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SOLUTIONS

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DANIEL SITARU

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Proposed by

Daniel Sitaru-Romania, George Apostolopoulos-Greece

Vasile Mircea Popa-Romania, D.M.Bătinețu-Giurgiu-Romania

Florică Anastase-Romania, Marian Ursărescu-Romania

Marin Chirciu-Romania, Alexandru Păun-Romania

Marius Drăgan-Romania, Neculai Stanciu-Romania

Laura Molea-Romania, George Florin Șerban-Romania

Gheorghe Molea-Romania, Raluca Maria Caraion-Romania

Ruxandra Daniela Tonilă-Romania

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Solutions by

Daniel Sitaru-Romania, George Apostolopoulos-Greece, Fao Ler-Iraq Vasile Mircea Popa-Romania, D.M.Bătinețu-Giurgiu-Romania, Rana Ranino-Algerie, Mohamed Amine Ben Ajiba-Morocco, Soumava Chakraborty-India Florică Anastase-Romania, Marian Ursărescu-Romania, Michael Sterghiou-Greece, Hikmat Mammadov-Azerbaijan, Bedri Hajrizi-Kosovo, Ertan Yildirim-Turkiye, Angel Plaza-Spain, Florentin Vișescu-Romania, Daoudi Abdessattar-Tunisia, Precious Itsuokor-Nigeria, Ashwni-Kumar-India, Sanong Huayrerai-Thailand, Geanina Tudose-Romania, Ravi Prakash-India, Alex Szoros-Romania Marin Chirciu-Romania, Alexandru Păun-Romania, Daniel Immarube-Nigeria Marius Drăgan-Romania, Neculai Stanciu-Romania, Tapas Das-India, Adrian Popa-Romania, Avishek Mitra-India, Nikos Ntorvas-Greece, Togrul Ehmedov-Azerbaijan, Ankush Kumar Parcha-India, Laura Molea-Romania, George Florin Șerban-Romania, Gheorghe Molea-Romania, Raluca Maria Caraion-Romania Ruxandra Daniela Tonilă-Romania, Aggeliki Papasyropoulou-Greece

PROBLEMS FOR JUNIORS

JP.451 Prove that if $m \geq 0, n \geq 1$, then in any ΔABC with usual notations is true the inequality

$$\sum_{cyc} \frac{b^n + c^n}{a^m} \geq 2^{n+1}3^{1-m-n}\left(5 - \frac{r}{R}\right)^m$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution by proposers

We denote $S_n = a^n + b^n + c^n$ so $b^n + c^n = S_n - a^n$, and other two similar. Therefore,

$(b^n + c^n, c^n + a^n, a^n + b^n)$ and $\left(\frac{1}{a^m}, \frac{1}{b^m}, \frac{1}{c^m}\right)$ are in the same order.

By Chebyshev's inequality, Jensen's inequality and well-known formulas we obtain that

$$\sum_{cyc} \frac{b^n + c^n}{a^m} \geq \frac{1}{3}\sum_{cyc}(S_n - a^n)\sum_{cyc}\frac{1}{a^m} = \frac{2}{3}S_n\sum_{cyc}\frac{1}{a^m} =$$

$$= \frac{2}{3}(a^n + b^n + c^n)\sum_{cyc}\left(\frac{1}{a}\right)^m \geq \frac{2}{3}\cdot 3^{1-n}(2s)^n 3^{1-m}\left(\sum_{cyc}\frac{1}{a}\right)^m =$$

$$= \frac{2}{3}\cdot 3^{1-n}(2s)^n 3^{1-m}\frac{(\sum ab)^m}{4^m R^m, r^m s^m} = 2^{n-2m+1}3^{1-m-n}s^{n-m}\left(\frac{s^2 + r^2 + 4Rr}{Rr}\right)^m ;(1)$$

By (1) and $s^2 \geq 16Rr - 5r^2 (Gerretsen's)$ we get

$$\sum_{cyc} \frac{b^n + c^n}{a^m} \geq 2^{n+1}3^{1-m-n}\left(5 - \frac{r}{R}\right)^m$$

JP.452 Solve for real numbers:

$$\begin{cases} \sqrt{x} - y^5 = 3 \\ \sqrt[5]{\sqrt{x} - 3} - \sqrt[5]{y^5 + 6} = -1 \end{cases}$$

Proposed by George Florin Şerban, Neculai Stanciu-Romania

Solution 1 by proposers

We note that $x > 0$. From the first equation we have $y = \sqrt[5]{3 - \sqrt{x}}$ which replaced in the second equation yields that $(*)$: $\sqrt[5]{y^5 + 6} = y + 1$. Since $y^5 + 6 = \sqrt{x} + 3 > 3$ we have

$y^4+2y^3+2y^2+y-1=0 \Leftrightarrow (y^2+y+\alpha)(y^2+y-\beta)=0$, where

$\alpha-\beta=\alpha\beta=1$. We have

$\alpha=\frac{\sqrt{5}+1}{2}$ and $\beta=\frac{\sqrt{5}-1}{2}$.

Equation $y^2+y+\alpha=0$ has not solutions, and positive root of the equation

$y^2+y-\beta=0$ is $y=\frac{\sqrt{2\sqrt{5}-1}-1}{2}$. So,

$$x=\left(y^5+3\right)^2=\left[\left(\frac{-1+\sqrt{2\sqrt{5}-1}}{2}\right)^5+3\right]^2=\frac{7+5\sqrt{5}}{2}$$

Hence,

$$(x,y)=\left(\frac{7+5\sqrt{5}}{2};\frac{-1+\sqrt{2\sqrt{5}-1}}{2}\right)$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\begin{cases}\sqrt{x}-y^5=3\\ \sqrt[5]{\sqrt{x}-3}-\sqrt[5]{y^5+6}=-1\end{cases}.\text{ Let }\sqrt{x}=y^5+3, x\ge 0\Rightarrow$$

$$y-\left(y^5+6\right)^{\frac{1}{5}}=-1\Rightarrow y^4+2y^3+2y^2+y-1=0$$

$$\left(y^2+y+\frac{1-\sqrt{5}}{2}\right)\left(y^2+y+\frac{1+\sqrt{5}}{2}\right)=0$$

$$y=\frac{-1+\sqrt{-1+2\sqrt{5}}}{2}\text{ and }x=\frac{7+5\sqrt{5}}{2}$$

Solution 3 by Bedri Hajrizi-Mitrovica-Kosovo

$$\text{Let}\begin{cases}\sqrt{x}=a^5+3\\ y^5=b^5-6\end{cases}\Rightarrow\begin{cases}a^5+3-b^5+6=3\\ a-b=-1\end{cases}\Rightarrow\begin{cases}b^5-a^5=6\\ b-a=1\end{cases}\Rightarrow b=a+1$$

$$5a^4+10a^3+10a^2+5a-5=0$$

$$a^4+2a^3+2a^2+a-1=0,\qquad (a^2+a)^2+(a^2+a)-1=0$$

$$a^2+a=\frac{-1\pm\sqrt{5}}{2},\qquad 4a^2+4a=-2\pm 2\sqrt{5}$$

$$4a^2+4a+1=-1\pm 2\sqrt{5},\qquad (2a+1)^2=-1\pm 2\sqrt{5}$$

$$2a+1=\pm\sqrt{-1+2\sqrt{5}}\Rightarrow a=\frac{-1\pm\sqrt{-1+2\sqrt{5}}}{2}$$

$$a^5+3\geq 0\Rightarrow a=\frac{-1+\sqrt{-1+2\sqrt{5}}}{2}$$

$$\sqrt{x}=a^5+3\Rightarrow x=\left(a^5+3\right)^2\Rightarrow b=\frac{1+\sqrt{-1+2\sqrt{5}}}{2}$$

$$y^5=b^5-6\Rightarrow y=\sqrt[5]{b^5-6}$$

Solution 4 by Ertan Yildirim-Izmir-Turkiye

$$\sqrt{x}-y^5=3\Rightarrow y^5=\sqrt{x}-3$$

$$\sqrt[5]{\sqrt{x}-3}-\sqrt[5]{\sqrt{x}+3}=\sqrt[5]{\sqrt{x}-3}-\sqrt[5]{y^5+6}=-1$$

$$\begin{cases}a^5=\sqrt{x}-3\\ b^5=\sqrt{x}+3\end{cases}\Rightarrow a-b=-1\Rightarrow b-a=1$$

$$\Rightarrow b^2+a^2-2ab=1\Rightarrow b^2+a^2=1+2ab$$

$$b^5-a^5=(b-a)(b^4+b^3a+b^2a^2+ba^3+a^4)=6$$

$$b^4+b^3a+b^2a^2+ba^3+a^4=6$$

$$b^4+b^2a^2+a^4+b^3a+ba^3=(b^2+a^2)^2-a^2b^2+ab(a^2+b^2)=6$$

$$=(b^2+a^2-ab)(b^2+a^2+ab)+ab(1+2ab)=6$$

$$(1+ab)(1+3ab)+ab(1+2ab)=6$$

Let $ab=m$, then

$$(1+m)(1+3m)+m(1+2m)=6$$

$$5m^2+5m=5\Rightarrow m^2+m=1\Rightarrow m^2+m+\frac{1}{4}=\frac{5}{4}$$

$$\left(m+\frac{1}{2}\right)^2=\frac{5}{4}\Rightarrow m_1=\frac{\sqrt{5}-1}{2},m_2=\frac{-\sqrt{5}-1}{2}$$

$$\Rightarrow ab=\frac{\sqrt{5}-1}{2}\Rightarrow a^5b^5=\left(\frac{\sqrt{5}-1}{2}\right)^5=(\sqrt{x}-3)(\sqrt{x}+3)=x-9$$

$$x=\left(\frac{\sqrt{5}-1}{2}\right)^5+9\Rightarrow y=\sqrt[5]{\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)^5+9}-3}$$

$$\Rightarrow ab=\frac{-\sqrt{5}-1}{2}\Rightarrow a^5b^5=\left(\frac{-\sqrt{5}-1}{2}\right)^5=(\sqrt{x}-3)(\sqrt{x}+3)=x-9$$

$$x=9-\left(\frac{\sqrt{5}-1}{2}\right)^5\Rightarrow y=\sqrt[5]{\sqrt{9-\left(\frac{\sqrt{5}-1}{2}\right)^5}-3}$$

JP.453 Prove that:

$$\prod_{cyc}\frac{(1+\sin A\sin B)(1+\sin A\sin C)}{1+\sin A\sqrt{\sin B\sin C}}\ge\left(1+\frac{1}{R}\cdot\sqrt[3]{\frac{F^2}{4R}}\right)^3$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

For $x,y,z>0$ we have:

$$(1+x)(1+y)\ge\left(1+\sqrt{xy}\right)^2\Leftrightarrow 1+x+y+xy\ge 1+2\sqrt{xy}+xy\Leftrightarrow$$

$$x+y-2\sqrt{xy}\ge 0\Leftrightarrow\left(\sqrt{x}-\sqrt{y}\right)^2\ge 0\text{ true.}$$

$$(1+x)(1+y)(1+z)\ge\left(1^3+\sqrt[3]{x^3}\right)\left(1^3+\sqrt[3]{y^3}\right)\left(1^3+\sqrt[3]{z^3}\right)\overset{Holder}{\ge}$$

$$\ge\left(1\cdot1\cdot1+\sqrt[3]{xyz}\right)^3=\left(1+\sqrt[3]{xyz}\right)^3;(*)$$

Now,

$$\frac{(1+\sin A\sin B)(1+\sin A\sin C)}{1+\sin A\sqrt{\sin B\sin C}}\ge\frac{\left(1+\sqrt{\sin A\sin B\cdot\sin A\sin C}\right)^2}{1+\sin A\sqrt{\sin B\sin C}}=$$

$$=1+\sin A\sqrt{\sin B\sin C}$$

Hence,

$$\prod_{cyc}\frac{(1+\sin A\sin B)(1+\sin A\sin C)}{1+\sin A\sqrt{\sin B\sin C}}\ge\prod_{cyc}\left(1+\sin A\sqrt{\sin B\sin C}\right)\overset{(*)}{\ge}$$

$$\ge\left(1+\sqrt[3]{\prod_{cyc}\sin^2 A}\right)^3$$

$$\prod_{cyc}\sin A=\frac{F}{2R^2}\Rightarrow\prod_{cyc}\sin^2 A=\frac{F^2}{4R^4}$$

Therefore,

$$\prod_{cyc}\frac{(1+\sin A\sin B)(1+\sin A\sin C)}{1+\sin A\sqrt{\sin B\sin C}}\ge\left(1+\frac{1}{R}\cdot\sqrt[3]{\frac{F^2}{4R}}\right)^3$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality we have:

$$(1+\sin A\sin B)(1+\sin A\sin C)\geq\left(\sqrt{1\cdot 1}+\sqrt{\sin A\sin B\cdot\sin A\sin C}\right)^2=$$
$$=\left(1+\sin A\sqrt{\sin B\sin C}\right)^2$$

Then

$$\frac{(1+\sin A\sin B)(1+\sin A\sin C)}{1+\sin A\sqrt{\sin B\sin C}}\geq 1+\sin A\sqrt{\sin B\sin C}$$

$$\prod_{cyc}\frac{(1+\sin A\sin B)(1+\sin A\sin C)}{1+\sin A\sqrt{\sin B\sin C}}\geq\prod_{cyc}\left(1+\sin A\sqrt{\sin B\sin C}\right)\overset{Holder}{\geq}$$

$$\geq\left(\sqrt[3]{1\cdot 1\cdot 1}+\sqrt[3]{\prod_{cyc}\sin A\sqrt{\sin B\sin C}}\right)^3=\left(1+\sqrt[3]{\sin A\sin B\sin C^2}\right)^3=$$

$$=\left(1+\sqrt[3]{\left(\frac{F}{2R^2}\right)^2}\right)^3$$

Therefore,

$$\prod_{cyc}\frac{(1+\sin A\sin B)(1+\sin A\sin C)}{1+\sin A\sqrt{\sin B\sin C}}\geq\left(1+\frac{1}{R}\cdot\sqrt[3]{\frac{F^2}{4R}}\right)^3$$

JP.454 If $a,b,c>0$ such that $a+b+c=1$, then prove that:

$$\sum_{cyc}ab(3a+2b+c)\leq\frac{2}{3}$$

Proposed by Laura and Gheorghe Molea-Romania

Solution 1 by proposers

$$\sum a(a-b)^2\geq 0\Leftrightarrow\sum a^3+\sum ab^2-2\sum a^2b\geq 0$$

$$\sum a^3+\sum a^2b+\sum ab^2\geq 3\sum a^2b\Leftrightarrow$$

$$\frac{\sum a^3+\sum a^2b+\sum ab^2}{3\prod a}\geq\frac{\sum a^2b}{\prod a}\Leftrightarrow\frac{(\sum a)(\sum a^2)}{3\prod a}\geq\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\Rightarrow$$

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\leq\frac{\sum a^2}{3abc}\Leftrightarrow 3(a^2b+b^2c+c^2a)\leq a^2+b^2+c^2;\ \ (*)$$

From Schur's inequality, we get:

$$2(ab+bc+ca) \le a^2+b^2+c^2+\frac{9abc}{a+b+c} \Leftrightarrow$$

$$2(ab+bc+ca) \le a^2+b^2+c^2+9abc; (**)$$

By adding, we have:

$$ab(3a+2)-3abc+bc(3b+2)-3abc+ca(3c+2)-3abc \le 2(a^2+b^2+c^2)$$

$$ab(3a+2-3c)+bc(3b+2-3a)+ca(3c+2-3b) \le$$

$$\le 2(a+b+c)^2-4(ab+bc+ca)$$

$$ab(3a+6-3c)+bc(3b+6-3a)+ca(3c+6-3b) \le 2$$

$$\sum ab(3a+6a+6b+6c-3c) \le 2$$

$$3\left(\sum ab\right)(3a+2b+c) \le 2 \Leftrightarrow \sum ab(3a+2b+c) \le \frac{2}{3}$$

Equality holds if $a=b=c=\frac{1}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Homogenizing the given inequality we get the equivalent expression

$$3\sum_{cyc} ab(3a+2b+c) \le 2\left(\sum_{cyc} a\right)^3$$

$$Or\ \ 9\sum_{cyc} a^2b+6\sum_{cyc} ab^2+9abc \le 2\sum_{cyc} a^3+6\sum_{cyc} ab(a+b)+12abc$$

$$Or\ \ 3\sum_{cyc} a^2b \le 2\sum_{cyc} a^3+3abc$$

$$Or\ \ 0 \le \sum_{cyc} a(a-b)(a-c)+\sum_{cyc} a(a-b)^2$$

Which is true from Schur's inequality.

$$Equality\ holds\ iff\ a=b=c=\frac{1}{3}.$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} ab(3a+2b+c) \le \frac{2}{3} \Leftrightarrow 9\sum_{cyc} a^2b+6\sum_{cyc} ab^2+9abc$$

$$\le 2 \overset{\because 1=a+b+c}{=} 2\left(\sum_{cyc} a\right)^3 \overset{\text{expanding and re-arranging}}{\Leftrightarrow} 2\sum_{cyc} a^3+3abc \overset{(*)}{\ge} 3\sum_{cyc} a^2b$$

$$\text{Now, Schur} \Rightarrow 2\sum_{cyc} a^3 + 3abc \overset{(**)}{\geq} \sum_{cyc} a^3 + \sum_{cyc} a^2b + \sum_{cyc} ab^2 \therefore (**)$$

$$\Rightarrow \text{in order to prove } (*), \text{it suffices to prove} : \sum_{cyc} a^3 + \sum_{cyc} a^2b + \sum_{cyc} ab^2 \geq 3\sum_{cyc} a^2b$$

$$\Leftrightarrow \sum_{cyc} a^3 + \sum_{cyc} ab^2 \geq 2\sum_{cyc} a^2b \to \text{true} \because a^3 + ab^2 \overset{A-G}{\geq} 2a^2b \text{ analogs} \therefore \sum_{cyc} ab(3a+2b+c) \leq \frac{2}{3}\ \forall a,b,c > 0 \mid a+b+c = 1 \text{ (QED)}$$

Solution 4 by Michael Sterghiou-Greece

$$\sum_{cyc} ab(3a+2b+c) \leq \frac{2}{3}; (1)$$

WLOG, assume $\alpha = \min\{a,b,c\}$, then $\alpha \in \left(0,\frac{1}{3}\right]$ or if we extend the conditions of the problem to $\left[0,\frac{1}{3}\right]$. By expanding (1) we get

$$3a^2b + 2a^2c + 2ab^2 + 3abc + 3ac^2 + 2b^2c + 2bc^2 - \frac{2}{3} \leq 0$$

This is a trinomial of $\alpha, f(\alpha)$ with coefficient of α^2 equal to $3b + 2c \geq 0$.

As $\alpha \in \left[0,\frac{1}{3}\right]$ it suffices that $\max\left\{f(0), f\left(\frac{1}{3}\right)\right\} \leq 0$. Now,

$f(0) = -b^3 - b^2 + 2b - \frac{2}{3} < 0, b \in [0,1]$. As $a = 0$ then $c = 1 - b$.

$f\left(\frac{1}{3}\right) = b^3 - 2b^2 + b - \frac{4}{27} = \frac{1}{27}(3b-4)(3b-1)^2 < 0$. As $a = \frac{1}{3}$ then $c = \frac{2}{3} - b$.

Equality holds for $a = b = c = \frac{1}{3}$.

JP.455 In ΔABC the following relationship holds:

$$\frac{(a^2+b)(a^2+c)(b^2+a)(b^2+c)(c^2+a)(c^2+b)}{(a+1)^2(b+1)^2(c+1)^2} \geq 1728r^6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

First we prove that:

$$\frac{(a^2+b)(b^2+a)}{(a+1)(b+1)} \geq ab;\ \ (1)$$

$$(a^2+b)(b^2+a)\ge ab(a+1)(b+1)\Leftrightarrow$$

$$a^2b^2+a^3+b^3+ba\ge ab(ab+a+b+1)$$

$$a^2b^2+a^3+b^3+ab\ge a^2b^2+a^2b+ab^2+ab$$

$$a^3+b^3-a^2b-ab^2\ge 0\Leftrightarrow (a-b)(a^2-b^2)\ge 0\Leftrightarrow (a-b)^2(a+b)\ge 0$$

By (1), we get:

$$\prod_{cyc}\frac{(a^2+b)(b^2+a)}{(a+1)(b+1)}\ge \prod_{cyc}ab=(abc)^2=(4Rrs)^2\overset{Euler}{\ge}$$

$$\ge (4\cdot 2r\cdot r\cdot s)^2=8^2r^4s^2\ge 64r^4\left(3\sqrt{3}r\right)^2\ge 64\cdot 27r^4\cdot r^2=1728r^6$$

Equality holds for $a=b=c$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$(a^2+b)(b^2+a)-ab(a+1)(b+1)=(a^3+b^3)-ab(a+b)=$$

$$=(a+b)(a-b)^2\ge 0$$

$$Then:\ (a^2+b)(b^2+a)\ge ab(a+1)(b+1)$$

$$Similarly\ we\ have:$$

$$(b^2+c)(c^2+b)\ge bc(b+1)(c+1)\ \&\ (c^2+a)(a^2+c)\ge ca(c+1)(a+1)$$

$$From\ these\ inequalities\ we\ get:$$

$$\frac{(a^2+b)(a^2+c)(b^2+a)(b^2+c)(c^2+a)(c^2+b)}{(a+1)^2(b+1)^2(c+1)^2}\ge (abc)^2=16s^2R^2r^2$$

$$By\ Mitrinovic's\ inequality\ \ s\ge 3\sqrt{3}r\ \ and\ \ Euler's\ inequality\ \ R\ge 2r\ we\ get:$$

$$\frac{(a^2+b)(a^2+c)(b^2+a)(b^2+c)(c^2+a)(c^2+b)}{(a+1)^2(b+1)^2(c+1)^2}\ge 16.27r^2.4r^2.r^2=1728r^6.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

JP.456 In ΔABC the following relationship holds:

$$b\left(\frac{a}{b}\right)^{\frac{2\sqrt{3}s}{9R}}+c\left(\frac{b}{c}\right)^{\frac{2\sqrt{3}s}{9R}}+a\left(\frac{c}{a}\right)^{\frac{2\sqrt{3}s}{9R}}\le 3\sqrt{3}R$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f(x)=x^a; x>0, 0<a\le 1$, then $f''(x)=a(a-1)x^{a-2}\le 0$;

f –concave, so by Jensen's inequality and $0<\frac{2\sqrt{3}s}{9R}\le 1(Mitrinovic)$, we have:

$$\frac{b}{2s}\left(\frac{a}{b}\right)^{\frac{2\sqrt{3}s}{9R}}+\frac{c}{2s}\left(\frac{b}{c}\right)^{\frac{2\sqrt{3}s}{9R}}+\frac{a}{2s}\left(\frac{c}{a}\right)^{\frac{2\sqrt{3}s}{9R}}\le\left(\frac{b}{2s}\cdot\frac{a}{b}+\frac{c}{2s}\cdot\frac{b}{c}+\frac{a}{2s}\cdot\frac{c}{a}\right)^{\frac{2\sqrt{3}s}{9R}}=$$

$$=\left(\frac{a+b+c}{2s}\right)^{\frac{2\sqrt{3}s}{9R}}=\left(\frac{2s}{2s}\right)^{\frac{2\sqrt{3}s}{9R}}=1$$

$$b\left(\frac{a}{b}\right)^{\frac{2\sqrt{3}s}{9R}}+c\left(\frac{b}{c}\right)^{\frac{2\sqrt{3}s}{9R}}+a\left(\frac{c}{a}\right)^{\frac{2\sqrt{3}s}{9R}}\le 2s\overset{Mitrinovic}{\le}2\cdot\frac{3\sqrt{3}}{2}R=3\sqrt{3}R$$

Equality holds for $a=b=c$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ Mitrinovic's\ inequality\ we\ have:\ \frac{2\sqrt{3}s}{9R}\le\frac{\sqrt{3}.3\sqrt{3}R}{9R}=1.$$

$$Let\ p=\frac{2\sqrt{3}s}{9R}\in(0,1]\ and\ f(x)=x^p,\ \ x>0.\ We\ have:\ f''(x)=p(p-1)x^{p-1}$$
$$\le 0,\forall x>0.$$

$$Then\ f\ is\ concave\ on\ (0,\infty)\ and\ by\ Jensen's\ inequality\ we\ have:$$

$$b\left(\frac{a}{b}\right)^{\frac{2\sqrt{3}s}{9R}}+c\left(\frac{b}{c}\right)^{\frac{2\sqrt{3}s}{9R}}+a\left(\frac{c}{a}\right)^{\frac{2\sqrt{3}s}{9R}}=b.f\left(\frac{a}{b}\right)+c.f\left(\frac{b}{c}\right)+a.f\left(\frac{c}{a}\right)\le$$

$$\le(b+c+a)f\left(\frac{b.\frac{a}{b}+c.\frac{b}{c}+a.\frac{c}{a}}{b+c+a}\right)=2s.f(1)=2s\overset{Mitrinovic}{\lesseqgtr}3\sqrt{3}R.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

