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SOLUTIONS

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JP.031. Let a, b, c be non-negative real numbers. Prove that

$$9(a+b+c) \geq \sqrt[4]{\frac{a^4+b^4+c^4}{3}} + 26\sqrt{\frac{ab+bc+ca}{3}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean:** a, b, c **números reales no negativos. Probar que:

$$9(a+b+c) \geq \sqrt[4]{\frac{a^4+b^4+c^4}{3}} + 26\sqrt[4]{\frac{(ab+bc+ca)^2}{9}}$$

***Desde que:** $a, b, c \geq 0$, **sea:** $a^4+b^4+c^4 = 3x^4$, $(ab+bc+ac)^2 = 9y^4$*

$\Rightarrow 9(a+b+c) \geq x + 26y$. ***Por la desigualdad de Holder:***

$$(x^4+26y^4)(1+26)(1+26)(1+26) \geq (x+26y)^4 \rightarrow \sqrt[4]{27^3(x^4+26y^4)} \geq x+26y$$

***Esto es suficeiente probar:** $9(a+b+c) \geq \sqrt[4]{27^3(x^4+26y^4)} \rightarrow$*

$$\rightarrow 3^8(a+b+c)^4 \geq 3^9(x^4+26y^4)$$

$$\Rightarrow 3(a+b+c)^4 \geq 3(a^4+b^4+c^4) + 26(ab+bc+ac)^2$$

$$\Rightarrow 3(a^4+b^4+c^4) + 18(a^2b^2+b^2c^2+c^2a^2) + 12ab(a^2+b^2) +$$

$$+12bc(b^2+c^2) + 12ca(c^2+a^2) + 36abc(a+b+c) \geq$$

$$\geq 3(a^4+b^4+c^4) + 26(a^2b^2+b^2c^2+c^2a^2) + 52abc(a+b+c)$$

$$\Rightarrow 12ab(a^2+b^2) + 12bc(b^2+c^2) + 12ca(c^2+a^2) \geq$$

$$\geq 8(a^2b^2+b^2c^2+c^2a^2) + 16abc(a+b+c)$$

$$\Rightarrow 12ab(a^2+b^2) + 12bc(b^2+c^2) + 12ca(c^2+a^2) \geq$$

$$\geq 24(a^2b^2+b^2c^2+c^2a^2) \geq 8(a^2b^2+b^2c^2+c^2a^2) + 16abc(a+b+c)$$

$\Rightarrow 16(a^2b^2+b^2c^2+c^2a^2) \geq 16abc(a+b+c)$ ... ***(LQQD)***

JP.032. Prove the following inequality holds for all non-negative real numbers a, b:

$$\frac{1}{4a+1} + \frac{1}{4b+1} + \frac{6}{2a+2b+1} \geq \frac{4}{3a+b+1} + \frac{4}{3b+a+1}$$

Proposed by Nguyen Viet Hung- Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar la siguiente desigualdad para todos los números reales non negativos: a, b:

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{6}{2a+2b+1}\geq\frac{4}{3a+b+1}+\frac{4}{3b+a+1}$$

Sea: $x=4a+1\geq 1,\ y=4b+1\geq 1,\ x+y=2(2a+2b+1)$. ***Además:***

$$12a+4b+4=3(4a+1)+(4b+1)=3x+y,$$

$$12b+4a+4=3(4b+1)+(4a+1)=3y+x$$

$$\Rightarrow:\frac{1}{x}+\frac{1}{y}+\frac{12}{x+y}\geq\frac{16}{3x+y}+\frac{16}{3y+x}\to\frac{x+y}{xy}+\frac{12}{x+y}\geq\frac{16(3y+x)+16(3x+y)}{(3x+y)(y+3x)}$$

$$\Rightarrow\frac{(x+y)^2+12xy}{xy(x+y)}\geq\frac{64(x+y)}{(3x^2+3y^2+10xy)}\to$$

$$\to[3(x^2+y^2)+10xy][(x+y)^2+12xy]\geq 64(x+y)^2xy$$

$$\Rightarrow 3(x^2+y^2)(x+y)^2+36xy(x^2+y^2)+10xy(x+y)^2+120x^2y^2\geq 64(x+y)^2xy$$

$$\Rightarrow 3(x^2+y^2)(x+y)^2+36xy(x^2+y^2)+10xy(x+y)^2+120x^2y^2\geq 64(x+y)^2xy$$

$$\Rightarrow 3(x^2+y^2)(x+y)^2+36xy(x^2+y^2)+120x^2y^2\geq 54(x+y)^2xy$$

$$\Rightarrow (x^2+y^2)(x+y)^2+12xy(x^2+y^2)+40x^2y^2\geq 18(x+y)^2xy$$

$$\Rightarrow (x^2+y^2)^2+14xy(x^2+y^2)+40x^2y^2\geq 18xy(x^2+y^2)+36x^2y^2$$

$$\Rightarrow (x^2+y^2)^2-4xy(x^2+^2)+4x^2y^2=\left((x^2+y^2)-(2xy)\right)^2=(x-y)^4\geq 0$$

La igualdad se alcanza cuando: $x=y=4a+1=4b+1\to a=b$

Solution 2 by Soumitra Mandal – Kolkata – India

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{6}{2a+2b+1}\geq\frac{4}{3a+b+1}+\frac{4}{a+3b+1}$$

$$\Leftrightarrow\int_0^1 x^{4a}\,dx+\int_0^1 x^{4b}\,dx+6\int_0^1 x^{2(a+b)}\,dx\geq 4\int_0^1 x^{3a+b}\,dx+4\int_0^1 x^{a+3b}\,dx$$

$$\Leftrightarrow A^4+B^4+6A^2B^2\geq 4AB(A^2+B^2)$$

$$\Leftrightarrow (A^2+B^2)^2-4AB(A^2+B^2)+4A^2B^2\geq 0\Leftrightarrow (A-B)^4\geq 0,$$ ***which is true***

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{6}{2a+2b+1}\geq\frac{4}{3a+b+1}+\frac{4}{a+3b+1}$$

(proved)

Solution 3 by Henry Ricardo - New York – USA

Noting that $\frac{1}{4a+1}=\int_0^1 t^{4a}\,dt$, *we see that the given inequality is equivalent to*

$$\int_0^1 t^{4a}+t^{4b}+6t^{2a+2b}dt \geq \int_0^1 4t^{3a+b}+4t^{3b+a}\,dt,$$

or $t^{4a}+t^{4b}+6t^{2a+2b}\geq 4t^{3a+b}+4t^{3b+a}$. *If we let* $t^a=x$ *and* $t^b=y$, *the inequality is equivalent to* $x^4+y^4+6x^2y^2\geq 4x^3y+4xy^3$, *or* $(x-y)^4\geq 0$, *which is true.*

Solution 4 by Imad Zak – Saida – Lebanon

Another attempt:

Let $A=\frac{1}{4a+1}+\frac{3}{2a+2b+1}+\frac{4}{3b+a+1}$ *and* $B=\frac{1}{4b+1}+\frac{3}{2a+2b+1}-\frac{4}{3b+a+1}$ *we want to prove* $A+B\geq 0$. *We find* $A=\frac{2(a-b)(5a-b+1)}{(4a+1)(3a+b+1)(2a+2b+2)}$ *and* $B=\frac{2(a-b)(a-5b-1)}{(4b+1)(3b+a+1)(2a+2b+1)}$ *and finally* $A+B=\frac{2(a-b)}{2a+2b+1}\cdot\left(\frac{5a-b+1}{(4a+1)(3a+b+1)}+\frac{a-5b-1}{(4b+1)(a+3b+1)}\right)=24(a-b)^{\frac{4}{D}}$ *where*

$$D=(4a+1)(4b+1)(3a+b+1)(a+3b+1)(2a+2b+1)$$

Clearly $+B\geq 0$ *. Q.E.D. and equality holds when* $a=b$.

JP.033. Let a,b,c be positive real numbers such that

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}+2abc=1$$

Prove that

$$\frac{\sqrt{bc}}{a}+\frac{\sqrt{ca}}{b}+\frac{\sqrt{ab}}{c}\geq 2(a+b+c)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c *números reales positivos de tal manera que:*

$a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ab}+2abc=1$. *Probar que:* $\frac{\sqrt{bc}}{a}+\frac{\sqrt{ac}}{b}+\frac{\sqrt{ab}}{c}\geq 2(a+b+c)$ ... *(A)*

Siendo: $A+B+C=\pi$. *En un triángulo* ABC, *se cumple:*

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$$

$\Rightarrow$ *Sea:* $a\sqrt{bc}=\cos^2 A$, $b\sqrt{ac}=\cos^2 B$, $c\sqrt{ab}=\cos^2 C$,

$abc = \cos A \cos B \cos C > 0$ ***(Δ acutángulo). Por lo tanto:***

$a = \frac{\cos^3 A}{\cos B \cos C} > 0, b = \frac{\cos^3 B}{\cos A \cos C} > 0, c = \frac{\cos^3 C}{\cos A \cos B} > 0$***. La desigualdad es equivalente en***

... (A): $\Rightarrow \frac{\cos^2 B \cos^2 C}{\cos^4 A} + \frac{\cos^2 A \cos^2 C}{\cos^4 B} + \frac{\cos^2 A \cos^2 B}{\cos^4 C} \geq 2\frac{\cos^3 A}{\cos B \cos C} + 2\frac{\cos^3 B}{\cos A \cos C} + 2\frac{\cos^3 C}{\cos A \cos B}$

De la siguiente desigualdad para todos x, y, z ***números reales, se cumple en un triángulo*** ABC***:*** $x^2 + y^2 + z^2 \geq 2xy\cos A + 2yz\cos B + 2zx\cos C$***. Siendo:***

$x = \frac{\cos A \cos C}{\cos^2 B} > 0, y = \frac{\cos B \cos A}{\cos^2 C} > 0, z = \frac{\cos B \cos C}{\cos^2 A} > 0 \to$ ***(Δ acutángulo)***

Se obtiene: $\Rightarrow \frac{\cos^2 B \cos^2 C}{\cos^4 A} + \frac{\cos^2 A \cos^2 C}{\cos^4 B} + \frac{\cos^2 A \cos^2 B}{\cos^4 C} \geq$

$\geq 2\frac{\cos^3 A}{\cos B \cos C} + 2\frac{\cos^3 B}{\cos A \cos C} + 2\frac{\cos^3 C}{\cos A \cos B}$ ***... (LQQD)***

JP.034. Find all pairs (x, y) of integers satisfying the equation

$$x^4 - (y+2)x^3 + (y-1)x^2 + (y^2+2)x + y = 2.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Nguyen Viet Hung – Hanoi – Vietnam

The equation is equivalent to $(x^2 - 2x - y)(x^2 - yx - 1) = 2$.

There are four possible cases as follows

Case 1: $\begin{cases} x^2 - 2x - y = 1, \\ x^2 - yx - 1 = 2, \end{cases} \Leftrightarrow \begin{cases} x^2 - 2x - 1 = y, \\ x(x-y) = 3. \end{cases}$

It's easy to find 3 pairs of (x, y) ***satisfying this system of equations as***

$(-1, 2), (1, -2), (3, 2)$.

Case 2: $\begin{cases} x^2 - 2x - y = 2, \\ x^2 - yx - 1 = 1, \end{cases} \Leftrightarrow \begin{cases} x^2 - 2x - 2 = y, \\ x(x-y) = 2. \end{cases}$

There is only one pair (x, y) ***satisfying this system of equations as*** $(-1, 1)$***.***

Case 3: $\begin{cases} x^2 - 2x - y = -1, \\ x^2 - yx - 1 = -2, \end{cases} \Leftrightarrow \begin{cases} (x-1)^2 = y, \\ x(x-y) = -1. \end{cases}$

There is no pair of (x, y) ***satisfying this system of equations.***

Case 4: $\begin{cases} x^2 - 2x - y = -2, \\ x^2 - yx - 1 = -1, \end{cases} \Leftrightarrow \begin{cases} x^2 - 2x + 2 = y, \\ x(x-y) = 0. \end{cases}$

We find 3 pairs (x, y) satisfying this system of equations as $(0,2),(1,1),(2,2)$.

So, there are 7 pairs (x, y) satisfying the requirement as

$$(-1,2),(1,-2),(3,2),(-1,1),(0,2),(1,1),(2,2).$$

JP.035. Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that

$$5+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+abc \geq 3(ab+bc+ca)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números reales no negativos de tal manera que:

$a+b+c=3$. ***Probare que:*** $5+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+abc \geq 3(ab+bc+ca)$

$$\Rightarrow 5+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+abc \geq (a+b+c)(ab+bc+ac)$$

$$\Rightarrow 5+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+abc \geq (a+b)(b+c)(c+a)+abc$$

$$\Rightarrow 15+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \geq 3(a+b)(b+c)(c+a)$$

$$\Rightarrow a^3+b^3+c^3+15+3\sqrt[3]{a}+3\sqrt[3]{b}+3\sqrt[3]{c} \geq a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$

$$\Rightarrow a^3+b^3+c^3+3\sqrt[3]{a}+3\sqrt[3]{b}+3\sqrt[3]{c} \geq (a+b+c)^3=27$$

$\Rightarrow a^3+b^3+c^3+3\sqrt[3]{a}+3\sqrt[3]{b}+3\sqrt[3]{c} \geq 12$. ***Desde que:*** $a, b, c \geq 0$. ***Por:*** $MA \geq MG$

$\Rightarrow a^3+\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a} \geq 4\sqrt[4]{a^4}=4a \rightarrow a^3+3\sqrt[3]{a} \geq 4a$ ***... (A)***

Análogamente: $b^3+3\sqrt[3]{b} \geq 4b$ ***... (B);*** $c^3+3\sqrt[3]{c} \geq 4c$ ***... (C)***

Sumando: (A) + (B) + (C): $(a^3+b^3+c^3)+3\sqrt[3]{a}+3\sqrt[3]{b}+3\sqrt[3]{c} \geq 4(a+b+c)=12$

JP.036. Prove the following inequality

$$[(x+y)(y+z)(z+x)]^4 \geq \frac{16^3}{27}(x+y+z)^3x^3y^3z^3$$

where x, y, z are positive real numbers.

Proposed by Andrei Bogdan Ungureanu – Romania

Solution 1 by Soumitra Mandal – Kolkata – India

$$9\prod_{cyc}(x+y) \geq 8(x+y+z)(xy+yz+zx)$$

$$\Rightarrow \left[\prod_{cyc}(x+y)\right]^4 \geq \frac{2^{12}}{9^4}(x+y+z)^4(xy+yz+zx)^4 =$$

$$= \frac{16^3}{9^4}(x+y+z)^3 \cdot (x+y+z)(xy+yz+zx) \cdot (xy+yz+zx)^3$$

$$\geq \frac{16^3}{9^4}(x+y+z)^3 \cdot 9xyz \cdot 27\, x^2y^2z^2 = \frac{16^3}{27}(x+y+z)^3x^3y^3z^3$$

(proved)

Solution 2 by Pham Quy – Vietnam

Lemma:

$$(x+y)(y+z)(z+x) \geq \frac{8}{9}(x+y+z)(xy+yz+zx) \overset{AM-GM}{\geq} \frac{8}{3}(x+y+z)\sqrt[3]{(xyz)^2}$$

$$\Rightarrow [(x+y)(y+z)(z+x)]^3 \geq \frac{8^3}{27}(x+y+z)^3(xyz)^2 \text{ (1)}$$

By AM-GM inequality

$$(x+y)(y+z)(z+x) \geq 2^2xyz \text{ (2)}$$

(1) & (2)

$$\Rightarrow [(x+y)(y+z)(z+x)]^4 \geq \frac{16^3}{27}(x+y+z)^3x^3y^3z^3 \text{ (q.e.d.)}$$

The equality holds at $x = y = z$

Solution 3 by Rustem Zeynalov – Baku – Azerbaidjian

$$x+y=a;\ y+z=b;\ z+x=c$$

$$a^4b^4c^4 \geq \frac{16^3}{27}\cdot\left(\frac{a+b+c}{2}\right)^3 \cdot \left[\frac{a+b-c}{2}\cdot\frac{a+c-b}{2}\cdot\frac{b+c-a}{2}\right]^3$$

$$a^4b^4c^4 \geq \frac{1}{27}[(a+b+c)(a+b-c)(a+c-b)(b+c-a)]^3$$

$$(a+b+c)(a+b-c)(a+c-b)(b+c-a) \leq \sqrt[3]{27a^4b^4c^4}$$

$$2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 \leq \sqrt[3]{27a^4b^4c^4}$$

$$a^4+b^4+c^4+\sqrt[3]{27a^4b^4c^4} \geq 2a^2b^2+2a^2c^2+2b^2c^2$$

Schur inequality

JP.037. Let x, y, z be positive real numbers such that:

$$16(a^2+b^2+c^2)+27=128abc$$

Find the maximum value of the expression:

$$E=\frac{1}{a^3+b^3+\frac{27}{64}}+\frac{1}{b^3+c^3+\frac{27}{64}}+\frac{1}{c^3+a^3+\frac{27}{64}}$$

Proposed by Iuliana Trașcă; Neculai Stanciu – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Sea: x, y, z ***números*** $\mathbb{R}^+$ ***de tal manera que:*** $16(a^2+b^2+c^2)+27=128abc$

Hallar el máximo valor de: $A=\frac{1}{a^3+b^3+\frac{27}{64}}+\frac{1}{b^3+c^3+\frac{27}{64}}+\frac{1}{c^3+a^3+\frac{64}{27}}$. ***Desde que:***

$$(4a-3)^2+(4b-3)^2+(4c-3)^2=16(a^2+b^2+c^2)+27-24(a+b+c)=$$

$$=128abc-24(a+b+c)\geq 0\Rightarrow\frac{128}{24}\geq\frac{a+b+c}{abc}\rightarrow\frac{16}{3}\geq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

Por: MA ≥ MG: $a^3+b^3+\frac{27}{64}\geq 3\sqrt[3]{\frac{27a^3b^3}{64}}\rightarrow\frac{1}{a^3+b^3+\frac{27}{64}}\leq\frac{4}{9ab}$

Por la tanto tenemos en ... (A): $A=\frac{1}{a^3+b^3+\frac{27}{64}}+\frac{1}{b^3+c^3+\frac{27}{64}}+\frac{1}{c^3+a^3+\frac{64}{27}}\leq\frac{4}{9ab}+\frac{4}{9bc}+\frac{4}{9ac}\leq$

$\leq\frac{4}{9}\times\frac{16}{3}=\frac{64}{27}$ ***... (LQQD)***

$A_{Máx}\leq\frac{64}{27}$. ***La igualdad se alcanza cuando:*** $a=b=c=\frac{3}{4}$

JP.038. Let $a, b, c>0$, prove that:

$$6\left(\sum ab\right)\left(\sum a^2\right)+7abc\left(\sum a\right)\geq 23abc\sqrt{3\left(\sum a^2\right)}\ (*)$$

Proposed by Soumitra Mandal – Kolkata – India

Solution by Ngo Minh Ngoc Bao – Vietnam

We have two lemma: Lemma 1: If $a, b, c>0$ ***then*** $\left(\sum\frac{a}{b}\right)\left(\sum a\right)\geq 3\sqrt{3\left(\sum a^2\right)}$

Prove: Use Cauchy – Schwarz

$$\sum\frac{a}{b}\geq\frac{\left(\sum a\right)^2}{\sum ab}\Rightarrow\left(\sum\frac{a}{b}\right)\left(\sum a\right)\geq\frac{\left(\sum a\right)^3}{\sum ab}.$$

We need to prove: $\frac{(\sum a)^3}{\sum ab} \ge 3\sqrt{3(\sum a^2)} \Leftrightarrow (\sum a)^6 \ge 27(\sum ab)^2(\sum a^2)$ ***(**).***

Use AM – GM inequality: $(\sum a)^6 = \left((\sum a^2) + (\sum ab) + (\sum ab)\right)^3 \ge 27(\sum a^2)(\sum ab)^2$

$\Rightarrow LHS$ ***(**)*** $\ge RHS$ ***(**).***

Lemma 2: Consider polynomial

$$f(x,y,z) = \sum x^4 + A\sum x^2y^2 + Bxyz\sum x + C\sum x^3y + D\sum xy^3$$

(with A, B, C, D ***is the constant)***

$$f(x,y,z) \ge 0 \Leftrightarrow \begin{cases} 1 + A + B + C + D \ge 0 \\ 3(1 + A) < C^2 + CD + D^2 \\ 5 + A + 2C + 2D \ge 0 \\ g(x) = (4 + C + D)(x^3 + 1) + (A + 2C - D - 1)x^2 + (A - C + 2D - 1)x \ge 0, \forall x \ge 0 \end{cases}$$

My solution

(*) $\Leftrightarrow 6\sum a^3b + 6\sum ab^3 + 13abc\sum a \ge 23abc\sqrt{3(\sum a^2)}$***, we need to prove:***

$$6\sum a^3b + 6\sum ab^3 + 13abc\sum a - \frac{23}{3}abc\left(\sum \frac{a}{b}\right)\left(\sum a\right) \ge 0$$

$$\Leftrightarrow 6\sum a^3b + 6\sum ab^3 + 13abc\sum a - \frac{23}{3}\left(\sum a^2b^2 + \sum ab^3 + abc\sum a\right) \ge 0$$

$$\Leftrightarrow 6\sum a^3b - \frac{5}{3}\sum ab^3 - \frac{23}{3}\sum a^2b^2 + \frac{16}{3}abc\sum a \ge 0 \text{ (***)}$$

Use lemma 2 with $A = -\frac{23}{3}, B = \frac{16}{3}, C = 6, D = -\frac{5}{3}$***.***

We have:

$$\begin{cases} 1 + A + B + C + D = 1 - \frac{23}{3} + \frac{16}{3} + 6 - \frac{5}{3} = 3 > 0 \\ 5 + A + 2C + 2D = 5 - \frac{23}{3} + 12 - \frac{10}{3} = 6 > 0 \\ 3(1 + A) = -\frac{60}{3} < 6^2 - 6\cdot\frac{5}{3} + \frac{25}{9} = 26 + \frac{25}{9} = C^2 + CD + D^2 \end{cases}$$

Considering function: $g(x) = \frac{25}{3}x^3 + 5x^2 - 18x + \frac{25}{3} \Rightarrow g'(x) = 25x^2 + 10x - 18$

$$g'(x) = 0 \Leftrightarrow 25x^2 + 10x - 18 = 0 \Leftrightarrow \left[\begin{array}{l} x = \frac{-1 + \sqrt{19}}{5} \\ x = \frac{-1 - \sqrt{19}}{5} \end{array}\right.$$

x	0		$\frac{-1+\sqrt{19}}{5}$		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$		↘		↗	

$$g(x) \geq g\left(\frac{-1+\sqrt{19}}{5}\right) > 0 \Rightarrow g(x) > 0, \forall x \geq 0$$

JP.039. In ABC triangle the following relationship holds:

$$3\left(a^a b^b c^c\right)^{\frac{1}{2s}} \geq \sqrt[9]{4RS} \sum \left(a^a b^b c^c\right)^{\frac{1}{3s}}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

If $\lambda \in (0,1); x, y, z \in \mathbb{R}; x + y + z = 1$ *then:* $\sum a^x b^y c^z \geq \sum a^{\lambda x + \frac{1-\lambda}{3}} b^{\lambda y + \frac{1-\lambda}{3}} c^{\lambda z + \frac{1-\lambda}{3}}$

For: $x = \frac{a}{a+b+c}; y = \frac{b}{a+b+c}; z = \frac{c}{a+b+c}$. *We have:* $x + y + z = 1; a + b + c = 2p$

$$\sum a^{\frac{a}{2s}} \cdot b^{\frac{b}{2s}} \cdot c^{\frac{c}{2s}} \geq \sum a^{\frac{\lambda a}{2s} + \frac{1-\lambda}{3}} b^{\frac{\lambda b}{2s} + \frac{1-\lambda}{3}} c^{\frac{\lambda c}{2s} + \frac{1-\lambda}{3}} \quad (1)$$

We take: $\lambda = \frac{2}{3}; \frac{\lambda a}{2s} + \frac{1-\lambda}{3} = \frac{2a}{6s} + \frac{1-\frac{2}{3}}{3} = \frac{a}{3s} + \frac{1}{9} = \frac{3a+s}{9s}$

and analogous: $\frac{\lambda b}{2s} + \frac{1-\lambda}{3} = \frac{3b+s}{9s}; \frac{\lambda c}{2s} + \frac{1-\lambda}{3} = \frac{3c+s}{9s}$. *The relationship (1) can be written:*

$$\sum \left(a^a \cdot b^b \cdot c^c\right)^{\frac{1}{2s}} \geq \sum a^{\frac{3a+s}{9s}} b^{\frac{3b+s}{9s}} c^{\frac{3c+s}{9s}} = \sum \left(a^{3a+s} \cdot b^{3b+s} \cdot c^{3c+s}\right)^{\frac{1}{9s}} =$$

$$= \sum (abc)^{\frac{1}{9}} \left(a^{3a} b^{3b} c^{3c}\right)^{\frac{1}{9s}} = \sqrt[9]{abc} \sum \left(a^a b^b c^c\right)^{\frac{1}{3s}} = \sqrt[9]{4RS} \sum \left(a^a b^b c^c\right)^{\frac{1}{3s}}$$

JP.040. Prove that if $a, b, c, d \in (0, \infty); \sqrt{3}(ad - bc) = ac + bd \neq 0$ then:

$$d\left(a + b\sqrt{3}\right) - c\left(b - a\sqrt{3}\right) > 4\sqrt[4]{abcd}$$

Proposed by Daniel Sitaru – Romania

Solution by Anas Adlany - El Jadida- Morroco

We have $\sqrt[3]{3}(ad - bc) := ac + bd \neq 0 \Rightarrow d(a\sqrt{3} - b) := c(a + b\sqrt{3})$

Also, we conclude that $ad > bc \Rightarrow \sqrt[4]{abcd} < \sqrt{abcd}$ **and** $a\sqrt{3} > b, 3a > b\sqrt{3}$.

Thus, $d(a + b\sqrt{3}) - c(b - a\sqrt{3}) := d(a + b\sqrt{3}) + \frac{c^2}{d}(a + b\sqrt{3}) := \left(\frac{c^2+d^2}{d}\right)(a + b\sqrt{3})$

Hence, we have to prove that $\left(\frac{c^2+d^2}{d}\right)(a + b\sqrt{3}) > 4\sqrt[4]{abcd}$. **But,**

$$a > \frac{bc}{d} \Rightarrow a + b\sqrt{3} > b\left(\sqrt{3} + \frac{c}{d}\right) \Rightarrow \left(\frac{c^2+d^2}{d}\right)(a + b\sqrt{3}) > \frac{b}{d}(c^2 + d^2)\left(\sqrt{3} + \frac{c}{d}\right)$$

$$\left(\frac{c^2 + d^2}{d}\right)(a + b\sqrt{3}) > 2bc\left(\sqrt{3} + \frac{c}{d}\right) > 4bc\sqrt{\frac{bc}{ad}} := 4\sqrt{\frac{b^3c^3}{ad}}$$

And note that $\sqrt{\frac{b^3c^3}{ad}} > \sqrt{abcd} \Leftrightarrow (ad)^2 > (bc)^2$

Which is true due to the first observation (see above).

Conclusion: From all those inequalities, we shall obtain the desired inequality.

Comment: this is a great problem for juniors, all thanks to sir DAN SITARU.

JP.041. Prove that in an ABC acute-angled triangle the following relationship holds:

$$\cos\left(\frac{\pi}{4} - A\right) + \cos\left(\frac{\pi}{4} - B\right) + \cos\left(\frac{\pi}{4} - C\right) > \frac{2S}{R^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC, la siguiente desigualdad:

$$\cos\left(\frac{\pi}{4} - A\right) + \cos\left(\frac{\pi}{4} - B\right) + \cos\left(\frac{\pi}{4} - C\right) > \frac{2S}{R^2}$$

Dado que es un triángulo acutángulo $0 < A, B, C < \frac{\pi}{2}$,

$\cos A, \cos B, \cos C > 0$, $\text{sen } A, \text{sen } B, \text{sen } C > 0$

Desde que: $S = 2R^2 \text{sen } A \text{ sen } B \text{ sen } C$, **se tiene la desigualdad:**

$$\Rightarrow \frac{\sqrt{2}}{2}\left((\cos A + \text{sen } A) + (\cos B + \text{sen } B) + (\cos C + \text{sen } C)\right) > 4 \text{ sen } A \text{ sen } B \text{ sen } C \ldots \text{(A)}$$

Probaremos lo siguiente:

$$\Rightarrow \frac{\sqrt{2}}{2}(\cos A + \operatorname{sen} A) > \operatorname{sen} 2A \to \cos A + \operatorname{sen} A > \sqrt{2}\operatorname{sen} A \to (\cos A + \operatorname{sen} A)^2 > 2\operatorname{sen}^2 A$$

$$\Rightarrow 1 + \operatorname{sen} 2A > 2\operatorname{sen}^2 2A \to 2\operatorname{sen}^2 2A - \operatorname{sen} 2A - 1 = (2\operatorname{sen} 2A + 1)(\operatorname{sen} 2A - 1) < 0$$

Lo cual es cierto ya que: $0 < 2A < \pi \to 0 < \operatorname{sen} 2A < 1$, ***por la tanto:***

$2\operatorname{sen} 2A + 1 > 0 \wedge \operatorname{sen} 2A - 1 < 0$. ***Por lo tanto se tendrá en ... (A):***

$$\Rightarrow \frac{\sqrt{2}}{2}\big((\cos A + \operatorname{sen} A) + (\cos B + \operatorname{sen} B) + (\cos C + \operatorname{sen} C)\big) >$$

$$> \operatorname{sen} 2A + \operatorname{sen} 2B + \operatorname{sen} 2C = 4\operatorname{sen} A \operatorname{sen} B \operatorname{sen} C \quad \textbf{(LQQD)}$$

Solution 2 by Nguyen Minh Triet - Quang Ngai – Vietnam

$(\forall)x$, ***we have:*** $\cos x + \sin x = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right)$. ***Hence,*** $LHS = \left[\sum_{cyc}(\cos A + \sin A)\right] \cdot \frac{1}{\sqrt{2}}$

Let $x = \cos A + \sin A$; $y = \cos B + \sin B$, $c = \cos C + \sin C$. ***Then*** $x, y, z \in \left(0, \sqrt{2}\right]$ ***and***

$LHS = \frac{1}{\sqrt{2}}(x + y + z)$. ***So*** $\left(\sqrt{2} - x\right)\left(x\sqrt{2} + 1\right) \geq 0 \Rightarrow x \geq x^2\sqrt{2} - \sqrt{2}$.

Similarly $y \geq y^2\sqrt{2} - \sqrt{2}; z \geq z^2\sqrt{2} - \sqrt{2}$

$$\Rightarrow x + y + z \geq \sqrt{2} \cdot (x^2 + y^2 + z^2 - 3) = \sqrt{2}\sum_{cyc}[(\sin A + \cos A)^2 - 1] =$$

$$= \sqrt{2} \cdot \sum_{cyc} \sin 2A = 4\sqrt{2} \cdot \sin A \sin B \sin C = \sqrt{2} \cdot \frac{a}{R} \cdot \frac{b}{R} \cdot \sin C = \frac{2S \cdot \sqrt{2}}{R^2}$$

$\Rightarrow \frac{1}{\sqrt{2}}(x + y + z) \geq \frac{2S}{R^2}$ ***or*** $LHS \geq RHS$. ***The equality doesn't hold, so:***

$$\sum_{cyc} \cos\left(\frac{\pi}{4} - A\right) > \frac{2S}{R^2}$$

(q.e.d.)

Solution 3 by Soumava Chakraborty – Kolkata – India

Given inequality $\Leftrightarrow \frac{1}{\sqrt{2}}\cos A + \frac{1}{\sqrt{2}}\sin A + \frac{1}{\sqrt{2}}\cos B + \frac{1}{\sqrt{2}}\sin B + \frac{1}{\sqrt{2}}\cos C + \frac{1}{\sqrt{2}}\sin C >$

$$> \frac{2}{R^2}(2R^2 \sin A \sin B \sin C)$$

$$\Leftrightarrow \sum \cos a + \sum \sin A > 4\sqrt{2}\left(2\sin\tfrac{A}{2}\cos\tfrac{A}{2}\right)\left(2\sin\tfrac{B}{2}\cos\tfrac{B}{2}\right)\left(2\sin\tfrac{C}{2}\cos\tfrac{C}{2}\right)$$

$$\Leftrightarrow 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$> 2\sqrt{2}\left(4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right) \Leftrightarrow 1 + x + y > 2\sqrt{2}xy, \text{ where}$$

$$x = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, y = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \Leftrightarrow 1 + x > y(2\sqrt{2}x - 1); 0 < x \le \frac{1}{2} \text{ and}$$

$$0 < y \le \frac{3\sqrt{3}}{2}$$

Case 1: $x \le \frac{1}{2\sqrt{2}} \Rightarrow 2\sqrt{2}x - 1 \le 0 \Rightarrow y(2\sqrt{2}x - 1) \le 0;\ (y > 0)$

$x > 0, 1 + x > 0$ ***and hence*** $1 + x > 0$ ***or any negative quantity***

$$\Rightarrow 1 + x > y(2\sqrt{2}x - 1)$$

Case 2: $x > \frac{1}{2\sqrt{2}} \Rightarrow 2\sqrt{2}x - 1 > 0$

$1 + x > y(2\sqrt{2}x - 1) \Leftrightarrow \frac{1+x}{2\sqrt{2}x-1} > y$. ***Now,*** $x \le \frac{1}{2} \Rightarrow 2\sqrt{2}x \le \sqrt{2} \Rightarrow 2\sqrt{2}x - 1 \le \sqrt{2} - 1$

$\Rightarrow \frac{1}{2\sqrt{2}x-1} \ge \frac{1}{\sqrt{2}-1};\ (2\sqrt{2}x - 1 > 0) = \sqrt{2} + 1$. ***Again,*** $1 + x > 1 + \frac{1}{2\sqrt{2}};\ x > \frac{1}{2\sqrt{2}}$

$$\frac{(1 + x)}{2\sqrt{2}x - 1} > \left(1 + \frac{1}{2\sqrt{2}}\right)(\sqrt{2} + 1) = \sqrt{2} + 1 + \frac{1}{2} + \frac{1}{2\sqrt{2}} = \frac{3}{2} + \frac{5\sqrt{2}}{4} > \frac{3\sqrt{3}}{2} \ge y$$

$\Rightarrow 1 + x > y(2\sqrt{2}x - 1)$ ***(Proved)***

Solution 4 by Myagmarsuren Yadamsuren – Darkhan – Mongolia

$$\frac{\sqrt{2}}{2}\cdot\left(\underbrace{\cos A + \cos B + \cos C}_{1+\frac{r}{R}} + \underbrace{(\sin A + \sin B + \sin C)}_{\frac{p}{R}}\right) = \frac{\sqrt{2}}{2}\left(1 + \frac{r}{R} + \frac{p}{R}\right) > \frac{2s}{R^2} \text{ (ASSURE)}$$

$$1 + \frac{r}{R} + \frac{p}{R} > \frac{2\sqrt{2}\cdot S}{R^2};\ R^2 + R\cdot r + R\cdot p > 2\sqrt{2}\cdot S;\ R \ge 2r; p \ge 3\sqrt{3}r; S = p\cdot r$$

$$R^2 + R\cdot r + R\cdot p \ge 2R\cdot r + Rr + 3\sqrt{3}\cdot R\cdot r > 2\sqrt{2}\cdot p\cdot r;\ (3 + 3\sqrt{3})\cdot R > 2\sqrt{2}\cdot p$$

$$\left(\frac{3+3\sqrt{3}}{2\sqrt{2}}\right)\cdot R > p \quad (*) - \text{(ASSURE)}$$

$$p = \frac{a + b + c}{2} = R\cdot(\sin A + \sin B + \sin C) \le R\cdot\frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}\cdot R$$

$$\frac{3\sqrt{3}}{2}\cdot R \ge p \quad (**)$$

$$(*); (**) \Rightarrow \frac{3+3\sqrt{3}}{2\sqrt{2}} > \frac{3\sqrt{3}}{2} \quad \text{(True)}\ p \le \frac{3\sqrt{3}}{2}\cdot R < \frac{3+3\sqrt{3}}{2\sqrt{2}}\cdot R$$

JP.042. Prove that in $\Delta\ ABC$:

$$\sum \frac{a^2(b^2+c^2-a^2)^3}{b^2c^2} \geq 64S^2(1-\cos^2 A-\cos^2 B-\cos^2 C)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Anas Adlany- El Jadida- Morroco

First, note that in any triangle. $1-\sum\cos^2(A) := 2\prod\cos(A)$

So, the original inequality is equivalent to $\sum a^2\frac{(b^2+c^2-a^2)^3}{(bc)^2} \geq 2\times 64\times S^2\prod\cos(A)$

Let's do it! From the cosine's law, we have $\cos(A) := \frac{b^2+c^2-a^2}{2bc}$

Now, if we use AM-GM, we shall obtain

$$\sum a^2\frac{(b^2+c^2-a^2)^3}{(bc)^2} \geq 3\sqrt[3]{\prod\left(a^2\frac{(b^2+c^2-a^2)^3}{(bc)^2}\right)} := 3abc\sqrt[3]{abc}\prod\cos(A)$$

Hence, it suffices to show that $2\times 64\times S^2 \geq 24abc\sqrt[3]{abc} \Longleftrightarrow 16S^2 \geq 3abc\sqrt[3]{abc}$

$\Longleftrightarrow (a+b+c)\prod(a+b-c) \geq 3abc\sqrt[3]{abc}$*. But this is true to AM-GM inequality and*

$\prod(a+b-c) \geq abc$*. Done!*

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triágulo ABC: $\sum\frac{a^2(b^2+c^2-a^2)^3}{b^2c^2} \geq 64S^2(1-\cos^2 A-\cos^2 B-\cos^2 C)$

Desde que: $A+B+C=\pi$

$$(1-\cos^2 A-\cos^2 B-\cos^2 C) = \sin^2 A-(1-\sin^2 B)-(1-\sin^2 C) =$$

$$= \sin^2 A+\sin^2 B+\sin^2 C-2$$

$$\Rightarrow (1-\cos^2 A-\cos^2 B-\cos^2 C) = \sin^2 A+\sin^2 B+\sin^2 C-2 =$$

$= 2\cos A\cos B\cos C$*. Además:*

$$b^2+c^2-a^2 = 4S\cot A,\ a^2+c^2-b^2 = 4S\cot B,\ a^2+b^2-c^2 = 4S\cot C,$$

$S = \frac{abc}{4R}$*. En un triángulo* ABC: $\sin A\sin B\sin C > 0 \wedge 1-8\cos A\cos B\cos C \geq 0$

$$\Rightarrow \sum\frac{a^2}{b^2c^2}64S^3\cot^3 A \geq 64S^2(2\cos A\cos B\cos C) \Rightarrow$$

$$\Rightarrow \sum \frac{a^2}{b^2c^2} S\, cot^3 A \geq 2\cos A \cos B \cos C \Rightarrow \sum \frac{a^2}{b^2c^2} S\, cot^3 A \geq 2\cos A \cos B \cos C \Rightarrow$$

$$\Rightarrow \sum \frac{a^3}{4Rbc} \cdot \frac{\cos^3 A}{\sin^3 A} \geq 2\cos A \cos B \cos C \Rightarrow \sum \frac{\cos^3 A}{2\sin B \sin C} \geq 2\cos A \cos B \cos C \Rightarrow$$

$$\Rightarrow 4\cos^3 A \sin A + 4\cos^3 B \sin B + 4\cos^3 C \sin C \geq 2\sin 2A \sin 2B \sin 2C$$

$$\Rightarrow \sin 2A\,(1+\cos 2A) + \sin 2B(1+\cos 2B) + \sin 2c\,(1+\cos 2C) \geq$$

$$\geq 2\sin 2A \sin 2B \sin 2C$$

$$\Rightarrow (\sin 2A + \sin 2B + \sin 2C) + (0,5)(\sin 4A + \sin 4B + \sin 4C) \geq$$

$$\geq 2\sin 2A \sin 2B \sin 2C$$

$$\Rightarrow 4\sin A \sin B \sin C - 2\sin 2A \sin 2B \sin 2C \geq 2\sin 2A \sin 2B \sin 2C \rightarrow$$

$$\rightarrow 4\sin A \sin B \sin C\,(1 - 8\cos A \cos B \cos C) \geq 0$$

Solution 3 by Soumava Chakraborty – Kolkata – India

$$\frac{a^2(b^2+c^2-a^2)^3}{b^2c^2} = \frac{a^2(2bc\cos A)^3}{b^2c^2} = 8a^2bc\cos^3 A$$

$$S = \Delta$$

$$= 8(4R^2\sin^2 A)(4R^2\sin B \sin C)\cos^3 A = 128R^4(\sin A \sin B \sin C)\sin A \cos^3 A$$

$$= 64R^2(2R^2\sin A \sin B \sin C)\sin A \cos^3 A = (64R^2 \cdot \sin A \cdot \cos^3 A)\Delta$$

Similarly, $\frac{b^2(c^2+a^2-b^2)^3}{c^2a^2} = (64R^2\sin B\cos^3 B)\Delta$

and $\frac{c^2(a^2+b^2-c^2)^3}{a^2b^2} = (64R^2\sin C\cos^3 C)\Delta$

given inequality $\Leftrightarrow R^2\sum(\sin A\cos^3 A) \geq \Delta(1-\cos^2 A-\cos^2 B-\cos^2 C)$

$$\sin A\cos^3 A = \frac{1}{4}(2\sin A\cos A)(2\cos^2 A) = \frac{1}{4}(\sin 2A)(1+\cos 2A)$$

$$= \frac{1}{4}(\sin 2A + \sin 2A\cos 2A) = \frac{1}{4}\sin 2A + \frac{1}{8}\sin 4A$$

$$R^2\sum(\sin A\cos^3 A) = \frac{R^2}{4}\sum \sin 2A + \frac{R^2}{8}\sum \sin 4A$$

$$\sum \sin 4A = \sin 4A + \sin 4B + \sin 4C$$

$$= 2\sin\big(2(A+b)\big)\cos\big(2(A-B)\big) + 2\sin 2C\cos 2C$$

$$= 2\sin(2\pi - 2C)\cos\big(2(A-B)\big) + 2\sin 2C\cos 2C$$

$$= -2\sin 2C\cos\big(2(A-B)\big) + 2\sin 2C\cos 2C = 2\sin 2C\left\{\cos 2C - \cos\big(2(A-B)\big)\right\}$$

$$= 4\sin 2C\sin(C+A-B)\sin(A-B-C) = 4\sin 2C\sin(\pi-2B)\sin(2A-\pi)$$

$$= -4\sin 2A\sin 2B\sin 2C$$

$$\frac{R^2}{8}\left(\sum \sin 4A\right) = \frac{R^2}{8}(-32\sin A\sin B\sin C\cos A\cos B\cos C)$$

$$= -2(2R^2\sin A\sin B\sin C)(\cos A\cos B\cos C) = -2\Delta\cos A\cos B\cos C$$

Again, $\frac{R^2}{4}(\sum \sin 2A) = R^2\sin A\sin B\sin C = \frac{\Delta}{2}$ ***given inequality*** $\Leftrightarrow$

$$\frac{\Delta}{2} - 2\Delta\cos A\cos B\cos C \geq \Delta(1-\cos^2 A - \cos^2 B - \cos^2 C)$$

$$\Leftrightarrow 1 - 4\cos A\cos B\cos C \geq 2 - (2\cos^2 A + 2\cos^2 B + 2\cos^2 C)$$

$$\Leftrightarrow -1 - 4\cos A\cos B\cos C \geq -(3 + \cos 2A + \cos 2B + \cos 2C)$$

$$= -3 + 1 + 4\cos A\cos B\cos C \Leftrightarrow 8\cos A\cos B\cos C \leq 1 \Leftrightarrow \cos A\cos B\cos C \leq \frac{1}{8}$$

which is true (proved)

JP.043. Let a, b, c, d be nonnegative real numbers such as $a+b+c+d=4$. Prove that:

a) $ab+bc+cd+da \leq 4$

b) $a^2bc+b^2cd+c^2da+d^2ab \leq 4$

c) $abc+bcd+cda+dab \leq 4$

d) $ab\sqrt{c}+bc\sqrt{d}+cd\sqrt{a}+da\sqrt{b} \leq 4$

Proposed by Nguyen Tuan Anh - Viet Nam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c, d ***números reales no negativos de tal manera que:*** $a+b+c+d=4$.

Probar que:

a) $ab+bc+cd+da \leq 4$

$$\Rightarrow b(a+c)+d(a+c) = (b+d)(a+c) \leq \frac{[(b+d)+(a+c)]^2}{4} \Rightarrow$$

$$\Rightarrow [(b+d)-(a+c)]^2 \geq 0$$

b) $a^2bc+b^2cd+c^2da+d^2ab \leq 4$

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Desde que:

$$\Rightarrow a^2bc + b^2cd + c^2da + d^2ab - (ab + cd)(ac + bd) = bd(b - d)(c - a)\dots \textbf{(A)}$$

$$\Rightarrow a^2bc + b^2cd + c^2da + d^2ab - (ac + bd)(ad + bc) = ac(a - c)(b - d)\dots \textbf{(B)}$$

Multiplicando (A) × (B):

$$\left(\sum a^2bc - (ab + cd)(ac + bd)\right)\left(\sum a^2bc - (ac + bd)(ad + bc)\right) =$$

$$= -(abcd)(b - d)^2(a - c)^2 \leq 0$$

Por la tanto se puede afirmar lo siguiente:

$$\Rightarrow a^2bc + b^2cd + c^2da + d^2ab \leq (ab + cd)(ac + bd) \text{ v}$$

$$a^2bc + b^2cd + c^2da + d^2ab \leq (ac + bd)(ad + bc)$$

Si: $a^2bc + b^2cd + c^2da + d^2ab \leq (ab + cd)(ac + bd) \leq$

$$\leq \frac{[(ab + cd) + (ac + bd)]^2}{4} \leq \frac{[(b + c)(a + d)]^2}{4} \leq \frac{16}{4} = 4$$

Si: $a^2bc + b^2cd + c^2da + d^2ab \leq (ac + bd)(ad + bc) \leq$

$$\leq \frac{[(ac + bd) + (ad + bc)]^2}{4} \leq \frac{[(c + d)(a + b)]^2}{4} \leq \frac{16}{4} = 4$$

c) $abc + bcd + cda + dab \leq 4$

Solo basta probar lo siguiente: $abc + bcd + cda + dab = ac(b + d) + bd(a + c) \leq$

$$\leq ab + bc + cd + da = (a + c)(b + d)$$

$$\Rightarrow 4ac(b + d) + 4bd(a + c) \leq (a + c)(b + d)[(b + d) + (a + c)]$$

$$\Rightarrow (b + d)^2(a + c) + (a + c)^2(b + d) \leq 4(a + c)bd + 4ac(b + d)$$

$$\Rightarrow (a + c)(b - d)^2 + (b + d)(a - c)^2 \geq 0$$

Por la tanto: $\Rightarrow abc + bcd + cda + dab \leq ab + bc + cd + da \leq 4$

d) $ab\sqrt{c} + bc\sqrt{d} + cd\sqrt{a} + da\sqrt{b} \leq 4$. ***Desde que:*** $a, b, c, d \geq 0$. ***Por:*** $MA \geq MG$

$$\Rightarrow ab\sqrt{c} + bc\sqrt{d} + cd\sqrt{a} + da\sqrt{b} \leq \frac{ab+abc}{2} + \frac{bc+bcd}{2} + \frac{cd+cda}{2} + \frac{da+dab}{2} \leq \frac{8}{2} = 4 \quad \textbf{(LQQD)}$$

JP.044. Let a, b, c, d be nonnegative real numbers such as

$a + b + c + d = 4.$

a) $a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \leq 4$

b) $\sqrt{abc} + \sqrt{bcd} + \sqrt{cda} + \sqrt{dab} \leq 4$

c) $\sqrt[n]{abc} + \sqrt[n]{bcd} + \sqrt[n]{cda} + \sqrt[n]{dab} \leq 4; (n \in \mathbb{N})$

d) $ab\sqrt[n]{c} + bc\sqrt[n]{d} + cd\sqrt[n]{a} + da\sqrt[n]{b} \leq 4; (n \in \mathbb{N})$

Proposed by Nguyen Tuan Anh – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c, d números reales no negativos, de tal manera que:

$$a + b + c + d = 4$$

a) $a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \leq 4$. *Desde que: $a, b, c, d \geq 0$. Por: MA ≥ MG*

$$\Rightarrow a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \leq \frac{a+abc}{2} + \frac{b+bcd}{2} + \frac{c+cda}{2} + \frac{d+dab}{2} \ldots (A)$$

Anteriormente ya se demostro lo siguiente: $\Rightarrow abc + bcd + cda + dab \leq 4$

$$\Rightarrow \text{Por lo tanto tenemos en (A): } \frac{a+abc}{2} + \frac{b+bcd}{2} + \frac{c+cda}{2} + \frac{d+dab}{2} \leq 4$$

$$\Rightarrow \text{Por transitividad: } a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \leq 4$$

b) $\sqrt{abc} + \sqrt{bcd} + \sqrt{cda} + \sqrt{dab} \leq 4 \Rightarrow$ *Por: MA ≥ MG*

$$\Rightarrow \sqrt{abc} + \sqrt{bcd} + \sqrt{cda} + \sqrt{dab} \leq \frac{ab+c}{2} + \frac{bc+d}{2} + \frac{cd+a}{2} + \frac{da+b}{2} \ldots (B)$$

Asimismo también ya se ha demostrado lo siguiente: $\Rightarrow ab + bc + cd + da \leq 4.$

Por lo tanto, por transitividad en (B): $\sqrt{abc} + \sqrt{bcd} + \sqrt{cda} + \sqrt{dab} \leq 4$

c) $\sqrt[n]{abc} + \sqrt[n]{bcd} + \sqrt[n]{cda} + \sqrt[n]{dab} \leq 4, n \in \mathbb{N}$

$$\text{Sea: } f(x) = x^{\frac{1}{n}} \ \forall\, x \in < 0, +\infty > \wedge \text{ considerando para: } n > 1$$

$$\text{Calculamos la primera y segunda derivada: } f'(x) = \frac{x^{\frac{1-n}{n}}}{n} \wedge f''(x) = \frac{x^{\frac{-2n+1}{n}}(-n+1)}{n^2} < 0$$

Desde que: $f''(x) < 0 \rightarrow$ entonces f es una función concava y se cumple:

Desigualdad Ponderada de Jensen:

$$f(abc) + f(bcd) + f(cda) + f(dab) \leq 4f\left(\frac{abc + bcd + cda + dab}{4}\right) =$$

$$= 4\left(\frac{abc+bcd+cda+dab}{4}\right)^{\frac{1}{n}} \le 4(1)^n = 4$$

d) $ab\sqrt[n]{c} + bc\sqrt[n]{d} + cd\sqrt[n]{a} + da\sqrt[n]{b} \le 4, n \in \mathbb{N}$

Siendo: $f(x) = x^{\frac{1}{n}}$ ***(Concava)*** $\forall\ x \in <0, +\infty> \wedge$ ***considerando para:*** $n > 1$

Desigualdad Ponderada de Jensen: $\Rightarrow abf(c) + bcf(d) + cdf(a) + daf(b) \le$

$$\le (ab+bc+cd+da)f\left(\frac{abc+bcd+cda+dab}{ab+bc+cd+da}\right) \le 4(1)^n = 4$$

Ya que: $f\left(\frac{abc+bcd+cda+dab}{ab+bc+cd+da}\right) = \left(\frac{abc+bcd+cda+dab}{ab+bc+cd+da}\right)^{\frac{1}{n}} \le (1)^n = 1$

Solution 2 by Soumava Chakraborty – Kolkata – India

$a, b, c, d \in \mathbb{R}^+ \cup \{0\}$***, then, given*** $a+b+c+d=4$,

a) $a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} \le 4$

b) $\sqrt{abc} + \sqrt{bcd} + \sqrt{cda} + \sqrt{dab} \le 4$

If 2 variables or 3 variables $=0$, LHS ***of (a) and*** LHS ***of (b) both*** $= 0 < 4$.

If 1 variable $=0$***, say*** $a=0$***, then,*** $b+c+d=4$***, with*** $0 < b, c, d < 4$.

So, $b\sqrt{cd} \overset{GM-AM}{\le} \frac{b(c+d)}{2} = \frac{b(4-b)}{2} \le \frac{2^2}{2} = 2 < 4 \left(\because \sqrt{b(4-b)} \le \frac{b+4-b}{2} = 2, as\ 4-b > 0\right)$

which proves (a). Also, if $a=0$, $\sqrt[3]{bcd} \le \frac{b+c+d}{3} = \frac{4}{3} \Rightarrow bcd \le \frac{64}{27} \Rightarrow \sqrt{bcd} \le \frac{8}{3\sqrt{3}} < 4$,

which proves (b). Now, let's consider $a, b, c, d > 0$

a) $a\sqrt{bc} + b\sqrt{cd} + c\sqrt{da} + d\sqrt{ab} = \sqrt{ab}\sqrt{ac} + \sqrt{bc}\sqrt{bd} + \sqrt{cd}\sqrt{ca} + \sqrt{da}\sqrt{bd}$

$$\overset{GM\le AM}{\le} \left(\frac{a+b}{2}\right)\left(\frac{a+c}{2}\right) + \frac{(b+c)(b+d)}{4} + \frac{(c+d)(c+a)}{4} + \frac{(d+a)(b+d)}{4}$$

$$= \frac{(a+c)}{4}(a+b+c+d) + \frac{(b+d)}{4}(b+c+d+a)$$

$$= \frac{(a+b+c+d)^2}{4} = \frac{16}{4} = 4 \text{ (equality at } a=b=c=d=1\text{)}$$

(Proved)

b) $\sqrt{abc} + \sqrt{bcd} + \sqrt{cda} + \sqrt{dab} = \sqrt{bc}(\sqrt{a} + \sqrt{d}) + \sqrt{da}(\sqrt{b} + \sqrt{c})$

$$\overset{GM\le AM}{\le} \left(\frac{b+c}{2}\right)\left(\frac{a+1}{2} + \frac{d+1}{2}\right) + \left(\frac{d+a}{2}\right)\left(\frac{b+1}{2} + \frac{c+1}{2}\right)$$

$$=\frac{2(ab+bd+ac+cd)+2(a+b+c+d)}{4}$$

$$=\frac{2(a+d)(b+c)+8}{4}\leq\frac{2(4)+8}{4}=\frac{16}{4}=4$$ ***(equality at** $a=b=c=d=1$**)***

$$\left(\because\sqrt{(a+d)(b+c)}\overset{GM\leq AM}{\leq}\frac{a+d+b+c}{2}=\frac{4}{2}=2\right)$$

(Proved)

$a,b,c,d\in\mathbb{R}^+\cup\{0\}$ ***and*** $a+b+c+d=4$***. Then,***

c) $\sqrt[n]{abc}+\sqrt[n]{bcd}+\sqrt[n]{cda}+\sqrt[n]{dab}\leq 4$

d) $ab\sqrt[n]{c}+bc\sqrt[n]{d}+cd\sqrt[n]{a}+da\sqrt[n]{b}\leq 4$

If exactly 2 or 3 variables $=0$, LHS ***of c), d)*** $=0<4$

If exactly 1 variable, say $a=0$***, then*** $b+c+d=4$

Let us first prove c) for $a=0$ ***and*** $0<b,c,d<4$

Case 1 : $n=1$

$$\sqrt[3]{bcd}\overset{G\leq A}{\leq}\frac{b+c+d}{3}=\frac{4}{3}\Rightarrow bcd\leq\frac{64}{27}<4$$

Case 2: $n=2$

$$\sqrt{bcd}\leq\frac{8}{3\sqrt{3}}<4$$

Case 3: $n=3$

$$\sqrt[3]{bcd}\leq\frac{4}{3}<4$$

Case 4: $n\geq 4$

$$\sqrt[n]{bcd}=\sqrt[n]{bcd\cdot\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{n-3}}$$

$$\overset{GM\leq AM}{\leq}\frac{b+c+d+n-3}{n}=\frac{4+n-3}{n}=\frac{1+n}{n}=1+\frac{1}{n}<4$$

Let us now prove d) for $a=0$ & $0<b,c,d<4$

$$bc\left(d^{\frac{1}{n}}\right)\overset{GM\leq AM}{\leq}\frac{(b+c)^2}{4}\left(d^{\frac{1}{n}}\right)=\frac{(4-d)^2\left(d^{\frac{1}{n}}\right)}{4}(\because b+c=4-d)$$

$$=\frac{(4-d)^{\frac{1}{n}}d^{\frac{1}{n}}\cdot(4-d)^{2-\frac{1}{n}}}{4}\quad\textit{(1)}$$

Now, $\sqrt{(4-d)d} \overset{GM\le AM}{\le} \frac{4+d+d}{2} = 2 \Rightarrow (4-d)d \le 4 \Rightarrow (4-d)^{\frac{1}{n}} d^{\frac{1}{n}} \le 4^{\frac{1}{n}}$ ***(i) (ii)***

Also $4-d < 4 (\because d > 0) \Rightarrow (4-d)^{2-\frac{1}{n}} < 4^{2-\frac{1}{n}} \left(\because 2 - \frac{1}{n} \ge 1\right)$

(i) × ***(ii)*** $\Rightarrow (4-d)^{\frac{1}{n}} d^{\frac{1}{n}} (4-d)^{2-\frac{1}{n}} < \left(4^{\frac{1}{n}}\right)\left(4^{2-\frac{1}{n}}\right) \Rightarrow \frac{(4-d)^{\frac{1}{n}} d^{\frac{1}{n}} (4-d)^{2-\frac{1}{n}}}{4} \overset{(2)}{<} \frac{4^2}{4} = 4$

$$\Rightarrow bc\left(d^{\frac{1}{n}}\right) \overset{(1)}{\le} \frac{(4-d)^{\frac{1}{n}} d^{\frac{1}{n}} (4-d)^{2-\frac{1}{n}}}{4} < 4$$

Hence, c), d) are proved for $a = 0$ ***and*** $0 < b, c, d < 4$

$\Rightarrow$ ***c), d) holds true if exactly 1 variable*** $= 0$

Let us now consider $0 < a, b, c, d < 4$***. Let us first prove (c)***

Case 1: $n = 1$

$$abc + bcd + cda + dab = bc(a+d) + da(b+c)$$

$$\le \frac{(b+c)^2(a+d)}{4} + \frac{(d+a)^2(b+c)}{4} \; (GM \le AM)$$

$$= \frac{(b+c)\{(b+c)(a+d)\} + (d+a)\{(d+a)(b+c)\}}{4}$$

$$\le \frac{4(b+c) + 4(d+a)}{4}\left(\because \sqrt{(b+c)(a+d)} \le \frac{b+c+d}{2} = \frac{4}{2} = 2\right)$$

$$= \frac{4(a+b+c+d)}{4} = a+b+c+d = 4$$

Case 2: $n = 2 \Rightarrow$ ***given inequality is:***

$\sqrt{abc} + \sqrt{bcd} + \sqrt{cda} + \sqrt{dab} \le 4$***, which is inequality (b), which was proved earlier.***

Case 3: $n = 3$

$$\sqrt[3]{abc} + \sqrt[3]{bcd} + \sqrt[3]{cda} + \sqrt[3]{dab}$$

$$\overset{GM\le AM}{\le} \frac{a+b+c}{3} + \frac{b+c+d}{3} + \frac{c+d+a}{3} + \frac{d+a+b}{3} = \frac{3(a+b+c+d)}{3} = 4$$

Case 4: $n \ge 4$

$$\left.\begin{array}{l}\sqrt[n]{abc}=\sqrt[n]{abc\cdot\underbrace{1\cdot1\cdot1\dots1}_{n-3}}\overset{GM\le AM}{\le}\dfrac{a+b+c+n-3}{n}\\ \sqrt[n]{bcd}=\sqrt[n]{bcd\cdot\underbrace{1\cdot1\cdot1\dots1}_{n-3}}\overset{G\le A}{\le}\dfrac{b+c+d+n-3}{n}\\ \sqrt[n]{cda}=\sqrt[n]{cda\cdot\underbrace{1\cdot1\cdot1\dots1}_{n-3}}\overset{G\le A}{\le}\dfrac{c+d+a+n-3}{n}\\ \sqrt[n]{dab}=\sqrt[n]{dab\cdot\underbrace{1\cdot1\cdot1\dots1}_{n-3}}\overset{G\le A}{\le}\dfrac{d+a+b+n-3}{n}\end{array}\right\}$$

$$\sqrt[n]{abc}+\sqrt[n]{bcd}+\sqrt[n]{cda}+\sqrt[n]{dab}\le\frac{3(a+b+c+d)+4n-12}{n}=\frac{4n}{n}=4$$

Let us prove (d) for $0<a,b,c,d<4$

Case 1: $n=1\Rightarrow$ ***(d) becomes*** $abc+bcd+cda+dab\le4$

$$abc+bcd+cda+dab$$

$$\le bc(a+d)+da(b+c)\overset{G\le A}{\le}\frac{(b+c)^2(a+d)}{4}+\frac{(d+a)^2(b+c)}{4}$$

$$=\left(\frac{b+c}{4}\right)\{(b+c)(a+d)\}+\frac{(d+a)}{4}\{(b+c)(d+a)\}$$

$$\le\frac{b+c}{4}\cdot4+\frac{d+a}{4}\cdot\left\{\because\sqrt{(b+c)(a+d)}\overset{G\le A}{\le}\frac{b+c+a+d}{2}=2\right\}$$

$$=a+b=c+d=4$$

Case 2: $n\ge2$

$$\sqrt[n]{c}=\sqrt[n]{c\cdot\underbrace{1\cdot1\cdot1\dots1}_{n-1}}\overset{G\le A}{\le}\frac{c+n-1}{n}$$

Similarly, $\sqrt[n]{d}\le\frac{d+n-1}{n},\sqrt[n]{a}\le\frac{a+n-1}{n},\sqrt[n]{b}\le\frac{b+n-1}{n}$

$$\therefore ab\sqrt[n]{c}+bc\sqrt[n]{d}+cd\sqrt[n]{a}+da\sqrt[n]{b}$$

$$\le\frac{abc+(n-1)ab+bcd+(n-1)bc+cda+(n-1)cd+dab+(n-1)da}{n}$$

$$=\frac{abc+bcd+cda+dab}{n}+\left(\frac{n-1}{n}\right)(ab+bc+cd+da)$$

$\le\frac{4}{n}+\frac{n-1}{n}(a+c)(b+d)$ ***(***$\because abc+bcd+cda+dab\le4$***, as proved in Case (1) above)***

$$\leq \frac{4}{n} + \left(\frac{n-1}{n}\right) 4 \left(\because \sqrt{(a+c)(b+d)} \overset{G\leq A}{\leq} \frac{a+c+b+d}{4} = 2\right) = \frac{4+4n-4}{n} = 4 \textit{ (Done)}$$

JP.045. If $a, b, c \geq \frac{1}{3}$ then:

$$\prod \left(a^2 + \sum a^3 + \sum ab - 3abc\right) \geq (a+b)^2(b+c)^2(c+a)^2$$

Proposed by Mihály Bencze – Romania

Solution by proposer

In inequality $x^3 + y^3 + z^3 - 3xyz \geq 0$ we take $x = a - \frac{1}{3}, y = b - \frac{1}{3}, z = c - \frac{1}{3}$

and we obtain $a^3 + b^3 + c^3 - 3abc \geq a^2 + b^2 + c^2 - ab - bc - ca$ or

$$a^3 + b^3 + c^3 + ab + bc + ca - 3abc \geq a^2 + b^2 + c^2 \text{ or}$$

$$a^2 + \sum a^3 + \sum ab - 3abc \geq 2a^2 + b^2 + c^2 = (a^2 + b^2) + (a^2 + c^2) \geq$$

$$\geq \frac{(a+b)^2}{2} + \frac{(a+c)^2}{2} \geq (a+b)(a+c) \text{ therefore}$$

$$\prod \left(a^2 + \sum a^3 + \sum ab - 3abc\right) \geq \prod (a+b)(a+c) = \prod (a+b)^2$$

SP.031. If $(a_n)_{n\geq 1} \subset (0, \infty)$ is a sequence that

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in (0, \infty)$$

find:

$$\Omega = \lim_{n\to\infty} \left(\sqrt[2n+2]{a_{n+1}} - \sqrt[2n]{a_n}\right)\sqrt{n}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution by Soumitra Mandal – Kolkata – India

Theorem: Let $(t, a) \in \mathbb{R}_+ \times \mathbb{R}_+^*$. If $(a_n)_{n\geq 1} \in S(\mathbb{R}_+^*)$ is a $B - (t+1, a)$ sequence, then $\left(\sqrt[n]{a_n}\right)_{n\geq 1}$ is a $L - \left(t, a(t+1), e^{-(t+1)}\right)$ sequence.

$$\lim_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} = a \in (0, \infty)$$

$$\Omega = \lim_{n\to\infty} \left(\sqrt[2n+2]{a_{n+1}} - \sqrt[2n]{a_n}\right)\sqrt{n} = \left\{\lim_{n\to\infty} \left(\sqrt[2n+2]{a_{n+1}} - \sqrt[2n]{a_n}\right)\right\}\left(\lim_{n\to\infty} \sqrt{n}\right)$$

$$= \{\lim_{n\to\infty}(\sqrt[n+1]{c_{n+1}} - \sqrt[n]{c_n})\}(\lim_{n\to\infty}\sqrt{n}) \text{ where } c_n = \sqrt{a_n} \text{ for all } n \ge 1$$

$$\lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n} = \left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)\left(\lim_{n\to\infty}\sqrt{\frac{a_{n+1}}{n\cdot a_n}}\right) = \sqrt{a}\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)$$

Hence, $(c_n)_{n\ge1}$ *is* $B-\left(1,\sqrt{a}\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)\right)$ *sequence, so by the above theorem*

$(\sqrt[n]{c_n})_{n\ge1}$ *is a* $L-\left(0,\sqrt{a}\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)\cdot 1\cdot e^{-1}\right)$ *sequence.*

$$\Omega = \frac{\sqrt{a}}{e}(\lim_{n\to\infty}\sqrt{n})\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right) = \frac{\sqrt{a}}{e}\lim_{n\to\infty}\left(\sqrt{n}\times\frac{1}{\sqrt{n}}\right) = \frac{\sqrt{a}}{e} \text{ (Ans:)}$$

SP.032. If $(a_n)_{n\ge1}; (b_n)_{n\ge1} \subset (0,\infty)$ such that

$$\lim_{n\to\infty}\frac{a_n+1}{na_n} = a \in (0,\infty); \lim_{n\to\infty}\frac{b_{n+1}}{nb_n} = b \in (0,\infty)$$

find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[2n+2]{a_{n+1}\cdot b_{n+1}} - \sqrt[2n]{a_n\cdot b_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal – Kolkata – India

Theorem: *Let* $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. *If* $(a_n)_{n\ge1}\in S(\mathbb{R}_+^*)$ *is a* $B-(t+1,a)$ *sequence, then* $(\sqrt[n]{a_n})_{n\ge1}$ *is a* $L-(t,a(t+1),e^{-(t+1)})$ *sequence.*

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = b;\ \Omega = \lim_{n\to\infty}\left(\sqrt[2n+2]{a_{n+1}b_{n+1}} - \sqrt[2n]{a_nb_n}\right)$$

$$= \lim_{n\to\infty}(\sqrt[n+1]{c_{n+1}} - \sqrt[n]{c_n}), \text{ where } c_n = \sqrt{a_nb_n} \text{ for all } n\ge1$$

$$\lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n} = \lim_{n\to\infty}\frac{\sqrt{a_{n+1}b_{n+1}}}{n\cdot\sqrt{a_nb_n}} = \left(\lim_{n\to\infty}\sqrt{\frac{a_{n+1}}{n\cdot a_n}}\right)\left(\lim_{n\to\infty}\sqrt{\frac{b_{n+1}}{n\cdot b_n}}\right) = \sqrt{ab}$$

Hence $(c_n)_{n\ge1}$ *is a* $B-(1,\sqrt{ab})$ *sequence, so by the above theorem*

$(\sqrt[n]{c_n})_{n\ge1}$ *is a* $L-(0,\sqrt{ab}\cdot1\cdot e^{-1})$ *sequence i.e.* $L-\left(0,\frac{\sqrt{ab}}{e}\right)$ *sequence. So ,*$\Omega = \frac{\sqrt{ab}}{3}$

SP.033. Let be: $r,s\in[0,\infty)$; $(a_n)_{n\geq1}$; $(b_n)_{n\geq1}\subset(0,\infty)$ such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^r}=a\in(0,\infty);\ \lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^{s+1}}=b\in(0,\infty);\ x_n=\sum_{k=1}^{n}\frac{1}{k}$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}-\sqrt[n]{a_nb_n}\right)e^{-(r+s)x_n}\right)$$

Proposed by D. M. Bătinețu – Giurgiu – Romania

Solution by Soumitra Mandal – Kolkata – India

Theorem: Let $(t,a)\in\mathbb{R}_+^*\times\mathbb{R}_+^*$. If $<a_n>_{n\geq1}\in S(\mathbb{R}_+^*)$ is a $B-(t+1,a)$ sequence then $<\sqrt[n]{a_n}>_{n\geq1}$ is a $L-\left(t,a(t+1),e^{-(t+1)}\right)$ sequence.

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n}=a\in(0,\infty)\text{ and }\lim_{n\to\infty}\frac{b_{n+1}}{n^{s+1}\cdot b_n}=b\in(0,\infty)$$

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{a_{n+1}b_{n+1}}-\sqrt[n]{a_nb_n}\right)e^{-(r+s)x_n}\right)$$

$$=\left\{\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}b_{n+1}}-\sqrt[n]{a_nb_n}\right)\right\}\left(\lim_{n\to\infty}e^{-(r+s)x_n}\right)$$

Let $c_n=a_nb_n$ for all $n\geq1$

$$\lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n}=\left(\lim_{n\to\infty}n^{r+s}\right)\left(\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n}\right)\left(\lim_{n\to\infty}\frac{b_{n+1}}{n^{s+1}\cdot b_n}\right)=ab\left(\lim_{n\to\infty}n^{r+s}\right)$$

Hence $<c_n>_{n\geq1}$ is a $B-\left(1,ab(\lim_{n\to\infty}n^{r+s})\right)$ is a sequence. Hence the above theorem yields $<\sqrt[n]{c_n}>_{n\geq1}$ a $L-(0,ab(\lim_{n\to\infty}n^{r+s})\cdot1\cdot e^{-1})$ sequence.

$$\Omega=\frac{ab}{e}\left(\lim_{n\to\infty}n^{r+s}\right)\left(\lim_{n\to\infty}e^{-(r+s)x_n}\right)=\frac{ab}{e}\left(\lim_{n\to\infty}n^{r+s}\,e^{-(r+s)(\gamma_n+\ln n)}\right)$$

Where $\gamma_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n$ is Euler's Constant $=\frac{ab}{e^{(r+s)\gamma_n+1}}$ (Ans :)

SP.034. Let $f:(0,\infty)\to(0,\infty)$ be a continuous function such that:

$$lim_{x\to\infty}\frac{f(x+1)}{xf(x)}=a\in(0,\infty)\text{,and it does exists:}$$

$$lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x}\text{, find:}$$

$$\Omega=\lim_{x\to\infty}\left(\frac{x+1}{\left(f(x+1)\right)^{\frac{1}{2x+2}}}-\frac{x}{\left(f(x)\right)^{\frac{1}{2x}}}\right)\cdot\sqrt{x}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution by Marian Ursărescu – Romania

Let $a_n=f(n)\Rightarrow\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\wedge\exists\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}$. *We must calculate:*

$$\Omega=\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[2n+2]{a_{n+1}}}-\frac{n}{\sqrt[2n]{a_n}}\right)\sqrt{n}\quad(1)$$

$$\Omega=\lim_{n\to\infty}\left(e^{\ln\frac{n+1}{\sqrt[2n+2]{a_{n+1}}}}-e^{\ln\frac{n}{\sqrt[2n]{a_n}}}\right)\sqrt{n}=\lim_{n\to\infty}e^{\frac{n}{\sqrt[2n]{a_n}}}\left(e^{\ln\frac{n+1}{\sqrt[2n+2]{a_{n+1}}}-\ln\frac{n}{\sqrt[2n]{a_n}}}-1\right)\sqrt{n}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{a_n}}\cdot n\cdot\left(e^{\ln\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{a_n}}{\sqrt[2n+2]{a_{n+1}}}\right)}-1\right)\quad(2)$$

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{a_n}}=\lim_{n\to\infty}\frac{\sqrt[2n]{n^n}}{\sqrt[2n]{a_n}}=\sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}}}\overset{C.D.}{=}\sqrt{\lim_{n\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n}}=$$

$$=\sqrt{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{a_n(n+1)}{a_{n+1}}}==\sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{a_n\cdot n}{a_{n+1}}\cdot\frac{n+1}{n}}=\sqrt{\frac{e}{a}}\quad(3)$$

$$\lim_{n\to\infty}n\left(e^{\ln\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{a_n}}{\sqrt[2n+2]{a_{n+1}}}\right)}-1\right)==\lim_{n\to\infty}\frac{\left(e^{\ln\frac{\sqrt[2n]{a_n}}{\sqrt[2n+1]{a_{n+1}}}}-1\right)}{\ln\frac{\sqrt[2n]{a_n}}{\sqrt[2n+2]{a_{n+1}}}}\cdot\ln\frac{\sqrt[2n]{a_n}}{\sqrt[2n+2]{a_{n+1}}}=$$

$$=\lim_{n\to\infty}n\ln\sqrt{\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}}=\lim_{n\to\infty}\frac{n}{2}\ln\left(\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}\right)=$$

$$=\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}\right)^n=\frac{1}{2}\ln\left(\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}\cdot\sqrt[n+1]{a_{n+1}}\right)\right)$$

$$=\frac{1}{2}\ln\left(\lim_{n\to\infty}\frac{na_n}{a_{n+1}}\cdot\frac{1}{n}\sqrt[n+1]{a_{n+1}}\right)=$$

$$=\frac{1}{2}\ln\left(\lim_{n\to\infty}\frac{na_n}{a_{n+1}}\cdot\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\right)=\frac{1}{2}\ln\left(\frac{1}{a}\cdot 1\cdot\frac{a}{e}\right)=\frac{-1}{2}\quad (4)$$

For (1)+(2)+(3)+(4)$\Rightarrow \Omega=-\frac{1}{2}\sqrt{\frac{e}{a}}$

SP.035. Evaluate

$$\lim_{n\to\infty}\frac{\lfloor\sqrt{44}\rfloor+\lfloor\sqrt{4444}\rfloor+\cdots+\overbrace{\lfloor\sqrt{44\ldots44}\rfloor}^{2n\text{ digits }4}}{10^n}$$

where $\lfloor x\rfloor$ denotes the integer part of x.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Henry Ricardo - New York – USA

To simplify things typographically, we introduce the notation $(mm\ldots mm)_k$ to denote the k - digit number each of whose digits is m.

First we see that for any positive integer k

$$(44\ldots44)_{2k}=(44\ldots44)_k\cdot 10^k+(44\ldots44)_k$$

$$=(44\ldots44)_k\cdot(10^k+1)=4(11\ldots11)_k\cdot(9(11\ldots11)_k+2)$$

$$=36\cdot(11\ldots11)_k^2+8(11\ldots11)_k=(66\ldots66)_k^2+8(11\ldots11)_k$$

$$<(66\ldots66)_k^2+8(11\ldots11)_k+\frac{4}{9}=\left((66\ldots66)_k+\frac{2}{3}\right)^2$$

Thus $(66\ldots66)_k^2<(44\ldots44)_{2k}<\left((66\ldots66)_k+\frac{2}{3}\right)^2$*, implying that*

$(66\ldots66)_k<\sqrt{(44\ldots44)_{2k}}<(66\ldots66)_k+\frac{2}{3}$ *and so* $\lfloor\sqrt{(44\ldots44)_{2k}}\rfloor=(66\ldots66)_k$*.*

Now we have $\frac{\sum_{k=1}^n\lfloor\sqrt{(44\ldots44)_{2k}}\rfloor}{10^n}=\frac{6\cdot\sum_{k=1}^n(11\ldots11)_k}{10^n}=\frac{6\cdot\sum_{k=1}^n\left(\frac{10^k-1}{9}\right)}{10^n}$

$$=\frac{2}{3}\left(\frac{\sum_{k=1}^{n}10^k-n}{10^n}\right)=\frac{2}{3}\frac{\left(\frac{10^{n+1}-1}{9}-n\right)}{10^n}=\frac{2}{27}\left(\frac{10^{n+1}-1-9n}{10^n}\right)=\frac{2}{27}\left(10-\frac{1}{10^n}-\frac{9n}{10^n}\right)\to\frac{20}{27}\text{ as }n\to\infty.$$

SP.036. Let a, b, c be positive real numbers such that

$$3(a+b)(b+c)(c+a)\geq\frac{8}{\sqrt[8]{a^3+b^3+c^3}}$$

Prove that $a+b+c\geq\sqrt[3]{9}$.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Anas Adlany - El Jadida- Morroco

We have known that $(a+b+c)^3\coloneqq\sum a^3+3\prod(a+b)\geq\sum a^3+\frac{8}{\sqrt[8]{a^3+b^3+c^3}}$

$$\geq 9\sqrt[9]{\left(\sum a^3\right)\left(\frac{1}{\sqrt[8]{a^3+b^3+c^3}}\right)^8}$$ ***. Thus,*** $a+b+c\geq\sqrt[3]{9}$***. Hence proved.***

Solution 2 by Soumitra Mandal – Kolkata – India

$$3(a+b)(b+c)(c+a)\geq\frac{8}{\sqrt[8]{a^3+b^3+c^3}}$$

$$\Rightarrow\sum_{cyc}a^3+3\prod_{cyc}(a+b)\geq\frac{8}{\sqrt[8]{a^3+b^3+c^3}}+(a^3+b^3+c^3)$$

$$\geq(8+1)\sqrt[9]{\left\{\frac{1}{\sqrt[8]{a^3+b^3+c^3}}\right\}^8(a^3+b^3+c^3)}=9$$

$\Rightarrow(a+b+c)^3\geq 9\Rightarrow a+b+c\geq\sqrt[3]{9}$ ***(proved). Equality at*** $a=b=c=\frac{1}{\sqrt[3]{3}}$

SP.037. Compute the limit

$$\lim_{n\to\infty}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}n\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta}d\theta$$

Proposed by Kunihiko Chikaya – Tokyo – Japan

Solution by Mirza Uzair Baig-Lahore-Pakistan

It is easy to prove the following asymptotic expansions

$$n\ln\left(1+\frac{a}{n}\right)^b=\left(\frac{a}{n}\right)^b\left(\frac{a^2b(3b+5)}{24n}-\frac{ab}{2}+n+O(n^{-2})\right)$$

$$=\frac{a^{2+b}b(3b+5)}{24n^{1+b}}-\frac{a^{1+b}b}{2n^b}+a^bn^{1-b}+O(n^{-2-b})$$

$$\left(1+\frac{a}{n}\right)^b=1+\frac{ab}{n}+O(n^{-2}).$$

Now now that

$$n\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}=n\left(\frac{\tan(x)+\sec(x)}{n}\right)^{\cos(x)}+O(n^{-\delta})$$

$$\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}=1+\frac{\cos\theta\cot\theta}{n}+O(n^{-2})$$

$$\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta}=1+\frac{\sin\theta\sec^2\theta\cot\theta}{n}+O(n^{-2})$$

For $x\in\left(\frac{\pi}{4},\frac{\pi}{3}\right)$ we have, $n^{1-\cos(x)}\to\infty, n\to\infty$.

Thus limit is $+\infty$.

SP.038. If $x,y,z\in\mathbb{R}\setminus\{1\}$ and $n\in\mathbb{N}$ then:

$$\frac{1}{3}\sum_{cyclic}(x-2y-2z)\left(\frac{x^{n+1}-1}{x-1}\right)+\sqrt{x^2+y^2+z^2}\sum_{k=0}^{n}\sqrt{x^{2k}+y^{2k}+z^{2k}}\geq 0$$

Proposed by Mihály Bencze – Romania

Solution by proposer

First we show that if $a,b,c,x,y,z\in\mathbb{R}$ then:

$$ax+by+cz+\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)}\geq\frac{2}{3}(a+b+c)(x+y+z)\quad \textit{(1)}$$

Let us $t=\sqrt{\frac{x^2+y^2+z^2}{a^2+b^2+c^2}}, x=pt, y=qt, z=rt\Rightarrow a^2+b^2+c^2=p^2+q^2+r^2$ and (1)

becomes $ap+bq+cr+a^2+b^2+c^2\geq\frac{2}{3}(c+b+c)(p+q+r)$ or

$4(a+b+c)(p+q+r) \le \left((a+b+c)+(p+q+r)\right)^2$ *it suffices to prove that:*

$(a+b)^2+(b+q)^2+(c+r)^2 \ge \frac{1}{3}\left((a+p)+(b+q)+(c+r)\right)^2$. *This is clearly true.*

In (1) we take $a = x^k, b = y^k, c = z^k \Rightarrow$

$$\sum_{k=0}^{n}\left(x^{k+1}+y^{k+1}+z^{k+1}+\sqrt{(x^2+y^2+z^2)(x^{2k}+y^{2k}+z^{2k})}\right) \ge$$

$$\ge \frac{2}{3}\sum_k^n (x+y+z)(x^k+y^k+z^k) \text{ or}$$

$$\frac{1}{3}\sum_{cyclic}(x-2y-2z)\left(\frac{x^{n+1}-1}{x-1}\right)+\sqrt{x^2+y^2+z^2}\sum_{k=0}^{n}\sqrt{x^{2k}+y^{2k}+z^{2k}} \ge 0$$

SP.039. Prove that if $a,b,c \in (1,\infty)$ then:

$$e^{\left|\ln\frac{ab}{c}\right|}\cdot e^{\left|\ln\frac{ac}{b}\right|}\cdot e^{\left|\ln\frac{bc}{a}\right|}\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3 \ge 27$$

Proposed by Daniel Sitaru – Romania

Solution by Nguyen Phuc Tang – Hanoi – Vietnam

We have $\left|\ln\frac{ab}{c}\right|+\left|\ln\frac{ac}{b}\right|+\left|\ln\frac{bc}{a}\right| \ge \left|\ln\frac{ab}{c}+\ln\frac{ac}{b}+\ln\frac{bc}{a}\right| = \ln(abc)$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \frac{3}{\sqrt[3]{abc}} \quad \text{(AM-GM)}$$

$$LHS = e^{\left|\ln\frac{ab}{c}\right|+\left|\ln\frac{ac}{b}\right|+\left|\ln\frac{bc}{a}\right|}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3 \ge e^{\ln(abc)}\cdot\frac{27}{abc} \ge abc\cdot\frac{27}{abc} = 27$$

Equality holds if $a=b=c$.

SP.040. Prove that if $a,b,c \in (\sqrt{3},\infty)$ then:

$$\frac{\ln(bc)}{\ln(ea^2)}+\frac{\ln(ac)}{\ln(eb^2)}+\frac{\ln(ab)}{\ln(ec^2)} \ge 2\sum\frac{\ln c}{1+2\sqrt{\ln a\ln b}}$$

Proposed by Daniel Sitaru – Romania

Solution by Nguyen Phuc Tang – Hanoi – Vietnam

We have $\ln a \ge 1, \ln b \ge 1, \ln c \ge 1$. *The given inequality is equivalent to*

$$\frac{\ln b+\ln c}{1+2\ln a}+\frac{\ln a+\ln c}{1+2\ln b}+\frac{\ln b+\ln a}{1+2\ln c}\ge\sum\frac{\ln c}{1+2\sqrt{\ln a\ln b}}$$

$$\Leftrightarrow\sum(\ln c)\left(\frac{1}{1+2\ln a}+\frac{1}{1+2\ln b}-\frac{2}{1+2\sqrt{\ln a\ln b}}\right)\ge 0\quad (*)$$

(*) is true, by the well – known inequality

$\frac{1}{1+x^2}+\frac{1}{1+y^2}\ge\frac{2}{1+xy}$ ***for all*** $x,y>0$ ***&*** $xy\ge 1$***. Equality holds if*** $a=b=c$***.***

SP.041. Let be $f:[0,1]\to\mathbb{R}$, f continuous on $[0,1]$. Compute:

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left((n-k+1)^2\sum_{j=1}^{k}f\left(\frac{j}{n}\right)\right)}{n^2(n+1)(2n+1)}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash – New Delhi – India

$$\sum_{k=1}^{n}(n-k+1)^2\sum_{j=1}^{k}f\left(\frac{j}{n}\right)=$$

$$=n^2f\left(\frac{1}{n}\right)+(n-1)^2\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)\right]+(n-2)^2\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)\right]+\cdots$$

$$\ldots+1^2\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right]=$$

$$=\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\left[1^1+2^2+\cdots+(n-k+1)^2\right]\quad (1)$$

We know $\frac{1}{3}k^3<1^2+2^2+\cdots+k^2<\frac{1}{3}(k+1)^3$

$$\frac{1}{3}(n-k+1)^3<\sum_{j=1}^{n-k+1}j^2<\frac{1}{3}(n-k+2)^3$$

Using (1), we get $\sum_{k=1}^{n}\frac{1}{3}\frac{(n-k+1)^3f\left(\frac{k}{n}\right)}{n^2(n+1)(2n+1)}<J<\sum_{k=1}^{n}\frac{1}{3}\frac{(n-k+2)^3f\left(\frac{k}{n}\right)}{n^2(n+1)(2n+1)}$ ***(2)***

When $J=\frac{\sum_{k=1}^{n}\left((n-k+1)^2\sum_{j=1}^{k}f\left(\frac{j}{n}\right)\right)}{n^2(n+1)(2n+1)}$***. Now,*** $\sum_{k=1}^{n}\frac{1}{3}\cdot\frac{(n-k+1)^3f\left(\frac{k}{n}\right)}{n^2(n+1)(2n+1)}=$

$$=\frac{1}{6}\sum_{k=1}^{n}\frac{1}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)}\left(1+\frac{1}{n}-\frac{k}{n}\right)^3f\left(\frac{k}{n}\right)$$

$$=\frac{1}{6}\cdot\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{n}\cdot\sum_{k=1}^{n}\left\{\left(1-\frac{k}{n}\right)^3 f\left(\frac{k}{n}\right)+\frac{3}{n}\left(1-\frac{k}{n}\right)^2 f\left(\frac{k}{n}\right)+\frac{3}{n^2}\left(1-\frac{k}{n}\right)f\left(\frac{k}{n}\right)+\frac{1}{n^3}f\left(\frac{k}{n}\right)\right\}$$

$$\to\frac{1}{6}\left[\int_0^1(1-x)^3 f(x)dx+(0)\int_0^1(1-x)^2 f(x)\,dx+(0)\int_0^1(1-x)f(x)\,dx+(0)\int_0^1 f(x)\,dx\right]=$$

$$=\frac{1}{6}\int_0^1(1-x)^3 f(x)dx$$

Similarly, expression on RHS of (2) approaches:

$$\frac{1}{6}\int_0^1(1-x)^3 f(x)dx;\ J\to\frac{1}{6}\int_0^1(1-x)^3 f(x)dx$$

as $n\to\infty$

SP.042. If $A,B\in M_2(R)$ then:

$$\det(xI_2+yAB+zBA)+\det(yI_2+zAB+xBA)+\det(zI_2+xAB+yBA)\ge$$
$$\ge(xy+yz+zx)\left(\left(1+Tr(AB)\right)^2+2\det(AB)-Tr(A^2B^2)\right)$$

for any $x,y,z\in\mathbb{R}$

Proposed by Mihály Bencze – Romania

Solution by proposer

With elementary calculus holds:

$$\det(xI_2+yAB+zBA)=x^2+x(y+z)Tr(AB)+(y^2+z^2)\det(AB)+$$

$+yz\left(\left(Tr(AB)\right)^2-Tr(A^2B^2)\right)$ ***and using the inequality*** $x^2+y^2+z^2\ge xy+yz+zx$

holds $\det(xI_2+yAB+zBA)+\det(yI_2+zAB+xBA)+\det(zI_2+xAB+yBA)=$

$$=(x^2+y^2+z^2)+2(xy+yz+zx)Tr(AB)+2(x^2+y^2+z^2)\det(AB)+$$

$$+(xy+yz+zx)\left(\left(Tr(AB)\right)^2-Tr(A^2B^2)\right)\ge(xy+yz+zx)\left(1+2Tr(AB)+Tr(AB)\right)^2+$$

$$+2\det(AB)-Tr(A^2B^2)=(xy+yz+zx)\left(\left(1+Tr(AB)\right)^2+2\det(AB)-Tr(A^2B^2)\right)$$

SP.043. If $x, y, z, a, b, c > 0$ then:

$$x^3y + y^3z + z^3x \geq \left(x^{3a+c}y^{3b+a}z^{3c+b}\right)^{\frac{1}{a+b+c}} + \left(y^{3a+c}z^{3b+a}x^{3c+b}\right)^{\frac{1}{a+b+c}} +$$
$$+\left(z^{3a+c}x^{3b+a}y^{3c+b}\right)^{\frac{1}{a+b+c}}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

We have: $ax^3y + by^3z + cz^3x \geq (a+b+c)\left((x^3y)^a(y^3z)(z^3x)^c\right)^{\frac{1}{a+b+c}} =$

$$= (a+b+c)\left(x^{3a+c}y^{3b+a}z^{3c+b}\right)^{\frac{1}{a+b+c}} \Rightarrow (a+b+c)\sum x^3y =$$

$$= \sum (ax^3y + by^3z + cz^3x) \geq (a+b+c)\sum \left(x^{3a+c}y^{3b+a}z^{3c+b}\right)^{\frac{1}{a+b+c}}$$

SP.044. In all convex quadrilateral $ABCD$ we have:

$$(-a+b+c+d)^\alpha + (a-b+c+d)^\alpha + (a+b-c+d)^\alpha + (a+b+c-d)^\alpha \geq$$
$$\geq \left(\frac{a+b+c}{3}+d\right)^\alpha + \left(\frac{b+c+d}{3}+a\right)^\alpha + \left(\frac{c+d+a}{3}+b\right)^\alpha + \left(\frac{d+a+b}{3}+c\right)^\alpha$$

for all $\alpha \geq 1$.

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\frac{x_1^\alpha + x_2^\alpha + x_3^\alpha}{3} \geq \left(\frac{x_1+x_2+x_3}{3}\right)^\alpha \text{ for all } x_1, x_2, x_3 > 0$$

If $x_1 = -a+b+c+d, x_2 = a-b+c+d, x_3 = a+b-c+d, x_4 = a+b+c-d$

and $x_1, x_2, x_3, x_4 > 0$ *then* $x_1^\alpha + x_2^\alpha + x_3^\alpha + x_4^\alpha = \sum_{cyclic}\frac{x_1^\alpha + x_2^\alpha + x_3^\alpha}{3} \geq \sum\left(\frac{x_1+x_2+x_3}{3}\right)^\alpha$

or $\sum(-a+b+c+d)^\alpha \geq \sum\left(\frac{a+b+c}{3}+d\right)^\alpha$ *etc.*

SP.045. If $a, b, c \in (0,1)$ then:

$$\frac{1}{\left(a(1-a^4)\right)^{4n}} + \frac{1}{\left(b(1-b^4)\right)^{4n}} + \frac{1}{\left(c(1-c^4)\right)^{4n}} \geq 3\left(\frac{3125}{256}\right)^n$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\text{We have: } \frac{1}{1-a^4} \geq \frac{5\sqrt[5]{5}}{4}a \Leftrightarrow a\sqrt[5]{4} = x, \frac{1}{5-x^4} \geq \frac{5x}{4} \Leftrightarrow$$

$$\Leftrightarrow (x-1)^2(x^3+2x^2+3x+4) \geq 0 \Rightarrow \frac{1}{a(1-a^4)} \geq \frac{5\sqrt[5]{5}}{4} \Rightarrow$$

$$\sum \frac{1}{\left(a(1-a^4)\right)^{4n}} \geq \sum \left(\frac{5\sqrt[5]{5}}{4}\right)^{4n} = 3\left(\frac{3125}{256}\right)^n$$

UP.031. If $A, B, C \in M_n(\mathbb{C})$; $n \geq 2$; $\det A \neq 0$; $AB = BA$; $AC = CA$; $A^2B + C = ABC$ then $BC = CB$.

Propsed by D.M. Bătinețu – Giurgiu – Romania

Solution by Soumitra Mandal – Kolkata – India

$$AB = BA; AC = CA \text{ and } A^2B + C = ABC$$

$$\text{Now } A^2B + C = ABC \Rightarrow (A^2B + C) \cdot B = ABCB \Rightarrow A^2B^2 + CB = ABCB$$

$$\Rightarrow A(AB)B + CB = ABCB \Rightarrow (AB)^2 + CB = ABCB \; [\because AB = BA]$$

$$\Rightarrow CB = AB(CB - AB) \ldots (1)$$

$$\text{Again, } A^2B + C = ABC \Rightarrow B \cdot (A^2B + C) = BABC \Rightarrow BA^2B + BC = BABC$$

$$\Rightarrow (BA)AB + BC = BABC \Rightarrow (BA)^2 + BC = BABC \; [\because AB = BA]$$

$$\Rightarrow BC = BA(BC - BA) \Rightarrow BC = AB(BC - AB)[\because AB = BA] \ldots (2)$$

$$\text{So, from (2) – (1): } BC - CB = AB(BC - CB)$$

$$\det(BC - CB) = \det(AB)\det(BC - CB) \Rightarrow \det(BC - CB)(1 - \det(AB)) = 0$$

Now, now $\det(AB) \neq 1$ *since if* $\det(AB) = 1$ *the* $AB = I_n \Rightarrow B = A^{-1}$ *so from relation* $A^2B + C = ABC$ *we would have got* $A = O_n$ *but* $\det(A) \neq 0$*, hence a contradiction. So,* $\det(AB) = 1$ *is neglected.*

$$\therefore \det(BC - CB) = 0 \Rightarrow BC = CB \text{ (proved)}$$

UP.032. If $x, y, z \in \mathbb{C}^*; A, B, C \in M_n(\mathbb{C}); n \geq 2; x^2A + B = xAB; y^2B + C = yBC;$

$z^2C + A = zCA$ then:

$$\frac{xy(yz+z)+1}{yz}A + \frac{yz(zx+2)+1}{zx}B + \frac{zx(xy+2)+1}{xy}C = 3ABC$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

***Theorem:** If $M, N \in M_n(\mathbb{C})$ such that $MN = I_n \Rightarrow NM = I_n$, then $MN = NM$*

$$x^2A + B = xAB \Rightarrow xAB - x^2 - B = O_n \Rightarrow xA(B - xI_n) - B + xI_n = xI_n$$

$$\Rightarrow (xA - I_n)(B - xI_n) = xI_4 \Rightarrow (B - xI_n)(xA - I_n) = xI_n \Rightarrow$$

$\Rightarrow xBA - B - x^2A + xI_n = I_n \Rightarrow xBA = x^2A + B \Rightarrow AB = BA$ *and similarly* $BC = CB$

and $AC = CA$. *We must show this:*

$$(x^2y^2z + 2x^2y + x)A + (xy^2z^2 + 2y^2z + y)B + (x^2yz^2 + 2z^2x + z)C = 3xyzABC \quad \textbf{(1)}$$

$$xAB = x^2A + B \Rightarrow xyzABC = x^2yzAC + yzBC = x^2y(z^2C + A) + z(y^2B + C) =$$

$$= x^2yz^2C + x^2yA + y^2zB + zC \quad \textbf{(2)}$$

$$yBC = y^2B + C \Rightarrow xyzABC = xy^2zAB + xzAC =$$

$$= y^2z(x^2A + B) + x(z^2C + A) = x^2y^2zA + y^2zB + xz^2C + xA \quad \textbf{(3)}$$

$$zCA = z^2C + A \Rightarrow xyz = xyz^2BC + xyAB = xz^2(y^2B + C) + y(x^2A + B) =$$

$$= xy^2z^2B + xz^2C + x^2yA + yB \quad \textbf{(4)}$$

From (2)+(3)+(4)$\Rightarrow 3xyzABC = (x^2y^2z + 2x^2y + x)A + (xy^2z^2 + 2y^2z + y)B +$

$+(x^2yz^2 + 2z^2x + z)C \Rightarrow$ ***(1) its true.***

UP.033. If $a, b > 0$ then:

$$2\left(\sqrt{a}+\sqrt{b}\right)^2 + \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \leq 3\sqrt[3]{\frac{a+b}{2}}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{\frac{2a+b}{3}}+\sqrt[3]{\frac{a+2b}{3}}\right)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

By the AM-GM inequality $\Rightarrow \begin{cases} \sqrt[3]{1\cdot\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}} \le \frac{1+\frac{2a}{a+b}+\frac{3a}{a+b+c}}{3} \\ \sqrt[3]{1\cdot 1\cdot\frac{3b}{a+b+c}} \le \frac{1+1+\frac{3b}{a+b+c}}{3} \\ \sqrt[3]{1\cdot\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}} \le \frac{1+\frac{2b}{a+b}+\frac{3c}{a+b+c}}{3} \end{cases}$

After addition holds $a+\sqrt[3]{ab\frac{a+b}{2}}+\sqrt[3]{abc} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}}$ **but**

$a+\sqrt{ab}+\sqrt[3]{abc} \le a+\sqrt[3]{ab\frac{a+b}{2}}+\sqrt[3]{abc}$ **therefore for** $a,b,c>0$ **holds:**

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}} \quad (1)$$

In (1) we take $c=a$ **and** $c=b$ **then:** $\frac{a+\sqrt{ab}+\sqrt[3]{a^2b}}{3} \le \sqrt[3]{a\frac{a+b}{2}\cdot\frac{2a+b}{3}}$ **and**

$\frac{a+\sqrt{ab}+\sqrt[3]{ab^2}}{3} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+2b}{3}}$ **and after addition holds the desired inequality.**

UP.034. Find the numbers $a,b,c\in\mathbb{N}^*$ knowing that:

$$\frac{a+1}{b}\in\mathbb{N},\frac{b+1}{c}\in\mathbb{N}\text{ and }\frac{c+1}{a}\in\mathbb{N}$$

Proposed by Gheorghe Alexe; George – Florin Șerban – Romania

Solution by SK Rejuan West Bengal – India

Case I: If three at a,b,c **are equal ie** $a=b=c\in\mathbb{N}^*$

$\frac{a+1}{b}=\frac{b+1}{c}=\frac{c+1}{a}=\frac{a+1}{a}$, **which belongs to** $\mathbb{N}$ **[given]**

Now, $\frac{a+1}{a}\in\mathbb{N}$ **if** $a=1\Rightarrow a=1=b=c$

$\therefore (1,1,1)=(a,b,c)$ **is a solution**

Case II: If two of them are equal. Let $a=b(\ne c)$

$\frac{a+1}{b},\frac{b+1}{c},\frac{c+1}{a}\in\mathbb{N}$ **[given]. Now,** $\frac{a+1}{b}=\frac{a+1}{a}$, **it belongs to** $\mathbb{N}$ **if** $a=1\Rightarrow a=1=b$

From $\frac{b+1}{c}\in\mathbb{N}$ **we get,** $\frac{1+1}{c}\in\mathbb{N}\ [\because b=1]\Rightarrow\frac{2}{c}\in\mathbb{N}\Rightarrow c=1$ **or** 2

but $c\ne(a=b)\Rightarrow c\ne 1\Rightarrow c=2$ **ie** $(a,b,c)=(1,1,2)$ **is a solution,**

similarly, by taking $a = c(\neq b)$ ***we get*** $(a,b,c) = (1,2,1)$ ***is a solution***

and by taking $a \neq (b = c)$ ***we get*** $(a,b,c) = (2,1,1)$ ***is a solution***

Case III: If three of them inequal, so in this case we get six possibilities ie

$a < b < c$ ***or*** $a < c < b$ ***or*** $b < c < a$ ***or*** $b < a < c$ ***or*** $c < a < b$ ***or*** $c < b < a$

Subcase I: When $a < b < c \Rightarrow (a+1) < b+1 < (c+1)$ ***(1)***

From (1) we get, $\frac{a+1}{b} < \frac{b+1}{b} = 1 + \frac{1}{b}$ $[\because b \in \mathbb{N}^*]$

$\because \frac{a+1}{b} \in \mathbb{N}$ ***and*** $\frac{a+1}{b} < 1 + \frac{1}{b} \Rightarrow \frac{a+1}{b} = 1 \Rightarrow a+1 = b$

$$\left[\because b \in N^* \therefore b \geq 1 \Rightarrow \frac{1}{b} \leq 1 \Rightarrow 1 + \frac{1}{b} \leq 2 \Rightarrow \frac{a+1}{b} < 2 \text{ and } \frac{a+1}{b} \in \mathbb{N}\right]$$

Subcase II: If $a < c < b \Rightarrow a+1 < c+1 < b+1$***. It is given*** $\frac{a+1}{b} \in \mathbb{N} \Rightarrow b|(a+1)$

Also by own assumption $a < b$ ***and by given condition*** $b|(a+1)$

$\Rightarrow a$ ***and*** b ***must be consecutive number in*** $\mathbb{N}^*$ ***and*** $a < b$

$\because a, b$ ***are consecutive numbers in*** $\mathbb{N}^*$ ***and*** $a < b$ ***so there exists no number*** c ***betwen*** a

and b which also belongs to $\mathbb{N}^*$ ***ie for*** $a < c < b$ ***and***

$\frac{a+1}{b}, \frac{b+1}{c}, \frac{c+1}{b} \in \mathbb{N}$ ***we get no solutions***

$\therefore$ ***No solutions for the case*** $a < c < b$ ***and*** $\frac{a+1}{b}, \frac{b+1}{c}, \frac{c+1}{a} \in \mathbb{N}$***.***

Subcase III: If $b < c < a$***. In this case, by similar calculation at subcase I we get***

$(a,b,c) = (3,1,2)$; $(a,b,c) = (5,3,4)$

Subcase IV: If $b < a < c$

In this case, by similar calculation at subcase II we get $\exists$ ***no solution.***

Subcase V: If $c < a < b$***, in this case by similar calculation at subcase I we get,***

$(a,b,c) = (2,3,1)$; $(a,b,c) = (4,5,3)$

Subcase VI:

If $c < b < a$***, in this case by similar calculation at subcase I we get,*** $\exists$ ***no solutions.***

Similarly from (1) we get, $\frac{b+1}{c} < \frac{c+1}{c} = 1 + \frac{1}{c}$

$\because \frac{b+1}{c} \in \mathbb{N}$ ***and*** $\frac{b+1}{c} < 1 + \frac{1}{c} \Rightarrow \frac{b+1}{c} = 1 \Rightarrow b+1 = c \Rightarrow a+1+1 = c \Rightarrow c = a+2$

[by similar asignment]

Now, $\frac{c+1}{a} \in \mathbb{N}$ *[given* $\Rightarrow \frac{a+2+1}{a} \in \mathbb{N} \Rightarrow \frac{a+3}{a} \in \mathbb{N} \Rightarrow 1 + \frac{3}{a} \in \mathbb{N}$

which is possible if $a = 1$ *or* 3

$$\left.\begin{array}{l} if\ a = 1 \Rightarrow b = 2 \Rightarrow c = 3 \\ if\ a = 3 \Rightarrow b = 4 \Rightarrow c = 5 \end{array}\right\} \begin{array}{l}(a,b,c) = (1,2,3) \\ (a,b,c) = (3,4,5)\end{array} \text{ is also solution}$$

Therefore the solutions are:

$$(a,b,c) \in \{(1,1,1)\ ,(1,1,2),(1,2,1),(2,1,1),(1,2,3),(3,4,5),(3,1,2),(5,3,4),(2,3,1),(4,5,3)\}$$

UP.035. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!} - \sqrt[3n]{(2n-1)!!}\right)\sqrt[3]{n^2}$$

Proposed by D. M. Bătinețu – Giurgiu – Romania

Solution by Soumitra Mandal – Kolkata – India

Theorem: *Let* $(t,a) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. *If* $(a_n)_{n\geq 1} \in S(\mathbb{R}_+^*)$ *is a* $B-(t+1,a)$ *sequence, then* $\left(\sqrt[n]{a_n}\right)_{n\geq 1}$ *is a* $L-\left(t, a(t+1), e^{-(t+1)}\right)$ *sequence.*

$$\Omega = \lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!} - \sqrt[3n]{(2n-1)!!}\right)\sqrt[3]{n^2} =$$

$$= \left\{\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)\right\}\left(\lim_{n\to\infty}\sqrt[3]{n^2}\right) \text{ where } a_n = \sqrt[3]{(2n-1)!!}$$

for all $n \geq 1$

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = \left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\left(\lim_{n\to\infty}\sqrt[3]{\frac{(2n+1)!!}{n\cdot(2n-1)!!}}\right) =$$

$$= \left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\left(\lim_{n\to\infty}\sqrt[3]{\frac{1}{n}\cdot\frac{(2n+1)!}{2^n\cdot n!}\cdot\frac{2^{n-1}\cdot(n-1)!}{(2n-1)!}}\right) =$$

$$= \left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\left(\lim_{n\to\infty}\sqrt[3]{2-\frac{1}{n}}\right) = \sqrt[3]{2}\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)$$

Hence, $(a_n)_{n\geq 1}$ *is a* $B-\left(1, \sqrt[3]{2}\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\right)$ *sequence so by the above theorem*

$$\left(\sqrt[n]{a_n}\right)_{n\geq 1} \text{ is a } L-\left(0, \sqrt[3]{2}\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\cdot 1\cdot e^{-1}\right) \text{ sequence}$$

$$\Omega = \frac{\sqrt[3]{2}}{3}\left(\lim_{n\to\infty}\sqrt[3]{n^2}\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right) = \frac{\sqrt[3]{2}}{e} \text{ (Ans:)}$$

UP.036. Let $(a_n)_{n\geq 1}$ be a positive real sequence such that:

$$\lim_{n\to\infty}(a_{n+1} - a_n) = r \in R_+^*, u, v \in R, u + v = 1.$$

We denote $a_n! = a_1 a_2 \dots a_n$, $G_n = (a_n!)^{\frac{1}{n}}, \forall\, n \in N^*$. Compute:

$$\lim_{n\to\infty}\left((n+1)^u \sqrt[n+1]{(G_{n+1}!)^v} - n^u \sqrt[n]{(G_n!)^v}\right)$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal – Kolkata – India

Let $(t, a) \in \mathbb{R}_+ \times \mathbb{R}_+^*$. If $< a_n >_{n\geq 1} \in S(\mathbb{R}_+^*)$ is a $B - (t+1, a)$ sequence then

$< \sqrt[n]{a_n} >_{n\geq 1}$ is a $L - \left(t, a(t+1), e^{-(t+1)}\right)$ sequence.

$\lim_{n\to\infty}(a_{n+1} - a_n) = r \in \mathbb{R}_+^*, u, v \in \mathbb{R}$ *and* $u + v = 1$

$a_n! = a_1 a_2 \dots a_n$ *and* $G_n = \sqrt[n]{a_n!}$ *where* $n \in \mathbb{N}^*$

$$\Omega = \lim_{n\to\infty}\left((n+1)^u \sqrt[n+1]{(G_{n+1})^v} - n^u \sqrt[n]{(G_n!)^v}\right)$$

$$= \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^{u(n+1)}(G_{n+1})^v} - \sqrt[n]{n^{nu}(G_n!)^v}\right)$$

Let $H_n = n^{nu}(G_n!)^v$ *where* $n \geq 1$ *and* $u + v = 1$

$$\lim_{n\to\infty}\frac{H_{n+1}}{n\cdot H_n} = \left(\lim_{n\to\infty}\frac{1}{n^{1-u}}\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nu}\right)\left(\lim_{n\to\infty}\frac{G_{n+1}!}{G_n!}\right)^v$$

$$= e^u\left(\lim_{n\to\infty}\frac{G_{n+1}}{n}\right)^v \text{ since, } G_{n+1}! = G_{n+1}G_n!$$

$$= e^u\left\{\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\sqrt[n+1]{a_{n+1}!}\right)\right\}^v = e^u\left\{\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{a_{n+2}!}{a_{n+1}!}\right)\right\}^v$$

[Cauchy D – Alembert's Theorem]

$$= e^u\left(\lim_{n\to\infty}\frac{a_{n+2}}{n}\right)^v = e^u\left(\lim_{n\to\infty}\frac{a_{n+3} - a_{n+2}}{n+1-n}\right)^v = e^u r^v$$

Hence, $< H_n >_{n\geq 1}$ is a $B - (1, e^u r^v)$ sequence. By the above theorem it yields that

$< \sqrt[n]{H_n} >_{n\geq 1}$ is a $L - (0, e^u r^v \cdot 1 \cdot e^{-1})$ sequence i.e. $L - (0, e^{u-1} r^v)$ sequence.

$$\Omega = e^{u-1} r^v \text{ (Ans :)}$$

UP.037. Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be positive real sequence such that

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a \in \mathbb{R}_+^*,$$

$$\lim_{n\to\infty}(b_{n+1}-b_n) = b \in \mathbb{R}_+^*, u, v \in \mathbb{R}$$

with $u + v = 1$. Calculate

$$\lim_{n\to\infty}\left(a_{n+1}^u \sqrt[n+1]{(b_1 b_2 \dots b_n b_{n+1})^v} - a_n^u \sqrt[n]{(b_1 b_2 \dots b_n)^v}\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal – Kolkata – India

***Theorem:* Let $(t, a) \in \mathbb{R}_+ \times \mathbb{R}_+^*$. If $< a_n >_{n\geq 1} \in S(\mathbb{R}_+^*)$ is a $B - (t+1, a)$ sequence then**

$< \sqrt[n]{a_n} >_{n\geq 1}$ is a $L - \left(t, a(t+1), e^{-(t+1)}\right)$ sequence.

$\lim_{n\to\infty}(a_{n+1}-a_n) = a \in \mathbb{R}_+^*$ *and* $\lim_{n\to\infty}(b_{n+1}-b_n) = b \in \mathbb{R}_+^*$ *and* $u + v = 1$

$$\Omega = \lim_{n\to\infty}\left(a_{n+1}^u \sqrt[n+1]{\left(\prod_{k=1}^{n+1} b_k\right)^v} - a_n^u \sqrt[n]{\left(\prod_{k=1}^{n} b_k\right)^v}\right)$$

Let $H_n = a_n^{nu}\left(\prod_{k=1}^n b_k\right)^v$ for all $n \geq 1$ and $u + v = 1$

$$\lim_{n\to\infty}\frac{H_{n+1}}{n^{u+1}\cdot H_n} = \left(\lim_{n\to\infty}\frac{1}{n^{2u+v}}\right)\left(\lim_{n\to\infty} a_{n+1}^u\right)\left(\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^{nu}\right)\left(\lim_{n\to\infty} b_{n+1}^v\right)$$

$$= \left(\lim_{n\to\infty}\frac{1}{n^{2u+v}}\right)\left(\lim_{n\to\infty}(a_{n+1}a_n)^u\right)\left(\lim_{n\to\infty} b_{n+1}^v\right)$$

Applying Cauchy – D'Alembert's theorem

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \left(\lim_{n\to\infty}\frac{a_{n+1}}{n}\right)^u\left(\lim_{n\to\infty}\frac{a_n}{n}\right)^u\left(\lim_{n\to\infty}\frac{b_{n+1}}{n}\right)^v$$

$$= \left(\lim_{n\to\infty}\frac{a_{n+2}-a_{n+1}}{n+1-n}\right)^u\left(\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n+1-n}\right)^u\left(\lim_{n\to\infty}\frac{b_{n+2}-b_{n+1}}{n+1-n}\right) = a^{2u}b^v$$

Hence $< H_n >_{n\geq 1}$ is $B - (u+2, a^{2u}b^v)$ sequence. Hence by the above theorem it yields $< \sqrt[n]{H_n} >_{n\geq 1}$ as a $L - \left(u+1, a^{2u}b^v(u+2)e^{-(u+2)}\right)$ sequence or

$L - \left(u+1, a^{2u}b^v(3u+2v)\cdot e^{-(3u+2v)}\right)$ sequence $\because u + v = 1$

$$\therefore \Omega = \frac{a^{2u}b^v(3u+2v)}{e^{3u+2v}} \quad \textit{(Ans :)}$$

UP.038. Let $(a_n)_{n\geq 1}$ be a positive real sequence such that

$$\lim_{n\to\infty}(a_{n+1}-a_n)=r\in\mathbb{R}_+^*$$

We denote $a_n!=a_1a_2\dots a_n,\ G_n=(a_n!)^{\frac{1}{n}},\ \forall\ n\in\mathbb{N}^*$. Evaluate

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{G_{n+1}!}}-\frac{n^2}{\sqrt[n]{G_n!}}\right)$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal – Kolkata – India

Theorem: Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. If $< a_n >_{n\geq 1}\in S(\mathbb{R}_+^*)$ is a $B-(t+1,a)$ sequence then

$$< \sqrt[n]{a_n} >_{n\geq 1} \text{ is a } L-\left(t,a(t+1),e^{-(t+1)}\right) \text{ sequence.}$$

$$\lim_{n\to\infty}(a_{n+1}-a_n)=r\in\mathbb{R}_+^*$$

$$a_n!=a_1a_2\dots a_n \text{ and } G_n=\sqrt[n]{a_n!} \text{ for all } n\in\mathbb{N}^*$$

$$\Omega=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{G_{n+1}!}}-\frac{n^2}{\sqrt[n]{G_n}}\right)=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{(n+1)^{2(n+1)}}{G_{n+1}!}}-\sqrt[n]{\frac{n^{2n}}{G_n!}}\right)$$

$$\text{Let } H_n=\frac{n^{2n}}{G_n!} \text{ for all } n\geq 1$$

$$\therefore\lim_{n\to\infty}\frac{H_{n+1}}{n\cdot H_n}=\left(\lim_{n\to\infty}n\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2(n+1)}\right)\left(\lim_{n\to\infty}\frac{G_n!}{G_{n+1}}\right)$$

$$=\left(\lim_{n\to\infty}n\right)e^2\left(\lim_{n\to\infty}\frac{1}{G_{n+1}}\right)=e^2\left(\lim_{n\to\infty}n\right)\left(\frac{1}{\lim\limits_{n\to\infty}\sqrt[n+1]{a_{n+1}}}\right)=e^2\left(\lim_{n\to\infty}n\right)\left(\frac{1}{\lim\limits_{n\to\infty}\frac{a_{n+2}!}{a_{n+1}!}}\right)$$

Applying Cauchy D-Alembert's Theorem

$$=e^2\left(\frac{1}{\lim\limits_{n\to\infty}\frac{a_{n+2}}{n}}\right)=e^2\left(\frac{1}{\lim\limits_{n\to\infty}\frac{a_{n+3}-a_{n+2}}{n+1-n}}\right)=\frac{e^2}{r}$$

hence $< H_n >_{n\geq 1}$ is a $B-\left(1,\frac{e^2}{r}\right)$ sequence. According to the above theorem it yields

$$< \sqrt[n]{H_n} >_{n\geq 1} \text{ is a } L-\left(0,\frac{e^2}{r}\cdot 1\cdot e^{-1}\right) \text{ sequence i.e. } L-\left(0,\frac{e}{r}\right) \text{ sequence.}$$

$$\Omega=\frac{e}{r}\ \textit{(Ans :)}$$

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UP.039. Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be positive real sequences with:

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a\in R_+^*, \lim_{n\to\infty}(b_{n+1}-b_n)=b\in R_+^*,$$

$$P_n=\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}, P_n!=P_1P_2\dots P_n,$$

$\forall n\in N^*, u,v\in R, u+v=1$. **Find**

$$\lim_{n\to\infty}\left(b_{n+1}^u\sqrt[n+1]{(P_{n+1}!)^v}-b_n^u\sqrt[n]{(P_n!)^v}\right)$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal – Kolkata – India

Theorem:* Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. *If* $<a_n>_{n\geq 1}\in S(\mathbb{R}_+^*)$ *is a

$B-(t+1,a)$ ***sequence then*** $<\sqrt[n]{a_n}>_{a\geq 1}$ ***is a***

$L-\left(t,a(t+1),e^{-(t+1)}\right)$ ***sequence.***

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a\in\mathbb{R}_+^*;\ \lim_{n\to\infty}(b_{n+1}-b_n)=b\in\mathbb{R}_+^*$$

$$P_n=\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}, P_n!=\prod_{k=1}^{n}P_k$$

$$\Omega=\lim_{n\to\infty}\left(b_{n+1}^u\sqrt[n+1]{(P_{n+1}!)^v}-b_n^u\sqrt[n]{(P_n!)^v}\right)\ \textit{where } u+v=1$$

$$=\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}^{u(n+1)}(P_{n+1}!)^v}-\sqrt[n]{b_n^{un}(P_n!)^v}\right)=\lim_{n\to\infty}\left(\sqrt[n+1]{c_{n+1}}-\sqrt[n]{c_n}\right)$$

where $c_n=b_n^{un}(P_n!)^v$ ***where*** $n\in\mathbb{N}^*$

$$\therefore\lim_{n\to\infty}\frac{c_{n+1}}{n^{u+1}\cdot c_n}=\lim_{n\to\infty}\frac{1}{n^{u+1}}\frac{b_{n+1}^{u(n+1)}(P_{n+1}!)^v}{b_n^{un}(P_n!)^v}=$$

$$=\left(\lim_{n\to\infty}\frac{b_{n+1}^u}{n^{2u+v}}\right)\left(\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right)^{nu}\right)\left(\lim_{n\to\infty}(P_{n+1})^v\right)==\left(\lim_{n\to\infty}\frac{b_{n+1}^u}{n^{2u+v}}\right)\left(\lim_{n\to\infty}b_n^u\right)\left(\lim_{n\to\infty}(P_{n+1})^v\right)=$$

Applying Cauchy – D Alembert's Theorem

$$=\left(\lim_{n\to\infty}\left(\frac{b_{n+1}}{n}\right)^u\right)\left(\lim_{n\to\infty}\left(\frac{b_n}{n}\right)^u\right)\left(\lim_{n\to\infty}\left(\frac{P_{n+1}}{n}\right)^v\right)=$$

$$=\left(\lim_{n\to\infty}\left(\frac{b_{n+2}-b_{n+1}}{n+1-n}\right)^{u}\right)\left(\lim_{n\to\infty}\left(\frac{b_{n+1}-b_{n}}{n+1-n}\right)^{u}\right)\left(\lim_{n\to\infty}\frac{1}{n^{v}}\right)\left(\lim_{n\to\infty}\left(\frac{\sum_{k=1}^{n}a_k^2}{n}\right)^{\frac{v}{2}}\right)=$$

$$=b^{2u}\left(\lim_{n\to\infty}\frac{1}{n^{v}}\right)\left(\lim_{n\to\infty}\left(\prod_{k=1}^{n}a_k\right)^{\frac{v}{n}}\right)$$

since

$$\lim_{n\to\infty}\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}=\lim_{n\to\infty}\sqrt[n]{a_1a_2\ldots a_n}=b^{2u}\left(\lim_{n\to\infty}\frac{1}{n^{v}}\right)\left(\lim_{n\to\infty}a_{n+1}^{v}\right)$$

Applying Cauchy – D' Alembert's Theorem

$$=b^{2u}\left(\lim_{n\to\infty}\left(\frac{a_{n+1}}{n}\right)^{v}\right)=b^{2u}\left(\lim_{n\to\infty}\left(\frac{a_{n+2}-a_{n+1}}{n+1-n}\right)^{v}\right)=b^{2u}a^{v}$$

Hence, $< c_n >_{n\ge1}$ ***constitues a*** $B-(u+2,b^{2u}a^{v})$ ***sequence by the above theorem*** $< \sqrt[n]{c_n} >_{n\ge1}$ ***constitues*** $L-\left(u+1,b^{2u}a^{v}(u+2)e^{-(u+2)}\right)$ ***sequence or*** $L-\left(u+1,b^{2u}a^{v}(3u+2v)e^{-(3u+2v)}\right)$ ***sequence.***

$$\Omega=\frac{b^{2u}a^{v}(3u+2v)}{e^{3u+2v}}\quad \textit{(Ans:)}$$

UP.040. Let $(a_n)_{n\ge1}$ be a positive real sequence such that

$$\lim_{n\to\infty}(a_{n+1}-a_n)=r\in R_+^*.$$

For any $x\in R_+^*$ we denote $M_n^{[x]}=\left(\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}\right)^{\frac{1}{x}}$ and $M_n^{[x]}!=M_1^{[x]}M_2^{[x]}\ldots M_n^{[x]},\forall\, n\in N^*$.

Find:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{M_{n+1}^{[x]}!}}-\frac{n^2}{\sqrt[n]{M_n^{[x]}!}}\right)$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal – Kolkata – India

Theorem: Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$***. If*** $< a_n >_{n\ge1}\in S(\mathbb{R}_+^*)$ ***is a***

$B-(t+1,a)$ *sequence then* $<\sqrt[n]{a_n}>_{n\geq 1}$ *is a* $L-\left(t,a(t+1),e^{-(t+1)}\right)$ *sequence.*

$$\lim_{n\to\infty}(a_{n+1}-a_n)=r\in\mathbb{R}_+^*$$

for any $x\in\mathbb{R}_+^*$ *we denote* $M_n^{[x]}=\sqrt[x]{\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}}$ *and*

$M_n^{[x]}!=M_1^{[x]}M_2^{[x]}\dots M_n^{[x]}$ *for all* $n\in\mathbb{N}^*$

$$\Omega=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{M_{n+1}^{[x]}}}-\frac{n^2}{\sqrt[n]{M_n^{[x]}}}\right)=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{(n+1)^{2(n+1)}}{M_{n+1}^{[x]}!}}-\sqrt[n]{\frac{n^{2n}}{M_n^{[x]}!}}\right)=$$

$$=\lim_{n\to\infty}\left(\sqrt[n+1]{c_{n+1}}-\sqrt[n]{c_n}\right)\text{ where } c_n=\frac{n^{2n}}{M_n^{[x]}!}\text{ for all } n\geq 1$$

$$\lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n}=\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{(n+1)^{2(n+1)}}{M_{n+1}^{[x]}!}\cdot\frac{M_n^{[x]}!}{n^{2n}}\right)=$$

$$=\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}\right)\left(\lim_{n\to\infty}(n+1)^2\right)\left(\lim_{n\to\infty}\frac{1}{M_{n+1}^{[x]}}\right)=$$

$$=e^2\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\right)\left(\lim_{n\to\infty}n\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt[x]{\frac{\sum_{k=1}^{n+1}a_k^x}{n+1}}}\right)=e^2\left(\lim_{n\to\infty}n\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{\prod_{k=1}^{n+1}a_k}}\right)$$

since

$$\lim_{n\to\infty}\sqrt[x]{\frac{\sum_{k=1}^{n}a_k^x}{n}}=\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}a_k}=e^2\left(\lim_{n\to\infty}n\right)\left(\lim_{n\to\infty}\frac{1}{\lim_{n\to\infty}a_{n+1}}\right)$$

Applying Cauchy – D Alembet's Theorem

$$=e^2\left(\frac{1}{\lim_{n\to\infty}\frac{a_{n+1}}{n}}\right)=\frac{e^2}{\lim_{n\to\infty}\frac{a_{n+2}-a_n}{n+1-n}}=\frac{e^2}{r}$$

hence, $<c_n>_{n\geq 1}$ *is a* $B-\left(0,\frac{e^2}{r}\right)$ *sequence and by the above theorem*

$<\sqrt[n]{c_n}>_{n\geq 1}$ *constitutes a* $L-\left(1,\frac{e^2}{r}\cdot e^{-1}\right)$ *sequence or,*

$L-\left(1,\frac{e}{r}\right)$ *sequence.* $\Omega=\frac{e}{r}$ *(Ans :)*

UP.041. Prove that:

$$\frac{3^{\frac{3}{2}}}{2}\cdot\sum_{0}^{\infty}\frac{(n!)^2}{(2n+1)!}=\pi$$

Proposed by Francis Fregeau – Quebec – Canada

Solution by proposer

Let $f(2n+1)=\int_0^\infty u^{2n+1}e^{-u^2}\,du$

We now consider another similar integral as a function of v such that:

$$[f(2n+1)]^2=\int_0^\infty u^{2n+1}e^{-u^2}\,du\cdot\int_0^\infty v^{2n+1}e^{-v^2}\,dv=\int_0^\infty\int_0^\infty (uv)^{2n+1}e^{-(u^2+v^2)}\,dudv$$

We now apply the change of variables: $u=r\cdot\cos\theta\,;v=r\cdot\sin\theta$

And our domain of integration is: $u\ge 0;v\ge 0\Rightarrow 0\le\theta\le\frac{\pi}{2};r\ge 0$

$$\Rightarrow[f(2n+1)]^2=\int_0^\infty\int_0^{\frac{\pi}{2}}[\sin(\theta)\cdot\cos(\theta)]^{2n+1}r^{4n+3}e^{-r^2}\,d\theta dr$$

$$=\int_0^{\frac{\pi}{2}}[\sin(\theta)\cdot\cos(\theta)]^{2n+1}\,d\theta\cdot f(2[2n+1]+1)$$

We now turn our attention back to $f(2n+1)$ and apply the substitution:

$$u^2=x\Rightarrow du=\frac{1}{2}\cdot x^{-\frac{1}{2}}dx$$

$$\therefore\int_0^\infty u^{2n+1}e^{-u^2}du=\frac{1}{2}\int_0^\infty x^n e^{-x}\,dx=\frac{\Gamma(n+1)}{2}=\frac{n!}{2}$$

$$\Rightarrow\int_0^{\frac{\pi}{2}}[\sin(\theta)\cdot\cos(\theta)]^{2n+1}\,d\theta=\frac{[f(2n+1)]^2}{f(2[2n+1]+1)}=\frac{1}{2}\cdot\frac{(n!)^2}{(2n+1)!}$$

Next:

$$\sum_0^\infty[\sin(\theta)\cdot\cos(\theta)]^{2n+1}=\frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2}$$

We consider: $\int \frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2} d\theta = -2\int \frac{\sin(2\theta)}{\sin^2(2\theta)-4} d\theta$

$$= -\int \frac{\sin(x)}{\sin^2(x)-4} dx = \int \frac{\sin(x)}{\cos^2(x)+3} dx$$

$$\cos(x) = y \Rightarrow dx = -\frac{dy}{\sin(x)}$$

$$\Rightarrow \int \frac{\sin(x)}{\cos^2(x)+3} dx = \int \frac{1}{3+y^2} dy = \frac{1}{\sqrt{3}}\arctan\left(\frac{y}{\sqrt{3}}\right)$$

Un-doing the substitutions yields:

$$\int \frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2} d\theta = \frac{\arctan\left(\frac{\cos(2\theta)}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2} d\theta = \frac{\pi}{3^{\frac{3}{2}}}$$

$$\sum_0^\infty \frac{1}{2}\cdot\frac{(n!)^2}{(2n+1)!} = \int_0^{\frac{\pi}{2}} \sum_0^\infty [\sin(\theta)\cdot\cos(\theta)]^{2n+1} = \int_0^{\frac{\pi}{2}} \frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2} d\theta = \frac{\pi}{3^{\frac{3}{2}}}$$

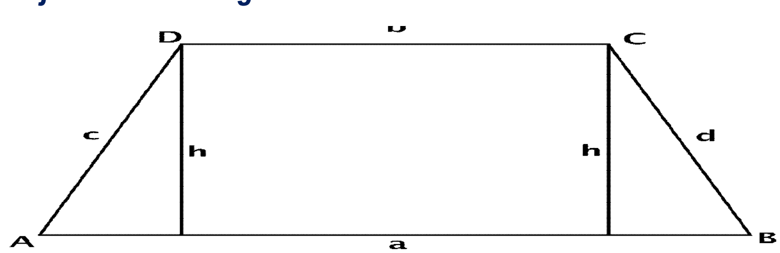
$$\therefore \frac{3^{\frac{3}{2}}}{2}\cdot\sum_0^\infty \frac{(n!)^2}{(2n+1)!} = \pi$$

UP.042. Let $ABCD$ be a trapeze where $AB \parallel CD$; $AB = a$; $CD = b$; $AD = c$; $BC = d$; $a > b$. Prove that

$$\text{Area }[ABCD] < \frac{(a+b)(a-b+c+d)^2}{16(a-b)}$$

Proposed by Daniel Sitaru – Romania

Solution by SK Rejuan - West Bengal – India



Let $ABCD$ be the trapeze and h be the hight of it. Area $[ABCD]=\frac{1}{2}h(a+b)$

Now, from picture, $h<c$ *and* $h<b$

$$\Rightarrow 2h<c+d=\frac{(a-b)(c+d)}{(a-b)}\ \textit{[as } a-b>0\textit{]}\Rightarrow 2h<\frac{(a-b)(c+d)}{(a-b)}=\frac{4(a-b)(c+d)}{4(a-b)}$$

$$\Rightarrow 2h<\frac{4(a-b)(c+d)}{(a-b)}<\frac{\{(a-b)+(c+d)\}^2}{4(a-b)}\ \textit{[by GM<AM]}\Rightarrow 2h<\frac{(a-b+c+d)^2}{4(a-b)}$$

$$\Rightarrow \frac{1}{4}(a+b)\cdot 2h<\frac{1}{4}(a+b)\frac{(a-b+c+d)^2}{4(a-b)}$$

$$[\because a+b>0]$$

$$\Rightarrow \frac{1}{2}h(a+b)<\frac{(a+b)(a-b+c+d)^2}{16(a-b)}\Rightarrow Area\,[ABCD]<\frac{(a+b)(a-b+c+d)^2}{16(a-b)}$$

[proved]

UP.043. Prove that in any $\Delta\,ABC$ we have:

$$2s+\sqrt{\sum(a^2+2ab\cos(A-B))}\geq\sum\sqrt{a^2+2ab\cos(A-B)+b^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Nguyen Phuc Tang – Hanoi – Vietnam

We have: $LHS-RHS=\sqrt{\sum(a^2+2ab\cos(A-B))}-(a+b+c)+2(a+b+c)-$

$$-\sum\sqrt{a^2+2ab\cos(A-B)+b^2}=$$

$$=\sum\frac{2ab[1-\cos(A-B)]}{\sqrt{a^2+2ab\cos(A-B)+b^2}+a+b}-\frac{\sum 2a\,[-\cos(A-B)+1]}{\sqrt{\sum(a^2+2ab\cos(A-B))}+(a+b+c)}$$

We prove that:

$$\sqrt{\sum(a^2+2ab\cos(A-B))}+a+b+c\geq\sqrt{(a^2+2ab\cos(A-B)+b^2}+a+b$$

$$\Leftrightarrow\sqrt{\sum(a^2+2ab\cos(A-B))}\geq\sqrt{(a^2+2ab\cos(A-B)+b^2}-c\quad (*)$$

$\oplus$ *case* $c\geq\sqrt{(a^2+2ab\cos(A-B))+b^2}$ *then (*) is true*

$\oplus$ *case* $c<\sqrt{(a^2+2ab\cos(A-B)+b^2}$

(*) $\Leftrightarrow 2bc\cos(B-C)+2ac\cos(A-C)\geq-2c\sqrt{(a^2+2ab\cos(A-B)+b^2)}$

$$-[b\cos(B-C)+a\cos(A-C)]\leq\sqrt{(a^2+2ab\cos(A-B))+b^2}\ (**)$$

if $b\cos(B-C)+a\cos(A-C)\geq 0\Rightarrow$ *(**) is true*

if $b\cos(B-C)+a\cos(A-C)<0\Rightarrow \sin B\cos(B-C)+\sin A\cos(A-C)<0$

$\Rightarrow \cos C\sin(A+B)+2\sin A\sin B\sin C<0\Rightarrow \cos C<0\Rightarrow C>A\ \&\ C>B$

(**) $\Leftrightarrow b^2(1-\cos^2(B-C))+c^2(1-\cos^2(A-C))+2ab[2\cos(A-B)-$

2cosA−CcosB−C≥0 (***)

(**) is true, because

$2\cos(A-B)-2\cos(A-C)\cos(B-C)=\cos(A-B)-\cos(A+B-2C)=$

$=2\sin(C-A)\sin(C-B)>0$*. Equality holds if* $a=b=c$*.*

UP.044. For all $n\in\mathbb{N}^*$ holds:

$$\left[\sqrt{n}+\sqrt{n+1}\right]+\left[\sqrt{n}+\sqrt{n+2}\right]+\left[\sqrt{n}+\sqrt{n+3}\right]=$$
$$=\left[\sqrt{4n+1}\right]+\left[\sqrt{4n+3}\right]+\left[\sqrt{4n+5}\right]$$

where $[\cdot]$ denote the integer part.

Proposed by Mihály Bencze – Romania

Solution by proposer

We prove that: $\sqrt{4n+1}\leq\sqrt{n}+\sqrt{n+1}<\sqrt{4n+2}$

$$\sqrt{4n+3}\leq\sqrt{n}+\sqrt{n+2}<\sqrt{4n+4}$$

$$\sqrt{4n+5}\leq\sqrt{n}+\sqrt{n+3}<\sqrt{4n+6}$$

1) If $\sqrt{4n+2},\sqrt{4n+4},\sqrt{4n+6}\notin\mathbb{N}$ *then:* $\left[\sqrt{4n+1}\right]=\left[\sqrt{4n+2}\right]$;

$\left[\sqrt{4n+3}\right]=\left[\sqrt{4n+4}\right];\left[\sqrt{4n+5}\right]=\left[\sqrt{4n+6}\right]$

2) If $\sqrt{4n+2},\sqrt{4n+4},\sqrt{4n+6}\in\mathbb{N}\Rightarrow\left[\sqrt{4n+2}\right]=\left[\sqrt{4n+1}\right]+1$;

$\left[\sqrt{4n+4}\right]=\left[\sqrt{4n+3}\right]+1;\left[\sqrt{4n+6}\right]=\left[\sqrt{4n+5}\right]+1$ *therefore*

$\left[\sqrt{n}+\sqrt{n+1}\right]=\left[\sqrt{4n+1}\right];\ \left[\sqrt{n}+\sqrt{n+2}\right]=\left[\sqrt{4n+3}\right];\ \left[\sqrt{n}+\sqrt{n+3}\right]=\left[\sqrt{4n+5}\right]$

After addion holds.

UP.045. Calculate:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\frac{1}{1+\sqrt[2]{\tan(x_1)}\sqrt[3]{\tan(x_2)}\dots\sqrt[n+1]{\tan(x_n)}}dx_1dx_2\dots dx_n$$

Proposed by Cornel Ioan Vălean – Romania

Solution by Hamza Mahmood – Lahore – Pakistan

Let

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{1}{1+\sqrt[2]{\tan(x_1)}\cdot\sqrt[3]{\tan(x_2)}\cdot\dots\cdot\sqrt[n]{\tan(x_{n-1})}\cdot\sqrt[n+1]{\tan(x_n)}}dx_1dx_2\dots dx_{n-1}dx_n$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\prod_{k=2}^{n+1}\sqrt[k]{\cos(x_{k-1})}}{\prod_{k=1}^{n+1}\sqrt[k]{\cos(x_{k-1})}+\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}dx_1dx_2\dots dx_{n-1}dx_n\dots(A)$$

Now by substitution $x_i\to\frac{\pi}{2}-x_i$*, we have:*

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}{\prod_{k=1}^{n+1}\sqrt[k]{\sin(x_{k-1})}+\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}dx_1dx_2\dots dx_{n-1}dx_n\dots(B)$$

Adding (A) and (B) gives:

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\prod_{k=2}^{n+1}\sqrt[k]{\cos(x_{k-1})}+\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}{\prod_{k=1}^{n+1}\sqrt[k]{\cos(x_{k-1})}+\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}dx_1dx_2\dots dx_{n-1}dx_n=$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(1)\,dx_1dx_2\dots dx_n\Rightarrow 2I=\left(\frac{\pi}{2}\right)^n\Rightarrow I=\frac{\pi^n}{2^{n+1}}$$