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PROBLEMS FOR JUNIORS

JP.436 In ΔABC the following relationship holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{18R}{ab+bc+ca}\cdot\sqrt[3]{w_a w_b w_c}\geq 6$$

Proposed by Alex Szoros-Romania

Solution 1 by proposer

Lemma. If $x,y,z>0$ then: $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{9\sqrt[3]{xyz}}{x+y+z}\geq 6;(1)$

Proof. We have: $3\left(\frac{x}{y}+\frac{y}{z}+zx\right)=\left(\frac{2x}{y}+\frac{y}{z}\right)+\left(\frac{2y}{z}+\frac{z}{x}\right)+\left(\frac{2z}{x}+\frac{x}{z}\right)\geq$

$$\geq 3\sqrt[3]{\frac{x}{y}\frac{x}{y}\frac{y}{z}}+3\sqrt[3]{\frac{y}{z}\frac{y}{z}\frac{z}{x}}+3\sqrt[3]{\frac{z}{x}\frac{z}{x}\frac{x}{z}}=3\sqrt[3]{\frac{x^2}{yz}}+3\sqrt[3]{\frac{y^2}{zx}}+3\sqrt[3]{\frac{z^2}{xz}}=$$

$$=\frac{3(x+y+z)}{\sqrt[3]{xyz}}$$

$$\frac{1}{3}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\geq\frac{x+y+z}{3\sqrt[3]{xyz}}+\frac{3\sqrt[3]{xyz}}{x+y+z}\geq 2\Rightarrow$$

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{9\sqrt[3]{xyz}}{x+y+z}\geq 6$$

Putting $x=h_b;\,y=h_a$and $z=h_c$, inequality (1) becomes as:

$$\frac{h_b}{h_a}+\frac{h_a}{h_c}+\frac{h_c}{h_b}+\frac{9\sqrt[3]{h_a h_b h_c}}{h_a+h_b+h_c}\geq 6\Rightarrow$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{9\sqrt[3]{h_a h_b h_c}}{h_a+h_b+h_c}\geq 6;(2)$$

But: $w_a w_b w_c\geq h_a h_b h_c$, so (2) becomes as:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{9\sqrt[3]{w_a w_b w_c}}{h_a+h_b+h_c}\geq 6;(3)$$

On the other hand, we have:

$$\sum_{cyc} h_a = \sum_{cyc} \frac{2F}{a} = 2F\sum_{cyc} \frac{1}{a} = \frac{2F}{abc}\sum_{cyc} ab = \frac{1}{2R}\sum_{cyc} ab\,;(4)$$

From (3) and (4) we get the conclusion:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{18R}{ab+bc+ca}\cdot\sqrt[3]{w_a w_b w_c}\geq 6$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The inequality in the statement can be rewritten as

$$:\ \frac{h_b}{h_a}+\frac{h_c}{h_b}+\frac{h_a}{h_c}+\frac{9\sqrt[3]{w_a w_b w_c}}{h_a+h_b+h_c}\geq 6.$$

Since $w_a \geq h_a$ *(And analogs), we have :* $\sqrt[3]{w_a w_b w_c} \geq \sqrt[3]{h_a h_b h_c}$

Also, by AM – GM inequality, we have : $\frac{h_b}{h_a}+\frac{h_b}{h_a}+\frac{h_a}{h_c}\geq 3\sqrt[3]{\frac{h_b}{h_a}\cdot\frac{h_b}{h_a}\cdot\frac{h_a}{h_c}}=\frac{3h_b}{\sqrt[3]{h_a h_b h_c}}$

Similarly, we have : $\frac{h_c}{h_b}+\frac{h_c}{h_b}+\frac{h_b}{h_a}\geq\frac{3h_c}{\sqrt[3]{h_a h_b h_c}}$ *and* $\frac{h_a}{h_c}+\frac{h_a}{h_c}+\frac{h_c}{h_b}\geq\frac{3h_a}{\sqrt[3]{h_a h_b h_c}}$

Summing up these inequalities, we obtain : $\frac{h_b}{h_a}+\frac{h_c}{h_b}+\frac{h_a}{h_c}\geq\frac{h_a+h_b+h_c}{\sqrt[3]{h_a h_b h_c}}.$

$$\textit{Then}\quad \frac{h_b}{h_a}+\frac{h_c}{h_b}+\frac{h_a}{h_c}+\frac{9\sqrt[3]{w_a w_b w_c}}{h_a+h_b+h_c}$$

$$\geq\frac{h_a+h_b+h_c}{\sqrt[3]{h_a h_b h_c}}+\frac{9\sqrt[3]{h_a h_b h_c}}{h_a+h_b+h_c}\overset{AM-GM}{\geq}2\sqrt{\frac{h_a+h_b+h_c}{\sqrt[3]{h_a h_b h_c}}\cdot\frac{9\sqrt[3]{h_a h_b h_c}}{h_a+h_b+h_c}}$$

$$=6.$$

$$\textit{Therefore,}\quad \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{18R}{ab+bc+ca}\cdot\sqrt[3]{w_a w_b w_c}\geq 6.$$

JP.437 If $a,b>0$ then:

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\geq 2ab+\sqrt{2ab(a^2+b^2)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be the function: $f:\mathbb{R}\to\mathbb{R}; f(x)=\left(x-\frac{2ab}{a+b}\right)\left(x-\frac{a+b}{2}\right)-(x-\sqrt{ab})\left(x-\sqrt{\frac{a^2+b^2}{8}}\right)$

$$f\left(\frac{2ab}{a+b}\right)=\left(\frac{2ab}{a+b}-\sqrt{ab}\right)\left(\frac{2ab}{a+b}-\sqrt{\frac{a^2+b^2}{8}}\right)>0$$

$$f\left(\frac{a+b}{2}\right)=\left(\frac{a+b}{2}-\sqrt{ab}\right)\left(\frac{a+b}{2}-\sqrt{\frac{a^2+b^2}{8}}\right)<0$$

$$f\left(\frac{2ab}{a+b}\right)\cdot f\left(\frac{a+b}{2}\right)<0$$

Hence, the equation $f(x)=0$ has real roots, then $\Delta\geq 0$

$$f(x)=2x^2-\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)x+ab+\sqrt{\frac{ab(a^2+b^2)}{2}}$$

$$\Delta\geq 0\Rightarrow\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)^2-8\left(ab+\sqrt{\frac{ab(a^2+b^2)}{8}}\right)\geq 0$$

$$\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\geq 8\left(ab+\sqrt{\frac{ab(a^2+b^2)}{8}}\right)$$

Therefore,

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\geq 2ab+\sqrt{2ab(a^2+b^2)}$$

Equality holds for $a=b$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM-GM inequality, we have: $(x+y)^2\geq 4xy;\forall x,y>0$

Then,

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\ge 4\left(\frac{ab}{a+b}+\frac{a+b}{4}\right)\left(\frac{\sqrt{ab}}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)$$

$$\frac{ab}{a+b}+\frac{a+b}{4}\ge 2\sqrt{\frac{ab}{a+b}\cdot\frac{a+b}{4}}=\sqrt{ab}$$

Therefore,

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\ge 4\sqrt{ab}\left(\frac{\sqrt{ab}}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)$$

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\ge 2ab+\sqrt{2ab(a^2+b^2)}$$

Solution 3 by Tapas Das-India

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2=\left[\left(\frac{ab}{a+b}+\frac{a+b}{4}\right)\left(\frac{\sqrt{ab}}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)\right]^2=$$

$$=\left(\frac{ab}{a+b}+\frac{a+b}{4}\right)^2+\left(\frac{\sqrt{ab}}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)^2+2\left(\frac{ab}{a+b}+\frac{a+b}{4}\right)\left(\frac{\sqrt{ab}}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)$$

$$\ge ab+\frac{\sqrt{2ab(a^2+b^2)}}{2}+2\left[ab+\frac{\sqrt{2ab(a^2+b^2)}}{4}\right]=2ab+\sqrt{2ab(a^2+b^2)}$$

Now,

$$\frac{ab}{a+b}+\frac{a+b}{4}\ge 2\left(\frac{ab}{a+b}\cdot\frac{a+b}{4}\right)^{\frac{1}{2}}=\sqrt{ab}$$

$$\left(\frac{ab}{a+b}+\frac{a+b}{4}\right)^2\ge ab$$

$$\left(\frac{\sqrt{ab}}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\ge 4\left(\frac{\sqrt{ab}}{2}\cdot\sqrt{\frac{a^2+b^2}{8}}\right)=\frac{4\sqrt{ab}}{2}\cdot\frac{\sqrt{2(a^2+b^2)}}{4}=\frac{\sqrt{2ab(a^2+b^2)}}{2}$$

$$\left(\frac{ab}{a+b}+\frac{a+b}{4}\right)\left(\frac{\sqrt{ab}}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)\geq\sqrt{ab}\left(\frac{\sqrt{ab}}{2}+\sqrt{\frac{2(a^2+b^2)}{16}}\right)=$$

$$=\sqrt{ab}\left(\frac{\sqrt{ab}}{2}+\frac{\sqrt{2(a^2+b^2)}}{4}\right)=\frac{ab}{2}+\frac{\sqrt{2ab(a^2+b^2)}}{4}$$

Solution 4 by Khaled Abd Imouti-Damascus-Syria

$$\sqrt{\frac{a^2+b^2}{2}}\geq\frac{a+b}{2}\geq\sqrt{ab}\ (AGM)$$

$$T=\sqrt{\frac{a^2+b^2}{2}};M=\frac{a+b}{2};G\sqrt{ab}$$

$$\left(\frac{1}{2}\cdot\frac{G^2}{M}+\frac{G}{2}+\frac{M}{2}+\frac{T}{2}\right)^2\overset{?}{\geq}2G^2+\sqrt{4G^2T}$$

$$\frac{1}{4}\left(\frac{G^2}{M}+G+M+T\right)^2\geq 2G^2+2GT$$

$$\left(\frac{G^2}{M}+G+M+T\right)^2\geq 8(G^2+GT)$$

$$\left(\frac{G^2}{M}+M+(G+T)\right)^2\geq 8G(G+T)$$

$$\left(\frac{G^2}{M}+M+(G+T)\right)^2\geq\big(2G+(G+T)\big)^2\geq\left(2\sqrt{2G(G+T)}\right)^2$$

$$\left(\frac{G^2}{M}+M+(G+T)\right)^2\geq 8G(G+T)$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$x\leq y\leq z\leq t\Rightarrow(x+y+z+t)^2\geq 8(xz+yt)$$

$$x=m_h,y=m_g,z=m_a,t=m_p$$

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\geq 2ab+\sqrt{2ab(a^2+b^2)}\Leftrightarrow$$

$$\left(\frac{m_h}{2}+\frac{m_g}{2}+\frac{m_a}{2}+\frac{m_p}{2}\right)^2 \geq 2\cdot\frac{2ab}{a+b}\cdot\frac{a+b}{2}+2\sqrt{ab}\cdot\sqrt{\frac{a^2+b^2}{2}} \Leftrightarrow$$

$$\left(m_h+m_g+m_a+m_p\right)^2 \geq 8\left(m_hm_a+m_gm_p\right)$$

Let $f(a)=2a^2+a\left(m_h+m_g+m_a+m_p\right)+m_hm_a+m_gm_p$

$$f(a)=(a+m_h)(a+m_a)\left(a+m_g\right)\left(a+m_p\right)$$

$$f(m_h)=\left(-m_h+m_g\right)\left(-m_h+m_p\right)\geq 0$$

$$f(-m_a)=\left(-m_a+m_g\right)\left(-m_a+m_p\right)\leq 0 \Rightarrow \Delta_f \geq 0$$

$$\Rightarrow \left(m_h+m_g+m_p\right)^2 \geq 8\left(m_hm_a+m_gm_p\right)$$

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2 \geq 2ab+\sqrt{2ab(a^2+b^2)}$$

Solution 6 by Ravi Prakash-New Delhi-India

Let $H=\frac{2ab}{a+b}, G=\sqrt{ab}, A=\frac{1}{2}(a+b), Q=\sqrt{\frac{a^2+b^2}{2}}$ then

$$HA=G^2, Q^2+G^2=2A^2$$

Inequality becomes as:

$$\left(\frac{H}{2}+\frac{G}{2}+\frac{A}{2}+\frac{Q}{2}\right)^2 \geq 2G^2+2GQ \Leftrightarrow$$

$$(H+G+A+Q)^2 \geq 8G(G+Q) \Leftrightarrow$$

$$(H+A)^2+2(H+A)(G+Q)+(G+Q)^2-8G(G+Q)\geq 0; (1)$$

$$LHS_{(1)} \geq 4HA+4\sqrt{HA}(G+Q)^2+(G+Q)^2-8G(G+Q)=$$

$$=4G^2+4G(G+Q)^2+(G+Q)^2-8G(G+Q)=$$

$$=(2G-G-Q)^2=(G-Q)^2\geq 0$$

Solution 7 by Vivek Kumar-India

$$\frac{ab}{a+b}+\frac{a+b}{4}\geq 2\sqrt{\left(\frac{ab}{a+b}\right)\left(\frac{a+b}{4}\right)}=\sqrt{ab}$$

So, we have to prove

$$\left(\frac{3\sqrt{ab}}{2}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\geq 2ab+\sqrt{2ab(a^2+b^2)}\Leftrightarrow$$

$$\left(\frac{3\sqrt{ab}+\sqrt{a^2+b^2}}{2\sqrt{2}}\right)^2\geq 2ab+\sqrt{2ab(a^2+b^2)}\Leftrightarrow$$

$$\left(3\sqrt{2ab}+\sqrt{a^2+b^2}\right)^2\geq 16ab+8\sqrt{2ab(a^2+b^2)}\Leftrightarrow$$

$$18ab+a^2+b^2+6\sqrt{2ab(a^2+b^2)}\geq 16ab+8\sqrt{2ab(a^2+b^2)}\Leftrightarrow$$

$$\left(\sqrt{a^2+b^2}\right)^2+\left(\sqrt{2ab}\right)^2-2\sqrt{2ab(a^2+b^2)}\geq 0\Leftrightarrow$$

$$\left(\sqrt{a^2+b^2}-\sqrt{2ab}\right)^2\geq 2$$

JP.438 Find $\lambda>0$ so that the following relationship holds in any ΔABC.

$$\frac{3R}{r}\geq\lambda\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{9\sqrt[3]{abc}}{\lambda(a+b+c)}\geq 6$$

Proposed by Alex Szoros-Romania

Solution 1 by proposer

If it holds that

$$\frac{3R}{r}\geq\lambda\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{9\sqrt[3]{abc}}{\lambda(a+b+c)}\geq 6,\text{then }\Delta ABC\text{ is equilateral.}$$

$$\Rightarrow\begin{cases}a=b=c=l\\R=2r\end{cases}$$

But $6\geq 3\lambda+\frac{9l}{3\lambda l}\geq 6\Rightarrow 6\geq 3\lambda+\frac{3}{\lambda}\geq 6\Rightarrow\lambda+\frac{1}{\lambda}=2\Rightarrow\lambda=1$

Next, we prove that $\lambda=1$.

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\left(\frac{2a}{b}+\frac{b}{c}\right)+\left(\frac{2b}{c}+\frac{c}{a}\right)+\left(\frac{2c}{a}+\frac{a}{b}\right)\geq$$

$$\geq 3\sqrt[3]{\frac{a}{b}\frac{a}{b}\frac{b}{c}}+3\sqrt[3]{\frac{b}{c}\frac{b}{c}\frac{a}{b}}+3\sqrt[3]{\frac{c}{a}\frac{c}{a}\frac{a}{b}}=3\sqrt[3]{\frac{a^2}{bc}}+3\sqrt[3]{\frac{b^2}{ac}}+3\sqrt[3]{\frac{c^2}{ab}}=$$

$$=\frac{3(a+b+c)}{\sqrt[3]{abc}}\Rightarrow\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\frac{a+b+c}{3\sqrt[3]{abc}}$$

$$\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{3\sqrt[3]{abc}}{a+b+c}\geq\frac{a+b+c}{3\sqrt[3]{abc}}+\frac{3\sqrt[3]{abc}}{a+b+c}\geq 2$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{9\sqrt[3]{abc}}{a+b+c}\geq 6;(1)$$

On the other hand, using $a^2+b^2+c^2\leq 9R^2$ $(Leibniz)$ and from CBS inequality

$$\frac{3R}{2r}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a};(2)$$

Now, from $R\geq 2r$ $(Euler)$ and AM-GM:

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc}\Rightarrow\frac{3R}{2r}\geq 3\geq\frac{9\sqrt[3]{abc}}{a+b+c}\Rightarrow\frac{3R}{2r}\geq\frac{9\sqrt[3]{abc}}{a+b+c};(3)$$

By adding (2),(3), we get:

$$\frac{3R}{r}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{9\sqrt[3]{abc}}{a+b+c};(4)$$

From (1) and (4), it follows that:

$$\frac{3R}{r}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{9\sqrt[3]{abc}}{a+b+c}\geq 6$$

Therefore, $\lambda=1$.

Solution 2 by George Florin Șerban-Romania

If ΔABC is equilateral, then $R=2r$ and

$$\frac{3R}{r}=6\geq 3\lambda+\frac{9a}{3\lambda a}=3\lambda+\frac{3}{\lambda}\geq 6$$

Hence, $3\lambda+\frac{3}{\lambda}=6$ or $3\lambda^2-6\lambda+3=0$, then $\lambda=1$.

We must to prove:

$$\frac{3R}{r}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{9\sqrt[3]{abc}}{a+b+c}\geq 6$$

We have:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{9\sqrt[3]{abc}}{a+b+c}\overset{CBS}{\leq}\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{b^2}\right)}+\frac{9\sqrt[3]{abc}}{a+b+c}\overset{Leibniz}{\leq}$$

$$\leq \sqrt{9R^2 \cdot \frac{1}{4r^2}} + \frac{9\sqrt[3]{abc}}{a+b+c} \leq \frac{3R}{2r} + \frac{9 \cdot \frac{a+b+c}{3}}{a+b+c} = \frac{3R}{2r} + 3 \overset{(1)}{\leq} \frac{3R}{r}$$

$$(1) \Leftrightarrow 3R + 6r \leq 6R \Leftrightarrow R \geq 2r \ (Euler).$$

Now, for $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{9\sqrt[3]{abc}}{a+b+c} \geq 6$ we prove the inequality

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b+c}{\sqrt[3]{abc}}$$

$$\frac{a}{b} + \frac{a}{b} + \frac{c}{a} \geq 3\sqrt[3]{\frac{a^2}{bc}} = \frac{3a}{\sqrt[3]{abc}}$$

$$\frac{c}{a} + \frac{b}{c} + \frac{b}{c} \geq 3\sqrt[3]{\frac{b^2c}{ac}} = \frac{3b}{\sqrt[3]{abc}}$$

$$\frac{a}{b} + \frac{c}{a} + \frac{c}{a} \geq 3\sqrt[3]{\frac{c^2}{ab}} = \frac{3c}{\sqrt[3]{abc}}$$

By adding, it follows

$$3\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq \frac{3(a+b+c)}{\sqrt[3]{abc}} \Leftrightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b+c}{\sqrt[3]{abc}}$$

So, we have:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{9\sqrt[3]{abc}}{a+b+c} \geq 2\sqrt{\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \cdot \frac{9\sqrt[3]{abc}}{a+b+c}} =$$

$$= 2\sqrt{\frac{a+b+c}{\sqrt[3]{abc}} \cdot \frac{9\sqrt[3]{abc}}{a+b+c}} = 6$$

$$\frac{3R}{r} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{9\sqrt[3]{abc}}{a+b+c} \geq 6$$

Therefore, $\lambda = 1$.

JP.439 If x, y, z are natural numbers such that $2x^x + y^y = 3z^z$, then find

$$\frac{2021x + 2022y + 2023z}{x+y+z}$$

Proposed by Neculai Stanciu-Romania

Solution 1 by proposer

If $y > z \Rightarrow y \geq z+1 \Rightarrow 3z^z = 2x^x + y^y > (z+1)^{z+1} \geq z^{z+1} + (z+1)z^z \geq 3z^z$

contradiction!

If $y < z \Rightarrow 2x^x = 3z^z - y^y > 2z^z \Rightarrow x > z \Rightarrow x \geq z+1$

$3z^z = 2x^x + y^y > 2(z+1)^{z+1} \geq 2z^{z+1} + (z+1)z^z > 3z^z$ contradiction!

So, $y = z \Rightarrow x^x = z^z \Leftrightarrow x = z$. Hence, $x = y = z$.

Therefore,

$$\frac{2021x + 2022y + 2023z}{x+y+z} = \frac{6066}{3} = 2022$$

Solution 2 by Rovsen Pirguliyev-Azerbaijan

$x, y, z \in \mathbb{N}$. Let $x \neq y \neq z$. $\max(x,y,z) \overset{?}{\geq} 3$

If $z = \max(x,y,z)$, then $2x^x + y^y < 2z^z + z^z = 3z^z$

If $y = \max(x,y,z)$, then $3z^z < y \cdot y^{y-1} = y^y$ since $y - 1 \geq z$, then $y \geq z+1$ and $y \geq 3$.

If $x = \max(x,y,z)$, then $3z^z < x \cdot x^{x-1} = x^x$.

In all cases we obtain a contradiction, then $x = y = z$.

Therefore,

$$\frac{2021x + 2022y + 2023z}{x+y+z} = \frac{6066}{3} = 2022$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\frac{2021x + 2022y + 2023z}{x+y+z} = \beta$$

$$\begin{cases} z \geq 1 \\ x \geq 1 \\ y \geq 1 \end{cases} \Rightarrow \begin{cases} z^z \geq 1 \\ x^x \geq 1 \\ y^y \geq 1 \end{cases} \Rightarrow \begin{cases} 3z^z \geq 3 \\ 2x^x + y^y \geq 3 \end{cases}$$

If $\begin{cases} 3z^z = 3 \\ 2x^x + y^y = 3 \end{cases}$ then $x = 1; y = 1$ and $z = 1$.

$$\beta = \frac{2021x + 2022y + 2023z}{x+y+z} = \frac{3 \cdot 2021 + 3}{3} = 2022$$

JP.440 Solve for positive integers the equation

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2$$

Proposed by Neculai Stanciu-Romania

Solution 1 by proposer

WLOG, we can assume that $a \le b \le c$ so, $1+\frac{1}{a} \ge 1+\frac{1}{b} \ge 1+\frac{1}{c}$, then taking account that the greater of the numbers $1+\frac{1}{a}, 1+\frac{1}{b}, 1+\frac{1}{c}$ is greater than G-M of them, we deduce that

$1+\frac{1}{a} \ge \sqrt[3]{2}$. Hence, $a \in \{2,3,4\}$.

If $a=2$, then $3(b+!)(c+1)=4bc \Rightarrow c=3+\frac{12}{b-3}$ and

$(a,b,c) \in \{(2,4,15),(2,5,9),(2,6,7)\}$.

If $a=3$, then $c=2+\frac{6}{b-2}$ and $(a,b,c) \in \{(3,3,8),(3,4,5)\}$.

If $a=4$, then we doesn't have solutions.

Therefore,

$(a,b,c) \in \{(2,4,15),(2,5,9),(2,6,7),(3,3,8),(3,4,5)\}$ and permutations.

Solution 2 by George Florin Șerban-Romania

If $a,b,c \coloneqq 4 \Rightarrow 2=\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \le \left(1+\frac{1}{4}\right)^3=\left(\frac{5}{4}\right)^3=\frac{125}{64}$

$\Rightarrow 128 \le 125$ impossible!

Let $c \in \{1,2,3\}$.

Case 1) If $c=1 \Rightarrow \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\cdot 2=2 \Rightarrow ab+a+b+1=ab$

$a+b+1=0$ impossible because $a+b+1 \ge 3 > 0$

Case 2) If $c=2 \Rightarrow \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\cdot\frac{3}{2}=2 \Rightarrow 3ab+3a+3b+3=4ab \Rightarrow ab-a-b=3$

$$a(b-3)=3b+3 \Rightarrow a=\frac{3b+3}{b-3} \in \mathbb{N} \Rightarrow b-3|3b+3; b-3|3b+3-3b+9$$

$\Rightarrow b-3|12 \Rightarrow b-3 \in \{1,2,3,4,6,12,-1,-2\}, b-3 \ge 1-3=-2 \Rightarrow$

$$b \in \{4,5,6,7,9,15,2,1\}$$

If $b = 1 \Rightarrow a = -3 \notin \mathbb{N}$ impossible.

If $b = 2 \Rightarrow a = -9 \notin \mathbb{N}$ impossible.

If $b = 4 \Rightarrow a = 15 \Rightarrow (a,b,c) = (15,4,2)$

If $b = 5 \Rightarrow a = 9 \Rightarrow (a,b,c) = (9,5,2)$

If $b = 6 \Rightarrow a = 7 \Rightarrow (a,b,c) = (7,6,2)$

If $b = 7 \Rightarrow a = 6 \Rightarrow (a,b,c) = (6,7,2)$

If $b = 9 \Rightarrow a = 5 \Rightarrow (a,b,c) = (5,9,2)$

If $b = 15 \Rightarrow a = 4 \Rightarrow (a,b,c) = (4,15,2)$

Case 3) If $c = 3 \Rightarrow \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\cdot\frac{4}{3} = 2 \Rightarrow$

$$2ab + 2a + 2b + 2 = 3ab \Rightarrow ab - 2a - 2b = 2$$

$$a(b-2) = 2b + 2 \Rightarrow a = \frac{2b+2}{b-2} \in \mathbb{N}$$

$$b-2|2b+2,\ b-2|2b-4 \Rightarrow b-2|2b+2-2b+4$$

$$b-2|6 \Rightarrow b \in \{1,2,3,6,-1\}$$

$$b-2 \ge 1-2 \Rightarrow b-2 \ge -1 \Rightarrow b \in \{3,4,5,8,1\}$$

If $b = 1 \Rightarrow a = -4 \notin \mathbb{N}$ impossible.

If $b = 3 \Rightarrow a = 8 \Rightarrow (a,b,c) = (8,3,3)$

If $b = 4 \Rightarrow a = 5 \Rightarrow (a,b,c) = (5,4,3)$

If $b = 5 \Rightarrow a = 4 \Rightarrow (a,b,c) = (4,5,3)$

If $b = 8 \Rightarrow a = 3 \Rightarrow (a,b,c) = (3,8,3)$.

$$S = \{(3,8,3),(8,3,3),(3,3,8),(3,4,5),(3,5,4),(4,5,3),(4,3,5),(5,3,4),$$
$$(5,4,3),(2,5,9),(2,9,5),(5,9,2),(5,2,9),(9,2,5),(9,5,2),(2,6,7),(2,7,6),$$
$$(6,2,7),(6,7,2),(7,2,6),(7,6,2),(2,4,15),(2,15,4),(4,2,15),(4,15,2),(15,2,4),(15,4,2)\}$$

JP.441 Prove that if $a,b,c > 0$ then holds:

$$\frac{a^2+2b^2}{b\sqrt{a^2+b^2}} + \frac{b^2+2c^2}{c\sqrt{b^2+c^2}} + \frac{c^2+2a^2}{a\sqrt{c^2+a^2}} > 6$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

We have the following transforms:

$$\frac{a^2+2b^2}{b\sqrt{a^2+b^2}}+\frac{b^2+2c^2}{c\sqrt{b^2+c^2}}+\frac{c^2+2a^2}{a\sqrt{c^2+a^2}}>6 \Leftrightarrow$$

$$\frac{\left(\frac{a}{b}\right)^2+2}{\sqrt{\left(\frac{a}{b}\right)^2+1}}+\frac{\left(\frac{b}{c}\right)^2+2}{\sqrt{\left(\frac{b}{c}\right)^2+1}}+\frac{\left(\frac{c}{a}\right)^2+2}{\sqrt{\left(\frac{c}{a}\right)^2+1}}>6$$

Let be the function $f:\mathbb{R}\to M, f(x)=\frac{x^2+2}{\sqrt{x^2+1}}$. We have:

$$f(x)=\frac{x^2+2}{\sqrt{x^2+1}}=\frac{\sqrt{(x^2+1)^2}+1}{\sqrt{x^2+1}}=\sqrt{x^2+1}+\frac{1}{\sqrt{x^2+1}}\overset{AGM}{>}2;\text{ where}$$

$$x^2+1>0, \forall x\in\mathbb{R}$$

So, $f(x)\geq 2; \forall x\in\mathbb{R}$. Remains to prove that $[2,\infty)\subset f(\mathbb{R})$.

Let be $u\in[2,\infty)$ such that $f(x)=u$, hence,

$$u=\frac{x^2+2}{\sqrt{x^2+1}}\Leftrightarrow u\sqrt{x^2+1}=x^2+2\Leftrightarrow x^4+(4-u^2)x^2+4-u^2=0$$

Solving the above equation, we find the next solution:

$$x=\sqrt{\frac{u^2-4+u\sqrt{u^2-1}}{2}}$$

So, for all $u\in[2,\infty)$ exists $x=\sqrt{\frac{u^2-4+u\sqrt{u^2-1}}{2}}$ such that $f(x)=u$, which means that $[2,\infty)\subset f(\mathbb{R})$. In this conditions the above problem has been solved.

Solution 2 by Daniel Văcaru-Romania

We have:

$$b\sqrt{a^2+b^2}=\sqrt{b^2(a^2+b^2)}\overset{AGM}{<}\frac{a^2+2b^2}{2}\Rightarrow$$

$$\frac{a^2+2b^2}{b\sqrt{a^2+b^2}}>2;\quad (1)$$

By adding relationships (1), we find:

$$\frac{a^2+2b^2}{b\sqrt{a^2+b^2}}+\frac{b^2+2c^2}{c\sqrt{b^2+c^2}}+\frac{c^2+2a^2}{a\sqrt{c^2+a^2}}>6$$

Solution 3 by Marin Chirciu-Romania

Lemma. If $x>0$ then $\frac{x^2+2}{\sqrt{x^2+1}}>2$

Proof. We have:

$\frac{x^2+2}{\sqrt{x^2+1}}=\sqrt{x^2+1}+\frac{1}{\sqrt{x^2+1}}\overset{AGM}{>}2$. **For** $x=\frac{a}{b}$ **we get the conclusion.**

Note by editor:

$$\frac{\left(\frac{a}{b}\right)^2+2}{\sqrt{\left(\frac{a}{b}\right)^2+1}}>2\Leftrightarrow\frac{\frac{a^2+2b^2}{b^2}}{\frac{\sqrt{a^2+b^2}}{b}}>2\Leftrightarrow\frac{a^2+2b^2}{b\sqrt{a^2+b^2}}>2\Rightarrow\sum_{cyc}\frac{a^2+2b^2}{b\sqrt{a^2+b^2}}>6$$

Remark. The problem it can be developed.

If $a,b,c>0$ then

$$\sum_{cyc}\frac{(b-c)^2+2a^2}{a\sqrt{(b-c)^2+a^2}}\ge 6$$

Proof. Using Lemma and putting $x=\frac{b-c}{a}$.

Equality holds for $a=b=c$.

JP.442 For $x,y,z>0$ prove that:

$$\sqrt{\frac{x^2z}{xy^2+yz^2}}+\sqrt{\frac{xy^2}{yz^2+x^2z}}+\sqrt{\frac{yz^2}{x^2z+xy^2}}>2$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

Let put $a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{x}$ **then we get:**

$$\sqrt{\frac{x^2z}{xy^2+yz^2}}=\sqrt{\frac{\frac{x}{y}}{\frac{xy+z^2}{zx}}}=\sqrt{\frac{\frac{x}{y}}{\frac{y}{z}+\frac{z}{x}}}=\sqrt{\frac{a}{b+c}}$$

Analogously, we can find that

$$\sqrt{\frac{xy^2}{yz^2+x^2z}}=\sqrt{\frac{b}{c+a}}\text{ și }\sqrt{\frac{yz^2}{x^2z+xy^2}}=\sqrt{\frac{c}{a+b}}$$

In this conditions, the problem becomes as:

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}>2$$

Using AM-GM inequality, we can written as:

$$\sqrt{\frac{a}{b+c}}=\frac{2}{\frac{b+c}{a}+1}=\frac{2a}{a+b+c}$$

Analogously, we can find that

$$\sqrt{\frac{b}{c+a}}\geq\frac{2b}{a+b+c}\text{ și }\sqrt{\frac{c}{a+b}}\geq\frac{2c}{a+b+c}$$

By adding the above relationships, it follows that

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq\frac{2a+2b+2c}{a+b+c}=2$$

Equality holds for $b+c=a, c+a=b, a+b=c \Leftrightarrow a=b=c=0$ impossible because

$a,b,c>0.$

Solution 2 by Marin Chirciu-Romania

Denoting $xy^2=a, yz^2=b, zx^2=c$ inequality becomes as

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}>2$$

Using AM-GM inequality, we can written as:

$$\sqrt{\frac{a}{b+c}}\geq\frac{2}{\frac{b+c}{a}+1}=\frac{2a}{a+b+c}$$

Analogously, we can find that

$$\sqrt{\frac{b}{c+a}} \ge \frac{2b}{a+b+c} \text{ şi } \sqrt{\frac{c}{a+b}} \ge \frac{2c}{a+b+c}$$

By adding the above relationships, it follows that

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > \frac{2a+2b+2c}{a+b+c} = 2$$

JP.443 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{w_a^4}{w_b^2 + w_c(w_a+w_b)}} + \sqrt[3]{\frac{w_b^4}{w_c^2 + w_a(w_b+w_c)}} + \sqrt[3]{\frac{w_c^4}{w_a^2 + w_b(w_c+w_a)}} \ge 3\sqrt[3]{3r^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Lemma. If $x, y, z > 0$ then:

$$\sqrt[3]{\frac{x^4}{y^2 + z(x+y)}} + \sqrt[3]{\frac{y^4}{z^2 + x(y+z)}} + \sqrt[3]{\frac{z^4}{x^2 + y(z+x)}} \ge \left(\sum_{cyc} x\right)^{\frac{2}{3}}$$

Proof. Using Radon's inequality, we get:

$$\sum_{cyc} \sqrt[3]{\frac{x^4}{y^2 + z(x+y)}} = \sum_{cyc} \frac{x^{\frac{4}{3}}}{[y^2 + z(x+y)]^{\frac{1}{3}}} \overset{Radon}{\ge} \frac{(\sum x)^{\frac{4}{3}}}{[\sum(x^2 + x(x+y))]^{\frac{1}{3}}} =$$

$$= \frac{(\sum x)^{\frac{4}{3}}}{(\sum x^2 + 2\sum yz)^{\frac{1}{3}}} = \frac{(\sum x)^{\frac{1}{3}}}{(\sum x^2 + 2\sum yz)^{\frac{1}{3}}} = \frac{(\sum x)^{\frac{4}{3}}}{[(\sum x)^2]^{\frac{1}{3}}} = (\sum x)^{\frac{2}{3}}$$

Denoting $x = w_a, y = w_b, z = w_c$ it is enough to prove that:

$(\sum w_a)^{\frac{2}{3}} \ge 3\sqrt[3]{3r^2}$, which follows from $\sum w_a \ge 9r$.

Remains to prove that $(\sum 9r)^{\frac{2}{3}} \ge 3\sqrt[3]{3r^2} \Leftrightarrow (9r)^2 \ge 27 \cdot 3r^2 \Leftrightarrow 81r^2 \ge 81r^2$.

Equality holds is and only if triangle is equilateral.

Solution 2 by George Florin Şerban-Romania

$$\sum_{cyc} \sqrt[3]{\frac{w_a^4}{w_b^2 + w_c(w_a+w_b)}} = \sum_{cyc} \frac{(w_a)^{\frac{4}{3}}}{\left(w_b^2 + w_c(w_a+w_b)\right)^{\frac{1}{3}}} =$$

$$= \sum_{cyc} \frac{(w_a)^{\frac{1}{3}+1}}{\left(w_b^2 + w_c(w_a + w_b)\right)^{\frac{1}{3}}} \overset{Radon}{\geq} \frac{(\sum w_a)^{\frac{4}{3}}}{\sum \left(w_b^2 + w_c(w_a + w_b)\right)^{\frac{1}{3}}} =$$

$$= \frac{(\sum w_a)^{\frac{4}{3}}}{(\sum w_a)^{\frac{2}{3}}} = \left(\sum_{cyc} w_a\right)^{\frac{2}{3}} = \sqrt[3]{\left(\sum_{cyc} w_a\right)^2} \overset{w_a \geq h_a}{\geq} \sqrt[3]{\left(\sum_{cyc} h_a\right)^2} \overset{(1)}{\geq} 3\sqrt[3]{3r^2}$$

$$(1) \Leftrightarrow \left(\sum_{cyc} h_a\right)^2 \geq 81r^2 \Leftrightarrow \sum_{cyc} h_a \geq 9r; (2)$$

$$\sum_{cyc} h_a = \frac{s^2 + r^2 + 4Rr}{2R} \overset{Gerretsen}{\geq} \frac{16Rr - 5r^2 + r^2 + 4Rr}{2R} = \frac{20Rr - 4r^2}{2R} \overset{(3)}{\geq} 9r$$

$$(3) \Leftrightarrow 20Rr - 4r^2 \geq 18Rr \Leftrightarrow 2Rr \geq 4r^2 \Leftrightarrow R \geq 2r(Euler).$$

Therefore,

$$\sum_{cyc} \sqrt[3]{\frac{w_a^4}{w_b^2 + w_c(w_a + w_b)}} \geq 3\sqrt[3]{3r^2}$$

JP. 444 If $a, b, c > 0$ such that $a + b + c = 1$ and $\lambda \geq 1$ then:

$$\frac{a}{\sqrt{b + \lambda c}} + \frac{b}{\sqrt{c + \lambda a}} + \frac{c}{\sqrt{a + \lambda b}} \geq \sqrt{\frac{3}{\lambda + 1}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Using Jensen's inequality for the convex function $f(x) = \frac{1}{\sqrt{x}}, x > 0$, we get:

$$af(b + \lambda c) + bf(c + \lambda a) + cf(a + \lambda b) \geq f\left(a(b + \lambda c) + b(c + \lambda a) + c(a + \lambda b)\right)$$

Hence,

$$LHS = \frac{a}{\sqrt{b + \lambda c}} + \frac{b}{\sqrt{c + \lambda a}} + \frac{c}{\sqrt{a + \lambda b}} = af(b + \lambda c) + bf(c + \lambda a) + cf(a + \lambda b) \geq$$

$$\geq f\left(a(b + \lambda c) + b(c + \lambda a) + c(a + \lambda b)\right) = f\left((\lambda + 1)\sum bc\right) =$$

$$=\frac{1}{\sqrt{(\lambda+1)\sum bc}}\geq\frac{1}{\sqrt{(\lambda+1)\cdot\frac{(\sum a)^2}{3}}}=\frac{1}{\sqrt{(\lambda+1)\cdot\frac{1}{3}}}=\sqrt{\frac{3}{\lambda+1}}$$

Equality holds for $a=b=c=\frac{1}{3}$.

Solution 2 by Daniel Văcaru-Romania

We have:

$$\frac{a}{\sqrt{b+\lambda c}}+\frac{b}{\sqrt{c+\lambda a}}+\frac{c}{\sqrt{a+\lambda b}}=\frac{a^2}{a\sqrt{b+\lambda c}}+\frac{b^2}{b\sqrt{c+\lambda a}}+\frac{c^2}{c\sqrt{a+\lambda b}}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a+b+c)^2}{a\sqrt{b+\lambda c}+b\sqrt{c+\lambda a}+c\sqrt{a+\lambda b}};\ (1)$$

But $t\to\sqrt{t}$ is concave, and we obtain

$$a\sqrt{b+\lambda c}+b\sqrt{c+\lambda a}+c\sqrt{a+\lambda b}\leq$$

$$\leq(a+b+c)\sqrt{\frac{a(b+\lambda c)+b(c+\lambda a)+c(a+\lambda b)}{a+b+c}}=$$

$$=\sqrt{(\lambda+1)(ab+bc+ca)};\ (2)$$

By SOS technology, we have:

$$1=1^2\geq 3(ab+bc+ca)\Rightarrow ab+bc+ca\leq\frac{1}{3};\ (3)$$

From (1),(2) and (3) we obtain:

$$\frac{a}{\sqrt{b+\lambda c}}+\frac{b}{\sqrt{c+\lambda a}}+\frac{c}{\sqrt{a+\lambda b}}\geq\sqrt{\frac{3}{\lambda+1}}$$

JP.445 If $a,b,c>0$ such that $a+b+c=\lambda$ and $\lambda>0$ then:

$$\sum_{cyc}\frac{\lambda+a}{\lambda-a}\leq 2\sum_{cyc}\frac{a}{b}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

We have:

$$\sum_{cyc}\frac{\lambda+a}{\lambda-a}\le 2\sum_{cyc}\frac{a}{b}\Leftrightarrow\sum_{cyc}\left(1+\frac{2a}{\lambda-a}\right)\le 2\sum_{cyc}\frac{a}{b}\Leftrightarrow$$

$$3+\sum_{cyc}\frac{2a}{b+c}\le 2\sum_{cyc}\frac{a}{b}\Leftrightarrow\sum_{cyc}\left(\frac{a}{b}-\frac{a}{b+c}\right)\ge\frac{3}{2}\Leftrightarrow\sum_{cyc}\frac{ac}{b(b+c)}\ge\frac{3}{2}$$

which follows from pqr −method. Denoting $p=a+b+c, q=ab+bc+ca, r=abc$,

we have $q^2=(ab+bc+ca)^2\ge 3abc(a+b+c)=3pr$, so $q^2\ge 3pr$; (1)

Hence, we get:

$$LHS=\sum_{cyc}\frac{ac}{b(b+c)}=\frac{1}{abc}\sum_{cyc}\frac{(ac)^2}{b+c}\overset{CBS}{\ge}\frac{1}{abc}\cdot\frac{(\sum ac)^2}{\sum(b+c)}=\frac{1}{abc}\cdot\frac{(\sum bc)^2}{2\sum a}=$$

$$=\frac{1}{r}\cdot\frac{q^2}{2p}\overset{(1)}{\ge}\frac{1}{r}\cdot\frac{3pr}{2p}=\frac{3}{2}=RHS$$

Equality holds for $a=b=c=\frac{1}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{\lambda+a}{\lambda-a}\le 2\sum_{cyc}\frac{a}{b}\leftrightarrow\sum_{cyc}\left(1+\frac{2a}{b+c}\right)\le 2\sum_{cyc}\frac{a}{b}\ \leftrightarrow\ 3+2\sum_{cyc}\frac{a}{b+c}\le 2\sum_{cyc}\frac{a}{b}$$

$$\textit{Multiplying the both sides by } ab+bc+ca, \textit{we get}:$$

$$3\sum_{cyc}ab+2\sum_{cyc}a^2+2\sum_{cyc}\frac{abc}{b+c}\le 2\sum_{cyc}a^2+2\sum_{cyc}ab+2\sum_{cyc}\frac{a^2c}{b}$$

$$\leftrightarrow\sum_{cyc}ab+2\sum_{cyc}\frac{abc}{b+c}\le 2\sum_{cyc}\frac{a^2c}{b}\quad(1)$$

$$\textit{By CBS inequality, we have}:\sum_{cyc}\frac{a^2c}{b}=\sum_{cyc}\frac{(ca)^2}{bc}\ge\frac{(ca+ab+bc)^2}{bc+ca+ab}$$

$$=\sum_{cyc}ab\quad\textit{and}\quad\frac{1}{b+c}\le\frac{1}{4}\left(\frac{1}{b}+\frac{1}{c}\right)\quad(\textit{And analogs})$$

$$\textit{Then}:\quad LHS_{(1)}\le\sum_{cyc}ab+2\sum_{cyc}\frac{abc}{4}\left(\frac{1}{b}+\frac{1}{c}\right)=2\sum_{cyc}ab\le 2\sum_{cyc}\frac{a^2c}{b}=RHS_{(1)}.$$

$$\textit{So the proof is completed. Equality holds when } a=b=c=\frac{\lambda}{3}.$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}=\frac{1+\frac{b}{a}}{\frac{b}{a}+\frac{c}{a}}+\frac{1+\frac{c}{b}}{\frac{c}{b}+\frac{a}{b}}+\frac{1+\frac{a}{c}}{\frac{a}{c}+\frac{b}{c}}=$$

$$=\frac{1+\frac{1}{x}}{\frac{1}{x}+z}+\frac{1+\frac{1}{y}}{\frac{1}{y}+x}+\frac{1+\frac{1}{z}}{\frac{1}{z}+y}=\frac{1+x}{1+zx}+\frac{1+y}{1+xy}+\frac{1+z}{1+yz}=$$

$$=\frac{y(1+x)}{1+y}+\frac{z(1+y)}{1+z}+\frac{x(1+z)}{1+x},\text{where }\frac{a}{b}=x;\frac{b}{c}=y;\frac{c}{a}=z$$

$$(x+y+z)(x+1)(y+1)(z+1)\ge$$

$$\ge x(1+z)^2(1+y)+y(1+x)^2(1+z)+z(1+y)^2(1+x)$$

$$(x+y+z)\left(2+x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge$$

$$\ge x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

$$x^2+y^2+z^2+\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\text{ true.}$$

Solution 4 by Khaled Abd Imouti-Damascus-Syria

$$\sum_{cyc}\frac{\lambda+a}{\lambda-a}\le 2\sum_{cyc}\frac{a}{b}\Leftrightarrow\sum_{cyc}\left(1+\frac{2a}{b+c}\right)\le 2\sum_{cyc}\frac{a}{b}\Leftrightarrow 3+2\sum_{cyc}\frac{a}{b+c}\le 2\sum_{cyc}\frac{a}{b}$$

$$2\left(\sum_{cyc}\frac{a}{b}-\frac{a}{b+c}\right)\ge 3\Leftrightarrow\sum_{cyc}\frac{ac}{b(b+c)}\ge\frac{3}{2}$$

Let $ab=x;bc=y;ca=z$, then

$$\frac{z}{b^2+y}+\frac{x}{c^2+z}+\frac{y}{a^2+x}\ge\frac{3}{2}\Leftrightarrow$$

$$\frac{z^2}{xy+zy}+\frac{x^2}{yz+zx}+\frac{y^2}{xz+yx}\overset{?}{\ge}\frac{3}{2}$$

We have:

$$\frac{z^2}{xy+zy}+\frac{x^2}{yz+zx}+\frac{y^2}{xz+yx}\ge\frac{(x+y+z)^2}{2(xy+yz+zx)}\ge\frac{3}{2}\Leftrightarrow$$

$x^2+y^2+z^2+xy+yz+zx\ge 0$ true $\forall x,y,z>0$

JP.446 Let $a>0$ fixed. Solve for real numbers:

$$x\left(1+\frac{1}{\sqrt{a^2x^2-1}}\right)=\frac{35}{12a}$$

Proposed by Laura and Gheorghe Molea-Romania

Solution 1 by proposers

Firstly, we observe that $x>0$ and $a^2x^2-1>0$, where $x>\frac{1}{a}$.

Denoting $\sqrt{a^2x^2-1}=t$, then $x=\frac{\sqrt{t^2+1}}{a}, t>0$. Equation becomes as:

$$\sqrt{t^2+1}\left(1+\frac{1}{t}\right)=\frac{35}{12}\Leftrightarrow\sqrt{t^2+1}=\frac{35t}{12(t+1)}$$

$$t^2+1=\frac{1225t^2}{144t^2+288t+144}\Leftrightarrow$$

$$144t^4+288t^3 0937t^2+288t+144=0\Leftrightarrow$$

$$t^2+2t-\frac{937}{144}+\frac{2}{t}+\frac{1}{t^2}=0$$

Let us denote: $t+\frac{1}{t}=k; k>0$, then $t^2+\frac{1}{t^2}=k^2-2$. Equation becomes:

$$k^2+2k-\frac{937}{144}=0\Leftrightarrow(k+1)^2=\frac{1369}{44}\Leftrightarrow k+1=\frac{37}{12}\Rightarrow k=\frac{25}{12}$$

$$t+\frac{1}{t}=\frac{25}{12}\Leftrightarrow 12t^2-25t+12=0\Rightarrow t_1=\frac{4}{3}; t_2=\frac{3}{4}$$

$$\sqrt{a^2x^2-1}=\frac{4}{3}\Rightarrow a^2x^2=\frac{25}{9}\Rightarrow x_1=\frac{5}{3a}>\frac{1}{a}$$

$$\sqrt{a^2x^2-1}=\frac{3}{4}\Rightarrow a^2x^2=\frac{25}{16}\Rightarrow x_2=\frac{5}{4a}>\frac{1}{a}$$

Hence, $S=\left\{\frac{5}{3a},\frac{5}{4a}\right\}$

Solution 2 by Amir Sofi-Kosovo

$$x\left(1+\frac{1}{\sqrt{a^2x^2-1}}\right)=\frac{35}{12a}\Leftrightarrow\frac{1}{\frac{1}{ax}}+\frac{1}{\sqrt{1-\frac{1}{a^2x^2}}}=\frac{35}{12}$$

$$\frac{1}{ax}=u,\sqrt{1-\frac{1}{a^2x^2}}=v;u,v>0\Rightarrow$$

$$\begin{cases}\frac{1}{u}+\frac{1}{v}=\frac{35}{12}\\ u^2+v^2=1\end{cases}\Rightarrow\begin{cases}uv=\frac{12}{35}(u+v)\\ (u+v)^2-2uv=1\end{cases}\Rightarrow\begin{cases}u+v=\frac{35}{12}uv\\ u+v=\frac{7}{5}\end{cases}$$

$$\begin{cases}uv=\frac{12}{25}\\ u+v=\frac{7}{5}\end{cases}\Rightarrow(u,v)\in\left\{\left(\frac{3}{5},\frac{4}{5}\right),\left(\frac{4}{5},\frac{3}{5}\right)\right\}\Rightarrow x\in\left\{\frac{5}{3a};\frac{5}{4a}\right\}$$

Solution 3 by Tapas Das-India

$$x\left(1+\frac{1}{\sqrt{a^2x^2-1}}\right)=\frac{35}{12a}\Rightarrow ax\left(1+\frac{1}{\sqrt{a^2x^2-1}}\right)=\frac{35}{12}.\text{Let }x=\sec\theta\Rightarrow$$

$$\sec\theta\left(1+\frac{1}{\sqrt{\sec^2\theta-1}}\right)=\frac{35}{12}\Rightarrow\sec\theta\left(1+\frac{\cos\theta}{\sin\theta}\right)=\frac{35}{12}$$

$$\frac{\sin\theta+\cos\theta}{2\sin\theta\cos\theta}=\frac{35}{24}\Rightarrow\frac{\sin\theta+\cos\theta}{\sin 2\theta}=\frac{35}{24}$$

$$\frac{(\sin\theta+\cos\theta)^2}{\sin^2 2\theta}=\frac{1225}{576}\Rightarrow\frac{1+\sin 2\theta}{\sin^2 2\theta}=\frac{1225}{576}$$

Let $\sin 2\theta=t$, then

$$\frac{1+t}{t^2}=\frac{1225}{576}\Leftrightarrow 1225t^2-576t-576=0$$

$$t=\frac{576\pm\sqrt{576^2-4\cdot1225\cdot(-576)}}{2\cdot1225}\Rightarrow t=\frac{1176}{1225}\Rightarrow\sin 2\theta=\frac{1176}{1225}$$

$$\cos^2 2\theta=1-\sin^2 2\theta=1-\left(\frac{1176}{1225}\right)^2\Rightarrow\cos 2\theta=\pm\frac{343}{1225}$$

$$2\cos^2\theta=1+\cos 2\theta=\frac{1568}{1225}\Rightarrow\cos^2\theta=\frac{784}{1225}\Rightarrow\cos\theta=\frac{4}{5}$$

$$\Rightarrow\sec\theta=\frac{5}{4}\Rightarrow ax=\frac{5}{4}\Rightarrow x=\frac{5}{4a}$$

$$\cos 2\theta=-\frac{343}{1225}\Rightarrow\cos^2\theta=\frac{441}{1225}\Rightarrow\cos\theta=\frac{3}{5}\Rightarrow\sec\theta=\frac{5}{3}\Rightarrow ax=\frac{5}{3}\Rightarrow x=\frac{5}{3a}$$

Solution 4 by Ravi Prakash-New Delhi-India

$$x\left(1+\frac{1}{\sqrt{a^2x^2-1}}\right)=\frac{35}{12a};(1)$$

As $a>0, RHS>0\Rightarrow LHS>0\Rightarrow x>0\Rightarrow ax>0.$

Put $ax=\sec\theta, 0<\theta<\frac{\pi}{2}$, then (1) can be written as

$$\sec^2\theta+\csc^2\theta+2\sec\theta\csc\theta=\frac{1225}{144}$$

$$\frac{4}{\sin^2 2\theta}+\frac{4}{\sin 2\theta}+1=\frac{1225}{144}+1$$

$$\left(\frac{2}{\sin 2\theta}+1\right)^2=\left(\frac{37}{12}\right)^2\Rightarrow\frac{2}{\sin 2\theta}+1=\frac{37}{12}$$

$$\frac{2}{\sin 2\theta}=\frac{25}{12}\Rightarrow\sin 2\theta=\frac{24}{25}\Rightarrow\frac{2\tan\theta}{1+\tan^2\theta}=\frac{24}{25}\Rightarrow$$

$$12\tan^2\theta-25\tan\theta+12=0$$

$$(3\tan\theta-4)(4\tan\theta-3)=0\Rightarrow\tan\theta\in\left\{\frac{4}{3},\frac{3}{4}\right\}$$

$$ax=\sec\theta=\sqrt{1+\tan^2\theta}\Rightarrow ax\in\left\{\frac{5}{3},\frac{5}{4}\right\}\Rightarrow x\in\left\{\frac{5}{3a},\frac{5}{4a}\right\}$$

JP.447 If $4R+r=1$, then prove that:

$$\sum_{cyc}\frac{\sqrt{r_ar_b}}{r_a+r_b}\le\frac{1}{2}\left(1+\frac{R}{r}\right)$$

Proposed by Laura and Gheorghe Molea-Romania

Solution 1 by proposers

We known that $r_a+r_b+r_c=4R+r$, but $4R+r=1$, then

$r_a+r_b+r_c=1$. We have:

$$\sqrt{r_ar_b}\le\frac{r_a+r_b}{2}\text{ and }\frac{4}{r_a+r_b}\le\frac{1}{r_a}+\frac{1}{r_b}\Leftrightarrow 0\le(r_a-r_b)^2$$

By multiplying, we have:

$$\frac{4\sqrt{r_ar_b}}{r_a+r_b}\le\frac{r_a+r_b}{2}\left(\frac{1}{r_a}+\frac{1}{r_b}\right)=\frac{1}{2}\left(2+\frac{r_a}{r_b}+\frac{r_b}{r_a}\right)$$

Hence,

$$\frac{\sqrt{r_a r_b}}{r_a+r_b}\le\frac{1}{8}\left(2+\frac{r_a}{r_b}+\frac{r_b}{r_a}\right)$$

$$\sum_{cyc}\frac{\sqrt{r_a r_b}}{r_a+r_b}\le\frac{1}{8}\left(6+\sum_{cyc}\frac{r_a+r_b}{r_c}\right)=\frac{1}{8}\left(6+\sum_{cyc}\frac{1-r_a}{r_a}\right)=\frac{1}{8}\left(3+\frac{1}{r}\right)=$$

$$=\frac{1}{8}\left(3+\frac{4R+r}{r}\right)=\frac{1}{8}\left(3+\frac{4R}{r}+1\right)=\frac{1}{2}\left(1+\frac{R}{r}\right)$$

Therefore,

$$\sum_{cyc}\frac{\sqrt{r_a r_b}}{r_a+r_b}\le\frac{1}{2}\left(1+\frac{R}{r}\right)$$

Equality holds if and only if ΔABC is equilateral.

Solution 2 by Daniel Văcaru-Romania

We have: $\frac{\sqrt{r_a r_b}}{r_a+r_b}\le\frac{1}{2}$

We obtain: $\sum_{cyc}\frac{\sqrt{r_a r_b}}{r_a+r_b}\le\frac{3}{2}$; (1)

On the other hand, by Euler, $\frac{R}{r}\ge 2$, it follows $\frac{1}{2}\left(1+\frac{R}{r}\right)\ge\frac{3}{2}$; (2)

By (1) and (2), we obtain:

$$\sum_{cyc}\frac{\sqrt{r_a r_b}}{r_a+r_b}\le\frac{1}{2}\left(1+\frac{R}{r}\right).$$

JP.448 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\ge\frac{3}{16}\left(\frac{s}{R}\right)^2$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

For $x,y,z>0$ we have: $\sum\frac{x^4}{y+z}\ge\frac{x^3+y^3+z^3}{2}$

WLOG, assume that $x^4\ge y^4\ge z^4\Rightarrow\frac{1}{y+z}\ge\frac{1}{z+x}\ge\frac{1}{x+y}$. From Chebyshev's inequality:

$$\sum_{cyc}\frac{x^4}{y+z}\ge\frac{1}{3}(x^4+y^4+z^4)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right);(1)$$

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\ge\frac{9}{2(x+y+z)};(2)$$

From (1) and (2), we get:

$$\sum_{cyc}\frac{x^4}{y+z}\ge\frac{3}{4}\cdot\frac{x^4+y^4+z^4}{x+y+z}\ge\frac{x^3+y^3+z^3}{2}$$

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\ge\frac{\cos^6\frac{A}{2}+\cos^6\frac{B}{2}+\cos^6\frac{C}{2}}{2};(3)$$

But: $\sum_{cyc}\cos^6\frac{A}{2}=\frac{(4R+r)^3-3s^2(2R+r)}{32R^3};(4)$

From (3) and (4), it follows:

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\ge\frac{(4R+r)^3-3s^2(2R+r)}{32R^3};(5)$$

$(4R+r)^2\ge 3s^2(Doucet's);(6)$. From (5) and (6), it follows

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\ge\frac{3s^2(4R+r-2R-r)}{32R^3}=\frac{3s^2\cdot 2R}{32R^3}=\frac{3s^2}{16R^2}$$

Solution 2 by Marin Chirciu-Romania

Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\ge\frac{1}{18}\left(2+\frac{r}{2R}\right)^3$$

Proof.

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\overset{Bergstrom}{\ge}\frac{\left(\sum\cos^2\frac{A}{2}\right)^4}{9\cdot\sum\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)}=\frac{\left(\sum\cos^2\frac{A}{2}\right)^4}{9\cdot 2\sum\cos^2\frac{A}{2}}=$$

$$=\frac{1}{18}\left(\sum_{cyc}\cos^2\frac{A}{2}\right)^3=\frac{1}{18}\left(2+\frac{r}{2R}\right)^3$$

Equality holds if and only if ΔABC is equilateral.

Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\overset{Lemma}{\geq}\frac{1}{18}\left(2+\frac{r}{2R}\right)^3\overset{(1)}{\geq}\frac{3}{32}\left(\frac{s}{R}\right)^2=RHS,$$

$$(1)\Leftrightarrow\frac{1}{18}\left(2+\frac{r}{2R}\right)^3\geq\frac{3}{32}\left(\frac{s}{R}\right)^2\Leftrightarrow 2(4R+r)^3\geq 27Rs^2$$

which follows from $s^2\leq\frac{R(4R+r)^3}{2(2R-r)}$ $(Blundon-Gerretsen)$

Remains to prove that:

$$2(4R+r)^2\geq 27R\cdot\frac{R(4R+r)^3}{2(2R-r)}\Leftrightarrow 5R^2-8Rr-4r^2\geq 0\Leftrightarrow$$

$(R-2r)(5R+2r)\geq 0$ **true from** $R\geq 2r$ $(Euler)$.

Equality holds if and only if ΔABC is equilateral.

Remark. The problem can be developed.

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\lambda\cos^2\frac{C}{2}}\geq\frac{3}{2(\lambda+1)}\left(\frac{s}{R}\right)^2$$

Marin Chirciu

Solution

Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\geq\frac{1}{18}\left(2+\frac{r}{2R}\right)^3$$

Proof.

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\overset{Bergstrom}{\geq}\frac{\left(\sum\cos^2\frac{A}{2}\right)^4}{9\cdot\sum\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)}=\frac{\left(\sum\cos^2\frac{A}{2}\right)^4}{9\cdot 2\sum\cos^2\frac{A}{2}}=$$

$$=\frac{1}{18}\left(\sum_{cyc}\cos^2\frac{A}{2}\right)^3=\frac{1}{18}\left(2+\frac{r}{2R}\right)^3$$

Equality holds if and only if ΔABC is equilateral.

Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\lambda\cos^2\frac{C}{2}}\overset{Lemma}{\geq}\frac{1}{9(\lambda+1)}\left(2+\frac{r}{2R}\right)^3\overset{(1)}{\geq}\frac{3}{2(\lambda+1)}\left(\frac{s}{R}\right)^2=RHS,$$

$$(1)\Leftrightarrow\frac{1}{18}\left(2+\frac{r}{2R}\right)^3\geq\frac{3}{2(\lambda+1)}\left(\frac{s}{R}\right)^2\Leftrightarrow 2(4R+r)^3\geq 27Rs^2$$

which follows from $s^2\leq\frac{R(4R+r)^3}{2(2R-r)}$ $(Blundon-Gerretsen)$

Remains to prove that:

$$2(4R+r)^2\geq 27R\cdot\frac{R(4R+r)^3}{2(2R-r)}\Leftrightarrow 5R^2-8Rr-4r^2\geq 0\Leftrightarrow$$

$(R-2r)(5R+2r)\geq 0$ true from $R\geq 2r$ $(Euler)$.

Equality holds if and only if ΔABC is equilateral.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Holder inequality, we have:

$$\left(\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\right)\left(\sum_{cyc}\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\right)\left(\sum_{cyc}1\right)^2\geq\left(\sum_{cyc}\cos^2\frac{A}{2}\right)^4$$

Then,

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\geq\frac{1}{18}\left(\sum_{cyc}\cos^2\frac{A}{2}\right)^3;(1)$$

By AM-GM inequality, we have:

$$\left(\sum_{cyc}\cos^2\frac{A}{2}\right)^3 \ge 3^3\cdot\prod_{cyc}\cos^2\frac{A}{2} = 27\left(\frac{s}{4R}\right)^2;(2)$$

From (1) and (2), we obtain:

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}} \ge \frac{1}{18}\cdot 27\left(\frac{s}{4R}\right)^2 = \frac{3}{32}\left(\frac{s}{R}\right)^2$$

Equality holds iff ΔABC is equilateral.

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}} \overset{\text{Bergstrom}}{\ge} \frac{1}{2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)}\left(\sum\cos^4\frac{A}{2}\right)^2 \overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{18\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)}\left(\sum\cos^2\frac{A}{2}\right)^4$$

$$\Rightarrow \sum\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}} \overset{(*)}{\ge} \frac{1}{18}\left(\sum\cos^2\frac{A}{2}\right)^3$$

Via A − G, $(a^2+b^2+c^2)^3 \ge 27a^2b^2c^2$

applying which on circumcevian triangle of the incenter of ABC,

$$\left(4R^2\sum\cos^2\frac{A}{2}\right)^3 \ge 27\prod\left(4R^2\cos^2\frac{A}{2}\right)$$

$$\Rightarrow\left(\sum\cos^2\frac{A}{2}\right)^3 \ge 27.\frac{s^2}{16R^2} \Rightarrow \frac{1}{18}\left(\sum\cos^2\frac{A}{2}\right)^3 \ge \frac{1}{18}.27.\frac{s^2}{16R^2}$$

$$\Rightarrow\frac{1}{18}\left(\sum\cos^2\frac{A}{2}\right)^3 \overset{(**)}{\ge} \frac{3}{32}\left(\frac{s}{R}\right)^2 \therefore (*),(**) \Rightarrow \sum\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}$$

$$\ge\frac{3}{32}\left(\frac{s}{R}\right)^2 \text{ (QED)}$$

JP.449 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cot^4\frac{A}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}\ge 3\sqrt{3}\left(\frac{4R}{r}-5\right)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

For $x,y,z>0$ we have: $\sum\frac{x^4}{y+z}\ge\frac{x^3+y^3+z^3}{2}$

WLOG, assume that $x^4\ge y^4\ge z^4\Rightarrow\frac{1}{y+z}\ge\frac{1}{z+x}\ge\frac{1}{x+y}$. From Chebyshev's inequality:

$$\sum_{cyc}\frac{x^4}{y+z}\ge\frac{1}{3}(x^4+y^4+z^4)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right);(1)$$

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\ge\frac{9}{2(x+y+z)};(2)$$

From (1) and (2), we get:

$$\sum_{cyc}\frac{x^4}{y+z}\ge\frac{3}{4}\cdot\frac{x^4+y^4+z^4}{x+y+z}\ge\frac{x^3+y^3+z^3}{2}$$

$$\sum_{cyc}\frac{\cot^4\frac{A}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}\ge\frac{\cot^3\frac{A}{2}+\cot^3\frac{B}{2}+\cot^3\frac{C}{2}}{2};(3)$$

$$\sum_{cyc}\cot^3\frac{A}{2}=\frac{s(s^2-12Rr)}{r^3};(4)$$

From (3) and (4), we get:

$$\sum_{cyc}\frac{\cot^4\frac{A}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}\ge\frac{s(s^2-12Rr)}{r^3};(5)$$

$$s\ge 3\sqrt{3}r\ (Mitrinovic's);\ s^2\ge 16Rr-5r^2(Gerretsen's);(6)$$

From (5) and (6) it follows that:

$$\sum_{cyc}\frac{\cot^4\frac{A}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}\ge\frac{3\sqrt{3}r(4Rr-5r^2)}{r^3}=\frac{3\sqrt{3}(4R-5r)}{r}=3\sqrt{3}\left(\frac{4R}{r}-5\right)$$

Solution 2 by Marin Chirciu-Romania

$$LHS=\sum_{cyc}\frac{\cot^4\frac{A}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}\overset{Bergstrom}{\geq}\frac{\left(\sum\cot^2\frac{A}{2}\right)^2}{2\sum\cot\frac{A}{2}}=\frac{\left(\frac{s^2-2r^2-8Rr}{r^2}\right)^2}{2\cdot\frac{s}{r}}\overset{Blundon}{\geq}$$

$$\geq\frac{\left(\frac{\frac{2r(2R-r)(4R+r)}{R}-2r^2-8Rr}{r^2}\right)^2}{2\cdot\frac{s}{r}}=\frac{\left(\frac{8R^2-6Rr-2r^2}{Rr}\right)^2}{\frac{2s}{r}}$$

$$=\frac{r(8R^2-6Rr-2r^2)^2}{2R^2r^2s}\overset{(1)}{\geq}3\sqrt{3}\left(\frac{4R}{r}-5\right)$$

$$(1)\Leftrightarrow\frac{r(8R^2-6Rr-2r^2)^2}{2R^2r^2s}\geq3\sqrt{3}\left(\frac{4R}{r}-5\right)\Leftrightarrow$$

$(8R^2-6Rr-2r^2)^2\geq3\sqrt{3}sR^2(4R-5r)$ which follows from

$$s\leq\frac{3R\sqrt{3}}{2}\,(Mitrinovic)$$

Remains to prove that:

$$(8R^2-6Rr-2r^2)\geq3\sqrt{3}\cdot\frac{3R\sqrt{3}}{3}\cdot R^2(4R-5r)\Leftrightarrow$$

$$2(8R^2-6Rr-2r^2)^2\geq27R^3(4R-5r)\Leftrightarrow20R^4-57R^3r+8R^2r^2+48Rr^3+8r^4\geq0$$

$(R-2r)(10R^3-17R^2r-26Rr^2-4r^3)\geq0$ which follows from $R\geq2r(Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. The problem can be developed.

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cot^4\frac{A}{2}}{\cot\frac{B}{2}+\lambda\cot\frac{C}{2}}\geq\frac{3\sqrt{3}}{\lambda+1}\left(\frac{4R}{r}-5\right);\lambda\geq0$$

Marin Chirciu

Solution.

$$LHS=\sum_{cyc}\frac{\cot^4\frac{A}{2}}{\cot\frac{B}{2}+\lambda\cot\frac{C}{2}}\overset{Bergstrom}{\geq}\frac{\left(\sum\cot^2\frac{A}{2}\right)^2}{(\lambda+1)\sum\cot\frac{A}{2}}=\frac{\left(\frac{s^2-2r^2-8Rr}{r^2}\right)^2}{(\lambda+1)\cdot\frac{s}{r}}\overset{Blundon}{\geq}$$

$$\geq\frac{\left(\frac{\frac{2r(2R-r)(4R+r)}{R}-2r^2-8Rr}{r^2}\right)^2}{(\lambda+1)\cdot\frac{s}{r}}=\frac{\left(\frac{8R^2-6Rr-2r^2}{Rr}\right)^2}{\frac{(\lambda+1)s}{r}}$$

$$=\frac{r(8R^2-6Rr-2r^2)^2}{(\lambda+1)R^2r^2s}\overset{(1)}{\geq}\frac{3\sqrt{3}}{(\lambda+1)}\left(\frac{4R}{r}-5\right)$$

$$(1)\Leftrightarrow\frac{r(8R^2-6Rr-2r^2)^2}{2R^2r^2s}\geq\frac{3\sqrt{3}}{\lambda+1}\left(\frac{4R}{r}-5\right)\Leftrightarrow$$

$(8R^2-6Rr-2r^2)^2\geq3\sqrt{3}sR^2(4R-5r)$ which follows from

$$s\leq\frac{3R\sqrt{3}}{2}(Mitrinovic)$$

Remains to prove that:

$$(8R^2-6Rr-2r^2)\geq3\sqrt{3}\cdot\frac{3R\sqrt{3}}{3}\cdot R^2(4R-5r)\Leftrightarrow$$

$$2(8R^2-6Rr-2r^2)^2\geq27R^3(4R-5r)\Leftrightarrow20R^4-57R^3r+8R^2r^2+48Rr^3+8r^4\geq0$$

$(R-2r)(10R^3-17R^2r-26Rr^2-4r^3)\geq0$ which follows from $R\geq2r(Euler)$.

Equality holds if and only if triangle is equilateral.

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\tan^4\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}\geq\frac{s}{9R}$$

Marin Chirciu

Solution. We have:

$$LHS=\sum_{cyc}\frac{\tan^4\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}\overset{Bergstrom}{\geq}\frac{\left(\sum\tan^2\frac{A}{2}\right)^2}{2\sum\tan\frac{A}{2}}=\frac{\left(\frac{(4R+r)^2}{s^2}-2\right)^2}{2\cdot\frac{4R+r}{r}}\overset{Blundon}{\geq}$$

$$\geq \frac{\left(\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}-2\right)^2}{2\cdot\frac{4R+r}{r}} = \frac{\left(\frac{2R-2r}{R}\right)^2}{\frac{2(4R+r)}{s}} = \frac{2s(R-r)^2}{R^2(4R+r)} \overset{(1)}{\geq} \frac{s}{9R} = RHS$$

$$(1) \Leftrightarrow \frac{2s(R-r)^2}{R^2(4R+r)} \geq \frac{s}{9R} \Leftrightarrow 18(R-r)^2 \geq R(4R+r) \Leftrightarrow$$

$14R^2-37Rr+18r^2 \geq 0 \Leftrightarrow (R-2r)(4R-9r) \geq 0$ **true from** $R \geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. The problem can be developed.

In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\tan^4\frac{A}{2}}{\tan\frac{B}{2}+\lambda\tan\frac{C}{2}} \geq \frac{2s}{9(\lambda+1)R};\lambda \geq 0$$

Marin Chirciu

Solution. **We have:**

$$LHS = \sum_{cyc} \frac{\tan^4\frac{A}{2}}{\tan\frac{B}{2}+\lambda\tan\frac{C}{2}} \overset{Bergstrom}{\geq} \frac{\left(\sum\tan^2\frac{A}{2}\right)^2}{(\lambda+1)\sum\tan\frac{A}{2}} = \frac{\left(\frac{(4R+r)^2}{s^2}-2\right)^2}{(\lambda+1)\cdot\frac{4R+r}{r}} \overset{Blundon}{\geq}$$

$$\geq \frac{\left(\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}-2\right)^2}{(\lambda+1)\cdot\frac{4R+r}{r}} = \frac{\left(\frac{2R-2r}{R}\right)^2}{\frac{(\lambda+1)(4R+r)}{s}} = \frac{4s(R-r)^2}{(\lambda+1)R^2(4R+r)} \overset{(1)}{\geq} \frac{s}{9R} = RHS$$

$$(1) \Leftrightarrow \frac{4s(R-r)^2}{(\lambda+1)R^2(4R+r)} \geq \frac{2s}{9(\lambda+1)R} \Leftrightarrow 18(R-r)^2 \geq R(4R+r) \Leftrightarrow$$

$14R^2-37Rr+18r^2 \geq 0 \Leftrightarrow (R-2r)(4R-9r) \geq 0$ **true from** $R \geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

JP.450 In ΔABC the following relationship holds:

$$\sum_{cyc} \sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}} \geq \frac{27r}{4R+r}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

For $x,y,z>0$ we have:

$$\sum_{cyc}\sqrt[3]{\frac{x^2+yz}{y^2+z^2}}\ge\frac{9\sqrt[3]{xyz}}{x+y+z}\Leftrightarrow\sum_{cyc}\frac{x^2+yz}{\sqrt[3]{xyz(x^2+yz)^2(y^2+z^2)}}\ge\frac{9}{x+y+z}$$

$$\frac{x^2+yz}{\sqrt[3]{xyz(x^2+yz)^2(y^2+z^2)}}\overset{AGM}{\ge}\frac{3(x^2+yz)}{\sum x^2y}$$

We must show:

$$\frac{3(x^2+y^2+z^2+xy+yz+zx)}{x^2y+y^2z+z^2x}\ge\frac{9}{(x+y+z)^3}\Leftrightarrow$$

$$x^3+y^3+z^3+3xyz\ge x^2y+y^2z+z^2x$$

Therefore,

$$\sum_{cyc}\sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}}\ge\frac{9\sqrt[3]{r_ar_br_c}}{r_a+r_b+r_c};(1),\qquad r_a+r_b+r_c=4R+r;(2)$$

$$\sqrt[3]{r_ar_br_c}=\sqrt[3]{s^2r}\ge\sqrt[3]{27r^3}=3r;(3)$$

From (1), (2) and (3) it follows that:

$$\sum_{cyc}\sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}}\ge\frac{27r}{4R+r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}}$$

$$=\sum\frac{r_a^2+r_br_c}{\sqrt[3]{(r_b^2+r_c^2)(r_a^2+r_br_c)(r_a^2+r_br_c)}}\overset{A-G}{\ge}\sum\frac{3(r_a^2+r_br_c)}{r_b^2+r_c^2+r_a^2+r_br_c+r_a^2+r_br_c}$$

$$\overset{A-G}{\ge}\sum\frac{3(r_a^2+r_br_c)}{2r_a^2+r_b^2+r_c^2+r_b^2+r_c^2}=\frac{3}{2(r_a^2+r_b^2+r_c^2)}\left(\sum r_a^2+\sum r_br_c\right)$$

$$=\frac{3((4R+r)^2-2s^2+s^2)}{2((4R+r)^2-2s^2)}\overset{?}{\ge}\frac{27r}{4R+r}$$

$$\Leftrightarrow(4R+r)^3-(4R+r)s^2\overset{?}{\ge}18r(4R+r)^2-36rs^2$$

$$\Leftrightarrow(4R+r)^3-18r(4R+r)^2\underset{(*)}{\overset{?}{\ge}}(4R-8r)s^2-27rs^2$$

$$\text{Now, RHS of } (*) \overset{\text{Gerretsen}}{\lessapprox} (4R-8r)(4R^2+4Rr+3r^2)$$

$$-27r(16Rr-3r^2) \overset{?}{\le} (4R+r)^3 - 18r(4R+r)^2$$

$$\Leftrightarrow 3t^3-14t^2+20t-8 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)^2(2t+t-2) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true} \therefore \sum \sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}} \ge \frac{27r}{4R+r} \text{ (QED)}$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The inequality in the statement can be rewritten as

$$\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} r_br_c\right)\left(\sum_{cyc} \sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}}\right) \ge 27r_ar_br_c$$

By Holder's inequality, we have:

$$S = \left(\sum_{cyc}(y+z)\right)\left(\sum_{cyc}x(y+z)\right)\left(\sum_{cyc}\sqrt[3]{\frac{x^2+yz}{y^2+z^2}}\right) \ge$$

$$\ge \left(\sum_{cyc}\sqrt[9]{(y+z)^3\cdot x^3(y+z)^3\cdot\frac{x^2+yz}{y^2+z^2}}\right)^3 ;\forall x,y,z>0$$

By AM-GM inequality, we have:

$$(y+z)^6 = [2yz+(y^2+z^2)]^2(y+z)^2 \ge 4\cdot 2yz(y^2+z^2)\cdot 4yz =$$

$= 32(yz)^2(y^2+z^2)$ and $x^2+yz \ge 2x\sqrt{yz}$. Then,

$$x^3(y+z)^6\cdot\frac{x^2+yz}{y^2+z^2} \ge x^3\cdot 32(yz)^2(y^2+z^2)\cdot\frac{2x\sqrt{yz}}{y^2+z^2} = 4^3\sqrt{x^3(xyz)^5}$$

Thus,

$$S \ge \left(\sum_{cyc}\sqrt[9]{4^3\sqrt{x^3(xyz)^5}}\right)^3 = 4\left(\sqrt[18]{(xyz)^5}\cdot\sum_{cyc}\sqrt[6]{x}\right)^3 \overset{AGM}{\ge}$$

$$\ge 4\left(\sqrt[18]{(xyz)^5}\cdot 3\sqrt[18]{xyz}\right)^3 = 4\cdot 27xyz$$

$$\text{Or } \left(\sum_{cyc}(y+z)\right)\left(\sum_{cyc}x(y+z)\right)\left(\sum_{cyc}\sqrt[3]{\frac{x^2+yz}{y^2+z^2}}\right) \ge 27xyz$$

Taking $x=r_a; y=r_b; z=r_c$ yields the desired result.
Equality holds if and only if ΔABC is equilateral.

Solution 4 by Marin Chirciu-Romania

$$\sum_{cyc}\sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}}=\sum_{cyc}\sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}\cdot 1\cdot 1}\overset{AGM}{\geq}\sum_{cyc}\frac{3}{\frac{r_b^2+r_c^2}{r_a^2+r_br_c}+1+1}=$$

$$=\sum_{cyc}\frac{3(r_a^2+r_br_c)}{2r_a^2+(r_b+r_c)^2}\overset{CBS}{\geq}\sum_{cyc}\frac{3(r_a^2+r_br_c)}{2r_a^2+2(r_b^2+r_c^2)}=\frac{3}{2}\cdot\frac{\sum r_a^2+\sum r_br_c}{\sum r_a^2}=$$

$$=\frac{3}{2}\cdot\frac{[(4R+r)^2-2s^2]+s^2}{(4R+r)^2-2s^2}=\frac{3}{2}\cdot\frac{(4R+r)^2-s^2}{(4R+r)^2-2s^2}\overset{(1)}{\geq}\frac{27r}{4R+r}$$

$$(1)\Leftrightarrow\frac{3}{2}\cdot\frac{(4R+r)^2-s^2}{(4R+r)^2-2s^2}\geq\frac{27r}{4R+r}\Leftrightarrow s^2(35r-4R)+(4R-17r)(4R+r)^2\geq 0$$

We distinguish the following cases:

Case 1) If $(35r-4R)>0$ applying Gerretsen's inequality $s^2\geq 16Rr-5r^2$, we mast to prove $(16Rr-5r^2)(35r-4R)+(4R-17r)(4R+r)^2\geq 0\Leftrightarrow$ $4R^3-19R^2r+28Rr^2-12r^3\geq 0\Leftrightarrow(R-2r)^2(4R-3r)\geq 0$ true from $R\geq 2r$ $(Euler)$.

Case 2) If $(35r-4R)\leq 0$ applying Blundon-Gerretsen:

$$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

Remains to prove that:

$$\frac{R(4R+r)^2}{2(2R-r)}(35r-4R)+(4R-17r)(4R+r)^2\geq 0\Leftrightarrow 12R^2-41Rr+34r^2\geq 0$$

$$\Leftrightarrow(R-2r)(12R-17r)\geq 0 \text{ which is true from } R\geq 2r\ (Euler).$$

Equality holds iff ΔABC is equilateral.

Remark. Let's replace r_a with h_a.

In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt[3]{\frac{h_a^2+h_bh_c}{h_b^2+h_c^2}}\geq\frac{6r}{R}$$

Marin Chirciu

Solution

$$\sum_{cyc}\sqrt[3]{\frac{h_a^2+h_bh_c}{h_b^2+h_c^2}}=\sum_{cyc}\sqrt[3]{\frac{h_a^2+h_bh_c}{h_b^2+h_c^2}\cdot 1\cdot 1}\overset{GHM}{\geq}\sum_{cyc}\frac{3}{\frac{h_b^2+h_c^2}{h_a^2+h_bh_c}+1+1}=$$

$$=\sum_{cyc}\frac{3(h_a^2+h_bh_c)}{2h_a^2+(h_b+h_c)^2}\overset{CBS}{\geq}\sum_{cyc}\frac{3(h_a^2+h_bh_c)}{2h_a^2+2(h_b^2+h_c^2)}=\frac{3}{2}\cdot\frac{\sum h_a^2+\sum h_bh_c}{\sum h_a^2}=$$

$$=\frac{3}{2}\cdot\frac{\left[\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-2\cdot\frac{2rs^2}{R}\right]+\frac{2rs^2}{R}}{\left[\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-2\cdot\frac{2rs^2}{R}\right]}$$

$$=\frac{3}{2}\cdot\frac{s^4+2s^2r^2+r^2(4R+r)^2}{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}\overset{(1)}{\geq}\frac{6r}{R}$$

$$(1)\Leftrightarrow\frac{s^4+2s^2r^2+r^2(4R+r)^2}{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}\geq\frac{4r}{R}$$

$$s^2[s^2(R-4r)+2r^2\,(17R-4r)]+r^2(4R+r)^2(R-4r)\geq 0$$

We distinguish the following cases:

Case 1) If $(R-4r)\geq 0$ inequality is obviously true.

Case 2) If $(R-4r)<0$ inequality can be rewritten as:

$r^2(4R+r)^2(R-4r)\geq s^2[s^2(4r-R)+2r^2(4r-17R)]$ **which follows from Blundon-Gerretsen inequality**

$$16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

Remains to prove:

$$r^2(4R+r)^2(R-4r)\geq\frac{R(4R+r)^2}{2(2R-r)}[(16Rr-5r^2)(4r-R)+2r^2(4r-17R)]$$

$$16R^3-31R^2r-6Rr^2+8r^3\geq 0\Leftrightarrow(R-2r)(16R^2+Rr-4r^2)\geq 0$$

Equality holds iff ΔABC is equilateral.

Remark. Inequality can be stronger.

$$\sum_{cyc}\sqrt[3]{\frac{h_a^2+h_bh_c}{h_b^2+h_c^2}}\ge\frac{27r}{4R+r}$$

Marin Chirciu

Solution.

$$\sum_{cyc}\sqrt[3]{\frac{h_a^2+h_bh_c}{h_b^2+h_c^2}}=\sum_{cyc}\sqrt[3]{\frac{h_a^2+h_bh_c}{h_b^2+h_c^2}\cdot 1\cdot 1}\overset{GHM}{\ge}\sum_{cyc}\frac{3}{\frac{h_b^2+h_c^2}{h_a^2+h_bh_c}+1+1}=$$

$$=\sum_{cyc}\frac{3(h_a^2+h_bh_c)}{2h_a^2+(h_b+h_c)^2}\overset{CBS}{\ge}\sum_{cyc}\frac{3(h_a^2+h_bh_c)}{2h_a^2+2(h_b^2+h_c^2)}=\frac{3}{2}\cdot\frac{\sum h_a^2+\sum h_bh_c}{\sum h_a^2}=$$

$$=\frac{3}{2}\cdot\frac{\left[\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-2\cdot\frac{2rs^2}{R}\right]+\frac{2rs^2}{R}}{\left[\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-2\cdot\frac{2rs^2}{R}\right]}$$

$$=\frac{3}{2}\cdot\frac{s^4+2s^2r^2+r^2(4R+r)^2}{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}\overset{(1)}{\ge}\frac{27r}{4R+r}$$

$$(1)\Leftrightarrow s^2[s^2(4R-17r)+2r^2(76R-17r)]+r^2(4R+r)^2(4R-17r)\ge 0$$

We distinguish the following cases:

Case 1) If $(4R-17r)\ge 0$ inequality is obviously true.

Case 2) If $(4R-17r)<0$ inequality can be rewritten as:

$r^2(4R+r)^2(4R-17r)\ge s^2[s^2(17r-4R)+2r^2(17r-76R)]$ **which follows from**

$$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$$

Remains to prove:

$$r^2(4R+r)^2(R-4r)\ge\frac{R(4R+r)^2}{2(2R-r)}[(16Rr-5r^2)(17R-4r)+2r^2(17r-76R))]$$

$$64R^3-124R^2r-25Rr^2+34r^3\ge 0\Leftrightarrow(R-2r)(64R^2+4Rr-17r^2)\ge 0$$

Which is true from $R\ge 2r$ $(Euler)$. Equality holds iff ΔABC is equilateral.

In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt[3]{\frac{h_a^2+h_bh_c}{h_b^2+h_c^2}}\ge\frac{27r}{4R+r}\ge\frac{6r}{R}$$

Marin Chirciu

Solution. See up these inequalities.

PROBLEMS FOR SENIORS

SP.436 In ΔABC the following relationship holds:

$$\frac{1}{(b+\sqrt{bc}+\sqrt{ca})h_a^3}+\frac{1}{(c+\sqrt{ca}+\sqrt{ab})h_b^3}+\frac{1}{(a+\sqrt{ab}+\sqrt{bc})h_c^3}\geq\frac{\sqrt{3}}{6F}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\sum_{cyc}\frac{1}{(b+\sqrt{bc}+\sqrt{ca})h_a^3}=\sum_{cyc}\frac{a^3}{(b+\sqrt{bc}+\sqrt{ca})(ah_a)^3}=$$

$$=\frac{1}{8F^3}\cdot\sum_{cyc}\frac{a^3}{b+\sqrt{bc}+\sqrt{ca}}\geq\frac{1}{8F^3}\cdot\sum_{cyc}\frac{a^3}{b+\frac{b+c}{2}+\frac{c+a}{2}}=$$

$$=\frac{1}{4F^3}\cdot\sum_{cyc}\frac{a^3}{2b+b+c+c+a}=\frac{1}{4F^3}\cdot\sum_{cyc}\frac{a^3}{a+3b+2c}=$$

$$=\frac{1}{4F^3}\cdot\sum_{cyc}\frac{a^4}{a^2+3ab+2ac}\overset{Bergstrom}{\geq}$$

$$\geq\frac{1}{4F^3}\cdot\frac{2(a^2+b^2+c^2)^2}{a^2+b^2+c^2+3(ab+bc+ca)+2(ab+bc+ca)}=$$

$$=\frac{1}{4F^3}\cdot\frac{2(a^2+b^2+c^2)^2}{a^2+b^2+c^2+5(ab+bc+ca)}\geq$$

$$\geq\frac{1}{4F^3}\cdot\frac{2(a^2+b^2+c^2)^2}{a^2+b^2+c^2+5(a^2+b^2+c^2)}=\frac{(a^2+b^2+c^2)^2}{4F^2\cdot 6(a^2+b^2+c^2)}\geq$$

$$\geq\frac{a^2+b^2+c^2}{24F^2}\overset{I-W}{\geq}\frac{4\sqrt{3}F}{24F^2}=\frac{\sqrt{3}}{6F}$$

Solution 2 by Nguyen Van Canh-Ben Tre-Vietnam

Let us denote: $p=\frac{a+b+c}{2}$

By AM-GM Inequality we have:

$$b+\sqrt{bc}+\sqrt{ca}\leq b+\frac{b+c}{2}+\frac{c+a}{2}=\frac{3b}{2}+c+\frac{a}{2};$$

Similary:

$$c+\sqrt{ca}+\sqrt{ab}\le\frac{3c}{2}+a+\frac{b}{2};$$

$$a+\sqrt{ab}+\sqrt{bc}\le\frac{3a}{2}+b+\frac{c}{2};$$

$$\Rightarrow b+\sqrt{bc}+\sqrt{ca}+c+\sqrt{ca}+\sqrt{ab}+a+\sqrt{ab}+\sqrt{bc}\le 3(a+b+c);$$

Now,

$$\frac{1}{(b+\sqrt{bc}+\sqrt{ca})h_a^3}+\frac{1}{(c+\sqrt{ca}+\sqrt{ab})h_b^3}+\frac{1}{(a+\sqrt{ab}+\sqrt{bc})h_c^3}$$

$$=\frac{1}{8F^3}\cdot\left(\frac{a^3}{b+\sqrt{bc}+\sqrt{ca}}+\frac{b^3}{c+\sqrt{ca}+\sqrt{ab}}+\frac{c^3}{a+\sqrt{ab}+\sqrt{bc}}\right)$$

$$\overset{\text{Holder}}{\ge}\frac{1}{3.8F^3}\cdot\frac{(a+b+c)^3}{b+\sqrt{bc}+\sqrt{ca}+c+\sqrt{ca}+\sqrt{ab}+a+\sqrt{ab}+\sqrt{bc}}$$

$$\ge\frac{1}{24F^3}\cdot\frac{(a+b+c)^3}{3(a+b+c)}=\frac{(a+b+c)^2}{72F^2}=\frac{2p.2p}{72F^2}\overset{p\ge3\sqrt{3}r}{\ge}\frac{12\sqrt{3}.pr}{72F^2}=\frac{12\sqrt{3}.F}{72F^2}=\frac{\sqrt{3}}{6F}$$

Proved.

Solution 3 by Marin Chirciu-Romania

$$b+\sqrt{bc}+\sqrt{ca}\overset{AGM}{\le}b+\frac{b+c}{2}+\frac{c+a}{2}=\frac{a+3b+2c}{2};\ \ (1)$$

$$LHS=\sum_{cyc}\frac{1}{(b+\sqrt{bc}+\sqrt{ca})h_a^3}\overset{(1)}{\ge}\sum_{cyc}\frac{1}{\left(\frac{a+3b+2c}{2}\right)h_a^3}=$$

$$=2\sum_{cyc}\frac{\frac{1}{h_a^3}}{a+3b+2c}\overset{Holder}{\ge}2\frac{\left(\sum\frac{1}{h_a}\right)^3}{3\sum(a+3b+2c)}=2\frac{\left(\sum\frac{1}{h_a}\right)^3}{3\cdot6\sum a}=$$

$$=\frac{\left(\sum\frac{a}{2F}\right)^3}{9\sum a}=\frac{\frac{1}{8F^3}\left(\sum a\right)^3}{9\sum a}=\frac{1}{8F^3}\frac{\left(\sum a\right)^2}{9}=\frac{s^2}{72F^3}=\frac{s^2}{18F^3}\overset{(2)}{\ge}\frac{\sqrt{3}}{6F^2}=RHS$$

Where $(2)\Leftrightarrow\frac{s^2}{18F^3}\ge\frac{\sqrt{3}}{6F^2}\Leftrightarrow s^2\ge3\sqrt{3}F\Leftrightarrow s\ge3\sqrt{3}r\ (Mitrinovic).$

Equality holds if and only if triangle is equilateral.

Remark. The problem can be developed.

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{(b+\sqrt{bc}+\sqrt{ca})h_a^n}\ge\left(\frac{s}{3}\right)^{n-2}\cdot\frac{1}{2F^n}, n\in\mathbb{N}, n\ge 2.$$

Solution. We have:

$$b+\sqrt{bc}+\sqrt{ca}\overset{AGM}{\le}b+\frac{b+c}{2}+\frac{c+a}{2}=\frac{a+3b+2c}{2};\ \ (1)$$

$$LHS=\sum_{cyc}\frac{1}{(b+\sqrt{bc}+\sqrt{ca})h_a^n}\overset{(1)}{\ge}\sum_{cyc}\frac{1}{\left(\frac{a+3b+2c}{2}\right)h_a^n}=2\sum_{cyc}\frac{\frac{1}{h_a^n}}{a+3b+2c}\overset{Holder}{\ge}$$

$$\ge 2\cdot\frac{\left(\sum\frac{1}{h_a}\right)^n}{3^{n-2}\sum(a+3b+2c)}=2\cdot\frac{\left(\sum\frac{1}{h_a}\right)^n}{3^{n-2}\cdot 6\sum a}=\frac{\left(\sum\frac{a}{2F}\right)^n}{3^{n-1}\sum a}=$$

$$=\frac{1}{2^nF^n}\frac{(\sum a)^{n-1}}{3^{n-1}\sum a}=\frac{s^{n-1}}{2\cdot 3^{n-1}F^n}=\left(\frac{s}{3}\right)^{n-2}\cdot\frac{1}{2F^n}=RHS$$

Equality holds if and only if triangle is equilateral.

SP.437 Prove that in acute ΔABC the following relationship holds:

$$\max\{h_a^2,h_b^2,h_c^2\}-\min\{h_a^2,h_b^2,h_c^2\}\ge\frac{1}{2}(s^2-2R^2-8Rr-3r^2)$$

Proposed by Cristian Miu-Romania

Solution 1 by proposer

Let us first prove for acute triangle the Erdos inequality:

$$\min\{h_a^2,h_b^2,h_c^2\}\le R+r\le\max\{h_a^2,h_b^2,h_c^2\}$$

It is easy to prove that

$$\sum_{cyc}a^2\cot A=4F \text{ and } \sum_{cyc}a\cot A=2(R+r)$$

So, we need to prove:

$$\min\{a,b,c\}\le\frac{4F}{2(R+r)}\le\max\{a,b,c\}$$

Because if x,y,z are real numbers and u,v,w are positive numbers

$$\min\left\{\frac{x}{u},\frac{y}{v},\frac{z}{w}\right\}\le\frac{x+y+z}{u+v+w}\le\max\left\{\frac{x}{u},\frac{y}{v},\frac{z}{w}\right\}$$

From $\min\{a,b,c\} \le \frac{4F}{2(R+r)} \le \max\{a,b,c\}$, we obtain:

$$\min\{h_a,h_b,h_c\} \le \frac{4F}{2(R+r)} \le \max\{h_a,h_b,h_c\}$$

Now, let us prove that:

$$\min\{h_a,h_b,h_c\} \le \frac{1}{2}\sqrt{a^2+b^2+c^2} \le \max\{h_a,h_b,h_c\}$$

It is easy to prove that

$$\sum_{cyc} a^2 \cot A = 4F \text{ and } \sum_{cyc} \cot A = \frac{a^2+b^2+c^2}{4F}$$

So, we obtain that

$$\min\{h_a,h_b,h_c\} \le \frac{1}{2}\sqrt{a^2+b^2+c^2} \le \max\{h_a,h_b,h_c\}$$

But: $\frac{1}{2}\sqrt{a^2+b^2+c^2} \ge R+r$ because it is Walker's inequality:

$$s^2 \ge 2R^2+8Rr+3r^2$$

$$\sum_{cyc} a^2 = 2(s^2-r^2-4Rr)$$

Now, we can write that

$$\min\{h_a,h_b,h_c\} \le R+r \le \frac{1}{2}\sqrt{a^2+b^2+c^2} \le \max\{h_a,h_b,h_c\}$$

Hence,

$$\min\{h_a^2,h_b^2,h_c^2\} \le (R+r)^2 \le \frac{1}{4}\sum_{cyc} a^2 \le \max\{h_a^2,h_b^2,h_c^2\}$$

Therefore,

$$\max\{h_a^2,h_b^2,h_c^2\} - \min\{h_a^2,h_b^2,h_c^2\} \ge \frac{1}{2}(s^2-2R^2-8Rr-3r^2)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$max\{{h_a}^2,{h_b}^2,{h_c}^2\} - min\{{h_a}^2,{h_b}^2,{h_c}^2\} \overset{(*)}{\overbrace{\ge}} \frac{1}{2}(s^2-2R^2-8Rr-3r^2)$$

$$We\ have:$$

$$(*)\ \leftrightarrow\ max\{h_a{}^2,h_b{}^2,h_c{}^2\}-min\{h_a{}^2,h_b{}^2,h_c{}^2\}\geq\frac{1}{4}(a^2+b^2+c^2)-(R+r)^2$$

So it suffices to prove :

$$max\{h_a{}^2,h_b{}^2,h_c{}^2\}\geq\frac{1}{4}(a^2+b^2+c^2)\ \ and\ \ min\{h_a{}^2,h_b{}^2,h_c{}^2\}\leq(R+r)^2$$

WLOG, we may assume that $a\geq b\geq c$. *Clearly, we have :* $max\{h_a{}^2,h_b{}^2,h_c{}^2\}=h_c{}^2$ *and* $min\{h_a{}^2,h_b{}^2,h_c{}^2\}=h_a{}^2$.

Since ΔABC *is acute, then there are positive numbers* x,y,z *such that :* $a=\sqrt{y+z},\quad b=\sqrt{z+x},\quad c=\sqrt{x+y}$.

$$We\ have:16F^2=2\sum_{cyc}a^2b^2-\sum_{cyc}a^4=2\sum_{cyc}(y+z)(z+x)-\sum_{cyc}(x+y)^2$$

$$=4\sum_{cyc}xy\ \ then:4F^2=xy+yz+zx$$

$$We\ need\ to\ prove:\ h_c{}^2=\frac{4F^2}{c^2}\geq\frac{1}{4}(a^2+b^2+c^2)\ or\ \frac{xy+yz+zx}{x+y}\geq\frac{x+y+z}{2}\ \ or\ \ yz+zx\geq x^2+y^2$$

Since $a\geq b\geq c$, *we have :* $z\geq y\geq x$ *then*

$$yz+zx\geq y.y+x.x=x^2+y^2.\ Thus,\ \ max\{h_a{}^2,h_b{}^2,h_c{}^2\}\geq\frac{1}{4}(a^2+b^2+c^2).$$

Now, since $h_a=min\{h_a,h_b,h_c\}$, *we have :*

$$h_a\leq\frac{1}{3}\sum_{cyc}h_a=\frac{1}{3}\sum_{cyc}\frac{bc}{2R}=\frac{s^2+r^2+4Rr}{6R}\overset{Gerretsen}{\leq}\frac{4R^2+8Rr+4r^2}{6R}$$

$$=\frac{(R+r)(2R+2r)}{3R}\overset{Euler}{\leq}\frac{(R+r).3R}{3R}=R+r.$$

Then : $min\{h_a{}^2,h_b{}^2,h_c{}^2\}=h_a{}^2\leq(R+r)^2$. *So the proof is completed.*

SP.438 In ΔABC the following relationship holds:

$$\frac{a^3}{b+\sqrt{bc}+\sqrt{ca}}+\frac{b^3}{c+\sqrt{ca}+\sqrt{ab}}+\frac{c^3}{a+\sqrt{ab}+\sqrt{bc}}\geq\frac{4\sqrt{3}}{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{a^3}{b+\sqrt{bc}+\sqrt{ca}}\geq\sum_{cyc}\frac{a^3}{b+\frac{b+c}{2}+\frac{c+a}{2}}=2\sum_{cyc}\frac{a^3}{2b+b+c+c+a}=$$

$$=2\sum_{cyc}\frac{a^3}{a+3b+2c}=2\sum_{cyc}\frac{a^4}{a^2+3ab+2ac}\overset{Bergstrom}{\geq}$$

$$\geq\frac{2(a^2+b^2+c^2)^2}{a^2+b^2+c^2+3(ab+bc+ca)+2(ab+bc+ca)}=$$

$$=\frac{2(a^2+b^2+c^2)^2}{a^2+b^2+c^2+5(ab+bc+ca)}\geq$$

$$\geq\frac{2(a^2+b^2+c^2)^2}{a^2+b^2+c^2+5(a^2+b^2+c^2)}=\frac{1}{3}(a^2+b^2+c^2)\overset{I-W}{\geq}$$

$$\geq\frac{1}{3}\cdot 4\sqrt{3}F=\frac{4\sqrt{3}}{3}\cdot F$$

Solution 2 by Marin Chirciu-Romania

$$b+\sqrt{bc}+\sqrt{ca}\overset{AGM}{\leq}b+\frac{b+c}{2}+\frac{c+a}{2}=\frac{a+3b+2c}{2};\ (1)$$

$$LHS=\sum_{cyc}\frac{a^3}{b+\sqrt{bc}+\sqrt{ca}}\overset{(1)}{\geq}\sum_{cyc}\frac{a^3}{\frac{a+3b+2c}{2}}=2\sum_{cyc}\frac{a^3}{a+3b+2c}\overset{Holder}{\geq}$$

$$\geq 2\frac{(\sum a)^3}{3\sum(a+3b+2c)}=2\frac{(\sum a)^3}{3\cdot 6\sum a}=\frac{(\sum a)^3}{9\sum a}=\frac{(\sum a)^2}{9}=\frac{4s^2}{9}\overset{(2)}{\geq}\frac{4\sqrt{3}}{3}F=RHS$$

$$(2)\Leftrightarrow\frac{4s^2}{9}\geq\frac{4\sqrt{3}}{3}F\Leftrightarrow s^2\geq 3\sqrt{3}F\Leftrightarrow s\geq 3\sqrt{3}r(Mitrinovic).$$

Equality holds if and only if triangle is equilateral.

Remark. The problem can be developed.

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^n}{b+\sqrt{bc}+\sqrt{ca}}\geq\left(\frac{2s}{3}\right)^{n-1},n\in\mathbb{N},n\geq 2$$

Solution.

$$b+\sqrt{bc}+\sqrt{ca}\overset{AGM}{\leq}b+\frac{b+c}{2}+\frac{c+a}{2}=\frac{a+3b+2c}{2};\ (1)$$

$$LHS=\sum_{cyc}\frac{a^n}{b+\sqrt{bc}+\sqrt{ca}}\overset{(1)}{\geq}\sum_{cyc}\frac{a^n}{\frac{a+3b+2c}{2}}=2\sum_{cyc}\frac{a^n}{a+3b+2c}\overset{Holder}{\geq}$$

$$\geq 2\frac{(\sum a)^n}{3^{n-2}\sum(a+3b+2c)}=2\frac{(\sum a)^n}{3^{n-2}\cdot 6\sum a}=\frac{(\sum a)^n}{3^{n-1}\sum a}=\frac{(\sum a)^{n-1}}{3^{n-1}}=\left(\frac{2s}{3}\right)^{n-1}=RHS$$

$$(2)\Leftrightarrow\frac{4s^2}{9}\geq\frac{4\sqrt{3}}{3}F\Leftrightarrow s^2\geq 3\sqrt{3}F\Leftrightarrow s\geq 3\sqrt{3}r(Mitrinovic).$$

Equality holds if and only if triangle is equilateral.

SP.439 If $x,y,z>0$, then:

$$\frac{1+x+x^2}{1+y+z+z^2}+\frac{1+y+y^2}{1+z+x+x^2}+\frac{1+z+z^2}{1+x+y+y^2}\ge\frac{9}{4}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$1+t+t^2\ge 3\sqrt[3]{1\cdot t\cdot t^2}=3t\Leftrightarrow t\ge\frac{1+t+t^2}{3};\forall t\in\mathbb{R}_+^*$$

$$\sum_{cyc}\frac{1+x+x^2}{1+y+z+z^2}\ge\sum_{cyc}\frac{1+x+x^2}{1+z+z^2+\frac{1+y+y^2}{3}}=$$

$$=3\sum_{cyc}\frac{1+x+x^2}{3(1+z+z^2)+1+y+y^2};(1)$$

Denote: $u=1+x+x^2;v=1+y+y^2;w=1+z+z^2$, inequality (1) becomes:

$$\sum_{cyc}\frac{1+x+x^2}{1+y+z+z^2}\ge 3\sum_{cyc}\frac{u}{v+3w}=3\sum_{cyc}\frac{u^2}{uv+3uw}\overset{Bergstrom}{\ge}$$

$$\ge 3\cdot\frac{(u+v+w)^2}{\sum(uv+3uw)}=\frac{3(u+v+w)^2}{4(uv+vw+wu)}\ge$$

$$\ge\frac{3\cdot 3(uv+vw+wv)}{4(uv+vw+wu)}=\frac{9}{4}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ a=1+x+x^2,\quad b=1+y+y^2,\quad c=1+z+z^2.$$

$$By\ AM-GM\ inequality, we\ have:\ \frac{a}{3}=\frac{1+x+x^2}{3}\ge\sqrt[3]{1.x.x^2}$$
$$=x\ \ (And\ analogs)$$

$$Then:\ \frac{1+x+x^2}{1+y+z+z^2}+\frac{1+y+y^2}{1+z+x+x^2}+\frac{1+z+z^2}{1+x+y+y^2}\ge\frac{a}{\frac{b}{3}+c}+\frac{b}{\frac{c}{3}+a}+\frac{c}{\frac{a}{3}+b}=$$

$$=3\left(\frac{a^2}{ab+3ca}+\frac{b^2}{bc+3ab}+\frac{c^2}{ca+3bc}\right)\overset{Bergström}{\ge}\frac{3(a+b+c)^2}{4(ab+bc+ca)}\ge$$
$$\ge\frac{3.3(ab+bc+ca)}{4(ab+bc+ca)}=\frac{9}{4}.$$

So the proof is completed. Equality holds iff $x=y=z=1$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{1+x+x^2}{1+y+z+z^2}+\frac{1+y+y^2}{1+z+x+x^2}+\frac{1+z+z^2}{1+x+y+y^2}$$

$$=\frac{(1+x+x^2)^2}{(1+x+x^2)(1+y+z+z^2)}+\frac{(1+y+y^2)^2}{(1+y+y^2)(1+z+x+x^2)}$$

$$+\frac{(1+z+z^2)^2}{(1+z+z^2)(1+x+y+y^2)}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{\text{cyc}}(1+x+x^2)\right)^2}{\sum_{\text{cyc}}(1+x+x^2)(1+y+y^2)+\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}$$

$$\geq\frac{3\sum_{\text{cyc}}(1+x+x^2)(1+y+y^2)}{\sum_{\text{cyc}}(1+x+x^2)(1+y+y^2)+\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}$$

$$=3.\frac{3\sum_{\text{cyc}}(1+x+x^2)(1+y+y^2)}{3\sum_{\text{cyc}}(1+x+x^2)(1+y+y^2)+3\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}$$

$$=3.\frac{3\sum_{\text{cyc}}(1+x+x^2)(1+y+y^2)+3\sum_{\text{cyc}}\left(y(1+x+x^2)\right)-3\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}{3\sum_{\text{cyc}}(1+x+x^2)(1+y+y^2)+3\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}$$

$$=3-3.\frac{\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}{\sum_{\text{cyc}}(1+x+x^2)(1+y+y^2)+\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}$$

$$=3-3.\frac{\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}{\sum_{\text{cyc}}\left(y(1+x+x^2)\right)+\sum_{\text{cyc}}(1+y^2)(1+x+x^2)+\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}$$

$$\overset{A-G}{\geq}3-\frac{\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}{2\sum_{\text{cyc}}\left(y(1+x+x^2)\right)+2\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}=3-\frac{\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}{4\sum_{\text{cyc}}\left(y(1+x+x^2)\right)}$$

$$=3-\frac{1}{4}=\frac{9}{4}\ (\text{QED})$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{1+x+x^2}{1+y+z+z^2}+\frac{1+y+y^2}{1+z+x+x^2}+\frac{1+z+z^2}{1+x+y+y^2}=$$

$$=\frac{1+z+x+x^2-z}{1+y+z+z^2}+\frac{1+x+y+y^2-x}{1+z+x+x^2}+\frac{1+y+z+z^2-y}{1+x+y+y^2}\geq\frac{9}{4}$$

$$\frac{1+x+x^2}{1+y+z+z^2}+\frac{1+y+y^2}{1+z+x+x^2}+\frac{1+z+z^2}{1+x+y+y^2}\ge$$

$$\ge\frac{9}{4}+\frac{x}{1+z+x+x^2}+\frac{y}{1+z+y+y^2}+\frac{z}{1+y+z+z^2}$$

Which is true, because

$$\frac{x}{1+z+x+x^2}+\frac{y}{1+z+y+y^2}+\frac{z}{1+y+z+z^2}\le\frac{3}{4}$$

$$\frac{x}{3x+z}+\frac{y}{3y+x}+\frac{z}{3z+y}\le\frac{3}{4}$$

$$4[x(3y+x)(3z+y)+y(3x+z)(3z+y)+z(3x+z)(3y+x)]\le$$

$$\le 3(3x+z)(3y+x)(3y+y)$$

$$4[x(9yz+3y^2+3zx+xy)+y(9xz+3xy+3z^2+yz)+z(9xy+3x^2+3yz+xz)]$$

$$\le$$

$$\le 3[(9xy+3x^2+3yz+zx)(3z+y)]$$

$$108xyz+24(xy^2+z^2y+zx^2)+4(x^2y+y^2z+z^2x)\le$$

$$\le 84xyz+27(x^2z+z^2y+y^2x)+9(x^2y+y^2z+z^2x)$$

$$27xyz\le 3(x^2z+z^2y+y^2x)+9(x^2y+y^2z+z^2x)$$

True because

$$\frac{1+z+x+x^2}{1+y+z+z^2}+\frac{1+x+y+y^2}{1+z+x+x^2}+\frac{1+y+z+z^2}{1+x+y+y^2}\ge 3$$

SP.440 If $a,b,c>0$, then prove that:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\ge\frac{\sqrt{3(a^2+b^2+c^2)}}{a+b+c}$$

Proposed by Neculai Stanciu-Romania

Solution 1 by proposer

We have: $a^2+b^2+c^2\ge ab+bc+ca$ and $\sqrt{3(a^2+b^2+c^2)}\ge a+b+c$, then writing the inequality from the statement as:

$$\frac{\sum a^2-\sum ab}{\sum ab}\ge\frac{\sqrt{3\sum a^2}-\sum a}{\sum a}\Leftrightarrow\frac{\sum a^2-\sum ab}{\sum ab}\ge\frac{\left(\sqrt{3\sum a^2}\right)^2-(\sum a)^2}{(\sum a)\left(\sqrt{3\sum a^2}+\sum a\right)}$$

$$\sum \frac{(a-b)^2}{2\sum ab} \geq \sum \frac{(a-b)^2}{\sum a\left(\sqrt{3\sum a^2}+\sum a\right)}$$

It is suffices to show that:

$$\frac{(a-b)^2}{2\sum ab} \geq \frac{(a-b)^2}{\sum a \cdot \left(\sqrt{3\sum a^2}+\sum a\right)}$$

If $a=b$ we have equality, if not, it is suffice to prove that

$$\frac{1}{2\sum ab} \geq \frac{1}{\sum a \cdot \left(\sqrt{3\sum a^2}+\sum a\right)} \Leftrightarrow \left(\sum a\right)^2 + \sum a \cdot \sqrt{3\sum a^2} \geq 2\sum ab$$

which is true, because $\left(\sum a\right)^2 \geq 3\sum ab$ and

$$\sum a \cdot \sqrt{3\sum a^2} \geq \sum a \cdot \sum a = \left(\sum a\right)^2$$

Solution 2 by Daniel Văcaru-Romania

We have:

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{\sqrt{3(a^2+b^2+c^2)}}{a+b+c} \Leftrightarrow (a^2+b^2+c^2)(a+b+c)^2 \geq 3(ab+bc+ca)^2$$

But $a^2+b^2+c^2 \geq ab+bc+ca$; (1) and $(a+b+c)^2 \geq 3(ab+bc+ca)$; (2)

Multiplying (1) and (2), we find:

$$(a^2+b^2+c^2)(a+b+c)^2 \geq 3(ab+bc+ca)^2$$

Solution 3 by Marin Chirciu-Romania

We have:

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{\sqrt{3(a^2+b^2+c^2)}}{a+b+c} \Leftrightarrow$$

$$(a^2+b^2+c^2)(a+b+c)^2 \geq 3(ab+bc+ca)^2$$

But, it is known that

$$a^2+b^2+c^2 \geq ab+bc+ca;\ (1) \text{ and } (a+b+c)^2 \geq 3(ab+bc+ca);\ (2)$$

By multiplying (1) and (2), we find:

$$(a^2+b^2+c^2)(a+b+c)^2 \geq 3(ab+bc+ca)^2$$

SP.441 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a}{r_a+2r_b} \leq \frac{3(4R-5r)}{4R+r}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\text{If } x,y,z>0,\text{then: }\sum_{cyc}\frac{x}{x+2y}\le\frac{3(x^2+y^2+z^2)}{(x+y+z)^2}$$

$$\Leftrightarrow\sum_{cyc}\left(1-\frac{y}{x+2y}\right)\ge\frac{3(x^2+y^2+z^2)}{(x+y+z)^2}$$

$$\Leftrightarrow\sum_{cyc}\frac{y}{x+2y}\cdot\sum_{cyc}y(x+2y)\ge(x+y+z)^2$$

We must show that:

$$(x+y+z)^4\ge 3(xy+yz+zx)(xy+yz+zx+2x^2+2y^2+2z^2)\Leftrightarrow$$

$$((x+y+z)^2-(xy+yz+zx)^2)^2\ge 0$$

$$\Rightarrow\sum_{cyc}\frac{r_a}{r_a+2r_b}\le\frac{3(r_a^2+r_b^2+r_c^2)}{(r_a+r_b+r_c)^2};(1)$$

$$r_a^2+r_b^2+r_c^2=(4R+r)^2-2s^2;(2)$$

$$r_a+r_b+r_c=4R+r;(3)$$

From (1),(2) and (3) it follows that:

$$\sum_{cyc}\frac{r_a}{r_a+2r_b}\le\frac{3((4R+r)^2-2s^2)}{(4R+r)^2}=3\left(1-\frac{2s^2}{(4R+r)^2}\right);(4)$$

$$2s^2\ge 6r(4R+r)(Doucet's)\Rightarrow -2s^2\le -6r(4R+r)\Rightarrow$$

$$1-\frac{2s^2}{(4R+r)^2}\le 1-\frac{6r}{4R+r}=\frac{4R-5r}{4R+r};(5)$$

From (4),(5) we get:

$$\sum_{cyc}\frac{r_a}{r_a+2r_b}\le\frac{3(4R-5r)}{4R+r}$$

Solution 2 by Marin Chirciu-Romania

Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a}{r_a+2r_b}\ge\frac{9r}{4R+r}$$

Proof. We have:

$$\sum_{cyc}\frac{r_a+2r_b-2r_b}{r_a+2r_b}\le\frac{3(4R-5r)}{4R+r}\Leftrightarrow 3-\sum_{cyc}\frac{2r_b}{r_a+2r_b}\le\frac{3(4R-5r)}{4R+r}$$

$$\sum_{cyc}\frac{r_b}{r_a+2r_b}\ge\frac{9r}{4R+r}$$

Now, using Lemma and Bergstrom's inequality, we have:

$$\sum_{cyc}\frac{r_b}{r_a+2r_b}=\sum_{cyc}\frac{r_b^2}{r_ar_b+2r_b^2}\ge\frac{(\sum r_b)^2}{\sum(r_ar_b+2r_b^2)}=\frac{(4R+r)^2}{s^2+2[(4R+r)^2-2s^2]}\overset{(1)}{\ge}$$

$$\ge\frac{9r}{4R+r}$$

$$(1)\Leftrightarrow\frac{(4R+r)^2}{s^2+2[(4R+r)^2-2s^2]}\ge\frac{9r}{4R+r}\Leftrightarrow 27rs^2+(4R+r)^3\ge 18r(4R+r)^2$$

which follows from $s^2\ge\frac{r(4R+r)^2}{R+r}$ $(Gerretsen)$

Remains to prove that:

$$27r\cdot\frac{r(4R+r)^2}{R+r}+(4R+r)^3\ge 18r(4R+r)^2\Leftrightarrow 4R^2-13Rr+10r^2\ge 0$$

$$\Leftrightarrow (R-2r)(4R-5r)\ge 0 \text{ true from } R\ge 2r\ (Euler).$$

Remark. The problem can be developed.

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}\cdot\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+r},\frac{1}{2}\le\lambda\le\frac{7}{2}$$

Solution. Lemma. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_b}{r_a+\lambda r_b}\ge\frac{27r}{(\lambda+1)(4R+r)}$$

Proof. We have:

$$\sum_{cyc}\frac{r_a+\lambda r_b-\lambda r_b}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}\cdot\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+R}\Leftrightarrow$$

$$3-\sum_{cyc}\frac{\lambda r_b}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}\cdot\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+r}\Leftrightarrow$$

$$\sum_{cyc}\frac{r_b}{r_a+\lambda r_b}\ge\frac{27r}{(\lambda+1)(4R+r)}$$

$$\sum_{cyc}\frac{r_b}{r_a+\lambda r_b}=\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}\ge\frac{(\sum r_b)^2}{\sum(r_ar_b+\lambda r_b^2)}=\frac{(4R+r)^2}{s^2+\lambda[(4R+r)^2-2s^2]}\overset{(1)}{\ge}$$

$$\frac{27r}{(\lambda+1)(4R+r)}$$

$$(1)\Leftrightarrow\frac{(4R+r)^2}{s^2+\lambda[(4R+r)^2-2s^2]}\ge\frac{27r}{(\lambda+1)(4R+r)}\Leftrightarrow$$

$(2\lambda-1)27rs^2+(\lambda+1)(4R+r)^3\ge 27\lambda r(4R+r)^2$ which follows from

$$s^2\ge\frac{r(4R+r)^2}{R+r}\ (Gerretsen)$$

Remains to prove that:

$$(2\lambda-1)27r\cdot\frac{r(4R+r)^2}{R+r}+(\lambda+1)(4R+R)^3\ge 27\lambda r(4R+r)^2\Leftrightarrow$$

$$4(\lambda+1)R^2+(5-22\lambda)Rr+(28\lambda-26)r^2\ge 0\Leftrightarrow$$

$(R-2r)[4(\lambda+1)R+(13-14\lambda)r]\ge 0$ true from $R\ge 2r(Euler)$ and $\lambda\le\frac{7}{2}$.

Equality holds if and only if triangle is equilateral.

Note. For $\lambda=2$ we get the Problem SP441. From RMM 30, Autumn 2023, proposed by Marian Ursărescu

SP.442 In ΔABC the following relationship holds:

$$\sqrt{\frac{a+c-b}{a}}+\sqrt{\frac{b+a-c}{b}}+\sqrt{\frac{c+b-a}{c}}\le\sqrt{8+\frac{2r}{R}}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

We must to prove:

$$\left(\sum_{cyc}\sqrt{\frac{a+c-b}{a}}\right)^2\le 8+\frac{2r}{R};(1)$$

$$\sqrt{\frac{a+c-b}{a}}=\sqrt{\frac{2R(\sin A+\sin B-\sin C)}{2R\sin A}}=\sqrt{\frac{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2\sin\frac{A}{2}\sin\frac{B}{2}}}=\sqrt{\frac{2\sin\frac{C}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}}}$$

$$\left(\sum_{cyc}\sqrt{\frac{a+c-b}{a}}\right)^2=\left(\sum_{cyc}\sqrt{\frac{2\sin\frac{C}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}}}\right)^2=$$

$$=\frac{\left(\cos\frac{B}{2}\sqrt{\sin C}+\cos\frac{C}{2}\sqrt{\sin A}+\cos\frac{A}{2}\sqrt{\sin B}\right)^2}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\overset{CBS}{\le}$$

$$\le\frac{(\sin A+\sin B+\sin C)\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{\frac{s}{R}\cdot\frac{4R+r}{2R}}{\frac{s}{4R}}=\frac{8(4R+r)}{R}$$

Therefore,

$$\sqrt{\frac{a+c-b}{a}}+\sqrt{\frac{b+a-c}{b}}+\sqrt{\frac{c+b-a}{c}}\le\sqrt{8+\frac{2r}{R}}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt{\frac{a+c-b}{a}}+\sqrt{\frac{b+a-c}{b}}+\sqrt{\frac{c+b-a}{c}}\overset{CBS}{\le}$$

$$\le\sqrt{\left(\frac{a+c-b}{ca}+\frac{b+a-c}{ba}+\frac{c+b-a}{ac}\right)(a+b+c)}=$$

$$=\sqrt{\left(\frac{2\cos^2\frac{B}{2}}{s}+\frac{2\cos^2\frac{C}{2}}{s}+\frac{2\cos^2\frac{A}{2}}{s}\right)\cdot 2s}=\sqrt{4\sum_{cyc}\cos^2\frac{A}{2}}=$$

$$=\sqrt{2\left(4+\frac{r}{R}\right)}=\sqrt{8+\frac{2r}{R}}$$

Therefore,

$$\sqrt{\frac{a+c-b}{a}}+\sqrt{\frac{b+a-c}{b}}+\sqrt{\frac{c+b-a}{c}}\le\sqrt{8+\frac{2r}{R}}$$

SP.443 Determine all functions $f:\mathbb{R}\to(0,+\infty)$ continuous such that

$$f(4x)\cdot f(3x)=2^x;\forall x\in\mathbb{R}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$f(4x)\cdot f(3x)=2^x;\forall x\in\mathbb{R}\Leftrightarrow \log_2 f(4x)+\log_2 f(3x)=\log_2 2^x=x$$

Let $g(x)=\log_2 f(x)$, then $g(4x)+g(3x)=x$

$$x\to\frac{x}{4}: g(x)+g\left(\frac{3}{4}x\right)=\frac{x}{4}$$

$$x\to\frac{3}{4}x: g\left(\frac{3}{4}x\right)+g\left(\left(\frac{3}{4}\right)^2x\right)=\frac{x}{4}\cdot\frac{3}{4}$$

Hence, $g(x)-g\left(\left(\frac{3}{4}\right)^2x\right)=\frac{x}{4}\left(1-\frac{3}{4}\right)=\frac{x}{16}$

Let $\left(\frac{3}{4}\right)^2=a; a\in(0,1)$, then

$$g(x)-g(ax)=\frac{x}{16}$$

$$g(ax)-g(a^2x)=\frac{1}{16}\cdot ax$$

… … … … … … … … … … …

$$g(a^{n-1}x)-g(a^nx)=\frac{1}{16}a^{n-1}x$$

Therefore,

$$g(x)-g(a^nx)=\frac{1}{16}x(1+a+a^2+\cdots+a^{n-1})$$

$$\lim_{n\to\infty}\left(g(x)-g(a^nx)\right)=\lim_{n\to\infty}\frac{1}{16}x\cdot\frac{a^n-1}{a-1}$$

$$\Rightarrow g(x)-g(0)=\frac{1}{16}\cdot\frac{x}{1-a}=\frac{1}{16}\cdot\frac{x}{1-\frac{9}{16}}=\frac{x}{7}$$

$$g(0)=\log_2 f(0)=\log_2 1$$

So, $g(x)=\frac{x}{7}$, then $f(x)=2^{\frac{x}{7}}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Replacing } x \textit{ by } \frac{x}{4}, \textit{we get}:\ f(x).f\left(\frac{3}{4}x\right)=2^{\frac{x}{4}},\ \ \forall x\in R\ \ (1)$$

$$\textit{Replacing } x \textit{ by } \frac{3x}{16}, \textit{we get}:\ f\left(\frac{3}{4}x\right).f\left(\frac{9}{16}x\right)=2^{\frac{3x}{16}},\ \ \forall x\in R\ \ (2)$$

$$\textit{From } (1) \textit{ and } (2) \textit{ we obtain}:\ \frac{f(x)}{f\left(\frac{9}{16}x\right)}=2^{\frac{x}{16}}, \forall x\in R\ \ (3)$$

Replacing x by $\left(\frac{9}{16}\right)^k x, (k \in N)$ *in* (3), *we get* : $\frac{f\left(\left(\frac{9}{16}\right)^k x\right)}{f\left(\left(\frac{9}{16}\right)^{k+1} x\right)} = 2^{\left(\frac{9}{16}\right)^k \cdot \frac{x}{16}}, \ \forall x \in R$

Multiplying this relation for $k = 0, 1, \dots, n-1, n \in N^*$, *we obtain* : $\frac{f(x)}{f\left(\left(\frac{9}{16}\right)^n x\right)}$

$$= 2^{\frac{x}{16} \cdot \sum_{k=0}^{n-1}\left(\frac{9}{16}\right)^k}, \ \forall x \in R$$

Thus, $f(x) = 2^{\frac{x}{16} \cdot \frac{1-\left(\frac{9}{16}\right)^n}{1-\frac{9}{16}}} \cdot f\left(\left(\frac{9}{16}\right)^n x\right) = 2^{\frac{x}{7}\left(1-\left(\frac{9}{16}\right)^n\right)} \cdot f\left(\left(\frac{9}{16}\right)^n x\right), \forall x \in R$

Since f is continuous on R then $f(x) = \lim_{n\to+\infty}\left(2^{\frac{x}{7}\left(1-\left(\frac{9}{16}\right)^n\right)} \cdot f\left(\left(\frac{9}{16}\right)^n x\right)\right) = 2^{\frac{x}{7}} \cdot f(0)$

$= c \cdot 2^{\frac{x}{7}}, \ \forall x \in R,$ *where* $c = f(0)$.

Replacing in the given equation, we obtain : $\left(c \cdot 2^{\frac{4x}{7}}\right) \cdot \left(c \cdot 2^{\frac{3x}{7}}\right) = 2^x$ *or* $c^2 = 1$ *then* $c = 1$ $(\because f : R \to (0, +\infty))$

Therefore, $f(x) = 2^{\frac{x}{7}}, \ \forall x \in R$.

Solution 3 by Ruxandra Daniela Tonilă-Romania

$$f(4x) \cdot f(3x) = 2^x \Leftrightarrow \log f(4x) + \log f(3x) = x \log 2$$

$$\log f(4x) + \log f(3x) - \frac{7x}{7}\log 2 = 0$$

$$\log f(4x) - \frac{4x}{7}\log 2 + \log f(3x) - \frac{3x}{7}\log 2 = 0$$

$$\log\left(\frac{f(4x)}{2^{\frac{4x}{7}}}\right) + \log\left(\frac{f(3x)}{2^{\frac{3x}{7}}}\right) = 0$$

Let be $g: (0, \infty) \to \mathbb{R}, g(x) = \log\left(\frac{f(x)}{2^{\frac{x}{7}}}\right)$, then

$$g(4x) + g\left(\frac{3}{4}x\right) = 0$$

$$x \to \frac{x}{4}: g\left(\frac{3x}{4}\right) + g\left(\left(\frac{3}{4}\right)^2 x\right) = 0 \overset{(-)}{\Rightarrow} g(x) - g\left(\left(\frac{3}{4}\right)^2 x\right) = 0$$

$$x \to \left(\frac{3}{4}\right)^2 x: \ g\left(\left(\frac{3}{4}\right)^2 x\right) - g\left(\left(\frac{3}{4}\right)^2 x\right) = 0$$

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$$x \to \left(\frac{3}{4}\right)^n x:\ g\left(\left(\frac{3}{4}\right)^n x\right) - g\left(\left(\frac{3}{4}\right)^{n+2} x\right) = 0 \overset{(+)}{\Rightarrow}$$

$$g(x) - g\left(\left(\frac{3}{4}\right)^{n+2} x\right) = 0 \Rightarrow g(x) = g\left(\left(\frac{3}{4}\right)^{n+2} x\right); \forall x \in (0,\infty), n \in \mathbb{N}$$

Since $g(x) = h(t(x))$, where $h(x) = \log x$ and $t(x) = \frac{f(x)}{2^{\frac{x}{7}}}$ are continuous, so $g(x)$ is continuous too, (1).

$$g(x) = g\left(\left(\frac{3}{4}\right)^{n+2} x\right) \Leftrightarrow g(x) = \lim_{n\to\infty} g\left(\left(\frac{3}{4}\right)^{n+2} x\right) \overset{(1)}{=}$$

$$= g\left(\lim_{n\to\infty}\left(\frac{3}{4}\right)^{n+2} x\right) = g(0) = \log(f(0))$$

$$f(4x)\cdot f(3x) = 2^x \Leftrightarrow (f(0))^2 = 1 \Leftrightarrow \log(f(0)) = 0$$

$$\text{Hence, } g(x) = 0, \forall x \in (0,\infty) \Leftrightarrow \log\left(\frac{f(x)}{2^{\frac{x}{7}}}\right) = 0 \Leftrightarrow f(x) = 2^{\frac{x}{7}}$$

Solution 4 by Ravi Prakash-New Delhi-India

Let $g(x) = \log_2 f(x), x \in \mathbb{R}$. As $f(x)$ is continuous on $\mathbb{R}$, $g(x)$ is continuous in $\mathbb{R}$.

Since, $f(4x)\cdot f(3x) = 2^x \Rightarrow \log_2 f(4x) + \log_2 f(3x) = x$

$$\Rightarrow g(4x) + g(3x) = x, \forall x \in \mathbb{R};\ (1)$$

$$g(0) = 0$$

$$x \to \frac{3}{4}x:\ g(3x) - g\left(\frac{3^2}{4}x\right) = \frac{3}{4}x; (2)$$

From (1) and (2) we get:

$$g(4x) - g\left(\frac{3^2}{4}x\right) = \frac{1}{4}x$$

$$g(x) - g\left(\left(\frac{3}{4}\right)^2 x\right) = \frac{1}{4^2}x$$

$$g(x) - g\left(\frac{9}{16}x\right) = \frac{1}{16}x;\ (3)$$

Putting $x \to \frac{9}{16}x, \left(\frac{9}{16}\right)^2 x, \dots, (9.16)^n x$ in (3), we get:

$$g\left(\frac{9}{16}x\right)-g\left(\left(\frac{9}{16}\right)^2x\right)=\frac{1}{16}\left(\frac{9}{16}x\right)$$

$$g\left(\left(\frac{9}{16}\right)^2x\right)-g\left(\left(\frac{9}{16}\right)^3x\right)=\frac{1}{16}\left(\frac{9}{16}\right)^2x$$

… … … … … … … … … … … … … … … … … …

$$g\left(\left(\frac{9}{16}\right)^nx\right)-g\left(\left(\frac{9}{16}\right)^{n+1}x\right)=\frac{1}{16}\left(\frac{9}{16}\right)^nx$$

Adding the above relationships, we get:

$$g(x)-g\left(\left(\frac{9}{16}\right)^{n+1}x\right)=\frac{x}{16}\left[1+\frac{9}{16}+\left(\frac{9}{16}\right)^2+\cdots+\left(\frac{9}{16}\right)^n\right]$$

Taking limit as $n\to\infty$, we get

$$g(x)=\frac{x}{16}\cdot\frac{1}{1-\frac{9}{16}}=\frac{x}{7}\Rightarrow \log_2 f(x)=\frac{x}{7}\Rightarrow f(x)=2^{\frac{x}{7}}$$

SP.444 If $a,b,c.0$ and $abc=1$ then:

$$\sum_{cyc}\frac{a^8+28a^6+70a^4+28a^2+1}{b^6+7b^4+7b^2+1}\ge 24$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Lemma. If $a>0$ then

$$\frac{a^8+28a^6+70a^4+28a^2+1}{a^6+7a^4+7a^2+1}\ge 8a$$

Proof. We have:

$$\frac{a^8+28a^6+70a^4+28a^2+1}{a^6+7a^4+7a^2+1}\ge 8a\Leftrightarrow$$

$$a^8-8a^7+28a^6-56a^5+70a^4-56a^3+28a^2-8a+1\ge 0\Leftrightarrow$$

$(a-1)^8\ge 0$. **Equality holds for $a=1$.**

$$\sum_{cyc}\frac{a^8+28a^6+70a^4+28a^2+1}{b^6+7b^4+7b^2+1}\overset{AGM}{\ge}3\sqrt[3]{\prod_{cyc}\frac{a^8+28a^6+70a^4+28a^2+1}{a^6+7a^4+7a^2+1}}\overset{Lemma}{\ge}$$

$$\geq 3\sqrt[3]{\prod_{cyc} 8a} = 24$$

Equality holds for $a = b = c = 1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, let's prove that;

$$\frac{a^8 + 28a^6 + 70a^4 + 28a^2 + 1}{a^6 + 7a^4 + 7a^2 + 1} \geq 8a, \forall a > 0; (1)$$

$$(1) \Leftrightarrow a^8 - 8a^7 + 28a^6 - 56a^5 + 70a^4 - 56a^3 + 28a^2 - 8a + 1 \geq 0$$

$\Leftrightarrow (a-1)^8 \geq 0$ which is true.

Then,

$$\sum_{cyc} \frac{a^8 + 28a^6 + 70a^4 + 28a^2 + 1}{b^6 + 7b^4 + 7b^2 + 1} \overset{AGM}{\geq} 3\sqrt[3]{\prod_{cyc} \frac{a^8 + 28a^6 + 70a^4 + 28a^2 + 1}{a^6 + 7a^4 + 7a^2 + 1}} \overset{(1)}{\geq}$$

$$\geq 3\sqrt[3]{\prod_{cyc} 8a} \overset{abc=1}{=} 24$$

Equality holds for $a = b = c = 1$.

SP.445 If $x, y, z > 0$ and $\lambda \geq 3$ then:

$$\sum_{cyc} \frac{x}{\sqrt{4y^2 + (\lambda^2 - 8)yz + 4z^2}} \geq \frac{3}{\lambda}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Using Holder's inequality, we have:

$$\sum_{cyc} \frac{x}{\sqrt{4y^2 + (\lambda^2 - 8)yz + 4z^2}} \sum_{cyc} \frac{x}{\sqrt{4y^2 + (\lambda^2 - 8)yz + 4z^2}} \sum_{cyc} x\left(4y^2 + (\lambda^2 - 8)yz + 4z^2\right)$$

$$\geq \left(\sum_{cyc} x\right)^3 \Leftrightarrow$$

$$\left(\sum_{cyc}\frac{x}{\sqrt{4y^2+(\lambda^2-8)yz+4z^2}}\right)^2\sum_{cyc}x\left(4y^2+(\lambda^2-8)yz+4z^2\right)\geq\left(\sum_{cyc}x\right)^3\Leftrightarrow$$

$$\left(\sum_{cyc}\frac{x}{\sqrt{4y^2+(\lambda^2-8)yz+4z^2}}\right)^2\geq\frac{\left(\sum_{cyc}x\right)^3}{\sum_{cyc}x\left(4y^2+(\lambda^2-8)yz+4z^2\right)}\overset{(1)}{\geq}\frac{9}{\lambda^2}$$

$$(1)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{\sum_{cyc}x(4y^2+(\lambda^2-8)yz+4z^2)}\geq\frac{9}{\lambda^2}\Leftrightarrow$$

$$\lambda^2(\sum x)^3\geq 9\sum x(4y^2+(\lambda^2-8)yz+4z^2)\Leftrightarrow$$

$$\lambda^2\sum x^3+(3\lambda^2-36)\sum yz(y+z)\geq(21\lambda^2-216)xyz\Leftrightarrow$$

$$\lambda^2\sum x^3+(216-21\lambda^2)xyz\geq(36-3\lambda^2)\sum yz(y+z)$$

which is true from $\sum x^3+3xyz\geq\sum yz(y+z);(Schur's)\Big|\cdot\lambda^2\Rightarrow$

$$\lambda^2\sum x^3+3\lambda^2xyz+(216-24\lambda^2)xyz\geq\lambda^2\sum yz(y+z)+(216-24\lambda^2)xyz\Leftrightarrow$$

$$\lambda^2\sum x^3+(216-21\lambda^2)xyz\geq\lambda^2\sum yz(y+z)+(216-24\lambda^2)xyz\Leftrightarrow$$

$$\lambda^2yz(y+z)+(216-24\lambda^2)xyz\geq(36-3\lambda^2)\sum yz(y+z)\Leftrightarrow$$

$$(4\lambda^2-36)\sum yz(y+z)+(216-24\lambda^2)xyz\geq 0\Leftrightarrow$$

$$4(\lambda^2-9)\sum yz(y+z)+24(9-\lambda^2)xyz\geq 0\Leftrightarrow$$

$4(\lambda^2-9)[\sum yz(y+z)-6xyz]\geq 0$which is true from $(\lambda^2-9)\geq 0$ and $\sum yz(y+z)\geq 6xyz(AM-GM)$. Equality holds for $x=y=z$.

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Applying Holder's inequality, we have:

$$\left(\sum_{cyc}\frac{x}{\sqrt{4y^2+(\lambda^2-8)yz+4z^2}}\right)^2\left\{\sum_{cyc}x\left(4y^2+(\lambda^2-8)yz+4z^2\right)\right\}\geq\left(\sum_{cyc}x\right)^3$$

$$\sum_{cyc}\frac{x}{\sqrt{4y^2+(\lambda^2-8)yz+4z^2}}\geq\sqrt{\frac{(x+y+z)^3}{4\sum xy(x+y)+3\left(\lambda^2-8\right)xyz}}$$

We need to prove that:

$$\frac{(x+y+z)^3}{4\sum xy(x+y)+3(\lambda^2-8)xyz}\geq\frac{9}{\lambda^2}\Leftrightarrow$$

$$\lambda^2(x+y+z)^3 \ge 36\sum_{cyc} xy(x+y) + 27(\lambda^2-8)xyz$$

$$(\lambda^2-9)\{(x+y+z)^3 - 27xyz\} + 9\{(x+y+z)^3 - 27xyz\} \ge$$

$$(\lambda^2-9)\{(x+y+z)^3 - 27xyz\} + 9(p^3 - 4pq + 9r) \ge 0$$

Which is true, where $p = x+y+z, q = xy+yz+zx$ and $p^3 - 4pq + 9r \ge 0$

Therefore,

$$\sum_{cyc} \frac{x}{\sqrt{4y^2 + (\lambda^2-8)yz + 4z^2}} \ge \frac{3}{\lambda}$$

Solution 3 by Michael Sterghiou-Greece

$$\sum_{cyc} \frac{x}{\sqrt{4y^2 + (\lambda^2-8)yz + 4z^2}} \ge \frac{3}{\lambda}; \quad (1)$$

Denote $(p,q,r) = (\sum x, \sum xy, xyz)$. We can assume WLOG, $p = 3$, then $q \le 3, r \le 1$.
The function $f(t) = \frac{1}{\sqrt{t}}$ is convex on $(0,\infty)$ so by applying generalized Jensen's inequality with weights x, y, z, we get:

$$\sum_{cyc} \frac{x}{\sqrt{4y^2 + (\lambda^2-8)yz + 4z^2}} \ge \frac{x+y+z}{\sqrt{\frac{\sum[x(4y^2 + (\lambda^2-8)yz + 4z^2]}{x+y+z}}}$$

$$= \frac{3}{\sqrt{\frac{4\cdot 3(q-r) + 3r(\lambda^2-8)}{3}}}$$

Because $\sum(xy^2 + xz^2) = pq - 3r$
It suffices to show after squaring that $\lambda^2 \ge 4q - 12r + r\lambda^2$ or
$f(\lambda) = 4q - 12r - (1-r)\lambda^2 \le 0 \Rightarrow f(\lambda) \searrow$. So,
$f(\lambda) \le f(3) = 4q - 3r - 9 \overset{?}{\le} 0$ which is true from 3 rd degree Schur's inequality.
Equality holds for $x = y = z$.

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{x}{\sqrt{4y^2 + (\lambda^2-8)yz + 4z^2}} = \sum \frac{x\sqrt{x}}{\sqrt{4xy^2 + (\lambda^2-8)xyz + 4z^2x}}$$

$$= \sum \frac{x^{\frac{3}{2}}}{(4xy^2 + (\lambda^2-8)xyz + 4z^2x)^{\frac{1}{2}}} \overset{Radon}{\ge} \frac{(\sum_{cyc} x)^{\frac{3}{2}}}{(4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz)^{\frac{1}{2}}} \overset{?}{\ge} \frac{3}{\lambda}$$

$$\Leftrightarrow \frac{\left(\sum_{cyc} x\right)^3}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz} \overset{?}{\geq} \frac{9}{\lambda^2}$$

$$\Leftrightarrow \frac{\sum_{cyc} x^3 + 3\sum_{cyc} xy^2 + 3\sum_{cyc} x^2y + 6xyz}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz} \overset{?}{\geq} \frac{9}{\lambda^2}$$

$$\Leftrightarrow \frac{\sum_{cyc} x^3 + 3\sum_{cyc} xy^2 + 3\sum_{cyc} x^2y + 6xyz - 4\sum_{cyc} xy^2 - 4\sum_{cyc} x^2y - 3(\lambda^2-8)xyz}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz} \overset{?}{\geq} \frac{9-\lambda^2}{\lambda^2}$$

$$\Leftrightarrow \frac{\sum_{cyc} x^3 + 3xyz - \sum_{cyc} xy^2 - \sum_{cyc} x^2y + 3xyz(9-\lambda^2)}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz} \overset{?}{\geq} \frac{9-\lambda^2}{\lambda^2}$$

$$\Leftrightarrow \frac{\sum_{cyc} x^3 + 3xyz - \sum_{cyc} xy^2 - \sum_{cyc} x^2y}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz}$$

$$+\frac{\lambda^2-9}{\lambda^2} \underset{(*)}{\overset{?}{\geq}} \frac{3(\lambda^2-9)xyz}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz}$$

$$\text{Now, Schur} \Rightarrow \sum_{cyc} x^3 + 3xyz - \sum_{cyc} xy^2 - \sum_{cyc} x^2y \geq 0$$

$$\therefore \frac{\sum_{cyc} x^3 + 3xyz - \sum_{cyc} xy^2 - \sum_{cyc} x^2y}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz} \geq 0 \Rightarrow \text{LHS of } (*)$$

$$\geq \frac{\lambda^2-9}{\lambda^2} \overset{?}{\geq} \frac{3(\lambda^2-9)xyz}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz}$$

$$\overset{\because \lambda\geq 3 \Rightarrow \lambda^2-9\geq 0}{\Leftrightarrow} \frac{1}{\lambda^2} \overset{?}{\geq} \frac{3xyz}{4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3(\lambda^2-8)xyz}$$

$$\Leftrightarrow 4\sum_{cyc} xy^2 + 4\sum_{cyc} x^2y + 3\lambda^2 xyz - 24xyz \overset{?}{\geq} 3\lambda^2 xyz$$

$$\Leftrightarrow \sum_{cyc} x^2y + \sum_{cyc} xy^2 \overset{?}{\geq} 6xyz \to \text{true}$$

$$\because \sum_{cyc} x^2y + \sum_{cyc} xy^2 \overset{A-G}{\geq} 3xyz + 3xyz = 6xyz \therefore (*) \text{ is true} \Rightarrow \forall\, x,y,z > 0 \text{ and } \lambda$$

$$\geq 3, \sum \frac{x}{\sqrt{4y^2 + (\lambda^2-8)yz + 4z^2}} \geq \frac{3}{\lambda} \text{ (QED)}$$

SP.446 If $x, y \geq 0, x + y > 0$ then in any convex quadrilateral with sides a, b, c, d and r –inradii, holds:

$$(a^2x + b^2y)^6 + (b^2x + c^2y)^6 + (c^2x + d^2y)^6 + (d^2x + a^2y)^6$$
$$\geq 4^7(x+y)^6 r^{12}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$LHS=\frac{(a^2x+b^2y)^6}{1^5}+\frac{(b^2x+c^2y)^6}{1^5}+\frac{(c^2x+d^2y)^6}{1^5}+\frac{(d^2x+a^2y)^6}{1^5}\overset{Radon}{\geq}$$

$$\geq\frac{(a^2x+b^2y+b^2x+c^2y+c^2x+d^2y+d^2x+a^2y)^6}{(1+1+1+1)^5}=$$

$$=\frac{\left(a^2(x+y)+b^2(x+y)+c^2(x+y)+d^2(x+y)\right)^6}{4^5}=$$

$$=\frac{(x+y)^6(a^2+b^2+c^2+d^2)^6}{4^5}=$$

$$=\frac{(x+y)^6}{4^5}\cdot\left(\frac{a^2}{1}+\frac{b^2}{1}+\frac{c^2}{1}+\frac{d^2}{1}\right)^6\overset{Bergstrom}{\geq}$$

$$\geq\frac{(x+y)^6}{4^5}\cdot\frac{(a+b+c+d)^{12}}{(1+1+1+1)^6}=\frac{(x+y)^6}{4^5}\cdot\frac{(2s)^{12}}{4^6}=$$

$$=\frac{(x+y)^6}{4^{11}}\cdot 4^6\cdot s^{12}\overset{Mitrinovic}{\geq}\frac{(x+y)^6}{4^5}\cdot\left(4r\cdot\tan\frac{\pi}{4}\right)^{12}=$$

$$=\frac{(x+y)^6}{4^5}\cdot 4^{12}\cdot r^{12}=4^7(x+y)^6\cdot r^{12}$$

Equality holds for square: $a=b=c=d; r=\frac{a}{2}$.

Solution 2 by Adrian Popa-Romania

$x,y\geq 0, x+y>0, ABCD$ —quadrilateral convex with sides a,b,c,d and r —inradii.

$$\frac{(a^2x+b^2y)^6}{1^5}+\frac{(b^2x+c^2y)^6}{1^5}+\frac{(c^2x+d^2y)^6}{1^5}+\frac{(d^2x+a^2y)^6}{1^5}\overset{Radon}{\geq}$$

$$\geq\frac{(a^2x+b^2y+b^2x+c^2y+c^2x+d^2y+d^2x+a^2y)^6}{(1+1+1+1)^5}=$$

$$=\frac{\left(a^2(x+y)+b^2(x+y)+c^2(x+y)+d^2(x+y)\right)^6}{4^5}=$$

$$=\frac{(x+y)^6(a^2+b^2+c^2+d^2)^6}{4^5}$$

We must to prove that:

$$(a^2+b^2+c^2+d^2)^6(x+y)^6\geq 4^{12}(x+y)^6\cdot r^{12}\,\Big|:(x+y)^6$$

$$(a^2+b^2+c^2+d^2)^6 \ge 4^{12}r^{12} \Leftrightarrow a^2+b^2+c^2+d^2 \ge 16r^2$$

$ABCD$ −circumscribed, then $F^2 = abcd = r^2s^2$

$$abcd = r^2 \cdot \frac{(a+b+c+d)^2}{4}$$

$$\frac{a+b+c+d}{4} \overset{AGM}{\ge} \sqrt[4]{abcd} \Rightarrow (a+b+c+d)^2 \ge 16\sqrt{abcd} \,\Big|\cdot \frac{r^2}{4}$$

$$\frac{r^2}{4}\cdot(a+b+c+d)^2 \ge 4r^2\sqrt{abcd} \Rightarrow abcd \ge 4r^2\sqrt{abcd}\,\Big| : \sqrt{abcd}$$

$$\Rightarrow \sqrt{abcd} \ge 4r^2$$

Hence,

$$a^2+b^2+c^2+d^2 \overset{AGM}{\ge} 4\sqrt{abcd} > 4\cdot 4r^2 = 16r^2 (true.)$$

SP.447 If $x \ge 0$ then:

$$e^x \cdot 2^{e^{-x}-1} + e^{-x}\cdot 2^{e^x-1} \ge \cosh x \cdot 2^{\operatorname{sech} x}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

If $f:[0,\infty)\to\mathbb{R}$ is convex function then by Jensen's inequality:

$$\frac{e^x}{e^x+e^{-x}}\cdot f(e^{-x}) + \frac{e^{-x}}{e^x+e^{-x}}\cdot f(e^x) \ge f\left(\frac{e^x}{e^x+e^{-x}}\cdot e^{-x} + \frac{e^{-x}}{e^x+e^{-x}}\cdot e^x\right) =$$

$$= f\left(\frac{2}{e^x+e^{-x}}\right) = f\left(\frac{1}{\cosh x}\right) = f(\operatorname{sech} x); \quad (1)$$

Let be $f:[0,\infty)\to\mathbb{R}, f(x) = 2^x$, then $f'(x) = 2^x\log 2$ and

$f''(x) = 2^x \log^2 2 > 0 \Rightarrow f$ −convex function hence, by (1) it follows that:

$$\frac{e^x}{e^x+e^{-x}}\cdot 2^{e^{-x}} + \frac{e^{-x}}{e^x+e^{-x}}\cdot 2^{e^x} \ge 2^{\operatorname{sech} x} \Leftrightarrow$$

$$\frac{e^x\cdot 2^{e^{-x}-1}}{\cosh x} + \frac{e^{-x}\cdot 2^{e^x-1}}{\cosh x} \ge 2^{\operatorname{sech} x} \Leftrightarrow$$

$$e^x\cdot 2^{e^{-x}-1} + e^{-x}\cdot 2^{e^x-1} \ge \cosh x \cdot 2^{\operatorname{sech} x}$$

Equality holds for $x = 0$.

Solution 2 by Tapas Das-India

$$e^x\cdot 2^{e^{-x}-1} + e^{-x}\cdot 2^{e^x-1} \overset{AGM}{\ge} 2\sqrt{e^x\cdot 2^{e^{-x}-1}\cdot e^{-x}\cdot 2^{e^x-1}} =$$

$$= 2\sqrt{e^x e^{-x} 2^{e^x+e^{-x}-2}} = 2\cdot 2^{-1}\cdot 2^{\frac{e^x+e^{-x}}{2}} = 2^{\cosh x} \geq \cosh x\,(\text{ because } 2^t > t)$$

$$\cosh x = 2^{\operatorname{sech} x}\cdot\cosh x = \cosh x\left(1 - 2^{\sinh x}\right)$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \geq \sqrt{e^x\cdot e^{-x}} = 1$$

$$\text{Let } f(x) = 1 - 2^{\operatorname{sech} x} \Rightarrow f'(x) = 2^{\operatorname{sech} x}\log 2\,(\operatorname{sech} x\tanh x)$$

$$f'(x) > 0, \forall x \geq 0 \Rightarrow f \nearrow \Rightarrow f(x) \geq f(0) \Rightarrow 1 - 2^{\operatorname{sech} x} \geq 0$$

$$\cosh x\left(1 - 2^{\sinh x}\right) > 0 \Rightarrow \cosh x > 2^{\sinh x}\cosh x$$

$$e^x\cdot 2^{e^{-x}-1} + e^{-x}\cdot 2^{e^x-1} \geq \cosh x$$

$$e^x\cdot 2^{e^{-x}-1} + e^{-x}\cdot 2^{e^x-1} \geq 2^{\operatorname{sech} x}\cdot\cosh x$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{The function } f(x) = 2^x \textit{ is convex on } (0,\infty)\quad \left(\because\ f''(x) = (\log 2)^2 . 2^x \geq 0\right)$$

$$\textit{By Jensen's inequality, we have}:$$

$$\frac{1}{\cosh x}\left(e^x . 2^{e^{-x}-1} + e^{-x} . 2^{e^x-1}\right) = \frac{e^x}{e^x + e^{-x}} . f(e^{-x}) + \frac{e^{-x}}{e^x + e^{-x}} . f(e^x) \geq$$

$$\geq f\left(\frac{e^x}{e^x + e^{-x}} . e^{-x} + \frac{e^{-x}}{e^x + e^{-x}} . e^x\right) = f(\operatorname{sech} x) = 2^{\operatorname{sech} x}.$$

$$\textit{Therefore,}\qquad e^x . 2^{e^{-x}-1} + e^{-x} . 2^{e^x-1} \geq \cosh x\, .\, 2^{\operatorname{sech} x},\qquad \forall x \geq 0.$$

Solution 4 by Ravi Prakash-New Delhi-India

Firstly, we show that:

$$2^t \geq t\cdot 2^{\frac{1}{t}}, \forall t \geq 1 \Leftrightarrow t\log 2 \geq \log t + \frac{1}{t}\log 2$$

$$\text{Let } f(t) = \left(t - \frac{1}{t}\right)\log 2 + \log t\,, t \geq 1, \text{then } f'(t) = \left(1 + \frac{1}{t^2}\right)\log 2 + \frac{1}{t} \geq 0, \forall t \geq 1$$

$$\Rightarrow f \text{ -strictly increasing on } [1,\infty) \Rightarrow f(t) \geq f(1) = 0, \forall t \geq 1.$$

Equality when $t = 1$. Now, for $x \geq 0$

$$e^x\cdot 2^{e^{-x}-1} + e^{-x}\cdot 2^{e^x-1} \overset{AGM}{\geq} 2\sqrt{e^x\cdot 2^{e^{-x}-1}\cdot e^{-x}\cdot 2^{e^x-1}} =$$

$$= 2\sqrt{e^x e^{-x} 2^{e^x+e^{-x}-2}} = 2^{\cosh x} \geq \cosh x\, e^{\frac{1}{\cosh x}} \Rightarrow$$

$$e^x\cdot 2^{e^{-x}-1} + e^{-x}\cdot 2^{e^x-1} \geq \cosh x\, e^{\cosh x}$$

SP.448 If $x,y,z>0$ and $\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=\frac{3}{2}$ then:

$$\frac{\sqrt{x^2+1}}{x^2-x+1}+\frac{\sqrt{y^2+1}}{y^2-y+1}+\frac{\sqrt{z^2+1}}{z^2-z+1}\le 3\sqrt{2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(x-1)^2\ge 0\Rightarrow x^2-2x+1\ge 0\Rightarrow x^2+1\le 2x^2-2x+2$$

$$\frac{1}{x^2+1}\ge\frac{1}{2x^2-2x+2}\Rightarrow\frac{1}{x^2-x+1}\le\frac{2}{x^2+1};\quad (1)$$

$$\frac{x+1}{x^2-x+1}=\frac{x\cdot 1+1\cdot 1}{x^2-x+1}\overset{CBS}{\le}\frac{\sqrt{(x^2+1^2)(1^2+1^2)}}{x^2-x+1}\overset{(1)}{\le}\frac{2}{x^2+1}\cdot\sqrt{2(x^2+1)}=\frac{2\sqrt{2}}{\sqrt{x^2+1}}$$

$$\frac{x+1}{x^2-x+1}\le\frac{2\sqrt{2}}{\sqrt{x^2+1}}\Rightarrow\frac{\sqrt{x^2+1}}{x^2-x+1}\le\frac{2\sqrt{2}}{x+1}$$

Therefore,

$$\sum_{cyc}\frac{\sqrt{x^2+1}}{x^2-x+1}\le 2\sqrt{2}\cdot\sum_{cyc}\frac{1}{x+1}=2\sqrt{2}\cdot\frac{3}{2}=3\sqrt{2}$$

Equality holds for $x=y=z=1$.

Solution 2 by Marin Chirciu-Romania

Lemma. If $x>0$ then:

$$\frac{\sqrt{x^2+1}}{x^2-x+1}\le\frac{2\sqrt{2}}{x+1}$$

Proof.

$$\frac{\sqrt{x^2+1}}{x^2-x+1}\le\frac{2\sqrt{2}}{x+1}\,\Big|^2\Leftrightarrow 7x^4-18x^3+22x^2-18x+7\ge 0\Leftrightarrow$$

$(x-1)^2(7x^2-4x+7)\ge 0$. Equality holds for $x=1$.

Now, using Lemma and summing, we get:

$$\sum_{cyc}\frac{\sqrt{x^2+1}}{x^2-x+1}\le 2\sqrt{2}\cdot\sum_{cyc}\frac{1}{x+1}=2\sqrt{2}\cdot\frac{3}{2}=3\sqrt{2}$$

Equality holds for $x=y=z=1$.

SP.449 If $a, b \geq 0$ then:

$$\sqrt{2^{a+b}} - 2^{\sqrt{ab}} + 3^{\sqrt{a+b}} - 3^{\sqrt{ab}} + \frac{1}{\sqrt{6^{a+b}}} - \frac{1}{6^{\sqrt{ab}}} \geq 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f: [0, \infty) \to \mathbb{R}, f(x) = 2^x + 3^x + 6^{-x}$, then

$f'(x) = 2^x \log 2 + 3^x \log 3 - 6^{-x} \log 6$ and

$f''(x) = 2^x \log^2 2 + 3^x \log^2 3 + 6^{-x} \log^2 6 \Rightarrow f''(x) > 0 \Rightarrow f'$ –increasing function,

then

$f'(x) \geq f'(0) \Rightarrow f'(x) \geq \log 2 + \log 3 - \log 6 = 0 \Rightarrow f$ –increasing.

$$\sqrt{ab} \leq \frac{a+b}{2} \Rightarrow f'(\sqrt{ab}) \leq f'\left(\frac{a+b}{2}\right) \Leftrightarrow$$

$$2^{\sqrt{ab}} + 3^{\sqrt{ab}} + 6^{-\sqrt{ab}} \leq 2^{\frac{a+b}{2}} + 3^{\frac{a+b}{2}} + 6^{-\frac{a+b}{2}} \Leftrightarrow$$

$$2^{\sqrt{ab}} + 3^{\sqrt{ab}} + \frac{1}{6^{\sqrt{ab}}} \leq 2^{\frac{a+b}{2}} + 3^{\frac{a+b}{2}} + \frac{1}{6^{\frac{a+b}{2}}}$$

Therefore,

$$\sqrt{2^{a+b}} - 2^{\sqrt{ab}} + 3^{\sqrt{a+b}} - 3^{\sqrt{ab}} + \frac{1}{\sqrt{6^{a+b}}} - \frac{1}{6^{\sqrt{ab}}} \geq 0$$

Solution 2 by Tapas Das-India

Let $f(x) = 2^x + 3^x + 6^{-x}$, then $f'(x) = 2^x \log 2 + 3^x \log 3 - 6^{-x} \log 6$ and

$f''(x) = 2^x \log^2 2 + 3^x \log^2 3 + 6^{-x} \log^2 6 \Rightarrow f''(x) > 0 \Rightarrow f'$ –increasing function.

Now, $f'(x) > f'(0), f'(0) = 0 \Rightarrow f'(x) > 0 \Rightarrow f$ –increasing.

$$\frac{a+b}{2} \geq \sqrt{ab}\ (AM - GM) \Rightarrow f\left(\frac{a+b}{2}\right) \geq f(\sqrt{ab})$$

$$2^{\sqrt{ab}} + 3^{\sqrt{ab}} + 6^{-\sqrt{ab}} \leq 2^{\frac{a+b}{2}} + 3^{\frac{a+b}{2}} + 6^{-\frac{a+b}{2}} \Leftrightarrow$$

$$2^{\sqrt{ab}} + 3^{\sqrt{ab}} + \frac{1}{6^{\sqrt{ab}}} \leq 2^{\frac{a+b}{2}} + 3^{\frac{a+b}{2}} + \frac{1}{6^{\frac{a+b}{2}}}$$

Therefore,

$$\sqrt{2^{a+b}}-2^{\sqrt{ab}}+3^{\sqrt{a+b}}-3^{\sqrt{ab}}+\frac{1}{\sqrt{6^{a+b}}}-\frac{1}{6^{\sqrt{ab}}}\ge 0$$

SP.450 In ΔABC the following relationship holds:

$$4s^2+2\left(s-\frac{3c}{2}\right)^2+\frac{3}{2}(a-b)^2\ge 108r^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$4s^2+2\left(s-\frac{3c}{2}\right)^2+\frac{3}{2}(a-b)^2=$$

$$=4\cdot\frac{(a+b+c)^2}{4}+2\left(\frac{a+b+c}{2}-\frac{3c}{2}\right)^2+\frac{3}{2}(a-b)^2=$$

$$=(a+b+c)^2+\frac{1}{2}(a+b-2c)^2+\frac{3}{2}(a-b)^2=$$

$$=a^2+b^2+c^2+2ab+2bc+2ca+\frac{a^2}{2}+\frac{b^2}{2}+2c^2+$$

$$+ab-2ac-2bc+\frac{3}{2}a^2+\frac{3}{2}b^2-3ab=$$

$$=a^2+2a^2+b^2+2b^2+3c^2=3(a^2+b^2+c^2)\overset{I-W}{\ge}$$

$$\ge 3\cdot 4\sqrt{3}\cdot F=12\sqrt{3}rs\overset{Mitrinovic}{\ge}12\sqrt{3}\cdot r\cdot 3\sqrt{3}\cdot r=108r^2$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu-Romania

Lemma. In ΔABC the following relationship holds:

$$4s^2+2\left(s-\frac{3c}{2}\right)^2+\frac{3}{2}(a-b)^2=3\sum_{cyc}a^2$$

Proof. We have:

$$4s^2+2\left(s-\frac{3c}{2}\right)^2+\frac{3}{2}(a-b)^2=4\left(\frac{a+b+c}{2}\right)^2+2\left(\frac{a+b+c-3c}{2}\right)^2+\frac{3}{2}(a-b)^2=$$

$$=\sum_{cyc}a^2+2\sum_{cyc}bc+2\sum_{cyc}a^2-2\sum_{cyc}bc=3\sum_{cyc}a^2$$

Using Lemma, inequality becomes as

$$3\sum_{cyc} a^2 \geq 108r^2 \Leftrightarrow \sum_{cyc} a^2 \geq 36r^2 (J.Neuberg, 1906)$$

Equality holds if and only if triangle is equilateral.

Remark. The problem can be developed.

In ΔABC the following relationship holds:

$$108r^2 \leq 4s^2 + 2\left(s - \frac{3c}{2}\right)^2 + \frac{3}{2}(a-b)^2 \leq 27R^2$$

Solution. Using Lemma, inequaity becomes:

$$108r^2 \leq 3\sum_{cyc} a^2 \leq 27R^2 \Leftrightarrow 26r^2 \leq \sum_{cyc} a^2 \leq 9R^2 (J.Neuberg, 1906)$$

Equality holds if and only if triangle is equilateral.

UNDERGRADUATE PROBLEMS

UP.436 Let $f, g: (0,\infty) \to (0,\infty)$ be continuous functions such that

$$g(x)\int_0^x f(t)\,dt = f(x)\int_0^x g(t)\,dt = 1; \forall x \in (0,\infty)$$

Prove that exists $a, b \in (0,\infty)$ such that

$$\frac{1}{3}\cdot\sum_{cyc}\left(f(\alpha) + g(\alpha)\right) \geq \frac{a+1}{\sqrt{2a+b}}, \text{where } \alpha + \beta + \gamma = 3$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

Because f, g are continuous functions, from hypothesis we deduce that f, g are differentiable functions, then:

$$(1)\begin{cases} f(x) = -\dfrac{g'(x)}{g^2(x)} \\ g(x) = -\dfrac{f'(x)}{f^2(x)} \end{cases}; \forall x > 0 \Rightarrow \frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}$$

$$\Rightarrow \exists a \in (0,\infty): \log f(x) = \log\left(a \cdot g(x)\right) \Rightarrow f(x) = a \cdot g(x)$$

From (1) we obtain that

$$a\cdot g^3(x)=-g'(x)\Leftrightarrow \frac{g'(x)}{g^3(x)}=-a$$

By integrating, we have:

$$\exists b\in(0,\infty):\frac{1}{2g^2(x)}=ax+\frac{b}{2},\Leftrightarrow g(x)=\frac{1}{\sqrt{2ax+b}} \text{ and } f(x)=\frac{a}{\sqrt{2ax+b}}$$

$$\sum_{cyc}\big(f(\alpha)+g(\alpha)\big)=(a+1)\sum_{cyc}\frac{1}{\sqrt{2a\alpha+b}}\overset{Bergstrom}{\geq}(a+1)\cdot\frac{9}{\sum\sqrt{2a\alpha+b}}\overset{CBS}{\geq}$$

$$\geq(a+1)\cdot\frac{9}{\sqrt{3(2a(\alpha+\beta+\gamma)+3b)}}=\frac{9(a+1)}{3\sqrt{2a+b}}=\frac{3(a+1)}{\sqrt{2a+b}}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{For any } x\in(0,\infty), \textit{we have}: \ \frac{f(x)}{\int_0^x f(t)dt}=\frac{g(x)}{\int_0^x g(t)dt}.$$

$$\textit{Integrating this relation, we obtain}: \ \log\left(\int_0^x f(t)dt\right)=\log\left(c.\int_0^x g(t)dt\right),\forall x\in(0,\infty) \textit{ where } c>0.$$

$$\textit{Then}: \int_0^x f(t)dt=c.\int_0^x g(t)dt,\forall x\in(0,\infty). \ \textit{Deriving this relation, we obtain}: \ f(x)=c.g(x),\forall x\in(0,\infty)$$

$$\textit{Replacing in } (1), \textit{we get}: \ cg(x)\int_0^x g(t)dt=1,\forall x\in(0,\infty).$$

Integrating this relation, we get :

$$\frac{c}{2}\left(\int_0^x g(t)dt\right)^2=x+d,\forall x\in(0,\infty), \textit{where } d>0.$$

$$\textit{Then}: \int_0^x g(t)dt=\sqrt{\frac{2(x+d)}{c}},\forall x\in(0,\infty). \ \textit{Deriving this relation, we obtain}: \ g(x)=\frac{1}{\sqrt{2c(x+d)}},\forall x\in(0,\infty)$$

$$\textit{Thus}, \quad g(x)=\frac{1}{\sqrt{2c(x+d)}} \ \textit{and} \ f(x)=c.g(x)=\sqrt{\frac{c}{2(x+d)}},\forall x\in(0,\infty)$$

$$\textit{Then,}\quad \frac{1}{3}\sum_{cyc}\big(f(\alpha)+g(\alpha)\big)=\frac{1}{3}\sum_{cyc}\left(\sqrt{\frac{c}{2(\alpha+d)}}+\frac{1}{\sqrt{2c(\alpha+d)}}\right)$$
$$=\frac{1}{3}\sum_{cyc}\frac{c+1}{\sqrt{2c(\alpha+d)}}$$

The function $h(x)=\frac{1}{\sqrt{x}}$ *is convex on* $(0,\infty)$, *then by Jensen's inequality, we have:*

$$\frac{1}{3}\sum_{cyc}\big(f(\alpha)+g(\alpha)\big)=\frac{c+1}{3}\sum_{cyc}h\big(2c(\alpha+d)\big)\geq (c+1).h\left(\frac{1}{3}\sum_{cyc}2c(\alpha+d)\right)$$
$$=(c+1).h(2c+2cd)=\frac{c+1}{\sqrt{2c+2cd}}.$$

Therefore, there exists $a,b\in(0,\infty)$ *such that*

$$\frac{1}{3}\sum_{cyc}\big(f(\alpha)+g(\alpha)\big)\geq\frac{a+1}{\sqrt{2a+b}}$$

Solution 3 by Adrian Popa-Romania

$$g(x)\cdot\int_0^x f(t)dt=1\Rightarrow\int_0^x f(t)dt=\frac{1}{g(x)},\forall x>0\Rightarrow$$
$$f(x)=-\frac{g'(x)}{g(x)}$$
$$f(x)\cdot\int_0^x g(t)dt=1\Rightarrow\int_0^x g(t)dt=\frac{1}{f(x)}\Rightarrow fg(x)=-\frac{f'(x)}{f(x)}$$
$$g'(x)=\frac{-f''(x)f^2(x)+2f(x)\big(f'(x)\big)^2}{f^4(x)}\text{ and } g^2(x)=\frac{\big(f'(x)\big)^2}{f^4(x)}$$
$$\Rightarrow f(x)=\frac{f''(x)f^2(x)-2f(x)\big(f'(x)\big)^2}{f^4(x)}\cdot\frac{f^4(x)}{\big(f'(x)\big)^2}$$
$$f(x)=\frac{f''(x)f^2(x)}{\big(f'(x)\big)^2}-2f(x)\Rightarrow 3=\frac{f''(x)f^2(x)}{\big(f'(x)\big)^2}$$
$$2=\frac{f''(x)f(x)-\big(f'(x)\big)^2}{\big(f'(x)\big)^2}\,\Big|\int\Rightarrow 2x+c=-\frac{f(x)}{f'(x)}$$
$$\frac{f'(x)}{f(x)}=-\frac{1}{2x+c}\,\Big|\int\Rightarrow\log f(x)=-\frac{1}{2}\log(2x+c)\Rightarrow f(x)=\frac{1}{\sqrt{2x+c}}$$

$$f'(x)=-\frac{\left(\sqrt{2x+c}\right)'}{2x+c}=-\frac{1}{(2x+c)\sqrt{2x+c}}$$

$$g(x)=-\frac{f'(x)}{f^2(x)}=\frac{1}{(2x+c)\sqrt{2x+c}}\cdot(2x+c)=\frac{1}{\sqrt{2x+c}}=f(x)$$

$$\frac{1}{3}\sum_{cyc}\big(f(\alpha)+g(\alpha)\big)=\frac{2}{3}\sum_{cyc}f(\alpha)=\frac{2}{3}\left(\frac{1}{\sqrt{2\alpha+c}}+\frac{1}{\sqrt{2\beta+c}}+\frac{1}{\sqrt{2\gamma+c}}\right)$$

$$\text{Let } f(x)=\frac{1}{\sqrt{2x+c}}\Rightarrow f'(x)=-(2x+c)^{-\frac{3}{2}}\Rightarrow f''(x)=3(2x+c)^{-\frac{5}{2}}>0,\forall x>0$$

$\Rightarrow f$ –convex function. So,

$$f(\alpha)+f(\beta)+f(\gamma)\geq 3f\left(\frac{\alpha+\beta+\gamma}{3}\right)$$

$$\frac{2}{3}\sum_{cyc}f(\alpha)\geq\frac{2}{3}\cdot 3\cdot\frac{1}{\sqrt{2\cdot\frac{\alpha+\beta+\gamma}{3}+c}}=\frac{2}{\sqrt{2+c}}\Rightarrow a=b=1.$$

UP. 437 Let $ABCD$ be a bicentric quadrilateral , R and r its exradius and inradius , a , b , c , and d its sides (in this order) and e , f its diagonals.

Prove that :

$$\frac{R^2-Rr+r^2\sqrt{2}}{r^2}\geq\frac{e^2}{f^2}+\frac{f^2}{e^2}$$

Proposed by Vasile Jiglău-Romania

Solution 1 by proposer

In order to prove the inequality , we'll use the following known formulas :

$$e+f=\frac{s}{2R}\left(\sqrt{4R^2+r^2}+r\right), ef=2r\left(\sqrt{4R^2+r^2}+r\right); (1)$$

We'll use the inequality of Blundon's inequality $s\leq\sqrt{4R^2+r^2}+r; (2)$

From the formulas (1) we get:

$$\frac{(e+f)^2}{ef}=\frac{s^2}{4R^2}\cdot\frac{\left(\sqrt{4R^2+r^2}+r\right)^2}{2r\left(\sqrt{4R^2+r^2}+r\right)}=\frac{s^2\left(\sqrt{4R^2+r^2}+r\right)}{8R^2r}$$

This imply:

$$\frac{e^2+f^2}{ef}+2=\frac{s^2\left(\sqrt{4R^2+r^2}+r\right)}{8R^2r}\Rightarrow 2+\frac{e^2+f^2}{ef}=\frac{s^2\left(\sqrt{4R^2+r^2}+r\right)}{8R^2r}$$

$$\Rightarrow 2+\sqrt{2+\frac{e^4+f^4}{e^2f^2}}=\frac{s^2(\sqrt{4R^2+r^2}+r)}{8R^2r}\le\frac{\left(\sqrt{4R^2+r^2}+r^2\right)^3}{8R^2r}$$

(I've used Blundon's inequality (2)). We obtain :

$$\sqrt{2+\frac{e^4+f^4}{e^2f^2}}\le\frac{(4R^2+r^2)\sqrt{4R^2+r^2}+3r(4R^2+r^2)+3r^2\sqrt{4R^2+r^2}+r^3}{8R^2r}-2=$$

$$=\frac{(R^2+r^2)\sqrt{4R^2+r^2}-R^2r+r^3}{2R^2r}$$

$$\Rightarrow\frac{e^2}{f^2}+\frac{f^2}{e^2}=\frac{e^4+f^4}{e^2f^2}\le\left[\frac{(R^2+r^2)\sqrt{4R^2+r^2}-R^2r+r^3}{2R^2r}\right]^2-2$$

which becomes after calculation equivalent to:

$$\frac{e^2}{f^2}+\frac{f^2}{e^2}\le\frac{2R^6+R^4r^2+2R^2r^4+r^6-r(R^4-r^4)\sqrt{4R^2+r^2}}{2R^4r^2}$$

It is sufficient to prove that:

$$\frac{2R^6+R^4r^2+2R^2r^4+r^6-r(R^4-r^4)\sqrt{4R^2+r^2}}{2R^4r^2}\le\frac{R^2-Rr+r^2\sqrt2}{r^2}$$

$$\Leftrightarrow 2R^6+R^4r^2+2R^2r^4+r^6-r(R^4-r^4)\sqrt{4R^2+r^2}\le 2R^6-2R^5r+2\sqrt2R^4r^2$$

$$\Leftrightarrow 2R^5-\left(2\sqrt2-1\right)R^4r+2R^2r^3+r^5\le(R^4-r^4)\sqrt{4R^2+r^2}$$

$$\Leftrightarrow\left[2R^2-\left(2\sqrt2-1\right)R^4r+2R^2r^3+r^2\right]^2\le\left[(R^4-r^4)\sqrt{4R^2+r^2}\right]^2$$

(because both terms are positive in the previous inequality) . After computation , this becomes equivalent to

$$(R-r\sqrt2)[(2\sqrt2-1)R^4-(2\sqrt2-2)R^3r+(2\sqrt2-2)R^2r^2-(\sqrt2-1)Rr^3+(\sqrt2-1)r^4]\ge 0$$

$$\left(\frac{R}{r\sqrt2}-1\right)[(8\sqrt2-4)\left(\frac{R}{r\sqrt2}-1\right)^4+(36\sqrt2-24)\left(\frac{R}{r\sqrt2}-1\right)^3+(64\sqrt2-52)\left(\frac{R}{r\sqrt2}-1\right)^2+$$

$$+(53\sqrt2-50)\left(\frac{R}{r\sqrt2}-1\right)+(18\sqrt2-19)]\ge 0$$

which is true because of the Euler's inequality for bicentric quadrilaterals $R \geq r\sqrt{2}$, and because all the coefficients in the second paranthesis are positive, qed .

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

With usual notation, the following identities are well known:

$$ef = 2r\left(\sqrt{4R^2+r^2}+r\right) \quad and \quad e+f = \frac{s\left(\sqrt{4R^2+r^2}+r\right)}{2R}$$

Using these identities, we get:

$$\frac{e^2}{f^2}+\frac{f^2}{e^2} = \left(\frac{(e+f)^2}{ef}-2\right)^2-2 = \left(\frac{s^2\left(\sqrt{4R^2+r^2}+r\right)}{8R^2r}-2\right)^2-2.$$

By W.J. Blundon and R.H. Eddy's inequality, we have: $s \leq \sqrt{4R^2+r^2}+r.$

$$Then: \frac{e^2}{f^2}+\frac{f^2}{e^2} \leq \left(\frac{\left(\sqrt{4R^2+r^2}+r\right)^3}{8R^2r}-2\right)^2-2$$

$$= \left(\frac{(R^2+r^2)\sqrt{4R^2+r^2}-r(R^2-r^2)}{2R^2r}\right)^2-2$$

$$= \frac{2R^6+R^4r^2+2R^2r^4+r^6-r(R^4-r^4)\sqrt{4R^2+r^2}}{2R^4r^2}$$

So it suffices to prove that:

$$\frac{R^2-Rr+\sqrt{2}r^2}{r^2} \geq \frac{2R^6+R^4r^2+2R^2r^4+r^6-r(R^4-r^4)\sqrt{4R^2+r^2}}{2R^4r^2} \quad (1)$$

Let $x := \frac{R}{\sqrt{2}r}$. *By L. Fejes Tóth inequality, we have:* $R \geq \sqrt{2}r$ *or* $x \geq 1$

Then, we have:

$$(1) \leftrightarrow 2x^2-\sqrt{2}x+\sqrt{2} \geq \frac{16x^6+4x^4+4x^2+1-(4x^4-1)\sqrt{8x^2+1}}{8x^4}$$

$$\leftrightarrow (4x^4-1)\left(\sqrt{8x^2+1}-3\right) \geq 8\sqrt{2}x^5-8(\sqrt{2}+1)x^4+4x^2+4$$

$$\leftrightarrow \frac{8(4x^4-1)(x-1)(x+1)}{\sqrt{8x^2+1}+3} \geq 4(x-1)(2\sqrt{2}x^4-2x^3-2x^2-x-1)$$

Since $x \geq 1$ *and* $\sqrt{8x^2+1} \leq 2\sqrt{2}x+1$, *so it suffices to prove:*

$$\frac{(4x^4-1)(x+1)}{\sqrt{2}x+2} \geq 2\sqrt{2}x^4-2x^3-2x^2-x-1$$

Or $2(2-\sqrt{2})x^4+2(2+\sqrt{2})x^3+(4+\sqrt{2})x^2+(\sqrt{2}-1)x+1 \geq 0$ *which is true.*

$$Therefore, \quad \frac{R^2-Rr+\sqrt{2}r^2}{r^2} \geq \frac{e^2}{f^2}+\frac{f^2}{e^2}.$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\text{If } m = \text{product of the lengths of the diagonals, then}: \frac{m}{4r^2} - \frac{4R^2}{m} = 1$$

$$\Rightarrow m^2 - 4mr^2 - 16R^2r^2 = 0 \Rightarrow m = \frac{4r^2 \pm \sqrt{16r^4 + 64R^2r^2}}{2}$$

$$= 2r^2 \pm 2r\sqrt{4R^2 + r^2}$$

$$= 2r\left(r + \sqrt{4R^2 + r^2}\right) \ (\because m > 0) \therefore m = ef \overset{(a)}{=} 2r\left(r + \sqrt{4R^2 + r^2}\right)$$

$$\text{Proof: Via Ptolemy's } 2^{nd} \text{ theorem}, \frac{e}{f} = \frac{ad + bc}{ab + cd} \text{ and hence}, \frac{f}{e} = \frac{ab + cd}{ad + bc} \Rightarrow \frac{e}{f} + \frac{f}{e}$$

$$= \frac{(ab + cd + ad + bc)^2 - 2(ab + cd)(ad + bc)}{(ab + cd)(ad + bc)}$$

$$= \frac{\left(a(b + d) + c(b + d)\right)^2 - 2(ab + cd)(ad + bc)}{(ab + cd)(ad + bc)}$$

$$= \frac{\left((a + c)(b + d)\right)^2 - 2(ab + cd)(ad + bc)}{(ab + cd)(ad + bc)} \overset{a+c=b+d=s}{=} \frac{s^4 - 2(ab + cd)(ad + bc)}{(ab + cd)(ad + bc)}$$

$$\Rightarrow \frac{e}{f} + \frac{f}{e} \overset{(*)}{=} \frac{s^4 - 2(ab + cd)(ad + bc)}{(ab + cd)(ad + bc)}$$

$$\text{Now}, (ab + cd)(ad + bc) = bd(a^2 + c^2) + ac(b^2 + d^2) = bd(s^2 - 2ac) + ac(s^2 - 2bd)$$

$$= s^2(ac + bd) - 4r^2s^2 \overset{\text{Ptolemy}}{=} s^2ef - 4r^2s^2 \overset{\text{via }(a)}{=} s^2.2r\left(r + \sqrt{4R^2 + r^2}\right) - 4r^2s^2$$

$$\Rightarrow (ab + cd)(ad + bc) \overset{(**)}{=} 2rs^2\left(\sqrt{4R^2 + r^2} - r\right) \therefore (*), (**) \Rightarrow \frac{e}{f} + \frac{f}{e}$$

$$= \frac{s^4 - 4rs^2\left(\sqrt{4R^2 + r^2} - r\right)}{2rs^2\left(\sqrt{4R^2 + r^2} - r\right)} = \frac{s^2 - 4r\left(\sqrt{4R^2 + r^2} - r\right)}{2r\left(\sqrt{4R^2 + r^2} - r\right)} \Rightarrow \frac{e^2}{f^2} + \frac{f^2}{e^2}$$

$$= \left(\frac{e}{f} + \frac{f}{e}\right)^2 - 2$$

$$= \left(\frac{s^2 - 4r\left(\sqrt{4R^2 + r^2} - r\right)}{2r\left(\sqrt{4R^2 + r^2} - r\right)}\right)^2 - 2$$

$$= \frac{s^4 + 8r^2\left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}\right) - 8s^2r\left(\sqrt{4R^2 + r^2} - r\right)}{4r^2\left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}\right)} \overset{\text{Blundon-Eddy}}{\leq}$$

$$\frac{s^2\left(4R^2 + 2r^2 + 2r\sqrt{4R^2 + r^2}\right) + 8r^2\left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}\right) - 8s^2r\left(\sqrt{4R^2 + r^2} - r\right)}{4r^2\left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}\right)}$$

$$= \frac{s^2\left(4R^2 + 10r^2 - 6r\sqrt{4R^2 + r^2}\right) + 8r^2\left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}\right)}{4r^2\left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}\right)}$$

$$\overset{\text{Blundon-Eddy}}{\le}\frac{\left(4R^2+2r^2+2r\sqrt{4R^2+r^2}\right)\left(4R^2+10r^2-6r\sqrt{4R^2+r^2}\right)+8r^2\left(4R^2+2r^2-2r\sqrt{4R^2+r^2}\right)}{4r^2\left(4R^2+2r^2-2r\sqrt{4R^2+r^2}\right)}$$

$$=\frac{2R^4+4R^2r^2+3r^4-r(2R^2+r^2)\sqrt{4R^2+r^2}}{r^2\left(2R^2+r^2-r\sqrt{4R^2+r^2}\right)}\overset{?}{\le}\frac{R^2-Rr+r^2\sqrt{2}}{r^2}$$

$$\Leftrightarrow 2R^3+3R^2r+Rr^2+3r^3-r\sqrt{2}(2R^2+r^2)\underset{(\bullet)}{\overset{?}{\le}}\sqrt{4R^2+r^2}\left(R^2+Rr+r^2-r^2\sqrt{2}\right)$$

$$\text{Now, }2R^3+3R^2r+Rr^2+3r^3-r\sqrt{2}(2R^2+r^2)\overset{\text{L.Fejes Toth,1948}}{\ge}2R^2.r\sqrt{2}+3R^2r+r\sqrt{2}.r^2$$
$$+3r^3-r\sqrt{2}(2R^2+r^2)=3R^2r+3r^3>0\ and\ R^2+Rr+r^2-r^2\sqrt{2}$$
$$\overset{\text{L.Fejes Toth,1948}}{\ge}R^2+r\sqrt{2}.r+r^2-r^2\sqrt{2}=R^2+r^2>0\ \therefore(\bullet)$$
$$\Leftrightarrow\left(2R^3+3R^2r+Rr^2+3r^3-r\sqrt{2}(2R^2+r^2)\right)^2$$
$$\le(4R^2+r^2)\left(R^2+Rr+r^2-r^2\sqrt{2}\right)^2$$
$$\Leftrightarrow(2R^3+3R^2r+Rr^2+3r^3)^2+2r^2(2R^2+r^2)^2$$
$$-2\sqrt{2}r(2R^2+r^2)(2R^3+3R^2r+Rr^2+3r^3)$$
$$\le(4R^2+r^2)\left((R^2+Rr+r^2)^2+2r^4-2\sqrt{2}r^2(R^2+Rr+r^2)\right)$$
$$\Leftrightarrow 2\sqrt{2}r(4R^5+2R^4r+4R^2r^3+2r^5)\ge r(4R^5+8R^4r+8R^3r^2+12R^2r^3+4Rr^4+8r^5)$$
$$\Leftrightarrow 8(4R^5+2R^4r+4R^2r^3+2r^5)^2-(4R^5+8R^4r+8R^3r^2+12R^2r^3+4Rr^4+8r^5)^2\ge 0$$
$$\Leftrightarrow 7t^{10}+4t^9-6t^8+2t^7-10t^6-12t^5-9t^4-14t^3-5t^2-4t-2$$
$$\ge 0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow(t^2-2)(7t^8+4t^7+8t^6+10t^5+6t^4+8t^3+3t^2+2t+1)\ge 0\to\text{true}$$
$$\because t\overset{\text{L.Fejes Toth,1948}}{\ge}\sqrt{2}\Rightarrow(\bullet)\text{ is true}\Rightarrow\frac{R^2-Rr+r^2\sqrt{2}}{r^2}\ge\frac{e^2}{f^2}+\frac{f^2}{e^2}\ \text{(QED)}$$

UP.438 If $t\in\mathbb{R}_+^*$ and $(a_n)_{n\ge1},(b_n)_{n\ge1}$ be positive real sequences such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in\mathbb{R}_+^*,\ \lim_{n\to\infty}\frac{b_{n+1}}{n^tb_n}=b\in\mathbb{R}_+^*.\ \text{Find:}$$

$$\Omega=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}-\frac{\sqrt[n]{a_nb_n}}{n^t}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

We have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_nb_n}}{n^{t+1}}=\lim_{n\to\infty}\sqrt[n]{\frac{a_nb_n}{n^{n(t+1)}}}=\lim_{n\to\infty}\frac{a_{n+1}b_{n+1}}{(n+1)^{(n+1)(t+1)}}\cdot\frac{n^{n(t+1)}}{a_nb_n}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{na_n}\cdot\frac{b_{n+1}}{n^tb_n}\left(\frac{n}{n+1}\right)^{(n+1)(t+1)}=ab\cdot\frac{1}{e^{t+1}}=\frac{ab}{e^{t+1}}$$

If $u_n=\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}\cdot\frac{n^t}{\sqrt[n]{a_nb_n}},\forall n\ge 2$ then $u_n=\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}\cdot\frac{n^{t+1}}{\sqrt[n]{a_nb_n}}\cdot\frac{n+1}{n}$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}b_{n+1}}{a_nb_n}\cdot\frac{n^{nt}}{(n+1)^{nt}}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}b_{n+1}}}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{na_n}\cdot\frac{b_{n+1}}{n^tb_n}\cdot\left(\frac{n}{n+1}\right)^n\cdot\frac{n^{t+1}}{\sqrt[n+1]{a_{n+1}b_{n+1}}}=ab\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{nt+t+1}\cdot\frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}b_{n+1}}}=$$

$$=abe^{-t}\cdot\lim_{n\to\infty}\frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}b_{n+1}}}=abe^{-t}\cdot\frac{e^{t+1}}{ab}=e$$

We deonte: $B_n=\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}-\frac{\sqrt[n]{a_nb_n}}{n^t}=\frac{\sqrt[n]{a_nb_n}}{n^t}(u_n-1)=$

$$=\frac{\sqrt[n]{a_nb_n}}{n^t}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n=\frac{\sqrt[n]{a_nb_n}}{n^{t+1}}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n;\forall n\ge 2$$

Hence,

$$\lim_{n\to\infty}B_n=\lim_{n\to\infty}\frac{\sqrt[n]{a_nb_n}}{n^{t+1}}\cdot\lim_{n\to\infty}\frac{u_n-1}{\log u_n}\cdot\log\left(\lim_{n\to\infty}u_n^n\right)=\frac{ab}{e^{t+!}}\cdot 1\cdot\log e=\frac{ab}{e^{t+1}}$$

Solution 2 by Marian Ursărescu-Romania

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{a_nb_n}}{n^t}\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}\cdot\frac{n^t}{\sqrt[n]{a_nb_n}}-1\right)=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{a_nb_n}}{n^{t+1}}\cdot n\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{\sqrt[n]{a_nb_n}}\cdot\left(\frac{n}{n+1}\right)^t-1\right);\quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{n}{n+1}\cdot\frac{a_{n+1}}{na_n}=$$

$$=\frac{1}{e}\cdot 1\cdot a=\frac{a}{e};\quad (2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^t}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^{nt}}}\overset{C-D}{=}\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{(n+1)t}}\cdot\frac{n^{nt}}{b_n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{tn}\cdot\frac{n}{n+1}\cdot\frac{b_{n+1}}{n^tb_n}=$$

$$=\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^n\right]^t\cdot\frac{n}{n+1}\cdot\frac{b_{n+1}}{n^tb_n}=\frac{1}{e^t}\cdot 1\cdot b=\frac{b}{e^t};\quad (3)$$

$$\lim_{n\to\infty} n\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{\sqrt[n]{a_nb_n}}\cdot\left(\frac{n}{n+1}\right)^t-1\right)=\lim_{n\to\infty} n\left(e^{\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{\sqrt[n]{a_nb_n}}\cdot\left(\frac{n}{n+1}\right)^t}-1\right)=$$

$$=\lim_{n\to\infty} n\frac{e^{\log\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{\sqrt[n]{a_nb_n}}\cdot\left(\frac{n}{n+1}\right)^t\right)}-1}{\log\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{\sqrt[n]{a_nb_n}}\cdot\left(\frac{n}{n+1}\right)^t\right)}\cdot\log\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{\sqrt[n]{a_nb_n}}\cdot\left(\frac{n}{n+1}\right)^t\right)=$$

$$=\lim_{n\to\infty} n\log\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{\sqrt[n]{a_nb_n}}\cdot\left(\frac{n}{n+1}\right)^t\right)=\lim_{n\to\infty}\log\left(\left(\frac{n}{n+1}\right)^{nt}\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{\sqrt[n]{a_nb_n}}\right)^n\right)=$$

$$=\log\left(\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^n\right)^t\frac{a_{n+1}b_{n+1}}{a_nb_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}b_{n+1}}}\right)=$$

$$=\log\left(\lim_{n\to\infty}\frac{1}{e^t}\cdot\frac{a_{n+1}}{na_n}\cdot\frac{b_{n+1}}{n^tb_n}\cdot\frac{n^{t+1}}{\sqrt[n+1]{a_{n+1}b_{n+1}}}\right)\overset{(2),(3)}{=}\log\left(\frac{1}{e^t}\cdot ab\cdot\frac{e}{a}\cdot\frac{e^t}{b}\right)=\log e=1;(4)$$

From (1) to (4), it follows that

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{a_nb_n}}{n^t}\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}\cdot\frac{n^t}{\sqrt[n]{a_nb_n}}-1\right)=\frac{ab}{e^{t+1}}$$

UP.439 If $f(x)=\frac{1-\sqrt{1-2x}}{2}$, $f_n^{-1}=\underbrace{f^{-1}\circ f^{-1}\circ\ldots\circ f^{-1}}_{n-times}$ then find:

$$\Omega=\lim_{n\to\infty}\int_0^{\frac{1}{2}}f_n^{-1}(x)\,dx$$

Proposed by Neculai Stanciu-Romania

Solution 1 by proposer

We have: $f'(x)=-2x^2-2x$ and by mathematical induction principle, we get:

$$f_n^{-1}(x)=\frac{1}{2}-2^{2^n-1}\left(x-\frac{1}{2}\right)^{2^n}$$

Indeed, for $n=1$, we have $f'(x)=\frac{1}{2}-2\left(x-\frac{1}{2}\right)^2$ true!

We assume that is true for $n=k$, and for $n=k+1$, we have:

$$f_{k+1}^{-1}(x)=f_k^{-1}(f(x))=\frac{1}{2}-2^{2^k-1}\left(\frac{1}{2}-2\left(x-\frac{1}{2}\right)^2-\frac{1}{2}\right)^{2^k}=$$

$$=\frac{1}{2}-2^{2^k-1}\left(-2\left(x-\frac{1}{2}\right)^2\right)^{2^k}=\frac{1}{2}-2^{2^{k+1}-1}\left(x-\frac{1}{2}\right)^{2^{k+1}}$$

Therefore,

$$\int_0^1 f_n^{-1}(x)\,dx=\left(\frac{1}{2}x-2^{2^k-1}\cdot\frac{1}{2^n+1}\left(x-\frac{1}{2}\right)^{2^k+1}\right)\Bigg|_0^{\frac{1}{2}}=\frac{1}{4}-\frac{1}{4(2^n+1)}$$

Hence $\Omega=\lim\limits_{n\to\infty}\int_0^{\frac{1}{2}} f_n^{-1}(x)\,dx=\frac{1}{4}$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\text{Say } x=f(s)\Rightarrow dx=\frac{ds}{1-\sqrt{1-2s}}\Rightarrow\Omega=\lim_{n\to\infty}\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-2s}}f_{n-1}^{-1}(s)\,ds\overset{s=f(u)}{=}$$

$$=\lim_{n\to\infty}\frac{1}{2^2}\int_0^{\frac{1}{2}}\frac{1}{(1-2u)^{\frac{1}{2^2}+\frac{1}{2}}}f_{n-2}^{-1}(u)du=\lim_{n\to\infty}\frac{1}{2^3}\int_0^{\frac{1}{2}}\frac{1}{(1-2s)^{\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2}}}f_{n-3}^{-1}(s)ds=$$

$$=\lim_{n\to\infty}\frac{1}{2^n}\int_0^{\frac{1}{2}}\frac{s}{(1-2s)^{\sum_{k=1}^{n}\frac{1}{2^k}}}ds\overset{u=2s}{=}\frac{1}{4}\lim_{n\to\infty}\frac{1}{2^n}\int_0^1\left[(1-u)^{\frac{1}{2^n}-1}-(1-u)^{\frac{1}{2^n}}\right]du=$$

$$=\frac{1}{4}\lim_{n\to\infty}\frac{1}{2^n}\left[\frac{1}{\frac{1}{2^n}}-\frac{1}{\frac{1}{2^n}+1}\right]=\frac{1}{4}\lim_{n\to\infty}\left(1-\frac{1}{2^n+1}\right)=\frac{1}{4}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\int_0^{\frac{1}{2}}f_n^{-1}(x)\,dx=\frac{1}{4}$$

UP.440 Find a closed form:

$$\Omega=\left(\int_0^1\frac{x^{29}-x^9}{x^{40}+1}dx\right)\left(\int_0^1\frac{x^{29}-2x^9}{x^{40}+4}dx\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\int_0^1\frac{x^{29}-x^9}{x^{40}+1}dx=\int_0^1\frac{x^{20}(x^9-x^{-11})}{x^{20}(x^{20}+x^{-20})}dx=\int_0^1\frac{x^{10-1}-x^{-10-1}}{(x^{10}+x^{-10})^2-2}dx=$$

$$=\frac{1}{10}\int_0^1\frac{(x^{10}+x^{-10})'}{(x^{10}+x^{-10})^2-2}dx=\frac{1}{20\sqrt{2}}\log\left|\frac{x^{10}+x^{-10}-\sqrt{2}}{x^{10}+x^{-10}+\sqrt{2}}\right|\Bigg|_0^1=$$

$$=\frac{1}{20\sqrt{2}}\log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)$$

$$\int_0^1\frac{x^{29}-2x^9}{x^{40}+4}dx=\int_0^1\frac{x^{20}(x^9-2x^{-11})}{x^{20}(x^{20}+4x^{-20})}dx=\int_0^1\frac{x^{10-1}-2x^{-10-1}}{(x^{10}+2x^{-10})^2-4}dx=$$

$$=\frac{1}{10}\int_0^1\frac{(x^{10}+2x^{-10})'}{(x^{10}+2x^{-10})^2-2^2}dx=\frac{1}{40}\log\left|\frac{x^{10}+2x^{-10}-2}{x^{10}+2x^{-10}+2}\right|\Bigg|_0^1=\frac{1}{40}\log\left(\frac{1}{5}\right)$$

Therefore,

$$\Omega=\frac{1}{800\sqrt{2}}\log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\cdot\log\left(\frac{1}{5}\right)$$

Solution 2 by Timson Folorunsho-Nigeria

$$I_1=\int_0^1\frac{x^{29}-x^9}{x^{40}+1}dx=\int_0^1\frac{x^9((x^{10})^2-1)}{(x^{10})^4+1}dx\overset{y=x^{10}}{=}\frac{1}{10}\int_0^1\frac{y^2-1}{y^4+1}dy=$$

$$=\frac{1}{10}\int_0^1\frac{1-\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy=\frac{1}{10}\int_0^1\frac{1-\frac{1}{y^2}}{\left(y+\frac{1}{y}\right)^2-2}dy\overset{u=y+\frac{1}{y}}{=}$$

$$=-\frac{1}{10}\int_2^\infty\frac{du}{(u-\sqrt{2})(u+\sqrt{2})}=-\frac{1}{20\sqrt{2}}\int_2^\infty\left(\frac{1}{u-\sqrt{2}}-\frac{1}{u+\sqrt{2}}\right)du=$$

$$=-\frac{1}{20\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|\Bigg|_2^\infty=\frac{1}{20\sqrt{2}}\log\left|\frac{2-\sqrt{2}}{2+\sqrt{2}}\right|$$

$$I_2=\int_0^1\frac{x^{29}-2x^9}{x^{40}+4}dx=\int_0^1\frac{x^9((x^{10})^2-2)}{(x^{10})^4+4}dx\overset{u=x^{10}}{=}$$

$$=\frac{1}{10}\int_0^1\frac{u^2-2}{u^4+4}du=\frac{1}{10}\int_0^1\frac{1-\frac{2}{u^2}}{u^2+\frac{4}{u^2}}du=\frac{1}{10}\int_0^1\frac{1-\frac{2}{u^2}}{\left(u+\frac{2}{u}\right)^2-4}du\overset{u+\frac{2}{u}}{=}$$

$$=\frac{1}{10}\int_\infty^3\frac{dv}{v^2-4}=\frac{1}{10}\int_\infty^3\frac{dv}{(v-2)(v+2)}=\frac{1}{40}\int_\infty^3\left(\frac{1}{v-2}-\frac{1}{v+2}\right)dv=$$

$$=\frac{1}{40}\log\left|\frac{v-2}{v+2}\right|\Bigg|_\infty^3=\frac{1}{40}\log\frac{1}{5}$$

$$\Omega = I_1 \cdot I_2 = \frac{1}{800\sqrt{2}}\log\left|\frac{2-\sqrt{2}}{2+\sqrt{2}}\right| \cdot \log\frac{1}{5}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\text{For } k \geq 1, \text{let } I(k) = \int_0^1 \frac{x^{29}-kx^9}{x^{40}+k^2}dx = \int_0^1 \frac{x^9(x^{20}-k)}{x^{40}+k^2}dx \overset{x^{10}=t}{=}$$

$$= \frac{1}{10}\int_0^1 \frac{t^2-k}{t^4+k^2}dt = \frac{1}{10}\int_0^1 \frac{1-\frac{k}{t^2}}{t^2+\frac{k^2}{t^2}}dt \overset{t+\frac{k}{t}=u}{=} \frac{1}{10}\int_\infty^{1+k}\frac{du}{u^2-2k} =$$

$$= \frac{1}{20\sqrt{2k}}\log\left|\frac{u-\sqrt{2k}}{u+\sqrt{2k}}\right|\Bigg|_\infty^{1+k} = \frac{1}{20\sqrt{2k}}\log\left|\frac{1+k-\sqrt{2k}}{1+k+\sqrt{2k}}\right|$$

Now,

$$\Omega = I(1)I(2) = \frac{1}{2\sqrt{2}}\log\left|\frac{2-\sqrt{2}}{2+\sqrt{2}}\right| \cdot \frac{1}{20\cdot 2}\log\left|\frac{3-2}{3+2}\right| =$$

$$= \frac{1}{800\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\log\frac{1}{5} = -\frac{1}{400\sqrt{2}}\log(\sqrt{2}-1)\log 5$$

Solution 4 by Yen Tung Chung-Taichung-Taiwan

$$\int_0^1 \frac{x^{29}-x^9}{x^{40}+1}dx = \int_0^1 \frac{x^9-x^{-11}}{x^{20}+x^{-20}}dx = \int_0^1 \frac{x^9-x^{-11}}{(x^{10}+x^{-10})^2-2}dx =$$

$$= \frac{1}{10}\int_0^1 \frac{1}{(x^{10}+x^{-10})^2-2}d(x^{10}+x^{-10}) =$$

$$= \frac{1}{20\sqrt{2}}\int_0^1\left(\frac{1}{x^{10}+x^{-10}-\sqrt{2}} - \frac{1}{x^{10}+x^{-10}+\sqrt{2}}\right)d(x^{10}+x^{-10}) =$$

$$= \frac{1}{20\sqrt{2}}\log\left|\frac{x^{10}+x^{-10}-\sqrt{2}}{x^{10}+x^{-10}+\sqrt{2}}\right|\Bigg|_0^1 = \frac{1}{20\sqrt{2}}\log\left|\frac{x^{20}-\sqrt{2}x^{10}+1}{x^{20}+\sqrt{2}x^{10}+1}\right|\Bigg|_0^1 =$$

$$= \frac{1}{20\sqrt{2}}\log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)$$

$$\int_0^1 \frac{x^{29}-2x^9}{x^{40}+4}dx = \int_0^1 \frac{x^9-2x^{-11}}{x^{20}+4x^{-20}}dx = \int_0^1 \frac{x^9-2x^{-11}}{(x^{10}+2x^{-10})^2-4}dx =$$

$$= \frac{1}{10}\int_0^1 \frac{1}{(x^{10}+2x^{-10})^2-4}d(x^{10}+2x^{-10}) =$$

$$=\frac{1}{40}\int_0^1\left(\frac{1}{x^{10}+2x^{-10}-2}-\frac{1}{x^{10}+2x^{-10}+2}\right)d(x^{10}+2x^{-10})=$$

$$=\frac{1}{40}\log\left|\frac{x^{10}+2x^{-10}-2}{x^{10}+2x^{-10}+2}\right|\Bigg|_0^1=\frac{1}{40}\log\frac{1}{5}=-\frac{1}{40}\log 5$$

UP.441 If $f:\mathbb{R}\to\left[-\frac{5}{2},\frac{5}{2}\right]$, f −continuous, then:

$$\int_{-\frac{5}{2}}^{\frac{5}{2}}\sqrt{50-8f^2(x)}\,dx+\int_{-\frac{5}{2}}^{\frac{5}{2}}f(x)\,dx\le\frac{75}{2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Denote $f(x)=y; y\in\left[-\frac{5}{2},\frac{5}{2}\right]$ then $2y\in[-5,5]$ and

$-2y\in[-5,5], 15-2y\in[10,20]$, hence $15-2y>0$.

We will start from $(5-6y)^2\ge 0; \forall y\in\left[-\frac{5}{2},\frac{5}{2}\right]\Leftrightarrow$

$$25+36y^2-60y\ge 0\Leftrightarrow 200-32y^2\le 225-60y+4y^2$$

$$200-32y^2\le(15-2y)^2\Leftrightarrow\sqrt{200-32y^2}\le 15-2y$$

$$\sqrt{200-32y^2}+2y\le 15\Leftrightarrow\sqrt{200-32f^2(x)}+2f(x)\le 15$$

$$2\sqrt{50-8f^2(x)}+2f(x)\le 15\Leftrightarrow\sqrt{50-8f^2(x)}+f(x)\le\frac{15}{2}$$

Therefore,

$$\int_{-\frac{5}{2}}^{\frac{5}{2}}\sqrt{50-8f^2(x)}\,dx+\int_{-\frac{5}{2}}^{\frac{5}{2}}f(x)\,dx\le\int_{-\frac{5}{2}}^{\frac{5}{2}}\frac{15}{2}dx=\frac{75}{2}$$

Equality holds for $f(x)=\frac{5}{6}; \forall x\in\left[-\frac{5}{2},\frac{5}{2}\right]$.

Solution 2 by Daniel Văcaru-Romania

With CBS inequality, we have:

$$\sqrt{50-8f^2(x)}+f(x)=\sqrt{2}\cdot\sqrt{25-4f^2(x)}+\frac{1}{2}[2f(x)]\le$$

$$\leq \sqrt{\left(\sqrt{2}\right)^2+\left(\frac{1}{2}\right)^2}\cdot\sqrt{25-4f^2(x)+[2f(x)]^2}=\frac{15}{2};\quad (*)$$

Integrating $(*)$ on $\left[-\frac{5}{2};\frac{5}{2}\right]$, we obtain:

$$\int_{-\frac{5}{2}}^{\frac{5}{2}}\sqrt{50-8f^2(x)}\,dx+\int_{-\frac{5}{2}}^{\frac{5}{2}}f(x)\,dx\leq\int_{-\frac{5}{2}}^{\frac{5}{2}}\frac{15}{2}dx=\frac{75}{2}$$

Equality holds for $f(x)=\frac{5}{6};\forall x\in\left[-\frac{5}{2},\frac{5}{2}\right]$.

UP.442 If $(H_n)_{n\geq1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$, then compute

$$\Omega=\lim_{n\to\infty}e^{-2H_n}\sum_{k=2}^{n}\sqrt[k]{k!}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Daniel Văcaru-Romania

We have:

$$\Omega=\lim_{n\to\infty}e^{-2H_n}\sum_{k=2}^{n}\sqrt[k]{k!}=\lim_{n\to\infty}\frac{\sum_{k=2}^{n}\sqrt[k]{k!}}{e^{2H_n}}\overset{C-S}{=}\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{e^{2H_{n+1}}-e^{2H_n}}.$$

We know that, if $x_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(n+1)$, then $(x_n)_{n\geq1}$ is convergent and his limit is γ. Furthermore, by Lagrange, for $x\to e^{2x}, x\in[H_n,H_{n+1}]$, we have

$$e^{2H_{n+1}}-e^{2H_n}=(2H_{n+1}-2H_n)2e^{c_n}=\frac{4e^{2c_n}}{n+1}, c_n\in(H_n,H_{n+1});(1)$$

But $x\to e^{2x}$ is increasing, and we obtain:

$$\frac{4e^{2H_n}}{n+1}\leq e^{2H_{n+1}}-e^{2H_n}\leq\frac{4e^{2H_{n+1}}}{n+1}$$

We obtain:

$$\frac{(n+1)\sqrt[n+1]{(n+1)!}}{4e^{2H_{n+1}}}\leq\frac{\sqrt[n+1]{(n+1)!}}{e^{2H_{n+1}}-e^{2H_n}}\leq\frac{(n+1)\sqrt[n+1]{(n+1)!}}{4e^{2H_n}};\quad (1)$$

But we have: $H_n=x_n+\log(n+1)\Rightarrow e^{2H_n}=(n+1)^2\cdot e^{2x_n}$; (2) and

$e^{2H_{n+1}}=(n+2)^2\cdot e^{2x_{n+1}}$; (3). Then (1) gets:

$$\frac{(n+1)\sqrt[n+1]{(n+1)!}}{4(n+2)^2e^{2x_{n+1}}} \le \frac{\sqrt[n+1]{(n+1)!}}{e^{2H_{n+1}}-e^{2H_n}} \le \frac{(n+1)\sqrt[n+1]{(n+1)!}}{4(n+1)^2e^{2x_n}}; \quad (3)$$

It is well known that

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$$

We have:

$$\lim_{n\to\infty}\frac{(n+1)\sqrt[n+1]{(n+1)!}}{4(n+2)^2e^{2x_{n+1}}}=\lim_{n\to\infty}\frac{(n+1)\sqrt[n+1]{(n+1)!}}{4(n+1)^2e^{2x_n}}=\frac{1}{4e^{2\gamma+1}}$$

UP.443 Determine all the derivable functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy

$$\frac{f(x^3)-f(0)}{f(x)-f(0)}=x^2, \forall x\neq 0$$

Proposed by Neculai Stanciu-Romania

Solution by proposer

We have:

$$\frac{f(x^3)-f(0)}{x^3}=\frac{f(x)-f(0)}{x}$$

Replacing x with $x^{\frac{1}{3}}$, we deduce that

$$\frac{f(x)-f(0)}{x}=\frac{f\left(x^{\frac{1}{3}}\right)-f(0)}{x^{\frac{1}{3}}}=\frac{f\left(x^{\frac{1}{9}}\right)-f(0)}{x^{\frac{1}{9}}}=\cdots=\frac{f\left(x^{\frac{1}{3^n}}\right)-f(0)}{x^{\frac{1}{3^n}}}$$

$$\lim_{n\to\infty}x^{\frac{1}{3^n}}=\begin{cases}1, \text{for } x>0\\ 0, \text{for } x=0\\ -1, \text{for } x<0\end{cases}$$

Hence, by continuity it follows that

$$\frac{f(x)-f(0)}{x}=\begin{cases}f(1)-f(0), \text{for } x>0\\ f(0)-f(-1), \text{for } x<0\end{cases}$$

$$f(x)=\begin{cases}f(0)+\big(f(1)-f(0)\big)x, \text{for } x>0\\ f(0) \quad , \text{for } x=0\\ f(0)-f(-1) \quad , \text{for } x<0\end{cases}$$

The derivability of f in 0 imply $f(1)-f(0)=f(0)-f(-1)=f'(0)$.

Hence, $f(x)=f'(0)x+f(0), \forall x\in\mathbb{R}$.

UP.444 Find:

$$\Omega = \lim_{n\to\infty}\left(e^{2H_{n+1}} - e^{2H_n}\right)\cdot\frac{1}{\sqrt[n]{n!}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} \overset{CDA}{=} \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

Denote: $\gamma_n = H_n - \log n\,;\ \lim_{n\to\infty}\gamma_n = \gamma$

$$\left(e^{2H_{n+1}} - e^{2H_n}\right)\cdot\frac{1}{\sqrt[n]{n!}} = e^{2H_n}\left(e^{2(H_{n+1}-H_n)} - 1\right)\cdot\frac{1}{\sqrt[n]{n!}} =$$

$$= e^{2H_n}\cdot\left(e^{\frac{2}{n+1}} - 1\right)\cdot\frac{1}{\sqrt[n]{n!}} = \frac{e^{2H_n}}{n^2}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{e^{\frac{2}{n+1}}-1}{\frac{2}{n+1}}\cdot\frac{2n}{n+1} =$$

$$= e^{2H_n}\cdot e^{-2\log n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{e^{\frac{2}{n+1}}-1}{\frac{2}{n+1}}\cdot\frac{2n}{n+1} =$$

$$= e^{2\gamma_n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{e^{\frac{2}{n+1}}-1}{\frac{2}{n+1}}\cdot\frac{2n}{n+1}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(e^{2H_{n+1}} - e^{2H_n}\right)\cdot\frac{1}{\sqrt[n]{n!}} = \lim_{n\to\infty}\left(e^{2\gamma_n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{e^{\frac{2}{n+1}}-1}{\frac{2}{n+1}}\cdot\frac{2n}{n+1}\right) = 2e^{1+2\gamma}$$

Solution 2 by Adrian Popa-Romania

$$\Omega = \lim_{n\to\infty}\left(e^{2H_{n+1}} - e^{2H_n}\right)\cdot\frac{1}{\sqrt[n]{n!}} = \lim_{n\to\infty}\frac{e^{2H_n}}{\sqrt[n]{n!}}\left(e^{2(H_{n+1}-H_n)} - 1\right) =$$

$$= \lim_{n\to\infty}\frac{e^{2H_n}}{\sqrt[n]{n!}}\left(e^{\frac{2}{n+1}} - 1\right) = \lim_{n\to\infty}\frac{e^{2H_n}}{\sqrt[n]{n!}}\cdot\frac{e^{\frac{2}{n+1}}-1}{\frac{2}{n+1}}\cdot\frac{2}{n+1} =$$

$$= \lim_{n\to\infty} \frac{n \cdot 2e^{2H_n}}{(n+1)\sqrt[n]{n!} \cdot n} = \lim_{n\to\infty} \frac{2e^{2H_n}}{n(n+1)} \cdot \sqrt[n]{\frac{n^n}{n!}}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} \overset{C-D}{=} \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)n!}{(n+1)!} \cdot \left(\frac{n+1}{n}\right)^n = e$$

Hence, we have:

$$\Omega = \lim_{n\to\infty} \frac{2e \cdot e^{2H_n}}{n(n+1)} = 2e \cdot \lim_{n\to\infty} \frac{e^{2H_n}}{n(n+1)} = 2e \cdot \lim_{n\to\infty} \frac{e^{2(\gamma+\log n)}}{n(n+1)} =$$

$$= 2e \cdot e^{2\gamma} \lim_{n\to\infty} \frac{n^2}{n(n+1)} = 2e^{2\gamma+1}$$

Solution 3 by Ruxandra Daniela Tonilă-Romania

$$\Omega = \lim_{n\to\infty} \left(e^{2H_{n+1}} - e^{2H_n}\right) \cdot \frac{1}{\sqrt[n]{n!}} = \lim_{n\to\infty} \frac{e^{2H_{n+1}} - e^{2H_n}}{(n+1)^2 - n^2} \cdot \frac{(n+1)^2 - n^2}{\sqrt[n]{n!}} = \Omega_1 \cdot \Omega_2$$

$$\Omega_1 = \lim_{n\to\infty} \frac{e^{2H_{n+1}} - e^{2H_n}}{(n+1)^2 - n^2} \overset{C-S}{=} \lim_{n\to\infty} \frac{e^{2H_n}}{n^2} = \lim_{n\to\infty} \frac{e^{2(\log n+\gamma)}}{n^2} =$$

$$= e^{2\gamma} \cdot \lim_{n\to\infty} \frac{e^{\log n^2}}{n^2} = e^{2\gamma}; \ (1)$$

$$\Omega_2 = \lim_{n\to\infty} \frac{(n+1)^2 - n^2}{\sqrt[n]{n!}} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n+1)^n}{n!}} \overset{C-D}{=} \lim_{n\to\infty} \frac{(2n+3)^{n+1}}{(n+1)!} \cdot \frac{n!}{(2n+1)^n} =$$

$$= \lim_{n\to\infty} \left(1 + \frac{2}{2n+1}\right)^n \cdot \frac{2n+3}{n+1} = 2e; \ (2)$$

From (1) and (2), we find that: $\Omega = 2e^{2\gamma+1}$

UP.445 Solve for real numbers:

$$e^2\Omega^2(a) - 6e\Omega(a) + 8 = 0, \text{ where}$$

$$\Omega(a) = \lim_{n\to\infty} \left(\sqrt[n]{a} - 1\right) \cdot \sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{CDA}{=} \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} =$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{2}{e}$$

$$\Omega(a)=\lim_{n\to\infty}\left(\sqrt[n]{a}-1\right)\cdot\sqrt[n]{(2n-1)!!}=\lim_{n\to\infty}\frac{e^{\frac{\log a}{n}}-1}{\frac{\log a}{n}}\cdot\log a\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=$$

$$=1\cdot\log a\cdot\frac{2}{e}=\frac{2\log a}{e}$$

Hence,

$$e^2\Omega^2(a)-6e\Omega(a)+8=0\Leftrightarrow e^2\cdot\frac{4\log^2 a}{e^2}-6e\cdot\frac{2\log a}{e}+8=0$$

$$\log^2 a-3\log a+2=0\Leftrightarrow\log a\in\{1;2\}$$

Therefore $a\in\{e;e^2\}$

Solution 2 by Daniel Văcaru-Romania

We have:

$$\Omega(a)=\lim_{n\to\infty}\left(\sqrt[n]{a}-1\right)\cdot\sqrt[n]{(2n-1)!!}=\lim_{n\to\infty}\frac{\sqrt[n]{a}-1}{\frac{1}{n}}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{a}-1}{\frac{1}{n}}\cdot\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\log a\cdot\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n};\quad(1)$$

For the second limit from (1), use D'Alembert, and we find:

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{D'A}{=}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{2}{e}\Rightarrow e\Omega(a)=2\log a$$

The equation $e^2\Omega^2(a)-6e\Omega(a)+8=0$ provides

$e\Omega(a)=2$ or $e\Omega(a)=4$. Therefore, $a\in\{e,e^2\}$.

UP.446 Find:

$$\Omega=\lim_{n\to\infty}n^3\cdot\sqrt[n]{n!}\cdot\sqrt[n]{(2n-1)!!}\cdot\sin\frac{1}{n^2}\cdot\sin\frac{1}{n^3}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{CDA}{=}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{CDA}{=}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{2}{e}$$

Therefore,

$$\Omega=\lim_{n\to\infty}n^3\cdot\sqrt[n]{n!}\cdot\sqrt[n]{(2n-1)!!}\cdot\sin\frac{1}{n^2}\cdot\sin\frac{1}{n^3}=$$

$$=\lim_{n\to\infty}\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}\cdot\frac{\sin\frac{1}{n^3}}{\frac{1}{n^3}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=1\cdot1\cdot\frac{1}{e}\cdot\frac{2}{e}=\frac{2}{e^2}$$

Solution 2 by Daniel Văcaru-Romania

$$\lim_{n\to\infty}n^2\sin\frac{1}{n^2}=\lim_{n\to\infty}\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}=1$$

We must calculate

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{D'A}{=}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{D'A}{=}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=\frac{2}{e}$$

Then we have with notation

$$x_n=\frac{\sqrt[n]{n}}{n}\text{ and }y_n=\frac{\sqrt[n]{(2n-1)!!}}{n}$$

$$\lim_{n\to\infty}n\cdot\sqrt[n]{n!}\cdot\sqrt[n]{(2n-1)!!}\cdot\sin\frac{1}{n^3}=\lim_{n\to\infty}\frac{\sin\frac{1}{n^3}}{\frac{1}{n^3}\cdot\frac{1}{x_n}\cdot\frac{1}{y_n}}=\lim_{n\to\infty}x_ny_n\cdot\lim_{n\to\infty}\frac{\sin\frac{1}{n^3}}{\frac{1}{n^3}}=\frac{2}{e^2}$$

Therefore, $\Omega=\frac{2}{e^2}$.

UP.447 Find:

$$\Omega(a)=\lim_{n\to\infty} e^{5H_n}\cdot\left(\sqrt[n^2]{a}-1\right)\cdot\sin^3\frac{1}{n};\ a>0$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by proposers

$$\Omega(a)=\lim_{n\to\infty} e^{5H_n}\cdot\left(\sqrt[n^2]{a}-1\right)\cdot\sin^3\frac{1}{n}=\lim_{n\to\infty}\frac{e^{5H_n}}{n^5}\cdot n^5\cdot\left(\sqrt[n^2]{a}-1\right)\cdot\sin^3\frac{1}{n}=$$

$$=\lim_{n\to\infty}\frac{e^{5H_n}}{e^{5\log n}}\cdot\frac{\sqrt[n^2]{a}-1}{\frac{1}{n^2}}\cdot\frac{\sin^3\frac{1}{n}}{\frac{1}{n^3}}=$$

$$=\lim_{n\to\infty} e^{5H_n-5\log n}\cdot\lim_{n\to\infty}\frac{\sqrt[n^2]{a}-1}{\frac{1}{n^2}}\cdot\lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)^3=$$

$$=\lim_{n\to\infty} e^{5\gamma_n}\cdot\log a\cdot 1^3=e^{5\gamma}\cdot\log a$$

UP.448 If $m\in[1,\infty)$, $a_k\in\mathbb{R}_+^*$, $k=\overline{1,n}$, $n\in\mathbb{N}^*-\{1,2\}$ and

$\sum_{k=1}^n a_k=s\in\mathbb{R}_+^*$, then:

$$\sum_{k=1}^n a_k^m\geq s\left(\prod_{k=1}^n a_k\right)^{\frac{m-1}{n}}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

By AM-GM inequality and by $\sum_{k=1}^n a_k=s$, we have:

$$s\geq n\cdot\sqrt[n]{\prod_{k=1}^n a_k}\Leftrightarrow\frac{s}{n}\geq\sqrt[n]{\prod_{k=1}^n a_k}\Leftrightarrow\left(\frac{s}{n}\right)^n\geq\prod_{k=1}^n a_k$$

By J. Radon's inequality, we have:

$$\sum_{k=1}^n a_k^m\geq\frac{1}{n^{m-1}}\left(\sum_{k=1}^n a_k\right)^m=\frac{s^m}{n^{m-1}}=s\cdot\left(\frac{s}{n}\right)^{m-1}\geq s\left(\prod_{k=1}^n a_k\right)^{\frac{m-1}{n}}$$

Solution 2 by Adrian Popa-Romania

$$\left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} = \sqrt[n]{\prod_{k=1}^{n} a_k} \overset{AGM}{\leq} \frac{a_1 + a_2 + \cdots + a_n}{n}$$

We must to prove that:

$$\sum_{k=1}^{n} a_k^m > (a_1 + a_2 + \cdots + a_n) \cdot \frac{(a_1 + a_2 + \cdots + a_n)^{m-1}}{n^{m-1}} = \frac{(a_1 + a_2 + \cdots + a_n)^m}{n^{m-1}}$$

But from Radon's inequality we have:

$$\sum_{k=1}^{n} a_k^m = \frac{\sum_{k=1}^{n} a_k^m}{1^{m-1}} > \frac{(a_1 + a_2 + \cdots + a_n)^m}{n^{m-1}} \text{ true.}$$

UP.449 Prove that:

$$\sum_{k=1}^{n+1} \frac{\sqrt[k]{n+1} + \sqrt[k]{n-1}}{2\sqrt[k]{n}} \leq \sum_{k=1}^{n} \sqrt[k]{k+1}, \text{ where } n, k \in \mathbb{N}^*$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

First, we prove that

$$n + 1 \leq \sum_{k=1}^{n} \sqrt[k]{k+1}; n, k \in \mathbb{N}$$

We have: $0 \leq (n-1)(k-1) = nk - (n-k+1) \Rightarrow n - k + 1 \leq nk$, and then

$$\prod_{i=1}^{k}(n-i+1) \leq \prod_{i=1}^{k} ni \Leftrightarrow \prod_{i=1}^{k}(n-i+1) \leq n^k \prod_{i=1}^{k} i$$

$$\frac{\prod_{i=1}^{k}(n-i+1)}{\prod_{i=1}^{k} i} \leq n^k$$

It is known that: $\frac{\prod_{i=1}^{k}(n-i+1)}{\prod_{i=1}^{k} i} = \binom{n}{k}$, then $\binom{n}{k} \leq n^k$. So, $\binom{n}{k}\left(\frac{1}{n}\right)^k \leq 1$.

In these conditions, we can write

$$\sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k \leq n + 1$$

From Newton's binomial theorem, we have:

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k=\left(1+\frac{1}{n}\right)^n\Rightarrow\left(1+\frac{1}{n}\right)^n\le n+1\Leftrightarrow 1+\frac{1}{n}\le\sqrt[n]{n+1};(1)$$

So,

$$n+1\le\sum_{k=1}^{n}\sqrt[k]{k+1};n,k\in\mathbb{N};(1)$$

Now, let be the function $f:[1,\infty)\to\mathbb{R},f(x)=\sqrt[k]{x+1}-\sqrt[k]{x}$, then:

$$f(x_1)-f(x_2)=\left(\sqrt[k]{x_1+1}-\sqrt[k]{x_1}\right)-\left(\sqrt[k]{x_2+1}-\sqrt[k]{x_2}\right)=$$

$$=\left(\sqrt[k]{1+x_1}-\sqrt[k]{1+x_2}\right)-\left(\sqrt[k]{x_1}-\sqrt[k]{x_2}\right)=$$

$$=\frac{x_1-x_2}{\sqrt[k]{(1+x_1)^{k-1}}+\sqrt[k]{(1+x_1)^{k-2}(1+x_2)}+\dots+\sqrt[k]{(1+x_1)(1+x_2)^{k-2}}+\sqrt[k]{(1+x_2)^{k-1}}}$$

$$-\frac{x_1-x_2}{\sqrt[k]{x_1^{k-1}}+\sqrt[k]{x_1^{k-2}\cdot x_2}+\dots+\sqrt[k]{x_1x_2^{k-2}}+\sqrt[k]{x_2^{k-1}}}$$

Because $sgn\big(f(x_1)-f(x_2)\big)=-sgn(x_1-x_2)$, then f is decreasing on $[0,\infty)$.

So, we have: $f(n)<f(n-1);\forall n\in\mathbb{N}^*\Leftrightarrow\sqrt[k]{n+1}-\sqrt[k]{n}<\sqrt[k]{n}-\sqrt[k]{n-1}$

$$\sqrt[k]{n+1}+\sqrt[k]{n-1}<2\sqrt[k]{n}\Leftrightarrow\sum_{k=1}^{n+1}\frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}<n+1;(2)$$

From (1) and (2), it follows that:

$$\sum_{k=1}^{n+1}\frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}\le\sum_{k=1}^{n}\sqrt[k]{k+1};n,k\in\mathbb{N}^*$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The function $f(x)=\sqrt[k]{x},k\ge 2$ is concave on $(0,\infty)$ because

$$f''(x)=-\frac{k-1}{k^2\cdot\sqrt[k]{x^{2k-1}}}\le 0,\forall x>0$$

By Jensen's inequality, we have

$$f(n+1)+f(n-1)\le 2f\left(\frac{(n+1)+(n-1)}{2}\right)=2f(n)$$

Then,

$$\sqrt[k]{n+1}+\sqrt[k]{n-1}\le 2\sqrt[k]{n} \text{ or } \frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}\le 1;\forall k\in\mathbb{N}, n\ge 2, n\in\mathbb{N}^*$$

Thus,

$$\sum_{k=1}^{n+1}\frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}=\frac{(n+1)+(n-1)}{2n}+\sum_{k=2}^{n+1}\frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}=1+\sum_{k=2}^{n+1}1=n+1$$

So, it is suffices to prove that

$$\sum_{k=1}^{n}\sqrt[k]{k+1}\ge n+1, \forall n\in\mathbb{N}^*$$

If $n=1$, we have equality. Assume that $n\ge 2$. We have:

$$\sqrt[k]{k+1}\ge 1, \forall k\in\mathbb{N}, k\ge 2\Rightarrow \sum_{k=1}^{n}\sqrt[k]{k+1}\ge 2\sum_{k=2}^{n}1=2+(n-1)=n+1$$

Therefore,

$$\sum_{k=1}^{n+1}\frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}\le\sum_{k=1}^{n}\sqrt[k]{k+1}$$

Solution 3 by Ravi Prakash-New Delhi-India

Let $f(x)=(x+1)^{\frac{1}{k}}$ and $g(x)=x^{\frac{1}{k}}, x\in[n-1,n]$

By Cauchy's mean value theorem, there exists some $c\in(n-1,n)$ such that

$$\frac{f(n)-f(n-1)}{g(n)-g(n-1)}=\frac{f'(c)}{g'(c)}\Rightarrow$$

$$\frac{(n+1)^{\frac{1}{k}}-n^{\frac{1}{k}}}{n^{\frac{1}{k}}-(n-1)^{\frac{1}{k}}}=\frac{(c+1)^{1-\frac{1}{k}}}{c^{1-\frac{1}{k}}}=\left(\frac{c}{c+1}\right)^{1-\frac{1}{k}}\le 1;\forall k\ge 1$$

$$\Rightarrow (n+1)^{\frac{1}{k}}-n^{\frac{1}{k}}\le n^{\frac{1}{k}}-(n-1)^{\frac{1}{k}}$$

$$(n+1)^{\frac{1}{k}}+(n-1)^{\frac{1}{k}}\le 2n^{\frac{1}{k}}\Rightarrow$$

$$\frac{(n+1)^{\frac{1}{k}}+(n-1)^{\frac{1}{k}}}{2n^{\frac{1}{k}}}\le 1$$

$$\sum_{k=1}^{n+1}\frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}\le n+1;\ \ (1)$$

For $k\ge 2,(1+k)^{\frac{1}{k}}\ge 1\Rightarrow$

$$\sum_{k=1}^{n}\sqrt[k]{k+1}=2+\sum_{k=2}^{n}\sqrt[k]{k+1}\ge 2+(n-1)=n+1;(2)$$

From (1) and (2) it follows that

$$\sum_{k=1}^{n+1}\frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}\le\sum_{k=1}^{n}\sqrt[k]{k+1}$$

UP.450 If $A_1A_2\ldots A_n(n\ge 3)$ is a convex polygon with the inradius r and the lengths sides $A_kA_{k+1}=a_k,k=\overline{1,n},A_{n+1}=A_1$. If $h_k\in\mathbb{R}_+^*$ such that $a_kh_k=(n-1)F,k=\overline{1,n}$, then:

$$\sum_{k=1}^{n}\frac{h_k-(n-1)r}{h_k+(n-1)r}\ge\frac{n(n-2)}{n+2}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

We have:

$$B_n=\sum_{k=1}^{n}\frac{h_k-(n-1)r}{h_k+(n-1)r}=\sum_{k=1}^{n}\frac{h_ka_k-(n-1)ra_k}{h_ka_k+(n-1)ra_k}=\sum_{k=1}^{n}\frac{(n-1)F-(n-1)ra_k}{(n-1)F+(n-1)ra_k}=$$

$$=\sum_{k=1}^{n}\frac{F-ra_k}{F+ra_k}$$

Hence,

$$B_n+n=\sum_{k=1}^{n}\left(\frac{F-ra_k}{F+ra_k}+1\right)=2F\sum_{k=1}^{n}\frac{1}{F+ra_k}\overset{Bergstrom}{\ge}2F\cdot\frac{(1+1+\cdots+1)^2}{\sum_{k=1}^{n}(F+ra_k)}=$$

$$=\frac{2n^2F}{nF+r\sum_{k=1}^{n}a_k}=\frac{2n^2F}{nF+2rs}=\frac{2n^2}{n+2}$$

Therefore,

$$B_n \geq \frac{2n^2}{n+2} - n = \frac{n(n-2)}{n+2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $s = \frac{1}{2}\sum_{k=1}^{n} a_k = \frac{F}{r}$. We have:

$$B_n = \sum_{k=1}^{n} \frac{h_k - (n-1)r}{h_k + (n-1)r} = \sum_{k=1}^{n} \frac{\frac{(n-1)sr}{a_k} - (n-1)r}{\frac{(n-1)sr}{a_k} + (n-1)r} = \sum_{k=1}^{n} \frac{s - a_k}{s + a_k} =$$

$$= \sum_{k=1}^{n} \left(\frac{2s}{s + a_k} - 1\right) \overset{CBS}{\geq} 2s \cdot \frac{n^2}{\sum_{k=1}^{n}(s + a_k)} - n = \frac{2s \cdot n^2}{ns + 2s} - n = \frac{n(n-2)}{n+2}$$

Therefore,

$$\sum_{k=1}^{n} \frac{h_k - (n-1)r}{h_k + (n-1)r} \geq \frac{n(n-2)}{n+2}$$