JP.457 If $a,b,c>0, a+b+c=3$ then:

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le 6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}, f(x)=\log\left(a+\frac{b(a+b)}{x}\right)$, then

$$f'(x)=\frac{-\frac{b(a+b)}{x^2}}{a+\frac{b(a+b)}{x}}=-\frac{b(a+b)}{ax^2+bx(a+b)}$$

$$f''(x)=b(a+b)\cdot\frac{2ax+b(a+b)}{\left(ax^2+xb(a+b)\right)^2}>0$$

f −convex function and by Jensen's inequality, we get:

$$2f\left(\frac{a+b}{2}\right)\le f(a)+f(b)\Leftrightarrow$$

$$2\log\left(a+\frac{b(a+b)}{\frac{a+b}{2}}\right)\le\log\left(a+\frac{b(a+b)}{a}\right)+\log\left(a+\frac{b(a+b)}{b}\right)$$

$$2\log(a+2b)\le\log\left(\left(a+b+\frac{b^2}{a}\right)(2a+b)\right)$$

$$(a+2b)^2\le\left(a+b+\frac{b^2}{a}\right)(2a+b)$$

$$\frac{(a+2b)^2}{2a+b}\le a+b+\frac{b^2}{a}$$

$$\sum_{cyc}\frac{(a+2b)^2}{2a+b}\le 2(a+b+c)+\sum_{cyc}\frac{b^2}{a}$$

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le 6$$

Equality holds for $a=b=c=1$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ and $a+b+c=3$ we will obtain that:

$$\frac{b^2}{a}+a+b\ge\frac{(a+2b)^2}{2a+b}\Leftrightarrow\frac{b^2+a^2+ab}{a}\ge\frac{a^2+4b^2+4ab}{2a+b}$$

$$(b^2+a^2+ab)(2a+b)\ge a(a^2+4b^2+4ab)$$

$$2ab^2+2a^3+2a^2b+b^3+a^2b+ab^2\ge a^3+4ab^2+4a^2b$$

$$a^3+b^3\ge ab^2+a^2b$$

$$\frac{b^2}{c}+b+c\ge\frac{(b+2c)^2}{2b+c}\text{ and }\frac{c^2}{a}+c+a\ge\frac{(c+2a)^2}{2c+a}$$

Hence,

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le 6$$

Solution 3 by Tapas Das-India

$$\frac{(a+2b)^2}{2a+b}-\left[(a+b)+\frac{b^2}{a}\right]=\frac{a(a+2b)^2-(a+b)a(2a+b)-b^2(2a+b)}{a(2a+b)}=$$

$$=\frac{(a^3+4a^2b+4ab^2)-(2a^3+3a^2b+ab^2)-(2ab^2+b^3)}{a(2a+b)}=$$

$$=\frac{ab^2+a^2b-(a^3+b^3)}{a(2a+b)}=\frac{ab^2+a^2b-(a+b)^3}{a(2a+b)}=$$

$$=\frac{(a+b)(ab-a^2+ab-b^2)}{a(2a+b)}=-\frac{(a+b)(a-b)^2}{a(2a+b)}\le 0$$

$$\frac{(a+2b)^2}{2a+b}\le (a+b)+\frac{b^2}{a}$$

Analogous, we have:

$$\frac{(b+2c)^2}{2b+c}\le (b+c)+\frac{c^2}{b}\ \text{and}\ \frac{(c+2a)^2}{2c+a}\le (a+c)+\frac{a^2}{c}$$

$$\frac{(a+2b)^2}{2a+b}+\frac{(b+2c)^2}{2b+c}+\frac{(c+2a)^2}{2c+a}\le 2(a+b+c)+\frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{c}$$

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le 6$$

Solution 4 by Nikos Ntorvas-Greece

It suffices to prove that:

$$\frac{(x+2y)^2}{2x+y}-\frac{y^2}{x}\le (x+y);\forall x,y>0;(I)$$

$$(I)\Leftrightarrow x(x+2y)^2-(2x+y)y^2\le x(2x+y)(x+y)$$

$$x(x^2+4xy+4y^2)-2xy^2-y^3\le 2x^3+x^2y+2x^2y+xy^2$$

$$x^3+4x^2y+4xy^2-2xy^2-y^3\le 2x^3+x^2y+2x^2y+xy^2$$

$$x^3+y^3-x^2y-xy^2\ge 0\Leftrightarrow (x+y)(x^2+xy+y^2)-xy(x+y)\ge 0$$

$(x+y)(x^2+y^2)\ge 0$ **true for all** $x,y>0$

For $a,b,c>0$ **from (I) we have that:**

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le(a+b)+(b+c)+(c+a)$$

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le 2(a+b+c)$$

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le 6$$

Solution 5 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ CBS\ inequality\ we\ have:$$

$$\frac{b^2}{a}+a+b\ge\frac{(b+a+b)^2}{a+a+b}\ \ then:\ \frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}\le a+b$$

$$Similarly\ we\ have:\ \frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}\le b+c\ \ and\ \ \frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le c+a$$

$$Summing\ up\ these\ inequalities\ we\ get:$$

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le 2(a+b+c)=6.$$

$$Equality\ holds\ iff\ a=b=c=1.$$

JP.458 If $a,b,c>0$ such that $a^2+b^2+c^2=1$ then prove:

$$\sum_{cyc} ab(3a^3b+4ab-2c^2+1)\le 2$$

Proposed by Gheorghe Molea-Romania

Solution 1 by proposer

$$\sum_{cyc} a^2(a^2-b^2)^2\ge 0\Leftrightarrow\sum_{cyc} a^6+\sum_{cyc} a^2b^4-2\sum_{cyc} a^4b^2\ge 0$$

$$\Leftrightarrow\sum_{cyc} a^6+\sum_{cyc} a^4b^2+\sum_{cyc} a^2b^4\ge 3\sum_{cyc} a^4b^2$$

$$\frac{\sum a^6+\sum a^4b^2+\sum a^2b^4}{3\prod a^2}\ge\frac{\sum a^4b^2}{\prod a^2}\Leftrightarrow$$

$$\frac{(\sum a^2)(\sum a^4)}{3\prod a^2}\ge\frac{a^2}{c^2}+\frac{b^2}{a^2}+\frac{c^2}{b^2}\Leftrightarrow\frac{a^2}{c^2}+\frac{b^2}{a^2}+\frac{c^2}{b^2}\le\frac{\sum a^4}{3a^2b^2c^2}$$

$$\Leftrightarrow 3(a^4b^2+b^4c^2+c^4a^2)\le a^4+b^4+c^4;\ \ (*)$$

From Schur's inequality we have:

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\le a^4+b^4+c^4+abc(a+b+c)$$

$$\Leftrightarrow\sum_{cyc}ab(1-c^2)\le\sum_{cyc}a^4+\prod_{cyc}a\sum_{cyc}a\Leftrightarrow$$

$$\sum_{cyc}ab-2\prod_{cyc}a\sum_{cyc}a\le\sum_{cyc}a^4;\ \ (**)$$

By adding $(*)$ and $(**)$, it follows that

$$\sum_{cyc}ab(3a^3b+1)-2\prod_{cyc}a\sum_{cyc}a\le 2\sum_{cyc}a^4\Leftrightarrow$$

$$\sum_{cyc}ab(3a^3b+1)-2\prod_{cyc}a\sum_{cyc}a\le 2-4\sum_{cyc}a^2b^2\Leftrightarrow$$

$$\sum_{cyc}ab(3a^3b+4ab+1)-2\prod_{cyc}a\sum_{cyc}a\le 2\Leftrightarrow$$

$$\sum_{cyc}ab(3a^3b+4ab-2c^2+1)\le 2$$

Equality holds for $a=b=c=\frac{\sqrt{3}}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Homogenizing the given inequality we get the equivalent expression

$$3\sum_{cyc}a^4b^2+\left(\sum_{cyc}a^2\right)\left(4\sum_{cyc}a^2b^2-2abc\sum_{cyc}a\right)+\left(\sum_{cyc}a^2\right)^2\left(\sum_{cyc}ab\right)\le 2\left(\sum_{cyc}a^2\right)^3$$

After expanding and simplifying we get the equivalent inequality

$$\sum_{cyc}a^4b^2+2\sum_{cyc}a^3b^3+\sum_{cyc}ab(a^4+b^4)\le 2\sum_{cyc}a^6+2\sum_{cyc}a^2b^4+abc\sum_{cyc}a^3$$

We have : $\sum_{cyc}a^6+abc\sum_{cyc}a^3-\sum_{cyc}ab(a^4+b^4)$

$$=\sum_{cyc}a^4(a-b)(a-c)\overset{Schur}{\ge}0\ \ (1)$$

By AM − GM inequality we have :

$$a^4b^2\le\frac{a^6+a^2b^4}{2}\ (\text{and analogs})\ \text{and}\ a^3b^3\le\frac{a^6+3a^2b^4}{4}\ (\text{and analogs})$$

$$Then:\ \sum_{cyc} a^4b^2 + 2\sum_{cyc} a^3b^3 \le \sum_{cyc}\frac{a^6+a^2b^4}{2} + 2\sum_{cyc}\frac{a^6+3a^2b^4}{4}$$
$$= \sum_{cyc} a^6 + 2\sum_{cyc} a^2b^4 \quad (2)$$

From (1) and (2) yields the required inequality.

So the proof is completed. Equality holds iff $a=b=c=\frac{\sqrt{3}}{3}$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} ab(3a^3b+4ab-2c^2+1) \le 2 \Leftrightarrow 3\sum_{cyc} a^4b^2 + 4\sum_{cyc} a^2b^2 - 2abc\sum_{cyc} a + \sum_{cyc} ab \le 2$$

$$\overset{\because 1=a^2+b^2+c^2}{\Leftrightarrow} 3\sum_{cyc} a^4b^2 + 4\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 2abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) + \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)^2 \le 2\left(\sum_{cyc} a^2\right)^3$$

$$\overset{\text{expanding and re-arranging}}{\Leftrightarrow} 2\sum_{cyc} a^6 - \sum_{cyc}\left(ab\left(\sum_{cyc} a^4 - c^4\right)\right) + abc\sum_{cyc} a^3 - \sum_{cyc} a^4b^2 + 2\sum_{cyc} a^2b^4 - 2\sum_{cyc} a^3b^3 \ge 0$$

$$\Leftrightarrow 2\sum_{cyc} a^6 - \left(\sum_{cyc} a^4\right)\left(\sum_{cyc} ab\right) + 2abc\sum_{cyc} a^3 - \sum_{cyc} a^4b^2 + 2\sum_{cyc} a^2b^4 - 2\sum_{cyc} a^3b^3 \overset{(*)}{\ge} 0$$

$$\text{Now, } \sum_{cyc} a^6 + \sum_{cyc} a^2b^4 \overset{A-G}{\ge} 2\sum_{cyc} a^4b^2 \Rightarrow \text{LHS of } (*)$$

$$\ge \sum_{cyc} a^6 - \left(\sum_{cyc} a^4\right)\left(\sum_{cyc} ab\right) + 2abc\sum_{cyc} a^3 + \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 - 2\sum_{cyc} a^3b^3$$

$$= \sum_{cyc} a^6 - \left(\sum_{cyc} a^4\right)\left(\sum_{cyc} ab\right) + 2abc\sum_{cyc} a^3 + \sum_{cyc} a^2b^2\left(\sum_{cyc} a^2 - c^2\right) - 2\sum_{cyc} a^3b^3 \overset{?}{\ge} 0$$

$$\Leftrightarrow \sum_{cyc} a^6 - \left(\sum_{cyc} a^4\right)\left(\sum_{cyc} ab\right) + 2abc\sum_{cyc} a^3 + \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 - 2\sum_{cyc} a^3b^3 \overset{?}{\ge} 0$$

$$\Leftrightarrow\left(\sum_{cyc}a^3\right)^2-\left(\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}ab\right)+2abc\sum_{cyc}a^3+\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$
$$-3a^2b^2c^2-4\sum_{cyc}a^3b^3\underset{(**)}{\overset{?}{\geq}}0$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum a=\sum x=2s\Rightarrow\sum a=s\Rightarrow a=s-x, b=s-y, c=s-z$

Via aforementioned substitutions, $\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab=s^2-2\sum_{cyc}(s-x)(s-y)$
$=s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2\overset{(i)}{=}s^2-8Rr-2r^2$ and

$$\sum_{cyc}a^3=\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)=s^3-3xyz=s^3-12Rrs$$

$\Rightarrow\sum_{cyc}a^3\overset{(ii)}{=}s(s^2-12Rr)$ and also $\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)$
$=(4Rr+r^2)^2-2r^2s^2$
$\Rightarrow\sum_{cyc}a^2b^2\overset{(iii)}{=}(4Rr+r^2)^2-2r^2s^2$ and moreover, $\sum_{cyc}a^3b^3$
$=\left(\sum_{cyc}ab\right)^3-3abc(a+b)(b+c)(c+a)=(4Rr+r^2)^3-3r^2s.4Rrs$
$\Rightarrow\sum_{cyc}a^3b^3\overset{(a)}{=}(4Rr+r^2)^3-12Rr^3s^2$

Via (i), (ii), (iii), (a), (**)

$$\Leftrightarrow s^2(s^2-12Rr)^2$$
$$-(4Rr+r^2)\left((s^2-8Rr-2r^2)^2-2\left((4Rr+r^2)^2-2r^2s^2\right)\right)$$
$$+2r^2s^2(s^2-12Rr)+(s^2-8Rr-2r^2)\left((4Rr+r^2)^2-2r^2s^2\right)$$
$$-3r^4s^2-4\left((4Rr+r^2)^3-12Rr^3s^2\right)\geq 0$$

$\Leftrightarrow s^6-(28Rr+r^2)s^4+r^2s^2(224R^2+64Rr+2r^2)-8r^3(4R+r)^3\overset{(\bullet)}{\geq}0$ and
$\because(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\therefore$ in order to

prove $(\bullet)$, it suffices to prove

$$: s^6-(28Rr+r^2)s^4+r^2s^2(224R^2+64Rr+2r^2)-8r^3(4R+r)^3 \geq (s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (20Rr-16r^2)s^4-r^2s^2(544R^2-544Rr+73r^2)+r^3(3584R^3-4224R^2r+1104Rr^2-133r^3)\overset{(\bullet\bullet)}{\geq} 0 \text{ and}$$

$$\because (20Rr-16r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (\bullet\bullet), \text{it suffices to prove}$$

$$: (96R^2-168Rr+87r^2)s^2 \overset{(\bullet\bullet\bullet)}{\geq} r(1536R^3-3072R^2r+1956Rr^2-267r^3)$$

Now, $\because 96R^2-168Rr+87r^2 = 12R^2+84R(R-2r)+87r^2 \overset{\text{Euler}}{\geq} 12R^2+87r^2 > 0$

$$\therefore \text{LHS of } (\bullet\bullet\bullet) \overset{\text{Rouche}}{\geq} (96R^2-168Rr+87r^2)\left(2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\right) \overset{?}{\geq} r(1536R^3-3072R^2r+1956Rr^2-267r^3)$$

$$\Leftrightarrow (R-2r)(96R^3-264R^2r+207Rr^2-45r^3) \overset{?}{\geq} (R-2r)(96R^2-168Rr+87r^2)\sqrt{R^2-2Rr}$$

$$\overset{\because R-2r\geq 0}{\Leftrightarrow} 96R^3-264R^2r+207Rr^2-45r^3 \underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\geq}} (96R^2-168Rr+87r^2)\sqrt{R^2-2Rr}$$

$$\because 96R^3-264R^2r+207Rr^2-45r^3 = (R-2r)(60R^2+36R(R-2r)+63r^2)+81r^3 \overset{\text{Euler}}{\geq} 81r^3 > 0 \therefore (\bullet\bullet\bullet\bullet) \Leftrightarrow$$

$$(96R^3-264R^2r+207Rr^2-45r^3)^2 \geq (R^2-2Rr)(96R^2-168Rr+87r^2)^2$$

$$\Leftrightarrow r^3(1152R^3+576R^2r-3492Rr^2+2025r^3) \geq 0$$

$$\Leftrightarrow r^3\left((R-2r)(1152R^2+2880Rr+2268r^2)+6561r^3\right) \geq 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (\bullet\bullet\bullet\bullet)$$

$$\Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} ab(3a^3b+4ab-2c^2+1) \leq 2 \ \forall\, a,b,c > 0 \ \middle|\ a^2+b^2+c^2 = 1 \text{ (QED)}$$

JP.459 If $x, y, z > 0$ then:

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq \max\left\{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}; \frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right\}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$\because a^2+b^2+c^2 \geq ab+bc+ca$$

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq \frac{x}{y}\cdot\frac{y}{z}+\frac{y}{z}\cdot\frac{z}{x}+\frac{z}{x}\cdot\frac{x}{y} = \frac{x}{z}+\frac{y}{x}+\frac{z}{y}$$

$$\frac{2x^2}{3y^2}+\frac{y^2}{6z^2}+\frac{z^2}{x^2}=\frac{1}{6}\left(\frac{x^2}{y^2}+\frac{x^2}{y^2}+\frac{x^2}{y^2}+\frac{x^2}{y^2}+\frac{z^2}{x^2}\right)\overset{AGM}{\geq}$$

$$\geq\frac{1}{6}\cdot 6\sqrt[6]{\left(\frac{x^2}{y^2}\right)^4\cdot\frac{y^2}{z^2}\cdot\frac{z^2}{x^2}}=\sqrt[6]{\frac{x^8\cdot y^2\cdot z^2}{x^2\cdot y^8\cdot z^2}}=\frac{x}{y}$$

Thus,

$$\sum_{cyc}\left(\frac{2x^2}{3y^2}+\frac{y^2}{6z^2}+\frac{z^2}{x^2}\right)\geq\sum_{cyc}\frac{x}{y}$$

$$\left(\frac{2}{3}+\frac{1}{6}+\frac{1}{6}\right)\sum_{cyc}\frac{x^2}{y^2}\geq\sum_{cyc}\frac{x}{y}$$

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

JP.460 If $a,b,c,m,n>0$ then:

$$\left(\frac{a}{\sqrt[3]{mb+nc}}+\frac{b}{\sqrt[3]{mc+na}}+\frac{c}{\sqrt[3]{ma+nb}}\right)^3\geq\frac{(a+b+c)^4}{(m+n)(a^2+b^2+c^2)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\left(\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}\right)\left(\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}\right)\left(\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}\right)\cdot\sum_{cyc}a(mb+nc)\overset{Holder}{\geq}$$

$$\geq\left(\sum_{cyc}a\right)^4$$

$$\left(\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}\right)^3\geq\frac{(a+b+c)^4}{\sum a(mb+nc)}=\frac{(a+b+c)^4}{(m+n)(ab+bc+ca)}\geq$$

$$\geq\frac{(a+b+c)^4}{(m+n)(a^2+b^2+c^2)}$$

Equality holds for $a=b=c$.

Solution 2 by Tapas Das-India

$$\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}=\sum_{cyc}\frac{a^{\frac{4}{3}}}{\sqrt[3]{mab+nac}}\geq\frac{(\sum a)^{\frac{4}{3}}}{[\sum(mab+nac)]^{\frac{1}{3}}}=\frac{(\sum a)^{\frac{4}{3}}}{[(m+n)(\sum ab)]^{\frac{1}{3}}}$$

Hence,

$$\left(\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}\right)^3\geq\frac{(a+b+c)^4}{\sum a(mb+nc)}=\frac{(a+b+c)^4}{(m+n)(ab+bc+ca)}\geq$$

$$\geq\frac{(a+b+c)^4}{(m+n)(a^2+b^2+c^2)}$$

Equality holds for $a=b=c$.

JP.461 If $m,n,p,x,y,z>0$ then in ΔABC with area F holds:

$$\frac{(m+n)(z+x)}{py}a^2b^2+\frac{(p+m)(y+z)}{mz}b^2c^2+\frac{(n+p)(x+y)}{nx}c^2a^2\geq 64F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{(m+n)(z+x)}{py}a^2b^2\geq 4\sum_{cyc}\frac{\sqrt{mnzx}}{py}a^2b^2\geq$$

$$\geq 4\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{mnzx}}{py}a^2b^2}=4\cdot 3\sqrt[3]{a^4b^4c^4}=\frac{4}{3}\left(3\sqrt[3]{a^2b^2c^2}\right)\overset{Carlitz}{\geq}$$

$$\geq\frac{4\cdot 16\cdot 3}{3}F^2=64F^2$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{(m+n)(z+x)}{py}a^2b^2\overset{AGM}{\geq}3\sqrt[3]{\frac{\prod 2\sqrt{mn}\cdot\prod 2\sqrt{zx}}{mnp\cdot xyz}\cdot\prod a^4}=$$

$$=3\sqrt[3]{\frac{64mnp\cdot xyz}{mnp\cdot xyz}\prod a^4}=3\cdot 4\sqrt[3]{(4Rrs)^4}$$

Need to show:

$$12\sqrt[3]{(4Rrs)^4} \ge 64F^2 \Rightarrow 12^3 \cdot 4^4(Rrs)^4 \ge 4^9(rs)^6 \Rightarrow 27R^4 \ge 16r^2s^2 \text{ which is true}$$

$$\text{from } R \ge 2r;\ (Euler) \text{ and } 3\sqrt{3}R \ge 2s;\ (Mitrinovic)$$

Solution 3 by Tapas Das-India

$$(m+n)(z+x) \overset{AGM}{\ge} 2\sqrt{mn} \cdot 2\sqrt{zx} = 4\sqrt{mn \cdot zx}$$

$$(p+m)(y+z) \ge 4\sqrt{pm \cdot yz}$$

$$(n+p)(x+y) \ge 4\sqrt{np \cdot xy}$$

$$\sum_{cyc} \frac{(m+n)(z+x)}{py} a^2b^2 \overset{AGM}{\ge} 3\sqrt[3]{\frac{\prod 2\sqrt{mn} \cdot \prod 2\sqrt{zx}}{mnp \cdot xyz} \cdot \prod a^4} =$$

$$= 3\sqrt[3]{\frac{64mnp \cdot xyz}{mnp \cdot xyz} \prod a^4} = 3 \cdot 4\sqrt[3]{(4Rrs)^4} = 12\sqrt[3]{(4Rrs)^4} \overset{Carlitz}{\ge}$$

$$\ge 12\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{4}{3}} = 12 \cdot \frac{16F^2}{3} = 64F^2$$

JP.462 In ΔABC the following relationship holds:

$$\left(\frac{1}{h_ah_b} + \frac{1}{h_bh_c} + \frac{1}{h_ch_a}\right)\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \ge \frac{9}{16F^2}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution 1 by proposer

$$\left(\sum_{cyc} \frac{1}{h_ah_b}\right)\left(\sum_{cyc} \frac{1}{(a+b)^2}\right) = \left(\sum_{cyc} \frac{ab}{ah_abh_b}\right)\sum_{cyc} \frac{1}{(a+b)^2} =$$

$$= \frac{1}{4F^2}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{1}{(a+b)^2}\right) \overset{Ji\ Chen}{\ge} \frac{1}{4F^2} \cdot \frac{9}{4} = \frac{9}{16F^2}$$

Equality holds iff ΔABC is equilateral.

Solution 2 by Alex Szoros-Romania

$$ah_a = 2F \Rightarrow \frac{1}{h_a} = \frac{a}{2F} \Rightarrow \frac{1}{h_ah_b} = \frac{ab}{4F^2} \Rightarrow \sum_{cyc} \frac{1}{h_ah_b} = \frac{\sum ab}{4F^2};(1)$$

$$\sum_{cyc}\frac{1}{(a+b)^2}\geq\frac{9}{4(ab+bc+ca)};(Iran\ 1996)\ (2)$$

From (1) and (2) it follows that:

$$\left(\sum_{cyc}\frac{1}{h_ah_b}\right)\left(\sum_{cyc}\frac{1}{(a+b)^2}\right)\geq\frac{9\sum ab}{16F^2\cdot\sum ab}=\frac{9}{16F^2}$$

Note by Editor:

Lemma. (Iran Inequality 1996) If $a,b,c>0$ then holds:

$$(ab+bc+ca)\left(\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\right)\geq\frac{9}{4}$$

Proof.

