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SOLUTIONS

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PROBLEMS FOR JUNIORS

JP.466 If $a,b,c,d\in\mathbb{R}$ are such that $(a^2+b^2)(c^2+d^2)=25$ then:

$$3bd+4ad+4bc\le 3ac+25$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $\Omega=(a+ib)(c+id)(3+4i)$.

$$|\Omega|=\sqrt{(a^2+b^2)(c^2+d^2)(3^2+4^2)}=\sqrt{25\cdot 25}=25$$

$$\Omega=\big(ac-bd+i(bc+ad)\big)(3+4i)=$$

$$=3(ac-bd)-4(bc+ad)+i\big(4(ac-bd)+3(bc+ad)\big)$$

$$|Re(\Omega)|=|3(ac-bd)-4(bc+ad)|=|3ac-3bd-4bc-4ad|$$

Hence:

$$|Re(\Omega)|\le|\Omega|$$

$$|3ac-3bd-4bc-4ad|\le 25$$

$$3ac-3bd-4bc-4ad\ge -25$$

$$3ac+25\ge 3bd+4ad+4bc$$

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

The inequality in the statement can be written as follows:

$$3bd+4ad+4bc-3ca\le 5\sqrt{(a^2+b^2)(c^2+d^2)}$$

If $3bd+4ad+4bc-3ca\le 0$ *then the inequality is true.*

Assume now that $3bd+4ad+4bc-3ca\ge 0$. *The inequality is equivalent to*

$$(3bd+4ad+4bc-3ca)^2\le 25(a^2+b^2)(c^2+d^2)$$

$$\Leftrightarrow 16b^2d^2+9a^2d^2+9b^2c^2+16c^2a^2-24abd^2-24b^2cd+24a^2cd+24abc^2-14abcd\ge 0$$

$$\Leftrightarrow a^2\big(9d^2+24cd+16c^2\big)-2ab\big(12d^2+7cd-12c^2\big)+b^2\big(16d^2-24cd+9c^2\big)\ge 0$$

$$\Leftrightarrow a^2(3d+4c)^2-2ab(3d+4c)(4d-3c)+b^2(4d-3c)^2\ge 0$$

$$\Leftrightarrow [a(3d+4c)-b(4d-3c)]^2\ge 0\ \textit{ which is true.}$$

$$Therefore,\ \ 3bd+4ad+4bc\le 3ca+25.$$

Solution 3 by Henry Ricardo-New York-USA

$$3bd+4ad+4bc-3ac=3(bd-ac)+4(ad+bc)\overset{CBS}{\le}$$

$$\le\sqrt{3^2+4^2}\cdot\sqrt{(bd-ac)^2+(ad+bc)^2}=$$

$$=5\sqrt{b^2d^2+a^2c^2+a^2d^2+b^2c^2}=5\cdot\sqrt{a^2+b^2}\cdot\sqrt{c^2+d^2}=25$$

JP.467 If $x,y>0$ then in ΔABC the following relationship holds:

$$\frac{xa^n+yb^n}{xa^{n-1}+yb^{n-1}}+\frac{xb^n+yc^n}{xb^{n-1}+yc^{n-1}}+\frac{xc^n+ya^n}{xc^{n-1}+yb^{n-1}}\ge a+b+c$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\frac{xa^n+yb^n}{xa^{n-1}+yb^{n-1}}\ge\frac{xa+yb}{x+y}\Leftrightarrow$$

$$(x+y)(xa^n+yb^n)\ge(xa+yb)(xa^{n-1}+yb^{n-1})\Leftrightarrow$$

$$x^2a^n+xyb^n+xya^n+y^2b^n\ge x^2a^n+y^2b^n+xyab^{n-1}+xya^{n-1}b\Leftrightarrow$$

$$a^n+b^n\ge a^{n-1}b+ab^{n-1}\Leftrightarrow a^{n-1}(a-b)-b^{n-1}(a-b)\ge 0\Leftrightarrow$$

$$(a-b)(a^{n-1}-b^{n-1})\ge 0$$

$(a-b)^2(a^{n-2}+a^{n-2}b+\dots+b^{n-2})\ge 0$ which is true.

Similarly,

$$\frac{xb^n+yc^n}{xb^{n-1}+yc^{n-1}}\ge\frac{xb+yc}{x+y}\ \text{ and }\ \frac{xc^n+ya^n}{xc^{n-1}+yb^{n-1}}\ge\frac{xc+ya}{x+y}$$

By adding, it follows that

$$\frac{xa^n+yb^n}{xa^{n-1}+yb^{n-1}}+\frac{xb^n+yc^n}{xb^{n-1}+yc^{n-1}}+\frac{xc^n+ya^n}{xc^{n-1}+yb^{n-1}}\ge a+b+c$$

Solution 2 by Mohamed Amine Ben Ajiba- Morocco

$$We\ have:\ \frac{xa^n+yb^n}{xa^{n-1}+yb^{n-1}}\overset{?}{\ge}\frac{xa+yb}{x+y}\Leftrightarrow(xa^n+yb^n)(x+y)$$
$$\ge(xa^{n-1}+yb^{n-1})(xa+yb)$$

$$\Leftrightarrow xy(a^n + b^n) \geq xy(a^{n-1}b + ab^{n-1}) \Leftrightarrow xy(a-b)(a^{n-1} - b^{n-1}) \geq 0$$

Which is true because $a - b$ *and* $a^{n-1} - b^{n-1}$ *have the same sign.*

$$\text{Then we have}: \frac{xa^n + yb^n}{xa^{n-1} + yb^{n-1}} \geq \frac{xa + yb}{x + y} \quad (\text{and analogs})$$

Summing up this inequality with similar ones we get:

$$\frac{xa^n + yb^n}{xa^{n-1} + yb^{n-1}} + \frac{xb^n + yc^n}{xb^{n-1} + yc^{n-1}} + \frac{xc^n + ya^n}{xc^{n-1} + ya^{n-1}} \geq \frac{xa + yb}{x + y} + \frac{xb + yc}{x + y} + \frac{xc + ya}{x + y}$$
$$= a + b + c.$$

Equality holds iff ΔABC *is equilateral.*

JP.468 Solve for real numbers:

$$x^{12} - 15x^3 + 14 = 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$x^{12} - 15x^3 + 14 = 0$$
$$x^{12} - 2x^9 + 2x^9 - 4x^6 + 4x^6 - 8x^3 - 7x^3 + 14 = 0$$
$$x^9(x^3 - 2) + 2x^6(x^3 - 2) - 7(x^3 - 2) = 0$$
$$(x^3 - 2)(x^9 + 2x^6 + 4x^3 - 7) = 0$$
$$x^3 - 2 = 0 \Rightarrow x^3 = 2 \Rightarrow x_1 = \sqrt[3]{2}, x_{2,3} \in \mathbb{C}$$
$$x^9 + 2x^6 + 4x^3 - 7 = 0$$
$$x^9 - x^6 + 3x^6 - 3x^3 + 7x^3 - 7 = 0$$
$$x^6(x^3 - 1) + 3x^3(x^3 - 1) + 7(x^3 - 1) = 0$$
$$(x^3 - 1)(x^6 + 3x^3 + 7) = 0$$
$$x^3 - 1 = 0 \Rightarrow x^3 = 1 \Rightarrow x_4 = 1, x_{5,6} \in \mathbb{C}$$
$$x^6 + 3x^3 + 7 = 0$$

Denote $x^3 = y \Rightarrow y^2 + 3y + 7 = 0; \Delta = 9 - 28 < 0 \Rightarrow y_{1,2} \in \mathbb{C}$

$$x_{7,8,\dots,12} \in \mathbb{C}$$

So, the real solutions are $S = \{\sqrt[3]{2}; 1\}$.

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\text{Let } k = x^3 \Rightarrow k^4 - 15k + 14 = 0$$

$$k^4 - k - 14k + 14 = 0 \Leftrightarrow k(k^3 - 1) - 14(k - 1) = 0$$

$$k(k-1)(k^2 + k + 1) - 14(k - 1) = 0$$

$$(k-1)(k^3 + k^2 + k + 1) = 0$$

$$k = 1 \text{ or } k^3 + k^2 + k - 14 = 0$$

$$(k-2)(k^2 + 3k + 7) = 0 \Leftrightarrow (k-1)(k-2)(k^2 + 3k + 7) = 0$$

$$k \in \left\{1, 2, \frac{-3 + \sqrt{9 - 4 \cdot 7}}{2}\right\} \cap \mathbb{R} \Rightarrow k \in \{1, 2\} \Rightarrow x \in \{1; \sqrt[3]{2}\}.$$

Solution 3 by Florentin Vișescu-Romania

$$x^{12} - 15x^3 + 14 = 0$$

$$x^3 = t \Rightarrow t^4 - 15t + 14 = 0$$

$$t^4 - t - 14t + 14 = 0 \Leftrightarrow t(t^3 - 1) - 14(t - 1) = 0$$

$$t(t-1)(t^2 + t + 1) - 14(t - 1) = 0$$

$$(t-1)(t^3 + t^2 + t - 14) = 0$$

$$t = 1 \Rightarrow x_1 = 1 \text{ or } t^3 + t^2 + t - 14 = 0 \Leftrightarrow$$

$$t^3 - 8 + t^2 - 4 + t - 2 = 0 \Leftrightarrow (t-2)(t^2 + 3t + 7) = 0$$

$$t = 2 \Rightarrow x = \sqrt[3]{2}$$

Solution 4 by Fayssal Abdelli-Algerie

$$y = x^3 \Rightarrow y^4 - 15y + 14 = 0$$

$$y^4 - y + 14 - 14y = 0 \Leftrightarrow y(y^3 - 1) - 14(y - 1) = 0$$

$$(y-1)(y^3 + y^2 + y - 14) = 0$$

$$y = 1 \text{ or } y^3 + y^2 + y - 14 = 0,\ \ y = 1 \Rightarrow x = 1$$

$$y^3 + y^2 + y - 14 = (y-2)(y^2 + 3y + 7) = 0 \Rightarrow y = 2 \Rightarrow x = \sqrt[3]{2}$$

$$x \in \{1; \sqrt[3]{2}\}$$

Solution 5 by Muhammad Afzal-Pakistan

$$y = x^3 \Rightarrow y^4 - 15y + 14 = 0$$

$$y^4 - y + 14 - 14y = 0 \Leftrightarrow y(y^3 - 1) - 14(y - 1) = 0$$

$$(y-1)(y^3+y^2+y-14)=0,\ \ y=1 \text{ or } y^3+y^2+y-14=0$$

$$y=1\Rightarrow x=1$$

$$y^3+y^2+y-14=(y-2)(y^2+3y+7)=0\Rightarrow y=2\Rightarrow x=\sqrt[3]{2}$$

$$x\in\{1;\sqrt[3]{2}\}$$

Solution 6 by Carlos Paiva-Brazil

$$x^{12}-15x^3+14=0, \text{let } y=x^3, \text{then:}$$

$$y^4-15y+14=0\Leftrightarrow (y^2-3y+2)(y^2+3y+7)=0\Leftrightarrow$$

$$y_1=1; y_2=2, \text{hence } x\in\{1;\sqrt[3]{2}\}$$

Solution 7 by Qudrat Muhammadi-Afghanistan

$$x^{12}-15x^3+14=0\Leftrightarrow (x-1)(x^3-2)(x^2+x+1)(x^6+3x^3+7)=0$$

$$x-1=0\Rightarrow x_1=1,\qquad x^3-2=0\Rightarrow x_2=\sqrt[3]{2}$$

$(x^2+x+1)=0;\ (x^6+3x^3+7)=0$ there is no real solution hence $x\in\{1;\sqrt[3]{2}\}$

Solution 8 by Fayssal Abdelli-Algerie

Let $y=x^3\Rightarrow y^4-15y+14=0$

$$y^4-16-15y+30=0\Leftrightarrow (y^4-2^4)-15(y-2)=0$$

$$(y^2-2^2)(y^2+2^2)-15(y-2)=0$$

$$(y-2)(y^3+2y^2+4y-7)=0$$

$y=2\Rightarrow x=\sqrt[3]{2}$ or $y^3+2y^2+4y-7=0\Leftrightarrow$

$$(y-1)(y^2+3y+7)=0\Rightarrow y=1\Rightarrow x=1$$

$y^3+3y+7=0$ there is no real solution.

Solution 9 by Angel Plaza-Spain

The only real solution to the proposed equation are $x=1$, and $x=\sqrt[3]{2}$.

By doing $x^3=y$, the equation becomes $y^4-15y+14=0$, with $y=1$ as a root.

Since $y^4-15y+14=(y-1)(y^3+y^2+y-14)=(y-1)(y-2)(y^2+3y+7)$.

Since $y^2+3y+7=\left(y+\frac{3}{2}\right)^2-\frac{9}{4}+7=\left(y+\frac{3}{2}\right)^2+\frac{19}{4}>0$, the equation does not have any more real roots.

Solution 10 by Henry Ricardo-New York-USA

Let $X = x^3$, our equation becomes

$$X^4 - 15X + 14 = (X-2)(X-1)(X^2+3X+7) = 0,$$

so that $X = 2, X = 1$ and $X = (-3 \pm 19i)/2$ are the solutions.

This translates to $x = \sqrt[3]{2}$ and $x = 1$ as the only real solutions to the original equation.

JP.469 Let $z_1, z_2, z_3 \in \mathbb{C}^*, A(z_1), B(z_2), C(z_3)$ different in pairs such that $|z_1| = |z_2| = |z_3| = 1$. If

$$\sum_{cyc} \sqrt{|(2z_1 - z_2 - z_3)(2z_2 - z_1 - z_3)|} = 9 \Rightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\sum_{cyc} \sqrt{|(2z_1 - z_2 - z_3)(2z_2 - z_1 - z_3)|} = 9 \Leftrightarrow$$

$$2\sum_{cyc} \sqrt{\left|\left(z_1 - \frac{z_2+z_3}{2}\right)\left(z_2 - \frac{z_1+z_3}{2}\right)\right|} = 9 \Leftrightarrow \sum_{cyc} \sqrt{m_a m_b} = \frac{9}{2}; (1)$$

$$\text{But } \sqrt{m_a m_b} \le \frac{m_a + m_b}{2} \Rightarrow \sum_{cyc} \sqrt{m_a m_b} \le m_a + m_b + m_c; (2)$$

$$m_a + m_b + m_c \le \frac{9R}{2}; (3)$$

From (2) and (3) we get

$$\sum_{cyc} \sqrt{m_a m_b} = \frac{9}{2}; (1)$$

From (1) and (4) we have equality if and only if ΔABC is equilateral.

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\sum_{cyc} \sqrt{|(2z_1 - z_2 - z_3)(2z_2 - z_1 - z_3)|} =$$

$$= \sum_{cyc} \sqrt{|(3z_1 - (z_1+z_2+z_3))(3z_2 - (z_1+z_2+z_3))|} =$$

$$= 3\sum_{cyc}\sqrt{|z_1 - z_G|\cdot|z_2 - z_G|} = 3\sum_{cyc}\sqrt{AG\cdot BG} =$$

$$= 3\cdot\frac{2}{3}\sum_{cyc}\sqrt{m_a\cdot m_b} \le 2\sum_{cyc}m_a \overset{AGM}{\le} 6\cdot\sqrt{\frac{1}{3}\sum_{cyc}m_a^2} =$$

$$= 6\cdot\sqrt{\frac{1}{12}\sum_{cyc}(2b^2+2c^2-a^2)} = 3\cdot\sqrt{\sum_{cyc}a^2} \overset{Leibniz}{\le} 3\cdot\sqrt{3R^2} = 9R = 9.$$

Equality holds for $a = b = c$.

JP.470 Let $z_1, z_2, z_3 \in \mathbb{C}, A(z_1), B(z_2), C(z_3)$ different in pairs such that $|z_1| = |z_2| = |z_3| = 1$.Prove that:

$$\sum_{cyc}\frac{1}{(|(z_1-z_2)|z_1-z_3| + (z_1-z_3)|z_1-z_2||)^2} = \frac{3}{(|z_1-z_2|+|z_2-z_3|+|z_3-z_1|)^2}$$

$$\Leftrightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$|z_1-z_2|+|z_2-z_3|+|z_3-z_1| = AB+BC+CA = a+b+c$$

$$\sum_{cyc}\frac{1}{\left(\left|(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2|\right|\right)^2} = \frac{3}{(|z_1-z_2|+|z_2-z_3|+|z_3-z_1|)^2}$$

$$\sum_{cyc}\frac{1}{\left(\frac{\left|(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2|\right|}{a+b+c}\right)^2} = 3 \Leftrightarrow \sum_{cyc}\frac{1}{\left(\frac{(b+c)z_1-bz_2-cz_3}{a+b+c}\right)^2}$$

$$= 3 \Leftrightarrow$$

$$\sum_{cyc}\frac{1}{\left(\frac{(b+c+a)z_1-az_1-bz_2-cz_3}{a+b+c}\right)^2} = 3 \Leftrightarrow \sum_{cyc}\frac{1}{\left|z_1-\frac{az_1+bz_2+cz_3}{a+b+c}\right|^2} = 3 \Leftrightarrow$$

$$\sum_{cyc}\frac{1}{AI^2} = 3;(1)$$

$$\text{But: } \sum_{cyc}\frac{1}{AI^2} = \sum_{cyc}\frac{\sin^2\frac{A}{2}}{r} = \frac{1}{r^2}\cdot\frac{2R-r}{2R} \ge \frac{3}{R^2} = 3;(2)$$

From (1) and (2) ΔABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$|z_1| = |z_2| = |z_3| = 1 \rightarrow \Delta ABC \in C(O, R = 1)$$

$$Let\ a = BC = |z_2 - z_3|,\ \ b = CA = |z_3 - z_1|,\ \ c = AB = |z_2 - z_1|.\ We\ have:$$

$$LHS_{(*)} = \sum_{cyc} \frac{1}{|(z_1 - z_2)b + (z_1 - z_3)c|^2} = \sum_{cyc} \frac{1}{|(a+b+c)z_1 - (az_1 + bz_2 + cz_3)|^2}$$

$$= \sum_{cyc} \frac{1}{\left(2s.\left|z_1 - \frac{az_1 + bz_2 + cz_3}{a+b+c}\right|\right)^2} =$$

$$= \frac{1}{(2s)^2}\sum_{cyc}\frac{1}{|z_1 - z_I|^2} = \frac{1}{(2s)^2}\sum_{cyc}\frac{1}{AI^2} = \frac{1}{(2s)^2}\sum_{cyc}\frac{\sin^2\frac{A}{2}}{r^2}$$

$$= \frac{1}{(2s)^2}\cdot\frac{1}{r^2}\left(1 - \frac{r}{2R}\right) \overset{Euler}{\geq} \frac{1}{(2s)^2}\cdot\frac{4}{R^2}\left(1 - \frac{1}{4}\right) \overset{R=1}{=} \frac{3}{(2s)^2} = RHS_{(*)}$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

$$Therefore,$$

$$\sum_{cyc} \frac{1}{\big|(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|\big|^2} = \frac{3}{(|z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|)^2}$$

$$\Leftrightarrow\ \ AB = BC = CA.$$

JP.471 In $\Delta ABC, AA_1, BB_1, CC_1$ internal bisectors and A_2, B_2, C_2 contact points with circumcircle of triangle ABC. Prove that:

$$A_1A_2 \cdot B_2C_2 + B_1B_2 \cdot A_2C_2 + C_1C_2 \cdot A_2B_2 \geq Rs$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$\rho(A_1) = AA_1 \cdot A_1A_2 = BA_1 \cdot A_1C$$

$$A_1A_2 = \frac{BA_1 \cdot A_1C}{A_1A_2}; (1)$$

From bisector theorem we have

$$BA_1 = \frac{ac}{a+c}, CA_1 = \frac{ab}{b+c}; (2)$$

$$AA_1 = wa = \frac{2bc}{b+c}\cos\frac{A}{2};(3)$$

From (1),(2) and (3) it follows

$$A_1A_2 = \frac{a^2}{2(b+c)\cos\frac{A}{2}};(4)$$

From Law of sines we have

$$\frac{B_2C_2}{\sin\left(\frac{B+C}{2}\right)} = 2R \Rightarrow B_2C_2 = 2R\cos\frac{A}{2};(5)$$

From (4) and (5) we get

$$AA_2 \cdot B_2C_2 = \frac{a^2R}{b+c}$$

$$\sum_{cyc} A_1A_2 \cdot B_2C_2 = R\sum_{cyc}\frac{a^2}{b+c} \overset{Bergstrom}{\geq} R \cdot \frac{(a+b+c)^2}{2(a+b+c)} = \frac{R(a+b+c)}{2} = Rs$$

JP.472 If $x, y, z > 0$ then:

$$\left(3x^3 - \frac{1}{x^2} + \frac{1}{x^5}\right)\left(3y^3 - \frac{1}{y^2} + \frac{1}{y^5}\right)\left(3z^3 - \frac{1}{z^2} + \frac{1}{z^5}\right) \geq (xy + yz + zx)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Denote: $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$, then

$$a^8 - a^5 + 3 - a^3 - 2 = a^5(a^3 - 1) - (a^3 - 1) = (a^3 - 1)(a^5 - 1) =$$

$$= (a-1)^2(a^2 + a + 1)(a^4 + a^3 + a^2 + a + 1) \geq 0$$

$$a^8 - a^5 + 3 - a^3 - 2 \geq 0 \Leftrightarrow a^8 - a^5 + 3 \geq a^3 + 2$$

$$\prod_{cyc}\left(3x^3 - \frac{1}{x^2} + \frac{1}{x^5}\right) = \prod_{cyc}\left(\frac{3}{a^3} - a^2 + a^5\right) = \prod_{cyc}\frac{1}{a^3}(a^8 - a^5 + 3) \geq$$

$$\geq \frac{1}{a^3b^3c^3}\prod_{cyc}(a^8 - a^5 + 3) \geq \frac{1}{a^3b^3c^3}\prod_{cyc}(a^3 + 2) =$$

$$= \frac{1}{a^3b^3c^3}\prod_{cyc}(a^3 + 1 + 1) \overset{Holder}{\geq} \frac{1}{a^3b^3c^3}(a \cdot 1 \cdot 1 + b \cdot 1 \cdot 1 + c \cdot 1 \cdot 1)^3 =$$

$$=\left(\frac{a+b+c}{abc}\right)^3=\left(xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\right)^3=(xy+yz+zx)^3$$

Equality holds for $x=y=z=1$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y,z>0$ we will have:

$$3x^3-\frac{1}{x^2}+\frac{1}{x}\geq 2x^3+1\Leftrightarrow x^3-\frac{1}{x^2}+\frac{1}{x^3}\geq 1$$

$$x^8-x^3+1\geq x^5\Leftrightarrow x^8-x^5-(x^3-1)\geq 0$$

$$x^5(x^3-1)-(x^3-1)\geq 0\Leftrightarrow (x^3-1)\left(x^5-1\right)\geq 0$$

$$(x-1)^2(x^2+x+1)(x^4+x^3+x^{22}+x+1)\geq 0$$

Therefore,

$$\prod_{cyc}\left(3x^3-\frac{1}{x^2}+\frac{1}{x^5}\right)\geq\prod_{cyc}(2x^3+1)\geq(xy+yz+zx)^3$$

Equality holds for $x=y=z=1$.

Solution 3 by Tapas Das-India

$$(x^3+1+x^3)^{\frac{1}{3}}(y^3+y^3+1)^{\frac{1}{3}}(1+z^3+z^3)^{\frac{1}{3}}\overset{Holder}{\geq}$$

$$\overset{Holder}{\geq}(x^3\cdot y^3\cdot 1)^{\frac{1}{3}}+(1\cdot y^3\cdot z^3)^{\frac{1}{3}}+(x^3\cdot 1\cdot z^3)^{\frac{1}{3}}=xy+yz+zx$$

$$(2x^3+1)(2y^3+1)(2z^3+1)\geq(xy+yz+zx)^3$$

$$3x^3-\frac{1}{x^2}+\frac{1}{x^5}-(2x^3+1)=\left(x^3+\frac{1}{x^5}\right)-\left(\frac{1}{x^2}+1\right)=$$

$$=(x^3-1)+\frac{1}{x^5}-\frac{1}{x^2}=(x^3-1)-\frac{x^3-1}{x^5}=$$

$$=\frac{(x^3-1)\left(x^5-1\right)}{x^5}=\frac{(x^3-1)[(x^3-1)(x^2+1)-x^3+x^2]}{x^5}=$$

$$=\frac{(x^3-1)[(x^3-1)(x^2+1)-x^2(x-1)]}{x^5}=$$

$$=\frac{(x-1)^2(x^2+x+1)(x^4+x^3+x^2+x+1)}{x^5}>0$$

$$\Rightarrow 3x^3-\frac{1}{x^2}+\frac{1}{x^5}\geq 2x^3+1$$

Similarly,

$$3y^3-\frac{1}{y^2}+\frac{1}{y^5}\ge 2y^3+1 \text{ and } 3z^3-\frac{1}{z^2}+\frac{1}{z^5}\ge 2z^3+1$$

Therefore,

$$\prod_{cyc}\left(3x^3-\frac{1}{x^2}+\frac{1}{x^5}\right)\ge\prod_{cyc}(2x^3+1)\ge(xy+yz+zx)^3$$

Equality holds for $x=y=z=1$.

