim_{n\to\infty}e^{H_n-\log n}=\frac{\log(a)}{e}\exp\left(\lim_{n\to\infty}(H_n-\log n)\right)$$
$$=\log(a)\,e^{\gamma-1}$$

which is the final required result of $\Omega(a)$ for all $a>0$.

Here H_n is n^{th} Harmonic number and we have used the definition of Euler-Mascheroni constant

$$\gamma=\lim_{n\to\infty}(H_n-\log n).$$

UP.432 Let $(b_n)_{n\geq1}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^{t+1}}=b>0;a>0;t\geq0;a,t-\text{fixed. Find:}$$

$$\Omega(a,b,t)=\lim_{n\to\infty}\frac{\left(\sqrt[n]{a}-1\right)\cdot\sqrt[n]{b_n}}{n^t}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution by proposers

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^{t+1}}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^{n(t+1)}}}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{(n+1)(t+1)}}\cdot\frac{n^{n(t+1)}}{b_n}=$$

$$=\lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^{t+1}}\cdot\left(\frac{n}{n+1}\right)^{(n+1)(t+1)}=b\cdot\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{(n+1)(t+1)}=\frac{b}{e^{t+1}};(1)$$

$$\lim_{n\to\infty}\frac{e^{\frac{1}{n}\log a}-1}{\frac{1}{n}\log a}=1;(2)$$

$$\Omega(a,b,t)=\lim_{n\to\infty}\frac{\left(\sqrt[n]{a}-1\right)\cdot\sqrt[n]{b_n}}{n^t}=\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^{t+1}}\cdot\lim_{n\to\infty}\frac{e^{\frac{1}{n}\log a}-1}{\frac{1}{n}\log a}\cdot\log a\overset{(1),(2)}{=}$$

$$=\frac{b}{e^{t+1}}\cdot1\cdot\log a$$

Therefore,

$$\Omega(a,b,t)=\lim_{n\to\infty}\frac{\left(\sqrt[n]{a}-1\right)\cdot\sqrt[n]{b_n}}{n^t}=\frac{b\log a}{e^{t+1}}$$

UP.433 If $m \geq 0$; m –fixed, $u, v > 0, u + v = 3$ then

$$\frac{1}{u^m}\left(\int_0^1 e^{x^3}\,dx\right)^{m+1} + \frac{1}{v^m}\left(\int_0^1 \sqrt[3]{\log x}\,dx\right)^{m+1} \geq \frac{1}{3^m}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\frac{1}{u^m}\left(\int_0^1 e^{x^3}\,dx\right)^{m+1} + \frac{1}{v^m}\left(\int_0^1 \sqrt[3]{\log x}\,dx\right)^{m+1} \overset{Radon}{\geq}$$

$$\geq \frac{1}{(u+v)^m}\left(\int_0^1 e^{x^3}\,dx + \int_0^1 \sqrt[3]{\log x}\,dx\right)^{m+1} \overset{Young}{\geq}$$

$$\geq \frac{1}{(u+v)^m} \cdot 1^{m+1} = \frac{1}{(u+v)^m} = \frac{1}{3^m}$$

Equality holds for $m = 0$.

Solution 2 by Iulian Cristi-Romania

$$\frac{1}{u^m}\left(\int_0^1 e^{x^3}\,dx\right)^{m+1} + \frac{1}{v^m}\left(\int_0^1 \sqrt[3]{\log x}\,dx\right)^{m+1} \overset{Radon}{\geq}$$

$$\geq \frac{1}{(u+v)^m}\left(\int_0^1 e^{x^3}\,dx + \int_0^1 \sqrt[3]{\log x}\,dx\right)^{m+1} \overset{Young}{\geq}$$

$$\geq \frac{1}{(u+v)^m} \cdot 1^{m+1} = \frac{1}{(u+v)^m} = \frac{1}{3^m}$$

Equality holds for $m = 0$.

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$\text{Let: } I_1 = \int_0^1 e^{x^3}\,dx \text{ and } I_2 = \int_0^1 \sqrt[3]{\log x}\,dx.$$

Let $f(x) = e^{x^3}, x \in \mathbb{R}$ then $f'(x) = 3x^2 e^{x^3}. f'(x) = 0 \Leftrightarrow x = 0, f$ –strictly increasing.