The appearance of $\sum_{cyc}\frac{1}{(b+c)^2}$ suggests the Cauchy Schwartz inequality (since it is the sum of squares). Let us the above idea to attack this inequality. By the Cauchy Schwartz inequality, we have

$$\left[\sum_{cyc}(ma+nb+nc)^2\right]\left[\sum_{cyc}\frac{1}{(b+c)^2}\right]\geq\left(\sum_{cyc}\frac{ma+nb+nc}{b+c}\right)^2$$

The equality holds if and only if

$$\frac{ma+nb+nc}{\frac{1}{b+c}}=\frac{mb+nc+na}{\frac{1}{c+a}}=\frac{mc+na+nb}{\frac{1}{a+b}}$$

We notice that the original inequality has an equality case for $a=b=1,c=0$, hence the above solution must be satisfied at this point, that is

$$\frac{m+n}{1}=\frac{m+n}{1}=\frac{2n}{\frac{1}{2}}\Leftrightarrow m=3n\Rightarrow m=3,n=1$$

And now, we have the solution as follows:

By the Cauchy Schwartz inequality, we have

$$\left(11\sum_{cyc}a^2+14\sum_{cyc}ab\right)\left[\sum_{cyc}\frac{1}{(a+b)^2}\right]=\left[\sum_{cyc}(3a+b+c)^2\right]\left[\sum_{cyc}\frac{1}{(b+c)^2}\right]\geq$$

$$\geq\left(\sum_{cyc}\frac{3a+b+c}{b+c}\right)^2=9\left(1+\sum_{cyc}\frac{a}{b+c}\right)^2$$

It suffices to prove that

$$4\left(1+\sum_{cyc}\frac{a}{b+c}\right)^2\geq\frac{11\sum a^2+14\sum ab}{\sum ab}$$

Due to the homogeneity, we may assume $a+b+c=1$, setting $q=\sum ab, r=abc$, then by Schur's inequality for third degree, we obtain $r\geq\max\left\{0,\frac{4q-1}{9}\right\}$.

The inequality becomes

$$4\left(\frac{1+q}{q-r}-1\right)^2\geq\frac{11-8q}{q}$$

If $1\geq 4q$, then:

$$4\left(\frac{1+q}{q-r}-2\right)^2-\frac{11-8q}{q}\geq 4\left(\frac{1+q}{q}-2\right)^2-\frac{11-8q}{q}=\frac{(4-3q)(1-4q)}{q^2}\geq 0$$

If $4q\geq 1$, then:

$$4\left(\frac{1+q}{q-r}-2\right)^2-\frac{11-8q}{q}\geq 4\left(\frac{1+q}{q-\frac{4q-1}{9}}-2\right)^2-\frac{11-8q}{q}=$$

$$=\frac{(1-2q)(4q-1)(11-17q)}{q(5q+1)^2}\geq 0$$

Equality holds if and only if $a=b=c$ or $a=b, c=0$ and any cyclic permutations.

JP.463 If in $\Delta ABC, I$ −incenter and R_a, R_b, R_c circumradius of $\Delta IBC, \Delta ICA, \Delta IAB$, then:

$$\frac{R_b^3R_c^3}{R_ah_a^2}+\frac{R_c^3R_a^3}{R_bh_b^2}+\frac{R_a^3R_b^3}{R_ch_c^2}\ge\frac{8R^2r}{3}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution by Daniel Sitaru-Romania

$$\frac{R_b^3R_c^3}{R_ah_a^2}+\frac{R_c^3R_a^3}{R_bh_b^2}+\frac{R_a^3R_b^3}{R_ch_c^2}=\sum_{cyc}\frac{R_b^3R_c^3}{R_a\cdot\frac{4F^2}{a^2}}=\frac{1}{4F^2}\sum_{cyc}\frac{a^2R_b^3R_c^3}{R_a}=$$

$$=\frac{1}{4F^2}\sum_{cyc}\frac{4R^2sin^2A\cdot\left(2Rsin\frac{B}{2}\right)^3\cdot\left(2Rsin\frac{C}{2}\right)^3}{2Rsin\frac{A}{2}}=$$

$$=\frac{1}{4F^2}\cdot(2R)^7\cdot4\sum_{cyc}sin\frac{A}{2}cos^2\frac{A}{2}sin^3\frac{B}{2}sin^3\frac{C}{2}=$$

$$=\frac{32R^7}{F^2}\cdot4sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\sum_{cyc}cos^2\frac{A}{2}sin^2\frac{B}{2}sin^2\frac{C}{2}=$$

$$=\frac{32R^7}{F^2}\cdot\frac{4r}{4R}\sum_{cyc}\frac{s(s-a)(s-a)(s-c)(s-a)(s-b)}{a^2b^2c^2}=$$

$$=\frac{8R^6r}{F^2}\cdot\frac{4}{16R^2F^2}\cdot F^2\sum_{cyc}(s-a)^2=\frac{4R^4r}{2F^2}(s^2-2r^2-8Rr)\ge$$

$$\overset{GERRETSEN}{\ge}\frac{2R^4r}{Frs}(16Rr-5r^2-2r^2-8Rr)=\frac{2R^4}{Fs}(8Rr-7r^2)=$$

$$=\frac{2R^4r(8R-7r)}{rs^2}=\frac{2R^4(8R-7r)}{s^2}\overset{MITRINOVIC}{\ge}\frac{2R^4(8R-7r)}{\frac{27R^2}{4}}=$$

$$=\frac{8R^2(8R-7r)}{27}\overset{EULER}{\ge}\frac{8R^2(8\cdot2r-7r)}{27}=\frac{8R^2r}{3}$$

Equality holds for $a=b=c$.

JP.464 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\cos^2 \frac{A}{2}}{2\cos^6 \frac{A}{2} + \cos^6 \frac{B}{2}} \le \frac{8R}{9r}$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$2\cos^6 \frac{A}{2} + \cos^6 \frac{B}{2} = \cos^6 \frac{A}{2} + \cos^6 \frac{A}{2} + \cos^6 \frac{B}{2} \ge 3\cos^4 \frac{A}{2} \cos^2 \frac{B}{2}$$

$$\frac{\cos^2 \frac{A}{2}}{2\cos^6 \frac{A}{2} + \cos^6 \frac{B}{2}} \le \frac{1}{3\cos^2 \frac{A}{2} \cos^2 \frac{B}{2}}$$

$$\sum_{cyc} \frac{\cos^2 \frac{A}{2}}{2\cos^6 \frac{A}{2} + \cos^6 \frac{B}{2}} \le \frac{1}{3} \sum_{cyc} \frac{1}{\cos^2 \frac{A}{2} \cos^2 \frac{B}{2}}$$

$$\text{But } \sum_{cyc} \frac{1}{\cos^2 \frac{A}{2} \cos^2 \frac{B}{2}} = \frac{8R(4R+r)}{s^2}; (2)$$

Note by editor:

$$\sum_{cyc} \frac{1}{\cos^2 \frac{A}{2} \cos^2 \frac{B}{2}} = \sum_{cyc} \frac{1}{\frac{s(s-a)}{bc} \cdot \frac{s(s-b)}{ca}} = \frac{1}{s^2} \sum_{cyc} \frac{abc^2}{(s-a)(s-b)} =$$

$$= \frac{abc}{s^2} \sum_{cyc} \frac{c}{(s-a)(s-b)} = \frac{4RF}{s \cdot s(s-a)(s-b)(s-c)} \sum_{cyc} c(s-c) =$$

$$= \frac{4RF}{s \cdot F^2} \left(s \sum_{cyc} c - \sum_{cyc} c^2 \right) = \frac{4R}{sF}(2s^2 - 2s^2 + 2r^2 + 8Rr) =$$

$$= \frac{4R}{s \cdot sr} \cdot 2r(4R+r) = \frac{8R(4R+r)}{s^2}$$

From (1) and (2) we get:

$$\sum_{cyc} \frac{\cos^2 \frac{A}{2}}{2\cos^6 \frac{A}{2} + \cos^6 \frac{B}{2}} \le \frac{8R(4R+r)}{3s^2}; (3)$$

But $s^2 \ge 3r(4R+r); (4)$ (Doucet)

Note by editor:

$$s^2 \overset{GERRETSEN}{\gtrless} 16Rr - 5r^2 = 12Rr + 4Rr - 5r^2 \overset{EULER}{\gtrless}$$

$$\geq 12Rr + 4 \cdot 2r - 5r^2 = 12Rr + 3r^2$$

From (3) and (4), it follows that:

$$\sum_{cyc} \frac{\cos^2 \frac{A}{2}}{2\cos^6 \frac{A}{2} + \cos^6 \frac{B}{2}} \leq \frac{8R(4R+r)}{9r(4R+r)} = \frac{8}{9} \cdot \frac{R}{r}$$

JP.465 In ΔABC the following relationship holds:

$$\sum_{cyc} (a^2 + b^2 - c^2)^2 + 8\sum_{cyc} a^2b^2 \geq 3888r^4$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(a^2 + b^2 - c^2)^2 + 4a^2c^2 + 4b^2c^2 = (a^2 + b^2 + c^2)^2$$

$$(b^2 + c^2 - a^2)^2 + 4b^2a^2 + 4c^2a^2 = (a^2 + b^2 + c^2)^2$$

$$(c^2 + a^2 - b^2)^2 + 4c^2b^2 + 4a^2b^2 = (a^2 + b^2 + c^2)^2$$

Hence, we get:

$$\sum_{cyc} (a^2 + b^2 - c^2)^2 + 8\sum_{cyc} a^2b^2 = 3(a^2 + b^2 + c^2)^2 \geq 3 \cdot \left(4\sqrt{3}F\right)^2 =$$

$$= 3 \cdot 16 \cdot 3F^2 = 9 \cdot 16r^2s^2 = 144r^2s^2 \overset{Mitrinovic}{\geq} 144r^2\left(3\sqrt{3}r\right)^2 =$$

$$= 144r^2 \cdot \left(3\sqrt{3}r\right)^2 = 144r^2 \cdot 27r^2 = 3888r^4$$

Equality holds for $a = b = c$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \left(a^2 + b^2 - c^2\right)^2 + 8\sum_{cyc} a^2b^2 = \sum_{cyc} a^4 + \sum_{cyc} \left(b^2 - c^2\right)^2 + 2\sum_{cyc} a^2\left(b^2 - c^2\right) + 8\sum_{cyc} a^2b^2$$

$$= \sum_{cyc} a^4 + \sum_{cyc} \left(b^4 + c^4 - 2b^2c^2\right) + 2\sum_{cyc} a^2b^2 - 2\sum_{cyc} a^2c^2 + 8\sum_{cyc} a^2b^2$$

$$=\sum_{cyc} a^4+\sum_{cyc} b^4+\sum_{cyc} c^4-2\sum_{cyc} a^2b^2+8\sum_{cyc} a^2b^2=3\left(\sum_{cyc} a^4+2\sum_{cyc} a^2b^2\right)$$

$$=3\left(\sum_{cyc} a^2\right)^2 \overset{\text{Ionescu-Weitzenbock}}{\geq} 3\left(4\sqrt{3}rs\right)^2 \overset{\text{Mitrinovic}}{\geq} 3\left(4\sqrt{3}r.3\sqrt{3}r\right)^2$$

$$=3888r^4 \text{ (QED)}$$

Solution 3 by Tapas Das-India

$$(a^2+b^2-c^2)^2=a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2$$

$$(a^2-b^2+c^2)^2=a^4+b^4+c^4-2a^2b^2+2b^2c^2-2c^2a^2$$

$$(-a^2+b^2+c^2)^2=a^4+b^4+c^4-2a^2b^2-2b^2c^2+2c^2a^2$$

$$\sum_{cyc}(a^2+b^2-c^2)^2=3\sum_{cyc} a^4-2\sum_{cyc} a^2b^2$$

$$\sum_{cyc}(a^2+b^2-c^2)^2+8\sum_{cyc} a^2b^2=3\sum_{cyc} a^4+6\sum_{cyc} a^2b^2=$$

$$=3\left[\sum_{cyc} a^4+2\sum_{cyc} a^2b^2\right]=3\left(\sum_{cyc} a^2\right)^2$$

Now, we have:

$$\frac{a^2+b^2+c^2}{3}\geq\frac{a+b+c}{3}\cdot\frac{a+b+c}{3}=\frac{4s^2}{9}$$

$$a^2+b^2+c^2\geq\frac{4}{3}s^2=\frac{4}{3}\left(3\sqrt{3}r\right)^2\geq 36r^2$$

$$\sum_{cyc}(a^2+b^2-c^2)^2+8\sum_{cyc} a^2b^2\geq 3(36r^2)^2=3\cdot 1296r^4=3888r^4$$

PROBLEMS FOR SENIORS

SP.451 If $ABCD$ is a convex quadrilateral such that $AC\cap BD=\{O\}, AE=EC, BF=FD$ with order $A-O-E-C$ respectively $B-F-O-D$,

$EF\cap AB=\{J\}, EF\cap CD=\{K\}, CJ\cap BK=\{L\}$ and M the point of KJ, then prove that O, M and L are collinear.

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution by proposers

We consider O the origin and we compute the vectors of $E, F, C, B, A, D, J, K, M$ and L; $E\to\vec{e}, F\to\vec{f}, C\to c\vec{e}, B\to b\vec{f}, (c, b>1)$

$$A\to(2-c)\vec{e}, D\to(2-b)\vec{f}.$$

Since $J\in(AB)\Rightarrow\exists\alpha,\beta\in\mathbb{R}$ such that $\alpha(2-c)\vec{e}+(1-\alpha)b\vec{f}=\beta\vec{e}+(1-\beta)\vec{f}$

$$\alpha=\frac{b-1}{b+c-2}\Rightarrow(b+c-2)\vec{j}=(2-c)(b-1)\vec{e}+b(c-1)\vec{f}$$

Analogous, the vector of K verify $(b+c-2)\vec{k}=(2-b)(c-1)\vec{f}+c(b-1)\vec{e}$

Since M is the midpoint of KJ, we have

$$\vec{m}=\frac{1}{b+c-2}\left[(b-1)\vec{e}+(c-1)\vec{f}\right];\quad(*)$$

Since $L\in(CJ), L\in(BK)\Rightarrow\exists\lambda,\mu\in\mathbb{R}$ such that

$$\lambda(b+c-2)b\vec{f}+(1-\lambda)(2-b)(c-1)\vec{f}+c(1-\lambda)(b-1)\vec{e}=$$
$$=\mu(b+c-2)c\vec{e}+(1-\mu)(2-c)(b-1)\vec{e}+b(1-\mu)(c-1)\vec{f}$$

Therefore, $\lambda=\frac{c-1}{b+c-1}$. So, the vector of L is

$$\vec{l}=\frac{bc}{b+c-1}\left[(b-1)\vec{e}+(c-1)\vec{f}\right];\quad(**)$$

From $(*)$ and $(**)$ we obtain that O, M and L are collinear.

SP.452 If $a, b, c>0$ then:

$$(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)\geq 256$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$1+a=1+\frac{a}{3}+\frac{a}{3}+\frac{a}{3}\overset{AGM}{\geq}4\sqrt[4]{\frac{a^3}{27}}$$

$$a+b=a+\frac{b}{3}+\frac{b}{3}+\frac{b}{3}\overset{AGM}{\geq}4\sqrt[4]{\frac{b^3a}{27}}$$

$$b+c=b+\frac{c}{3}+\frac{c}{3}+\frac{c}{3}\overset{AGM}{\geq}4\sqrt[4]{\frac{c^3b}{27}}$$

$$c+81=c+27+27+27\overset{AGM}{\geq}4\sqrt[4]{c\cdot 27^3}$$

Hence:

$$(1+a)(a+b)(b+c)(c+81)\geq 4^4\cdot\sqrt[4]{\frac{a^3}{27}\cdot\frac{b^3a}{27}\cdot\frac{c^3b}{27}\cdot c\cdot 27^3}$$

$$\frac{(1+a)(a+b)(b+c)(c+81)}{abc}\geq\frac{256}{abc}\cdot\sqrt[4]{a^4b^4c^4}$$

$$(1+a)\cdot\frac{a+b}{a}\cdot\frac{b+c}{b}\cdot\frac{c+81}{c}\geq\frac{256}{abc}\cdot abc$$

Therefore,

$$(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)\geq 256$$

Equality holds for $a=3,b=9,c=27$.

Solution 2 by Ertan Yildirim-Izmir-Turkiye

$$\sqrt[4]{(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)}\geq 1+\sqrt[4]{a\cdot\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{81}{c}}=4$$

Hence,

$$(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)\geq 4^4$$

$$(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)\geq 256$$

$$\left(\frac{b}{a}=a\ and\ \frac{81}{c}=\frac{c}{b}\ and\ \sqrt{b}=\frac{9}{\sqrt{b}}\right)\Rightarrow\ (b=9,c=27,a=3)$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$(1+a)\left(1+\frac{b}{a}\right)=1+\frac{b}{a}+a+b\overset{A-G}{\geq}1+b+2.\sqrt{\frac{b}{a}.a}=1+b+2\sqrt{b}=\left(1+\sqrt{b}\right)^2$$

$$\Rightarrow\sqrt{(1+a)\left(1+\frac{b}{a}\right)}\overset{(i)}{\geq}1+\sqrt{b}$$

Again, $\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)=1+\frac{81}{c}+\frac{c}{b}+\frac{81}{b}\overset{A-G}{\geq}1+\frac{81}{b}+2.\sqrt{\frac{81}{c}.\frac{c}{b}}=1+\frac{81}{b}+\frac{18}{\sqrt{b}}$

$$=\left(1+\frac{9}{\sqrt{b}}\right)^2\Rightarrow\sqrt{\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)}\overset{(ii)}{\geq}1+\frac{9}{\sqrt{b}}$$

$$\therefore (i).(ii)\Rightarrow\sqrt{(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)}\geq\left(1+\sqrt{b}\right)\left(1+\frac{9}{\sqrt{b}}\right)$$

$$=10+\sqrt{b}+\frac{9}{\sqrt{b}}\overset{A-G}{\geq}10+2.\sqrt{\sqrt{b}.\frac{9}{\sqrt{b}}}=16$$

$$\Rightarrow(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)\geq 256\ \forall a,b,c$$

>0, equality iff $\left(\frac{b}{a}=a \text{ and } \frac{81}{c}=\frac{c}{b} \text{ and } \sqrt{b}=\frac{9}{\sqrt{b}}\right)$

$\Rightarrow$ iff $(b=9, c=27, a=3)$ (QED)

Solution 4 by Tapas Das-India

$$\sqrt[4]{(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)}\geq\sqrt[4]{1\cdot1\cdot1\cdot1}+\sqrt[4]{a\cdot\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{81}{c}}=$$

$$=1+\sqrt[4]{81}=4$$

Hence,

$$(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)\geq 4^4$$

$$(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right)\geq 256$$

Equality occurs when: $a=\frac{a}{b}=\frac{c}{b}=\frac{81}{c}=k$

$$a=k, b=ak=k^2, c=bk=k^3, 81=ck=k^4\Rightarrow k=3$$

$$a=3, b=9, c=27$$

SP.453 Let $\varepsilon_i, i=\overline{1,n}$ be roots of equation $z^{n+1}=1, \varepsilon_i\neq1, \forall i=\overline{1,n}$

Solve for natural numbers:

$$n^2+\sum_{k=1}^{n}\frac{3\varepsilon_k-2}{1-\varepsilon_k}+\frac{3}{2}=0, n\in\mathbb{N}^*$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$S_n=\sum_{k=1}^{n}\frac{3\varepsilon_k-2}{1-\varepsilon_k}=\sum_{k=1}^{n}\frac{3(\varepsilon_k-1)+1}{1-\varepsilon_k}=-3n+\sum_{k=1}^{n}\frac{1}{1-\varepsilon_k}$$

Let be $\varepsilon_k=\cos\frac{2k\pi}{n+1}+i\sin\frac{2k\pi}{n+1}, k=\overline{1,n}$, then

$$S_n=-3n+\sum_{k=1}^{n}\frac{1}{1-\cos\frac{2k\pi}{n+1}-i\sin\frac{2k\pi}{n+1}}=$$

$$=-3n+\sum_{k=1}^{n}\frac{1}{2\sin^2\frac{k\pi}{n+1}-2i\sin\frac{k\pi}{n+1}\cos\frac{k\pi}{n+1}}=$$

$$=-3n+\sum_{k=1}^{n}\frac{1}{-2i\sin\frac{k\pi}{n+1}\left(\cos\frac{k\pi}{n+1}+i\sin\frac{k\pi}{n+1}\right)}=-3n+\sum_{k=1}^{n}\frac{i\left(\cos\frac{k\pi}{n+1}-i\sin\frac{k\pi}{n+1}\right)}{2\sin\frac{k\pi}{n+1}}$$

$$=$$

$$=-3n+\frac{n}{2}+\frac{i}{2}\sum_{k=1}^{n}\cot\frac{k\pi}{n+1}=-\frac{5}{2}n+\frac{1}{2}i\sum_{k=1}^{n}\cot\frac{k\pi}{n+1}=-\frac{5n}{2}$$

$$\text{Because }\sum_{k=1}^{n}\cot\frac{k\pi}{n+1}=0;\forall n\in\mathbb{N}.$$

So, we have:

$$n^2+\sum_{k=1}^{n}\frac{3\varepsilon_k-2}{1-\varepsilon_k}+\frac{3}{2}=0\Leftrightarrow n^2-\frac{5}{2}n+\frac{3}{2}=0\Leftrightarrow 2n^2-5n+3=0$$

$$\Leftrightarrow(2n-3)(n-1)=0\Leftrightarrow n=1.$$

Solution 2 by Adrian Popa-Romania

$$\varepsilon_i\text{ −roots of the equation }z^{n+1}=1$$

$$z^{n+1}-1=0\Rightarrow(z-1)(z^n+z^{n-1}+\cdots+z+1)=0$$

$$z\neq 0\Rightarrow z^n+z^{n-1}+\cdots+z+1=0$$

Let $f(z)=z^n+z^{n-1}+\cdots+z+1=(z-\varepsilon_1)(z-\varepsilon_2)\cdot\ldots\cdot(z-\varepsilon_n)$

$$\frac{f'(z)}{f(z)}=\frac{1}{z-\varepsilon_1}+\frac{1}{z-\varepsilon_2}+\cdots+\frac{1}{z-\varepsilon_n}$$

$$f(1)=n+1$$

$$f'(z)=nz^{n-1}+(n-1)z^{n-2}+\cdots+2z+1$$

$$f'(1)=n+(n-1)+\cdots+2+1=\frac{n(n+1)}{2}$$

So,

$$\frac{f'(1)}{f(1)}=\frac{n}{2}$$

Hence

$$S_n=\sum_{k=1}^{n}\frac{3\varepsilon_k-2}{1-\varepsilon_k}=\sum_{k=1}^{n}\frac{3(\varepsilon_k-1)+1}{1-\varepsilon_k}=-3n+\sum_{k=1}^{n}\frac{1}{1-\varepsilon_k}=-\frac{5}{2}n$$

Therefore,

$$n^2+\sum_{k=1}^{n}\frac{3\varepsilon_k-2}{1-\varepsilon_k}+\frac{3}{2}=0\Leftrightarrow n^2-\frac{5}{2}n+\frac{3}{2}=0\Leftrightarrow 2n^2-5n+3=0$$

$$\Leftrightarrow(2n-3)(n-1)=0\Leftrightarrow n=1.$$

Solution 3 by Ravi Prakash-New Delhi-India

ε_i –roots of the equation $z^{n+1}=1$

$$z^{n+1}-1=(z-\varepsilon_1)(z-\varepsilon_2)\cdot\ldots\cdot(z-\varepsilon_n)=1+z+z^2+\cdots+z^n$$

$$\log(z-\varepsilon_1)+\log(z-\varepsilon_2)+\cdots+\log(z-\varepsilon_n)=\log(1+z+z^2+\cdots+z^n)$$

Differentiating w.r.t z, we get:

$$\frac{1}{z-\varepsilon_1}+\frac{1}{z-\varepsilon_2}+\cdots+\frac{1}{z-\varepsilon_n}=\frac{1+2z+\cdots+nz^{n-1}}{1+z+\cdots+z^n}$$