Solution 4 by Hikmat Mammadov-Azerbaijan

Let $(x,y,z)\to\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$. We need to prove that:

$$\prod_{cyc}\left(x^5-x^2+\frac{3}{x^3}\right)\ge\left(\frac{x+y+z}{xyz}\right)^3,\qquad\prod_{cyc}(x^8-x^5+3)\ge(x+y+z)^3$$

$$x^8-x^5+3=\left(5\frac{x^8}{5}+\frac{3}{5}\right)-x^5+\frac{18}{5}\overset{AGM}{\ge}8\sqrt[8]{\frac{x^{40}}{5^8}}-x^5+\frac{18}{5}=\frac{3}{5}\left(x^5+4\right)=$$

$$=\frac{3}{5}\left[\left(3\cdot\frac{x^5}{3}+\frac{2}{3}\right)+\frac{10}{3}\right]\ge\frac{3}{5}\left(5\sqrt[5]{\frac{x^{15}}{3^5}}+\frac{10}{3}\right)=\frac{3}{5}\left(\frac{5}{3}x^3+\frac{10}{3}\right)=x^3+2$$

We need t prove that:

$$(x^3+2)(y^3+2)(z^3+2)\ge(x+y+z)^3$$

WLOG: $(y^3-1)(z^3-1)\ge 0\Rightarrow y^3z^3+1\ge y^3+z^3$

We have:

$$(x^3+2)(y^3+2)(z^3+2)\ge(x^3+2)(y^3z^3+1+2(y^3+z^3)+3)\ge$$

$$\ge(x^3+2)[3(y^3+z^3)+3]=3(x^3+1+1)(1+y^3+z^3)\ge$$

$$\ge(1+1+1)(x^3+1+1)(1+y^3+z^3)\ge(x+y+z)^3$$

JP.473 If $a_k>0,k=\overline{1,5}$ then prove that exists $i,j\in\overline{1,5}$ such that:

$$0\le\frac{a_j-a_i}{1+a_ia_j}\le\sqrt{2}-1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$a_i > 0 \Rightarrow (\exists) b_i \in \left(0, \frac{\pi}{2}\right);\ a_i = \tan b_i\,, i = \overline{1,5}$$

$$\left(0, \frac{\pi}{2}\right) = \left(0, \frac{\pi}{8}\right) \cup \left[\frac{\pi}{8}, \frac{2\pi}{8}\right) \cup \left[\frac{2\pi}{8}, \frac{3\pi}{8}\right) \cup \left[\frac{3\pi}{8}, \frac{\pi}{2}\right)$$

4-intervals, 5 points hence exists an interval which contains two values

$$b_i, b_j; b_j \geq b_i \Rightarrow 0 \leq b_j - b_i \leq \frac{\pi}{8}$$

$$\tan 0 \leq \tan(b_i - b_j) \leq \tan \frac{\pi}{8}$$

$$0 \leq \frac{\tan b_j - \tan b_i}{1 + a_i a_j} \leq \sqrt{2} - 1$$

Solution 2 by Mohamed Amine Ben Ajiba- Morocco

$$Let\ a_k = \tan(b_k)\,, where\ b_k \in \left(0, \frac{\pi}{2}\right), k = \overline{1,5}.$$

$$Dividing\ the\ interval\ \left(0, \frac{\pi}{2}\right)\ into\ 4\ disjoint\ subintervals\ as$$
$$: \left(0, \frac{\pi}{8}\right], \left(\frac{\pi}{8}, \frac{\pi}{4}\right], \left(\frac{\pi}{4}, \frac{3\pi}{8}\right], \left(\frac{3\pi}{8}, \frac{\pi}{2}\right).$$

$$By\ Pigeonhole\ Principle, there\ must\ be\ at\ least\ two\ of\ b_k\ that$$

$$are\ in\ the\ same\ interval, say\ b_j \geq b_i.$$

$$Then\ we\ have:\ \ 0 \leq b_j - b_i \leq \frac{\pi}{8}.$$

$$Since:\ \frac{a_j - a_i}{1 + a_i a_j} = \frac{\tan(b_j) - \tan(b_i)}{1 + \tan(b_j)\tan(b_i)} = \tan(b_j - b_i)$$

$$Then\ we\ get:\ 0 \leq \frac{a_j - a_i}{1 + a_i a_j} \leq \tan \frac{\pi}{8} = \sqrt{2} - 1.$$

$$Hence, there\ exists\ i, j \in \overline{1,5}\ such\ that:\ \ 0 \leq \frac{a_j - a_i}{1 + a_i a_j} \leq \sqrt{2} - 1.$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$a_k = \tan \beta_k\,, \beta_k \in \left[0, \frac{\pi}{2}\right]$$

$$x \to \tan x \text{ bijection, } \left[0; \frac{\pi}{2}\right] \to [0, \infty)$$

$$a_k - a_i = \frac{\tan\beta_k - \tan\beta_i}{1 + \tan\beta_k \tan\beta_i} = \tan(\beta_k - \beta_i) \Leftrightarrow \exists(i;j) \text{such that:}$$

$$0 \le \tan(\beta_j - \beta_i) \le \sqrt{2} - 1$$

$$\left[0; \frac{\pi}{2}\right] = \bigcup_{k=0}^{3} \left[\frac{k\pi}{8}; \frac{(k+1)\pi}{8}\right)$$

$$\exists i, j \text{ such that } \beta_i = \beta_j \Leftrightarrow \tan(\beta_i - \beta_j) = 0 \in [0; \sqrt{2} - 1)$$

Suppose that $\forall (i;j) \in [1;5]; \beta_j \ne \beta_i \Rightarrow \exists (i;j); \exists k \in \left[0; \frac{\pi}{2}\right]$ such that

$$\beta_i, \beta_j \in \left[\frac{k\pi}{8}; \frac{(k+1)\pi}{8}\right)$$

$$0 \le |\beta_j - \beta_i| \le \frac{\pi}{8} \text{ and } 0 \le \tan|\beta_j - \beta_i| \le \tan\left(\frac{\pi}{8}\right)$$

$$\tan\frac{\pi}{8} = \frac{2\sin^2\left(\frac{\pi}{8}\right)}{\sin\left(2 \cdot \frac{\pi}{8}\right)} = \frac{1 - \cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2} - 1 \Rightarrow \exists(i;j) \text{ such that:}$$

$$0 \le \frac{\tan\beta_i - \tan\beta_j}{1 + \tan\beta_i \tan\beta_j} = \frac{a_i - a_j}{1 + a_i a_j} \le \sqrt{2} - 1$$

Therefore,

$$0 \le \frac{a_i - a_j}{1 + a_i a_j} \le \sqrt{2} - 1$$

JP.474 If $0 < b \le a$ then:

$$\sqrt{a^2 + ab} + \sqrt{a^2 + \left(\frac{a+b}{2}\right)^2} \le 2a + (\sqrt{2} - 1)\left(\sqrt{ab} + \frac{a+b}{2}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

We will prove that if $a \ge x$ then:

$$\sqrt{a^2 + x^2} \le a + (\sqrt{2} - 1)x; \quad (1)$$

$$a^2 + x^2 \le a^2 + 2ax(\sqrt{2} - 1) + (\sqrt{2} - 1)^2 x^2$$

$$x^2 \le 2ax(\sqrt{2} - 1) + 2x^2 - 2\sqrt{2}x^2 + x^2$$

$$2ax(\sqrt{2}-1)\geq 2x^2(1-\sqrt{2})\Leftrightarrow a\geq x \text{ true!}$$

For $x=\sqrt{ab}\Rightarrow a\geq\sqrt{ab}\Leftrightarrow a\geq b$. Replace in (1):

$$\sqrt{a^2+ab}\leq a+(\sqrt{2}-1)\sqrt{ab};\quad (2)$$

For $x=\frac{a+b}{2}\Rightarrow a\geq\frac{a+b}{2}\Leftrightarrow a\geq b$. Replace in (1):

$$\sqrt{a^2+\left(\frac{a+b}{2}\right)^2}\leq a+(\sqrt{2}-1)\cdot\frac{a+b}{2};\quad (3)$$

By adding (2) and (3):

$$\sqrt{a^2+ab}+\sqrt{a^2+\left(\frac{a+b}{2}\right)^2}\leq 2a+(\sqrt{2}-1)\left(\sqrt{ab}+\frac{a+b}{2}\right)$$

Solution 2 by Mohamed Amine Ben Ajiba- Morocco

By CBS inequality we have :

$$\sqrt{a^2+ab}+\sqrt{a^2+\left(\frac{a+b}{2}\right)^2}\leq\sqrt{(1+1)\left[(a^2+ab)+\left(a^2+\left(\frac{a+b}{2}\right)^2\right)\right]}=\frac{3a+b}{\sqrt{2}}.$$

So it suffices to prove that :

$$\frac{3a+b}{\sqrt{2}}\leq 2a+(\sqrt{2}-1)\left(\sqrt{ab}+\frac{a+b}{2}\right)\ \ or$$

$$\frac{3-2\sqrt{2}}{2}(a-b)+(\sqrt{2}-1)(\sqrt{ab}-b)\geq 0$$

Which is true because $b\leq a$. Equality holds iff $a=b$.

Solution 3 by Hikmat Mammadov-Azerbaijan

We have: $\sqrt{a^2+ab}-a=\frac{ab}{\sqrt{a^2+ab}+a}\leq\frac{ab}{\sqrt{2ab}+\sqrt{ab}}=(\sqrt{2}-1)\sqrt{ab};(1)$

$$\sqrt{a^2+\left(\frac{a+b}{2}\right)^2}-a=\frac{(a+b)^2}{4\left(\sqrt{a^2+\left(\frac{a+b}{2}\right)^2}+a\right)}\leq$$

$$\le \frac{(a+b)^2}{4\left(\sqrt{2}\cdot\frac{a+b}{2}+\frac{a+b}{2}\right)} = (\sqrt{2}-1)\left(\frac{a+b}{2}\right); (2)$$

Hence, $\sqrt{a^2+ab}+\sqrt{a^2+\left(\frac{a+b}{2}\right)^2} \le 2a+(\sqrt{2}-1)\left(\sqrt{ab}+\frac{a+b}{2}\right)$

From (1) and (2), it follows that:

$$\sqrt{a^2+ab}+\sqrt{a^2+\left(\frac{a+b}{2}\right)^2} \le 2a+(\sqrt{2}-1)\left(\sqrt{ab}+\frac{a+b}{2}\right)$$

JP.475 If $x,y,z>0$ such that $x+y+z=3$ and $\lambda\ge 0$ then:

$$\text{(i)}\ \frac{1}{(x+\lambda)^2}+\frac{1}{(y+\lambda)^2}+\frac{1}{(z+\lambda)^2}\ge\frac{3}{(\lambda+1)^2}$$

$$\text{(ii)}\ \frac{x}{(y+\lambda)^2}+\frac{y}{(z+\lambda)^2}+\frac{z}{(x+\lambda)^2}\ge\frac{3}{(\lambda+1)^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Lemma. If $x\ge 0$ then

$$\frac{1}{(x+\lambda)^2}\ge\frac{\lambda+3-2x}{(\lambda+1)^2}$$

Proof. Using Tangent Line Method for the function $f:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{(x+\lambda)^2}$ in $x_0=1$, we have: $f'(x)=\frac{-2}{(x+\lambda)^3}, f'(1)=\frac{-2}{(\lambda+1)^2}$. Equation of the tangent in point $x_0=1$ is

$$y-\frac{1}{(\lambda+1)^2}=\frac{-2}{(\lambda+1)^2}(x-1)\Leftrightarrow y=\frac{-2x}{(x+\lambda)^3}+\frac{\lambda+3}{(x+\lambda)^3}$$

We show that:

$$f(x)=\frac{1}{(x+\lambda)^2}\ge\frac{\lambda+3-2x}{(\lambda+1)^3}\Leftrightarrow 2x^3-(\lambda+3)x^2-6\lambda x+2\lambda^2+3\lambda+1\ge 0\Leftrightarrow$$

$(x-1)^2(2x+3\lambda+1)\ge 0$. Equality holds for $x=1$.

Now, we have:

$$\text{(i)}\ \sum_{cyc}\frac{1}{(x+\lambda)^2}\overset{Lemma}{\ge}\sum_{cyc}\frac{\lambda+3-2x}{(\lambda+1)^3}=\frac{3(\lambda+3)-2\sum x}{(\lambda+1)^3}=\frac{3(\lambda+3)-2\cdot 3}{(\lambda+1)^3}=\frac{3}{(\lambda+1)^2}$$

Equality holds for $x = y = z = 1$.

$$\text{(ii)} \sum_{cyc} \frac{x}{(y+\lambda)^2} \overset{Lemma}{\geq} \sum_{cyc} x \cdot \frac{\lambda + 3 - 2y}{(\lambda+1)^3} = \frac{(\lambda+3)\sum x - \sum xy}{(\lambda+1)^3} \overset{(1)}{\geq}$$

$$\geq \frac{(\lambda+3)\cdot 3 - 3}{(\lambda+1)^3} = \frac{3}{(\lambda+1)^2}$$

$(1) \Leftrightarrow \sum x = 3$ **and** $\sum xy \leq 3 \Leftrightarrow 3\sum x \leq (\sum x)^2$.

Equality holds for $x = y = z = 1$.

Solution 2 by Tapas Das-India

$$\frac{1}{(x+\lambda)^2} + \frac{1}{(y+\lambda)^2} + \frac{1}{(z+\lambda)^2} = \frac{1^3}{(x+\lambda)^2} + \frac{1^3}{(y+\lambda)^2} + \frac{1^3}{(z+\lambda)^2} \overset{Radon}{\geq}$$

$$\geq \frac{(1+1+1)^3}{(x+y+z+3\lambda)^2} = \frac{27}{(3+3\lambda)^2} = \frac{3}{(\lambda+1)^2}$$

$$\frac{x}{(x+\lambda)^2} + \frac{y}{(y+\lambda)^2} + \frac{z}{(z+\lambda)^2} = \frac{x^3}{(xy+\lambda x)^2} + \frac{y^3}{(yz+\lambda y)^2} + \frac{z^3}{(zx+\lambda z)^2} \overset{Radon}{\geq}$$

$$\geq \frac{(x+y+z)^3}{\big(xy+yz+zx+\lambda(x+y+z)\big)^2} \geq \frac{(x+y+z)^3}{\left[\frac{(x+y+z)^2}{3} + \lambda(x+y+z)\right]^2} =$$

$$= \frac{27}{(3+3\lambda)^2} = \frac{3}{(\lambda+1)^2}$$

Solution 3 by Nikos Ntorvas-Greece

We need to show that:

$$\frac{3+\lambda-2t}{(1+\lambda)^3} \leq \frac{1}{(t+\lambda)^2}, \forall t > 0, \lambda \geq 0; (I)$$

$$(I) \Rightarrow 2t^3 + t62\,(3\lambda - 3) - 6\lambda t + 1 + 3\lambda \geq 0$$

$3(t-1)^2(\lambda+1) \geq 0$ **true for all** $t > 0, \lambda \geq 0$

(i) For $x, y, z > 0$, from (I) we have that:

$$\sum_{cyc} \frac{1}{(x+\lambda)^2} \geq \sum_{cyc} \frac{3+\lambda-2x}{(1+\lambda)^3} = \frac{9+3\lambda-2(x+y+z)}{(1+\lambda)^3} = \frac{3(1+\lambda)}{(1+\lambda)^3} = \frac{3}{(1+\lambda)^2}$$

Equality holds iff $x = y = z = 1$.

(ii) For $x, y, z > 0$, from (I) we have that

$$\sum_{cyc}\frac{y}{(x+\lambda)^2}\geq\frac{-\frac{2}{3}(\sum x)^2+(\lambda+3)(\sum x)}{(\lambda+1)^3}$$

$$\sum_{cyc}\frac{y}{(x+\lambda)^2}\geq\frac{-\frac{18}{3}+3(\lambda+3)}{(\lambda+1)^3},\quad \sum_{cyc}\frac{y}{(x+\lambda)^2}\geq\frac{3(\lambda+1)}{(\lambda+1)^3}$$

$$\sum_{cyc}\frac{y}{(x+\lambda)^2}\geq\frac{3}{(\lambda+1)^2}$$

Equality holds iff $x=y=z=1$.

JP.476 In ΔABC the following relationship holds:

$$\frac{a}{(b+\lambda c)^{n+1}s_a^n}+\frac{b}{(c+\lambda a)^{n+1}s_b^n}+\frac{c}{(a+\lambda b)^{n+1}s_c^n}\geq\frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{sR}\right)^n,\lambda\geq 0,n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Using Radon's inequality and Nesbitt's inequality, we get:

$$\sum_{cyc}\frac{a}{(b+\lambda c)^{n+1}s_a^n}=\sum_{cyc}\frac{\frac{a^{n+1}}{(b+c)\lambda^{n+1}}}{a^n s_a^n}=\sum_{cyc}\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(as_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+\lambda c}\right)^{n+1}}{(\sum as_a)^n}\overset{Nesbitt}{\geq}$$

$$\geq\frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{(3Rs)^n}=\frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{sR}\right)^n$$

$$\because \sum as_a\leq\sum am_a\leq 3Rs$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba- Morocco

$$We\ have\ :\ \frac{a}{(b+\lambda c)^{n+1}{s_a}^n}+\frac{b}{(c+\lambda a)^{n+1}{s_a}^n}+\frac{c}{(a+\lambda b)^{n+1}{s_c}^n}$$

$$=\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(as_a)^n}+\frac{\left(\frac{b}{c+\lambda a}\right)^{n+1}}{(bs_b)^n}+\frac{\left(\frac{c}{a+\lambda b}\right)^{n+1}}{(cs_c)^n}\geq$$

$$\overset{Radon}{\geq}\frac{\left(\frac{a}{b+\lambda c}+\frac{b}{c+\lambda a}+\frac{c}{a+\lambda b}\right)^{n+1}}{(as_a+bs_b+cs_c)^n}\overset{CBS\ \&\ s_a\leq m_a}{\geq}$$

$$\geq \frac{\left(\frac{(a+b+c)^2}{a(b+\lambda c)+b(c+\lambda a)+c(a+\lambda b)}\right)^{n+1}}{(am_a+bm_b+cm_c)^n} \geq$$

$$\overset{Panaitopol}{\geq} \frac{\left(\frac{(a+b+c)^2}{(\lambda+1)(ab+bc+ca)}\right)^{n+1}}{\left(a.\frac{Rh_a}{2r}+b.\frac{Rh_b}{2r}+c.\frac{Rh_c}{2r}\right)^n} \geq \frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{(sR+sR+sR)^n}$$

$$= \frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{sR}\right)^n, \text{ as desired. Equality holds iff } \Delta ABC \text{ is equilateral.}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\boxed{\text{Case 1}}\ n = 0 \text{ and then, LHS} = \frac{a}{b+\lambda c}+\frac{b}{c+\lambda a}+\frac{c}{a+\lambda b}$$

$$= \frac{a^2}{ba+\lambda ca}+\frac{b^2}{cb+\lambda ab}+\frac{c^2}{ac+\lambda bc} \overset{Bergstrom}{\geq} \frac{(a+b+c)^2}{ab+bc+ca+\lambda(ab+bc+ca)}$$

$$\geq \frac{3(ab+bc+ca)}{(ab+bc+ca)(\lambda+1)} = \frac{3}{\lambda+1}$$

$$\Rightarrow \frac{a}{(b+\lambda c)^{n+1}s_a^n}+\frac{b}{(c+\lambda a)^{n+1}s_b^n}+\frac{c}{(a+\lambda b)^{n+1}s_c^n} \geq \frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{sR}\right)^n \text{ is true for n}$$

$$= 0; \text{equality iff } \Delta\ ABC \text{ is equilateral}$$

$$\boxed{\text{Case 2}}\ n > 0 \text{ and } n \in \mathbb{N} \text{ and then, } \frac{a}{(b+\lambda c)^{n+1}s_a^n}+\frac{b}{(c+\lambda a)^{n+1}s_b^n}+\frac{c}{(a+\lambda b)^{n+1}s_c^n}$$

$$= \sum_{cyc}\frac{a^{n+1}}{(b+\lambda c)^{n+1}.(a^n s_a^n)} = \sum_{cyc}\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(as_a)^n} \overset{Radon}{\geq} \frac{\left(\sum_{cyc}\frac{a}{b+\lambda c}\right)^{n+1}}{\left(\sum_{cyc}as_a\right)^n}$$

$$= \frac{\left(\sum_{cyc}\frac{a^2}{ba+\lambda ca}\right)^{n+1}}{\left(\sum_{cyc}as_a\right)^n}$$

$$\overset{Bergstrom}{\geq} \frac{\left(\frac{(a+b+c)^2}{ab+bc+ca+\lambda(ab+bc+ca)}\right)^{n+1}}{\left(\sum_{cyc}as_a\right)^n} \geq \frac{\left(\frac{3(ab+bc+ca)}{(ab+bc+ca)(\lambda+1)}\right)^{n+1}}{\left(\sum_{cyc}as_a\right)^n}$$

$$= \frac{3^{n+1}}{(\lambda+1)^{n+1}}.\frac{1}{\left(\sum_{cyc}as_a\right)^n}$$

$$= \frac{3^{n+1}}{(\lambda+1)^{n+1}}.\frac{1}{\left(\sum_{cyc}a.\frac{2bc}{b^2+c^2}.m_a\right)^n} \overset{A-G}{\geq} \frac{3^{n+1}}{(\lambda+1)^{n+1}}.\frac{1}{\left(\sum_{cyc}am_a\right)^n}$$

$$\therefore \boxed{\frac{a}{(b+\lambda c)^{n+1}s_a^n}+\frac{b}{(c+\lambda a)^{n+1}s_b^n}+\frac{c}{(a+\lambda b)^{n+1}s_c^n} \overset{(*)}{\geq} \frac{3^{n+1}}{(\lambda+1)^{n+1}}.\frac{1}{\left(\sum_{cyc}am_a\right)^n}}$$

$$\text{Now,} \sum_{cyc} am_a \overset{CBS}{\le} \sqrt{3}.\sqrt{\sum_{cyc} a^2m_a^2} = \frac{\sqrt{3}}{2}.\sqrt{\sum_{cyc} a^2(2b^2+2c^2-a^2)} = \frac{\sqrt{3}}{2}.\sqrt{4\sum_{cyc} a^2b^2 - \sum_{cyc} a^4}$$

$$= \frac{\sqrt{3}}{2}.\sqrt{\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right) + 2\sum_{cyc} a^2b^2} = \frac{\sqrt{3}}{2}.\sqrt{16r^2s^2 + 2\sum_{cyc} a^2b^2}$$

$$\overset{Goldstone}{\le} \frac{\sqrt{3}}{2}.\sqrt{16r^2s^2 + 8R^2s^2} \overset{Euler}{\le} \frac{\sqrt{3}}{2}.\sqrt{4R^2s^2 + 8R^2s^2} = \frac{\sqrt{3}}{2}.\sqrt{12R^2s^2} = 3Rs$$

$$\therefore \boxed{\sum_{cyc} am_a \overset{(\bullet)}{\le} 3Rs}$$

$$\therefore \text{via } (*), (\bullet), \frac{a}{(b+\lambda c)^{n+1}s_a^n} + \frac{b}{(c+\lambda a)^{n+1}s_b^n} + \frac{c}{(a+\lambda b)^{n+1}s_c^n} \ge \frac{3^{n+1}}{(\lambda+1)^{n+1}}.\frac{1}{(3Rs)^n}$$

$$= \frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{sR}\right)^n \text{; equality iff } \Delta\, ABC \text{ is equilateral}$$

$\therefore$ combining cases 1 and 2, In any $\Delta\, ABC$, $\frac{a}{(b+\lambda c)^{n+1}s_a^n} + \frac{b}{(c+\lambda a)^{n+1}s_b^n} + \frac{c}{(a+\lambda b)^{n+1}s_c^n}$

$$\ge \frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{sR}\right)^n \ \forall \lambda \ge 0 \text{ and } n$$

$\in \mathbb{N}$, with equality iff $\Delta\, ABC$ is equilateral (QED)

JP.477 Let $a > 1, b > 1$ fixed. Solve for real numbers:

$$a^{\log_{2b}\left(x+\frac{b^2}{x}\right)} = \frac{(a+2b)x - b^2 - x^2}{x}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

From conditions of existence $x \in (0, \infty)$. For $x > 0$ we have:

$$x + \frac{b^2}{x} \overset{AGM}{\ge} 2b \Rightarrow \log_{2b}\left(x + \frac{b^2}{x}\right) \ge \log_{2b}(2b) = 1$$

$$a^{\log_{2b}\left(x+\frac{b^2}{x}\right)} \ge a; (1)$$

From (1), the given equation becomes as:

$$\frac{(a+2b)x - b^2 - x^2}{x} \ge a \Leftrightarrow ax + 2bx - b^2 - x^2 \ge ax \Leftrightarrow$$

$$x^2 + b^2 - 2bx \le 0 \Leftrightarrow (x-b)^2 \le 0 \Leftrightarrow x = b.$$

Solution 2 by Florentin Vișescu-Romania

$$\text{Let } x+\frac{b^2}{x}=t \Rightarrow a^{\log_{2b}t}=a+2b-t \Leftrightarrow$$

$$a^{\log_{2b}t}+t=a+2b$$

$$\begin{cases}2b>1\\ a>1\end{cases} \Rightarrow f(t)=a^{\log_{2b}t} \text{ –increasing function and } g(t)=t \text{ increasing function.}$$

So, the function $h(t)=a^{\log_{2b}t}+t$ is strictly increasing, therefore the equation

$h(t)=a+2b$ has maximum two solutions.

$$t=2b$$

$$x+\frac{b^2}{x}=2b \Leftrightarrow x^2-2bx+b^2=0 \Leftrightarrow (x-b)^2=0 \Leftrightarrow x=b.$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$a^{\log_{2b}\left(x+\frac{b^2}{x}\right)}=\frac{(a+2b)x-b^2-x^2}{x}, \qquad a^{\log_{2b}\left(x+\frac{b^2}{x}\right)}=\frac{(a+2b)x}{x}-\frac{b^2+x^2}{x}$$

$$a^{\log_{2b}\left(x+\frac{b^2}{x}\right)}=a+2b-\frac{b^2+x^2}{x}, \text{ Let } \frac{b^2+x^2}{x}=t \Rightarrow a^{\log_{2b}t}=a+2b-t$$

$$a^{\frac{\log t}{\log(2b)}}=a+2b-t; (*)$$

If $t=2b \Rightarrow (*)$ is true, but $2b-t<0 \Rightarrow a+2b-t<a$, so (*) is impossible.

If $t<2b \Rightarrow \log(t)<\log(2b) \Rightarrow \frac{\log(t)}{\log(2b)}<1 \Rightarrow a^{\frac{\log(t)}{\log(2b)}}<a$, but $2b-t>0 \Rightarrow$

$a+2b-t>a^{\log(2b)} \Rightarrow (*)$ impossible. Therefore, $t=2b \Rightarrow S=\{b\}$.