So, $f: \mathbb{R} \to \mathbb{R}, f(x) = e^{x^3}$ is bijective function, then f –invertible with

$f^{-1}: \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \sqrt[3]{\log x}$. Therefore,

$$\int_0^1 e^{x^3}\,dx = \int_0^1 \sqrt[3]{\log x}\,dx$$

$$u+v=3, u\le 3; v\le 3 \Rightarrow u^m\le 3^m, v^m\le 3^m \Rightarrow \frac{1}{u^m}\ge\frac{1}{3^m} \text{ and } \frac{1}{v^m}\ge\frac{1}{3^m}$$

$$\frac{1}{u^m}\left(\int_0^1 e^{x^3}\,dx\right)^{m+1}+\frac{1}{v^m}\left(\int_0^1 \sqrt[3]{\log x}\,dx\right)^{m+1}\ge$$

$$\ge\frac{2}{3^m}\left(\int_0^1 e^{x^3}\,dx\right)^{m+1}$$

$$\because e^{x^3}\ge 1+x^3\Rightarrow\int_0^1 e^{x^3}\,dx\ge\int_0^1(1+x^3)\,dx=1+\frac{1}{4}$$

$$\left(\int_0^1 e^{x^3}\,dx\right)^{m+1}\ge\left(1+\frac{1}{4}\right)^{m+1}\ge 1+(m+1)\cdot\frac{1}{4}$$

$$\left(\int_0^1 e^{x^3}\,dx\right)^{m+1}\ge 1+\frac{m+1}{4}$$

$$2\left(\int_0^1 e^{x^3}\,dx\right)^{m+1}\ge 1+\frac{m+1}{2}\ge 1$$

$$\frac{1}{u^m}\left(\int_0^1 e^{x^3}\,dx\right)^{m+1}+\frac{1}{v^m}\left(\int_0^1 \sqrt[3]{\log x}\,dx\right)^{m+1}\ge\frac{1}{3^m}$$

UP.434 If $a,b>0$; a,b –fixed, find:

$$\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}\,dx$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}\,dx\overset{y=-x}{=}\int_{a}^{-a}\frac{(-y)^{2022}}{b+\sqrt{b^2+(-y)^2}+(-y)}(-dy)=$$

$$=-\int_{-a}^{a}\frac{y^{2022}}{b+\sqrt{b^2+y^2}-y}\,dy=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}-x}\,dx$$

$$2\Omega(a,b)=\int_{-a}^{a}\left(\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}+\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}-x}\,dx\right)dx=$$

$$=\int_{-a}^{a}x^{2022}\cdot\frac{b+\sqrt{b^2+x^2}-x+b+\sqrt{b^2+x^2}+x}{\left(b+\sqrt{b^2+x^2}\right)^2-x^2}\,dx=$$

$$=\int_{-a}^{a}\frac{2\left(b+\sqrt{b^2+x^2}\right)x^{2022}}{2b\left(b+\sqrt{b^2+x^2}\right)}dx=\int_{-a}^{a}\frac{x^{2022}}{b}dx=\frac{1}{b}\left(\frac{a^{2023}}{2023}-\frac{(-a)^{2023}}{2023}\right)$$

$$2\Omega=\frac{2a^{2023}}{2023b},\Omega=\frac{a^{2023}}{2023b}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}dx\,;(1)$$

$$x=-t\Rightarrow\ \Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}-x}dx\,;(2)$$

By adding (1) and (2), we get:

$$2\Omega(a,b)=\int_{-a}^{a}\frac{2x^{2022}\left(b+\sqrt{b^2+x^2}\right)}{\left(b+\sqrt{b^2+x^2}\right)-x^2}dx$$

$$\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}\left(b+\sqrt{b^2+x^2}\right)}{b^2+b^2+x^2+2b\sqrt{b^2+x^2}-x^2}dx=$$

$$=\int_{-a}^{a}\frac{x^{2022}\left(b+\sqrt{b^2+x^2}\right)}{2b\left(b+\sqrt{b^2+x^2}\right)}dx=\frac{1}{b}\int_{0}^{a}x^{2022}\,dx=\frac{a^{2023}}{2023b}$$