Putting $z=1$, we have:

$$\frac{1}{1-\varepsilon_1}+\frac{1}{1-\varepsilon_2}+\cdots+\frac{1}{1-\varepsilon_n}=\frac{n}{2}\Rightarrow\sum_{k=1}^{n}\frac{1}{1-\varepsilon_k}=\frac{n}{2}$$

$$\sum_{k=1}^{n}\left(-3+\frac{1}{1-\varepsilon_k}\right)=-3n+\frac{n}{2}$$

$$\sum_{k=1}^{n}\left(\frac{3\varepsilon_k-2}{1-\varepsilon_k}\right)=-\frac{5}{2}n$$

Therefore,

$$n^2+\sum_{k=1}^{n}\frac{3\varepsilon_k-2}{1-\varepsilon_k}+\frac{3}{2}=0\Leftrightarrow n^2-\frac{5}{2}n+\frac{3}{2}=0\Leftrightarrow 2n^2-5n+3=0$$

$$\Leftrightarrow (2n-3)(n-1)=0\Leftrightarrow n=1.$$

SP.454 Let $(z_k)_{k=\overline{1,n-1}}$ be roots of the unity by order $n, z_k\neq 1$, $\forall k=\overline{1,n-1}, n\in\mathbb{N}^*, n\geq 3$.Find:

$$\Omega=\sum_{\substack{l=\overline{1-n-2}\\ k=\overline{2,n-1}, l<k}}\frac{z_l z_k}{(1-z_l)(1-z_k)}+\frac{n-2}{3}\cdot\sum_{k=1}^{n-1}\frac{z_k}{1-z_k}$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\text{Let } u_p=\frac{z_p}{1-z_p}\Rightarrow z_p=\frac{u_p}{1-u_p}, p=\overline{1,n-1}.$$

u_k –are roots of the equation $(u+1)^n-u^n=0\Leftrightarrow$

$$\binom{n}{1}u^{n-1}+\binom{n}{2}u^{n-2}+\cdots+1=0$$

$$\sum_{k=1}^{n-1}\frac{z_k}{1-z_k}=\sum_{k=1}^{n-1}u_k=-\frac{\binom{n}{2}}{\binom{n}{1}}=-\frac{\binom{n}{2}}{n}$$

$$\text{Now, let } u_p=\frac{z_p}{1-z_p}\Rightarrow z_p=\frac{u_p}{1-u_p}, p=\overline{1,n-1}.$$

u_k –are roots of the equation $(u+1)^n-u^n=0\Leftrightarrow$

$$\binom{n}{1}u^{n-1}+\binom{n}{2}u^{n-2}+\cdots+1=0$$

$$\sum_{\substack{l=\overline{1-n-2}\\ k=\overline{2,n-1}, l<k}}\frac{z_l z_k}{(1-z_l)(1-z_k)}=\sum_{\substack{l=\overline{1-n-2}\\ k=\overline{2,n-1}, l<k}}u_l u_k=\frac{\binom{n}{3}}{\binom{n}{1}}=\frac{(n-1)(n-2)}{6}$$

Therefore,

$$\Omega=\sum_{\substack{l=\overline{1-n-2}\\ k=\overline{2,n-1}, l<k}}\frac{z_l z_k}{(1-z_l)(1-z_k)}+\frac{n-2}{3}\cdot\sum_{k=1}^{n-1}\frac{z_k}{1-z_k}=0$$

SP.455 Let $(\varepsilon_i)_{i=\overline{1,n}}$ – be roots of unity by order n, n –even number. Solve for natural numbers:

$$\sum_{1\le i<j\le n}\left(\varepsilon_i-\varepsilon_j\right)^n=3n-2, n\in\mathbb{N}^*$$

Proposed by Florică Anastase, Alexandru Păun -Romania

Solution by proposers

If $\varepsilon_0=1$, then $\varepsilon_k=\varepsilon^{k-1}; \forall k=\overline{2,2m}$, where $n=2m$. Hence,

$$S=\sum_{1\le i<j\le n}\left(\varepsilon_i-\varepsilon_j\right)^{2m}=$$

$$=(2m-1)(1-\varepsilon)^{2m}+(2m-2)(1-\varepsilon^2)^{2m}+\cdots+(1-\varepsilon^{2m-1})^{2m}$$

But $\left(1-\varepsilon^k\right)^{2m}=\left(\varepsilon^{2m}-\varepsilon^k\right)^{2m}=\left[\varepsilon^k\left(\varepsilon^{2m-k}-1\right)\right]^{2m}=\left(1-\varepsilon^{2m-k}\right)^{2m}$

Hence,

$$S=m\left[4m-2-\binom{2m}{1}(\varepsilon+\varepsilon^2+\cdots+\varepsilon^{2m-1})+\cdots+\binom{2m}{2m-1}(\varepsilon+\varepsilon^2+\cdots+\varepsilon^{2m-1})\right]$$

$$=$$

$$=m\left[4m-\left(1-\binom{2m}{1}+\binom{2m}{2}-\binom{2m}{3}+\cdots-\binom{2m}{2m-1}+\binom{2m}{2m}\right)\right]=4m^2=n^2$$

Therefore,

$$\sum_{1\le i<j\le n}\left(\varepsilon_i-\varepsilon_j\right)^n=3n-2, n\in\mathbb{N}^*\Leftrightarrow n^2=3n-2\Leftrightarrow n\in\{1,2\}$$

But n –is even, so $n=2$.

SP.456 Let $\varepsilon_i, i=\overline{1,n}$ be roots of the equation $z^{n+1}=1, \varepsilon_i\neq 1, \forall i=\overline{1,n}$.

Solve for complex numbers:

$$z^{2n}+\frac{4}{5n}\sum_{k=1}^{n}\frac{3\varepsilon_k-2}{1-\varepsilon_k}\cdot z^n+4i(z^n-1)=0$$

Proposed by Florică Anastase, Raluca Maria Caraion-Romania

Solution 1 by proposer

$$S_n=\sum_{k=1}^{n}\frac{3\varepsilon_k-2}{1-\varepsilon_k}=\sum_{k=1}^{n}\frac{3(\varepsilon_k-1)+1}{1-\varepsilon_k}=-3n+\sum_{k=1}^{n}\frac{1}{1-\varepsilon_k}$$

Let be $\varepsilon_k=\cos\frac{2k\pi}{n+1}+i\sin\frac{2k\pi}{n+1}$, $k=\overline{1,n}$, then

$$S_n=-3n+\sum_{k=1}^{n}\frac{1}{1-\cos\frac{2k\pi}{n+1}-i\sin\frac{2k\pi}{n+1}}=$$

$$=-3n+\sum_{k=1}^{n}\frac{1}{2\sin^2\frac{k\pi}{n+1}-2i\sin\frac{k\pi}{n+1}\cos\frac{k\pi}{n+1}}=$$

$$=-3n+\sum_{k=1}^{n}\frac{1}{-2i\sin\left(\cos\frac{k\pi}{n+1}+i\sin\frac{k\pi}{n+1}\right)}=-3n+\sum_{k=1}^{n}\frac{i\left(\cos\frac{k\pi}{n+1}-i\sin\frac{k\pi}{n+1}\right)}{2\sin\frac{k\pi}{n+1}}$$

$$=-3n+\frac{n}{2}+\frac{i}{2}\sum_{k=1}^{n}\cot\frac{k\pi}{n+1}=-\frac{5}{2}n+\frac{1}{2}i\sum_{k=1}^{n}\cot\frac{k\pi}{n+1}=-\frac{5n}{2}$$

$$\text{Because }\sum_{k=1}^{n}\cot\frac{k\pi}{n+1}=0;\forall n\in\mathbb{N}.$$

It follows that:

$$z^{2n}-2z^n=4i(1-z^n)$$

Let $t=z^n\Rightarrow t^2-2t(1-2i)-4i=0$, then

$t_1=1-(2+\sqrt{3})i$ and $t_2=1-(2-\sqrt{3})i$. Thus,

$z_1^n=1-(2+\sqrt{3})i$ and $z_2^n=1-(2-\sqrt{3})i$

Because: $\begin{cases}2+\sqrt{3}=\tan\frac{5\pi}{12}\\2-\sqrt{3}=\tan\frac{\pi}{12}\end{cases}\Rightarrow\begin{cases}z_1^n=r_1\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)\\z_2^n=r_2\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)\end{cases}$.

$$\begin{cases}z_{k_1}=\sqrt[n]{r_1}\left(\cos\frac{\frac{19\pi}{12}+2k\pi}{n}+i\sin\frac{\frac{19\pi}{12}+2k\pi}{n}\right)\\z_{k_2}=\sqrt[n]{r_2}\left(\cos\frac{\frac{11\pi}{12}+2k\pi}{n}+i\sin\frac{\frac{11\pi}{12}+2k\pi}{n}\right)\end{cases};k=\overline{0,n-1}$$

$$r_{1,2}=2\sqrt{2\mp\sqrt{3}}$$

Solution 2 by Ravi Prakash-New Delhi-India

ε_i −roots of the equation $z^{n+1}=1$

$$z^{n+1}-1=(z-\varepsilon_1)(z-\varepsilon_2)\cdot\ldots\cdot(z-\varepsilon_n)=1+z+z^2+\cdots+z^n$$

$$\log(z-\varepsilon_1)+\log(z-\varepsilon_2)+\cdots+\log(z-\varepsilon_n)=\log(1+z+z^2+\cdots+z^n)$$

Differentiating w.r.t z, we get:

$$\frac{1}{z-\varepsilon_1}+\frac{1}{z-\varepsilon_2}+\cdots+\frac{1}{z-\varepsilon_n}=\frac{1+2z+\cdots+nz^{n-1}}{1+z+\cdots+z^n}$$

Putting $z=1$, we have:

$$\frac{1}{1-\varepsilon_1}+\frac{1}{1-\varepsilon_2}+\cdots+\frac{1}{1-\varepsilon_n}=\frac{n}{2}\Rightarrow\sum_{k=1}^{n}\frac{1}{1-\varepsilon_k}=\frac{n}{2}$$

$$\sum_{k=1}^{n}\left(-3+\frac{1}{1-\varepsilon_k}\right)=-3n+\frac{n}{2}$$

$$\sum_{k=1}^{n}\left(\frac{3\varepsilon_k-2}{1-\varepsilon_k}\right)=-\frac{5}{2}n$$

The equation becomes

$$z^{2n}+\frac{4}{5n}\left(-\frac{5}{2}n\right)z^n+4i(z^n-1)=0$$

$$z^{2n}-2(1-2i)z^n-4i=0$$

Put $z^n=t$, then:

$$t^2-2(1-2i)t-4i=0$$

$$t=\frac{1}{2}\left[2(1-2i)\pm2\sqrt{(1-2i)^2+4i}\right]=1-2i\pm\sqrt{3}i$$

Let $t=1-(2-\sqrt{3})i\Rightarrow z^n=1-\tan\left(\frac{\pi}{12}\right)i=\frac{1}{\cos\left(\frac{\pi}{12}\right)}\left[\cos\left(\frac{\pi}{12}\right)-i\sin\left(\frac{\pi}{12}\right)\right]$

$$z^n=\frac{1}{\cos\left(\frac{\pi}{12}\right)}\left[\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right]=$$

$$= \frac{1}{\cos\left(\frac{\pi}{12}\right)}\left[\cos\left(2k\pi + \frac{11\pi}{12}\right) + i\sin\left(2k\pi + \frac{11\pi}{12}\right)\right];\ \ k \in \mathbb{Z}$$

By the DeMoivre's theorem, the roots are:

$$z_k = \frac{1}{\sqrt[n]{\cos\left(\frac{\pi}{12}\right)}}\left[\cos\left(\frac{2k\pi}{n} + \frac{11\pi}{12n}\right) + i\sin\left(\frac{2k\pi}{n} + \frac{11\pi}{12n}\right)\right]; k = \overline{0, (n-1)}$$

$$z_k = \frac{1}{\sqrt[n]{\cos\left(\frac{\pi}{12}\right)}} e^{\left(2k + \frac{11}{12}\right)\frac{\pi i}{n}}; k = \overline{0, (n-1)}$$

Let $t = 1 - (2+\sqrt{3})i \Rightarrow z^n = 1 - \tan\left(\frac{5\pi}{12}\right)i = \frac{1}{\cos\left(\frac{5\pi}{12}\right)}\left[\cos\left(\frac{5\pi}{12}\right) - i\sin\left(\frac{5\pi}{12}\right)\right]$

$$z^n = \frac{1}{\cos\left(\frac{5\pi}{12}\right)}\left[\cos\left(2\pi - \frac{5\pi}{12}\right) + i\sin\left(2\pi - \frac{5\pi}{12}\right)\right] =$$

$$= \frac{1}{\cos\left(\frac{5\pi}{12}\right)}\left[\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right] =$$

$$= \frac{1}{\cos\left(\frac{5\pi}{12}\right)}\left[\cos\left(2k\pi + \frac{19\pi}{12}\right) + i\sin\left(2k\pi + \frac{19\pi}{12}\right)\right];\ \ k \in \mathbb{Z}$$

By the DeMoivre's theorem, the roots are:

$$z_k = \frac{1}{\sqrt[n]{\cos\left(\frac{5\pi}{12}\right)}}\left[\cos\left(\frac{2k\pi}{n} + \frac{19\pi}{12n}\right) + i\sin\left(\frac{2k\pi}{n} + \frac{19\pi}{12n}\right)\right]; k = \overline{0, (n-1)}$$

$$z_k = \frac{1}{\sqrt[n]{\cos\left(\frac{5\pi}{12}\right)}} e^{\left(2k + \frac{19}{12}\right)\frac{\pi i}{n}}; k = \overline{0, (n-1)}$$

SP.457 If $a, b, c, k > 0$ such that $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = k$, then:

$$27\prod_{cyc}(a+b) + 6k\left(\sum_{cyc} a\right)^2 \ge 14k^3$$

Proposed by Gheorghe Molea-Romania

Solution 1 by proposer

$$\text{Let } P=\prod_{cyc}(a+b)\Rightarrow P^2=\prod_{cyc}(a+b)^2$$
$$(a+b)(b+c)=b^2+ac+b(a+c)\geq b^2+ac+2\sqrt{ac}=$$
$$=\left(b+\sqrt{ac}\right)^2;(and\ analogs)$$
$$\Rightarrow P^2\geq\prod_{cyc}\left(b+\sqrt{ac}\right)^2\Rightarrow P\geq\prod_{cyc}\left(b+\sqrt{ac}\right)$$

From Schur's inequality, we have:

$$(d+e+f)^3+9def\geq 4(d+e+f)(de+ef+fg)$$

Let $d=a+\sqrt{bc}, e=b+\sqrt{ac}, f=c+\sqrt{ab}$, then:

$$9P\geq 4\left(\sum_{cyc}a+\sum_{cyc}\sqrt{ab}\right)\left(\sum_{cyc}ab+a\sum_{cyc}\sqrt{ab}+b\sum_{cyc}\sqrt{ab}+c\sum_{cyc}\sqrt{ab}\right)-$$
$$-\left(\sum_{cyc}a+\sum_{cyc}\sqrt{ab}\right)^3$$
$$9P\geq 4\left(\sum_{cyc}a+k\right)\left(\sum_{cyc}ab+k\sum_{cyc}a\right)-\left(\sum_{cyc}a+k\right)^3$$
$$9P\geq\left(\sum_{cyc}a+k\right)\left[4\sum_{cyc}ab+4k\sum_{cyc}a-\left(\sum_{cyc}a\right)^2-k^2-2k\sum_{cyc}a\right]$$
$$9P\geq\left(\sum_{cyc}a+k\right)\left[2k\sum_{cyc}a+4\sum_{cyc}ab-\left(\sum_{cyc}a\right)^2-k^2\right]$$

$$\text{But }\sum_{cyc}a\geq\sum_{cyc}\sqrt{ab}\Rightarrow\sum_{cyc}a\geq k\text{ and }\sum_{cyc}ab\geq\frac{1}{3}\left(\sum_{cyc}\sqrt{ab}\right)^2=\frac{k^2}{3}$$
$$\sum_{cyc}ab\geq\frac{k^2}{3}$$

We have:

$$9P\geq(k+k)\left[2k\cdot k+4\cdot\frac{k^2}{3}-\left(\sum_{cyc}a\right)^2-k^2\right]$$

$$9P \ge 2k\left[\frac{7k^2}{3}-\left(\sum_{cyc} a\right)^2\right] \Leftrightarrow 27P+6k\left(\sum_{cyc} a\right)^2 \ge 14k^3$$

Equality holds for $a=b=c=\frac{k}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \ a+b+c \overset{AM-GM}{\ge} \sqrt{ab}+\sqrt{bc}+\sqrt{ca}=k$$

$$ab+bc+ca \overset{CBS}{\ge} \frac{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2}{3}=\frac{k^2}{3}$$

$$Also\ by\ Cesaro\ inequality\ we\ have:\ \prod_{cyc}(a+b) \ge 8abc$$

$$Then:\ 9\prod_{cyc}(a+b) \ge 8\prod_{cyc}(a+b)+8abc=8(a+b+c)(ab+bc+ca) \ge$$

$$\ge 8.k.\frac{k^2}{3}=\frac{8k^3}{3}$$

$$Therefore,\ \ 27\prod_{cyc}(a+b)+6k\left(\sum_{cyc} a\right)^2 \ge 3.\frac{8k^3}{3}+6k.k^2=14k^3.$$

$$Equality\ holds\ iff\ a=b=c=\frac{k}{3}.$$

SP.458 Let $\Delta A_1B_1C_1, \Delta A_2B_2C_2$ be triangles with sides a_1, b_1, c_1, circumradii R_1, respectively a_2, b_2, c_2, R_2. Prove that:

$$\left(\frac{1}{a_1^3}+\frac{1}{b_1^3}+\frac{1}{c_1^3}\right)\left(\frac{1}{a_2^5}+\frac{1}{b_2^5}+\frac{1}{c_2^5}\right) \ge \frac{1}{9R_1^3R_2^5}$$

Proposed by D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\left(\frac{1}{a_1^3}+\frac{1}{b_1^3}+\frac{1}{c_1^3}\right)\left(\frac{1}{a_2^5}+\frac{1}{b_2^5}+\frac{1}{c_2^5}\right)=$$

$$=\left(\frac{\left(\frac{1}{a_1}\right)^3}{1^2}+\frac{\left(\frac{1}{b_1}\right)^3}{1^2}+\frac{\left(\frac{1}{c_1}\right)^3}{1^2}\right)\left(\frac{\left(\frac{1}{a_2}\right)^5}{1^4}+\frac{\left(\frac{1}{b_2}\right)^5}{1^4}+\frac{\left(\frac{1}{c_2}\right)^5}{1^4}\right)\overset{Radon}{\geq}$$

$$\geq\frac{\left(\frac{1}{a_1}+\frac{1}{b_1}+\frac{1}{c_1}\right)^3}{(1+1+1)^2}+\frac{\left(\frac{1}{a_2}+\frac{1}{b_2}+\frac{1}{c_2}\right)^5}{(1+1+1)^4}\overset{Ionescu-Tiu}{\geq}$$

$$\geq\frac{\left(\frac{\sqrt{3}}{R_1}\right)^3}{3^2}\cdot\frac{\left(\frac{\sqrt{3}}{R_2}\right)^5}{3^4}=\frac{1}{3^6}(\sqrt{3})^8\cdot\frac{1}{R_1^3R_2^5}=\frac{1}{9R_1^3R_2^5}$$

Equality holds for $a_1=b_1=c_1;\ a_2=b_2=c_2$.

Solution 2 by Adrian Popa-Romania

$$\frac{1^4}{a_1^3}+\frac{1^4}{b_1^3}+\frac{1^4}{c_1^3}\overset{Radon}{\geq}\frac{(1+1+1)^4}{(a_1+b_1+c_1)^3}=\frac{3^4}{2^3s_1^3}\overset{Mitrinovic}{\geq}\frac{3^4}{2^3\cdot\frac{3^3\cdot3\sqrt{3}R_1^3}{2^3}}=\frac{1}{\sqrt{3}R_1^3};(1)$$

$$\frac{1^6}{a_1^5}+\frac{1^6}{b_1^5}+\frac{1^6}{c_1^5}\overset{Radon}{\geq}\frac{(1+1+1)^6}{(a_1+b_1+c_1)^5}=\frac{3^6}{2^5s_2^5}\overset{Mitrinovic}{\geq}\frac{3^6}{2^5\cdot\frac{3^5\cdot3^2\sqrt{3}R_2^5}{2^3}}=\frac{1}{\sqrt{3}R_2^3};(2)$$

By multiplying (1) and (2), we get:

$$\left(\frac{1}{a_1^3}+\frac{1}{b_1^3}+\frac{1}{c_1^3}\right)\left(\frac{1}{a_2^5}+\frac{1}{b_2^5}+\frac{1}{c_2^5}\right)\geq\frac{1}{9R_1^3R_2^5}$$

SP.459 If $x,y\in\mathbb{R}_+$ and $x+y=4$ then in ΔABC holds:

$$\frac{a^4+b^4}{c^y}w_c^x+\frac{b^4+c^4}{a^y}w_a^x+\frac{c^4+a^4}{b^y}w_b^x\geq3\cdot2^{x+1}F^x$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

Because $w_a\geq h_a$ and analogous, then we get:

$$\sum_{cyc}\frac{a^4+b^4}{c^y}w_c^x\geq\sum_{cyc}\frac{a^4+b^4}{c^y}h_c^x=\sum_{cyc}\frac{a^4+b^4}{c^{x+y}}(ch_c)^x=$$

$$=2^x\cdot F^x\sum_{cyc}\frac{a^4+b^4}{c^4}=2^x\cdot F^x\sum_{cyc}\left(\frac{a^4+b^4}{c^4}+1\right)-3\cdot2^xF^x=$$

$$= 2^x F^x(a^4+b^4+c^4)\sum_{cyc}\frac{1}{c^4} - 3\cdot 2^x F^x \overset{Bergstrom}{\geq}$$

$$\geq 2^x F^x(a^4+b^4+c^4)\cdot\frac{(1+1+1)^2}{a^4+b^4+c^4} - 3\cdot 2^x F^x =$$

$$= 9\cdot 2^x\cdot F^x - 3\cdot 2^x\cdot F^x = 6\cdot 2^x F^x = 3\cdot 2^{x+1}\cdot F^x$$

Solution 2 by Tapas Das-India

$$\sum_{cyc}\frac{a^4+b^4}{c^y}w_c^x \geq \sum_{cyc}\frac{2a^2b^2}{c^y}w_c^x \geq 3\sqrt[3]{\prod_{cyc}\frac{2a^2b^2}{c^y}w_c^x} =$$

$$= 3\sqrt[3]{2^3\cdot\frac{(abc)^4}{(abc)^y}(w_a w_b w_c)^x} = 6(abc)^{\frac{4-y}{3}}\cdot(w_a w_b w_c)^{\frac{x}{3}} \geq 6(abc)^{\frac{4-y}{3}}\cdot(h_a h_b h_c)^{\frac{x}{3}}$$

$$= 6(abc)^{\frac{x}{3}}\cdot\left(\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\right)^{\frac{x}{3}} = 6(abc)^{\frac{x}{3}}\cdot\left(\frac{8F^3}{abc}\right)^{\frac{x}{3}} = 6\cdot 2^x\cdot F^x$$