JP.478 Let $m,n\geq 0$ and $ABC, A_1B_1C_1$ triangles with areas F, F_1 respectively, then

$$\frac{a^{m+2}\cdot a_1^{n+1}}{h_a^m}+\frac{b^{m+2}\cdot b_1^{n+1}}{h_b^n}+\frac{c^{m+2}\cdot c_1^{n+1}}{h_c^m}\geq 2^{m+n+1}\cdot\left(\sqrt[4]{3}\right)^{1-2m-n}\cdot F\cdot\left(\sqrt{F_1}\right)^{n+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Constantin Cocea-Romania

Solution 1 by proposers

$$\sum_{cyc}\frac{a^{m+2}\cdot a_1^{n+1}}{h_a^m}=\sum_{cyc}\frac{(a^2)^{m+1}\cdot a_1^{n+1}}{(ah_a)^m}=\frac{1}{2^mF^m}\cdot\sum_{cyc}(a^2)^{m+1}\cdot a_1^{n+1}\geq$$

$$\geq \frac{1}{2^m \cdot F^m} \cdot 3\sqrt[3]{\prod_{cyc}(a^2)^{m+1} \cdot a_1^{n+1}} = \frac{1}{2^m \cdot F^m} \cdot 3\left(\sqrt[3]{a^2b^2c^2}\right)^{m+1}\left(\sqrt[3]{a_1b_1c_1}\right)^{n+1} =$$

$$= \frac{1}{2^m \cdot F^m}\left(\sqrt[3]{a^2b^2c^2}\right)^{m+1}\left(\sqrt[3]{a_1^2b_1^2c_1^2}\right)^{\frac{n+1}{2}} =$$

$$= \frac{3}{2^m \cdot F^m} \cdot \left(3\sqrt[3]{a^2b^2c^2}\right)^{m+1}\left(3\sqrt[3]{a_1^2b_1^2c_1^2}\right)^{\frac{n+1}{2}} \cdot 3^{-m-1} \cdot 3^{-\frac{n+1}{2}} \overset{Carliz}{\geq}$$

$$\geq \left(\sqrt{3}\right)^{2-2m-2-n-1} \cdot \left(4\sqrt{3}F\right)^{m+1} \cdot \left(4\sqrt{3}F_1\right)^{\frac{n+1}{2}} \cdot \frac{1}{2^m \cdot F^m} =$$

$$= 2^{m+n+3} \cdot \left(\sqrt[4]{3}\right)^{n+1-2m-2n} \cdot F \cdot \left(\sqrt{F_1}\right)^{n+1} = 2^{m+n+1} \cdot \left(\sqrt[4]{3}\right)^{1-2m-n} \cdot F \cdot \left(\sqrt{F_1}\right)^{n+1}$$

Solution 2 by Tapas Das-India

$$\frac{a^{m+2} \cdot a_1^{n+1}}{h_a^m} = \frac{a^{m+2} \cdot a_1^{n+1}}{\left(\frac{2F}{a}\right)^m} = \frac{a^{2m+2} \cdot a_1^{n+1}}{(2F)^m}$$

Similarly, we get:

$$\frac{b^{m+2} \cdot b_1^{n+1}}{h_b^n} = \frac{b^{2m+2} \cdot b_1^{n+1}}{(2F)^m} \text{ and } \frac{c^{m+2} \cdot c_1^{n+1}}{h_c^m} = \frac{c^{2m+2} \cdot c_1^{n+1}}{(2F)^m}$$

$$\sum_{cyc} \frac{a^{m+2} \cdot a_1^{n+1}}{h_a^m} = \sum_{cyc} \frac{a^{2m+2} \cdot a_1^{n+1}}{(2F)^m} \overset{AGM}{\geq} 3\sqrt[3]{\prod_{cyc} \frac{a^{2m+2} \cdot a_1^{n+1}}{(2F)^m}} \geq$$

$$\geq \frac{3}{2^m \cdot F^m} \cdot \frac{2^{2m+2} \cdot F^{m+1}}{\left(\sqrt{3}\right)^{m+1}} \cdot \frac{2^{n+1}\left(\sqrt{F_1}\right)^{n+1}}{\left(\sqrt{3}\right)^{\frac{n+1}{2}}} = \frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^{2m+n+3}} \cdot 2^{m+n+1} \cdot F\left(\sqrt{F_1}\right)^{n+1} \cdot 2^2 \geq$$

$$\geq \left(\sqrt[4]{3}\right)^{1-2m-n} \cdot 2^{m+n+1} \cdot F \cdot \left(\sqrt{F_1}\right)^{n+1}$$

JP.479 If $a,b,c,d > 0; ab = cd; a < b, c < d; x,z \in [a,b]$ and $y,t \in [c,d]$, then:

$$ab(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right) \leq (a+b+c+d)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$x\in[a,b]\Rightarrow(x-a)(x-b)\le 0\Rightarrow x^2-(a+b)x+ab\le 0$$

$$x^2+ab\le(a+b)x\Rightarrow x+\frac{ab}{x}\le a+b;(1)$$

Analogous,

$$z+\frac{ab}{z}\le a+b;(2)$$

$$y\in[c,d]\Rightarrow(y-c)(y-d)\le 0\Rightarrow y^2-(c+d)y+cd\le 0$$

$$y^2+cd\le(c+d)y\Rightarrow y+\frac{cd}{y}\le c+d;(3)$$

Analogous,

$$t+\frac{cd}{t}\le c+d;(4)$$

$$x+y+z+t+ab\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\overset{AGM}{\ge}2\sqrt{(x+y+z+t)ab\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)};(5)$$

By adding (1),(2),(3) and (4), it follows that

$$x+y+z+t+ab\left(\frac{1}{x}+\frac{1}{z}\right)+cd\left(\frac{1}{y}+\frac{1}{t}\right)\le 2(a+b+c+d);(6)$$

By (5) and (6):

$$2\sqrt{(x+y+z+t)ab\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)}\le 2(a+b+c+d)$$

$$ab(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\le(a+b+c+d)^2$$

Equality holds for $x=t=z=t;ab=2(a+b)$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}:\ (x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\overset{ab=cd}{\triangleq}$$

$$=\left(\frac{x}{\sqrt{ab}}+\frac{y}{\sqrt{cd}}+\frac{z}{\sqrt{ab}}+\frac{t}{\sqrt{cd}}\right)\left(\frac{\sqrt{ab}}{x}+\frac{\sqrt{cd}}{y}+\frac{\sqrt{ab}}{z}+\frac{\sqrt{cd}}{t}\right)\le$$

$$\overset{AM-GM}{\leq}\frac{1}{4}\left[\left(\frac{x}{\sqrt{ab}}+\frac{y}{\sqrt{cd}}+\frac{z}{\sqrt{ab}}+\frac{t}{\sqrt{cd}}\right)+\left(\frac{\sqrt{ab}}{x}+\frac{\sqrt{cd}}{y}+\frac{\sqrt{ab}}{z}+\frac{\sqrt{cd}}{t}\right)\right]^2=$$

$$=\frac{1}{4}\left[\left(\frac{x}{\sqrt{ab}}+\frac{\sqrt{ab}}{x}\right)+\left(\frac{y}{\sqrt{cd}}+\frac{\sqrt{cd}}{y}\right)+\left(\frac{z}{\sqrt{ab}}+\frac{\sqrt{ab}}{z}\right)+\left(\frac{t}{\sqrt{cd}}+\frac{\sqrt{cd}}{t}\right)\right]^2.$$

$$\textit{Since } x\in[a,b] \textit{ then we have}:$$

$$(x-a)(x-b)\leq 0 \Leftrightarrow x^2+ab\leq x(a+b) \Leftrightarrow \frac{x}{\sqrt{ab}}+\frac{\sqrt{ab}}{x}\leq\frac{a+b}{\sqrt{ab}}$$

$$\textit{Similarly we have}:$$

$$\frac{y}{\sqrt{cd}}+\frac{\sqrt{cd}}{y}\leq\frac{c+d}{\sqrt{cd}},\quad \frac{z}{\sqrt{ab}}+\frac{\sqrt{ab}}{z}\leq\frac{a+b}{\sqrt{ab}},\quad \frac{t}{\sqrt{cd}}+\frac{\sqrt{cd}}{t}\leq\frac{c+d}{\sqrt{cd}}.$$

$$\textit{Then}:$$

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\leq\frac{1}{4}\left(2.\frac{a+b}{\sqrt{ab}}+2.\frac{c+d}{\sqrt{cd}}\right)^2\overset{ab=cd}{\cong}\frac{(a+b+c+d)^2}{ab}.$$

$$\textit{Therefore},\ \ ab(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\leq(a+b+c+d)^2.$$

JP.480 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}\geq 9$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Using the well-known identity:

$$\sum_{cyc}\cot\frac{A}{2}=\prod_{cyc}\cot\frac{A}{2}=\frac{s}{r} \text{ and let } \cot\frac{A}{2}=a;\cot\frac{B}{2}=b;\cot\frac{C}{2}=c, \text{ we get:}$$

If $a,b,c>0, a+b+c=abc$, then

$$\sqrt{a^2+b^2+3}+\sqrt{b^2+c^2+3}+\sqrt{c^2+a^2+3}\geq 9$$

Proof. Let us denote: $x=\frac{b+c}{a}; y=\frac{c+a}{b}; z=\frac{a+b}{c}$, **then** $\sqrt{b^2+c^2+3}=\sqrt{x(y+z)+1}$

$$\sqrt{x(y+z)+1}\overset{AGM}{\geq}\sqrt{x\cdot 2\sqrt{yz}+1}=\sqrt{2\sqrt{x}\cdot\sqrt{xyz}+1}\overset{(1)}{\geq}\sqrt{4\sqrt{2}x+1},\text{ where}$$

$$(1)\Leftrightarrow xyz\geq 8,\text{ which follows from:}$$

$$xyz=\frac{b+c}{a}\cdot\frac{c+a}{b}\cdot\frac{a+b}{c}\overset{Cesaro}{\geq}8.\text{ Therefore, }\sqrt{x(y+z)+1}\geq\sqrt{4\sqrt{2}x+1}$$

$$\sum_{cyc}\sqrt{b^2+c^2+3}\geq\sum_{cyc}\sqrt{4\sqrt{2}x+1}\overset{AGM}{\geq}3\sqrt[3]{\prod_{cyc}\sqrt{4\sqrt{2}x+1}}=$$

$$=3\sqrt[6]{\prod_{cyc}\left(4\sqrt{2}x+1\right)}\overset{(2)}{\geq}3\sqrt[6]{769}=9$$

$$\prod_{cyc}\left(4\sqrt{2}x+1\right)=128\sqrt{2xyz}+32\sum_{cyc}\sqrt{yz}+4\sum_{cyc}\sqrt{2x}+1\overset{xyz\geq 8}{\geq}$$

$$\geq 128\sqrt{2\cdot 8}+32\sum_{cyc}\sqrt{yz}+4\sum_{cyc}\sqrt{2x}+1=$$

$$=512+96\cdot\sqrt[3]{xyz}+4\cdot 3\sqrt[3]{\sqrt{8xyz}}+1\overset{xyz\geq 8}{\geq}$$

$$\geq 512+96\cdot\sqrt[3]{8}+4\cdot 3\cdot\sqrt[3]{\sqrt{8\cdot 8}}+1=729$$

Equality holds for $x=y=z=2\Leftrightarrow a=b=c=\sqrt{3}$.

Solution 2 by Alex Szoros-Romania

Let us denote: $\cot\frac{A}{2}=x,\cot\frac{B}{2}=y,\cot\frac{C}{2}=z$, where $x,y,z>0$. Thus,

$$\sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}=\sum_{cyc}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}\cdot\frac{(s-a)(s-b)}{s(s-c)}}=\sum_{cyc}\frac{s-a}{s}=1$$

$$x+y+z=\sum_{cyc}\cot\frac{A}{2}=\sum_{cyc}\frac{1}{\tan\frac{A}{2}}=\frac{\sum\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}=$$

$$=\frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=xyz$$

Hence,

$$x+y+z=xyz\overset{not.}{=}\lambda$$

$$\frac{x+y+z}{3}\ge\sqrt[3]{xyz}\Rightarrow\frac{\lambda}{3}\ge\sqrt[3]{\lambda}\Rightarrow\lambda^3\ge 27\lambda\Rightarrow\lambda^2\ge 27\Rightarrow\lambda\ge 3\sqrt{3};\ (*)$$

On the other hand,

$$\sqrt{\frac{x^2+y^2+\left(\sqrt{3}\right)^2}{3}}\ge\frac{x+y+\sqrt{3}}{3}\Rightarrow\sqrt{x^2+y^2+3}\ge\frac{1}{\sqrt{3}}\left(x+y+\sqrt{3}\right)$$

$$\sum_{cyc}\sqrt{x^2+y^2+3}\ge\frac{1}{3}\sum_{cyc}\left(x+y+\sqrt{3}\right)=\frac{2(x+y+z)+3\sqrt{3}}{\sqrt{3}}=\frac{2\lambda+3\sqrt{3}}{\sqrt{3}}$$

$$\sum_{cyc}\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}\overset{(*)}{\ge}\frac{6\sqrt{3}+3\sqrt{3}}{\sqrt{3}}=9$$

Solution 3 by Tapas Das-India

$$\cot^2\frac{A}{2}+\cot^2\frac{B}{2}\ge\left(\frac{\cot\frac{A}{2}+\cot\frac{B}{2}}{\sqrt{2}}\right)^2$$

$$\sum_{cyc}\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}\ge\sum_{cyc}\sqrt{\left(\frac{\cot\frac{A}{2}+\cot\frac{B}{2}}{\sqrt{2}}\right)^2+3}\ge$$

$$\ge\sqrt{\left(\frac{2}{\sqrt{2}}\sum_{cyc}\cot\frac{A}{2}\right)^2+\left(3\sqrt{3}\right)^2}=\sqrt{2\left(\sum_{cyc}\cot\frac{A}{2}\right)^2+27}\ge$$

$$\ge\sqrt{2\cdot\left(3\sqrt{3}\right)^2+27}=\sqrt{81}=9$$

$$\because\sum_{cyc}\cot\frac{A}{2}=\sum_{cyc}\frac{1}{\tan\frac{A}{2}}=\frac{\sum\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}=\frac{1}{\frac{r}{s}}=\frac{s}{r}\overset{Mitrinovic}{\ge}\frac{3\sqrt{3}r}{r}=3\sqrt{3}$$

Solution 4 by Sanong Huayrerai- Thailand

$$\sum_{cyc}\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}\ge\sqrt{3\left(\sum_{cyc}\cot\frac{A}{2}\right)^2+3\cdot(1+1+1)^3}\ge 9$$

$$3\left(\sum_{cyc}\cot\frac{A}{2}\right)^2+3\cdot 3^2\geq 81,\quad 8\left(\sum_{cyc}\cot\frac{A}{2}\right)^2\geq 54\Leftrightarrow\left(\sum_{cyc}\cot\frac{A}{2}\right)^2\geq 27\Leftrightarrow$$

$$\sum_{cyc}\cot\frac{A}{2}\geq 3\sqrt{3}\Leftrightarrow 3\cot\left(\frac{\sum\frac{A}{2}}{3}\right)\geq 3\sqrt{3},\text{because}$$

$$f(x)=\cot x\Rightarrow f'(x)=-\frac{1}{\sin^2 x}, f''(x)=\frac{2\cos x}{\sin x}\Rightarrow f\ \text{–convex function on}\ \left[0,\frac{\pi}{2}\right]$$

$$\sum_{cyc}\cot\frac{A}{2}\geq 3\cot\left(\frac{\sum\frac{A}{2}}{3}\right)$$

PROBLEMS FOR SENIORS

SP.466 Let $A,B\in M_4(\mathbb{R})$. If $AB+BA=O_4$ then $\det(A^4+A^2+B^2)\geq 0$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\det\left[\left(A^2+i(A+B)\right)\left(A^2-i(A+B)\right)\right]\geq 0;(1)$$

$$\det\left[\left(A^2+i(A+B)\right)\left(A^2-i(A+B)\right)\right]=$$

$$=\det(A^4-iA^3-iA^2B+iA^3+iBA^2+(A+B)^2)=$$

$$=\det(A^4-iA^2B+iBA^2+(A+B)^2);(2)$$

$$(A+B)^2=(A+B)(A+B)=A^2+AB+BA+B^2=A^2+B^2;(3)$$

$$AB=-BA\Rightarrow A^2B=-ABA\Rightarrow$$

$$-i(A^2B-BA^2)=-i(ABA-BA^2)=i(AB+BA)A=O_4;(4)$$

From (2),(3) and (4) we get

$$\det(A^4+A^2+B^2)=\det\left[\left(A^2+i(A+B)\right)\left(A^2-i(A+B)\right)\right];(5)$$

From (1) and (5) it follows: $\det(A^4+A^2+B^2)\geq 0$.

Solution 2 by Ravi Prakash- India

We first show that, if $C,D\in M_4(\mathbb{R})$ and $CD=DC$, then: $\det(C^2+D^2)\geq 0$

We have:

$$(C+iD)(C-iD)=C^2-iCD+iDC+D^2=C^2+D^2-i(CD-DC)=C^2+D^2$$

$$\det(C^2+D^2)=\det(C+iD)(C-iD)=\det(C+iD)\cdot\det(C-iD)=$$
$$=\det(C+iD)\cdot\overline{\det(C+\imath D)}=|\det(C+iD)|^2$$

Now,

$$A^4+A^2+B^2=A^4+A^2+B^2+AB+BA=A^4+(A+B)^2=C^2+D^2$$

where $C=A^2$ and $D=A+B$

Note that:

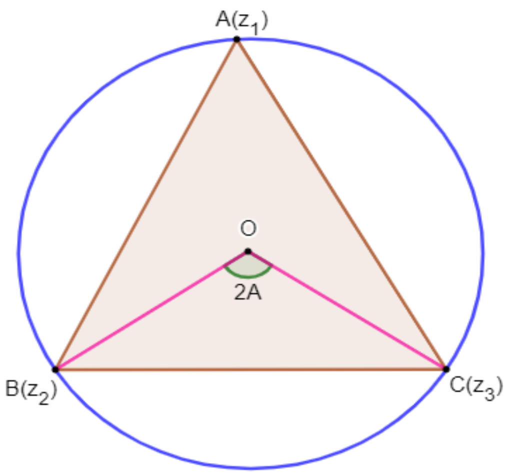
$$CD=A^2(A+B)=A^3+A(AB)=A^3-(AB)A=A^3—BAA=$$
$$=A^3+BA^2=(A+B)A^2=DC$$

Thus, by what we have proved above, we get: $\det(A^4+A^2+B^2)=\det(C^2+D^2)\geq 0$

SP. 467 Let $z_1,z_2,z_3\in\mathbb{C}^*, A(z_1),B(z_2),C(z_3)$ different in pairs such that

$|z_1|=|z_2|=|z_3|$ If

$$\left|\frac{z_1+z_2}{z_1-z_2}\right|^2+\left|\frac{z_2+z_3}{z_2-z_3}\right|^2+\left|\frac{z_3+z_1}{z_3-z_1}\right|^2=1\Rightarrow AB=BC=CA.$$



Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\left|\frac{z_1+z_2}{z_1-z_2}\right|^2=\frac{|z_1+z_2+z_3-z_3|^2}{|z_1-z_2|^2}=\frac{|z_H-z_C|^2}{|z_1-z_2|^2}=\frac{HC}{AB}=\frac{HC}{c}$$

Similarly,

$$\left|\frac{z_2+z_3}{z_2-z_3}\right|^2=\frac{HA}{a},\qquad \left|\frac{z_3+z_1}{z_3-z_1}\right|^2=\frac{HB}{b}$$

$$\frac{HA^2}{a^2}+\frac{HB^2}{b^2}+\frac{HC^2}{c^2}=1;(1)$$

$$HA^2=4R^2-a^2;(2)$$

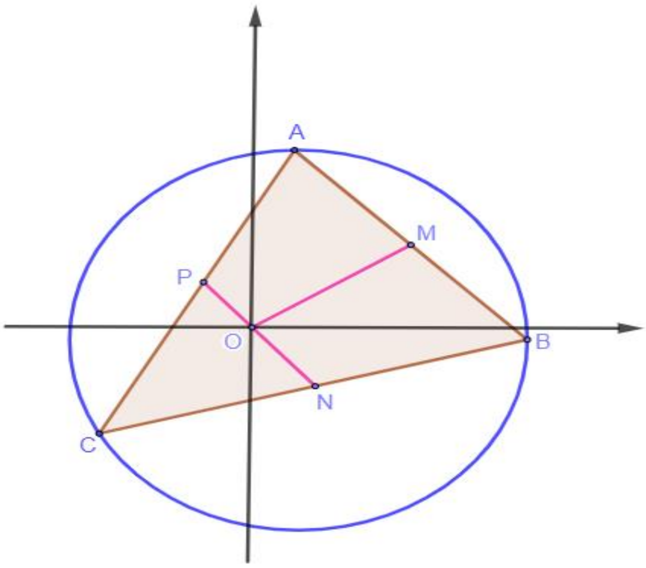
From (1) and (2) we get

$$\sum_{cyc}\frac{4R^2-a^2}{a^2}=1\Leftrightarrow 4R^2\sum_{cyc}\frac{1}{a^2}-3=1\Leftrightarrow \sum_{cyc}\frac{1}{a^2}=\frac{1}{R^2};(3)$$

$$\text{But }\sum_{cyc}\frac{1}{a^2}\ge\frac{1}{2Rr}\ge\frac{1}{R^2}(Steining);(4)$$

From (3) and (4) ΔABC is equilateral.

Solution 2 by Geanina Tudose-Romania



If $M,N,P-$ be midpoints of AB,BC,CA, then:

$$PM=\frac{|z_1+z_2|}{2};ON=\frac{|z_2+z_3|}{2};OP=\frac{|z_1+z_3|}{2}$$

The equality can be written:

$$\sum_{cyc}\frac{4OM^2}{AB^2}=1\Leftrightarrow\sum_{cyc}\frac{2(R^2+R^2)-AD^2}{AB^2}=1\Leftrightarrow$$

$$\sum_{cyc}\frac{4R^2}{AB^2}-3=1\Leftrightarrow\sum_{cyc}\frac{1}{AB^2}=\frac{1}{R^2}\Leftrightarrow\sum_{cyc}\frac{1}{4R^2\sin^2 A}=\frac{1}{R^2}\Leftrightarrow\sum_{cyc}\frac{1}{\sin^2 A}=4$$

Let be $f:(0,\pi)\to\mathbb{R}, f(x)=\frac{1}{\sin^2 x}; f'(x)=-\frac{2\cos x}{\sin^3 x}; f''(x)=2\cdot\frac{\sin^2 x+3\cos^2 x}{\sin 4x}>0\Rightarrow$

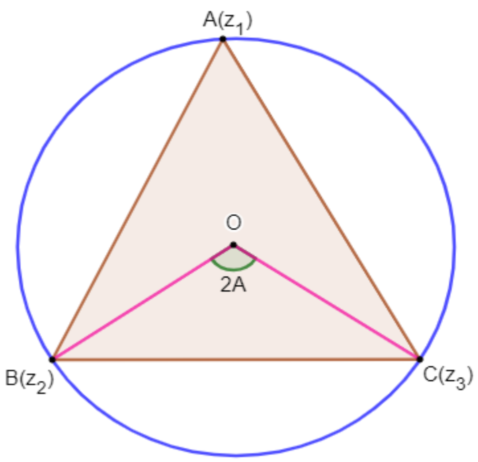
f –convex function and from Jensen's inequality:

$$\frac{f(A)+f(B)+f(C)}{3}\ge f\left(\frac{A+B+C}{3}\right)$$

$$\sum_{cyc}\frac{1}{\sin^2 A}=4$$

Equality holds for $A=B=C=\frac{\pi}{3}\Leftrightarrow AB=BC=CA.$

Solution 3 by Ravi Prakash- India



$$\frac{z_3}{z_2}=\frac{|z_3|}{|z_2|}e^{2iA}=e^{2iA}\Rightarrow\frac{\left|1+\frac{z_3}{z_2}\right|}{\left|1-\frac{z_3}{z_2}\right|}=\left|\frac{1+\cos(2A)+i\sin(2A)}{1-\cos(2A)-i\sin(2A)}\right|\Rightarrow$$

$$\frac{|z_2+z_3|^2}{|z_2-z_3|^2}=\frac{(1-\cos(2A))^2+\sin^2(2A)}{(1-\cos(2A))^2+\sin^2(2A)}=\frac{1-\cos(2A)}{1-\cos(2A)}=\cot^2 A$$

The given equation becomes: $\cot^2 A + \cot^2 B + \cot^2 C = 1$

Let $f(x) = \cot^2 x, x \in \left(0, \frac{\pi}{2}\right), f'(x) = -2\cot x \csc^2 x,$

$f''(x) = 2\csc^4 x + 4\csc^2 x \cot^2 x > 0$. By Jensen's inequality, we have:

$$f(A) + f(B) + f(C) \geq 3f\left(\frac{A+B+C}{3}\right)$$

$$\cot^2 A + \cot^2 B + \cot^2 C \geq 3\cot^2\frac{\pi}{3} = 1$$

Equality holds when $A = B = C \Leftrightarrow AB = BC = CA$.

Solution 4 by Ruxandra Daniela Tonilă-Romania

$$\left|\frac{z_1+z_2}{z_1-z_2}\right|^2 + \left|\frac{z_2+z_3}{z_2-z_3}\right|^2 + \left|\frac{z_3+z_1}{z_3-z_1}\right|^2 =$$

$$= \left|\frac{z_1+z_2+z_3-z_3}{z_1-z_2}\right|^2 + \left|\frac{z_2+z_3+z_1-z_1}{z_2-z_3}\right|^2 + \left|\frac{z_3+z_1+z_2-z_2}{z_3-z_1}\right|^2 =$$

$$= \frac{|z_H-z_3|^2}{|z_1-z_2|^2} + \frac{|z_H-z_1|^2}{|z_2-z_3|^2} + \frac{|z_H-z_2|^2}{|z_3-z_1|^2} = \left(\frac{CH}{AB}\right)^2 + \left(\frac{AH}{BC}\right)^2 + \left(\frac{BH}{AC}\right)^2 =$$

$$= \sum_{cyc}\left(\frac{2R\cos A}{2R\sin A}\right)^2 = \sum_{cyc}\left(\frac{\cos^2 A}{\sin^2 A} + 1\right) - 3 = \sum_{cyc}\frac{1}{\sin^2 A} - 3 \overset{Radon}{\geq}$$

$$\geq \frac{27}{(\sum \sin A)^2} - 3; \quad (1)$$

$$\sum_{cyc}\frac{1}{\sin^2 A} = \sum_{cyc}\frac{a}{2R} = \frac{s}{R} \overset{Mitrinovic}{\leq} \frac{3\sqrt{3}}{2}; (2)$$

From (1) and (2), it follows: $\sum_{cyc}\frac{\cos^2 A}{\sin^2 A} \geq \frac{27\cdot 4}{27} - 3 = 1$. Equality holds for $a = b = c$.

SP.468 In ΔABC the following relationship holds:

$$\frac{3\sqrt{3}}{2}k \leq \sum_{cyc}\frac{\sin^2 A}{\sin B + \sin C} \leq \frac{\sqrt{6}}{12}\left(\frac{4}{k} + 1\right)\sqrt{1-k}, \qquad k \in \left(0, \frac{1}{2}\right]$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

Let $AB = c, BC = a, AB = c$ be the lengths of the sides of ΔABC and let R, r be the circumradius, inradius respectively of ΔABC. We have:

$$\frac{1}{b+c} \le \frac{1}{4}\left(\frac{1}{b}+\frac{1}{c}\right) \Rightarrow \frac{a^2}{b+c} \le \frac{1}{4}\left(\frac{a^2}{b}+\frac{a^2}{c}\right)$$

Similarly we have:

$$\frac{b^2}{c+a} \le \frac{1}{4}\left(\frac{b^2}{c}+\frac{b^2}{a}\right) \text{ and } \frac{c^2}{a+b} \le \frac{1}{4}\left(\frac{c^2}{a}+\frac{c^2}{b}\right)$$

Adding up these inequalities, we get:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \le \frac{1}{4}\left(\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c}\right)$$

Now, using the Cauchy-Schwarz inequality, we have:

$$((b^2+c^2)^2+(c^2+a^2)^2+(a^2+b^2)^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

$$\ge \left(\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c}\right)^2$$

$$\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c} \le$$

$$\le \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} \cdot \sqrt{2(a^4+b^4+c^4)+2(a^2b^2+b^2c^2+c^2a^2)}$$

$$\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c} \le$$

$$\le \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} \cdot \sqrt{2(a^4+b^4+c^4)+2(a^4+b^4+c^4)}$$

$$\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c} \le \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} \cdot 2\sqrt{a^4+b^4+c^4}$$

Now, we we'll prove that:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \le \frac{1}{4r^2}$$

We have: $(b-c)^2 \geq 0 \Leftrightarrow a^2-(b-c)^2 \leq a^2 \Leftrightarrow \frac{1}{a^2} \leq \frac{1}{a^2-(b-c)^2}$ or

$$\frac{1}{a^2} \leq \frac{1}{(a-b+c)(a+b-c)}$$

Let $2s = a+b+c$ is the perimeter of ΔABC, then

$$\frac{1}{a^2} \leq \frac{1}{2(2-b)2(s-c)}$$

Similarly, we have:

$$\frac{1}{b^2} \leq \frac{1}{4(s-c)(s-a)} \text{ and } \frac{1}{c^2} \leq \frac{1}{4(s-a)(s-b)}$$

So, we get:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \leq \frac{1}{4}\left(\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}\right)=$$

$$=\frac{1}{4}\cdot\frac{s-a+s-b+s-c}{(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{(3s-2s)s}{s(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{s^2}{(rs)^2}=\frac{1}{4r^2}$$

Because $[ABC] = rs = \sqrt{s(s-a)(s-b)(s-c)}$ $(Heron)$. So,

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \leq \frac{1}{4r^2}$$

Also, we'll prove that:

$$a^4+b^4+c^4 \leq \frac{8}{3}R(R-r)(4R+r)^2$$

We know that

$$a^4+b^4+c^4 = 2(s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2)$$

We must to prove

$$s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2 \leq \frac{4}{3}R(R-r)(4R+r)^2 \Leftrightarrow$$

$$s^2(s^2-8Rr-6r^2) \leq \frac{(4R+r)^2}{3}(4R(R-r)-3r^2)$$

$$3s^2(s^2-8Rr-6r^2) \leq (4R+r)^2(4R^2-4Rr-3r^2); (1)$$

But $3s^2 \leq (4R+r)^2; (2)(Doucet)$. From (1) and (2) we must to prove that

$$s^2-8Rr-6r^2 \leq 4R^2-4Rr-3r^2 \Leftrightarrow s^2 \leq 4R^2+4Rr+3r^2 (Gerretsen)$$

So, $a^4+b^4+c^4 \leq \frac{8}{3}R(R-r)(4R+r)^2$.

Namely,

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{1}{4}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\cdot 2\sqrt{a^4+b^4+c^4}\le$$

$$\le\frac{1}{2}\sqrt{\frac{1}{4r^2}}\cdot\sqrt{\frac{8}{3}R(R-r)(4R+r)^2}=\frac{1}{4r}\cdot\frac{2\sqrt{2}}{\sqrt{3}}(4R+r)\sqrt{R(R-r)}=$$

$$=\frac{\sqrt{6}}{6}\cdot\frac{4R+r}{r}\sqrt{R(R-r)}$$

Also, we have:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge\frac{(a+b+c)^2}{2(a+b+c)}=\frac{a+b+c}{2}=s\ge 3\sqrt{3}r$$

Hence,

$$3\sqrt{3}r\le\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{\sqrt{6}}{6}\cdot\frac{4R+r}{r}\sqrt{R(R-r)}$$

Using the Law of sins in ΔABC, we get:

$$3\sqrt{3}r\le\sum_{cyc}\frac{\sin^2 A}{\sin B+\sin C}\le\frac{\sqrt{6}}{6}\cdot\frac{4R+r}{2Rr}\sqrt{R(R-r)}$$

$$3\sqrt{3}r\le\sum_{cyc}\frac{\sin^2 A}{\sin B+\sin C}\le\frac{\sqrt{6}}{12}\cdot\left(4\frac{R}{r}+1\right)\sqrt{1-\frac{r}{R}}$$

Putting $\frac{r}{R}=k\le 2$, we have:

$$\frac{3\sqrt{3}}{2}k\le\sum_{cyc}\frac{\sin^2 A}{\sin B+\sin C}\le\frac{\sqrt{6}}{12}\left(\frac{4}{k}+1\right)\sqrt{1-k}$$

Equality holds for $k=\frac{1}{2}$ and the ΔABC is an equilateral triangle.

SP.469 In ΔABC the following relationship holds:

$$\frac{y+z}{x\cdot w_a^4}+\frac{z+x}{y\cdot w_b^4}+\frac{x+y}{z\cdot w_c^4}\ge\frac{32}{27R^4},\qquad x,y,z>0$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Using $m_a \geq w_a$ we get: $\sum_{cyc} \frac{y+z}{x \cdot w_a^4} \geq \sum_{cyc} \frac{y+z}{x \cdot m_a^4} \geq \frac{32}{27R^4}$

Remains to prove that: $\sum_{cyc} \frac{y+z}{x \cdot m_a^4} \geq \frac{32}{27R^4}$

Lemma. For $x, y, z > 0$ and $f: D \to \mathbb{R}$ positive function, holds:

$$\sum_{cyc} \frac{y+z}{x} f(a) \geq 2 \sum_{cyc} \sqrt{f(b)f(c)}$$

Proof. We have:

$$\sum_{cyc} \frac{y+z}{x} f(a) = \sum_{cyc} \left(\frac{y+z}{x} + 1 - 1\right) f(a) = \sum_{cyc} \frac{x+y+z}{x} f(a) - \sum_{cyc} f(a) \overset{CBS}{\geq}$$

$$\geq (x+y+z) \frac{\left(\sum \sqrt{f(a)}\right)^2}{x+y+z} - \sum_{cyc} f(a) = \left(\sum_{cyc} \sqrt{f(a)}\right)^2 - \sum_{cyc} f(a) =$$

$$= \sum_{cyc} f(a) + 2 \sum_{cyc} \sqrt{f(b)f(c)} - \sum_{cyc} f(a) = 2 \sum_{cyc} \sqrt{f(b)f(c)}$$

Equality holds if and only if triangle is equilateral.Using Lemma for $f(a) = \frac{1}{m_a^4}$, we get:

$$\sum_{cyc} \frac{y+z}{x \cdot m_a^4} \overset{Lemma}{\geq} 2 \sum_{cyc} \sqrt{\frac{1}{m_b^4} \cdot \frac{1}{m_c^4}} = 2 \sum_{cyc} \frac{1}{m_b^2 m_c^2} \overset{(1)}{\geq} \frac{32}{27R^4}$$

$$(1) \Leftrightarrow 2 \sum_{cyc} \frac{1}{m_b^2 m_c^2} \geq \frac{32}{27R^4} \Leftrightarrow \sum_{cyc} \frac{1}{m_b^2 m_c^2} \geq \frac{16}{27R^4}$$

Which follows from:

$$\sum_{cyc} \frac{1}{m_b^2 m_c^2} = \frac{1}{m_a^2 m_b^2 m_c^2} \cdot \sum_{cyc} m_a^2 = \frac{1}{m_a^2 m_b^2 m_c^2} \cdot \frac{3}{4} \sum_{cyc} a^2 =$$

$$= \frac{1}{m_a^2 m_b^2 m_c^2} \cdot \frac{3}{4} \cdot 2(s^2 - r^2 - 4Rr) = \frac{3(s^2 - r^2 - 4Rr)}{2 m_a^2 m_b^2 m_c^2}$$

Using $m_a m_b m_c \leq \frac{Rs^2}{2}$, we get:

$$\sum_{cyc} \frac{1}{m_b^2 m_c^2} = \frac{3(s^2 - r^2 - 4Rr)}{2 m_a^2 m_b^2 m_c^2} \geq \frac{3(s^2 - r^2 - 4Rr)}{2\left(\frac{Rs^2}{2}\right)^2} = \frac{6(s^2 - r^2 - 4Rr)}{R^2 s^4}$$

Remains to prove that:

$$\frac{6(s^2-r^2-4Rr)}{R^2s^4}\ge\frac{16}{27R^4}\Leftrightarrow s^2(81R^2-8s^2)\ge 81R^2r(4R+r)$$

which follows from $16Rr-5r^2\le s^2\le 4R62+4Rr+3r^2(Gerretsen)$

Remains to prove:

$$(16Rr-5r^2)[81R^2-8(4R^2+4Rr+3r^2)]\ge 81R^2r(4R+r)\Leftrightarrow$$

$$230R^3-419R^2r-112Rr^2+60r^3\ge 0\Leftrightarrow(R-2r)(230R^2+41Rr-30r^2)\ge 0$$

which is true from $R\ge 2r(Euler)$**. Equality holds if and only if triangle is equilateral.**

Solution 2 by Myagmarsuren Yadamsuren- Mongolia

$$\sum_{cyc}\left(\frac{y+z}{xw_a^4}+\frac{1}{w_a^4}\right)-\sum_{cyc}\frac{1}{w_a^4}=\sum_{cyc}\frac{x+y+z}{xw_a^4}-\sum_{cyc}\frac{1}{w_a^4}=$$

$$=(x+y+z)\cdot\frac{\left(\sum\frac{1}{w_a^2}\right)^2}{x}-\sum_{cyc}\frac{1}{w_a^4}\overset{CBS}{\ge}(x+y+z)\cdot\frac{\left(\sum\frac{1}{w_a^2}\right)^2}{x+y+z}-\sum_{cyc}\frac{1}{w_a^4}=$$

$$=\left(\sum_{cyc}\frac{1}{w_a^2}\right)^2-\sum_{cyc}\frac{1}{w_a^4}=2\sum_{cyc}\frac{1}{w_a^2w_b^2}\overset{w_a^2\le s(s-a)}{\ge}$$

$$\ge 2\sum_{cyc}\frac{1}{s^2(s-a)(s-b)}=\frac{2}{s^2}\cdot\frac{s^2}{s(s-a)(s-b)(s-c)}=\frac{2}{F^2}=$$

$$=\frac{2}{s^2r^2}\ge\frac{2}{\frac{27}{4}R^2\cdot\frac{R^2}{4}}=\frac{32}{27}\cdot\frac{1}{R^4}$$

SP.470 In $\Delta ABC, o_a$ –circumcevian, holds:

$$\frac{6r}{R}\le\frac{r_a}{o_a}+\frac{r_b}{o_b}+\frac{r_c}{o_c}\le\frac{2R}{r}-1$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. In ΔABC the following relationship holds:

$$o_a=\frac{h_a}{\cos(B-C)}$$

Proof.

Let $D\in(BC),\{D\}=AO\cap BC$. We have: $AD=o_a$. Let $OE\perp AC, E$ –middle point of $[AC]$.

In $\Delta AOE,\mu(E)=\frac{\pi}{2}$, we have $\mu(\widehat{AOE})=B$ and $\mu(\widehat{OAE})=\frac{\pi}{2}-B\Rightarrow\mu(\widehat{DAC})=\frac{\pi}{2}-B$.

The angle ADB is exterior to ΔADC, so

$$\mu(\widehat{ADB})=\mu(\widehat{DAC})+\mu(\widehat{ACD})=\frac{\pi}{2}-(B-C)$$

Applying Law of sines in ΔABD:

$$\frac{AB}{\sin(DAB)}=\frac{AD}{\sin B}\Leftrightarrow\frac{c}{\sin(B-C)}=\frac{o_a}{\sin B}\Leftrightarrow o_a=\frac{c\sin B}{\cos(B-C)}$$

$$h_a=c\sin B\Rightarrow o_a=\frac{h_a}{\cos(B-C)}$$

Using Lemma, we get:

$$\sum_{cyc}\frac{r_a}{o_a}=\frac{1}{2}\sum_{cyc}\frac{a}{s-a}\cos(B-C)=\sum_{cyc}\frac{r_a}{h_a}\cos(B-C)=\frac{1}{2}\sum_{cyc}\frac{a}{s-a}\cos(B-C)$$

$$\sum_{cyc}\frac{r_a}{o_a}=\frac{1}{2}\sum_{cyc}\frac{a}{s-a}\cos(B-C)\le\frac{1}{2}\sum_{cyc}\frac{a}{s-a}=\frac{1}{2}\cdot\frac{2(2R-r)}{r}=\frac{2R}{r}-1$$

Equality holds if and only if triangle is equilateral.

$$\sum_{cyc}\frac{r_a}{o_a}=\frac{1}{2}\sum_{cyc}\frac{a}{s-a}\cos(B-C)\overset{AGM}{\ge}\frac{1}{2}\cdot3\sqrt[3]{\prod_{cyc}\frac{a}{s-a}\prod_{cyc}\cos(B-C)}=$$

$$=\frac{1}{2}\cdot3\sqrt[3]{\frac{4Rrs}{r^2s}\cdot\frac{s^2(s^2+2r^2-8Rr-6R^2)+8R^4+24R^3r+22R^2r^2+8Rr^3+r^4}{8R^4}}=$$

$$=\frac{1}{2}\cdot3\sqrt[3]{\frac{s^2(s^2+2r^2-8Rr-6R^2)+8R^4+24R^3r+22R^2r^2+8Rr^3+r^4}{2R^3r}}=$$

$$=\frac{1}{2R}\cdot3\sqrt[3]{\frac{s^2(s^2+2r^2-8Rr-6R^2)+8R^4+24R^3r+22R^2r^2+8Rr^3+r^4}{2r}}\overset{Gerrtesen}{\ge}$$

$$\geq \frac{3}{2R}\sqrt[3]{\frac{(16Rr-5r^2)(16Rr-5r^2+2r^2-8Rr-6R^2)+8R^4+24R^3r+22R^2r^2+8Rr^3+r^4}{2r}}$$

$$=\frac{3}{2R}\sqrt[3]{\frac{(16Rr-5r^2)(8Rr-3r^2-6R^2)+8R^4+24R^3r+22R^2r^2+8Rr^3+r^4}{2r}}=$$

$$=\frac{3}{2R}\sqrt[3]{\frac{8R^4-72R^3r+180R^2r^2-80Rr^3+16r^4}{2r}}=$$

$$=\frac{3}{2R}\sqrt[3]{\frac{4R^4-36R^3r+90R^2r^2-40Rr^3+8r^4}{r}}\overset{Euler}{\geq}$$

$$\geq \frac{3}{2R}\sqrt[3]{\frac{64r^4}{r}}=\frac{3}{2R}\sqrt[3]{64r^3}=\frac{6r}{R}$$

$$\because \prod_{cyc}\cos(B-C)=\frac{s^2(s^2+2r^2-8Rr-6R^2)+8R^4+24R^3r+22R^2r^2+8Rr^3+r^4}{8R^4}$$

Equality holds if and only if triangle is equilateral.

SP.471 In ΔABC the following relationship holds:

$$\frac{3}{2}\cdot\sqrt[6]{\frac{4r^5}{R^2}}\leq \sum_{cyc}\sqrt{m_a\cos\frac{B}{2}\cos\frac{C}{2}}\leq \frac{4R+r}{\sqrt{2R}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

$$\sum_{cyc}\sqrt{m_a\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AGM}{\geq} 3\sqrt[6]{\prod_{cyc}\cos^2\frac{A}{2}\prod_{cyc}m_a}\geq 3\sqrt[6]{\left(\frac{s}{4R}\right)\cdot 27r^3}\geq \frac{9}{2}\sqrt[6]{\frac{4r^5}{R^2}}$$

Equality holds if and only if triangle is equilateral.

$$\left(\sum_{cyc}\sqrt{m_a\cos\frac{B}{2}\cos\frac{C}{2}}\right)^2\overset{CBS}{\leq}\sum_{cyc}\cos\frac{B}{2}\cos\frac{C}{2}\sum_{cyc}m_a\leq \sum_{cyc}\cos^2\frac{A}{2}(4R+r)=$$

$$=\frac{4R+r}{2R}(4R+r)=\frac{(4R+r)^2}{2R}$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Myagmarsuren Yadamsuren- Mongolia

$$(*): \sum_{cyc} m_a \le 4R + r, \sum_{cyc} \cos\frac{A}{2}\cos\frac{B}{2} \le 2 + \frac{r}{2R}$$

$$\sum_{cyc}\sqrt{m_a\cdot\cos\frac{B}{2}\cos\frac{C}{2}} \overset{CBS}{\le} \sqrt{\sum_{cyc} m_a\cdot\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}} \overset{(*)}{\le}$$

$$\le \sqrt{(4R+r)\left(2+\frac{r}{2R}\right)} = \frac{4R+r}{\sqrt{2R}}, \quad (**): m_a \ge \frac{b+c}{a}\cos\frac{A}{2}$$

$$\sum_{cyc}\sqrt{m_a\cdot\cos\frac{B}{2}\cos\frac{C}{2}} \overset{AGM}{\ge} 3\sqrt[3]{\prod_{cyc} m_a\cdot\left(\prod_{cyc}\cos\frac{A}{2}\right)^2} \overset{(**)}{\ge}$$

$$\ge 3\sqrt[6]{\frac{\prod(b+c)}{8}\left(\prod_{cyc}\cos\frac{A}{2}\right)^3} \overset{AGM}{\ge} 3\sqrt[6]{abc\left(\frac{s}{4R}\right)^3} =$$

$$= 3\sqrt[3]{\frac{4s^4r}{4^3R^2}} = \frac{3}{2}\sqrt[3]{\frac{4s^4r}{R^2}} \overset{Mitrinovic}{\ge} \frac{3}{2}\sqrt[6]{\frac{\left(3\sqrt{3}\right)^4 r^5\cdot 4}{R^2}} = \frac{9}{2}\cdot\sqrt[6]{\frac{4r^5}{R^2}}$$

Solution 3 by Tapas Das-India

$$\sum_{cyc}\sqrt{m_a\cdot\cos\frac{B}{2}\cos\frac{C}{2}} \ge 3\cdot\sqrt[6]{\prod_{cyc} m_a\cdot\prod_{cyc}\cos^2\frac{A}{2}} \ge$$

$$\ge 3\cdot\sqrt[6]{\prod_{cyc} h_a\cdot\frac{s^2}{16R^2}} = \frac{3}{2}\cdot\sqrt[6]{27r^3\cdot\frac{4s^2}{R^2}} = \frac{9}{2}\cdot\sqrt[6]{\frac{4r^5}{R^2}} \ge \frac{3}{2}\cdot\sqrt[6]{\frac{4r^5}{R^2}}$$

$$\because m_am_bm_c \ge h_ah_bh_c \ge 27r^3$$

$$\because s^2 \ge 27r^2 \text{ and } m_a + m_b + m_c \le 4R + r$$

$$\sum_{cyc}\sqrt{m_a\cdot\cos\frac{B}{2}\cos\frac{C}{2}} \overset{CBS}{\le} \sqrt{\sum_{cyc} m_a\cdot\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}} \le$$

$$\le \sqrt{\sum_{cyc} m_a\cdot\sum_{cyc}\cos^2\frac{A}{2}} \le \sqrt{\left(2+\frac{r}{2R}\right)\sum_{cyc} m_a} \le \sqrt{\frac{4R+r}{2R}(4R+r)} = \frac{4R+r}{\sqrt{2R}}$$

SP.472 If $x, y, z > 0, x + y + z = 1$ then:

$$\left(x+\frac{1}{y}\right)^5+\left(y+\frac{1}{z}\right)^5+\left(z+\frac{1}{x}\right)^5 \ge \frac{100.000}{81}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\sum_{cyc}\left(x+\frac{1}{y}\right)^5 \overset{AGM}{\ge} 3\cdot\sqrt[3]{\prod_{cyc}\left(x+\frac{1}{y}\right)^5} = 3\left(\prod_{cyc}\left(x+\frac{1}{y}\right)\right)^{\frac{5}{3}} =$$

$$= 3\left(\prod_{cyc}\left(x+9\cdot\frac{1}{9y}\right)\right)^{\frac{5}{3}} \overset{AGM}{\ge} 3\left(10\prod_{cyc}\sqrt[10]{x\cdot\left(\frac{1}{9y}\right)^9}\right)^{\frac{5}{3}} =$$

$$= 3\cdot 10^{\frac{5}{3}\cdot 3}\left(\prod_{cyc}\frac{x^{\frac{1}{10}}}{9^{\frac{9}{10}}\cdot y^{\frac{9}{10}}}\right)^{\frac{5}{3}} = 3\cdot 10^5\cdot\frac{1}{9^{\frac{9}{10}\cdot 3\cdot\frac{5}{3}}}\left(\prod_{cyc}\frac{1}{x^{\frac{8}{10}}}\right)^{\frac{5}{3}} \overset{AGM}{\ge}$$

$$\ge 3\cdot 10^5\cdot\frac{1}{9^{\frac{9}{2}}}\cdot\frac{1}{\left(\frac{x+y+z}{3}\right)^{3\cdot\frac{8}{10}\cdot\frac{5}{3}}} = 3\cdot 10^5\cdot\frac{1}{3^9}\cdot\frac{1}{\left(\frac{1}{3}\right)^4} = 3\cdot 10^4\cdot\frac{1}{3^5} =$$

$$= \frac{10^5}{81} = \frac{100.000}{81}.$$ Equality holds for $x = y = z = \frac{1}{3}$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0$ and $x + y + z = 1$ we will obtain that:

$$\left(x+\frac{1}{y}\right)^5+\left(y+\frac{1}{z}\right)^5+\left(z+\frac{1}{x}\right)^5 =$$

$$= \left(x+1+\frac{x}{y}+\frac{z}{y}\right)^5+\left(y+1+\frac{x}{z}+\frac{y}{z}\right)^5+\left(z+1+\frac{y}{x}+\frac{z}{x}\right)^5 \ge$$

$$\ge \frac{\left(x+y+z+1+1+1+\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}\right)^5}{3\cdot 3^3} =$$

$$= \frac{(1+3+6)^5}{3^4} = \frac{10^5}{3^4} = \frac{100.000}{81}$$

Equality holds for: $x = y = z = \frac{1}{3}$.

Solution 3 by Tapas Das-India

$$x+\frac{1}{y}=x+\frac{x+y+z}{y}=x+\frac{x}{y}+\frac{z}{y}+1$$

$$\left(x+\frac{1}{y}\right)^5=\left(x+\frac{x}{y}+\frac{z}{y}+1\right)^5$$

Similarly,

$$\left(y+\frac{1}{z}\right)^5=\left(y+\frac{y}{z}+\frac{x}{z}+1\right)^5,\qquad \left(z+\frac{1}{x}\right)^5=\left(z+\frac{y}{x}+\frac{z}{x}+1\right)^5$$

$$\sum_{cyc}\left(x+\frac{1}{y}\right)^5=\sum_{cyc}\frac{\left(x+\frac{x}{y}+\frac{z}{y}+1\right)^5}{1^4}\overset{Radon}{\geq}\frac{\left(\sum x+\frac{1}{y}\sum x+\frac{1}{z}\sum x+\frac{1}{x}\sum x\right)^5}{(1+1+1)^4}=$$

$$=\frac{\left(1+\sum\frac{1}{x}\right)^5}{3^4}=\frac{(1+9)^5}{3^4}=\frac{100.000}{81}.$$ Equality holds for: $x=y=z=\frac{1}{3}$.

Solution 4 by Nikos Ntorvas-Greece

Let be the function $f(t)=t^5, t\in(0,1), f$ −convex function,

$f''(x)=20t^3>0, \forall t\in(0,1)$. From Jensen's inequality:

$$\sum_{cyc}f\left(x+\frac{1}{y}\right)\geq 3f\left(\frac{\sum x+\sum\frac{1}{x}}{3}\right)\Leftrightarrow\sum_{cyc}f\left(x+\frac{1}{y}\right)\geq 3f\left(\frac{1+\sum\frac{1}{x}}{3}\right)\Leftrightarrow$$

$$\sum_{cyc}f\left(x+\frac{1}{y}\right)\geq 3\cdot\frac{\left(1+\sum\frac{1}{x}\right)^5}{3^5}\overset{AGM}{\geq}3\cdot\frac{10^5}{3^5}=\frac{100.000}{81}$$

Equality holds for: $x=y=z=\frac{1}{3}$.

Solution 5 by Vivek Kumar-India

Let $x+\frac{1}{y}=a, y+\frac{1}{z}=b, z+\frac{1}{x}=c\Rightarrow$

$$a+b+c=\sum x+\sum\frac{1}{x}=1+\sum\frac{1}{x}\overset{CBS}{\geq}1+\frac{9}{\sum x}\Rightarrow a+b+c\geq 10$$

We have to prove:

$$\sum a^5\geq\frac{100.000}{81},\qquad \sum a^5\sum a\geq\left(\sum a^3\right)^2\Rightarrow\sum a^5\geq\frac{(\sum a^3)^2}{\sum a};(1)\ and$$

$$\sum a^3 \sum a \geq \left(\sum a^2\right)^2 \geq \left(\frac{1}{3}\left(\sum a\right)^2\right)^2; (2)$$

From (1), (2):

$$\sum a^5 \geq \frac{\left(\frac{1}{9}(\sum a)^3\right)^2}{\sum a} = \frac{1}{81}\left(\sum a\right)^5 \Rightarrow \sum a^5 \geq \frac{1}{81}\left(\sum a\right)^5 \geq \frac{100.000}{81}$$

Equality holds for: $x = y = z = \frac{1}{3}$.

Solution 6 by Henry Ricardo-New York-USA

The power means inequality yields:

$$\left(\frac{1}{3}\sum_{cyc}\left(x+\frac{1}{y}\right)^5\right)^{\frac{1}{5}} \geq \frac{1}{3}\sum_{cyc}\left(x+\frac{1}{y}\right) = \frac{1}{3}\left(1+\frac{xy+yz+zx}{xyz}\right)$$

$$\sum_{cyc}\left(x+\frac{1}{y}\right)^5 \geq 3\left(\frac{1+\frac{xy+yz+zx}{xyz}}{3}\right)^5$$

With MacLaurin's inequality and AM-GM inequality:

$$\sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz} \Leftrightarrow xy+yz+zx \geq 3\sqrt[3]{(xyz)^2}$$

$$\frac{xy+yz+zx}{xyz} \geq \frac{3\sqrt[3]{(xyz)^2}}{xyz} = \frac{3}{\sqrt[3]{xyz}} \geq \frac{3}{\frac{x+y+z}{3}} = 9$$

$$\sum_{cyc}\left(x+\frac{1}{y}\right)^5 \geq 3\left(\frac{1+\frac{xy+yz+zx}{xyz}}{3}\right)^5 \geq 3\left(\frac{1+9}{3}\right)^5 = \frac{100.000}{81}$$

Equality holds for: $x = y = z = \frac{1}{3}$.

SP.473 If $x, y, z > 0, \Delta ABC$ and $A_1 \in (BC), B_1 \in (CA), C_1 \in (AB)$ such that

$A_1B = xA_1C, B_1C = yB_1A, C_1A = zC_1B$, then holds:

$$aa_1 + bb_1 + cc_1 \geq 4\sqrt{3} \cdot \sqrt{\frac{xyz+1}{(x+1)(y+1)(z+1)}} \cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution 1 by proposers

We prove that:

$$[A_1B_1C_1] = \frac{xyz+1}{(x+1)(y+1)(z+1)} \cdot F$$

From Tsintsifas' theorem we have:

$$aa_1 + bb_1 + cc_1 \geq 4\sqrt{3} \cdot \sqrt{F} \cdot \sqrt{[A_1B_1C_1]};\ (1)$$

$$A_1B = xA_1C \Rightarrow a = BC = A_1B + A_1C = xA_1C + A_1C \Rightarrow A_1C = \frac{a}{x+1}$$

$$A_1B = a - B_1C = a - \frac{a}{x+1} = \frac{ax}{x+1}$$

Analogous,

$$B_1C = \frac{b}{y+z}, B_1A = \frac{by}{y+1}, C_1A = \frac{c}{z+1}, C_1B = \frac{cz}{z+1}$$

$$\frac{[CA_1B_1]}{[ABC]} = \frac{A_1C \cdot B_1C}{BC \cdot CA} = \frac{a}{x+1} \cdot \frac{by}{y+1} \cdot \frac{1}{ab} = \frac{y}{(x+1)(y+1)}$$

$$[CA_1B_1] = \frac{y}{(x+1)(y+1)} \cdot F$$

Similarly, we have:

$$[AB_1C_1] = \frac{z}{(y+1)(z+1)} \cdot F \text{ and } [BC_1A_1] = \frac{x}{(z+1)(x+1)} \cdot F$$

$$[A_1B_1C_1] = [ABC] - \frac{x}{(z+1)(x+1)} \cdot F + \frac{y}{(y+1)(z+1)} \cdot F + \frac{z}{(y+1)(z+1)} \cdot F =$$

$$= \frac{(x+1)(y+1)(z+1) - x(y+1) - y(z+1) - z(x+1)}{(x+1)(y+1)(z+1)} \cdot F =$$

$$= \frac{xyz+1}{(x+1)(y+1)(z+1)} \cdot F$$

Therefore, (1) becomes:

$$aa_1+bb_1+cc_1\ge 4\sqrt{3}\cdot\sqrt{F}\cdot\sqrt{\frac{xyz+1}{(x+1)(y+1)(z+1)}}\cdot\sqrt{F}=$$

$$=4\sqrt{3}\cdot\sqrt{\frac{xyz+1}{(x+1)(y+1)(z+1)}}\cdot F$$

Solution 2 by Adrian Popa-Romania

Using Routh's theorem, we have:

$$\frac{[A_1B_1C_1]}{[ABC]}=\frac{xyz}{(x+1)(y+1)(z+1)}$$

$$a_1a+b_1b+c_1c\overset{(?)}{\ge}4\sqrt{3}\cdot\sqrt{\frac{F_1}{F}}\cdot F=4\sqrt{3}\cdot\sqrt{F_1F}$$

$$a^2+b^2+c^2>4\sqrt{3}F \text{ and } a_1^2+b_1^2+c_1^2\ge 4\sqrt{3}\cdot F_1$$

$$(a^2+b^2+c^2)\left(a_1^2+b_1^2+c_1^2\right)\overset{CBS}{\ge}(aa_1+bb_1+cc_1)^2$$

$$(aa_1+bb_1+cc_1)^2\le 4\sqrt{3}F\cdot 4\sqrt{3}F_1=\left(4\sqrt{3}\right)^2FF_1$$

Therefore,

$$aa_1+bb_1+cc_1\le 4\sqrt{3}\cdot\sqrt{FF_1}$$

SP.474 If $m\ge 0$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{x^{m+1}\cdot a^{2m}}{(y+z)^{m+1}\cdot h_a^2}+\frac{y^{m+1}\cdot b^{2m}}{(z+x)^{m+1}\cdot h_b^2}+\frac{z^{m+1}\cdot c^{2m}}{(x+y)^{m+1}\cdot h_c^2}\ge 2^{m-1}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^{m-1}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution 1 by proposers

$$\sum_{cyc}\frac{x^{m+1}\cdot a^{2m}}{(y+z)^{m+1}\cdot h_a^2}=\sum_{cyc}\frac{x^{m+1}\cdot a^{2m+2}}{(y+z)^{m+1}\cdot(ah_a)^2}=\frac{1}{4F^2}\sum_{cyc}\left(\frac{xa^2}{y+z}\right)^{m+1}\overset{Radon}{\ge}$$

$$\ge\frac{1}{4F^2}\cdot\frac{1}{3^m}\left(\sum_{cyc}\frac{x}{y+z}a^2\right)^{m+1}\overset{Tsintsifas}{\ge}\frac{1}{4\cdot 3^m\cdot F^2}\left(2\sqrt{3}\cdot F\right)^{m+1}=$$

$$=\frac{2^{m+1}\left(\sqrt{3}\right)^{m+1}}{4\cdot 3^m}\cdot F^{m-1}=2^{m-1}\left(\sqrt{3}\right)^{1-m}\cdot F^{m-1}$$

Solution 2 by Tapas Das-India

$$\sum_{cyc}\frac{x^{m+1}}{(y+z)^{m+1}}\cdot\frac{a^{2m}}{h_a^2}=\sum_{cyc}\frac{x^{m+1}}{(y+z)^{m+1}}\cdot\frac{a^{2m}\cdot a^2}{4F^2}=$$

$$=\frac{1}{4F^2}\sum_{cyc}\frac{x^{m+1}a^{2(m+1)}}{(y+z)^{m+1}}\overset{Radon}{\geq}\frac{1}{4F^2}\cdot\frac{1}{3^m}\left(\sum_{cyc}\frac{x}{y+z}\cdot a^2\right)^{m+1}\geq$$

$$\overset{Tsintsifas}{\geq}\frac{1}{4F^2}\cdot\frac{1}{3^m}\cdot\left(2\sqrt{3}F\right)^{m+1}=\frac{2^{m+1}\cdot\left(\sqrt{3}\right)^{m+1}\cdot F^{m+1}}{\left(\sqrt{3}\right)^{2m}}=2^{m-1}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^{m-1}$$

SP.475 In ΔABC the following relationship holds:

$$\frac{m_a^2\cdot a^3}{\sqrt{m_bm_c}}+\frac{m_b^2\cdot b^3}{\sqrt{m_cm_a}}+\frac{m_c^2\cdot c^3}{\sqrt{m_am_b}}\geq 8\sqrt{3}\cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution 1 by proposers

Because $m_a\geq h_a$, we have:

$$\sum_{cyc}\frac{m_a^2\cdot a^3}{\sqrt{m_bm_c}}\geq\sum_{cyc}\frac{h_a\cdot m_a\cdot a^3}{\sqrt{m_bm_c}}=2F\cdot\sum_{cyc}\frac{m_a}{\sqrt{m_bm_c}}\cdot a^2\ ;(1)$$

So,

$$\sum_{cyc}\frac{m_a^2\cdot a^3}{\sqrt{m_bm_c}}\geq 2F\cdot\sum_{cyc}\frac{m_a}{\sqrt{m_bm_c}}\cdot a^2\geq 2F\cdot 3\sqrt[3]{\prod_{cyc}\frac{m_a}{\sqrt{m_bm_c}}\cdot a^3}=$$

$$=2F\cdot 3\sqrt[3]{a^2b^2c^2}\overset{Carliz}{\geq}2F\cdot 4\sqrt{3}\cdot F=8\sqrt{3}\cdot F^2$$

Solution 2 by proposers

$$\sum_{cyc}\frac{m_a^2\cdot a^3}{\sqrt{m_bm_c}}\geq 2F\cdot\sum_{cyc}\frac{m_a}{\sqrt{m_bm_c}}\cdot a^2\geq 2F\cdot 2\sum_{cyc}\frac{m_a}{m_b+m_c}\cdot a^2\overset{Tsintsifas}{\geq}$$

$$\geq 4F\cdot 2\sqrt{3}\cdot F=8\sqrt{3}\cdot F^2$$

Solution 3 by Tapas Das-India

$$\sum_{cyc}\frac{m_a^2\cdot a^3}{\sqrt{m_bm_c}}\ge 3\sqrt[3]{\frac{m_a^2m_b^2m_c^2(abc)^3}{m_am_bm_c}}=3\sqrt[3]{m_am_bm_c(abc)^3}\ge$$

$$\ge 3\sqrt[3]{(h_ah_bh_c)(abc)^3}=3\sqrt[3]{8F^3(abc)^2}\ge$$

$$\ge 3\sqrt[3]{8F^3\left(\frac{4F}{\sqrt{3}}\right)^3}=8\sqrt{3}F^2$$

Solution 4 by Myagmarsuren Yadamasuren- Mongolia

$$\sum_{cyc}\frac{m_a^2\cdot a^3}{\sqrt{m_bm_c}}\ge 3abc\cdot\sqrt[3]{m_am_bm_c}\overset{m_a\ge\sqrt{s(s-a)}}{\ge}$$

$$\ge 3abc\cdot\sqrt[3]{s\sqrt{s(s-a)(s-b)(s-c)}}=3abc\cdot\sqrt[3]{sF}=$$

$$=12\cdot\frac{abc}{4R}\cdot R\sqrt[3]{sF}=12F\cdot\sqrt[3]{sFR^3}\overset{Euler}{\ge}12F\cdot\sqrt[3]{F^2\cdot 2R^2}=$$

$$=12F\cdot\sqrt[3]{F^2\cdot 2\cdot R\cdot R}\ge 12F\cdot\sqrt[3]{F^2\cdot sr\cdot\frac{8}{3\sqrt{3}}}=8\sqrt{3}F^2$$

SP.476 If $x,y,z>0$ and $0\le\lambda\le\frac{1}{25}$ then:

$$\sum_{cyc}\frac{x}{\sqrt[3]{\lambda y^3+xyz+\lambda z^3}}\ge\frac{3}{\sqrt[3]{2\lambda+1}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Using Radon's inequality, we have:

$$\sum_{cyc}\frac{x}{\sqrt[3]{\lambda y^3+xyz+\lambda z^3}}=\sum_{cyc}\sqrt[3]{\frac{x^3}{\lambda y^3+xyz+\lambda z^3}}=\sum_{cyc}\sqrt[3]{\frac{x^4}{\lambda xy^3+x^2yz+\lambda xz^3}}=$$

$$=\sum_{cyc}\frac{x^{\frac{4}{3}}}{(\lambda xy^3+x^2yz+\lambda xz^3)^{\frac{1}{3}}}\overset{Radon}{\ge}\frac{(\sum x)^{\frac{4}{3}}}{[\sum(\lambda xy^3+x^2yz+\lambda xz^3)]^{\frac{1}{3}}}\overset{(1)}{\ge}\frac{3}{\sqrt[3]{2\lambda+1}}$$

$$(1) \Leftrightarrow \frac{(\sum x)^{\frac{4}{3}}}{[\sum(\lambda xy^3 + x^2yz + \lambda xz^3)]^{\frac{1}{3}}} \geq \frac{3}{\sqrt[3]{2\lambda+1}} \Leftrightarrow$$

$$\sqrt[3]{2\lambda+1}(\sum x)^{\frac{4}{3}} \geq 3\left[\sum(\lambda xy^3 + x^2yz + \lambda xz^3)\right]^{\frac{1}{3}} \Leftrightarrow$$

$$(2\lambda+1)(\sum x)^4 \geq 27\sum(\lambda xy^3 + x^2yz + \lambda xz^3) \Leftrightarrow$$

$$(2\lambda+1)[\sum x^4 + 6\sum y^2z^2 + 4\sum x^2\sum yz + 8xyz\sum x] \geq 27\sum(\lambda xy^3 + x^2yz + \lambda xz^3)$$

$$(2\lambda+1)[\sum x^4 + 6\sum y^2z^2 + 4\sum x^2\sum yz + 8xyz\sum x] \geq$$

$$\geq 27\lambda\sum yz(y^2+z^2) + 27xyz\sum x \Leftrightarrow$$

$$(2\lambda+1)\sum x^4 + 6(2\lambda+1)\sum y^2z^2 + 4(2\lambda+1)\sum x^2\sum yz + 8(2\lambda+1)xyz\sum x \geq$$

$$\geq 27\lambda\sum yz(y^2+z^2) + 27xyz\sum x \Leftrightarrow$$

$$(2\lambda+1)\sum x^4 + 6(2\lambda+1)\sum y^2z^2 + 4(2\lambda+1)\sum x^2\sum yz + (16\lambda-19)xyz\sum x \geq$$

$\geq 27\lambda\sum yz(y^2+z^2)$, which follows from Schur's inequality:

$x^r(x-y)(x-z) \geq 0, \forall x,y,z \geq 0$ and $r > 0$.

For $r = 2$ we have:

$$x^4 + y^4 + z^4 + xyz(x+y+z) \geq xy(x^2+y^2) + yz(y^2+z^2) + zx(z^2+x^2)$$

$$\sum x^4 + xyz\sum x \geq \sum yz(y^2+z^2) \Rightarrow$$

$$27\lambda\sum x^4 + 27\lambda xyz\sum x \geq 27\lambda\sum yz(y^2+z^2)$$

It is enough to prove:

$$(2\lambda+1)\sum x^4 + 6(2\lambda+1)\sum y^2z^2 + 4(2\lambda+1)\sum x^2\sum yz + (16\lambda-19)xyz\sum x \geq$$

$$\geq 27\lambda\sum yz(y^2+z^2) \Leftrightarrow$$

$$(1-25\lambda)\sum x^4 + 6(2\lambda+1)\sum y^2z^2 + 4(2\lambda+1)\sum x^2\sum yz \geq (19+11\lambda)xyz\sum x$$

which follows from $0 \leq \lambda \leq \frac{1}{25}$ and the inequalities:

$\sum x^4 \geq \sum y^2z^2$; $\sum y^2z^2 \geq xyz\sum x$ and $\sum x^2\sum yz \geq (\sum yz)^2 \geq 3xyz\sum x$

We get:

$$(1-25\lambda)\sum x^4 + 6(2\lambda+1)\sum y^2z^2 + 4(2\lambda+1)\sum x^2\sum yz \geq$$

$$\geq (1-25\lambda)xyz\sum x + 6(2\lambda+1)xyz\sum x + 4(2\lambda+1)3xyz\sum x =$$

$= (19+11\lambda)xyz\sum x$. Equality holds for $x = y = z$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p=x+y+z, q=xy+yz+zx, r=xyz,\ q\le\frac{p^2}{3}=3,\ r\le\left(\frac{p}{3}\right)^3=1$

$$\left(\sum_{cyc}\frac{x}{\sqrt[3]{\lambda y^3+xyz+\lambda z^3}}\right)^3\sum_{cyc}x(\lambda y^3+xyz+\lambda z^3)\overset{HOLDER}{\geq}(x+y+z)^4=81$$

$$\sum_{cyc}x(y^3+z^3)=p^2q-2q^2-pr=9q-2q^2-3r$$

$$\left(\sum_{cyc}\frac{x}{\sqrt[3]{\lambda y^3+xyz+\lambda z^3}}\right)^3\geq\frac{81}{\lambda(9q-2q^2-3r)+3r}\overset{to\ prove}{\geq}\left(\frac{3}{\sqrt[3]{2\lambda+1}}\right)^3\Leftrightarrow$$

$$\Leftrightarrow 3(2\lambda+1)\geq\lambda(9q-2q^2-3r)+3r\Leftrightarrow$$

$$\Leftrightarrow 3(1-r)+\lambda(-9q+2q^2+3r+6)\geq 0 \qquad (1)$$

$$(A)\qquad -9q+2q^2+3r+6\geq 0\Rightarrow\sum_{cyc}\frac{x}{\sqrt[3]{\lambda y^3+xyz+\lambda z^3}}\geq 0$$

$$(B)\quad -9q+2q^2+3r+6\geq 0\Rightarrow$$

$$\sum_{cyc}\frac{x}{\sqrt[3]{\lambda y^3+xyz+\lambda z^3}}\geq 3(1-r)+\frac{1}{25}(-9q+2q^2+3r+6)=$$

$$=\frac{-9q+2q^2-72r+81}{25}\geq\frac{-9q+2q^2-24q+81}{25}=\frac{(3-q)(27-2q)}{25}\geq 0$$

Equality holds for $x=y=z$.

SP.477 If $a,b,c,d\geq 1$ then:

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}+\sqrt{d-1}\leq\sqrt{2(ab+cd)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}+\sqrt{d-1}\overset{CBS}{\leq}$$

$$\sqrt{\left(\left(\sqrt{a-1}\right)^2+1^2\right)\left(\left(\sqrt{b-1}\right)^2+1^2\right)}+\sqrt{\left(\left(\sqrt{c-1}\right)^2+1^2\right)\left(\left(\sqrt{d-1}\right)^2+1^2\right)}=$$

$$=\sqrt{(a-1+1)(b-1+1)}+\sqrt{(c-1+1)(d-1+1)}=$$

$$= \sqrt{ab} + \sqrt{cd} \overset{CBS}{\leq} \sqrt{\left(\left(\sqrt{ab}\right)^2 + \left(\sqrt{cd}\right)^2\right)(1^2 + 1^2)} = \sqrt{2(ab + cd)}$$

Equality holds for $a = b = c = d = 2$.

Solution 2 by Ravi Prakash- India

Let $a = \sec^2 \alpha, b = \sec^2 \beta, c = \sec^2 \gamma, d = \tan \delta$.

As $a, b, c, d \geq 1 \Rightarrow \alpha, \beta, \gamma, \delta \in \left(0; \frac{\pi}{2}\right)$

The inequality becomes:

$$\tan \alpha + \tan \beta + \tan \gamma + \tan \delta \leq \sqrt{2(\sec^2 \alpha \sec^2 \beta + \sec^2 \gamma \sec^2 \delta)} =$$

$$= \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} + \frac{\sin(\gamma + \delta)}{\cos \gamma \cos \delta} \leq \sec \alpha \sec \beta + \sec \gamma \sec \delta$$

It is sufficient to show that:

$$\sec \alpha \sec \beta + \sec \gamma \sec \delta \leq \sqrt{2(\sec^2 \alpha \sec^2 \beta + \sec^2 \gamma \sec^2 \delta)} \Leftrightarrow$$

$$(\sec \alpha \sec \beta + \sec \gamma \sec \delta)^2 \leq 2(\sec^2 \alpha \sec^2 \beta + \sec^2 \gamma \sec^2 \delta) \Leftrightarrow$$

$$2 \sec \alpha \sec \beta \sec \gamma \sec \delta \leq \sec^2 \alpha \sec^2 \beta + \sec^2 \gamma \sec^2 \delta$$

$$(\sec \alpha \sec \beta - \sec \gamma \sec \delta)^2 \geq 0 \ true!$$

Solution 3 by Tapas Das-India

We will prove that

$$\sqrt{x - 1} + \sqrt{y - 1} \leq \sqrt{xy}, \forall x, y > 0$$

Using CBS for $a_1 = \sqrt{x - 1}, a_2 = 1, b_1 = 1, b_2 = \sqrt{y - 1} \Rightarrow$

$$[(x - 1) + 1][1 + (y - 1)] \geq \left(\sqrt{x - 1} + \sqrt{y - 1}\right)^2$$

$$xy \geq \left(\sqrt{x - 1} + \sqrt{y - 1}\right)^2$$

Now,

$$\sqrt{a - 1} + \sqrt{b - 1} + \sqrt{c - 1} + \sqrt{d - 1} \leq \sqrt{ab} + \sqrt{cd} \leq 2\sqrt{\frac{ab + cd}{2}} = \sqrt{2(ab + cd)}$$

Solution 4 by Sanong Huayrerai- Thailand

$$\sqrt{a - 1} + \sqrt{b - 1} + \sqrt{c - 1} + \sqrt{d - 1} \leq \sqrt{2(ab + cd)}$$

Let $x^2 = a - 1, y^2 = b - 1, z^2 = c - 1, t^2 = d - 1 \Rightarrow$

$$x+y+z+t\le\sqrt{2[(x^2+1)(y^2+1)+(z^2+1)(t^2+1)]}$$

$$x+y+z+t\le\sqrt{(x^2+1)(y^2+1)}+\sqrt{(z^2+1)(t^2+1)}$$

Because:

$$x+y\le\sqrt{(x^2+1)(y^2+1)}\Leftrightarrow x^2+y^2+2xy\le x^2+y^2+x^2y^2+1\Leftrightarrow$$

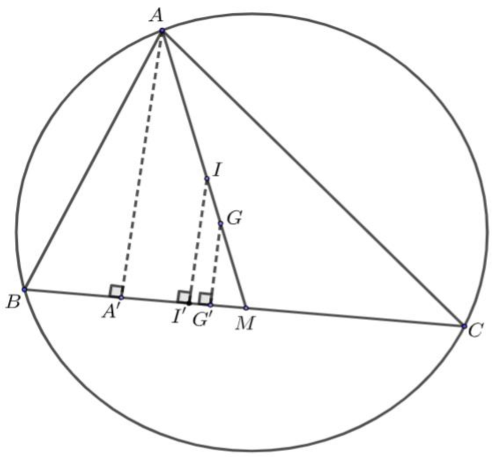
$$2xy\le x^2y^2+1 \text{ and } 2zt\le z^2t^2+1$$

SP.478 Let a,b,c –be the sides lengths of ΔABC, I –incenter, G –centroid. If $IG\perp BC$ and $b\ne c$ then:

$$\frac{b}{c+a}+\frac{c}{a+b}+\frac{ab+bc+ca}{a^2+b^2+c^2}<\frac{13}{6}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer



Let $M\in(BC)$, M –middle point of $BC\Rightarrow\overrightarrow{AM}=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})$; (1)

$$\frac{AG}{GM}=2\Rightarrow\overrightarrow{AG}=\frac{2}{3}\overrightarrow{AM}\overset{(1)}{\Rightarrow}\overrightarrow{AG}=\frac{1}{3}(\overrightarrow{AB}+\overrightarrow{AC})$$

I –incenter, then $\overrightarrow{AI}=\frac{b\overrightarrow{AB}+c\overrightarrow{AC}}{a+b+c}$; (2)

$$\overrightarrow{AI}+\overrightarrow{IG}=\overrightarrow{AG}\Rightarrow\overrightarrow{IG}=\overrightarrow{AG}-\overrightarrow{AI}=\frac{1}{3}(\overrightarrow{AB}+\overrightarrow{AC})-\frac{b\overrightarrow{AB}+c\overrightarrow{AC}}{a+b+c}$$

$$\overrightarrow{IG}=\frac{1}{3(a+b+c)}[(a+b+c-3b)\overrightarrow{AB}+(a+b+c-3c)\overrightarrow{AC}]$$

$$\overrightarrow{IG}=\frac{1}{3(a+b+c)}\left((a+c-2b)\overrightarrow{AB}+(a+b-2c)\overrightarrow{AC}\right);(3)$$

We have:

$$\overrightarrow{AB}\cdot\overrightarrow{AC}=\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AC}\right|\cdot\cos A=bc\cdot\cos A=$$

$$=bc\cdot\frac{b^2+c^2-a^2}{2bc}=\frac{b^2+c^2-a^2}{2};(4)$$

Hence,

$$\overrightarrow{IG}\cdot\overrightarrow{BC}=\overrightarrow{IG}\cdot\left(\overrightarrow{BA}+\overrightarrow{AC}\right)=\overrightarrow{IG}\cdot\left(\overrightarrow{AC}-\overrightarrow{AB}\right)=\overrightarrow{IG}\cdot\overrightarrow{AC}-\overrightarrow{IG}\cdot\overrightarrow{AB}\overset{(3)}{=}$$

$$=\frac{1}{3(a+b+c)}\left((a+c-2b)\overrightarrow{AB}\cdot\overrightarrow{AC}+(a+b-2c)\overrightarrow{AC}\cdot\overrightarrow{AC}\right)-$$

$$-\frac{1}{3(a+b+c)}\left((a+c-2b)\overrightarrow{AB}\cdot\overrightarrow{AB}+(a+b-2c)\overrightarrow{AC}\cdot\overrightarrow{AB}\right)\overset{(4)}{=}$$

$$=\frac{1}{3(a+b+c)}\left((a+c-2b)\frac{b^2+c^2-a^2}{2}+(a+b-2c)c^2\right)-$$

$$-\frac{1}{3(a+b+c)}\left((a+c-2b)c^2+(a+b-2c)\frac{b^2+c^2-a^2}{2}\right)=$$

$$=\frac{1}{6(a+b+c)}(a+b+c)(c-b)(b+c-3a)=\frac{1}{6}(c-b)(b+c-3a);(5)$$

$$IG\perp BC\Leftrightarrow\overrightarrow{IG}\cdot\overrightarrow{BC}=0\Leftrightarrow(c-b)(b+c-3a)=0,(\because b\neq c)\Rightarrow b+c=3a;(6)$$

Now, we have:

$$\frac{b}{c+a}+\frac{c}{a+b}+\frac{ab+bc+ca}{a^2+b^2+c^2}<\frac{13}{6}$$

$$\frac{b}{c+a}+\frac{c}{a+b}+\frac{ab+bc+ca}{a^2+b^2+c^2}<\frac{5}{2}-\frac{1}{3}$$

$$\frac{a}{3a}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{ab+bc+ca}{a^2+b^2+c^2}<\frac{5}{2}$$

So, we have to prove that:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{ab+bc+ca}{a^2+b^2+c^2}<\frac{5}{2}\Leftrightarrow\sum_{cyc}\left(1-\frac{a}{b+c}\right)>\frac{1}{2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\Leftrightarrow$$

$$\sum_{cyc}\frac{b+c-a}{b+c}>\frac{(a+b+c)^2}{2(a^2+b^2+c^2)}$$

By CBS inequality, we have:

$$2\left(\sum_{cyc} a^2\right)\left(\sum_{cyc}\frac{b+c-a}{b+c}\right)=\left[\sum_{cyc}(b+c)(b+c-a)\right]\left(\sum_{cyc}\frac{b+c-a}{b+c}\right)\geq$$

$$\geq\left(\sum_{cyc}(b+c-a)\right)^2=(a+b+c)^2$$

Equality holds if and only if $a=b=c$ but $(b\neq c)$ or $a=b, c=0$ (impossible!)

Solution 2 by Mohamed Amine Ben Ajiba- Morocco

$$\textit{For any point } M \textit{ we have}: \ (a+b+c)\overrightarrow{MI}=a\overrightarrow{MA}+b\overrightarrow{MB}+c\overrightarrow{MC}.$$

$$\textit{For } M\equiv G \textit{ we get}: \ (a+b+c)\overrightarrow{GI}=a\overrightarrow{GA}+b\overrightarrow{GB}+c\overrightarrow{GC}$$

$$\textit{Since}: \ \overrightarrow{GA}=-\frac{1}{3}\left(\overrightarrow{AB}+\overrightarrow{AC}\right), \ \overrightarrow{GB}=-\frac{1}{3}\left(\overrightarrow{BC}+\overrightarrow{BA}\right) \ \textit{and} \ \overrightarrow{GC}=-\frac{1}{3}\left(\overrightarrow{CA}+\overrightarrow{CB}\right)$$

$$\textit{Then we get}: \ (a+b+c)\overrightarrow{GI}=-\frac{1}{3}\left((a-b)\overrightarrow{AB}+(b-c)\overrightarrow{BC}+(c-a)\overrightarrow{CA}\right).$$

$$\textit{Now we have}: \ IG \,\square\, BC \ \Leftrightarrow \ \overrightarrow{GI}.\overrightarrow{BC}=0$$

$$\Leftrightarrow \ \left((a-b)\overrightarrow{AB}+(b-c)\overrightarrow{BC}+(c-a)\overrightarrow{CA}\right).\overrightarrow{BC}=0$$

$$\Leftrightarrow \ (a-b)\left(-\frac{c^2+a^2-b^2}{2}\right)+(b-c)a^2+(c-a)\left(-\frac{a^2+b^2-c^2}{2}\right)=0$$

$$\Leftrightarrow \ (a-b)(b-c)(b+c)+3(b-c)a^2-(c-a)(b-c)(b+c)=0$$

$$\Leftrightarrow \ (b-c)[3a^2+2a(b+c)-(b+c)^2]=0$$

$$\Leftrightarrow \ (b-c)(a+b+c)(3a-b-c)=0 \ \textit{then}: \ \boxed{3a=b+c}.$$

$$\textit{Now let } p:=(b+c)^2, \ \ s:=bc \ \textit{and} \ t:=\frac{p}{s}.$$

$$\textit{Since } a,b,c \textit{ are the sides lengths of } \Delta ABC \textit{ then we have}:$$

$$a+b>c \textit{ and } c+a>b$$

$$\Leftrightarrow \ 4b+c>3c \textit{ and } 4c+b>3b \ \Leftrightarrow \ 2b>c \textit{ and } 2c>b \ \Rightarrow \ \frac{1}{2}<\frac{b}{c}<2 \ \Rightarrow$$

$$4\overset{AM-GM}{\lesssim} t=\frac{b}{c}+\frac{c}{b}+2<\frac{9}{2}.$$

$$We\ have\ now: \frac{b}{c+a}+\frac{c}{a+b}+\frac{ab+bc+ca}{a^2+b^2+c^2}$$
$$=\frac{3b}{3c+3a}+\frac{3c}{3a+3b}+\frac{9a(b+c)+9bc}{(3a)^2+9(b^2+c^2)}=$$

$$=\frac{3b}{4c+b}+\frac{3c}{4b+c}+\frac{3p+9s}{p+9(p-2s)}=3\left(\frac{4p-6s}{4p+9s}+\frac{p+3s}{10p-18s}\right)$$
$$=\frac{3(44p^2-111ps+135s^2)}{2(20p^2+9ps-81s^2)}\overset{?}{\lesssim}\frac{13}{6}$$

$$\Leftrightarrow -34p^2+279ps-567s^2>0 \Leftrightarrow -34t^2+279t-567>0$$
$$\Leftrightarrow 2\left(\frac{9}{2}-t\right)[17(t-4)+5]>0$$

$$Which\ is\ true\ because\ \ 4<t<\frac{9}{2}\ and\ the\ proof\ is\ completed.$$

SP.479 In ΔABC the following relationship holds:

$$\frac{R^2}{r}\geq\frac{s}{16}\left(\frac{a}{w_b}+\frac{b}{w_a}\right)\left(\frac{b}{w_c}++\frac{c}{w_b}\right)\left(\frac{a}{w_a}+\frac{c}{w_c}\right)\geq\frac{8r^2}{R}$$

Proposed by Alex Szoros-Romania

Solution 1 by proposer

$$\frac{a}{w_b}+\frac{b}{w_a}\leq\frac{a}{h_b}+\frac{b}{h_a}=\frac{ab}{2F}+\frac{ab}{2F}=\frac{ab}{F}=\frac{4R}{c}$$
$$\frac{4R}{c}\geq\frac{a}{w_b}+\frac{b}{w_a};(1)$$

Using Padoa's inequality, we have:

$$abc\geq 8(s-a)(s-b)(s-c);(2)$$

$$c=(s-a)+(s-b)\overset{AGM}{\geq}2\sqrt{(s-a)(s-b)};(3)$$

From (2) and (3), it follows:

$$abc^2\geq 16(s-a)(s-b)(s-c)\sqrt{(s-a)(s-b)}$$

$$abc^2\geq 16\frac{F^2}{s^2}s\sqrt{(s-a)(s-b)}\geq 16r^2w_aw_b$$

$$2\cdot\sqrt{\frac{ab}{w_aw_b}}\ge\frac{8r}{c};(4)$$

$$\frac{a}{w_b}+\frac{b}{w_a}\overset{AGM}{\ge}2\sqrt{\frac{ab}{w_aw_b}};(5)$$

From (4) and (5), we get:

$$\frac{a}{w_b}+\frac{b}{w_a}\ge\frac{8r}{c};(6)$$

From (1) and (6), we get:

$$\frac{4R}{c}\ge\frac{a}{w_b}+\frac{b}{w_a}\ge\frac{8r}{c};(7)$$

From (7), it follows that:

$$\frac{64R^3}{abc}\ge\prod_{cyc}\left(\frac{a}{w_b}+\frac{b}{w_a}\right)\ge\frac{512r^3}{abc};\quad(8)$$

But $abc=4Rrs$, then (8) becomes:

$$\frac{16R^2}{rs}\ge\prod_{cyc}\left(\frac{a}{w_b}+\frac{b}{w_a}\right)\ge\frac{128r^2}{Rs}$$

$$\frac{R^2}{r}\ge\frac{s}{16}\prod_{cyc}\left(\frac{a}{w_b}+\frac{b}{w_a}\right)\ge\frac{8r^2}{R}$$

Solution 2 by Tapas Das-India

$$*\ w_a\le\sqrt{r_br_c}\Rightarrow w_aw_bw_c\le r_ar_br_c$$

$$*\ r_ar_br_c\le\frac{3}{8}abc\sqrt{3}\Rightarrow w_aw_bw_c\le\frac{3}{8}abc\sqrt{3}\ \text{ because}$$

$$r_a=\frac{F}{s-a}(and\ analogs)\Rightarrow r_ar_br_c=\frac{F^3}{(s-a)(s-b)(s-c)}=\frac{F^3s}{F^2}=Fs=$$

$$=\frac{abc}{4R}\cdot s\le\frac{abc}{4R}\cdot\frac{3\sqrt{3}}{2}R\le\frac{3\sqrt{3}abc}{8}$$

$$\frac{s}{16}\prod_{cyc}\left(\frac{a}{w_b}+\frac{b}{w_a}\right)\overset{AGM}{\ge}\frac{s}{16}\cdot\prod_{cyc}2\sqrt{\frac{ab}{w_aw_b}}=\frac{s}{2}\cdot\frac{abc}{w_aw_bw_c}\ge\frac{s}{2}abc\cdot\frac{8}{3abc\sqrt{3}}=$$

$$= 8 \cdot \frac{4}{3\sqrt{3}} \geq \frac{3\sqrt{3}r \cdot 4}{3\sqrt{3}} = 4r$$

$$* \, s^2 \geq 27r^2 \ (Mitrinovic)$$

We need to prove

$$4r \geq \frac{8r^2}{R} \Leftrightarrow R \geq 2r \ (Euler)$$

Hence,

$$\frac{s}{16}\prod_{cyc}\left(\frac{a}{w_b}+\frac{b}{w_a}\right) \geq \frac{8r^2}{R}$$

$$\frac{s}{16}\prod_{cyc}\left(\frac{a}{w_b}+\frac{b}{w_a}\right) \leq \frac{s}{16}\prod_{cyc}\left(\frac{a}{h_b}+\frac{b}{h_a}\right) =$$

$$* \, h_a \leq w_a, h_b \leq w_b, h_c \leq w_c$$

$$= \frac{s}{16}\prod_{cyc}\left(\frac{ab}{2F}+\frac{ab}{2F}\right) = \frac{s}{16} \cdot \frac{(abc)^2}{F^3} = \frac{s}{16} \cdot \frac{(4RF)^2}{F^3} = \frac{s}{16} \cdot \frac{16R^2}{F} = \frac{R^2}{r}$$

Solution 3 by Ertan Yildirim-Izmir-Turkiye

$$* \, h_a \leq w_a \leq \sqrt{s(s-a)}$$

$$\frac{a}{w_b}+\frac{b}{w_a} \leq \frac{a}{h_b}+\frac{b}{h_a} = \frac{a \cdot 2R}{ac}+\frac{b}{bc} \cdot 2R = \frac{2R}{c} \cdot 2 = \frac{4R}{c} \ (and \ analogs)$$

$$P = \frac{s}{16}\prod_{cyc}\left(\frac{a}{w_b}+\frac{b}{w_a}\right) \leq \frac{s}{16} \cdot \prod_{cyc}\frac{4R}{a} = \frac{64R^3}{4rs \cdot R} \cdot \frac{s}{16} = \frac{R^2}{r}$$

$$\frac{a}{w_b}+\frac{b}{w_a} \overset{AGM}{\geq} 2\sqrt{\frac{ab}{w_a w_b}}$$

$$P \geq \frac{s}{16}\prod_{cyc} 2\sqrt{\frac{ab}{w_a w_b}} = \frac{s}{16} \cdot \frac{8abc}{w_a w_b w_c} \geq \frac{s}{2} \cdot \frac{4sr \cdot R}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{2s^2 rR}{s^2 r}$$

$$= 2R \overset{?}{\geq} \frac{8r^2}{R}, \qquad 2R^2 \geq 8r^2 \Leftrightarrow R^2 \geq 4r^2 \Leftrightarrow R \geq 2r \ (Euler).$$

SP.480 In $\Delta ABC, F$ –area and the points $M \in (BC), N \in (CA), P \in (AB)$, then:

$$(a^2 + b \cdot BM)(b^2 + c \cdot CP)(c^2 + a \cdot AM) \geq 36\sqrt{3} \cdot F^3$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

Because $AM \geq h_a, BN \geq h_b, CP \geq h\_c$, then:

$$\prod_{cyc}(a^2 + b \cdot BN) \geq \prod_{cyc}(a^2 + b \cdot h_a) = \prod_{cyc}(a^2 + 2F); (1)$$

$$\prod_{cyc}(x^2 + t^2) \geq \frac{9}{4} \cdot t^4(xy + yz + zx); \forall\, t, x, y, z > 0; \quad (2)$$

So, in (2) we take: $x = a, y = b, z = c, t = \sqrt{2F}$, we get:

$$\prod_{cyc}(a^2 + 2F) \geq \frac{9}{4} \cdot 4F^2(ab + bc + ca) = 9F^2(ab + bc + ca) \overset{Gordon}{\geq}$$

$$\geq F^2 \cdot 4\sqrt{3}F = 36\sqrt{3}F^3$$

Let's prove inequality (2). We have:

$$(i): (x^2 + t^2)(y^2 + t^2) \geq \frac{3}{4}t^2((x + y)^2 + t^2) \Leftrightarrow$$

$$4x^2y^2 + 4t^2(x^2 + y^2) + 4t^4 \geq 3t^2(x^2 + y^2) + 6t^2xy + 3t^4 \Leftrightarrow$$

$$4x^2y^2 - 4t^2xy + t^4 + t^2(x^2 + y^2 - 2xy) \geq 0 \Leftrightarrow$$

$(2xy - t^2)^2 + t^2(x - y)^2 \geq 0$ which is clearly true. Equality holds for $2xy - t^2$ and $x = y$.

$(ii): (y^2 + t^2)(z^2 + t^2) \geq t^2(y + z)^2 \Leftrightarrow y^2z^2 + t^2(y^2 + z^2) + t^4 \geq t^2(x^2 + y^2) + 2t^2$ $\Leftrightarrow y^2z^2 - 2t^2yz + t^4 \geq 0 \Leftrightarrow (yz - t^2)^2 \geq 0$ which is clearly true. Equality holds for $t^2 = yz$. Therefore,

$$\prod_{cyc}(x^2 + t^2) \overset{(i)}{\geq} \frac{3}{4}t^2((x + y)^2 + t^2)(z^2 + t^2) \geq \frac{3}{4}t^2 \cdot t^2\big((x + y) + z\big)^2 =$$

$$= \frac{3}{4}t^4(x + y + z)^2 \geq \frac{9}{4}t^4(xy + yz + zx)$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$BN \geq h_b, CP \geq h_c, AM \geq h_a$$

$$\prod_{cyc}(a^2 + b \cdot BN) \geq \prod_{cyc}(a^2 + b \cdot h_b) = \prod_{cyc}(a^2 + 2F) =$$

$$= (a^2 + F + F)(b^2 + F + F)(c^2 + F + F) \overset{Holder}{\geq}$$

$$\geq \left(\sqrt[3]{a^2F^2} + \sqrt[3]{b^2F^2} + \sqrt[3]{c^2F^2}\right)^3 = F^2\left(\sqrt[3]{a^2} + \sqrt[3]{b^2} + \sqrt[3]{c^2}\right) \overset{AGM}{\geq}$$

$$\geq F^2\left(3\sqrt[3]{a^2b^2c^2}\right)^3 = 27F^2 \cdot \sqrt[3]{a^2b^2c^2} = 27F^2\sqrt[3]{16R^2F^2} \geq$$

$$\geq 27F^2 \cdot \sqrt[3]{16 \cdot \frac{2}{3\sqrt{3}} \cdot s \cdot 2rF^2} = \frac{108F^3}{\sqrt{3}} = 36\sqrt{3}F^3$$

Solution 3 by Tapas Das-India

$$BN \geq h_b, CP \geq h_c, AM \geq h_a$$

$$\prod_{cyc}(a^2 + b \cdot BN) \geq \prod_{cyc}(a^2 + b \cdot h_b) = \prod_{cyc}(a^2 + 2F) =$$

$$= (a^2 + F + F)(b^2 + F + F)(c^2 + F + F) \overset{Holder}{\geq}$$

$$\geq \left(\sqrt[3]{a^2F^2} + \sqrt[3]{b^2F^2} + \sqrt[3]{c^2F^2}\right)^3 = F^2\left(\sqrt[3]{a^2} + \sqrt[3]{b^2} + \sqrt[3]{c^2}\right) \overset{AGM}{\geq}$$

$$\geq F^2\left(3\sqrt[3]{a^2b^2c^2}\right)^3 \geq 27\left(\frac{4F}{\sqrt{3}}\right) \cdot F^2 = 27 \cdot \frac{4F}{\sqrt{3}}F^2 = 36\sqrt{3}F^3$$

UNDERGRADUATE PROBLEMS

UP.466 Find:

$$\Omega = \int_0^{\pi}\left(\frac{x\cos x}{1 + \sin x}\right)^2 dx$$

Proposed by Florică Anastase-Romania

Solution 1 by Benjamin Bamidele-Nigeria

$$\Omega=\int_0^{\pi}\left(\frac{x\cos x}{1+\sin x}\right)^2 dx=\int_0^{\pi}\frac{x^2\cos^2 x}{(1+\sin x)^2}dx=\int_0^{\pi}\frac{x^2\cos^2 x}{\left(2\sin^2\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)^2}dx \overset{t=\frac{x}{2}+\frac{\pi}{4}}{=}$$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(2t-\frac{\pi}{2}\right)^2\cos^2\left(2t-\frac{\pi}{2}\right)\csc^4 t\,dt=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(2t-\frac{\pi}{2}\right)^2\sin^2(2t)\csc^4 t\,dt=$$

$$=2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(2t-\frac{\pi}{2}\right)^2\cot^2 t\,dt\overset{IBP}{=}-2\left(2t-\frac{\pi}{2}\right)^2(t+\cot t)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(16t-4\pi)(t+\cot t)\,dt=$$

$$=2\pi^2-\frac{3\pi^3}{2}+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}t(16t-4\pi)+(16t-4\pi)\cot t\,dt=$$

$$=2\pi^2-\frac{3\pi^3}{2}+\frac{7\pi^3}{6}+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(16t-4\pi)\cot t\,dt\overset{IBP}{=}$$

$$=2\pi^2-\frac{\pi^3}{3}+(16t-4\pi)\log(\sin t)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}-16\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\log(\sin t)\,dt=$$

$$=2\pi^2-\frac{\pi^3}{3}+8\pi\log\left(\frac{1}{\sqrt{2}}\right)-16\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\log(\sin t)\,dt$$

Let: $\Phi=16\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\log(\sin t)\,dt=-16\log 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}dt-16\sum_{k=1}^{\infty}\frac{1}{k}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos(2kt)\,dt=$

$$=-8\pi\log 2-8\sum_{k=1}^{\infty}\left(\frac{\sin\left(\frac{3k\pi}{2}\right)}{k^2}-\frac{\sin\left(\frac{k\pi}{2}\right)}{k^2}\right)=-8\pi\log 2-8\sum_{k=1}^{\infty}\left(\frac{(-1)^k}{(2k-1)^2}-\frac{(-1)^{k-1}}{(2k-1)^2}\right)=$$

$$=-8\pi\log 2-16\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}=-8\pi\log 2+16G$$

Therefore $\Omega=\int_0^{\pi}\left(\frac{x\cos x}{1+\sin x}\right)^2dx=2\pi^2-\frac{\pi^3}{3}+4\pi\log 2-16G$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\pi}\left(\frac{x\cos x}{1+\sin x}\right)^2dx=\underbrace{\int_0^{\frac{\pi}{2}}x^2\left(\frac{\cos x}{1+\sin x}\right)^2dx}_{x\to\frac{\pi}{2}-x}+\underbrace{\int_{\frac{\pi}{2}}^{\pi}x^2\left(\frac{\cos x}{1+\sin x}\right)^2dx}_{x\to\frac{\pi}{2}+x}=$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)^2\tan^2\left(\frac{x}{2}\right)dx+\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}+x\right)^2\tan^2\left(\frac{x}{2}\right)dx=$$

$$=\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\tan^2\left(\frac{x}{2}\right)dx+2\int_0^{\frac{\pi}{2}}x^2\tan^2\left(\frac{x}{2}\right)dx\overset{x=2x}{=}\pi^2\underbrace{\int_0^{\frac{\pi}{4}}\tan^2 x\,dx}_{A}+16\underbrace{\int_0^{\frac{\pi}{4}}x^2\tan^2 x\,dx}_{B}$$

$$A=[\tan x-x]_0^{\frac{\pi}{4}}=1-\frac{\pi}{4}$$

$$B=[x^2\tan x-x^3]_0^{\frac{\pi}{4}}-2\int_0^{\frac{\pi}{4}}x\tan x\,dx+2\int_0^{\frac{\pi}{4}}x^2\,dx=$$

$$=\frac{\pi^2}{16}-\frac{\pi^3}{192}+2[x\log(\cos x)]_0^{\frac{\pi}{4}}-2\underbrace{\int_0^{\frac{\pi}{4}}\log(\cos x)\,dx}_{\frac{G}{2}-\frac{\pi}{4}\log 2},B=\frac{\pi^2}{16}-\frac{\pi^3}{192}+\frac{\pi}{4}\log 2-G$$

$$\Omega=\int_0^{\pi}\left(\frac{x\cos x}{1+\sin x}\right)^2dx=2\pi^2-\frac{\pi^3}{3}+4\pi\log 2-16G$$

Solution 3 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\pi}\frac{x^2\cos^2 x}{(1+\sin x)^2}dx=\int_0^{\pi}\frac{x^2(1-\sin^2 x)}{(1+\sin x)^2}dx=\int_0^{\pi}\frac{x^2(1-\sin x)}{1+\sin x}dx=-\frac{\pi^3}{3}+2\int_0^{\pi}\frac{x^2}{1+\sin x}dx$$

$$M=\int_0^{\pi}\frac{x^2}{1+\sin x}dx=\int_0^{\pi}\frac{x^2}{\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)^2}dx=\frac{1}{2}\int_0^{\pi}\frac{x^2}{\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}dx=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{\left(2x-\frac{\pi}{2}\right)^2}{\sin^2 x}dx$$

$$=\left[-\left(2x-\frac{\pi}{2}\right)^2\operatorname{ctg}x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}+4\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(2x-\frac{\pi}{2}\right)\operatorname{ctg}x\,dx$$

$$=\pi^2+8\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}x\operatorname{ctg}x\,dx-2\pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\operatorname{ctg}x\,dx=\pi^2+8\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}x\operatorname{ctg}x\,dx$$

$$=\pi^2+8\left[[x\log(\sin x)]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\log(\sin x)\,dx\right]$$

$$=\pi^2+8\left[-\frac{\pi}{4}\log(2)-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\log(\sin x)\,dx\right]=\pi^2-2\pi\log(2)-8\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\log(\sin x)\,dx$$

$$N=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\log(\sin x)\,dx=-\frac{\pi}{2}\log(2)-\sum_{k=1}^{\infty}\frac{1}{k}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos(2kx)\,dx$$

$$=-\frac{\pi}{2}\log(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}[\sin(2kx)]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$=-\frac{\pi}{2}\log(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{3\pi k}{2}\right)}{k^2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{\pi k}{2}\right)}{k^2}$$

$$=-\frac{\pi}{2}\log(2)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}=-\frac{\pi}{2}\log(2)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}$$

$$=-\frac{\pi}{2}\log(2)+G$$

$$M=\int_0^{\pi}\frac{x^2}{1+\sin x}dx=\pi^2-2\pi\log(2)-8\left(-\frac{\pi}{2}\log(2)+G\right)=\pi^2+2\pi\log(2)-8G$$

$$I=\int_0^{\pi}\frac{x^2\cos^2 x}{(1+\sin x)^2}dx=-\frac{\pi^3}{3}+2\left(\pi^2+2\pi\log(2)-8G\right)$$

UP.467 Find:

$$\Omega=\int_0^{\pi}\frac{x^2\cos^3 x}{(1+\sin^2 x)^2}dx$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$\text{Let: } f(x)=\frac{x^2\sin x}{1+\sin^2 x};\, x\in[0,\pi], \text{then:}$$

$$f'(x)=\frac{(x^2\sin x)'(1+\sin^2 x)-x^2\sin x(1+\sin^2 x)'}{(1+\sin^2 x)^2}=$$

$$=\frac{(2x\sin x+x^2\cos x)(1+\sin^2 x)-x^2\sin x\cdot 2\sin x\cos x}{(1+\sin^2 x)^2}=$$

$$=\frac{2x\sin x+2x\sin^3 x+x^2\cos x+x^2\sin^2 x\cos x-2x^2\sin^2 x\cos x}{(1+\sin^2 x)^2}=$$

$$=\frac{2x\sin x+2x\sin^3 x+x^2\cos x-x^2\sin^2 x\cos x}{(1+\sin^2 x)^2}=$$

$$=\frac{2x\sin x+2x\sin^3 x+x^2\cos x(1-\sin^2 x)}{(1+\sin^2 x)^2}=$$

$$=\frac{2x\sin x+2x\sin^3 x+x^2\cos^3 x}{(1+\sin^2 x)^2}=\frac{2x\sin x\,(1+\sin^2 x)}{(1+\sin^2 x)^2}+\frac{x^2\cos^3 x}{(1+\sin^2 x)^2}=$$

$$=\frac{2x\sin x}{1+\sin^2 x}+\frac{x^2\cos^3 x}{(1+\sin^2 x)^2}$$

So, we have:

$$\frac{2x\sin x}{1+\sin^2 x}+\frac{x^2\cos^3 x}{(1+\sin^2 x)^2}=\left(\frac{x^2\sin x}{1+\sin^2 x}\right)'$$

$$\int_0^\pi \frac{2x\sin x}{1+\sin^2 x}dx+\int_0^\pi \frac{x^2\cos^3 x}{(1+\sin^2 x)^2}dx=\frac{x^2\sin x}{1+\sin^2 x}\bigg|_0^\pi$$

$$\int_0^\pi \frac{2x\sin x}{1+\sin^2 x}dx+\int_0^\pi \frac{x^2\cos^3 x}{(1+\sin^2 x)^2}dx=0$$

$$\int_0^\pi \frac{x^2\cos^3 x}{(1+\sin^2 x)^2}dx=-2\int_0^\pi \frac{x\sin x}{1+\sin^2 x}dx;\,(1)$$

$$I=\int_0^\pi \frac{x\sin x}{1+\sin^2 x}dx=\int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\sin^2(\pi-x)}dx=$$

$$=\pi\underbrace{\int_0^\pi \frac{\sin x}{1+\sin^2 x}dx}_{I_1}-\int_0^\pi \frac{x\sin x}{1+\sin^2 x}dx=\pi I_1-I$$

$$I=\frac{\pi}{2}I_1=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{2-\cos^2 x}dx=\frac{\pi}{2}\int_0^\pi \frac{d(\cos x)}{\cos^2 x-2}=\frac{\pi}{4\sqrt{2}}\log\left|\frac{\cos x-\sqrt{2}}{\cos x+\sqrt{2}}\right|\Bigg|_0^\pi=$$

$$=\frac{\pi}{4\sqrt{2}}\log\left(\sqrt{2}+1\right);\,(2)$$

From (1) and (2) it follows that:

$$\int_0^\pi \frac{x^2\cos^3 x}{(1+\sin^2 x)^2}dx=-2\int_0^\pi \frac{x\sin x}{1+\sin^2 x}dx=-\frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2}+1\right)=$$

$$=-\pi\sqrt{2}\log\left(1+\sqrt{2}\right)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\int \frac{\cos^3 x}{(1+\sin^2 x)^2}dx \overset{t=\sin x}{=} \int \frac{1-t^2}{(1+t^2)^2}dt=\int \frac{1+t^2-2t^2}{(1+t^2)^2}dt=$$

$$=\int d\left(\frac{t}{1+t^2}\right)=\frac{t}{1+t^2}$$

$$\int \frac{\cos^3 x}{(1+\sin^2 x)^2}dx = \frac{\sin x}{1+\sin^2 x} + C$$

$$\Omega \overset{IBP}{=} \frac{x^2 \sin x}{1+\sin^2 x}\Bigg|_0^\pi - 2\int_0^\pi \frac{x\sin x}{1+\sin^2 x}dx \overset{x\to\pi-x}{=} -2\pi\int_0^\pi \frac{\sin x}{1+\sin^2 x}dx - \Omega$$

$$\Omega = \pi\int_0^\pi \frac{\sin x}{1+\sin^2 x}dx = -2\pi\int_0^{\frac{\pi}{2}} \frac{\sin x}{2-\cos^2 x}dx \overset{t=\cos x}{=} -2\pi\int_0^1 \frac{dt}{2-t^2}$$

$$\Omega = -\frac{\pi}{\sqrt{2}}\log\left(\frac{\sqrt{2}+t}{\sqrt{2}-t}\right)\Bigg|_0^1 = -\pi\sqrt{2}\log\left(1+\sqrt{2}\right)$$

Solution 3 by Ankush Kumar Parcha-India

$$I_1 = \int_0^\pi \frac{x\cos^3 x}{(1+\sin^2 x)^2}dx \overset{x\to\pi-x}{=} \int_0^\pi \frac{-(\pi-x)\cos^3 x}{(1+\sin^2 x)^2}dx$$

$$I_1 = I_1 - \pi\int_0^\pi \frac{\cos^3 x}{(1+\sin^2 x)^2}dx \Rightarrow \int_0^\pi \frac{\cos^3 x}{(1+\sin^2 x)^2}dx = 0;\ \ (1)$$

$$\Omega = \int_0^\pi \frac{x^2\cos^3 x}{(1+\sin^2 x)^2}dx \overset{x\to\pi-x}{=} -\pi^2\int_0^\pi \frac{\cos^3 x}{(1+\sin^2 x)^2}dx -$$

$$-\int_0^\pi \frac{x^2\cos^3 x}{(1+\sin^2 x)^2}dx + 2\pi\int_0^\pi \frac{x\cos^3 x}{(1+\sin^2 x)^2}dx\left(\text{ by using }(1)\right)$$

$$\Omega = -\Omega + 2\pi\int_0^\pi \frac{x\cos^3 x}{(1+\sin^2 x)^2}dx \Rightarrow \Omega = \pi\int_0^\pi \frac{x\cos^3 x}{(1+\sin^2 x)^2}dx;\ \ (2)$$

$$\text{Let } I = \int \frac{\cos^3 x}{(1+\sin^2 x)^2}dx \overset{\sin x = y}{=} \int \frac{1-y^2}{(1+y^2)^2}dy \overset{y=\tan z}{=} \int \frac{1-\tan z}{\sec^4 z}\cdot \sec^2 z\,dz =$$

$$= \int (\cos^2 z - \sin^2 z)dz$$

$$I = \sin z\cos z \Rightarrow I = \int \frac{\cos^3 x}{(1+\sin^2 x)^2}dx = \frac{\sin x}{1+\sin^2 x} + C\left(\text{by using }(2)\right)$$

$$\Omega = \pi\left[\frac{x\sin x}{1+\sin^2 x}\Bigg|_0^\pi - \int_0^\pi \frac{\sin x}{1+\sin^2 x}dx\right] = -\pi\int_0^\pi \frac{\sin x}{1+\sin^2 x}dx$$

by using Weierstrass substitution:

$$\Omega = -\pi\int_0^\infty \frac{\frac{2t}{1+t^2}}{1+\frac{4t^2}{(1+t^2)^2}}\cdot\frac{2t}{1+t^2}dt = -4\pi\int_0^\infty \frac{t(1+t^2)^2}{t^4+1+6t^2}\cdot\frac{dt}{(1+t^2)^2} \overset{t^2=k}{=}$$

$$=-2\pi\int_0^{\infty}\frac{dk}{k^2+6k+1}=-2\pi\int_0^{\infty}\frac{dk}{(k+3)^2-\left(2\sqrt{2}\right)^2}=$$

$$=\frac{\pi}{2\sqrt{2}}\log\left(\frac{k+3-2\sqrt{2}}{k+3+2\sqrt{2}}\right)\Bigg|_0^{\infty}=\frac{\pi}{2\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2=-\pi\sqrt{2}\log\left(1+\sqrt{2}\right)$$

UP.468 Let $m\in\mathbb{N}$ and $0<x<m$. If $i=\left[\frac{(m+1)x}{x+1}\right]$, prove that

$2\binom{m}{i}\geq x^{m-i}$, where $[x]$ is integer part of x and $\binom{m}{i}$ is a binomial coefficient.

Proposed by Ovidiu Pop-Romania

Solution by proposer

Let L_m be the Bleimann-Butzer-Hahn operator, defined by

$(L_mf)(x)=(1+x)^{-m}\sum_{k=0}^{m}\binom{m}{k}x^kf\left(\frac{k}{m-k+1}\right)$, for any $f\in C_B([0,\infty)), x\in[0,\infty)$ and $m\in\mathbb{N}$.

If $x\in[0,\infty)$, we consider the functions $\varphi_x:[0,\infty)\to[0,\infty), \varphi_{x(t)}=|t-x|, t\in[0,\infty)$

and

$e_0,e_1:[0,\infty)\to R, e_0(x)=1, e_1(x)=x, x\in[0,\infty)$. It is well known that

(1) $(L_me_0)(x)=1$,

(2) $(L_me_1)(x)=x-x\left(\frac{x}{1+x}\right)^m$

and

(3) $0<\frac{1}{m}<\frac{2}{m-1}<\frac{3}{m-2}<\ldots<\frac{m-1}{2}<\frac{m}{1}$,

where $x\in[0,\infty), m\in\mathbb{N}$.

Because the operators $(L_m)_{m\geq1}$ are linear and positive, we have

(4) $(L_m\varphi_x)(x)\geq0$

for any any $x\in[0,\infty), m\in\mathbb{N}$.

Lemma. Let $m\in N=\{1,2,\ldots\}$ and $0\leq x<m$. If $i=\left[\frac{(m+1)x}{x+1}\right]$, then

(5) $(L_m\varphi_x)(x)=(1+x)^{-m}\left\{2\binom{m}{i}x^{i+1}-x^{m+1}\right\}$.

Proof. Because $0 \le x < m$, it result that $i = \left[\frac{(m+1)x}{x+1}\right] \le \frac{(m+1)x}{x+1} < m$, so $i < m$.

Relation $i = \left[\frac{(m+1)x}{x+1}\right]$ is equivalent to $i \le \frac{(m+1)x}{x+1} < i+1$, equivalent to

$\frac{i}{m-i+1} \le x < \frac{i+1}{m-i}$, and taking (3) into account, we have

$$0 < \frac{1}{m} < \frac{2}{m-1} <\ldots< \frac{i}{m-i+1} \le x < \frac{i+1}{m-i} <\ldots< \frac{m-1}{2} < \frac{m}{1}.$$

If $\ge \frac{1}{m}$, taking that the function $\frac{x}{x+1}$ is increasing on $[0,\infty)$, it result that $\frac{x}{x+1} \ge \frac{1}{m+1}$, from where $\frac{(m+1)x}{x+1} \ge 1$, so $i \ge 1$. We have that

$$(L_m\varphi_x)(x) = (1+x)^{-m}\sum_{k=0}^{m}\binom{m}{k}x^k\left|\frac{k}{m-k+1} - x\right| =$$

$$= (1+x)^{-m}\sum_{k=0}^{i}\binom{m}{k}x^k\left(x - \frac{k}{m-k+1}\right) + (1+x)^{-m}\sum_{k=i+1}^{m}\binom{m}{i}x^k\left(\frac{k}{m-k+1} - x\right) =$$

$$= 2(1+x)^{-m}\sum_{k=0}^{i}\binom{m}{k}x^k\left(x - \frac{k}{m-k+1}\right) + (1+x)^{-m}\sum_{k=0}^{m}\binom{m}{i}x^k\left(\frac{k}{m-k+1} - x\right) =$$

$$= 2(1+x)^{-m}\left\{x\sum_{k=0}^{i}\binom{m}{k}x^k - \sum_{k=0}^{i}\binom{m}{k}x^k\frac{k}{m-k+1}\right\} +$$

$$+(1+x)^{-m}\sum_{k=0}^{m}\binom{m}{k}x^k\frac{k}{m-k+1} - x(1+x)^{-m}\sum_{k=0}^{m}\binom{m}{k}x^k =$$

$$= 2(1+x)^{-m}\left\{x + \sum_{k=1}^{i}\binom{m}{k}x^{k+1} - \sum_{k=1}^{i}\binom{m}{k-1}x^k\right\} + (L_me_1)(x) - x(L_me_0)(x)$$

and taking (1) and (2) into account, we have

$$(L_m\varphi_x)(x) = 2(1+x)^{-m}\left\{x + \sum_{k=1}^{i}\left(\binom{m}{k}x^{k+1} - \binom{m}{k-1}x^k\right)\right\}x\left(\frac{x}{1+x}\right)^m$$

$$= 2(1+x)^{-m}\left\{x + \binom{m}{1}x^2 - \binom{m}{0}x^1 + \binom{m}{2}x^3 - \binom{m}{1}x^2 + \cdots + \binom{m}{i}x^{i+1} - \binom{m}{i-1}x^i\right\}$$
$$- x^{m+1}(1+x)^{-m}$$

from where, relation (5) follows.

If $< \frac{1}{m}$, taking that the function $\frac{x}{x+1}$ is increasing on $[0,\infty)$, it result that $\frac{x}{x+1} < \frac{1}{m+1}$, from where $\frac{(m+1)x}{x+1} < 1$, so $i = 0$. We have that:

$$(L_m\varphi_x)(\mathrm{x}) = (1+x)^{-m}\left\{x + \sum_{k=0}^{m}\binom{m}{k}x^k\left(\frac{k}{m-k+1} - x\right)\right\}$$

$$= (1+x)^{-m}\left\{2x + \sum_{k=0}^{m}\binom{m}{k}x^k\left(\frac{k}{m-k+1} - x\right)\right\} =$$

$$= 2x(1+x)^{-m} + (L_m e_1)(x) - x(L_m e_0)(x) = (1+x)^{-m}(2x - x^{m+1}).$$

From the cases above, results that the relation (5) is true.

From (4) and (5), the inequality from hypothesis follows.

UP.469 Prove that:

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt[n]{(n!)^2}} < \frac{\pi^2 e}{6}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt[n]{(n!)^2}} = \sum_{n=1}^{\infty}\sqrt[n]{\frac{1}{1^2}\cdot\frac{1}{2^2}\cdot\ldots\cdot\frac{1}{n^2}} = \sum_{n=1}^{\infty}\sqrt[n]{\frac{\frac{1}{1^2}\cdot(1+1)^2\cdot\frac{1}{2^2}\cdot\frac{(2+1)^2}{(1+1)^2}\cdots\frac{1}{n^2}\cdot\frac{(n+1)^n}{n^n}}{(1+1)\cdot\frac{(2+1)^2}{(1+1)^2}\cdot\ldots\cdot\frac{(n+1)^n}{n^n}}}$$

$$=$$

$$= \sum_{n=1}^{\infty}\frac{1}{n+1}\cdot\sqrt[n]{\frac{1}{1^2}\cdot(1+1)^2\cdot\frac{1}{2^2}\cdot\frac{(2+1)^2}{(1+1)^2}\cdots\frac{1}{n^2}\cdot\frac{(n+1)^n}{n^n}} \overset{AM-GM}{\leq}$$

$$\leq \sum_{n=1}^{\infty}\frac{1}{n+1}\cdot\frac{\frac{1}{1^2}\cdot(1+1)^2 + \frac{1}{2^2}\cdot\frac{(2+1)^2}{(1+1)^2} + \cdots + \frac{1}{n^2}\cdot\frac{(n+1)^n}{n^n}}{n} \leq$$

$$\le \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\cdot\sum_{k=1}^{n}\frac{1}{k^2}\cdot\frac{(k+1)^k}{k^{k-1}}\le\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\frac{(k+1)^k}{k^{k-1}}\cdot\sum_{n=k}^{\infty}\frac{1}{n(n+1)}=$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\frac{(k+1)^k}{k^{k-1}}\cdot\sum_{n=k}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=$$

$$=\sum_{k=1}^{\infty}\frac{(k+1)^k}{k^{k-1}}\cdot\frac{1}{k^2}\cdot\frac{1}{k}=\sum_{k=1}^{\infty}\frac{1}{k^2}\left(1+\frac{1}{k}\right)^k<e\cdot\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2\cdot e}{6}$$

Solution 2 by Henry Ricardo-New York-USA

We use Carleman's inequality for a sequence of positive real numbers $a_1, a_2, a_3, ..$

$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{\frac{1}{n}}<e\sum_{n=1}^{\infty}a_n$$

Letting $a_k=\frac{1}{k^2}$ we have:

$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{\frac{1}{n}}=\sum_{n=1}^{\infty}\frac{1}{(1\cdot1\cdot2\cdot2\cdot\ldots\cdot n\cdot n)^{\frac{1}{n}}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt[n]{(n!)^2}}<e\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2e}{6}$$

where we use the well-known value of Riemann's zeta function.

UP.470 If $(a_n)_{n\ge1}, a_0>0, a_{n+1}=(2n+1)a_n, \forall n\in\mathbb{N}^*$ then find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by proposer

$a_1=1\cdot a_0, a_2=3\cdot1\cdot a_0$ and from mathematical induction, we obtain:

$a_n=(2n-1)!!, \forall n\in\mathbb{N}^*$. So, it follows that:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\,a_0}-\sqrt[n]{(2n-1)!!\,a_0}\right)=$$

$$=\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!}-\sqrt[n]{(2n-1)!!}\right)=\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\,(u_n-1)=$$

$$=\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n=\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n,\forall n\ge2$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n = \frac{2}{e}, \qquad u_n = \frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}, \forall n \geq 2$$

Hence,

$$\lim_{n\to\infty} u_n = 1 \Rightarrow \lim_{n\to\infty} \frac{u_n - 1}{\log u_n} = 1$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = 2 \cdot \frac{e}{2} = e$$

Therefore,

$$\Omega = \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = \frac{2}{e} \cdot 1 \cdot \log e = \frac{2}{e}$$

Solution 2 by Marian Ursărescu-Romania

$$a_1 = 1 \cdot a_0,$$

$$a_2 = 3 \cdot 1 \cdot a_0$$

$$\vdots$$

$$a_n = (2n-1)!!, \forall n \in \mathbb{N}^*$$

So, it follows that:

$$\Omega = \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = \lim_{n\to\infty} \left(\sqrt[n+1]{(2n+1)!!\, a_0} - \sqrt[n]{(2n-1)!!\, a_0}\right) =$$

$$= \lim_{n\to\infty} \left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right) = \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} \cdot n \left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} - 1\right); (1)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} =$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n = \frac{2}{e}; \ (2)$$

$$\lim_{n\to\infty} n \left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} - 1\right) = \lim_{n\to\infty} \frac{\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} - 1}{\log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)} \cdot n \cdot \log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) =$$

$$=\lim_{n\to\infty}\frac{e^{\log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)}-1}{\log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)}\cdot\log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^n=\lim_{n\to\infty}\log\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^n=$$

$$=\log\left(\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^n\right)=\log\left(\lim_{n\to\infty}\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)=$$

$$=\log\left(\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\right)=\log\left(2\cdot\frac{e}{2}\right)=\log e=1;(3)$$

From (1), (2) and (3): $\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\frac{2}{e}\cdot 1\cdot\log e=\frac{2}{e}$

UP.471 Let $m\ge 0$ and $H_n=\sum_{k=1}^{n}\frac{1}{n}, n\in\mathbb{N}^*$. Find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1}-\left(\sqrt[n]{n!}\right)^{m+1}\right)\cdot e^{-mH_n}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by proposer

$$B_n=\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1}-\left(\sqrt[n]{n!}\right)^{m+1}\right)\cdot e^{-mH_n}=$$

$$=\left(\sqrt[n]{n!}\right)^{m+1}(u_n-1)\cdot e^{-mH_n}=\left(\sqrt[n]{\frac{n!}{n^n}}\right)^{m+1}\cdot(u_n-1)n^{m+1}\cdot e^{-mH_n}=$$

$$=\left(\sqrt[n]{\frac{n!}{n^n}}\right)^{m+1}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n\cdot e^{m\log n-mH_n}=\left(\sqrt[n]{\frac{n!}{n^n}}\right)^{m+1}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n\cdot e^{-m\gamma_n}$$

where $\gamma_n\to\gamma,\gamma$ –Euler-Mascheroni constant.

$$u_n=\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+1}=\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}\right)^{m+1},\forall n\ge 2$$

Hence, we have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

$$\lim_{n\to\infty} u_n = \left(\frac{1}{e}\cdot e\cdot 1\right)^{m+1} = 1, \qquad \lim_{n\to\infty}\frac{u_n - 1}{\log u_n} = 1$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right)^{m+1} = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{m+1} = e^{m+1}$$

$$\lim_{n\to\infty} e^{-m\gamma_n} = e^{-m\gamma} = \frac{1}{e^{m\gamma}}$$

$$\Omega = \lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1} - \left(\sqrt[n]{n!}\right)^{m+1}\right)\cdot e^{-mH_n} = 1\cdot\log e^{m+1}\cdot\frac{1}{e^{m\gamma}} = \frac{m+1}{e^{m\gamma+m+1}}$$

Solution 2 by Asmat Qatea-Afghanistan

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}, \lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1} = \frac{1}{e}, \lim_{n\to\infty}(H_n - ln(n)) = \gamma$$

$$\Omega = \lim_{n\to\infty}\left(\left(\frac{n+1}{e}\right)^{m+1} - \left(\frac{n}{e}\right)^{m+1}\right)\left(e^{\gamma+lnn}\right)^{-m}$$

$$\Omega = \lim_{n\to\infty}\frac{1}{e^{m+1}}((n+1)^{m+1} - n^{m+1})e^{-\gamma m}n^{-m}, \qquad \Omega = e^{-\gamma m-m-1}\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{m+1} - 1}{\frac{1}{n}}$$

$$\Omega = e^{-\gamma m-m-1}\lim_{n\to\infty}\frac{(m+1)\left(1+\frac{1}{n}\right)^{m}\left(-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} = \frac{m+1}{e^{m\gamma+m+1}}$$

UP.472 Find:

$$\Omega = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^x\cdot\sin x\,(x + x\cot x + 1)\,dx$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

Solution 1 by proposers

Let $f:\left[\frac{\pi}{3},\frac{\pi}{6}\right]\to\mathbb{R}, f(x) = x\cdot e^x\cdot\sin x$, then $f'(x) = e^x\cdot\sin x\,(x + x\cot x + 1)$

$$\Omega = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^x\cdot\sin x\,(x + x\cot x + 1)\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f'(x)dx = f(x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= e^{\frac{\pi}{3}}\cdot\frac{\pi}{3}\cdot\frac{\sqrt{3}}{2} - e^{\frac{\pi}{6}}\cdot\frac{\pi}{6}\cdot\frac{1}{2} = \frac{\pi}{12}\left(2\sqrt{3}e^{\frac{\pi}{3}} - e^{\frac{\pi}{6}}\right) = \frac{\pi\cdot e^{\frac{\pi}{6}}}{12}\left(2\sqrt{3}\cdot e^{\frac{\pi}{3}} - 1\right)$$

Solution 2 by Tapas Das-India

$$e^x\cdot\sin x\,(x+x\cot x+1)=e^x(x\sin x+x\cos x+\sin x)$$

Let $f(x)=x\cdot\sin x$, then $f'(x)=\sin x+x\cos x$

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\cdot\sin x\,(x+x\cot x+1)\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\big(f(x)+f'(x)\big)dx=$$

$$=[e^xf(x)]_{\frac{\pi}{6}}^{\frac{\pi}{3}}==e^{\frac{\pi}{3}}\cdot\frac{\pi}{3}\cdot\frac{\sqrt{3}}{2}-e^{\frac{\pi}{6}}\cdot\frac{\pi}{6}\cdot\frac{1}{2}=\frac{\pi}{12}\left(2\sqrt{3}e^{\frac{\pi}{3}}-e^{\frac{\pi}{6}}\right)=\frac{\pi\cdot e^{\frac{\pi}{6}}}{12}\left(2\sqrt{3}\cdot e^{\frac{\pi}{3}}-1\right)$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\cdot\sin x\,(x+x\cot x+1)\,dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}xe^x\sin x\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}xe^x\cos x\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,dx$$

$$\because\int e^{ax}\sin(bx)\,dx=\frac{e^{ax}}{a^2+b^2}[a\sin(bx)-b\cos(bx)]+C$$

$$\because\int e^{ax}\cos(bx)\,dx=\frac{e^{ax}}{a^2+b^2}[a\cos(bx)+b\sin(bx)]+C$$

$$\Omega=\frac{xe^x(\sin x-\cos x)}{2}\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{e^x(\sin x-\cos x)}{2}dx+$$

$$+\frac{xe^x(\cos x+\sin x)}{2}\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{e^x(\cos x+\sin x)}{2}dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,dx=$$

$$=xe^x\sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,dx=\frac{\pi}{12}\left(2\sqrt{3}e^{\frac{\pi}{3}}-e^{\frac{\pi}{6}}\right)=$$

$$=\frac{\pi\cdot e^{\frac{\pi}{6}}}{12}\left(2\sqrt{3}\cdot e^{\frac{\pi}{3}}-1\right)$$

Solution 4 by Yen Tung Chung- Taiwan

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\cdot\sin x\,(x+x\cot x+1)\,dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}x(e^x\sin x+e^x\cos x)dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}xd(e^x\sin x)+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,dx=$$

$$=xe^x\sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,dx=\frac{\pi}{12}\left(2\sqrt{3}e^{\frac{\pi}{3}}-e^{\frac{\pi}{6}}\right)=$$

$$=\frac{\pi\cdot e^{\frac{\pi}{6}}}{12}\left(2\sqrt{3}\cdot e^{\frac{\pi}{3}}-1\right)$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\cdot\sin x\,(x+x\cot x+1)\,dx\overset{(*)}{=}$$

$$\frac{d}{dx}(xe^x\sin x)=e^x\sin x\,(x+x\cot x+1)$$

$$\overset{(*)}{=}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^x\sin x\,(x+x\cot x+1)dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}d(e^x x\sin x)=e^x x\sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=$$

$$=e^{\frac{\pi}{3}}\cdot\frac{\pi}{3}\cdot\frac{\sqrt{3}}{2}-e^{\frac{\pi}{6}}\cdot\frac{\pi}{6}\cdot\frac{1}{2}=\frac{\pi}{12}\left(2\sqrt{3}e^{\frac{\pi}{3}}-e^{\frac{\pi}{6}}\right)=\frac{\pi\cdot e^{\frac{\pi}{6}}}{12}\left(2\sqrt{3}\cdot e^{\frac{\pi}{3}}-1\right)$$

UP.473 Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{\left((n+1)!\right)^2}{(2n+1)!!}}-\sqrt[n]{\frac{(n!)^2}{(2n-1)!!}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposer

$$B_n=\sqrt[n+1]{\frac{\left((n+1)!\right)^2}{(2n+1)!!}}-\sqrt[n]{\frac{(n!)^2}{(2n-1)!!}}=\sqrt[n]{\frac{(n!)^2}{(2n-1)!!}}(u_n-1)=$$

$$=\left(\frac{\sqrt[n]{n!}}{n}\right)^2\cdot\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{u_n-1}{\log u_n}\cdot n\cdot\log u_n=$$

$$=\left(\frac{\sqrt[n]{n!}}{n}\right)^2\cdot\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{u_n-1}{\log u_n^n}\cdot\log u_n^n,n\geq 2,$$

We have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\frac{e}{2}$$

$$u_n=\sqrt[n+1]{\frac{\left((n+1)!\right)^2}{(2n+1)!!}}\cdot\sqrt[n]{\frac{(2n-1)!!}{(n!)^2}}=$$

$$=\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\right)^2\left(\frac{n}{\sqrt[n]{n!}}\right)^2\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\frac{n}{n+1};\forall n\geq 2$$

$$\lim_{n\to\infty}u_n=\left(\frac{1}{e}\right)^2\cdot e^2\cdot\frac{e}{2}\cdot\frac{2}{e}\cdot 1=1\Rightarrow\lim_{n\to\infty}\frac{u_n-1}{\log u_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{\left((n+1)!\right)^2}{(2n+1)!!}\cdot\frac{(2n-1)!!}{(n!)^2}\cdot\sqrt[n+1]{\frac{(2n+1)!!}{\left((n+1)!\right)^2}}=$$

$$=\lim_{n\to\infty}\frac{(n+1)^2}{2n+1}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{\left(\sqrt[n+1]{(n+1)!}\right)^2}=\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^2\cdot\frac{(2n-1)!!}{(n!)^2}\cdot\frac{n+1}{2n+1}=$$

$$=e^2\cdot\frac{2}{e}\cdot\frac{1}{2}=e$$

Therefore,

$$\lim_{n\to\infty}B_n=\frac{1}{e^2}\cdot\frac{e}{2}\cdot 1\cdot\log e=\frac{1}{2e}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\frac{\left((n+1)!\right)^2}{(2n+1)!!}=\frac{\left((n+1)!\right)^2}{(2n+1)(2n-1)\cdot\ldots\cdot 3\cdot 1}=\frac{2^{n+1}\left((n+1)!\right)^3}{(2n+2)!}\text{ and}$$

$$\frac{(n!)^2}{(2n-1)!!}=\frac{2^n(n!)^3}{(2n)!}$$

$$\sqrt{2\pi m}m^me^{-m}e^{\frac{1}{12m+1}}<m!<\sqrt{2\pi m}m^me^{-m}e^{\frac{1}{12m}}$$

$$(2\pi)^{\frac{3}{2}}(n+1)^{\frac{3}{2}}(n+1)^{3(n+1)}e^{-3(n+1)}e^{\frac{3}{12n+3}}<\left((n+1)!\right)^3(2\pi)^{\frac{3}{2}}(n+1)^{\frac{3}{2}}(n+1)^{3(n+1)}e^{-3(n+1)}e^{\frac{1}{4(n+1)}}$$

$$(2\pi)^{-\frac{1}{2}}2^{-\frac{1}{2}}(n+1)^{-\frac{1}{2}}2^{-2(n+1)}(n+1)^{-2(n+1)}e^{2(n+1)}e^{-\frac{1}{24(n+1)}}<\frac{1}{(2n+2)!}<$$

$$< (2\pi)^{-\frac{1}{2}}2^{-\frac{1}{2}}(n+1)^{-\frac{1}{2}}2^{-2(n+1)}(n+1)^{-2(n+1)}e^{2(n+1)}e^{-\frac{1}{23n+25}}$$

$$\sqrt{2}\pi 2^{-(n+1)}(n+1)^{n+2}e^{-(n+1)}e^{\frac{3}{12n+13}-\frac{1}{24(n+1)}} < \frac{2^{n+1}\left((n+1)!\right)^3}{(2n+2)!} <$$

$$< \sqrt{2}\pi 2^{-(n+1)}(n+1)^{n+2}e^{-(n+1)}e^{\frac{1}{4(n+1)}-\frac{1}{24n+25}}$$

$$2^{\frac{1}{2(n+1)}-1}\pi^{\frac{1}{n+1}}(n+1)^{\frac{n+2}{n+1}}e^{-1}e^{\left(\frac{1}{12n+13}-\frac{1}{24(n+1)}\right)\cdot\frac{1}{n+1}} < \sqrt[n+1]{\frac{2^{n+1}\left((n+1)!\right)^3}{(2n+2)!}} <$$

$$< 2^{\frac{1}{2(n+1)}-1}\pi^{\frac{1}{n+1}}(n+1)^{\frac{n+2}{n+1}}e^{-1}e^{\left(\frac{1}{4(n+1)}-\frac{1}{24n+25}\right)\cdot\frac{1}{n+1}}$$

$$-2^{\frac{1}{2n}-1}\pi^{\frac{1}{n}}n^{\frac{1+n}{n}}e^{-1}e^{\left(\frac{1}{4n}-\frac{1}{24n+1}\right)\cdot\frac{1}{n}} < -\sqrt[n]{\frac{2^n(n!)^3}{(2n)!}} < -2^{\frac{1}{2n}-1}\pi^{\frac{1}{n}}n^{\frac{n+1}{n}}e^{-1}e^{\left(\frac{3}{12n+1}-\frac{1}{24n}\right)\cdot\frac{1}{n}}$$

$$\Omega \le \lim_{n\to\infty}\left(2^{\frac{1}{2(n+1)}-1}\pi^{\frac{1}{n+1}}(n+1)^{\frac{n+2}{n+1}}e^{-1}e^{\left(\frac{1}{12n+13}-\frac{1}{24(n+1)}\right)\cdot\frac{1}{n+1}}\right.$$

$$\left.-2^{\frac{1}{2n}}2^{-1}\pi^{\frac{1}{n}}n^{\frac{n+1}{n}}e^{-1}e^{\left(\frac{1}{4n}-\frac{1}{24n+1}\right)\cdot\frac{1}{n}}\right) = \frac{1}{2e}\lim_{n\to\infty}\left((n+1)^{\frac{n+2}{n+1}} - n^{\frac{n+1}{n}}\right)$$

Similarly,

$$\Omega \ge \frac{1}{2e}\lim_{n\to\infty}\left((n+1)^{\frac{n+2}{n+1}} - n^{\frac{n+1}{n}}\right)$$

Thus,

$$\Omega = \frac{1}{2e}\lim_{n\to\infty}\left((n+1)^{\frac{n+2}{n+1}} - n^{\frac{n+1}{n}}\right)$$

$$\left(\frac{1}{x}\right)^{1+x} = \frac{1}{x}e^{-x\log x} = \frac{1}{x}\left(1 - x\log x + O(x^{2-\epsilon})\right), \epsilon > 0$$

$$\Omega = \frac{1}{2e}\lim_{n\to\infty}\left[(n+1)\left(1 - \frac{\log(n+1)}{n+1} + O\left(\frac{1}{n^{2-\epsilon}}\right)\right) - n\left(1 - \frac{\log(n)}{n} + O\left(\frac{1}{n^{2-\epsilon}}\right)\right)\right] =$$

$$= \frac{1}{2e}\lim_{n\to\infty}\left(1 + \log(n) - \log(n+1) + O\left(\frac{1}{n^{1-\epsilon}}\right)\right) =$$

$$= \frac{1}{2e}\lim_{n\to\infty}\left(1 - \log\left(\frac{n+1}{n}\right) + O\left(\frac{1}{n^{1-\epsilon}}\right)\right) = \frac{1}{2e}$$

UP.474. Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, a_n, b_n \in \mathbb{R}_+^* = (0,\infty)$ such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a$,

$\lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n\sqrt[n]{a_n}} = b, a, b \in \mathbb{R}$. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$B_n = \sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n} = \sqrt[n]{b_n}\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} - 1\right) = \sqrt[n]{b_n}(u_n - 1) =$$

$$= \sqrt[n]{b_n}\cdot\frac{u_n - 1}{\log u_n}\cdot\log u_n = \frac{\sqrt[n]{b_n}}{n}\cdot\frac{u_n - 1}{\log u_n}\cdot\log u_n^n, \forall n \geq 1,$$

$$u_n = \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}, \forall n \geq 2$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = \lim_{n\to\infty}\frac{b_{n+1}}{b_n\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{a_n}}{n} = b\cdot\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = b\cdot\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}} \overset{C-D'A}{=}$$

$$= b\cdot\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n} = b\cdot\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}\left(\frac{n}{n+1}\right)^{n+1} = \frac{ab}{e}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n} = \lim_{n\to\infty}\frac{b_{n+1}}{b_n\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{a_n}}{n}\left(\frac{n}{n+1}\right)^n =$$

$$= b\cdot\frac{a}{e}\cdot\frac{1}{e} = \frac{ab}{e^2}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{n+1}{n} = \frac{ab}{e^2}\cdot\frac{e^2}{ab}\cdot 1 = 1$$

$$\lim_{n\to\infty}\frac{u_n - 1}{\log u_n} = 1$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{b_{n+1}}{b_n}\cdot\frac{1}{\sqrt[n+1]{b_{n+1}}} = \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}} = \frac{ab}{e}\cdot\frac{e^2}{ab}\cdot 1 = e$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \frac{ab}{e^2}\cdot 2\cdot\log e = \frac{ab}{e^2}$$

Solution 2 by Adrian Popa-Romania

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)=\lim_{n\to\infty}\sqrt[n]{b_n}\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}-1\right)=$$

$$\lim_{n\to\infty}\sqrt[n]{b_n}\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}-1\right)=\lim_{n\to\infty}\sqrt[n]{b_n}\cdot\frac{\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}-1}{\log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}}\cdot\log u_n=$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}\cdot\frac{\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}-1}{\log u_n}\cdot\log\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^n,\forall n\geq 1$$

Hence,

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{b_n\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{a_n}}{n}\left(\frac{n}{n+1}\right)^n=$$

$$=b\cdot\frac{a}{e}\cdot\frac{1}{e}=\frac{ab}{e^2}$$

and

$$\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{b_n\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{a_n}}{n}=b\cdot\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=b\cdot\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D'A}{=}$$

$$=b\cdot\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=b\cdot\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}\left(\frac{n}{n+1}\right)^{n+1}=\frac{ab}{e}$$

Now, we have:

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{n+1}{n}=\frac{ab}{e^2}\cdot\frac{e^2}{ab}\cdot 1=1$$

$$\lim_{n\to\infty}\frac{\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}-1}{\log\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}}=1$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^n=\lim_{n\to\infty}\frac{b_{n+1}}{b_n}\cdot\frac{1}{\sqrt[n+1]{b_{n+1}}}=\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}=\frac{ab}{e}\cdot\frac{e^2}{ab}\cdot 1=e$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \frac{ab}{e^2}\cdot 2\cdot\log e = \frac{ab}{e^2}$$

UP.475 Let $(a_n)_{n\geq 1}$ be sequence of real numbers such that $a_1 = 1$ and $(n+1)^2(a_{n+1}-a_n)-(a_{n+1}+n+1)=0$. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{1+a_n}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

We have:

$$(n+1)^2(a_{n+1}-a_n)-(a_{n+1}+n+1)=0$$

$$(n+1)^2(a_{n+1}-a_n)-a_{n+1}-(n+1)=0$$

$$[(n+1)^2-1]a_{n+1}-(n+1)^2a_n = n+1$$

$$(n^2+2n)a_{n+1}-(n+1)^2a_n = (n+1)$$

$$n(n+2)a_{n+1}-(n+1)^2a_n = n+1$$

$$\frac{n+2}{n+1}\cdot a_{n+1}-\frac{n+1}{n}\cdot a_n = \frac{1}{n}$$

Summing after $k\in\overline{1,n}$ in relation:

$$\frac{k+2}{k+1}\cdot a_{k+1}-\frac{k+1}{k}\cdot a_k = \frac{1}{k}\Rightarrow$$

$$\frac{n+1}{n+1}\cdot a_{n+1}-2a_1 = H_n, \text{where } H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

Hence,

$$a_{n+1} = \frac{n+1}{n+2}\cdot(H_n+2)\Rightarrow a_n = \frac{n}{n+1}(H_{n-1}+2)\Rightarrow \lim_{n\to\infty}a_n = +\infty$$

$$\Omega = \lim_{n\to\infty}\sqrt[n]{1+a_n} = \lim_{n\to\infty}(1+a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[(1+a_n)^{\frac{1}{a_n}}\right]^{\frac{a_n}{n}} = e^{\lim\limits_{n\to\infty}\frac{H_{n-1}+2}{n+1}} \overset{L.C-S}{=} e^0 = 1$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ n\in N_{\geq 1}.\ We\ have:\ \ (n+1)^2(a_{n+1}-a_n)-(a_{n+1}+n+1)=0$$

$$\Leftrightarrow n(n+2)a_{n+1}-(n+1)^2a_n=n+1\Leftrightarrow \frac{n+2}{n+1}.a_{n+1}-\frac{n+1}{n}.a_n=\frac{1}{n},\forall n\ge 1\ \ (1)$$

$$Then:\ \sum_{k=1}^{n-1}\left(\frac{k+2}{k+1}.a_{k+1}-\frac{k+1}{k}.a_k\right)=\sum_{k=1}^{n-1}\frac{1}{k}\Leftrightarrow \frac{n+1}{n}.a_n-2a_1=H_{n-1}$$

$$Hence,\ \ a_n=\frac{n}{n+1}(H_{n-1}+2),\ \ \forall n\ge 1\ \ and\ \ \lim_{n\to\infty}a_n=+\infty\ \ (2)$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{1+a_n}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{1+a_{n+1}}{1+a_n}\overset{(1)}{=}\lim_{n\to\infty}\frac{1+\frac{n+1}{n+2}\left(\frac{n+1}{n}.a_n+\frac{1}{n}\right)}{1+a_n}=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n(n+2)}+\frac{1}{(n+2)(1+a_n)}\right)\overset{(2)}{=}1.\ \ Therefore,\ \ \Omega=1.$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$a_{n+1}=\frac{(n+1)^2}{(n+1)^2-1}a_n+\frac{n+1}{(n+1)^2-1}$$

$$\text{Let: } b_n=\frac{a_n}{n},b_1=1\Rightarrow b_{n+1}=\frac{n(n+1)}{n(n+2)}b_n+\frac{1}{n(n+1)}=\frac{n+1}{n+2}b_n+\frac{1}{n(n+2)}$$

$$\frac{b_{n+1}}{\prod_{j=1}^{n}\frac{j+1}{j+2}}=\frac{\frac{n+1}{n+2}}{\prod_{j=1}^{n}\frac{j+1}{j+2}}b_n+\frac{\frac{1}{n(n+2)}}{\prod_{j=1}^{n}\frac{j+1}{j+2}}$$

$$\text{Let: } c_n=\frac{b_n}{\prod_{j=1}^{n}\frac{j+1}{j+2}}\Rightarrow c_{n+1}=c_n+\frac{\frac{1}{n(n+2)}}{\prod_{j=1}^{n}\frac{j+1}{j+2}}\Rightarrow c_n=\frac{b_n}{\prod_{j=1}^{n-1}\frac{j+1}{j+2}},c_1=b_1=1$$

$$\prod_{j=1}^{n-1}\frac{j+1}{j+2}=\frac{2}{n+1}\Rightarrow c_n=\frac{b_n}{\frac{2}{n+1}}\Rightarrow\frac{\frac{1}{n(n+2)}}{\prod_{j=1}^{n}\frac{j+1}{j+2}}=\frac{\frac{1}{n(n+2)}}{\frac{2}{n+2}}=\frac{1}{2n}$$

$$c_{n+1}=c_n+\frac{1}{2n}=c_{n-1}+\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n-1}\right)=c_{n-2}+\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}\right)\dots$$

$$c_{n+1}=c_1+\frac{1}{2}H_n=1+\frac{1}{2}H_n;\left(H_n=\sum_{j=1}^{n}\frac{1}{j}\right)\Rightarrow c_n=1+\frac{1}{2}H_{n-1}$$

$$b_n=\frac{2}{n+1}\left(1+\frac{1}{2}H_n\right)$$

$$H_{n-1}\le\int_1^v\frac{1}{v}dv=\log(n)\Rightarrow H_n\le 1+\log(n)$$

$$1\le\sqrt[n]{1+a_n}\overset{Bernoulli}{\le}1+\frac{1}{n}a_n=1+b_n\le 1+\frac{2+1+\log(n-1)}{n+1}\xrightarrow{n\to\infty}\infty$$

Therefore, $\Omega=1$.

UP.476 If $(a_n)_{n\ge1}, a_n\in\mathbb{R}_+^*=(0,\infty), n\in\mathbb{N}^*$ and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot\sqrt[n]{n!}}=a>0$. Find:

$$\Omega(a)=\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$B_n=\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}=\sqrt[n]{a_n}\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}-1\right)=\sqrt[n]{a_n}\cdot(u_n-1)=$$

$$=\sqrt[n]{a_n}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n=\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n,\forall n\ge 2$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n\cdot\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n}}{n}\cdot\left(\frac{n}{n+1}\right)^{n+1}\right)=$$

$$=\frac{a}{e}\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{a}{e}\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}\frac{a}{e}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=$$

$$=\frac{a}{e}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{a}{e^2}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{a_n}}\cdot\frac{n+1}{n}=\frac{a}{e^2}\cdot\frac{e^2}{a}\cdot 1=1$$

$$\lim_{n\to\infty}\frac{u_n-1}{\log u_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{a_{n+1}}}=a\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{n}{n+1}=a\cdot\frac{e^2}{a}\cdot\frac{1}{e}\cdot 1=e$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = \frac{a}{e^2}\cdot 1\cdot \log e = \frac{a}{e^2}$$

Solution 2 by Adrian Popa-Romania

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = \lim_{n\to\infty}\sqrt[n]{a_n}\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1\right) =$$

$$= \lim_{n\to\infty}\sqrt[n]{a_n}\cdot\frac{\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1}{\log\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)}\cdot\log\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right) =$$

$$= \lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\cdot\frac{e^{\log\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)} - 1}{\log\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)}\cdot\log\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n, \forall n \ge 2$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n} = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n\cdot\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n}}{n}\cdot\left(\frac{n}{n+1}\right)^{n+1}\right) =$$

$$= \frac{a}{e}\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{a}{e}\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{C-D'A}{=} \frac{a}{e}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} =$$

$$= \frac{a}{e}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{a}{e^2}$$

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} = \lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{a_n}}\cdot\frac{n+1}{n} = \frac{a}{e^2}\cdot\frac{e^2}{a}\cdot 1 = 1$$

$$\lim_{n\to\infty}\frac{e^{\log\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)} - 1}{\log\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)} = 1$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{a_{n+1}}} = a\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{n}{n+1} =$$

$$= a\cdot\frac{e^2}{a}\cdot\frac{1}{e}\cdot 1 = e$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right) = \frac{a}{e^2}\cdot 1\cdot\log e = \frac{a}{e^2}$$

UP.477 If f is nonnegative function on $[0,1]$ and $f'(x)\ge 1$ then:

$$\int_0^x f^n(t)\,dt \ge x^{n-3}\left(\int_0^x f(t)\,dt\right)^{n-1};\ n\in\mathbb{N}, n\ge 2$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\int_0^x f^n(t)\,dt = \int_0^x f^{n-3}(t)\cdot f^3(t)\,dt \overset{Chebyshev}{\ge} x^{n-3}\left(\int_0^x f(t)\,dt\right)^{n-3}\cdot\int_0^x f^3(t)\,dt;\ (1)$$

$$\int_0^x f^3(t)\,dt - \left(\int_0^x f(t)\,dt\right)^2 \ge f^2(0)\int_0^x f(t)\,dt;\ (2)$$

When $x=1$ we get

Let $F:[0,1]\to\mathbb{R}$ be the function define by:

$$F(x) = \int_0^x f^3(t)\,dt - \left(\int_0^x f(t)\,dt\right)^2 \text{ and } g:[0,1]\to\mathbb{R}$$

$$g(x) = f^2(x) - 2\int_0^x f(t)\,dt$$

$$g'(x) = 2f(x)(f'(x)-1)\ge 0$$

Hence, g –increases and it follows that $g(x)\ge g(0)=f^2(0)$.

$$F'(x)\ge f^2(0)f(x) \Leftrightarrow \left[F(x)-f^2(0)\int_0^x f(t)dt\right]'\ge 0$$

Therefore, the function $h(x)=F(x)-f^2(0)\int_0^x f(t)dt$ is increases, so we get:

$$F(x)-f^2(0)\int_0^x f(t)dt \ge F(0)=0$$

From (1) and (2), it follows that:

$$\int_0^x f^n(t)\,dt \ge x^{n-3}\left(\int_0^x f(t)\,dt\right)^{n-3}\cdot\left(\int_0^x f(t)\,dt\right)^2$$

$$\int_0^x f^n(t)\,dt \ge x^{n-3}\left(\int_0^x f(t)\,dt\right)^{n-1};n\in\mathbb{N},n\ge 2$$

UP.478 If f is nonnegative function on $[0,1]$ and $f'(x)\ge 1$ then:

$$\int_0^x f^n(t)\,dt \ge x^{n-3}\left(\left(\int_0^x f(t)\,dt\right)^{n-1}+f^2(0)\left(\int_0^x f(t)\,dt\right)^{n-2}\right);n\in\mathbb{N},n\ge 2$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\int_0^x f^n(t)\,dt=\int_0^x f^{n-3}(t)\cdot f^3(t)\,dt \overset{Chebyshev}{\ge} x\int_0^x f^{n-3}(t)\,dt\cdot\int_0^x f^3(t)\,dt \overset{Chebyshev}{\ge}\dots$$

$$\overset{Chebyshev}{\ge} x^{n-3}\left(\int_0^x f(t)\,dt\right)^{n-3}\cdot\int_0^x f^3(t)\,dt;(1)$$

$$\int_0^x f^3(t)\,dt-\left(\int_0^x f(t)\,dt\right)^2\ge f^2(0)\int_0^x f(t)\,dt;(2)$$

Let $F:[0,1]\to\mathbb{R}$ be the function define by:

$$F(x)=\int_0^x f^3(t)\,dt-\left(\int_0^x f(t)\,dt\right)^2 \text{ and } g:[0,1]\to\mathbb{R}$$

$$g(x)=f^2(x)-2\int_0^x f(t)\,dt$$

$$g'(x)=2f(x)(f'(x)-1)\ge 0$$

Hence, g –increases and it follows that $g(x)\ge g(0)=f^2(0)$.

$$F'(x)\ge f^2(0)f(x)\Leftrightarrow\left[F(x)-f^2(0)\int_0^x f(t)dt\right]'\ge 0$$

Therefore, the function $h(x)=F(x)-f^2(0)\int_0^x f(t)dt$ is increases, so we get:

$$F(x)-f^2(0)\int_0^x f(t)dt\ge F(0)=0$$

From (1) and (2), it follows that:

$$\int_0^x f^n(t)\,dt\ge x^{n-3}\left(\int_0^x f(t)\,dt\right)^{n-3}\cdot\int_0^x f^3(t)\,dt\ge$$

$$\geq x^{n-3}\left(\int_0^x f(t)\,dt\right)^{n-3}\cdot\left(\left(\int_0^x f(t)\,dt\right)^2+f^2(0)\int_0^x f(t)\,dt\right)=$$

$$=x^{n-3}\left(\left(\int_0^x f(t)\,dt\right)^{n-1}+f^2(0)\left(\int_0^x f(t)\,dt\right)^{n-2}\right)$$

UP.479 Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with circumradius R and inradius r. Let r_a, r_b, r_c be the exradii of the triangle.

Prove that:

$$72\frac{r^4}{R^3}\leq\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\leq\frac{9}{8}\frac{R^4-8r^4}{r^3}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let $a=BC, b=CA, c=AB$ be the lengths of the sides of ΔABC. We have:

$$(b+c-a)(b+c+a)=(b+c)^2-a^2=4m_a^2;\ (2s=a+b+c)$$

$$b+c-a\leq\frac{4m_a^2}{2s}\Leftrightarrow(b+c-a)m_a^2\leq\frac{4m_a^4}{2s}$$

Similarly,

$$(c+a-b)m_b^2\leq\frac{4m_b^4}{2s}\text{ and }(a+b-c)m_c^2\leq\frac{4m_c^4}{2s}$$

Also, we know that:

$$r_a=\frac{rs}{s-a}=\frac{2rs}{b+c-a}\Leftrightarrow b+c-a=\frac{2rs}{r_a}$$

Similarly,

$$c+a-b=\frac{2rs}{r_b}\text{ and }a+b-c=\frac{2rs}{r_c}.\text{ So,}$$

$$\frac{3rs}{r_a}m_a^2\leq\frac{4m_a^2}{2s}\Leftrightarrow\frac{m_a^2}{r_a}\leq\frac{m_a^4}{rs^2}$$

Similarly,

$$\frac{m_b^2}{r_b}\leq\frac{m_b^4}{rs^2}\text{ and }\frac{m_c^2}{r_c}\leq\frac{m_c^4}{rs^2}$$

By adding up these three inequalities, we have:

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\le\frac{1}{rs^2}(m_a^4+m_b^4+m_c^4)$$

It is well-known that:

$$m_a^4+m_b^4+m_c^4=\frac{9}{16}(a^4+b^4+c^4).\text{ So,}$$

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\le\frac{9}{16rs^2}(a^4+b^4+c^4)$$

Now, we we'll prove that: $a^4+b^4+c^4\le 54R^3(R-r)$

It is well-known that: $a^4+b^4+c^4\le 2(a^2b^2+b^2c^2+c^2a^2)-16F^2$, where F denotes the area of ΔABC. We have:

$$a^4+b^4+c^4=2\left((ab+bc+ca)^2-2abc(a+b+c)\right)-16F^2$$

Also, we have: $ab+bc+ca=s^2+r^2+4Rr$, so that, with a short calculation, $a^4+b^4+c^4=2(s^4-2r(4R+3r)s^2+r^2(4R+r)^2)$ and the inequality to be proved becomes: $s^4-2r(4R+3r)s^2+r^2(4R+r)^2-27R^3(R-r)\le 0$.

The left hand sides is a quadratic in s^2 which written as $(s^2-\alpha)(s^2-\beta)$ with $\alpha=r(4R+3r)-\sqrt{\delta}$ and $\beta=r(4R+3r)+\sqrt{\delta}$, where the number δ being

$$8r^3(2R+r)+27R63\ (R-r)$$

We are reduced to proving that $\alpha\le s^2\le\beta$. Now, we'll use Gerretsen's inequality. The inequality $\alpha\le s^2$ follows from $16Rr-5r^2\le s^2\ (Gerretsen)$ since $\alpha\le 3r^2+4Rr\le 16Rr-5r^2\le s^2$. As for the inequality $s^2\le\beta$, using Gerretsen's second inequality $s^2\le 4R^2+4Rr+3r^2$, we see that it is sufficient to prove $4R^2\le\sqrt{\delta}$

or

$8r^4+16r^3R-27rR^3+11R^4\ge 0$. But setting $x=\frac{2R}{r}\overset{Euler}{\ge}1$, this rewrites as

$22x^4-27x^3+4x+1\ge 0$, that is $(x-1)\left(11x^3+(x-1)(11x^2+6x+1)\right)\ge 0$.

So, the later inequality holds and we are done. So,

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\le\frac{9}{16rs^2}\left(52R^3(R-r)\right)$$

We know that $s \geq 3\sqrt{3}r$, so

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\leq\frac{9}{16\left(3\sqrt{3}r\right)^2 r}\left(54R^3(R-r)\right)=\frac{9}{8r^3}R^3(R-r)=$$

$$=\frac{9}{8}\cdot\frac{R^4-R^3r}{r^3}\leq\frac{9}{8}\frac{R^4-8r^4}{r^3}$$

Now, for the left inequality, we have:

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\geq\frac{(m_a+m_b+m_c)^2}{r_a+r_b+r_c};(*)$$

Now we'll prove that $m_a+m_b+m_c\geq\frac{a^2+b^2+c^2}{2R}$.

We have: $AM\cdot MM_1=BM\cdot MC$ or

$$m_a\cdot MM_1=\frac{a}{2}\cdot\frac{a}{2}$$

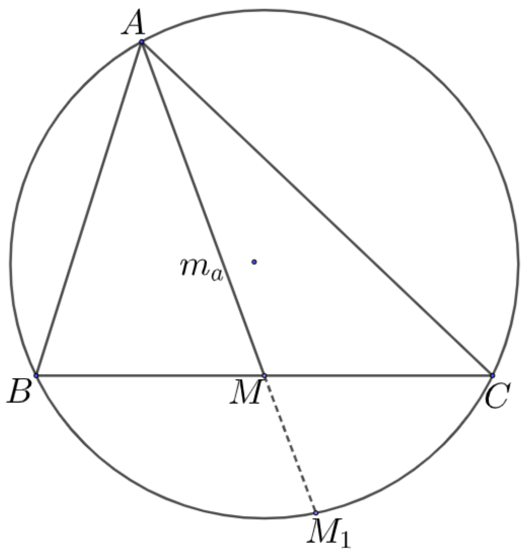
Also, we have:

$$MM_1=AM_1-AM\leq 2R-m_a$$

So, $m_a(2R-m_a)\geq\frac{a^2}{4}$

$$m_a\geq\frac{m_a^2+\frac{a^2}{4}}{2R}\text{ and similarly}$$

$$m_b\geq\frac{m_b^2+\frac{b^2}{4}}{2R}\text{ and }m_c\geq\frac{m_c^2+\frac{c^2}{4}}{2R}$$



Namely,

$$m_a+m_b+m_c\geq\frac{m_a^2+m_b^2+m_c^2+\frac{a^2+b^2+c^2}{4}}{2R}=$$

$$=\frac{\frac{3}{4}(a^2+b^2+c^2)+\frac{1}{4}(a^2+b^2+c^2)}{2R}=\frac{a^2+b^2+c^2}{2R}$$

So, $m_a+m_b+m_c\geq\frac{a^2+b^2+c^2}{2R}$. Also we know that $a^2+b^2+c^2\geq 36r^2$. Now, $(*)$ gives:

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\geq\frac{\left(\frac{36r^2}{2R}\right)^2}{4R+r}\geq\frac{\frac{36^2r^4}{4R^2}}{4R+\frac{R}{2}}=72\frac{r^4}{R^3}$$

Therefore,

$$72\frac{r^4}{R^3} \le \frac{m_a^2}{r_a} + \frac{m_b^2}{r_b} + \frac{m_c^2}{r_c} \le \frac{9}{8}\frac{R^4 - 8r^4}{r^3}$$

Equality holds if and only if triangle ABC is equilateral.

Solution 2 by Marin Chirciu-Romania

Lemma. In ΔABC the following relationship holds:

$$\frac{m_a^2}{r_a} + \frac{m_b^2}{r_b} + \frac{m_c^2}{r_c} = \frac{s^2 - 4r^2 - 7Rr}{r}$$

Proof. We have:

$$\sum \frac{m_a^2}{r_a} = \sum \frac{m_a^2}{\frac{F}{s-a}} = \frac{1}{F}\sum (s-a)m_a^2 = \frac{1}{rs}\cdot s(s^2 - 4r^2 - 7Rr) = \frac{s^2 - 4r^2 - 7Rr}{r}$$

$$\because \sum (s-a)m_a^2 = s(s^2 - 4r^2 - 7Rr)$$

$$\text{For LHS: } \sum \frac{m_a^2}{r_a} \le \frac{9}{16}\cdot\frac{R^4}{r^3}$$

$$\sum \frac{m_a^2}{r_a} = \frac{s^2 - 4r^2 - 7Rr}{r} \overset{Gerretsen}{\le} \frac{4R^2 + 4Rr + 3r^2 - 4r^2 - 7Rr}{r} =$$

$$= \frac{4R^2 - 3Rr - r^2}{r} \overset{(1)}{\le} \frac{9}{16}\cdot\frac{R^4}{r^3}$$

$$(1) \Leftrightarrow \frac{4R^2 - 3Rr - r^2}{r} \le \frac{9}{16}\cdot\frac{R^4}{r^3} \Leftrightarrow 9R^4 - 64R^2r^2 + 48Rr^3 + 16r^4 \ge 0 \Leftrightarrow$$

$(R - 2r)(9R^3 + 18R^2r - 28Rr^2 - 8r^3) \ge 0$, which is true from $R \ge 2r$ $(Euler)$.

Equality holds for ΔABC equilateral.

$$\text{For RHS: } \sum \frac{m_a^2}{r_a} \ge 72\frac{r^4}{R^3}$$

$$\sum \frac{m_a^2}{r_a} = \frac{s^2 - 4r^2 - 7Rr}{r} \overset{Gerretsen}{\ge} \frac{16Rr - 5r^2 - 4r^2 - 7Rr}{r} =$$

$$= \frac{9Rr - 9r^2}{r} = 9r \overset{(2)}{\ge} 72\frac{r^4}{R^3}$$

$$(2) \Leftrightarrow 9r \ge 72\frac{r^4}{R^3} \Leftrightarrow R^3 \ge 8r^3 \Leftrightarrow R \ge 2r\ (Euler).$$

Equality holds for ΔABC equilateral.

UP.480 If $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ are sequence of real numbers with

$$a_n \neq a_{n+1}, b_n \neq b_{n+1}, n \geq 1, \lim_{n\to\infty} a_n = a \in \mathbb{R}, \lim_{n\to\infty} b_n = b \in \mathbb{R},$$

$$\lim_{n\to\infty}\left(n(a_{n+1}-a_n)\right) = c \in \mathbb{R}, \lim_{n\to\infty}\left(n(b_{n+1}-b_n)\right) = d \in \mathbb{R} \text{ and}$$

$f, g: \mathbb{R} \to \mathbb{R}$ are differentiable functions with continuous derivative on $\mathbb{R}$,

then find in terms of a, b, c, d:

$$\Omega = \lim_{n\to\infty}\left(n\left(f(a_{n+1})g(b_{n+1}) - f(a_n)g(b_n)\right)\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by proposers

Applying Lagrange's theorem for function f on each interval $[a_n, a_{n+1}]$ yields that there exists x_n between a_n and a_{n+1} such that:

$$f(a_{n+1}) - f(a_n) = (a_{n+1}-a_n)f'(x_n), \forall n \in \mathbb{N}^*; \quad (1)$$

Since x_n is between a_n and $a_{n+1}, \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = a$, we have that

$$\lim_{n\to\infty} x_n = a.$$

By (1) we have: $n\left(f(a_{n+1}) - f(a_n)\right) = n(a_{n+1}-a_n)f'(x_n), \forall n \in \mathbb{N}^*$, so we deduce that

$$\lim_{n\to\infty}\left(n\left(f(a_{n+1}) - f(a_n)\right)\right) = c \cdot \lim_{n\to\infty} f'(x_n) = cf'\left(\lim_{n\to\infty} x_n\right) = cf'(a); \quad (2)$$

Analogously, we deduce that:

$$\lim_{n\to\infty}\left(n\left(g(x_{n+1}) - g(x_n)\right)\right) = d \cdot g'(b); \quad (3)$$

Also we have:

$$n\left(f(a_{n+1})g(b_{n+1}) - f(a_n)g(b_n)\right) = n(\left(f(a_{n+1}) - f(a_n)\right)\left(g(b_{n+1}) - g(b_n)\right)$$

Hence taking limit with $n \to \infty$ and taking account by (2),(3) we obtain:

$$\lim_{n\to\infty}\left(n\left(f(a_{n+1})g(b_{n+1}) - f(a_n)g(b_n)\right)\right) = c \cdot f'(a) \cdot g(b) + d \cdot f(a) \cdot g'(b)$$