Solution 3 by Vivek Kumar-India

$$\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}dx\overset{x=-x}{=}\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}-x}dx$$

$$2\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}dx+\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}-x}dx=$$

$$\int_{-a}^{a}\frac{x^{2022}\left(b+\sqrt{b^2+x^2}\right)}{b^2+b^2+x^2+2b\sqrt{b^2+x^2}-x^2}dx=$$

$$=\int_{-a}^{a}\frac{x^{2022}\left(b+\sqrt{b^2+x^2}\right)}{2b\left(b+\sqrt{b^2+x^2}\right)}dx=\frac{1}{b}\int_{0}^{a}x^{2022}\,dx=\frac{a^{2023}}{2023b}$$

Solution 4 by Tapas Das-India

$$\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}dx=\int_{-a}^{a}\frac{x^{2022}\left(b+x-\sqrt{b^2+x^2}\right)}{(b+x)^2-(b^2+x^2)}dx=$$

$$=\frac{1}{2b}\int_{-a}^{a} b\cdot x^{2021}\,dx+\frac{1}{2b}\int_{-a}^{a} x^{2022}\,dx-\int_{-a}^{a} x^{2021}\sqrt{b^2+x^2}\,dx$$

$$=\frac{1}{b}\cdot 2\int_{0}^{a} x^{2022}\,dx=\frac{a^{2023}}{2023b}$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}dx=\int_{-a}^{a}\frac{x^{2022}\left(b+x-\sqrt{b^2+x^2}\right)}{(b^2+2bx+x^2)-(b^2+x^2)}dx=$$

$$=\int_{-a}^{a}\frac{x^{2022}\left(b+x-\sqrt{b^2+x^2}\right)}{2bx}dx=$$

$$=\frac{1}{2b}\int_{-a}^{a} b\cdot x^{2021}\,dx+\frac{1}{2b}\int_{-a}^{a} x^{2022}\,dx-\int_{-a}^{a} x^{2021}\sqrt{b^2+x^2}\,dx$$

$$=\frac{1}{b}\cdot 2\int_{0}^{a} x^{2022}\,dx=\frac{a^{2023}}{2023b}$$

UP.435

If $\Omega(n)=\lim_{t\to\infty}\int_0^t\frac{dx}{(x^2+n^2)(x^2+n^4)(x^2+n^6)}$;

$n\in\mathbb{N}, n\geq 2$**. Find:**

$$\Omega=\lim_{n\to\infty}\frac{n^\alpha}{\Omega(n)}\cdot\int_0^1\sqrt[n]{1+x+x^n}\,dx\,;\alpha\in\mathbb{R}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

For $x\in[0,1]$ **and** $n\geq 2$**, we have:**

$$\left|\sqrt[n]{1+x+x^n}-\sqrt[n]{1+x}\right|=$$

$$=\left|\frac{x^n}{\left(\sqrt[n]{1+x+x^n}\right)^{n-1}+\left(\sqrt[n]{1+x+x^n}\right)^{n-2}\cdot\left(\sqrt[n]{1+x}\right)+\dots+\left(\sqrt[n]{1+x}\right)^{n-1}}\right|\leq x^n$$

Where, $\sqrt[n]{1+x}-x^n\leq\sqrt[n]{1+x+x^n}\leq\sqrt[n]{1+x}+x^n$**. Integrating on** $[0,1]$**, we have:**

$$\frac{2^{\frac{1}{n}}+1}{\frac{1}{n}+1}-\frac{1}{\frac{1}{n}+1}-\frac{1}{n+1}\leq I_n\leq\frac{2^{\frac{1}{n}}+1}{\frac{1}{n}+1}-\frac{1}{\frac{1}{n}+1}+\frac{1}{n+1}\Rightarrow I_n\to 1;(1)$$

$$\int_0^t \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\int_0^t \left(\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right)dx =$$

$$= \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{t}{a}-\frac{1}{b}\tan^{-1}\frac{t}{b}\right)$$

Hence,

$$\lim_{t\to\infty}\int_0^t \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{ab(a+b)}\cdot\frac{\pi}{2}$$