SP.460 In $\Delta ABC, M\in(BC), N\in(CA), P\in(AB)$ such that

$$\frac{MB}{MC} = \frac{NC}{NA} = \frac{PA}{PB} = x > 0.\text{Prove that:}$$

$$(MN^4+NP^4+PM^4)(x+1)^4 \geq 16(x^2-x+1)^2F^2$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$a = BM + MC = xMC + MC = (x+1)MC$$

$$MC = \frac{a}{x+1};\ MB = a - MC = a - \frac{a}{x+1} = \frac{ax}{x+1}$$

Analogous,

$$NC = \frac{bx}{x+1}; PA = \frac{cx}{x+1}; NA = \frac{b}{x+1}; PB = \frac{c}{x+1}$$

$$[CMN] = \frac{1}{2}MC\cdot NC\cdot\sin C = \frac{1}{2}\cdot\frac{a}{x+1}\cdot\frac{bx}{x+1}\cdot\sin C =$$

$$= \frac{x}{(x+1)^2}\cdot\frac{ab\cdot\sin C}{2} = \frac{x}{(x+1)^2}\cdot F$$

Analogous,

$$[ANP]=[BMP]=\frac{x}{(x+1)^2}\cdot F$$

$$[MNP]=F-\frac{3x}{(x+1)^2}\cdot F=\frac{x^2-x+1}{(x+1)^2}\cdot F$$

By Goldner's inequality in ΔMNP:

$$MN^4+NP^4+PM^4\geq 16([MNP])^2=\frac{16(x^2-x+1)^2}{(x+1)^4}\cdot F^2$$

Therefore,

$$(MN^4+NP^4+PM^4)(x+1)^4\geq 16(x^2-x+1)^2F^2$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ F'\ be\ the\ area\ of\ triangle\ MNP.$$

$$By\ Routh's\ theorem\ we\ have:\ \ F'=\frac{x^3+1}{(x+1)^3}.F=\frac{x^2-x+1}{(x+1)^2}.F.$$

$$From\ the\ inequality\ \ (MN^2-NP^2)^2+(NP^2-PM^2)^2+(PM^2-MN^2)^2\geq 0$$

$$we\ get:\ MN^4+NP^4+PM^4\geq$$

$$\geq 2(MN^2.NP^2+NP^2.PM^2+PM^2.MN^2)-(MN^4+NP^4+PM^4)=16F'^2$$

$$Then:\ MN^4+NP^4+PM^4\geq\frac{16(x^2-x+1)^2}{(x+1)^4}.F^2$$

$$Therefore,\ \ (MN^4+NP^4+PM^4)(x+1)^4\geq 16(x^2-x+1)^2F^2.$$

SP.461 In ΔABC the following relationship holds:

$$\frac{m_aw_a-r^2}{h_ah_b+r^2}+\frac{m_bw_c-r^2}{h_bh_c+r^2}+\frac{m_cw_a-r^2}{h_ch_a+r^2}\geq\frac{12}{5}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\sum_{cyc}\frac{m_aw_a-r^2}{h_ah_b+r^2}\geq\sum_{cyc}\frac{h_ah_b-r^2}{h_ah_b+r^2}=\sum_{cyc}\frac{ab(h_ah_b-r^2)}{ab(h_ah_b+r^2)}=$$

$$= -3 + \sum_{cyc}\left(1 + \frac{ah_a \cdot bh_b - r^2ab}{ah_a \cdot bh_b + r^2ab}\right) = -3 + \sum_{cyc}\frac{2ah_a \cdot bh_b}{ah_a \cdot bh_b + r^2ab} =$$

$$= -3 + 8F^2\sum_{cyc}\frac{1}{4F^2 + r^2ab} \overset{Bergstrom}{\geq} 8F^2 \cdot \frac{9}{12F^2 + r^2(ab + bc + ca)} - 3 \geq$$

$$\geq \frac{72F^2}{12F^2 + r^2 \cdot \frac{(a+b+c)^2}{3}} - 3 = \frac{72F^2}{12F^2 + r^2 \cdot \frac{4s^2}{3}} - 3 =$$

$$= \frac{216F^2}{36F^2 + 4F^2} - 3 = \frac{216}{40} - 3 = \frac{12}{5}$$

Equality holds for $a = b = c$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ \ m_a \geq h_a\ and\ w_b \geq h_b\ \ then\ we\ have:\ m_a w_b \geq h_a h_b\ \ (and\ analogs)$$

$$Then:$$

$$\frac{m_a w_b - r^2}{h_a h_b + r^2} + \frac{m_b w_c - r^2}{h_b h_c + r^2} + \frac{m_c w_a - r^2}{h_c h_a + r^2} \geq \frac{h_a h_b - r^2}{h_a h_b + r^2} + \frac{h_b h_c - r^2}{h_b h_c + r^2} + \frac{h_c h_a - r^2}{h_c h_a + r^2} =$$

$$= \frac{(2sr)^2 - ab.r^2}{(2sr)^2 + ab.r^2} + \frac{(2sr)^2 - bc.r^2}{(2sr)^2 + bc.r^2} + \frac{(2sr)^2 - ca.r^2}{(2sr)^2 + ca.r^2}$$

$$= \frac{4s^2 - ab}{4s^2 + ab} + \frac{4s^2 - bc}{4s^2 + bc} + \frac{4s^2 - ca}{4s^2 + ca} =$$

$$= \left(\frac{8s^2}{4s^2 + ab} - 1\right) + \left(\frac{8s^2}{4s^2 + bc} - 1\right) + \left(\frac{8s^2}{4s^2 + ca} - 1\right)$$

$$= 8s^2\left(\frac{1}{4s^2 + ab} + \frac{1}{4s^2 + bc} + \frac{1}{4s^2 + ca}\right) - 3 \geq$$

$$\overset{CBS}{\geq} \frac{8s^2.9}{3.4s^2 + (ab + bc + ca)} - 3 \geq \frac{72s^2}{12s^2 + \frac{1}{3}(a+b+c)^2} - 3 =$$

$$= \frac{3.72s^2}{3.12s^2 + 4s^2} - 3 = \frac{27}{5} - 3 = \frac{12}{5}.$$

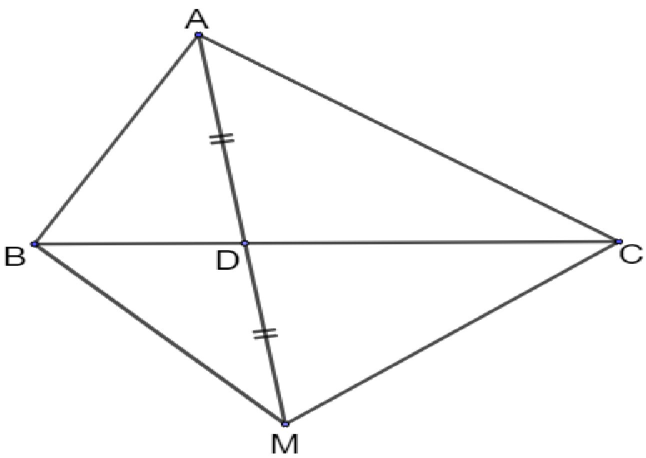
$$Therefore,\ \frac{m_aw_b-r^2}{h_ah_b+r^2}+\frac{m_bw_c-r^2}{h_bh_c+r^2}+\frac{m_cw_a-r^2}{h_ch_a+r^2}\ge\frac{12}{5}.$$

Equality holds for $a=b=c$.

SP.462 Let ABC be an triangle, D be a point on side BC and M be the symmetrical of A with respect to D. If $\frac{BM^2}{AB}+\frac{CM^2}{AC}=AB+AC$, then prove that AD is the bisector of the angle $\widehat{A}$, or is the height from the vertex A.

Proposed by Neculai Stanciu-Romania

Solution 1 by proposer



We denote $a=BC, b=CA, c=AB$, $p=BM, q=CM$ and $\frac{BD}{DC}=x$.

We have: $BD=\frac{ax}{x+1}$ and $DC=\frac{a}{x+1}$.

By Stewart's theorem, we obtain that:

$AB^2\cdot DC-AD^2\cdot BC+AC^2\cdot BD=BD\cdot DC\cdot BC$.

So,

$$AD^2=\frac{b^2x+c^2}{x+1}-\frac{a^2x}{(x+1)^2}$$

Since BD is median in ΔABM and CD is median in ΔACM, we have:

$$BD^2=\frac{AB^2+BM^2}{2}-\frac{AM^2}{4}\Leftrightarrow$$

$$\frac{a^2x^2}{(x+1)^2}=\frac{p^2+c^2}{2}-\frac{b^2x+c^2}{x+1}+\frac{a^2x}{(x+1)^2};(1)$$

Similarly, we obtain:

$$\frac{a^2}{(x+1)^2}=\frac{q^2+b^2}{2}-\frac{b^2x+c^2}{x+1}+\frac{a^2x}{(x+1)^2};(2)$$

By adding $(1)\times b$ with $(2)\times c$, taking account that the relation from hypothesis (which is equivalent with $p^2b+q^2c=bc(b+c)$), we deduce that:

$$a^2bx^2+a^2c=bc(b+c)(x+1)^2-(x+1)(b^3x+bc^2+b^2cx+c^3)+a^2x(b+c)\Leftrightarrow$$

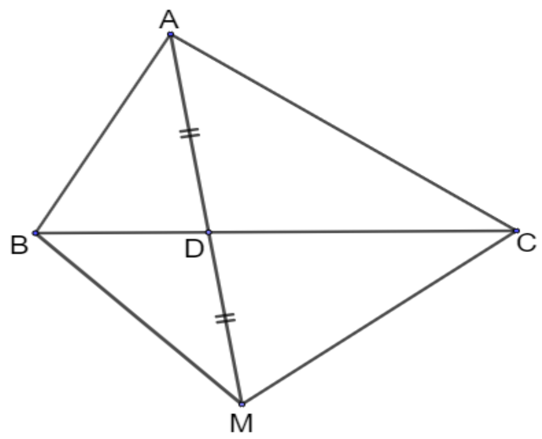
$$bx^2(a^2-c(b+c)+b^2+bc)-x\left(2bc(b+c)-bc^2-c^3-b^3-b^2c+a^2(b+c)\right)+$$
$$+a^2c-bc(b+c)+bc^2+c^3=0$$
$$bx^2(a^2+b^2-c^2)-x(b+c)(2bc-b^2-c^2+a^2)+c(a^2-b^2+c^2)=0$$

But last quadratic equation has solution $x_1=\frac{c}{b}, x_2=\frac{a^2-b^2+c^2}{a^2+b^2-c^2}$.

In the case $x_1=\frac{c}{b}$, with the converse bisector's theorem AD in the bisector from A.

By the cosine law $x_2=\frac{c\cos B}{b\cos C}$, so in this case AD is the height from A.

Solution 2 by Aggeliki Papaspyropoulou-Greece



Suppose that AD is not height h_a of the triangle ABC. Then we haveȘ

$$AB=c, AC=b, BC=a.$$

Using Stewart's theorem for ΔABC, we get:

$$b^2\cdot BD+c^2\cdot DC=$$
$$=AD^2\cdot a+a\cdot BD\cdot DC$$
$$\frac{b^2\cdot BD}{a}+\frac{c^2\cdot DC}{a}=AD^2+BD\cdot DC;(1)$$

Using Stewart's theorem for ΔMBC:

$$MB^2\cdot DC+MC^2\cdot BD=MD^2\cdot a+BD\cdot DC\cdot a$$
$$\frac{MB^2\cdot DC}{a}+\frac{MC^2\cdot BD}{a}=MD^2+BD\cdot DC=AD^2+BD\cdot DC;(2)$$
$$(1)+(2)\Rightarrow\frac{MB^2\cdot DC}{a}+\frac{MC^2\cdot BD}{a}=\frac{b^2\cdot BD}{a}+\frac{c^2\cdot DC}{a}$$
$$MB^2\cdot DC+MC^2\cdot BD=b^2\cdot BD+c^2\cdot DC$$
$$(MB^2-c^2)DC=(b^2-MC^2)BD$$

Since in this case $b\neq MC,(MB\neq c)$, we have:

$$\frac{DC}{DB}=\frac{b^2-MC^2}{MB^2-c^2};(3)$$

We'll prove that AD is the bisector of angle $\widehat{A}$, to prove it is enough to show that:

$$\frac{DC}{BD}=\frac{AC}{AB}=\frac{b}{c};(4)(bisector\ theorem)$$

By (3) and (4) we have to show:

$$\frac{b^2-MC^2}{MB^2-c^2}=\frac{b}{c}\Leftrightarrow cb^2-c\cdot MC^2=b\cdot MB^2-bc^2\Leftrightarrow$$

$$bc^2+cb^2=b\cdot MB^2+c\cdot MC^2\Leftrightarrow\frac{bc^2}{bc}+\frac{cb^2}{bc}=\frac{MB^2}{c}+\frac{MC^2}{b}\Leftrightarrow$$

$$c+b=\frac{MB^2}{c}+\frac{MC^2}{b}$$

Case II) Suppose $MB=c$, then in ΔAMB, MB and BD are medians according to AM $AD\perp BD$, AD −height of ΔABC from the vertex A and in ΔMCA we have $MC=b$.

$$AD\text{ −height}\Leftrightarrow\frac{MB^2}{c}+\frac{MC^2}{b}=\frac{c^2}{c}+\frac{b^2}{b}=c+b.$$

SP.463 If K −Lemoine's point in ΔABC, then:

$$\sum_{cyc}\frac{aKA+bKB-cKC}{2aKA+cKC}\geq 1$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

By Murray-Klamkin's principle in ΔABC, aKA, bKB, cKC can be sides in a triangle, hence

$aKA+bKB-cKC>0$ and permutations.

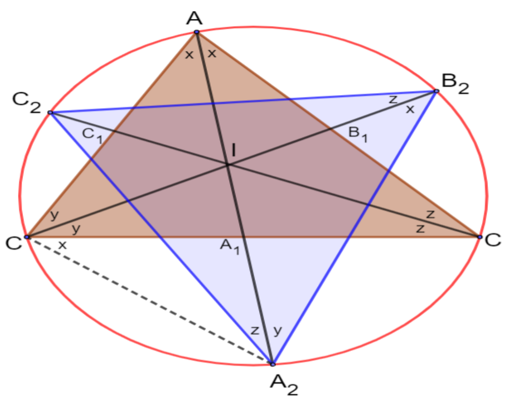
$$\sum_{cyc}\frac{aKA+bKB-cKC}{2aKA+cKC}=\sum_{cyc}\frac{(aKA+bKB-cKC)^2}{(2aKA+cKC)(aKA+bKB-cKC)}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(\sum aKA)^2}{(\sum aKA)^2}=1$$

Equality holds for $a=b=c$.

SP.464 In ΔABC, AA_1, BB_1, CC_1 internal bisector, A_2, B_2, C_2 contact point with circumcircle of triangle ABC. Prove that:

$$A_1A_2\cdot B_2C_2+B_1B_2\cdot A_2C_2+C_1C_2\cdot A_2B_2\geq Rs$$



Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\rho(A_1) = AA_1 \cdot A_1A_2 = BA_1 \cdot A_1C \Leftrightarrow A_1A_2 = \frac{BA_1 \cdot A_1C}{A_1A_2}; (1)$$

From bisector theorem, we have:

$$BA_1 = \frac{ac}{b+c}, CA_1 = \frac{ab}{b+c}; (2)$$

$$AA_1 = w_a = \frac{2bc}{b+c}\cos\frac{A}{2}; (3)$$

From (1),(2) and (3) we get

$$A_1A_2 = \frac{a^2}{2(b+c)\cos\frac{A}{2}}; (4)$$

From Law of Sines:

$$\frac{B_2C_2}{\sin\frac{B+C}{2}} = 2R \Rightarrow B_2C_2 = 2R\cos\frac{A}{2}; (5)$$

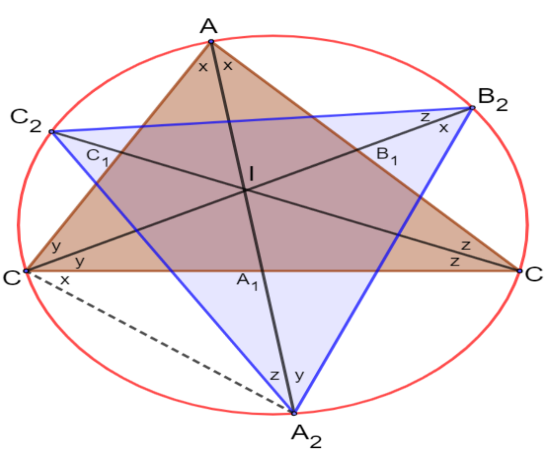
From (4) and (5) we have:

$$A_1A_2 \cdot B_2C_2 = \frac{Ra^2}{b+c}$$

Therefore,

$$\sum_{cyc} A_1A_2 \cdot B_2C_2 = R\sum_{cyc}\frac{a^2}{b+c} \geq \frac{R(a+b+c)^2}{2(a+b+c)} = \frac{R(a+b+c)}{2} = Rs$$

Solution 2 by Geanina Tudose-Romania



We have:

$$\frac{BA_1}{A_1C}=\frac{c}{b}\Rightarrow\frac{BA_1}{BC}=\frac{c}{b+c}\Rightarrow BA_1=\frac{ac}{b+c}$$

$$\text{In } \Delta BA_1A_2: \ \frac{A_1A_2}{\sin x}=\frac{BA_1}{\sin C}\Rightarrow A_1A_2=\frac{BA_1\cdot\sin x}{\sin C}=\frac{\frac{ac}{b+c}\sin x}{c}\cdot 2R$$

$$A_1A_2=\frac{2Ra\cdot\sin x}{b+c}$$

$$\text{In } \Delta A_2B_2C: \frac{B_2C_2}{\sin(x+y)}=2R\Rightarrow B_2C_2=2R\sin(x+y)$$

Hence,

$$A_1A_2\cdot B_2C_2=\frac{4R^2a\cdot\sin\frac{A}{2}\cdot\sin\frac{B+C}{c}}{2R(\sin B+\sin C)}=\frac{aR\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin B+\sin C}=\frac{aR\cdot\sin A}{\sin B+\sin C}$$

$$\sum_{cyc}A_1A_2\cdot B_2C_2=\sum_{cyc}\frac{2R^2\sin^2 A}{\sin B+\sin C}\overset{CBS}{\geq}2R^2\cdot\frac{(\sin A+\sin B+\sin C)^2}{2(\sin A+\sin B+\sin C)}=$$

$$=R^2(\sin A+\sin B+\sin C)=R\cdot\frac{a+b+c}{2}=Rs$$

SP.465 If $x,y,z>0, x+y+z=3\sqrt{3}$ then:

$$\frac{x^3y}{(x^2+1)^2}+\frac{y^3z}{(y^2+1)^2}+\frac{z^3x}{(z^2+1)^2}\leq\frac{27}{16}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

Let be $f:(0,\infty)\to\mathbb{R}, f(x)=\frac{x^3}{(x^2+1)^2}$, then

$$f'(x)=\frac{x^2(3-x^2)}{(x^2+1)^4}$$

$$f'(x)=0\Leftrightarrow x=\sqrt{3}$$

$$f(x)\leq f(\sqrt{3})=\frac{3\sqrt{3}}{16},\forall x>0$$

$$\frac{x^3}{(x^2+1)^2}\leq\frac{3\sqrt{3}}{16}\Rightarrow\frac{x^3y}{(x^2+1)^2}\leq\frac{3\sqrt{3}}{16}y$$

Therefore,

$$\sum_{cyc}\frac{x^3y}{(x^2+1)^2}\le\frac{3\sqrt{3}}{16}(x+y+z)=\frac{3\sqrt{3}}{16}\cdot 3\sqrt{3}=\frac{27}{16}$$

Equality holds for $x=y=z=\sqrt{3}$.

UNDERGRADUATE PROBLEMS

UP.451 Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=0}^{2n}(-1)^k\cdot\frac{4n+1}{4n-2k+1}\binom{2n}{k}}$$

Proposed by Ruxandra Daniela Tonilă-Romania

Solution 1 by proposer

$$\sum_{k=0}^{2n}(-1)^k\cdot\frac{4n+1}{4n-2k+1}\binom{2n}{k}=(4n+1)\sum_{k=0}^{2n}(-1)^k\cdot\frac{1}{4n-2k+1}\cdot\binom{2n}{k}=$$

$$=(4n+1)\sum_{k=0}^{2n}(-1)^n\binom{2n}{k}\cdot\int_0^1(x^2)^{2n-k}\,dx=$$

$$=(4n+1)\int_0^1\left(\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(x^2)^{2n-k}\right)dx=(4n+1)\int_0^1(x^2-1)^{2n}\,dx;\quad(1)$$

$$\text{Let } I_n=\int_0^1(x^2-1)^n dx\,,n\in\mathbb{N}^*.\text{ We have:}$$

$$I_n=x(x^2-1)\Big|_0^1-\int_0^1x[(x^2-1)^n]'\,dx=-2n\int_0^1x^2(x^2-1)\,dx=$$

$$=-2n\int_0^1(x^2-1+1)(x^2-1)^{n-1}\,dx=-2n(I_n+I_{n-1})$$

$$I_n=-\frac{2n}{2n+1}\cdot I_{n-1}\Rightarrow I_n=\left(\prod_{k=2}^{n}\frac{-2k}{2k+1}\right)I_1$$

$$I_n=\frac{(-1)^{n-1}\cdot 2^{n-1}\cdot n!}{(2n+1)!!}\cdot 3I_1$$

$$I_1=\int_0^1(x^2-1)dx=-\frac{2}{3}$$

Hence,

$$I_n=\frac{(-1)^n\cdot 2^n\cdot n!}{(2n+1)!!}$$

Therefore,

$$I_{2n}=\frac{(-1)^{2n}\cdot 2^{2n}\cdot(2n)!}{(4n+1)!!}=\frac{2^{2n}\cdot(2n)!}{(4n+1)!!};\quad (2)$$

From (1) and (2) it follows that:

$$\sum_{k=0}^{2n}(-1)^k\cdot\frac{4n+1}{4n-2k+1}\binom{2n}{k}=\frac{(4n+1)\cdot 2^{2n}\cdot(2n)!}{(4n+1)!!}=\frac{2^{2n}\cdot(2n)!}{(4n-1)!!}$$

So, we get

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=0}^{2n}(-1)^k\cdot\frac{4n+1}{4n-2k+1}\binom{2n}{k}}\overset{C-D}{=}\lim_{n\to\infty}\frac{2^{2n+2}}{2^{2n}}\cdot\frac{(2n+2)!}{(2n)!}\cdot\frac{(4n-1)!!}{(4n+3)!!}=$$

$$=4\lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(4n+1)(4n+3)}=1$$

Solution 2 by Ravi Prakash-New Delhi-India

$$I_n=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\cdot\frac{1}{4n-2k+1}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\int_0^1(x^2)^{(2n-k)}dx=$$

$$=\int_0^1\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(x^2)^{2n-k}\,dx=\int_0^1(1-x^2)^{2n}dx\overset{x=\sin\theta}{=}$$

$$=\int_0^{\frac{\pi}{2}}\cos^{4n+1}\theta\,d\theta=\frac{4n}{4n+1}\cdot\frac{4n-2}{4n-1}\cdot\ldots\cdot\frac{2}{3}$$

$$(4n+1)I_n=\frac{4n}{4n-1}\cdot\frac{4n-2}{4n-3}\cdot\ldots\cdot\frac{2}{1}$$

$$\lim_{n\to\infty}\sqrt[n]{(4n+1)I_n}=\lim_{n\to\infty}\frac{(4n+3)I_{n+1}}{(4n+1)I_n}=\lim_{n\to\infty}\frac{4n+4}{4n+3}\cdot\frac{4n+2}{4n+1}=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{4n+3}\right)\left(1+\frac{1}{4n+1}\right)=1$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\sum_{k=1}^{2n}(-1)^k\cdot\frac{4n+1}{4n-2k+1}\binom{2n}{k}=\sum_{k=0}^{2n}(-1)^k\frac{4n+1}{2k+1}\binom{2n}{k}=$$

$$=(4n+1)\sum_{k=0}^{2n}\int_0^1(-x^2)^k\binom{2n}{k}dx=(4n+1)\int_0^1(-x^2)^k\binom{2n}{k}dx=$$

$$=(4n+1)\int_0^1(1-x^2)^{2n}dx=(4n+1)\int_0^1(1-t)^{2n}\cdot t^{-\frac{1}{2}}\frac{dt}{2}=$$

$$=\left(2n+\frac{1}{2}\right)\beta\left(2n+1,\frac{1}{2}\right)=\frac{\Gamma(2n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2n+1+\frac{1}{2}\right)}=\frac{\Gamma(2n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2n+\frac{1}{2}\right)}$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=0}^{2n}(-1)^k\cdot\frac{4n+1}{4n-2k+1}\binom{2n}{k}}=\lim_{n\to\infty}\left(\frac{\Gamma(2n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2n+\frac{1}{2}\right)}\right)^{\frac{1}{n}}$$

$$\because\ \Gamma\left(2n+\frac{1}{2}\right)\sim\sqrt{2\pi}(2n)^{2n}e^{-2n}$$

$$\Gamma(2n+1)\sim\sqrt{2\pi}(2n+1)^{2n+\frac{1}{2}}e^{-2n}$$

$$\Omega=\lim_{n\to\infty}\pi^{\frac{1}{2n}}\left(\frac{2n+1}{2n}\right)^2(2n+1)^{\frac{1}{2n}}=\lim_{n\to\infty}e^{\frac{\log(1+2n)}{2n}}=1$$

UP.452 If $0<a\leq b$ then:

$$\int_a^b\int_a^b\frac{dxdy}{x+y}\leq\frac{37}{72}(b-a)\log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\int_a^b\int_a^b\frac{dxdy}{x+y}\leq\frac{37}{72}(b-a)\log\left(\frac{b}{a}\right)\Leftrightarrow$$

$$72\int_a^b\int_a^b\frac{dxdy}{x+y}\leq 37(b-a)\log\left(\frac{b}{a}\right)\Leftrightarrow$$

$$144\int_a^b\int_a^b\frac{dxdy}{x+y}\le 74(b-a)(\log b-\log a)\Leftrightarrow$$

$$144\int_a^b\int_a^b\frac{dxdy}{x+y}\le 25\int_a^b\int_a^b\frac{1}{y}dxdy+49\int_a^b\int_a^b\frac{1}{x}dxdy\Leftrightarrow$$

$$\int_a^b\int_a^b\frac{144}{x+y}dxdy\le\int_a^b\int_a^b\frac{25}{y}dxdy+\int_a^b\int_a^b\frac{49}{x}dxdy\Leftrightarrow$$

$$\frac{144}{x+y}\le\frac{25}{y}+\frac{49}{x}\Leftrightarrow 144xy\le 25x(x+y)+49y(x+y)\Leftrightarrow$$

$$25x^2+25xy+49xy+49y^2-144xy\ge 0\Leftrightarrow$$

$$25x^2-70xy+49y^2\ge 0\Leftrightarrow(5x-7y)^2\ge 0$$

Equality holds for $a=b$.

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\int_a^b\int_a^b\frac{dxdy}{x+y}=\frac{1}{2}\int_a^b\int_a^b\frac{2dxdy}{\frac{1}{x}+\frac{1}{y}}\overset{AH-AM}{\le}\frac{1}{2}\int_a^b\int_a^b\frac{\frac{1}{x}+\frac{1}{y}}{2}dxdy=$$

$$=\frac{1}{4}\int_a^b\int_a^b\left(\frac{1}{x}+\frac{1}{y}\right)dxdy=\frac{1}{4}\int_a^b\left(\log x+\frac{x}{y}\right)\Big|_a^b dy=$$

$$=\frac{1}{4}\int_a^b\left(\log\left(\frac{b}{a}\right)+\frac{b-a}{4}\right)dy=\frac{1}{4}(\log\left(\frac{b}{a}\right)y\Big|_a^b+(b-a)\log y|_a^b=$$

$$=\frac{1}{4}\left(\log\left(\frac{b}{a}\right)(b-a)+(b-a)\log\left(\frac{b}{a}\right)\right)=\frac{1}{2}(b-a)\log\left(\frac{b}{a}\right)$$

Therefore,

$$\int_a^b\int_a^b\frac{dxdy}{x+y}\le\frac{1}{2}(b-a)\log\left(\frac{b}{a}\right)\le\frac{37}{72}(b-a)\log\left(\frac{b}{a}\right)$$

Equality holds for $a=b$.

UP.453 If $m>0$ then find:

$$\Omega(m)=\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{m+1}{x+1}}-\left(\Gamma(x+1)\right)^{\frac{m+1}{x}}\right)\sin^m\left(\frac{\pi}{x}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{\sqrt[n]{\Gamma(n+1)}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{CDA}{=}$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

$$\text{Let be } u(x)=\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{m+1}=\left(\frac{(\Gamma(x+2))^{\frac{1}{x}}}{x+1}\cdot\frac{x}{(\Gamma(x+1))^{\frac{1}{x}}}\cdot\frac{x+1}{x}\right)^{m+1}$$

$$\lim_{x\to\infty}u(x)=\left(\frac{1}{e}\cdot e\cdot 1\right)^{m+1}=1$$

$$\lim_{x\to\infty}\frac{u(x)-1}{\log x}=1$$

$$\lim_{x\to\infty}(u(x))^x=\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{m+1}=\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+1))^{\frac{1}{x+1}}}\right)^{m+1}=e^{m+1}$$

$$\lim_{x\to\infty}\left(\frac{\sin\frac{\pi}{x}}{\frac{\pi}{x}}\right)^m=1^m=1$$

$$\Omega(m)=\lim_{x\to\infty}(\Gamma(x+1))^{\frac{m+1}{x}}(u(x)-1)\sin^m\left(\frac{\pi}{x}\right)=$$

$$=\lim_{x\to\infty}\left(\frac{\Gamma(x+1)}{x}\right)^{m+1}x^{m+1}\log u(x)\cdot\frac{u(x)-1}{\log u(x)}\cdot\sin^m\left(\frac{\pi}{x}\right)=$$

$$=\left(\frac{1}{e}\right)^{m+1}\cdot\lim_{x\to\infty}\frac{u(x)-1}{\log u(x)}\cdot\log(u(x))^x\cdot\frac{\sin^m\left(\frac{\pi}{x}\right)}{\frac{\pi}{x}}\cdot\pi^m=$$

$$=\left(\frac{1}{e}\right)^{m+1}\cdot 1\cdot\log e^{m+1}\cdot 1\cdot\pi^m=\left(\frac{1}{e}\right)^{m+1}\cdot(m+1)\cdot\pi^m=$$

$$=\frac{(m+1)\pi^m}{e^{m+1}}$$

Solution 2 by Angel Plaza-Gran Canaria-Spain

$$\because\ \Gamma(x)=\sqrt{2\pi}e^{-x}x^{x-\frac{1}{2}}\left(1+O\left(\frac{1}{x}\right)\right)$$

$$\Omega(m)=\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{m+1}{x+1}}-\left(\Gamma(x+1)\right)^{\frac{m+1}{x}}\right)\sin^m\left(\frac{\pi}{x}\right)=$$

$$=\lim_{x\to\infty}\left((2\pi)^{\frac{m+1}{2(x+1)}}e^{-(m+1)\frac{(x+2)}{x+!}}(x+2)^{(m+1)\left(\frac{x+\frac{3}{2}}{x+1}\right)}\right.$$

$$\left.-(2\pi)^{\frac{m+1}{2x}}e^{-(m+1)\frac{(x+1)}{x}}(x+1)^{(m+1)\left(\frac{x+\frac{1}{2}}{x+1}\right)}\right)\left(\frac{\pi}{x}\right)^m$$

$$=e^{-(m+1)}\lim_{x\to\infty}((x+2)^{m+1}-(x+1)^{m+1})\left(\frac{\pi}{x}\right)^m=$$

$$=e^{-(m+1)}\pi^m\lim_{x\to\infty}x\left(\left(\frac{x+2}{x}\right)^{m+1}-\left(\frac{x+1}{x}\right)^{m+1}\right)=$$

$$=e^{-(m+1)}\pi^m\lim_{x\to\infty}x\left(\left(1+\frac{2}{x}\right)^{m+1}-\left(1+\frac{1}{x}\right)^{m+1}\right)=$$

$$=e^{-(m+1)}\pi^m\lim_{x\to\infty}\left(1+\frac{2(m+1)}{x}-\left(1+\frac{m+1}{x}\right)\right)=$$

$$=(m+1)e^{-(m+1)}\pi^m=\frac{m+1}{e^{m+1}}\pi^m$$

UP.454 Let $A,B\in M_4(\mathbb{R})$ such that $AB+BA=O_4$, then prove:

$$\det(A^4+A^2+B^2)\geq 0$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\det\left[\left(A^2+i(A+B)\right)\left(A^2-i(A+B)\right)\right]\geq 0;(1)$$

$$\det\left[\left(A^2+i(A+B)\right)\left(A^2-i(A+B)\right)\right]=$$

$$=\det(A^4-iA^3-iA^2B+iA^3+iBA^2+(AB)^2);(2)$$

$$(A+B)^2=(A+B)(A+B)=A^2+AB+BA+B^2=A^2+B^2;(3)$$

$$A\cdot\,|\,AB=-BA\Rightarrow A^2B=-ABA\Rightarrow -i(A^2B-BA^2)=-i(-ABA-BA^2)=$$

$$= i(AB + BA) \cdot A = O_4; (4)$$

From (2),(3) and (4) we get:

$$\det\left[\left(A^2 + i(A+B)\right)\left(A^2 - i(A+B)\right)\right] = \det(A^4 + A^2 + B^2); (5)$$

From (1) and (5) it follows that

$$\det(A^4 + A^2 + B^2) \geq 0$$

Solution 2 by Florentin Vișescu-Romania

$$(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + b^2$$

$$A^4 + A^2 + B^2 = A^4 + (A+B)^2$$

$$\left(A^2 - i(A+B)\right)\left(A^2 + i(A+B)\right) = A^4 + iA^2(A+B) - i(A+B)A^2 + (A+B)^2 =$$

$$= A^4 + iA^3 + iA^2B - iA^3 - iBA^2 + (A+B)^2 =$$

$$= A^4 + i(A^2B - BA^2) + (A+B)^2 = A^4 + i(A(-BA) - BA^2) + (A+B)^2 =$$

$$= A^4 + i(-ABA - BA^2) + (A+B)^2 = A^4 + i(BA^2 - BA^2) + (A+B)^2 = A^4 + (A+B)^2$$

Hence, we get:

$$\det(A^4 + A^2 + B^2) = \det(A^4 + (A+B)^2) =$$

$$= \det\left(A^2 - i(A+B)\right)\left(A^2 + i(A+B)\right) =$$

$$= \overline{\det(A^2 + \imath(A+B))} \cdot \det\left(A^2 + i(A+B)\right) =$$

$$= \left|\det\left(A^2 + i(A+B)\right)\right|^2 \geq 0$$

Solution 3 by Ravi Prakash-New Delhi-India

We first show that $A^2B = BA^2$. We have:

$$A^2B = A(AB) = A(-BA) = -(AB)A = -(-BA)A = BA^2$$

Also, we have:

$$A^2 + B^2 = A^2 + B^2 + AB + BA = A(A+B) + B(A+B) = (A+B)^2$$

Next, note that

$$A^2(A+B) = A^3 + A^2B = A^3 + BA^2 = (A+B)A^2$$

Let $X = A^2, Y = A + B$ then $XY = YX$.

We have:

$$\det(X^2+Y^2)=\det\big((X+iY)(X-iY)\big)=\det(X+iY)\det(X-iY)=$$

$$=\det(X+iY)\det\overline{(X+\imath Y)}=|\det(X+iY)|^2$$

UP.455 Let ω be a root of the equation $x^4+(x-1)^4+1=0$.

Find: $\Omega=\omega^{300}+\omega^{303}$.

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$x^4+(x-1)^4+1=x^4+x^4-4x^3+6x^2-4x+1+1=$$

$$=2x^4-4x^3+6x^2-4x+2=2(x^4-2x^3+3x^2-2x+1)=$$

$$=2(x^4-2x^3+x^2+2x^2-2x+1)=2(x^2-x+1)^2$$

$$2(x^2-x+1)^2=0\Leftrightarrow x^2-x+1=0$$

$$\Rightarrow\omega^2-\omega+1=0\Rightarrow(\omega+1)(\omega^2-\omega+1)=0$$

$$\omega^3+1=0\Rightarrow\omega^3=-1$$

$$\Omega=\omega^{300}+\omega^{303}=(\omega^3)^{100}+(\omega^3)^{101}=(-1)^{100}+(-1)^{101}=1-1=0.$$

Solution 2 by Ashwni Kumar-India

$$x^4+(x-1)^4+1=0\Leftrightarrow x^4+(x^2-2x+1)^2+1=0$$

$$x^4-2x^3+3x^2-2x+1=0$$

$$(x^4-x^3+x^2)-(x^3-x^2+x)+(x^2-x+1)=0$$

$$x^2(x^2-x+1)-x(x^2-x+1)+(x^2-x+1)=0$$

$$(x^2-x+1)^2=0\Leftrightarrow x^2-x+1=0$$

$$x_{1,2}=x_{3,4}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

Let $\omega=\frac{1}{2}+\frac{\sqrt{3}}{2}i$ then $\omega^3+1=0\Rightarrow\omega^3=-1$

$$\Omega=\omega^{300}+\omega^{303}=(\omega^3)^{100}+(\omega^3)^{101}=(-1)^{100}+(-1)^{101}=1-1=0.$$

Solution 3 by Angel Plaza-Gran Canaria-Spain

Since $x^4+(x-1)^4+1=2(x^2-x+1)^2$, the roots of the given equation are

$$\omega_1=\frac{1}{2}+\frac{\sqrt{3}}{2}i=e^{\frac{\pi}{3}i}\text{ and }\omega_2=\frac{1}{2}-\frac{\sqrt{3}}{2}i=e^{\frac{-\pi}{3}i}$$

Both hold that $\omega_k^{300}=1$ and $\omega_k^{303}=-1$, for $k=1,2$. So,

$$\Omega = \omega^{300} + \omega^{303} = 0.$$

UP.456 In ΔABC the following relationship holds:

$$(m_b^2 + m_c^2)\sin A + (m_c^2 + m_a^2)\sin B + (m_a^2 + m_b^2)\sin C \geq 54\sqrt{3}\cdot\frac{r^3}{R}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let $a = BC, b = CA, c = AB$ be the lengths of the sides of ΔABC and $2s = a + b + c$ is the semiperimeter. We have:

$$m_a^2 = \frac{2(b^2+c^2)-a^2}{4} \geq \frac{(b+c)^2-a^2}{4} = \frac{(b+c+a)(b+c-a)}{4} = \frac{2s(2s-2a)}{4}$$

$$s(s-a) = \frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)} = \frac{F^2}{(s-b)(s-c)} = \frac{F}{s-b}\cdot\frac{F}{s-c} = r_b r_c$$

We have:

$$\frac{1}{m_a^2} + \frac{1}{m_b^2} \leq \frac{1}{r_b r_c} + \frac{1}{r_c r_a} = \frac{1}{r_c}\left(\frac{1}{r_a} + \frac{1}{r_b}\right) = \frac{1}{r_c}\left(\frac{s-a}{F} + \frac{s-b}{F}\right) =$$

$$= \frac{1}{r_c}\cdot\frac{2s-a-b}{F} = \frac{c}{r_c F}$$

So, $\frac{1}{m_a^2} + \frac{1}{m_b^2} \leq \frac{c}{r_c F}$. Also, we have:

$$(m_a^2 + m_b^2)\left(\frac{1}{m_a^2} + \frac{1}{m_b^2}\right) \geq 4 \Leftrightarrow m_a^2 + m_b^2 \geq \frac{4}{\frac{1}{m_a^2} + \frac{1}{m_b^2}}$$

So, $c(m_a^2 + m_b^2) \geq 4r_c F$. Similarly, we have: $a(m_b^2 + m_c^2) \geq 4r_a F$ and $b(m_c^2 + m_a^2) \geq 4r_b F$.

By adding these inequalities, we have:

$$a(m_b^2 + m_c^2) + b(m_c^2 + m_a^2) + c(m_a^2 + m_b^2) \geq 4F(r_a + r_b + r_c)$$

We know that $r_a + r_b + r_c = 4R + r, R \geq 2r (Euler)$ and $F = rs, s \geq 3\sqrt{3}R (Mitrinovic)$.

So, using the Law of sines, we get:

$$\left(m_b^2+m_c^2\right)\sin A+\left(m_c^2+m_a^2\right)\sin B+\left(m_a^2+m_b^2\right)\sin C\ge\frac{4rs(4R+r)}{2R}\ge$$

$$\ge\frac{4r\left(3\sqrt{3}r\right)(4\cdot 2r+r)}{2R}=54\sqrt{3}\cdot\frac{r^3}{R}$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Marian Ursărescu-Romania

$$m_b^2+m_c^2\ge 2m_bm_c$$

We must show:

$$\sum_{cyc}m_bm_c\sin A\ge 27\sqrt{3}\frac{r^3}{R};(1)$$

$$\sum_{cyc}m_bm_c\sin A\ge 3\sqrt[3]{(m_am_bm_c)^2\cdot\sin A\sin B\sin C};(2)$$

From (1) and (2) we must show:

$$\sqrt[3]{(m_am_bm_c)^2\cdot\sin A\sin B\sin C}\ge 9\sqrt{3}\frac{r^3}{R};(3)$$

$$m_a\ge\sqrt{s(s-a)}\Rightarrow m_am_bm_c\ge sF=s^2r;(4)$$

$$\sin A\sin B\sin C=\frac{sr}{2R^2};(5)$$

From (3), (4) and (5) we must show:

$$s^4r^2\cdot\frac{sr}{2R^2}\ge 9^3\cdot 3\sqrt{3}\cdot\frac{r^9}{R^3}\Leftrightarrow Rs^5\ge 2\cdot 3^7\sqrt{3}\cdot r^6$$

$R\ge 2r\ (Euler)$. **We must show:**

$$s^5\ge 3^7\sqrt{3}r^5;(6)$$

But $s\ge 3\sqrt{3}r(Mitrinovic)\Rightarrow s^5\ge\left(3\sqrt{3}\right)^5r^5=3^7\sqrt{3}r^5\Rightarrow(6)$ **its true.**

Solution 3 by Tapas Das-India

$$m_b^2+m_c^2=\frac{1}{4}(2a^2+2c^2-b^2+2a^2+2b^2-c^2)=\frac{1}{4}(4a^2+b^2+c^2)$$

$$\left(m_b^2+m_c^2\right)\sin A=\frac{1}{4}(4a^2+b^2+c^2)\frac{a}{2R}=\frac{a}{8R}(4a^2+b^2+c^2)$$

$$\left(m_c^2+m_a^2\right)\sin B=\frac{c}{8R}(4b^2+a^2+c^2)$$

$$\left(m_a^2+m_b^2\right)\sin C=\frac{c}{8R}\left(4c^2+a^2+b^2\right)$$

$$\sum_{cyc}\left(m_b^2+m_c^2\right)\sin A=\frac{1}{8R}\sum_{cyc}\left(4a^3+ac^2+ab^2\right)\geq$$

$$\geq\frac{1}{8R}\cdot\left(3\sqrt[3]{4a^3\cdot 4b^3\cdot 4c^3}+6\sqrt[6]{a^6b^6c^6}\right)=\frac{1}{8R}(12abc+6abc)=\frac{9}{4R}abc=$$

$$=\frac{9}{4R}\cdot 4RF\geq 54\sqrt{3}\frac{r^3}{R}$$

Solution 4 by Ertan Yildirim-Izmir-Turkiye

$$m_a\geq\sqrt{s(s-a)};(1)$$

$$a^2+b^2+c^2\geq 4\sqrt{3}F;(2)$$

$$\sum_{cyc}\left(m_b^2+m_c^2\right)\sin A\geq\sum_{cyc}[s(s-b)+s(s-c)]\sin A=$$

$$=\sum_{cyc}s(2s-b-c)\cdot\frac{a}{2R}=\sum_{cyc}sa\cdot\frac{a}{2R}=\frac{s}{2R}\sum_{cyc}a^2$$

$$\frac{s}{2R}\sum_{cyc}a^2\geq\frac{s}{2R}\cdot 4\sqrt{3}F=\frac{s}{2R}\cdot 4\sqrt{3}\cdot sr=\frac{r}{R}2\sqrt{3}s^2$$

$$\frac{r}{R}2\sqrt{3}\cdot s^2\overset{Mitrinovic}{\geq}\frac{2\sqrt{3}r}{R}\left(3\sqrt{3}r\right)^2=54\sqrt{3}\frac{r^3}{R}$$

UP.457 If $a,b,c,d>0$ such that $ab+bc+cd+da=1$ and $\lambda\geq 0$ then:

$$a\sqrt{b^2+\lambda}+b\sqrt{c^2+\lambda}+c\sqrt{d^2+\lambda}+d\sqrt{a^2+\lambda}\geq\sqrt{1+4\lambda}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

$$a\sqrt{b^2+\lambda}+b\sqrt{c^2+\lambda}+c\sqrt{d^2+\lambda}+d\sqrt{a^2+\lambda}=$$

$$=\sum_{cyc}a\sqrt{b^2+\lambda}=\sum_{cyc}a\left|b+\sqrt{\lambda}i\right|=\sum_{cyc}\left|ab+a\sqrt{\lambda}i\right|\geq$$

$$\geq\left|ab+bc+ca+ad+(a+b+c+d)\sqrt{\lambda}i\right|=$$

$$=\sqrt{(ab+bc+cd+da)^2+\lambda(a+b+c+d)^2}\overset{(1)}{\geq}$$

$$\geq\sqrt{(ab+bc+cd+da)^2+4\lambda(ab+bc+cd+da)}=\sqrt{1+4\lambda}$$

$$(1) \Leftrightarrow (a+b+c+d)^2 \geq 4(ab+bc+cd+da) \Leftrightarrow (a-b+c-d)^2 \geq 0$$

Equality holds for $a+c=b+d, a=b=c=d=\frac{1}{2}$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$a\sqrt{b^2+\lambda}+b\sqrt{c^2+\lambda}+c\sqrt{d^2+\lambda}+d\sqrt{a^2+\lambda}=$$

$$=\sqrt{(ab)^2+\lambda a^2}+\sqrt{(bc)^2+\lambda b^2}+\sqrt{(ca)^2+\lambda c^2}+\sqrt{(da)^2+\lambda d^2} \geq \sqrt{1+4\lambda}$$

$$\sqrt{(ab+bc+cd+da)^2+\left(\sqrt{\lambda}(a+b+c+d)\right)^2} \geq \sqrt{1+4\lambda}$$

$(ab+bc+cd+da)^2+\left(\sqrt{\lambda}(a+b+c+d)\right)^2 = 1+4\lambda$ true, because

$ab+bc+cd+da=1$ and

$$\left(\sqrt{\lambda}(a+b+c+d)\right)^2 = \lambda(a+b+c=d)^2 \geq 4\lambda(ab+bc+ca+ab)$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$(ab+bc+cd+da)^2 = \sum_{cyc} a^2b^2 + 2(ab.bc+ab.cd+ab.da+bc.cd+bc.da+cd.da)$$

$$= \sum_{cyc} a^2b^2 + 4abcd + 2\left((ab^2c+cd^2a)+(a^2bd+bc^2d)\right)$$

$$\Rightarrow (ab+bc+cd+da)^2 \stackrel{(*)}{=} \sum_{cyc} a^2b^2 + 4abcd + 2ac(b^2+d^2) + 2bd(a^2+c^2)$$

$$\text{Via CBS}, \left(b^2+(\sqrt{\lambda})^2\right)\left(c^2+(\sqrt{\lambda})^2\right)$$

$$\geq \left(bc+\sqrt{\lambda}.\sqrt{\lambda}\right)^2 \stackrel{\because b^2+\lambda, c^2+\lambda, bc+\lambda>0 \text{ as } \lambda\geq 0 \text{ and } b,c>0}{\Rightarrow} \sqrt{b^2+\lambda}.\sqrt{c^2+\lambda}$$

$$\geq bc+\lambda \text{ and analogs} \to (1)$$

$$\text{Also}, \left(\sum_{cyc} a\right)^2 \geq 4\sum_{cyc} ab \Leftrightarrow \sum_{cyc} a^2 + 2\sum_{cyc} ab + 2ac + 2bd \geq 4\sum_{cyc} ab$$

$$\Leftrightarrow (a^2+c^2+2ac)+(b^2+d^2+2bd) \geq 2\sum_{cyc} ab \Leftrightarrow (a+c)^2+(b+d)^2$$

$$\geq 2(ab+bc+cd+da)$$

$$\Leftrightarrow (a+c)^2+(b+d)^2 \geq 2(a+c)(b+d) \to \text{true via A} - \text{G} \Rightarrow \left(\sum_{cyc} a\right)^2$$

$$\geq 4\sum_{cyc} ab \stackrel{ab+bc+cd+da=1}{=} 4 \Rightarrow \left(\sum_{cyc} a\right)^2 \stackrel{(**)}{\geq} 4$$

$$\text{Now,}\left(a\sqrt{b^2+\lambda}+b\sqrt{c^2+\lambda}+c\sqrt{d^2+\lambda}+d\sqrt{a^2+\lambda}\right)^2$$

$$=a^2(b^2+\lambda)+b^2(c^2+\lambda)+c^2(d^2+\lambda)+d^2(a^2+\lambda)+2ab.\sqrt{b^2+\lambda}.\sqrt{c^2+\lambda}$$

$$+2ac.\sqrt{b^2+\lambda}.\sqrt{d^2+\lambda}$$

$$+2ad.\sqrt{b^2+\lambda}.\sqrt{a^2+\lambda}+2bc.\sqrt{c^2+\lambda}.\sqrt{d^2+\lambda}+2bd.\sqrt{c^2+\lambda}.\sqrt{a^2+\lambda}$$

$$+2cd.\sqrt{d^2+\lambda}.\sqrt{a^2+\lambda}\overset{\text{via }(1)}{\geq}\sum_{cyc}a^2b^2+\lambda\sum_{cyc}a^2+2ab(bc+\lambda)$$

$$+2ac(bd+\lambda)+2ad(ab+\lambda)$$

$$+2bc(cd+\lambda)+2bd(ac+\lambda)+2cd(ad+\lambda)$$

$$=\left(\sum_{cyc}a^2b^2+4abcd+2ac(b^2+d^2)+2bd(a^2+c^2)\right)$$

$$+\left(\lambda\sum_{cyc}a^2+2\lambda\sum_{sym}ab\right)\overset{\text{via }(*)}{=}(ab+bc+cd+da)^2+\lambda\left(\sum_{cyc}a\right)^2$$

$$\overset{ab+bc+cd+da=1}{=}1+\lambda\left(\sum_{cyc}a\right)^2\overset{\text{via }(**)\text{ and }\because\lambda\geq 0}{\geq}1+4\lambda$$

$$\Rightarrow a\sqrt{b^2+\lambda}+b\sqrt{c^2+\lambda}+c\sqrt{d^2+\lambda}+d\sqrt{a^2+\lambda}\geq\sqrt{1+4\lambda}\ \forall\, a,b,c,d$$
$$>0\,\big|\,ab+bc+cd+da=1\text{ and }\lambda\geq 0\text{ (QED)}$$

Solution 4 by Daoudi Abdessattar-Tunisia

$$ab+bc+cd+da=(a+c)(b+d)=1$$

$$\sum_{cyc}\sqrt{(ab)^2+a^2\lambda}\overset{Minkowski}{\geq}\sqrt{(a+c)^2(b+d)^2+(a+b+c+d)^2}\overset{AGM}{\geq}$$

$$\geq\sqrt{1+4(a+c)(b+d)\lambda}=\sqrt{1+4\lambda}$$

Solution 5 by Tapas Das-India

$$\sum_{i=1}^{n}\sqrt{a_i^2+b_i^2}\geq\sqrt{\left(\sum_{i=1}^{n}a_i\right)^2+\left(\sum_{i=1}^{n}b_i\right)^2};\forall a_i,b_i>0,i\in\overline{1,n};(Minkowski)$$

$$ab+bc+cd+da=1\Rightarrow(a+c)(b+d)=1$$

$$ab+bc+cd+da=(a+c)(b+d)=1$$

$$\sum_{cyc}\sqrt{(ab)^2+a^2\lambda}\overset{Minkowki}{\geq}\sqrt{(a+c)^2(b+d)^2+(a+b+c+d)^2}\overset{AGM}{\geq}$$

$$\geq\sqrt{1+4(a+c)(b+d)\lambda}=\sqrt{1+4\lambda}$$

UP.458 Find:

$$\Omega = \int_0^\infty \frac{x\log x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by proposer

$$\text{Let } A = \int_0^1 \frac{x\log x}{(x+1)(x^2+1)}dx \text{ and } B = \int_1^\infty \frac{x\log x}{(x+1)(x^2+1)}dx$$

We have, successively:

$$A = \int_0^1 \frac{(1-x)x\log x}{1-x^4}dx = \int_0^1 \frac{x\log x}{1-x^4}dx - \int_0^1 \frac{x^2\log x}{1-x^4}dx =$$

$$= \int_0^1 \sum_{n=0}^\infty x^{4n+1}\log x\,dx - \int_0^1 \sum_{n=0}^\infty x^{4n+2}\log x\,dx =$$

$$= \sum_{n=0}^\infty \left(\int_0^1 x^{4n+1}\log x\,dx - \int_0^1 x^{4n+1}\log x\,dx\right)$$

We will to use the following relationship:

$$\int_0^1 x^a \log x\,dx = -\frac{1}{(a+1)^2}, \text{where } a \in \mathbb{R}, a \geq 0$$

$$A = \sum_{n=0}^\infty \left[\frac{1}{(4n+3)^2} - \frac{1}{(4n+2)^2}\right] = \sum_{n=0}^\infty \left[\frac{\frac{1}{16}}{\left(n+\frac{3}{4}\right)^2} - \frac{\frac{1}{16}}{\left(n+\frac{2}{4}\right)^2}\right]$$

We use the following relationship:

$$\psi_1(x) = \sum_{n=0}^\infty \frac{1}{(x+n)^2}, \text{where } \psi_1(x) - \text{is trigamma function.}$$

We obtained the value of A:

$$A = \frac{1}{16}\left[\psi_1\left(\frac{3}{4}\right) - \psi_1\left(\frac{1}{2}\right)\right]$$

In the integral B we make the variable change: $x = \frac{1}{y}$. We obtain:

$$B = -\int_0^1 \frac{\log y}{(y+1)(y^2+1)}dy$$

By proceeding similarly to the integral A, we obtain:

$$B=\frac{1}{16}\left[\psi_1\left(\frac{1}{4}\right)-\psi_1\left(\frac{1}{2}\right)\right]$$

Result:

$$\Omega=A+B=\frac{1}{16}\left[\psi_1\left(\frac{1}{4}\right)+\psi_1\left(\frac{3}{4}\right)-2\psi_1\left(\frac{1}{2}\right)\right]$$

We use the reflection formula:

$$\psi_1(x)+\psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$$

We obtain: $\psi_1\left(\frac{1}{4}\right)+\psi_1\left(\frac{3}{4}\right)=2\pi^2$

The following special value is known: $\psi_1\left(\frac{1}{2}\right)=\frac{\pi^2}{2}\Rightarrow\psi_1\left(\frac{1}{2}\right)=\frac{\pi^2}{2}$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^\infty\frac{x\log x}{(1+x)(1+x^2)}dx=\int_0^1\frac{x\log x}{(1+x)(1+x^2)}dx+\overbrace{\int_1^\infty\frac{x\log x}{(1+x)(1+x^2)}dx}^{x\to\frac{1}{x}}=$$

$$=\int_0^1\frac{(x-1)\log x}{(1+x)(1+x^2)}dx=\int_0^1\left(\frac{x}{1+x^2}-\frac{1}{1+x}\right)\log x\,dx\overset{IBP}{=}$$

$$=\left[\left(\frac{1}{2}\log(1+x^2)-\log(1+x)\right)\log x\right]_0^1+\int_0^1\frac{\log(1+x)}{x}dx-\frac{1}{2}\overbrace{\int_0^1\frac{\log(1+x^2)}{x}dx}^{x^2\to x}=$$

$$=\frac{3}{4}\int_0^1\frac{\log(1+x)}{x}dx=-\frac{3}{4}Li_2(-1)=\frac{\pi^2}{16}$$

Therefore,

$$\Omega=\int_0^\infty\frac{x\log x}{(1+x)(1+x^2)}dx=\frac{\pi^2}{16}$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int_0^\infty\frac{x\log x}{(1+x)(1+x^2)}dx=\int_0^1\frac{x\log x}{(1+x)(1+x^2)}dx+\overbrace{\int_1^\infty\frac{x\log x}{(1+x)(1+x^2)}dx}^{x\to\frac{1}{x}}=$$

$$= \int_0^1 \frac{(x-1)\log x}{(1+x)(1+x^2)}dx = \int_0^1 \left(\frac{x}{1+x^2} - \frac{1}{1+x}\right)\log x\,dx =$$

$$= \int_0^1 \frac{x\log x}{1+x^2}dx - \int_0^1 \frac{\log x}{1+x}dx \overset{x^2=y}{=} \int_0^1 \frac{\log(\sqrt{y})}{1+(\sqrt{y})^2}\frac{dy}{2} - \int_0^1 \frac{\log x}{1+x}dx =$$

$$= \frac{1}{4}\int_0^1 \frac{\log y}{1+y}dy - \int_0^1 \frac{\log x}{1+x}dx = -\frac{3}{4}\int_0^1 \frac{\log x}{1+x}dx = -\frac{3}{4}\int_0^1 \log x\sum_{n=0}^{\infty}(-x)^n =$$

$$= -\frac{3}{4}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\log x\,dx = -\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(-1)(0!)}{(n+1)^2}$$

$$\because \int_0^1 x^m\log^n x\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, m \neq -1, n > -1$$

$$\Omega = \frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} = \frac{3}{4}\eta(2) = \frac{3}{4}(1-1^{1-2})\zeta(2)$$

$$\because \eta(2) = (1-2^{1-s})\zeta(s)$$

$$\Omega = \int_0^\infty \frac{x\log x}{(1+x)(1+x^2)}dx = \frac{\pi^2}{16}$$

Solution 4 by Fao Ler-Iraq

$$\Omega = \int_0^\infty \frac{x\log x}{(1+x)(1+x^2)}dx = \int_0^1 \frac{x\log x}{(1+x)(1+x^2)}dx + \overbrace{\int_1^\infty \frac{x\log x}{(1+x)(1+x^2)}dx}^{x\to\frac{1}{x}} =$$

$$= \int_0^1 \frac{x\log x}{(1+x)(1+x^2)}dx + \int_0^1 \frac{\frac{1}{x}\log\left(\frac{1}{x}\right)}{\left(\frac{1}{x}+1\right)\left(\frac{1}{x^2}+1\right)}d\left(\frac{1}{x}\right) =$$

$$= \int_0^1 \frac{x\log x}{(1+x)(1+x^2)}dx - \int_0^1 \frac{\log x}{(1+x)(1+x^2)}dx = \int_0^1 \frac{(x-1)\log x}{(1+x)(1+x^2)}dx =$$

$$= \Im\left((1+i)\int_0^1 \frac{\log x}{(x+1)(x+i)}dx\right) =$$

$$= \Im\left(\frac{1+i}{1-i}\int_0^1 \log x\left(\frac{1}{x+i} - \frac{1}{x+i}\right)dx\right) =$$

$$=\Im\left(i\left(Li_2\left(-\frac{1}{i}\right)-Li_2(-1)\right)\right)=\Im\left(iLi_2(i)+\frac{\pi^2}{12}i\right)=$$

$$=\Re\left(Li_2(i)\right)+\frac{\pi^2}{12}=-\frac{\pi^2}{48}+\frac{\pi^2}{12}=\frac{\pi^2}{16}$$

Solution 5 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^\infty\frac{x\log x}{(1+x)(1+x^2)}dx=\int_0^1\frac{x\log x}{(1+x)(1+x^2)}dx+\int_1^\infty\frac{x\log x}{(1+x)(1+x^2)}dx$$

$$=I_1+I_2$$

$$I_1=\int_0^1\frac{x\log x}{(1+x)(1+x^2)}dx=\frac{\pi^2}{32}-\frac{1}{2}G$$

$$I_2=\int_1^\infty\frac{x\log x}{(1+x)(1+x^2)}dx\overset{x=\frac{1}{y}}{=}\frac{1}{2}\underbrace{\int_0^1\frac{y\log y}{y^2+1}dy}_{I_{2a}}-\frac{1}{2}\underbrace{\int_0^1\frac{\log y}{1+y}dy}_{I_{2b}}-\frac{1}{2}\underbrace{\int_0^1\frac{\log y}{y^2+1}dy}_{I_{2c}}$$

$$I_{2a}=\int_0^1\frac{y\log y}{y^2+1}dy=-\frac{\pi^2}{48}$$

$$I_{2b}=\int_0^1\frac{\log y}{1+y}dy=-\frac{\pi^2}{12}$$

$$I_{2c}=\int_0^1\frac{\log y}{y^2+1}dy=-G$$

$$I_2=\frac{1}{2}(I_{2a}-I_{2b}-I_{2c})=\frac{\pi^2}{32}+\frac{1}{2}G\Rightarrow\Omega=I_1+I_2=\frac{\pi^2}{16}$$

UP.459 Calculate the integral:

$$\Omega=\int_0^\infty\frac{\log(x+1)}{x^3+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by proposer

$$\text{Let: } I(a)=\int_0^\infty\frac{\log(1+ax)}{x^3+1}dx, a\in(0,1)$$

We have, by derivation under integral sign:

$$I'(a)=\int_0^{\infty}\frac{x}{(x^3+1)(1+ax)}dx$$

We have:

$$\frac{x}{(x+1)(x^2-x+1)(1+ax)}=\frac{A}{x+1}+\frac{B}{1+ax}+\frac{Cx+D}{x^2-x+1}$$

We calculate the coefficients and we obtain:

$$A=\frac{1}{3(a-1)};B=-\frac{a^2}{a^3-1};C=\frac{1-a}{3(a^2+a+1)};D=\frac{1+2a}{3(a^2+a+1)}$$

We have:

$$\int\frac{A}{x+1}dx=A\log(x+1)+K,$$

where K is a arbitrary constant of integration.

$$\int\frac{B}{1+ax}dx=\frac{B}{a}\log(1+ax)+K$$

$$\int\frac{Cx+D}{x^2-x+1}dx=\frac{C}{2}\log(x^2-x+1)+\frac{2D+C}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+K$$

Let us denote:

$$P(x)=\int\frac{x}{(1+x)(1+ax)(x^2-x+1)}dx=$$

$$=\int\frac{A}{x+1}dx+\int\frac{B}{1+ax}dx+\int\frac{Cx+D}{x^2-x+1}dx$$

We have:

$$\int_0^{\infty}\frac{x}{(x^3+1)(1+ax)}dx=\lim_{x\to\infty}P(x)-P(0)$$

We can write:

$P(x)=Q(x)+R(x)$, where $Q(x)=\frac{2D+C}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$ and

$$R(x)=A\log(x+1)+\frac{B}{a}\log(1+ax)+\frac{C}{2}\log(x^2-x+1)$$

Let us denote:

$$\Delta_1=\lim_{x\to\infty}Q(x)-Q(0)$$

We have:

$$\lim_{x\to\infty} Q(x) = \frac{2D+C}{\sqrt{3}}\cdot\frac{\pi}{2}; Q(0) = -\frac{2D+C}{\sqrt{3}}\cdot\frac{\pi}{6}$$

So,

$$\Delta_1 = \frac{2D+C}{\sqrt{3}}\cdot\frac{\pi}{2}+\frac{2D+C}{\sqrt{3}}\cdot\frac{\pi}{6}; \Delta_1 = \frac{2D+C}{\sqrt{3}}\cdot\frac{2\pi}{3}$$

But we have: $2D+C=\frac{a+1}{a^2+a+1}$. Result:

$$\Delta_2 = \lim_{x\to\infty} R(x) - R(0)$$

We have:

$$R(x) = A\log(x+1)+\frac{B}{a}\log(1+ax)+\frac{C}{2}\log(x^2-x+1)$$

So, $R(0)=0$. We can write $R(x)$ in the following equivalent form:

$$R(x) = A\log\frac{x+1}{x}+\frac{B}{a}\log\frac{1+ax}{ax}+\frac{C}{2}\log\frac{x^2-x+1}{x^2}+\left(A+\frac{B}{a}+C\right)\log x+\frac{B}{a}\log a$$

But: $A+\frac{B}{a}+C=0$. Result: $\Delta_2 = -\frac{a\log a}{a^3-1}$.

We have:

$$\int_0^\infty \frac{x}{(x^3+1)(1+ax)}dx = \Delta_1+\Delta_2 = \frac{2\pi\sqrt{3}}{9}\cdot\frac{a+1}{a^2+a+1}-\frac{a\log a}{a^3-1}$$

We have:

$$\Omega = \lim_{\substack{a\to 1\\ a<1}} \mathrm{I}(a) = \int_0^1 I'(a)da\,, \text{because} \lim_{\substack{a\to 0\\ a>0}} I(a) = 0$$

$$\Omega = \frac{2\pi\sqrt{3}}{9}\int_0^1 \frac{a+1}{a^2+a+1}da - \int_0^1 \frac{a\log a}{a^3-1}da\,; (1)$$

The first integral in the above relation are integral of rational function and are calculated easily. She has the following values:

$$I_1 = \int_0^1 \frac{a+1}{a^2+a+1}da = \frac{1}{2}\log 3+\frac{1}{18}\pi\sqrt{3}; (2)$$

We calculate the second integral:

$$I_2 = \int_0^1 \frac{a\log a}{a^3-1}da$$

For this, we consider the function:

$$f(x)=\frac{x}{x^3-1}$$

We develop the function in power series. We have for $x\in(0,1)$:

$$f(x)=x-x^4-x^7-x^{10}-x^{13}-x^{16}-x^{19}-\cdots$$

$$I_2=\int_0^1 f(x)\log x\,dx=\frac{1}{4}+\frac{1}{25}+\frac{1}{64}+\frac{1}{121}+\frac{1}{196}+\frac{1}{289}+\frac{1}{400}+\cdots$$

$$=\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{8^2}+\frac{1}{11^2}+\frac{1}{17^2}+\frac{1}{20^2}+\cdots$$

Now, we will use the trigamma function, which is defined by the relationship:

$$\psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

We obtain:

$$I_2=\frac{1}{9}\psi_1\left(\frac{2}{3}\right)$$

But, we have the equality:

$$\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{2}{3}\right)=\frac{4\pi^2}{3}\ (the\ reflection\ formula)$$

So, we have:

$$I_2=\frac{4}{27}\pi^2-\frac{1}{9}\psi\left(\frac{1}{3}\right);(3)$$

Replacing the relationships (2),(3) in the relation (1), we have:

$$\Omega=\frac{1}{9}\left[-\pi^2+\pi\sqrt{3}\log 3+\psi_1\left(\frac{1}{3}\right)\right]$$

Solution 2 by Togrul Ehmedov-Azerbaijan

Let us denote:

$$I=\int_0^{\infty}\frac{\log(1+x)}{x^3+1}dx$$

We have:

$$I=\int_0^{\infty}\frac{\log(1+x)}{x^3+1}dx=\int_0^{\infty}\int_0^1\frac{x}{(x^3+1)(1+xy)}dydx=$$

$$= \int_0^1 \int_0^\infty \frac{x}{(x^3+1)(1+xy)} dxdy = \int_0^1 \frac{y \log y}{1-y^3} dy - \frac{2\pi}{3\sqrt{3}} \int_0^1 \frac{y^2-1}{1-y^3} dy =$$

$$= I_1 - \frac{2\pi}{3\sqrt{3}} I_2$$

We calculate I_1:

$$I_1 = \int_0^1 \frac{y \log y}{1-y^3} dy =$$

$$= \frac{1}{3}\left[\int_0^1 \frac{y \log y}{y^2+y+1} dy + \int_0^1 \frac{\log y}{1-y} dy - \int_0^1 \frac{\log y}{y^2+y+1} dy\right] = \frac{1}{3}(I_{1a} + I_{1b} - I_{1c})$$

$$I_{1a} = \int_0^1 \frac{y \log y}{y^2+y+1} dy = -\frac{1}{9}\left\{\frac{7\pi^2}{6} - \varphi'\left(\frac{1}{3}\right)\right\}$$

$$I_{1b} = \int_0^1 \frac{\log y}{1-y} dy = -\frac{\pi^2}{6}$$

$$I_{1c} = \int_0^1 \frac{\log y}{y^2+y+1} dy = \frac{2}{9}\left\{\frac{2\pi^2}{3} - \varphi'\left(\frac{1}{3}\right)\right\}$$

$$I_1 = \frac{1}{3}(I_{1a} + I_{1b} - I_{1c}) = \frac{1}{3}\left[-\frac{4\pi^2}{9} + \frac{1}{3}\varphi'\left(\frac{1}{3}\right)\right]$$

We calculate I_2:

$$I_2 = \int_0^1 \frac{y^2-1}{1-y^3} dy = -\frac{\log 3}{2} - \frac{\pi}{6\sqrt{3}}$$

We obtain:

$$I = I_1 - \frac{2\pi}{3\sqrt{3}} I_2 = \frac{1}{9}\left\{\varphi'\left(\frac{1}{3}\right) + \pi\sqrt{3}\log 3 - \pi^2\right\}$$

Solution 3 by Daniel Immarube-Nigeria

$$\Omega = \int_0^\infty \frac{\log(1+x)}{x^3+1} dx$$

We consider for $0 < a < 1$:

$$\Omega(\mathrm{a}) = \int_0^\infty \frac{\log(a+x)}{x^3+1} dx \Rightarrow \frac{d\Omega}{da} = \int_0^\infty \frac{1}{(a+x)(x^3+1)} dx$$

We have:

$$\frac{1}{(a+x)(x^3+1)}=$$

$$=-\frac{1}{a^3-1}\cdot\frac{1}{a+x}+\frac{1}{a^3-1}\cdot\frac{x^2}{x^3+1}-\frac{a}{a^3-1}\cdot\frac{x}{x^3+1}+\frac{a^2}{a^3-1}\cdot\frac{1}{x^3+1}$$

$$\frac{d\Omega}{da}=-\frac{1}{a^3-1}\int_0^\infty\left(\frac{1}{a+x}-\frac{x^2}{x^3+1}\right)dx-\frac{a}{a^3-1}\int_0^\infty\frac{x}{x^3+1}dx+\frac{a^2}{a^3-1}\int_0^\infty\frac{1}{x^3+1}dx$$

We obtain:

$$\frac{d\Omega}{da}=-\frac{1}{a^3-1}(-\log a)-\frac{a}{a^3-1}\cdot\frac{2\pi}{3\sqrt{3}}+\frac{a^2}{a^3-1}\cdot\frac{2\pi}{3\sqrt{3}}$$

$$\frac{d\Omega}{da}=-\frac{\log a}{1-a^3}+\frac{2\pi}{3\sqrt{3}}\cdot\frac{a-a^2}{1-a^3}$$

$$\Omega(1)-\Omega(0_+)=-\int_0^1\frac{\log a}{1-a^3}da+\frac{2\pi}{3\sqrt{3}}\int_0^1\frac{a-a^2}{1-a^3}da$$

Therefore,

$$\Omega(1)-\Omega(0_+)=-\left(-\frac{1}{9}\psi_1\left(\frac{1}{3}\right)\right)+\frac{2\pi}{3\sqrt{3}}\cdot\frac{1}{18}\left(-\pi\sqrt{3}+9\log 3\right)$$

$$\Omega(1)-\Omega(0_+)=\frac{1}{9}\psi_1\left(\frac{1}{3}\right)-\frac{\pi^2}{27}+\frac{\pi\log 3}{3\sqrt{3}}$$

But, we have:

$$\Omega(0_+)=-\frac{2\pi^2}{27}$$

$$\Omega(1)=\frac{1}{9}\psi_1\left(\frac{1}{3}\right)-\frac{\pi^2}{9}+\frac{\pi\log 3}{3\sqrt{3}}$$

So, it follows that: $\Omega=\Omega(1)=\frac{1}{9}\left[-\pi^2+\pi\sqrt{3}\log 3+\psi_1\left(\frac{1}{3}\right)\right]$

UP.460 In $\Delta ABC, I$ –incenter, O –circumcenter, the following relationship holds:

$$\sum_{cyc}\left(\cos\widehat{IOA}-\sin\frac{A}{2}\right)^2\geq\frac{3r^2}{R^2}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$\cos\widehat{IOA}=\frac{AI^2+OA^2-OI^2}{2AI\cdot OA}=\frac{\frac{r^2}{\sin^2\frac{A}{2}}+R^2-R^2+2Rr}{2R\cdot\frac{r}{\sin\frac{A}{2}}}=$$

$$=\frac{\frac{r}{\sin^2\frac{A}{2}}+2R}{\frac{2R}{\sin\frac{A}{2}}}=\frac{r}{2R\sin\frac{A}{2}}+\sin\frac{A}{2}$$

$$\cos\widehat{IOA}-\sin\frac{A}{2}=\frac{r}{2R\sin\frac{A}{2}}$$

$$\sum_{cyc}\left(\cos\widehat{IOA}-\sin\frac{A}{2}\right)^2=\frac{r^2}{4R^2}\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}}=\frac{r^2}{4R^2}\cdot\frac{s^2+r^2-8Rr}{r^2}=$$

$$=\frac{s^2+r^2-8Rr}{4R^2}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2-8Rr+r^2}{4R^2}=\frac{8Rr-4r^2}{4R^2}\overset{Euler}{\geq}$$

$$\geq\frac{8\cdot 2r\cdot r-4r^2}{4R^2}=\frac{12r^2}{4R^2}=\frac{3r^2}{R^2}$$

Equality holds for $a=b=c$.

UP.461 Find:

$$\Omega_n=\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$\frac{k^3+k^2-3k-2}{(k+2)!}=\frac{k^3+2k^2-k^2-2k-k-2}{(k+2)!}=\frac{(k^2-k-1)(k+2)}{(k+2)!}=\frac{k^2-k-1}{(k+1)!}$$

$$\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}=\sum_{k=1}^{n}\frac{k^2-k-1}{(k+1)!}=\sum_{k=1}^{n}\frac{(k^2-1)-k}{(k+1)!}=$$

$$=\sum_{k=1}^{n}\left(\frac{k-1}{k!}-\frac{k}{(k+1)!}\right)=-\frac{n}{n+1}$$

Hence,

$$\Omega_n=\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}\right)=-\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{n}{(n+1)!}=$$

$$=2-\left(1+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\right)=2-e$$

Solution 2 by Adrian Popa-Romania

$$\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}=\sum_{k=1}^{n}\frac{(k^2-k-1)(k+2)}{(k+2)!}=\sum_{k=1}^{n}\frac{k^2-k-1}{(k+1)!}=$$

$$=\sum_{k=1}^{n}\frac{k(k+1)-2(k+1)+1}{(k+1)!}=\sum_{k=1}^{n}\left(\frac{1}{(k-1)!}-\frac{2}{k!}+\frac{1}{(k+1)!}\right)=$$

$$=\frac{1}{(n+1)!}-\frac{2}{n!}+\frac{1}{n!}=\frac{1}{(n+1)!}-\frac{1}{n!}=-\frac{n}{(n+1)!}$$

Hence,

$$\Omega_n=\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}\right)=-\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{n}{(n+1)!}=$$

$$=2-\left(1+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\right)=2-e$$

Solution 3 by Precious Itsuokor-Nigeria

$$\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}=\sum_{k=1}^{n}\frac{(k^2-k-1)(k+2)}{(k+2)!}=\sum_{k=1}^{n}\frac{k^2-k-1}{(k+1)!}=$$

$$=\sum_{k=1}^{n}\frac{k(k+1)-2(k+1)+1}{(k+1)!}=\sum_{k=1}^{n}\left(\frac{1}{(k-1)!}-\frac{2}{k!}+\frac{1}{(k+1)!}\right)=$$

$$=\frac{1}{(n+1)!}-\frac{2}{n!}+\frac{1}{n!}=\frac{1}{(n+1)!}-\frac{1}{n!}=-\frac{n}{(n+1)!}$$

Hence,

$$\Omega_n=\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}\right)=-\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{n}{(n+1)!}=$$

$$=2-\left(1+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\right)=2-e$$

Solution 4 by Angel Plaza-Gran Canaria-Spain

By induction, it may be proved that:

$$\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}=-\frac{n}{(n+1)!}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{n}{(n+1)!}\right)=-\sum_{n=1}^{\infty}\frac{1}{(n+1)!}=2-e.$$

UP.462 Prove that:

$$\prod_{p=1}^{n}\left(1+\left(\sum_{k=1}^{n}\frac{2k}{k^4+(2p-1)k^2+p^2}\right)^2\right)<4$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$k^4+(2p-1)k^2+p^2=(k^2+p)^2-k^2=(k^2+p+k)(k^2+p-k)=$$
$$=(k^2-k+p)((k+1)^2-(k+1)+p)$$

$$\sum_{k=1}^{n}\frac{2k}{k^4+(2p-1)k^2+p^2}=\sum_{k=1}^{n}\frac{2k}{(k^2-k+p)(k^2+k+p)}=$$

$$=\sum_{k=1}^{n}\frac{(k^2+k+p)-(k^2-k+p)}{(k^2-k+p)(k^2+k+p)}=\sum_{k=1}^{n}\left(\frac{1}{k^2-k+p}-\frac{1}{k^2+k+p}\right)=$$

$$=\sum_{k=1}^{n}\left(\frac{1}{k^2-k+p}-\frac{1}{(k+1)^2+(k+1)+p}\right)=\frac{1}{p}-\frac{1}{n^2+n+p}<\frac{1}{p}$$

Hence,

$$\prod_{p=1}^{n}\left(1+\left(\sum_{k=1}^{n}\frac{2k}{k^4+(2p-1)k^2+p^2}\right)^2\right)<\prod_{p=1}^{n}\left(1+\frac{1}{p^2}\right)=$$

$$=\left(1+\frac{1}{1^2}\right)\prod_{p=2}^{n}\left(1+\frac{1}{p^2}\right)<2\prod_{p=2}^{n}\left(1-\frac{1}{p^2-1}\right)=2\prod_{p=2}^{n}\frac{p^2}{p^2-1}=$$

$$=2\prod_{p=2}^{n}\frac{p}{p-1}\cdot\prod_{p=2}^{n}\frac{p}{p+1}=2n\cdot\frac{2}{n+1}=4\cdot\left(1-\frac{1}{n+1}\right)<4$$

Solution 2 by Adrian Popa-Romania

$$\sum_{k=1}^{n}\frac{2k}{k^4+(2p-1)k^2+p^2}=\sum_{k=1}^{n}\frac{2k}{(k^2-k+p)(k^2+k+p)}=$$

$$=\sum_{k=1}^{n}\frac{(k^2+k+p)-(k^2-k+p)}{(k^2-k+p)(k^2+k+p)}=\sum_{k=1}^{n}\left(\frac{1}{k^2-k+p}-\frac{1}{k^2+k+p}\right)=$$

$$=\sum_{k=1}^{n}\left(\frac{1}{k^2-k+p}-\frac{1}{(k+1)^2+(k+1)+p}\right)=\frac{1}{p}-\frac{1}{n^2+n+p}$$

Hence,

$$\prod_{p=1}^{n}\left(1+\left(\sum_{k=1}^{n}\frac{2k}{k^4+(2p-1)k^2+p^2}\right)^2\right)<\prod_{p=1}^{n}\left(1+\frac{1}{p^2}\right)<\frac{\sinh\pi}{\pi}=3,67<4$$

$$\because \sinh x=x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right)\Rightarrow\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)=\frac{\sinh\pi}{\pi}$$

UP.463 Let $\lambda\geq\frac{7}{4}$. Solve for real numbers:

$$\begin{cases}\dfrac{x}{\sqrt{x^2-\lambda x+\lambda^2}}=\log_{\lambda^2-\lambda}(\lambda^2-y)\\ \dfrac{y}{\sqrt{y^2-\lambda y+\lambda^2}}=\log_{\lambda^2-\lambda}(\lambda^2-z)\\ \dfrac{z}{\sqrt{z^2-\lambda z+\lambda^2}}=\log_{\lambda^2-\lambda}(\lambda^2-x)\end{cases}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Let be the function $f:(-\infty,2\lambda)\to\mathbb{R}, f(x)=\frac{x}{\sqrt{x^2-\lambda x+\lambda^2}}$, then

$$f'(x)=\frac{2\lambda^2-\lambda x}{2(x^2-\lambda x+\lambda^2)^{\frac{3}{2}}};x\in(-\infty,2\lambda)$$

Because $f'(x)>0;\forall x\in(-\infty,2\lambda)$, hence f –stictly increasing on $(-\infty,2\lambda)$.

For $\lambda\geq\frac{7}{4}$ we have: $\lambda^2-\lambda>1\Rightarrow g(x)=\log_{\lambda^2-\lambda}x$ is increasing.

We distinguish the following cases:

Case I) If $x<\lambda, f\nearrow$, we have: $f(x)<f(\lambda)\Rightarrow\frac{x}{\sqrt{x^2-\lambda x+\lambda^2}}<1\Rightarrow\log_{\lambda^2-\lambda}(\lambda^2-y)<1\Rightarrow$

$y>\lambda$ and $f\nearrow$, then $f(y)>f(\lambda)\Rightarrow\frac{y}{\sqrt{y^2-\lambda y+\lambda^2}}>1\Rightarrow\log_{\lambda^2-\lambda}(\lambda^2-z)>1\Rightarrow$

$z<\lambda$ and $f\nearrow$, then $f(z)<f(\lambda)\Rightarrow\frac{z}{\sqrt{z^2-\lambda z+\lambda^2}}<1\Rightarrow\log_{\lambda^2-\lambda}(\lambda^2-x)<1$

$\Rightarrow x>\lambda$ impossible because $x<\lambda$.

Case II) If $x>\lambda$, similarly we get $x<\lambda$. So, $x=\lambda$ and hence

$x=y=z=\lambda$. Finally, we have $(x,y,z)=(\lambda,\lambda,\lambda)$.

Note: For $\lambda=2$ we obtain the Proposed problem by George Florin Şerban and Neculai Stanciu form R.M.M. no 7/2021

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\lambda^2-\lambda>1\Leftrightarrow\lambda^2-\lambda-1>0\Leftrightarrow\lambda>\frac{1+\sqrt{1^2-4.1(-1)}}{2}\Leftrightarrow\lambda>\frac{1+\sqrt{5}}{2}\to\text{true}\because\lambda\geq\frac{7}{4}$$

$$>\frac{1+\sqrt{5}}{2}\therefore\lambda^2-\lambda>1\Rightarrow ln(\lambda^2-\lambda)\overset{(*)}{>}0$$

Assuming $\boxed{x>\lambda}$, $\lambda(x-\lambda)>0\Rightarrow x\lambda-\lambda^2>0\Rightarrow x^2-x^2+x\lambda-\lambda^2>0\Rightarrow x^2>x^2-x\lambda+\lambda^2$

$$\Rightarrow x>\sqrt{x^2-x\lambda+\lambda^2}\left(\because x>\lambda\geq\frac{7}{4}>0\Rightarrow|x|=x\right)\Rightarrow\frac{x}{\sqrt{x^2-x\lambda+\lambda^2}}>1$$

$$\overset{\text{via (i)}}{\Rightarrow}\log_{\lambda^2-\lambda}(\lambda^2-y)>1\Rightarrow\frac{\ln(\lambda^2-y)}{\ln(\lambda^2-\lambda)}>1\Rightarrow\frac{\ln(\lambda^2-y)-\ln(\lambda^2-\lambda)}{\ln(\lambda^2-\lambda)}$$

$$>0\overset{\text{via }(*)}{\Rightarrow}\ln(\lambda^2-y)-\ln(\lambda^2-\lambda)>0\Rightarrow\ln(\lambda^2-y)>\ln(\lambda^2-\lambda)\Rightarrow\lambda^2-y$$

$$>\lambda^2-\lambda\boxed{\Rightarrow y<\lambda}$$

$$\Rightarrow y<0<\lambda\text{ or }0<y<\lambda\text{ and if }y<0,\frac{y}{\sqrt{y^2-y\lambda+\lambda^2}}<0<1\text{ and if }0<y<\lambda,\lambda(y-\lambda)<0$$

$$\Rightarrow y\lambda-\lambda^2<0\Rightarrow y^2-y^2+y\lambda-\lambda^2<0\Rightarrow y^2<y^2-y\lambda+\lambda^2$$

$$\Rightarrow y<\sqrt{y^2-y\lambda+\lambda^2}\ (\because y>0\Rightarrow|y|=y)\Rightarrow\frac{y}{\sqrt{y^2-y\lambda+\lambda^2}}<1\text{ and }\therefore y<\lambda\Rightarrow\frac{y}{\sqrt{y^2-y\lambda+\lambda^2}}$$

$$<1\overset{\text{via (ii)}}{\Rightarrow}\log_{\lambda^2-\lambda}(\lambda^2-z)<1\Rightarrow\frac{\ln(\lambda^2-z)}{\ln(\lambda^2-\lambda)}<1\Rightarrow\frac{\ln(\lambda^2-z)-\ln(\lambda^2-\lambda)}{\ln(\lambda^2-\lambda)}$$

$$<0$$

$\overset{\text{via }(*)}{\Rightarrow} \ln(\lambda^2 - z) - \ln(\lambda^2 - \lambda) < 0 \Rightarrow \ln(\lambda^2 - z) < \ln(\lambda^2 - \lambda) \Rightarrow \lambda^2 - z < \lambda^2 - \lambda \boxed{\Rightarrow z > \lambda}$

$\Rightarrow \lambda(z - \lambda) > 0 \Rightarrow z\lambda - \lambda^2 > 0 \Rightarrow z^2 - z^2 + z\lambda - \lambda^2 > 0 \Rightarrow z^2 > z^2 - z\lambda + \lambda^2$

$\Rightarrow z > \sqrt{z^2 - z\lambda + \lambda^2} \left(\because z > \lambda \geq \frac{7}{4} > 0 \Rightarrow |z| = z\right) \Rightarrow \frac{z}{\sqrt{z^2 - z\lambda + \lambda^2}}$

$> 1 \overset{\text{via (ii)}}{\Rightarrow} \log_{\lambda^2-\lambda}(\lambda^2 - x) > 1 \Rightarrow \frac{\ln(\lambda^2 - x)}{\ln(\lambda^2 - \lambda)} > 1 \Rightarrow \frac{\ln(\lambda^2 - x) - \ln(\lambda^2 - \lambda)}{\ln(\lambda^2 - \lambda)}$

> 0

$\overset{\text{via }(*)}{\Rightarrow} \ln(\lambda^2 - x) - \ln(\lambda^2 - \lambda) > 0 \Rightarrow \ln(\lambda^2 - x) > \ln(\lambda^2 - \lambda) \Rightarrow \lambda^2 - x > \lambda^2 -$

$\lambda \boxed{\Rightarrow x < \lambda}$ but $x > \lambda$ by assumption and $\therefore$ we face a contradiction $\therefore x \ngtr \lambda$

and proceeding in a similar fashion, if we assume $x < \lambda$, we would arrive at $x > \lambda$

$\therefore$ we face a contradiction $\Rightarrow x \nless \lambda \therefore \boxed{x \ngtr \lambda \text{ and } x \nless \lambda}$

$\Rightarrow \boxed{x = \lambda}$ and analogously,

if we assume $y > \lambda$, we would arrive at $y < \lambda \Rightarrow$ a contradiction and if we assume $y < \lambda$, we would arrive at $y > \lambda \Rightarrow$ a contradiction $\Rightarrow \boxed{y \ngtr \lambda \text{ and } y \nless \lambda}$

$\Rightarrow \boxed{y = \lambda}$ and also, if we

assume $z > \lambda$, we would arrive at $z < \lambda \Rightarrow$ contradiction and if we assume $z < \lambda$, we would arrive at $z > \lambda \Rightarrow$ contradiction $\Rightarrow \boxed{z \ngtr \lambda \text{ and } z \nless \lambda}$

$\Rightarrow \boxed{z = \lambda} \therefore x = y = z = \lambda$ (ans)

Solution 3 by Michael Sterghiou-Greece

$$\begin{cases} \dfrac{x}{\sqrt{x^2 - \lambda x + \lambda^2}} = \log_{\lambda^2-\lambda}(\lambda^2 - y)\,;(1) \\ \dfrac{y}{\sqrt{y^2 - \lambda y + \lambda^2}} = \log_{\lambda^2-\lambda}(\lambda^2 - z)\,;(2) \\ \dfrac{z}{\sqrt{z^2 - \lambda z + \lambda^2}} = \log_{\lambda^2-\lambda}(\lambda^2 - x)\,;(3) \end{cases}$$

First, $\lambda^2 - \lambda > 1$ and $x, y, z < \lambda^2$. We can easily show that

$$\frac{t}{\sqrt{t^2 - \lambda t + \lambda^2}} \geq 1 \Leftrightarrow t \geq \lambda \text{ and } \frac{t}{\sqrt{t^2 - \lambda t + \lambda^2}} \leq 1 \Leftrightarrow t \leq \lambda; \forall t \in \mathbb{R}$$

$$\text{Let } x \geq \lambda \Rightarrow LHS_{(1)} \geq 1 \Rightarrow \lambda^2 - y \geq \lambda^2 - \lambda \Rightarrow y \leq \lambda$$

$$LHS_{(2)} \leq 1 \Rightarrow \lambda^2 - z \leq \lambda^2 - \lambda \Rightarrow z \geq \lambda \Rightarrow LHS_{(3)} \geq 1$$

$$\lambda^2 - x \geq \lambda^2 - \lambda \Rightarrow x \leq \lambda(contradiction)$$

In the same manner we get also a contradiction if we assume $x \leq \lambda$. Hence, $x = \lambda$.

But then $LHS_{(1)} = 1 \Rightarrow \lambda^2 - y = \lambda^2 - \lambda \Rightarrow y = \lambda$ and then $z = \lambda$.

This is only solution of the system: $(x, y, z) = (\lambda, \lambda, \lambda)$.

UP.464 In ΔABC the following relationship holds:

$$\frac{a^3}{(5a+7b)(7a+5b)} + \frac{b^3}{(5b+7c)(7c+5b)} + \frac{c^3}{(5c+7a)(7c+5a)} \geq \frac{\sqrt{3}r}{24}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

If $x, y > 0$ then $(x-y)^2(2x+y) \geq 0 \Leftrightarrow$

$$(x^2 - 2xy + y^2)(2x+y) \geq 0 \Leftrightarrow$$

$$2x^3 + x^2y - 4x^2y - 2xy^2 + 2xy^2 + y^3 \geq 0 \Leftrightarrow$$

$$2x^3 - 3x^2y + y^3 \geq 0 \Leftrightarrow 2x^3 \geq 3x^2y - y^3$$

$$4x^3 \geq 2x^3 - x^2y + 4x^2y - 2xy^2 + 2xy^2 - y^3 \Leftrightarrow$$

$$4x^3 \geq (2x-y)(x^2+2xy+y^2) \Leftrightarrow 4x^3 \geq (2x-y)(x+y)^2$$

$$\frac{4x^3}{(x+y)^2} \geq 2x - y; (1)$$

If $x, y > 0$ then $(x-y)^2 \geq 0 \Leftrightarrow x^2 - 2xy + y^2 \geq 0 \Leftrightarrow$

$$x^2 + y^2 - 2xy \geq 0 \Leftrightarrow 36x^2 + 36y^2 - 2xy \geq 35x^2 + 35y^2 \Leftrightarrow$$

$$36x^2 + 36y^2 + (72-74)xy \geq 35x^2 + 35y^2 \Leftrightarrow$$

$$35x^2 + 35y^2 + 74xy \leq 36x^2 + 36y^2 + 72xy \Leftrightarrow$$

$$35x^2 + 35y^2 + 25xy + 49xy \leq 36(x+y)^2 \Leftrightarrow$$

$$(5x+7y)(5y+7x) \leq 36(x+y)^2 \Leftrightarrow \frac{1}{(5x+7y)(7x+5y)} \geq \frac{1}{36(x+y)^2}$$

$$\frac{x^3}{(5x+7y)(7x+5y)} \geq \frac{x^3}{36(x+y)^2}; (2)$$

From (1) and (2) and $a, b, c -$ sides of ΔABC, it follows that:

$$\sum_{cyc}\frac{a^3}{(5a+7b)(5b+7a)}\ge\sum_{cyc}\frac{a^3}{36(a+b)^2}=\frac{1}{144}\sum_{cyc}\frac{4a^3}{(a+b)^2}\overset{(1)}{\ge}$$

$$\ge\frac{1}{144}\sum_{cyc}(2a-b)=\frac{1}{144}\left(2\sum_{cyc}a-\sum_{cyc}b\right)=\frac{a+b+c}{144}=$$

$$=\frac{2s}{144}\overset{Mitrinovic}{\ge}\frac{2\cdot 3\sqrt{3}r}{144}=\frac{\sqrt{3}r}{24}$$

Equality holds for $a=b=c$.

Solution 2 by Avishek Mitra-West Bengal-India

$$\sqrt[6]{(5a+7b)(7a+5b)(5b+7c)(7c+5c)(5c+7a)(7c+5a)}\overset{AGM}{\le}$$

$$\le\frac{24(a+b+c)}{6}=4\cdot 2s=8s$$

$$\sum_{cyc}\frac{a^3}{(5a+7b)(7a+5b)}\overset{AGM}{\ge}3\sqrt[3]{\frac{(abc)^3}{\prod(5a+7b)(7a+5b)}}\ge$$

$$\ge\frac{3abc}{(8s)^2}=\frac{12Rrs}{64s^2}=\frac{3Rr}{16s}\overset{Mitrinovic}{\ge}\frac{3\cdot\frac{2s}{3\sqrt{3}}r}{16s}=\frac{r}{8\sqrt{3}}=\frac{\sqrt{3}r}{24}$$

Solution 3 by Tapas Das-India

$$\frac{a^3}{(5a+7b)(7a+5b)}\ge\frac{a^3}{\left(\frac{5a+7b+7a+5b}{2}\right)^2}=\frac{a^3}{(12a+12b)^2}$$

Similarly,

$$\frac{b^3}{(5b+7c)(7b+5c)}\ge\frac{4b^3}{(12b+12c)^2}\text{ and }\frac{c^3}{(5c+7a)(7c+5a)}\ge\frac{4c^3}{(12c+12a)^2}$$

$$\sum_{cyc}\frac{a^3}{(5a+7b)(7a+5b)}\ge 4\sum_{cyc}\frac{a^3}{(12a+12b)^2}\overset{Radon}{\ge}$$

$$\ge 4\cdot\frac{(a+b+c)^3}{\left(24(a+b+c)\right)^2}=\frac{4\cdot 2s}{24^2}\ge\frac{8\cdot 3\sqrt{3}r}{24^2}=\frac{\sqrt{3}r}{24}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$F=sr\Rightarrow sr=\sqrt{s(s-a)(s-b)(s-c)}$$

$$s^2r^2 = s(s-a)(s-b)(s-c)$$

$$x = b+c-a, y = a+c-b, z = a+b-c$$

$$4(x+y+z)r = xyz \leq \frac{(x+y+z)^3}{27}$$

$$4 \cdot 27r^2 \leq (x+y+z)^2$$

$$2 \cdot 3\sqrt{3}r \leq x+y+z$$

$$3\sqrt{3}r \leq \left(\frac{x+y+z}{2}\right) = \frac{1}{2}\left(\frac{x+y}{2} + \frac{y+z}{2} + \frac{z+x}{2}\right) = s$$

Hence,

$$\sum_{cyc} \frac{a^3}{(5a+7b)(7a+5b)} \geq \frac{\sqrt{3}r}{24}$$

$$12 \cdot 27 \sum_{cyc} \frac{a^4}{(12a+12a+12b)^3} \geq \frac{\sqrt{3}r}{24}$$

$$\frac{12 \cdot 27}{36^3} \cdot \frac{(a+b+c)^4}{(a+b+c)^3} \geq \frac{\sqrt{3}r}{24}$$

$$\frac{a+b+c}{4 \cdot 36} \geq \frac{\sqrt{3}r}{24} \Leftrightarrow s \geq 3\sqrt{3}r(Mitrinovic)$$

UP.465 In ΔABC the following relationship holds:

$$2^4(R+r-d)^3[4(R-r+d)^3 + R^3] \leq a^6 + b^6 + c^6 \leq$$

$$\leq 2^4(R+r+d)^3[4(R-r-d)^3 + R^3]$$

where we denote by $d = \sqrt{R^2 - 2Rr}$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

Solution by proposers

We use the fundamental inequality or Blundon's inequality.

For any triangle ABC the inequality $s_1 \leq s \leq s_2$ hold where s_1, s_2 represent the semiperimeter of two isosceles triangles $A_1B_1C_1$ and $A_2B_2C_2$ which have the same circumradius R and inradius r as the triangle ABC with sides

$$a_1 = 2\sqrt{(R+r-d)(R-r+d)};\ b_1 = c_1 = \sqrt{2R(R+r-d)}$$

$$a_2 = 2\sqrt{(R+r+d)(R-r-d)};\ b_2 = c_2 = 2\sqrt{2R(R+r+d)},\text{ where } d = \sqrt{R^2-2Rr}$$

From the identity $\sum x^3 = 3xyz + \sum x\left(\sum x^2 - \sum yz\right)$ if we replacing

$x = a^2, y = b^2, z = c^2$, then we obtain $\sum a^6 = 3a^2b^2c^2 + \sum a^2\left(\sum a^4 - \sum b^2c^2\right)$.

Consider the functions $f, g, h, F: [s_1; s_2] \to \mathbb{R}$, $f(s) = 3a^2b^2c^2 = 3(4Rrs)^2$,

$$g(s) = 2(s^2 - r^2 - 4Rr),$$

$$h(s) = \sum_{cyc} a^4 - \sum_{cyc}(bc)^2 = \left(\sum_{cyc} a\right)^4 - 4\sum_{cyc} ab\left(\sum_{cyc} a\right)^2 + 6abc\sum_{cyc} a + \left(\sum_{cyc} ab\right)^2 =$$

$$= s^4 - 2r(4R+7r)s^2 + (4R+r)^2r^2,\ \text{with } h'(s) = 4s(s^2 - 4Rr - 7r^2)$$

$$F(s) = f(s) + g(s)h(s).$$

Since $s^2 \ge s_1^2 \ge 16Rr - 5r^2 \ge 4Rr + 7r^2$ it result that h is increasing on $[s_1, s_2]$.

Also f and g are increasing on $[s_1, s_2]$. Hence, F is increasing on $[s_1, s_2]$, then

$F(s_1) \le F(s) \le F(s_2)$ or $a_1^6 + b_1^6 + c_1^6 \le a^6 + b^6 + c^6 \le a_2^6 + b_2^6 + c_2^6$; (1)

If we replacing the values of $a_1, b_1, c_1, a_2, b_2, c_2$ from the lemma in (1), the we obtain the desired inequality and we are done.