Now, we have:

$$\int_0^t \frac{dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} =$$

$$= \frac{1}{c^2-a^2}\left(\int_0^t \frac{dx}{(x^2+a^2)(x^2+b^2)} - \int_0^t \frac{dx}{(x^2+b^2)(x^2+c^2)}\right)$$

$$\Omega(a,b,c) = \lim_{t\to\infty}\int_0^t \frac{dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} =$$

$$= \frac{1}{c^2-a^2}\left(\frac{1}{ab(a+b)}-\frac{1}{bc(b+c)}\right) = \frac{a+b+c}{abc(a+b)(b+c)(c+a)}$$

So, we have:

$$\Omega(n,n^2,n^3) = \frac{n+n^2+n^3}{n^6(n+n^2)(n^2+n^3)(n^3+n)} = \frac{n^2+n+1}{n^9(n+1)^2(n^2+1)}; (2)$$

From (1) and (2), we get:

$$\Omega = \lim_{n\to\infty}\frac{n^\alpha}{\Omega(n)}\cdot\int_0^1 \sqrt[n]{1+x+x^n}\,dx = \lim_{n\to\infty}\frac{n^\alpha n^9(n+1)^2(n^2+1)}{n^2+n+1} =$$

$$= \lim_{n\to\infty}\frac{n^{\alpha+11}\left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{n^2}\right)}{1+\frac{1}{n}+\frac{1}{n^2}} = \begin{cases} 0; & \alpha > -11 \\ 1; & \alpha = -11 \\ +\infty; & \alpha < -11 \end{cases}$$

Solution 2 by Adrian Popa-Romania

$$\frac{1}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{Ax+b}{(x^2+a^2)}+\frac{Cx+D}{(x^2+b^2)}+\frac{Ex+F}{(x^2+c^2)}$$

$$A+C+E=0$$

$$B+D+F=0$$

$$An^6+An^4+Cn^6+Cn^2+En^4+En^2=0$$

$$Bn^6+Bn^4+Dn^6+Dn^2+Fn^4+Fn^2=0$$

$$An^{10}+Cn^8+Fn^6=0$$

$$Bn^{10}+Dn^8+Fn^6=1\Rightarrow A=C=E=0.$$

$$F(1-n^4)=D(n^2-1)\Rightarrow D=-(1+n^2)F$$

$$F(n^6-n^8-n^{10}-n^{10}+n^{12}+n^{10})=1\Rightarrow F=\frac{1}{n^{12}-n^{10}-n^8+n^6}$$

$$D=-\frac{n^2+1}{n^{12}-n^{10}-n^8+n^6},B=\frac{n^2}{n^{12}-n^{10}-n^8+n^6}$$

$$\Omega(n)=$$

$$=\lim_{t\to\infty}\left(\frac{n^2}{n^{12}-n^{10}-n^8+n^6}\cdot\frac{1}{n}\tan^{-1}\left(\frac{x}{n}\right)\Big|_0^t-\frac{n^2-1}{n^{12}-n^{10}-n^8+n^6}\cdot\frac{1}{n^2}\tan^{-1}\left(\frac{x}{n^2}\right)\Big|_0^t+\frac{1}{n^{12}-n^{10}-n^8+n^6}\cdot\frac{1}{n^3}\tan^{-1}\left(\frac{x}{n^3}\right)\Big|_0^t\right)$$

$$=\frac{n^4-n^3+n+1}{n^3(n^{12}-n^{10}-n^8+n^6)}$$

Now, we have:

$$\int_0^1\sqrt[n]{1+x+x^n}\,dx\overset{x^n\to 0}{=}\int_0^1\sqrt[n]{1+x}\,dx=\int_0^1(1+x)^{\frac{1}{n}}dx=$$

$$=\frac{(1+x)^{\frac{n+1}{n}}}{\frac{n+1}{n}}\Bigg|_0^1=\frac{n}{n+1}\cdot 2^{\frac{n}{n+1}}-\frac{n}{n+1}\xrightarrow{n\to\infty}1$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{n^\alpha n^3(n^{12}-n^{10}-n^8+n^6)}{n^4-n^3+n+1}=\begin{cases}0; & \alpha>-11\\ 1; & \alpha=-11\\ +\infty; & \alpha<-11\end{cases}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru