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PROBLEMS FOR JUNIORS

JP.481 For $x,y,z\in(0,1), x+y+z=1$ prove that:

$$\sum_{cyc}\frac{y+z}{x+z}\cdot\frac{x+2y-3xy}{x+2z-3xz}\ge 3$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{y+z}{x+z}\cdot\frac{x+2y-3xy}{x+2z-3xz}=\sum_{cyc}\frac{x+2y-3xy}{x+z}\cdot\frac{x+y}{x+2z-3xz}=$$

$$=\sum_{cyc}\frac{x-xy+2y-2xy}{x+z}\cdot\frac{x+y}{x-xz+2z-2xz}=$$

$$=\sum_{cyc}\frac{x(1-y)+2y(1-x)}{x+z}\cdot\frac{x+y}{x(1-z)+2z(1-x)}=$$

$$=\sum_{cyc}\frac{x(z+x)+2y(y+z)}{(y+z)(z+x)}\cdot\frac{(y+z)(x+y)}{x(x+y)+2z(y+z)}=$$

$$=\sum_{cyc}\frac{\frac{x}{y+z}+2\frac{y}{z+x}}{\frac{x}{y+z}+2\frac{z}{x+y}}=\sum_{cyc}\frac{a+2b}{a+2c}$$

where, $a=\frac{x}{y+z};b=\frac{y}{z+x};c=\frac{z}{x+y}$, then we have:

$$\frac{a+2b}{a+2c}+\frac{b+2c}{b+2a}+\frac{c+2a}{c+2b}=$$

$$=\frac{(a+2b)^2}{(a+2c)(a+2b)}+\frac{(b+2c)^2}{(b+2a)(b+2c)}+\frac{(c+2a)^2}{(c+2b)(c+2a)}\overset{CBS}{\ge}$$

$$\overset{CBS}{\ge}\frac{(a+2b+b+2c+c+2a)^2}{(a+2b)(a+2c)+(b+2c)(b+2a)+(c+2a)(c+2b)}=$$

$$=\frac{9(a+b+c)^2}{a^2+b^2+c^2+8ab+8bc+8ca}$$

We have: $\frac{9(a+b+c)^2}{a^2+b^2+c^2+8ab+8bc+8ca}\ge 3\Leftrightarrow$

$$3(a^2+b^2+c^2+2ab+2bc+2ca)\geq a^2+b^2+c^2+8ab+8bc+8ca \Leftrightarrow$$

$$a^2+b^2+c^2\geq ab+bc+ca \Leftrightarrow (a-b)^2+(b-c)^2+(c-a)^2\geq 0 \text{ true } (\forall)a,b,c>0$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\frac{x+2y-3xy}{x+2z-3xz}=\frac{x(1-y)+2y(1-x)}{x(1-z)+2z(1-x)}=\frac{x(z+x)+2y(y+z)}{x(x+y)+2z(y+z)},$$

$$Then:\ \frac{x+2y-3xy}{x+2z-3xz}=\frac{x+z}{x+y}\cdot\frac{\frac{x}{y+z}+2.\frac{y}{z+x}}{\frac{x}{y+z}+2.\frac{z}{x+y}}$$

$$Let\ a:=\frac{x}{y+z},\quad b:=\frac{y}{z+x},\quad c:=\frac{z}{x+y}.\ We\ have:$$

$$\sum_{cyc}\frac{x+y}{x+z}\cdot\frac{x+2y-3xy}{x+2z-3xz}=\sum_{cyc}\frac{a+2b}{a+2c}=\sum_{cyc}\frac{a}{a+2c}+2\sum_{cyc}\frac{b}{a+2c}\geq$$

$$\overset{CBS}{\geq}\frac{(a+b+c)^2}{\sum_{cyc}a(a+2c)}+\frac{2(a+b+c)^2}{\sum_{cyc}b(a+2c)}\geq\frac{(a+b+c)^2}{(a+b+c)^2}+\frac{2.3(ab+bc+ca)}{3(ab+bc+ca)}=3.$$

$$So\ the\ proof\ is\ complete.\ Equality\ holds\ iff\ a=b=c\ \Leftrightarrow x=y=z=\frac{1}{3}.$$

JP.482 If $x,y,z\in\left[0,\frac{\pi}{2}\right]$ then:

$$(\sin x)^{\cos^2 x}\cdot(\cos x)^{\sin^2 x}+(\sin y)^{\cos^2 y}\cdot(\cos y)^{\sin^2 y}+(\sin z)^{\cos^2 z}\cdot(\cos z)^{\sin^2 z}\leq\frac{3\sqrt{2}}{2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(\sin^2 x)^{\cos^2 x}\cdot(\cos^2 x)^{\sin^2 x}\overset{AM-GM}{\leq}\left(\frac{\sin^2 x\cos^2 x+\cos^2 x\sin^2 x}{\sin^2 x+\cos^2 x}\right)^{\sin^2 x+\cos^2 x}=$$

$$=2\sin^2 x\cos^2 x=\frac{1}{2}\sin^2(2x)\leq\frac{1}{2}$$

Hence,

$$\left((\sin x)^{\cos^2 x}\cdot(\cos x)^{\sin^2 x}\right)^2\leq\frac{1}{2},\qquad (\sin x)^{\cos^2 x}\cdot(\cos x)^{\sin^2 x}\leq\frac{1}{\sqrt{2}}$$

$$\sum_{cyc}(\sin x)^{\cos^2 x}\cdot(\cos x)^{\sin^2 x}\le\sum_{cyc}\frac{1}{\sqrt{2}},\quad \sum_{cyc}(\sin x)^{\cos^2 x}\cdot(\cos x)^{\sin^2 x}\le\frac{3\sqrt{2}}{2}$$

Equality holds for $x=y=z=\frac{\pi}{4}$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{For } x=0 \text{ or } x=\frac{\pi}{2},(\sin x)^{\cos^2 x}.(\cos x)^{\sin^2 x}=0<\frac{\sqrt{2}}{2}$$

$$\text{and we now shift our attention to } x\in\left(0,\frac{\pi}{2}\right) \text{ when } \sin x,\cos x>0 \text{ and then :}$$

$$(\sin x)^{\cos^2 x}.(\cos x)^{\sin^2 x}\overset{\because \cos^2 x+\sin^2 x=1}{=}\sqrt[\cos^2 x+\sin^2 x]{(\sin x)^{\cos^2 x}.(\cos x)^{\sin^2 x}}$$

$$\overset{\text{weighted GM}\le\text{weighted AM}}{\le}\frac{(\sin x)\cos^2 x+(\cos x)\sin^2 x}{\cos^2 x+\sin^2 x}=(\sin x.\cos x)(\sin x+\cos x)$$

$$=\left(\frac{1}{2}\sin(2x)\right)\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)\right)=\left(\frac{1}{2}\sin(2x)\right)\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$

$$\le\frac{1}{2}.\sqrt{2}\ \left(\because \sin(2x),\sin\left(x+\frac{\pi}{4}\right)\le 1\right)\Rightarrow(\sin x)^{\cos^2 x}.(\cos x)^{\sin^2 x}\le\frac{\sqrt{2}}{2}$$

$$\therefore \text{combining}, \forall\, x\in\left[0,\frac{\pi}{2}\right],(\sin x)^{\cos^2 x}.(\cos x)^{\sin^2 x}\le\frac{\sqrt{2}}{2}$$

$$\text{and analogously}, \forall\, y\in\left[0,\frac{\pi}{2}\right],(\sin y)^{\cos^2 y}.(\cos y)^{\sin^2 y}\le\frac{\sqrt{2}}{2} \text{ and}$$

$$\forall\, z\in\left[0,\frac{\pi}{2}\right],(\sin z)^{\cos^2 z}.(\cos z)^{\sin^2 z}\le\frac{\sqrt{2}}{2} \text{ and summing up,}$$

$$(\sin x)^{\cos^2 x}.(\cos x)^{\sin^2 x}+(\sin y)^{\cos^2 y}.(\cos y)^{\sin^2 y}+(\sin z)^{\cos^2 z}.(\cos z)^{\sin^2 z}$$

$$\le\frac{3\sqrt{2}}{2}\ \forall\, x,y,z\in\left[0,\frac{\pi}{2}\right], \text{with equality iff } x=y=z=\frac{\pi}{4}\ \text{(QED)}$$

JP.483 If $0<a,b,c\le 1$ then:

$$(a+b-ab)\cdot a^b+(b+c-bc)\cdot b^c+(c+a-ca)\cdot c^a\ge a+b+c$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$a^{1-b}=(1+a-1)^{1-b}\overset{Bernoulli}{\le}1+(a-1)(1-b)=$$

$$= 1 + a - 1b - 1 + b = a + b - ab$$

$$a^{1-b} \le a + b - ab \Rightarrow \frac{a}{a^b} \le a + b - ab$$

$$\Rightarrow a \le a^b(a + b - ab) \Rightarrow (a + b - ab) \cdot a^b \ge a$$

Hence, we have:

$$\sum_{cyc} (a + b - ab) \cdot a^b \ge a + b + c$$

Equality holds for $a = b = c = 1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{By Bernoulli's inequality we have:}$$

$$\frac{1}{a^b} = \left[1 + \left(\frac{1}{a} - 1\right)\right]^b \le 1 + b.\left(\frac{1}{a} - 1\right) = \frac{a + b - ab}{a}.$$

$$\textit{Then}: \ (a + b - ab).a^b \ge a.$$

$$\textit{Similarly we have}: \ (b + c - bc).b^c \ge b \ \textit{and} \ (c + a - ca).c^a \ge c.$$

$$\textit{Adding these inequalities we get}:$$

$$(a + b - ab).a^b + (b + c - bc).b^c + (c + a - ca).c^a \ge a + b + c.$$

$$\textit{Equality holds iff } a = b = c = 1.$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$b > 0 \Rightarrow 1 - b < 1 \text{ and } \because a > 0 \Rightarrow a - 1 > -1$$

$$\therefore a^{1-b} = \left(1 + (a - 1)\right)^{1-b} \overset{\text{Bernoulli}}{\le} 1 + (a - 1)(1 - b)$$

$$= 1 + a - ab - 1 + b = a + b - ab \Rightarrow \frac{a}{a^b} \le a + b - ab$$

$$\Rightarrow (a + b - ab).a^b \overset{(*)}{\ge} a \left(\because a + b - ab = a + b(1 - a) \overset{1 \ge a}{\ge} a > 0\right)$$

$$\text{and analogously}, (b + c - bc).b^c \overset{(**)}{\ge} b$$

$$\text{and } (c + a - ca).c^a \overset{(***)}{\ge} c \therefore (*) + (**) + (***)$$

$$\Rightarrow (a+b-ab).a^b+(b+c-bc).b^c+(c+a-ca).c^a \ge a+b+c$$

$$\forall\, a,b,c \in (0,1], \text{equality iff } a=b=c=1 \text{ (QED)}$$

JP.484 If $a,b,c>0$, then:

$$\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2}+\frac{(4b^2+3)(4c^2+3)}{(b+c+1)^2}+\frac{(4c^2+3)(4a^2+3)}{(c+a+1)^2} \ge 12$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\left(a^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2\right)\left(\left(\frac{1}{2}\right)^2+b^2+\left(\frac{1}{\sqrt{2}}\right)^2\right) \overset{CBS}{\ge} \left(a\cdot\frac{1}{2}+\frac{1}{2}\cdot b+\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right)^2$$

$$\left(a^2+\frac{1}{4}+\frac{1}{2}\right)\left(\frac{1}{4}+b^2+\frac{1}{2}\right) \ge \left(\frac{a+b+1}{2}\right)^2$$

$$\frac{4a^2+3}{4}\cdot\frac{4b^2+3}{4} \ge \frac{(a+b+1)^2}{4}$$

$$\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2} \ge 4$$

$$\sum_{cyc}\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2} \ge 4+4+4=12$$

$$\text{Equality holds for } \frac{a}{\frac{1}{2}}=\frac{\frac{1}{2}}{b}=\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \Leftrightarrow a=b=\frac{1}{2}.$$

Solution 2 by Marin Chirciu-Romania

Lemma .

If $a,b>0$ **then**

$$\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2} \ge 4.$$

Proof:

Denote $a+\frac{1}{2}=x, b+\frac{1}{2}=y$, $4a^2+3=4(x^2-x+1)$, $4b^2+3=4(y^2-y+1)$.

$$\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2}\geq 4$$

$$\frac{4(x^2-x+1)\cdot 4(y^2-y+1)}{(x+y)^2}\geq 4\Leftrightarrow$$

$$\Leftrightarrow 4(x^2-x+1)(y^2-y+1)\geq (x+y)^2\Leftrightarrow$$

$$3(x^2+y^2)+4x^2y^2-4xy(x+y)+2xy-4(x+y)+4\geq 0.$$

Denote $x+y=S, xy=P$ **:**

$$3S^2+4P^2-4SP-4S-4P+4\geq 0\Leftrightarrow (S-2)^2+(S-2P)^2+(S^2-4P)\geq 0,$$

which results by $(S-2)^2\geq 0,(S-2P)^2\geq 0\ S^2\geq 4P\Leftrightarrow (x-y)^2\geq 0$**, with equality for** $x=y=1$**,**

Hence $\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2}\geq 4$**, With equality for:** $a=b=\frac{1}{2}$**.**

By lemma:

$$Ms=\sum\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2}\overset{Lema}{\geq}\sum 4=12=Md.$$

Equality holds for: $a=b=c=\frac{1}{2}$**.**

Solution 3 by Soumava Chakraborty-Kolkata-India

$$(4a^2+3)(4b^2+3)-4(a+b+1)^2$$
$$=16a^2b^2+8a^2-8ab+8b^2-8a-8b+5$$
$$=(16a^2b^2-8ab+1)+8a^2+8b^2-8a-8b+4$$
$$=(4ab-1)^2+2(4a^2-4a+1)+2(4b^2-4b+1)$$
$$=(4ab-1)^2+2(2a-1)^2+2(2b-1)^2\geq 0,$$

with equality iff $a=b=\frac{1}{2}\therefore (4a^2+3)(4b^2+3)\geq 4(a+b+1)^2$

$$\Rightarrow\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2}\geq 4\ \textit{and analogs}$$

$$\overset{\text{summing up}}{\Rightarrow}$$

$$\frac{(4a^2+3)(4b^2+3)}{(a+b+1)^2}+\frac{(4b^2+3)(4c^2+3)}{(b+c+1)^2}+\frac{(4c^2+3)(4a^2+3)}{(c+a+1)^2}\geq 12,$$

with equality iff $a=b=c=\frac{1}{2}$ (QED)

JP.485 If $x, y, z > 0, x + y + z = 3$ then:

$$\frac{x^2 + 6xy + y^2}{\sqrt{xy}} + \frac{y^2 + 6yz + z^2}{\sqrt{yz}} + \frac{z^2 + 6zx + x^2}{\sqrt{xz}} \geq 24$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

We will prove that:

$$\frac{x^2 + 6xy + y^2}{\sqrt{xy}} \geq 4(x + y); (1)$$

$$(1) \Leftrightarrow x^2 + 6xy + y^2 \geq 4\sqrt{xy}(x + y) \Leftrightarrow (x^2 + 6xy + y^2)^2 \geq 16xy(x + y)^2 \Leftrightarrow$$

$$(x^2 + 6xy + y^2)^2 \geq 16xy(x^2 + 2xy + y^2)$$

Denote: $\alpha = x^2 + y^2; \beta = xy$, then:

$$(\alpha + 6\beta)^2 \geq 16\beta(\alpha + 2\beta) \Leftrightarrow$$

$$\alpha^2 + 36\beta^2 + 12\alpha\beta \geq 16\alpha\beta + 32\beta^2 \Leftrightarrow \alpha^2 - 4\alpha\beta + 4\beta^2 \geq 0 \Leftrightarrow (\alpha - 2\beta)^2 \geq 0 \text{ (true)}$$

By (1), we get:

$$\sum_{cyc} \frac{x^2 + 6xy + y^2}{\sqrt{xy}} \geq \sum_{cyc} 4(x + y) = 8\sum_{cyc} x = 24$$

Equality holds for $x = y = z = 1$.

Solution 2 by Marin Chirciu-Romania

Lemma .

If $x, y > 0$ then

$$\frac{x^2 + 6xy + y^2}{\sqrt{xy}} \geq 4(x + y).$$

Proof:

$$\frac{x^2 + 6xy + y^2}{\sqrt{xy}} \geq 4(x + y) \Leftrightarrow x^2 + 6xy + y^2 \geq 4\sqrt{xy}(x + y) \Leftrightarrow$$

$$(x^2 + 6xy + y^2)^2 \geq 16xy(x + y)^2 \; x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \geq 0 \Leftrightarrow (x - y)^4 \geq 0,$$

equality for $x = y$.

By Lemma and $x+y+z=3$:

$$Ms = \sum \frac{x^2+6xy+y^2}{\sqrt{xy}} \overset{Lema}{\geq} \sum 4(x+y) = 8\sum x = 8\cdot 3 = 24 = Md.$$

Equality holds for $x = y = z = 1$.

JP.486 If $a, b, c > 0$ with $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{64}{9}$, the prove that:

$$\frac{a^4+b^3}{a^3+b^3} + \frac{b^4+c^3}{b^3+c^3} + \frac{c^4+a^3}{c^3+a^3} \geq \frac{21}{8}$$

Proposed by Titu Zvonaru-Romania

Solution 1 by proposer

We note that for $a = b = c = \frac{3}{4}$ we have equality. First by AM-GM inequality and then by HM-AM, we obtain:

$$\sum_{cyc} \frac{a^4+b^4}{a^3+b^3} = \sum_{cyc} \frac{\frac{a^4}{3}+\frac{a^4}{3}+\frac{a^4}{3}+\frac{27}{256}+b^3-\frac{27}{256}}{a^3+b^3} \overset{AM-GM}{\geq}$$

$$\geq \sum_{cyc} \frac{4\sqrt[4]{\frac{a^4}{3}\cdot\frac{a^4}{3}\cdot\frac{a^4}{3}\cdot\frac{27}{256}}+b^3-\frac{27}{256}}{a^3+b^3} =$$

$$= \sum_{cyc} \frac{a^3+b^3-\frac{27}{256}}{a^3+b^3} = 3 - \frac{27}{256}\sum_{cyc}\frac{1}{a^3+b^3} \overset{HM-AM}{\geq} 3 - \frac{27}{256}\cdot\frac{1}{4}\sum_{cyc}\left(\frac{1}{a^3}+\frac{1}{b^3}\right) =$$

$$= 3 - \frac{27}{256}\cdot\frac{1}{4}\cdot 2\sum_{cyc}\frac{1}{a^3} = 3 - \frac{27}{256}\cdot\frac{1}{2}\cdot\frac{64}{9} = 3 - \frac{3}{8} = \frac{21}{8}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM-GM\ inequality\ we\ have:$$

$$a^4 + \frac{27}{256} = 3.\frac{a^4}{3} + \frac{27}{256} \geq 4\sqrt[4]{\left(\frac{a^4}{3}\right)^3 \cdot \frac{27}{256}} = a^3.$$

$$Then: \frac{a^4+b^3}{a^3+b^3} \geq \frac{\left(a^3-\frac{27}{256}\right)+b^3}{a^3+b^3} = 1 - \frac{27}{256(a^3+b^3)} \overset{CBS}{\geq} 1 - \frac{27}{256.4}\left(\frac{1}{a^3}+\frac{1}{b^3}\right).$$

$$Thus,\ \frac{a^4+b^3}{a^3+b^3} \geq 1 - \frac{27}{1024}\left(\frac{1}{a^3}+\frac{1}{b^3}\right)\ (and\ analogs)$$

$$Therefore,\ \frac{a^4+b^3}{a^3+b^3}+\frac{b^4+c^3}{b^3+c^3}+\frac{c^4+a^3}{c^3+a^3}\geq 3-\frac{27}{1024}\cdot 2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=\frac{21}{8}.$$

$$Equality\ holds\ iff\ a=b=c=\frac{3}{4}.$$

JP.487 In acute $\Delta ABC, A', B', C'$ are the contact points by the altitudes with the circle circumscribed of the triangle ABC. Prove that:

$$\sqrt[3]{(BA'+A'C)(CB'+B'A)(AC'+C'B)}\geq 4r$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

From Law of sines, we have:

$$\frac{BA'}{\sin\left(\frac{\pi}{2}-B\right)}=2R\Rightarrow BA'=2R\cos B,\ A'C=2R\cos C\ (and\ analogs)$$

$$(BA'+A'C)(CB'+B'A)(AC'+C'B)=$$
$$=8R^3(\cos B+\cos C)(\cos C+\cos A)(\cos A+\cos B);(1)$$

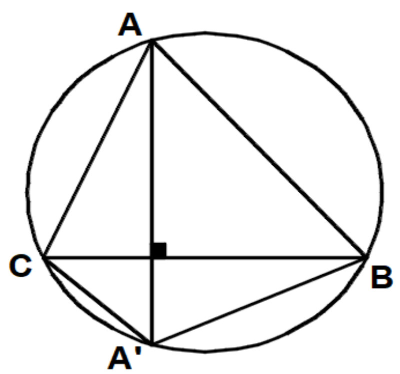
$$\text{But: }(\cos B+\cos C)(\cos C+\cos A)(\cos A+\cos B)=\frac{r(s^2+r^2+2Rr)}{4R^3};(2)$$

From (1) and (2), we get:

$$(BA'+A'C)(CB'+B'A)(AC'+C'B)=2r(s^2+r^2+2Rr)\overset{Gerretsen}{\geq}$$
$$\geq 2r(18Rr-4r^2)=4r^2(9R-2r)\overset{Euler}{\geq}4r^2\cdot 16r=64r^3$$

$$\text{Therefore: }\sqrt[3]{(BA'+A'C)(CB'+B'A)(AC'+C'B)}\geq 4r$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ have:$$

$$BA' = 2R_{\Delta A'BC}.\sin BCA' = 2R.\sin\left(\frac{\pi}{2} - B\right) = 2R\cos B.$$

$$Similarly\ we\ have: A'C = 2R\cos C.$$

$$Then:\ BA' + A'C = 2R(\cos B + \cos C) = 2R.2\sin\frac{A}{2}.\cos\frac{B-C}{2} =$$

$$\overset{Mollweide}{\cong} 4R\sin\frac{A}{2}.\frac{b+c}{a}\sin\frac{A}{2} = 4R.\frac{b+c}{a}.\sin^2\frac{A}{2}.$$

$$Therefore,\ \sqrt[3]{(BA'+A'C)(CB'+B'A)(AC'+C'B)} = \sqrt[3]{\prod_{cyc}\left(4R.\frac{b+c}{a}.\sin^2\frac{A}{2}\right)} =$$

$$= \sqrt[3]{(4R)^3.\frac{(a+b)(b+c)(c+a)}{abc}.\left(\frac{r}{4R}\right)^2} \overset{Cesaro\ \&\ Euler}{\geq} \sqrt[3]{4Rr^2.8.\frac{2r}{R}} = 4r.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

JP.488 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a+b}{ab}\cdot h_c \geq \frac{a+b+c}{R}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)h_c \geq \frac{a+b+c}{R};(1)$$

Using Pham Huu Duc's inequality:

$$x(a+b)+y(b+c)+z(c+a) \geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)}, \forall a,b,c,x,y,z > 0$$

We get:

$$\sum_{cyc}h_c\left(\frac{1}{a}+\frac{1}{b}\right) \geq 2\sqrt{\left(\sum_{cyc}h_ah_b\right)\left(\sum_{cyc}\frac{1}{ab}\right)};(2)$$

$$\sum_{cyc} h_a h_b = \frac{2s^2 r}{R} \text{ and } \sum_{cyc} \frac{1}{ab} = \frac{1}{2Rr}; (3)$$

From (2) and (3), it follows:

$$\sum_{cyc} h_c \left(\frac{1}{a} + \frac{1}{b}\right) \geq 2\sqrt{\frac{s^2}{R^2}} = \frac{2s}{R} = \frac{a+b+c}{R}$$

Solution 2 by Marin Chirciu-Romania

$$\sum \frac{a+b}{ab} \cdot h_c = \sum \frac{a+b}{ab} \cdot \frac{2F}{c} = \frac{2F}{abc} \sum (a+b) = \frac{2F}{4RF} \cdot 2\sum a = \frac{a+b+c}{R}.$$

JP.489 If $a, b, c > 0, a+b+c=3$ then:

$$\frac{(5a+b)(5a+4b)}{8a+b} + \frac{(5b+c)(5b+4c)}{8b+c} + \frac{(5c+a)(5c+4a)}{8c+a} \geq 18$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(a-b)^2 \geq 0 \Leftrightarrow a^2 - 2ab + b^2 \geq 0 \Leftrightarrow$$

$$25a^2 - 24a^2 + 25ab - 27ab + 4b^2 - 3b^2 \geq 0$$

$$25a^2 + 25ab + 4b^2 \geq 24a^2 + 27ab + 3b^3$$

$$25a^2 + 25ab + 20ab + 4b^2 \geq 24a^2 + 3ab + 24ab + 3b^2$$

$$5a(5a+b) + 4b(5a+b) \geq 3a(8a+b) + 3b(8a+b)$$

$$(5a+b)(5a+4b) \geq (8a+b)(3a+3b)$$

$$3(a+b)(8a+b) \leq (5a+b)(5a+4b)$$

$$\frac{(5a+b)(5a+4b)}{8a+b} \geq 3(a+b)$$

$$\sum_{cyc} \frac{(5a+b)(5a+4b)}{8a+b} \geq \sum_{cyc} 3(a+b) = 6(a+b+c) = 18$$

Equality holds for $a = b = c = 1$.

Solution 2 by Marin Chirciu-Romania

<u>**Lema .**</u>

If $a, b, c > 0$ then

$$\frac{(5a+b)(5a+4b)}{8a+b}\geq 3(a+b).$$

Proof:

$$\frac{(5a+b)(5a+4b)}{8a+b}\geq 3(a+b)\Leftrightarrow (5a+b)(5a+4b)\geq 3(a+b)(8a+b)$$

$$\Leftrightarrow a^2-2ab+b^2\geq 0\Leftrightarrow (a-b)^2\geq 0,\text{ equality for } a=b.$$

By Lemma and $a+b+c=3$:

$$Ms=\sum\frac{(5a+b)(5a+4b)}{8a+b}\overset{Lema}{\geq}\sum 3(a+b)=6\sum a=6\cdot 3=18=Md.$$

Equality holds for $a=b=c=1$.

JP.490 If $0<x,y,z\leq\frac{\pi}{2}$ then:

$$\frac{(1+\sin^2 x)^2(1+\sin^2 y)^2(1+\sin^2 z)^2}{(\sin^2 x+\sin^2 y)(\sin^2 y+\sin^2 z)(\sin^2 z+\sin^2 x)}\geq 8$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$0<x,y,z\leq\frac{\pi}{2}\Rightarrow \sin^2 x,\sin^2 y,\sin^2 z\in(0,1]$$

$$\sin^2 x\leq 1,\sin^2 y\leq 1,\sin^2 z\leq 1\Rightarrow(\sin^2 x-1)(\sin^2 y-1)\geq 0$$

$$\sin^2 x\sin^2 y-\sin^2 x-\sin^2 y+1\geq 0$$

$$\sin^2 x\sin^2 y+\sin^2 x+\sin^2 y+1\geq 2(sin62\ x+\sin^2 y)$$

$$(1+\sin^2 x)(1+\sin^2 y)\geq 2(\sin^2 x+\sin^2 y)$$

$$\frac{(1+\sin^2 x)(1+\sin^2 y)}{\sin^2 x+\sin^2 y}\geq 2$$

Therefore,

$$\prod_{cyc}\frac{(1+\sin^2 x)(1+\sin^2 y)}{\sin^2 x+\sin^2 y}\geq 8$$

Equality holds for $\sin x=\sin y=\sin z=1$

$$x=y=z=\frac{\pi}{2}$$

Solution 2 by Marin Chirciu-Romania

Lemma .

If $0 < x, y \le \frac{\pi}{2}$ then

$$\frac{\left(1+\sin^2 x\right)\left(1+\sin^2 y\right)}{\left(\sin^2 x+\sin^2 y\right)} \ge 2.$$

Proof.

$$\frac{\left(1+\sin^2 x\right)\left(1+\sin^2 y\right)}{\left(\sin^2 x+\sin^2 y\right)} \ge 2 \Leftrightarrow \left(1+\sin^2 x\right)\left(1+\sin^2 y\right) \ge 2\left(\sin^2 x+\sin^2 y\right) \Leftrightarrow$$

$$\Leftrightarrow 1+\sin^2 x\sin^2 y \ge \sin^2 x+\sin^2 y \Leftrightarrow \left(1-\sin^2 x\right)\left(1-\sin^2 y\right) \ge 0 \Leftrightarrow \cos^2 x\cos^2 y \ge 0,$$

equality for $x = y = \frac{\pi}{2}$.

By Lemma:

$$LHS = \frac{\left(1+\sin^2 x\right)^2\left(1+\sin^2 y\right)^2\left(1+\sin^2 z\right)^2}{\left(\sin^2 x+\sin^2 y\right)\left(\sin^2 y+\sin^2 z\right)\left(\sin^2 z+\sin^2 x\right)} = \prod \frac{\left(1+\sin^2 x\right)\left(1+\sin^2 y\right)}{\left(\sin^2 x+\sin^2 y\right)} \overset{Lema}{\ge} \prod 2 =$$

$$= 8 = RHS.$$

Equality holds for $x = y = z = \frac{\pi}{2}$.

JP.491 If A, B, C are the angles of a triangle, solve in real numbers x, y, z the system:

$$\begin{cases} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\sqrt{xy+yz+zx} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \\ \frac{xyz}{(x+y)(y+z)(z+x)} = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \end{cases}$$

Proposed by Cristian Miu-Romania

Solution 1 by proposer

Let us first prove the following statement:

If ABC and UVW are two triangles and

$$\begin{cases} \prod_{cyc} \cos A = \prod_{cyc} \cos U \\ \prod_{cyc} \sin A = \prod_{cyc} \sin U \end{cases} \text{ then } \Delta ABC \text{ and } \Delta UVW \text{ are similar.}$$

Proof. From $\prod \cos A = \prod \cos U$ and $\prod \sin A = \prod \sin U$, we obtain that

$$\prod \cot A = \prod \cot U$$

$$1 + \prod \cos A = 1 + \prod \cos U$$

So, we obtain:

$$\frac{1 + \prod \cos A}{\prod \sin A} = \frac{1 + \prod \cos U}{\prod \sin U} \Leftrightarrow \sum \cot A = \sum \cot U$$

Because in any triangle $XYZ, \sum \cot X = \frac{1+\prod \cos X}{\prod \sin X}$.

It is easy to prove this statement.

$$\sum \cot A \cot B = \sum \cot U \cot V = 1$$

So, the numbers $\cot A, \cot B, \cot C, \cot U, \cot V, \cot W$ are the roots of the same equation of third degree. This fact is possible is and only if

$$\{\cot A, \cot B, \cot C\} = \{\cot U, \cot V, \cot W\}$$

So, ΔABC and ΔUVW are similar.

Now, let's solve the problem.

$\sum xy > 0$ so from $\left(\sum \frac{1}{x}\right)\sqrt{\sum xy} = \prod \cot \frac{A}{2}$, we obtain $xyz > 0$

$$\sum xy > 0 \Rightarrow x^2 + xy + yz + zx > 0 \Leftrightarrow$$

$x(x+y) + z(x+y) > 0 \Leftrightarrow (x+y)(x+z) > 0$ and in the same way $(x+z)(y+z) > 0$ and $(y+z)(z+x) > 0$. That means the numbers $x+y, y+z, z+x$ are all positive or they are all negative. But from the second equation:

$\frac{\prod x}{\prod(x+y)} = \prod \sin \frac{A}{2}$ we obtain: $\prod(x+y) > 0$ so, $x+y > 0, y+z > 0, z+x > 0$.

We obtain that $\begin{cases} \sum x > 0 \\ \sum xy > 0 \\ \prod x > 0 \end{cases} \Rightarrow x, y, z > 0.$

Let us consider now MNP the triangle with sides $\sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ and XYZ the triangle with sides $\cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2}$. The system can be written as:

$\begin{cases} \cos M \cos N \cos P = \cos X \cos Y \cos Z \\ \sin M \sin N \sin P = \sin X \sin Y \sin Z \end{cases}$, because $X = \frac{\pi - A}{2}, Y = \frac{\pi - B}{2}, Z = \frac{\frac{\pi}{C}}{2}$.

It is easy to prove this statement, because

$$\cos^2\frac{A}{2}=\cos^2\frac{B}{2}+\cos^2\frac{C}{2}-2\cos\frac{B}{2}\cos\frac{C}{2}\cos\left(\frac{\pi-A}{2}\right)$$

We obtain that MNP and XYZ are similar, so

$$\frac{\sqrt{x+y}}{\cos\frac{A}{2}}=\frac{\sqrt{y+z}}{\cos\frac{B}{2}}=\frac{\sqrt{z+x}}{\cos\frac{C}{2}}=k$$

$$\begin{cases}x+y=k^2\cos^2\frac{A}{2}\\ y+z=k^2\cos^2\frac{B}{2}\\ z+x=k^2\cos^2\frac{C}{2}\end{cases}\Leftrightarrow\begin{cases}x=\frac{k^2}{2}\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\\ y=\frac{k^2}{2}\left(\cos^2\frac{B}{2}-\cos^2\frac{C}{2}+\cos^2\frac{A}{2}\right)\\ z=\frac{k^2}{2}\left(-\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\end{cases}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } (x,y,z) \text{ a solution of the system. We must have}: xy+yz+zx\ge 0,$$

$$\text{then } xyz>0 \text{ and } (x+y)(y+z)(z+x)>0.$$

$$\text{If we assume that } x>0 \text{ and } y,z<0, \text{we have}:\ (x+y)(z+x)=x^2+xy+yz+zx>0,$$

$$\text{then}: (x+y)(z+x)(y+z)<0 \text{ contradiction. Hence, } x,y,z>0.$$

$$\text{Now if } (x,y,z) \text{ a solution of the system and } k>0 \text{ then } (kx,ky,kz) \text{ is also a solution.}$$

$$\text{So if } (x,y,z) \text{ a solution of the system such that } xy+yz+zx=q,$$

$$\text{then } (x',y',z')=\left(\frac{x}{\sqrt{q}},\frac{y}{\sqrt{q}},\frac{z}{\sqrt{q}}\right) \text{ is also a solution with } x'y'+y'z'+z'x'=1.$$

$$\text{So we can assume that}:\ xy+yz+zx=1.$$

$$\text{We have}:\ xyz=\frac{\sqrt{(xy+yz+zx)^3}}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}=\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \text{ and}$$

$$x+y+z=(x+y+z)(xy+yz+zx)=(x+y)(y+z)(z+x)+xyz=$$

$$=\left(\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}+1\right)xyz=\left(\frac{4R}{r}+1\right).\frac{r}{s}=\frac{4R+r}{s}=\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}.$$

$$\text{Hence, } x,y,z \text{ are the solutions of the equation}:$$

$$t^3-\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)t^2+t-\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=0$$

$$\Leftrightarrow\left(t-\tan\frac{A}{2}\right)\left(t-\tan\frac{B}{2}\right)\left(t-\tan\frac{C}{2}\right)=0\Leftrightarrow t=\tan\frac{A}{2}\ or\ t=\tan\frac{B}{2}\ or\ t=\tan\frac{C}{2}.$$

$$Therefore, the\ solutions\ of\ the\ system\ are:$$

$$\left(k\tan\frac{A}{2},k\tan\frac{B}{2},k\tan\frac{C}{2}\right)=$$

$$=\begin{pmatrix}k'\left(-\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right),k'\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right),\\ k'\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}\right)\end{pmatrix},$$

$$k'=k.\frac{2R}{s}>0,\ \ and\ their\ permutation.$$

JP.492 In ΔABC the following relationship holds:

$$2F\le 2Rr+(3\sqrt{3}-4)r^2+\frac{(4R+r)\left(2R+(3\sqrt{3}-4)r\right)}{9}$$

Proposed by Tran Quoc Anh-Vietnam

Solution 1 by proposer

We use Blundon's inequality:

$$s\le 2R+(3\sqrt{3}-4)r\Rightarrow F=sr\le 2Rr+(3\sqrt{3}-4)r^2;\ \ (1)$$

But $F=\sqrt[3]{r_a(s-a)r_b(s-b)r_c(s-c)}=\sqrt[3]{r_ar_br_c(s-a)(s-b)(s-c)};\ \ (2)$

We use Cauchy's inequality and Blundon's inequaltity:

$$\sqrt[3]{r_ar_br_c}\le\frac{r_a+r_b+r_c}{3}=\frac{4R+r}{3};\ \ (3)$$

$$\sqrt[3]{(s-a)(s-b)(s-c)}\le\frac{(s-a)+(s-b)+(s-c)}{3}=\frac{s}{3}\le\frac{2R+(3\sqrt{3}-4)r}{3};\ \ (4)$$

From (2), (3) and (4) we have:

$$F\le(\frac{(4R+r)\left(2R+(3\sqrt{3}-4)r\right)}{9};\ \ (5)$$

From (1) and (5), we have:

$$2F\le 2Rr+(3\sqrt{3}-4)r^2+\frac{(4R+r)\left(2R+(3\sqrt{3}-4)r\right)}{9}$$

Equality holds if and only if triangle ABC is equilateral.

Solution 2 by Marin Chirciu-Romania

By Blundon's inequality $p \le 2R + \left(3\sqrt{3}-4\right)r$:

$$2F = 2rp \overset{Blundon}{\le} 2r\left(2R+\left(3\sqrt{3}-4\right)r\right) \overset{(1)}{\le} 2Rr + \left(3\sqrt{3}-4\right)r^2 + \frac{(4R+r)\left(2R+\left(3\sqrt{3}-4\right)r\right)}{9},$$

(1) $\Leftrightarrow 2r\left(2R+\left(3\sqrt{3}-4\right)r\right) \le 2Rr + \left(3\sqrt{3}-4\right)r^2 + \frac{(4R+r)\left(2R+\left(3\sqrt{3}-4\right)r\right)}{9} \Leftrightarrow$

$$\Leftrightarrow 2Rr + \left(3\sqrt{3}-4\right)r^2 \le \frac{(4R+r)\left(2R+\left(3\sqrt{3}-4\right)r\right)}{9} \Leftrightarrow$$

$\Leftrightarrow 9r\left(2R+\left(3\sqrt{3}-4\right)r\right) \le (4R+r)\left(2R+\left(3\sqrt{3}-4\right)r\right) \Leftrightarrow 9r \le 4R+r \Leftrightarrow 2r \le R$, **(Euler).**

Equality holds if and only if triangle ABC is equilateral.

JP.493 Find $x, y, z > 0$ such that $x + y + z = 9$ and

$$\frac{x^3+y^3}{(x+y)^3} + \frac{y^3+z^3}{(y+z)^3} + \frac{z^3+x^3}{(z+x)^3} = \frac{4}{3}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

We will prove that for $x, y > 0$ we have:

$$\frac{x^3+y^3}{(x+y)^3} \ge \frac{1}{4}; \quad (1)$$

$$\Leftrightarrow 4(x^3+y^3) \ge (x+y)^3 \Leftrightarrow 4x^3 + 4y^3 \ge x^3 + y^3 + 3x^2y + 3xy^2$$

$$\Leftrightarrow 3x^3 + 3y^3 - 3x^3y - 3xy^2 \ge 0 \Leftrightarrow x^3 + y^3 - x^2y - xy^2 \ge 0$$

$$\Leftrightarrow x^2(x-y) - y^2(x-y) \ge 0 \Leftrightarrow (x-y)(x^2-y^2) \ge 0$$

$$\Leftrightarrow (x-y)^2(x+y) \ge 0 \ (true!)$$

$$\text{By } (1): \sum_{cyc} \frac{x^3+y^3}{(x+y)^3} \ge \frac{3}{4}$$

Equality holds for $x = y = z$. By $x + y + z = 9 \Rightarrow x = y = z = 3$.

Solution 2 by Marin Chirciu-Romania

Lemma .

If $x, y > 0$ then

$$\frac{x^3+y^3}{(x+y)^3} \ge \frac{1}{4}.$$

Proof:

$$\frac{x^3+y^3}{(x+y)^3} \ge \frac{1}{4} \Leftrightarrow \frac{x^2-xy+y^2}{(x+y)^2} \ge \frac{1}{4} \Leftrightarrow 4(x^2-xy+y^2) \ge (x+y)^2 \Leftrightarrow 3(x-y)^2 \ge 0,$$

equality for $x = y$

By Lemma: $\sum \frac{x^3+y^3}{(x+y)^3} \ge \sum \frac{1}{4} = \frac{3}{4}$, equality for $x = y = z$.

By $x+y+z=9$ and $x = y = z$ results $x = y = z = 3$.

Solution: $(x, y, z) = (3,3,3)$.

JP.494 If $a, b, c > 0, a+b+c = 3$ then:

$$\frac{ab+bc+ca}{(a+b)(b+c)(c+a)} \le \frac{3}{8}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

We start with:

$$b(a-c)^2 + c(a-b)^2 + a(b-c)^2 \ge 0 \Leftrightarrow$$

$$b(a^2-2ac+c^2) + c(a^2-2ab+b^2) + a(b^2-2bc+c^2) \ge 0$$

$$a^2b + ab^2 - 6abc + a^2c + ac^2 + b^2c + bc^2 \ge 0$$

$$9a^2b - 8a^2b + 9ab^2 - 8ab^2 + 9b^2c - 8b^2c + 9c^2b - 8c^2b +$$

$$+9a^2c - 8a^2c + 9abc + 9abc - 8abc - 8abc - 8abc \ge 0$$

$$9(ab+bc+ca+b^2)(c+a) \ge 8(ab+bc+ca)(a+b+c)$$

$$9(a+b)(b+c)(c+a) \ge 8(ab+bc+ca)(a+b+c)$$

$$9(a+b)(b+c)(c+a) \ge 8(ab+bc+ca)\cdot 3$$

$$\frac{ab+bc+ca}{(a+b)(b+c)(c+a)} \le \frac{3}{8}$$

Equality holds for $a = b = c = 1$.

Solution 2 by Marin Chirciu-Romania

Lemma.

If $a, b, c > 0$ then

$$(a+b)(b+c)(c+a) \ge \frac{8}{9}(a+b+c)(ab+bc+ca)$$

Proof.

$$9(a+b)(b+c)(c+a) \ge 8(a+b+c)(ab+bc+ca) \Leftrightarrow$$

$$\Leftrightarrow 9\left(\sum bc\,(b+c)+2abc\right) \ge \left(\sum bc(b+c)+3abc\right) \Leftrightarrow \sum bc\,(b+c) \ge 6abc \Leftrightarrow$$

$$\Leftrightarrow \sum a(b-c)^2 \ge 0, \text{ equality for } a=b=c.$$

By Lemma and $a+b+c=3$:

$$(a+b)(b+c)(c+a) \ge \frac{8}{9}(a+b+c)(ab+bc+ca) \overset{a+b+c=3}{\Leftrightarrow}$$

$$(a+b)(b+c)(c+a) \ge \frac{8}{9}\cdot 3(ab+bc+ca) \Leftrightarrow$$

$$\Leftrightarrow (a+b)(b+c)(c+a) \ge \frac{8}{3}(ab+bc+ca) \Leftrightarrow$$

$$\Leftrightarrow \frac{ab+bc+ca}{(a+b)(b+c)(c+a)} \le \frac{3}{8}$$

Equality holds for $a=b=c=1$.

Solution 3 by Tran Quoc Thinh-Vietnam

$$(a+b)(b+c)(c+a) \ge \frac{8}{9}(a+b+c)(ab+bc+ca) \quad (*)$$

$$(*) \Leftrightarrow 9(a+b)(b+c)(c+a) \ge 8(a+b+c)(ab+bc+ca) \quad (1)$$

Using AM-GM, we have:

$$(a+b)(b+c)(c+a) \ge 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca} = 8abc$$

$$\Rightarrow 9(a+b)(b+c)(c+a) \ge 8(a+b)(b+c)(c+a)+8abc$$

$$\Rightarrow 9(a+b)(b+c)(c+a) \ge 8[(a+b)(b+c)(c+a)+abc] \quad (2)$$

$$(a+b)(b+c)(c+a)+abc = a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+abc$$

$$(a+b+c)(ab+bc+ca) = a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+abc$$

$$\Rightarrow (a+b)(b+c)(c+a)+abc = (a+b+c)(ab+bc+ca) \quad (3)$$

(2),(3) $\Rightarrow$ (1) true $\Rightarrow$ (*) true

When $a+b+c=3 \Rightarrow \frac{ab+bc+ca}{(a+b)(b+c)(c+a)} \le \frac{3}{8}$

Solution 4 by Angel Plaza-Spain

$$\text{Since } a+b+c=3, \text{then inequality may be written as:}$$
$$9(a+b)(b+c)(c+a)-8(a+b+c)(ab+bc+ca)\geq 0$$
$$\text{After some calculation, we get:}$$
$$a^2b+ab^2+b^2c+bc^2+c^2a+ca^2\geq 6abc, \text{which follows}$$
$$\text{from } AM-GM \text{ inequality.}$$

JP.495 If $t>0$ then in any ABC triangle with the area F the following inequality holds:

$$(r_a^2+t)(r_b^2+t)(r_c^2+t)\geq \frac{27\sqrt{3}}{4}t^2F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by proposer

We have

$$(u^2+1)(v^2+1)\geq \tfrac{3}{4}((u+v)^2+1), \forall u,v>0 \text{ with equality} \Leftrightarrow 2uv=1 \text{ and } u=v \quad (1)$$

and

$$(v^2+1)(w^2+1)\geq (v+w)^2, \forall v,w>0 \text{ with equality} \Leftrightarrow vw=1 \quad (2)$$

Indeed, we have:

$$(u^2+1)(v^2+1)\geq \frac{3}{4}((u+v)^2+1)\Leftrightarrow$$
$$\Leftrightarrow 4u^2v^2+4(u^2+v^2)+4\geq 3(u^2+v^2)+6uv+3\Leftrightarrow$$
$$\Leftrightarrow 4u^2v^2-4uv+1+u^2+v^2-2uv\geq 0\Leftrightarrow$$
$$\Leftrightarrow (2uv-1)^2+(u-v)^2\geq 0 \text{ with equality} \Leftrightarrow 2uv=1 \text{ and } u=v$$

Also we have:

$$(v^2+1)(w^2+1)\geq (v+w)^2\Leftrightarrow v^2w^2+v^2+w^2+1\geq v^2+w^2+2uv\Leftrightarrow$$
$$\Leftrightarrow (uv-1)^2\geq 0 \text{ with equality} \Leftrightarrow vw=1$$

Hence

$$\prod_{cyc}(u^2+1)\overset{(1)}{\geq}\tfrac{3}{4}((u+v)^2+1)(w^2+1)\overset{(2)}{\geq}\tfrac{3}{4}(u+v+w)^2 \quad (3)$$

Hence

$$\prod_{cyc}(r_a^2+t)=t^3\prod_{cyc}\left(\left(\frac{r_a}{\sqrt{t}}\right)^2+1\right)\overset{(3)}{\geq}t^3\cdot\frac{3}{4}\cdot\sum_{cyc}\left(\frac{r_a}{\sqrt{t}}\right)^2=$$

$$=\frac{3}{4}\cdot t^2\cdot(r_a+r_b+r_c)^2=\frac{3}{4}\cdot t^2\cdot(4R+r)^2\overset{Doucet}{\geq}$$

$$\geq\frac{3}{4}\cdot t^2\cdot\left(s\sqrt{3}\right)^2=\frac{9}{4}\cdot t^2\cdot s^2\overset{Mitrinovic}{\geq}\frac{9}{4}\cdot t^2\cdot s\cdot 3\sqrt{3}r=$$

$=\frac{27\cdot\sqrt{3}}{4}\cdot t^2\cdot s\cdot r=\frac{27\sqrt{3}}{4}\cdot t^2\cdot F$ where s is the semiperimeter of ABC triangle

Solution 2 by Marin Chirciu-Romania

By Hőlder:

$$Ms=\left(r_a^2+t\right)\left(r_b^2+t\right)\left(r_c^2+t\right)\overset{Holder}{\geq}\left(\sqrt[3]{r_a^2r_b^2r_c^2}+\sqrt[3]{t^3}\right)^3=\left(\sqrt[3]{\left(rp^2\right)^2}+t\right)^3\overset{(1)}{\geq}\frac{27\sqrt{3}}{4}t^2F=Md,$$

$$\textbf{(1)}\Leftrightarrow\left(\sqrt[3]{\left(rp^2\right)^2}+t\right)^3\geq\frac{27\sqrt{3}}{4}t^2\cdot rp\Leftrightarrow\left(p\sqrt[3]{r^2p}+t\right)^3\geq\frac{27\sqrt{3}}{4}t^2\cdot rp\Leftrightarrow$$

$\Leftrightarrow\left(p\sqrt[3]{r^2p}+t\right)^3\geq\frac{27\sqrt{3}}{4}t^2\cdot rp$, **which results by Mitrinovic** $p\geq 3\sqrt{3}r$.

Remains to prove:

$$\left(p\sqrt[3]{r^2\cdot 3\sqrt{3}r}+t\right)^3\geq\frac{27\sqrt{3}}{4}t^2\cdot rp\Leftrightarrow\left(p\sqrt[3]{\left(r\sqrt{3}\right)^3}+t\right)^3\geq\frac{27\sqrt{3}}{4}t^2\cdot rp\Leftrightarrow$$

$$\Leftrightarrow\left(p\cdot r\sqrt{3}+t\right)^3\geq\frac{27\sqrt{3}}{4}t^2\cdot rp\Leftrightarrow\left(F\sqrt{3}+t\right)^3\geq\frac{27\sqrt{3}}{4}t^2F.$$

PROBLEMS FOR SENIORS

SP.481 If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{ax^2}{r_a}+\frac{yb^2}{r_b}+\frac{cz^2}{r_c}\geq\frac{2\sqrt{3}(xy+yz+zx)}{2R-r}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution 1 by proposer

We have:

$$\sum_{cyc}\frac{ax^2}{r_a}=\sum_{cyc}\frac{(ax)^2}{ar_a}\overset{Bergstrom}{\geq}\frac{(ax+by+cz)^2}{ar_a+br_b+cr_c}=\frac{(ax+by+cz)^2}{2s(2R-r)}=$$

$$=\frac{\left(xa_1^2+yb_1^2+zc_1^2\right)^2}{2s(2R-r)}\overset{\substack{Klamkin\\ in\ \Delta A_1B_1C_1}}{\geq}\frac{16(xy+yz+zx)F_1^2}{2s(2R-r)};\ \ (1)$$

To the triangle ABC we associated the triangle $A_1B_1C_1$ with the sides:

$$a_1=\sqrt{a},b_1=\sqrt{b},c_1=\sqrt{c},F_1=\frac{1}{2}\sqrt{r(r_a+r_b+r_c)}$$

So, we obtain:

$$F_1=\frac{1}{2}\sqrt{r(4R+r)}\overset{Doucet}{\geq}\frac{1}{2}\sqrt{r\cdot s\sqrt{3}}=\frac{\sqrt[4]{3}}{2}\cdot\sqrt{sr}=\frac{\sqrt[4]{3}}{2}\cdot\sqrt{F};\ \ (2)$$

From (1) and (2) we deduce that:

$$\sum_{cyc}\frac{ax^2}{r_a}\geq\frac{16(xy+yz+zx)\sqrt{3}F}{4\cdot 2s(2R-r)}=\frac{2(xy+yz+zx)\sqrt{3}\cdot sr}{s(2R-r)}=$$

$$=\frac{2\sqrt{3}(xy+yz+zx)r}{2R-r}$$

Equality holds for $a=b=c,x=y=z$.

Solution 2 by Marin Chirciu-Romania

By Bergstrőm:

$$Ms=\sum\frac{ax^2}{r_a}=\sum\frac{x^2}{\frac{r_a}{a}}\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{\sum\frac{r_a}{a}}\overset{SOS}{\geq}\frac{3\sum xy}{\frac{p^2+(4R+r)^2}{4Rp}}=\frac{4Rp\cdot 3\sum xy}{p^2+(4R+r)^2}\overset{(1)}{\geq}\frac{2\sqrt{3}r\cdot\sum xy}{2R-r},$$

$$(1)\Leftrightarrow\frac{4Rp\cdot 3\sum xy}{p^2+(4R+r)^2}\geq\frac{2\sqrt{3}r\cdot\sum xy}{2R-r}\Leftrightarrow\frac{6Rp}{p^2+(4R+r)^2}\geq\frac{\sqrt{3}r}{2R-r},$$

Which results by Mitrinovic $p\geq 3\sqrt{3}r$.

Remains to prove:

$$\frac{6R\cdot 3\sqrt{3}r}{p^2+(4R+r)^2}\geq\frac{\sqrt{3}r}{2R-r}\Leftrightarrow\frac{18R}{p^2+(4R+r)^2}\geq\frac{1}{2R-r}\Leftrightarrow 18R(2R-r)\geq p^2+(4R+r)^2\Leftrightarrow$$

$$\Leftrightarrow 36R^2-18Rr\geq p^2+16R^2+8Rr+r^2\Leftrightarrow p^2\leq 20R^2-26Rr-r^2,$$

Which results by Gerretsen's $p^2\leq 4R^2+4Rr+3r^2$.

Remains to prove :

$$4R^2+4Rr+3r^2\leq 20R^2-26Rr-r^2\Leftrightarrow 8R^2-15Rr-2r^2\geq 0\Leftrightarrow (R-2r)(8R+r)\geq 0,$$

true by Euler $R \geq 2r$.

Equality holds for $a = b = c, x = y = z$.

SP.482 In $\Delta ABC, D, E \in (BC)$ such that $[BD] \equiv [EC]$. Prove that:

$$AD^2 + DE^2 + EA^2 + 1.5(BC^2 - DE^2) \geq 2(s^2 + r^2 - 5Rr)$$

Proposed by Gheorghe Molea-Romania

Solution 1 by proposer

We can have the order $B - D - E - C$ or $B - E - D - C$, the problem has different proof but same result. Let be the order: $B - D - E - C$ and M –middle point of BC, then M –midlle point of DE. Using theorem of medians in ΔABC and ΔADE:

$$\begin{cases} AM^2 = \dfrac{2(AB^2 + AC^2) - BC^2}{4} \\ AM^2 = \dfrac{2(AD^2 + AE^2) - DE^2}{4} \end{cases} \Rightarrow$$

$$2(AB^2 + AC^2) - BC^2 = 2(AD^2 + AE^2) - DE^2 \,\big|\, + (3BC^2 + 3DE^2)$$

$$2(AB^2 + AC^2 + BC^2) + 3DE^2 = 2(AD^2 + AE^2 + DE^2) + 3BC^2$$

$$AD^2 + AE^2 + DE^2 + 1.5BC^2 = AB^2 + AC^2 + BC^2 + 1.5DE^2$$

By Schur's inequality:

$$AB^2 + AC^2 + BC^2 + \frac{9AB \cdot AC \cdot BC}{AB + BC + AC} \geq 1.5DE^2 + 2(AB \cdot BC + AB \cdot AC + BC \cdot AC); (*)$$

But $AB + BC + CA = s$; $AB \cdot AC \cdot BC = 4RF$; $\frac{F}{s} = r$ and

$$AB \cdot BC + AB \cdot AC + BC \cdot AC = s^2 + r^2 + 4Rr$$

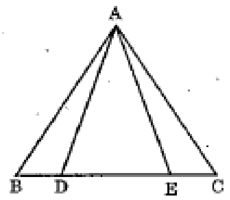
Inequality (*) becomes:

$$AD^2 + DE^2 + EA^2 + 1.5(BC^2 - DE^2) \geq 2(s^2 + r^2 + 4Rr) - 18Rr$$

$$AD^2 + DE^2 + EA^2 + 1.5(BC^2 - DE^2) \geq 2(s^2 + r^2 - 5Rr)$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India



$$\text{Let } m(\overline{BD}) = m(\overline{CE}) = x$$
Via Stewart's theorem on Δ ABC with AD as cevian,
$$AB^2.DC + AC^2.BD = BC.(AD^2 + BD.CD)$$
$$\Rightarrow c^2(a-x) + b^2x \overset{(i)}{=} a\left(AD^2 + x(a-x)\right) \text{ and,}$$
via Stewart's theorem on Δ ABC with AE as cevian,
$$AB^2.EC + AC^2.BE = BC.(AE^2 + BE.CE) \Rightarrow c^2x + b^2(a-x) \overset{(ii)}{=} a\left(AE^2 + x(a-x)\right)$$
$$\text{Now, (i) + (ii)} \Rightarrow c^2(a-x+x) + b^2(a-x+x) = a\left(AD^2 + EA^2 + 2x(a-x)\right)$$
$$\Rightarrow AD^2 + EA^2 = b^2 + c^2 - 2x(a-x)$$
$$\Rightarrow AD^2 + DE^2 + EA^2 + 1.5(BC^2 - DE^2)$$
$$= b^2 + c^2 - 2x(a-x) + (a-2x)^2 + \frac{3}{2}\left(a^2 - (a-2x)^2\right)$$
$$= b^2 + c^2 - 2x(a-x) + a^2 + 4x^2 - 4ax + \frac{3}{2}(2a-2x).2x$$
$$= b^2 + c^2 - 2ax + 2x^2 + a^2 + 4x^2 - 4ax + 6ax - 6x^2 = \sum_{cyc} a^2$$
$$= 2(s^2 - 4Rr - r^2) = 2s^2 - 10Rr + 2r^2 + 2r(R-2r) \overset{\text{Euler}}{\geq} 2s^2 - 10Rr + 2r^2$$
$$= 2(s^2 + r^2 - 5Rr) \text{ (QED)}$$

SP.483 If $x, y, z > 0, x^2 + y^2 + z^2 = \sqrt{3}$, then:

$$\sqrt{x^4 + x^2y^2 + y^4} + \sqrt{y^4 + y^2z^2 + z^4} + \sqrt{z^4 + z^2x^2 + x^4} \geq 3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(x^2 - y^2)^2 \geq 0 \Rightarrow x^4 + y^4 \geq 2x^2y^2$$
$$4x^4 + 4y^4 - 3x^4 - 3y^4 \geq 6x^2y^2 - 4x^2y^2$$
$$4x^4 + 4y^4 + 4x^2y^2 \geq 3x^4 + 3y^4 + 6x^2y^2$$
$$4(x^4 + x^2y^2 + y^4) \geq 3(x^4 + y^4 + x^2y^2)$$
$$\frac{x^4 + x^2y^2 + y^4}{x^4 + 2x^2y^2 + y^4} \geq \frac{3}{4}$$

$$\sqrt{\frac{x^4+x^2y^2+y^4}{(x^2+y^2)^2}} \ge \frac{\sqrt{3}}{2}$$

$$\sqrt{x^4+x^2y^2+y^4} \ge \frac{\sqrt{3}}{2}(x^2+y^2); (1)$$

Analogous,

$$\sqrt{y^4+y^2z^2+z^4} \ge \frac{\sqrt{3}}{2}(y^2+z^2); (2)$$

$$\sqrt{z^4+z^2x^2+x^4} \ge \frac{\sqrt{3}}{2}(z^2+x^2); (3)$$

By adding (1), a(2) and (3):

$$\sum_{cyc}\sqrt{x^4+x^2y^2+y^4} \ge \frac{\sqrt{3}}{2}\cdot 2\sum_{cyc}x^2 = \sqrt{3}\cdot\sqrt{3} = 3$$

Equality holds for $x=y=z=\frac{1}{\sqrt[4]{3}}$.

Solution 2 by Marin Chirciu-Romania

Lemma:

If $a,b>0$ **then:**

$$\sqrt{a^2+ab+b^2} \ge \frac{\sqrt{3}}{2}(a+b).$$

Proof:

$$\sqrt{a^2+ab+b^2} \ge \frac{\sqrt{3}}{2}(a+b) \Leftrightarrow 4(a^2+ab+b^2) \ge 3(a+b)^2 \Leftrightarrow (a-b)^2 \ge 0,$$

equality for $a=b$. **By Lemma for** $(a,b)=(x^2,y^2)$ **and analogs:**

$$Ms=\sum\sqrt{x^4+x^2y^2+y^4} \overset{Lema}{\ge} \sum\frac{\sqrt{3}}{2}(x^2+y^2)=\frac{\sqrt{3}}{2}\cdot 2\sum x^2=\sqrt{3}\sum x^2=\sqrt{3}\cdot\sqrt{3}=Md.$$

Equality holds for: $x=y=z=\frac{1}{\sqrt[4]{3}}$.

SP.484 If $a,b,c>0$ and $a^x+2b^x+3c^x \le a+2b+3c, (\forall)x\in\mathbb{R}$, then:

$$a^a\cdot b^{2b}\cdot c^{3c}=1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f:\mathbb{R}\to\mathbb{R}, f(x)=a^x+2b^x+3c^x$, **then**

$$f'(x) = a^x \log a + 2b^x \log b + 3c^x \log c, f(1) = a + 2b + 3c$$

$f(x) \geq f(1) \Rightarrow x = 1$ minimum point. By Fermat's theorem:

$$f'(1) = 0 \Rightarrow a \log a + 2b \log b + 3c \log c = 0 \Leftrightarrow$$

$$\log(a^a) + \log(b^{2b}) + \log(c^{3c}) = 0 \Leftrightarrow \log(a^a \cdot b^{2b} \cdot c^{3c}) = \log 1 \Leftrightarrow$$

$$a^a \cdot b^{2b} \cdot c^{3c} = 1$$

Solution 2 by Marin Chirciu-Romania

Let be $f: \mathbf{R} \to \mathbf{R}$, $f(x) = a^x + 2b^x + 3c^x$.

$a^x + 2b^x + 3c^x \leq a + 2b + 3c$, $\forall x \in \mathbf{R}$ can be written $f(x) \leq f(1)$, $\forall x \in \mathbf{R}$, $x_0 = 1$ is minimum point of function f derivable. By Fermat's theorem: $f'(1) = 0$.

$$f'(x) = a^x \ln a + 2b^x \ln b + 3c^x \ln c,$$

$$f'(1) = a \ln a + 2b \ln b + 3c \ln c = \ln a^a + \ln b^{2b} + \ln c^{3c} = \ln\left(a^a b^{2b} c^{3c}\right).$$

By $f'(1) = 0 \Leftrightarrow \ln\left(a^a b^{2b} c^{3c}\right) = 0$ results $a^a b^{2b} c^{3c} = 1$.

SP.485 If $a, b, c, d > 0$ and $a^{\log x} + b^{\log x} + c^{\log x} + d^{\log x} \geq 4; (\forall) x \in (0, \infty)$, then: $abcd = 1$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f: (0, \infty) \to \mathbb{R}, f(x) = a^{\log x} + b^{\log x} + c^{\log x} + d^{\log x}$

$$f(1) = a^{\log 1} + b^{\log 1} + c^{\log 1} + d^{\log 1} = 4$$

$f(x) \geq f(1); (\forall) x \in (0, \infty) \Rightarrow x = 1$ minimum point.

By Fermat's theorem $f'(1) = 0$

$$f'(x) = \frac{1}{x} a^{\log x} \log a + \frac{1}{x} b^{\log x} \log b + \frac{1}{x} c^{\log x} \log c + \frac{1}{x} d^{\log x} \log d$$

$$f'(1) = \log a + \log b + \log c + \log d = \log(abcd)$$

$f'(1) = \log(abcd)$ but $f'(1) = 0 \Rightarrow \log(abcd) = \log 1$

$$abcd = 1$$

Solution 2 by Marin Chirciu-Romania

Let be $f: (0, \infty) \to \mathbf{R}$, $f(x) = a^{\log x} + b^{\log x} + c^{\log x} + d^{\log x}$.

Inequality $a^{\log x}+b^{\log x}+c^{\log x}+d^{\log x}\geq 4$, $\forall x\in\mathbf{R}$ **can be written** $f(x)\geq f(1)$, $\forall x\in\mathbf{R}$, $x_0=1$ **is a minimum point of function** f **-derivable. By Fermat's theorem** $f'(1)=0$.

$$f'(x)=\frac{1}{x}a^{\log x}\ln a+\frac{1}{x}b^{\log x}\ln b+\frac{1}{x}c^{\log x}\ln c+\frac{1}{x}d^{\log x}\ln d,$$

$$f'(1)=\ln a+\ln b+\ln c+\ln d=\ln(abcd).$$

By $f'(1)=0\Leftrightarrow\ln(abcd)=0$ **result** $abcd=1$.

SP.486 If $A,B\in M_3(\mathbb{R})$ are such that:

$$AB=BA=BC=CB=CA=AC=O_3\text{, then:}$$

$$\det(I_3+3A+4B+5C+9A^2+16B^2+25C^2)\geq 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Lemma 1. If $X\in M_3(\mathbb{R})$ then:

$$I_3+X+X^2=(X-\omega I_3)(X-\omega^2 I_3)\text{, where }\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

Proof. **Its obvious that:** $\omega^3=1;\ \omega^2+\omega+1=0;\ \omega^2=\overline{\omega}$

$$(X-\omega I_3)(X-\omega^2 I_3)=X^2-\omega^2X-\omega X+\omega^3 I_3=$$

$$=X^2-(\omega^2+\omega)+I_3=X^2+X+I_3$$

Lemma 2. If $X\in M_3(\mathbb{C});\ X=(a_{ij})_{1\leq i,j\leq 3};\ \overline{X}(\overline{a}_{ij})_{1\leq i,j\leq 3}$, then:

$$\det(X\cdot\overline{X})\geq 0$$

Proof. $\det(X\cdot\overline{X})=\det X\cdot\det\overline{X}=\det X\cdot\overline{\det X}=(\det X)^2\geq 0$

Denote: $D=3A+4B+5C$, **then:**

$$D^2=(3A+4B+5C)^2=9A^2+16B^2+25C^2\text{ because:}$$

$$AB=BA=BC=CB=CA=AC=O_3$$

$$\det(I_3+3A+4B+5C+9A^2+16B^2+25C^2)=$$

$$=\det(I_3+D+D^2)\overset{Lemma\ 1.}{=}\det\left((D-\omega I_3)(D-\omega^2 I_3)\right)=$$

$$=\det(D-\omega I_3)\cdot\det(D-\omega^2 I_3)=\det(D-\omega I_3)\cdot\det\overline{(D-\omega I_3)}=$$

$$=\det(D-\omega I_3)\cdot\overline{\det(D-\omega I_3)}\overset{Lemma\ 2.}{\geq}0$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Firstly we have}: \ (3A+4B+5C)^2 =$$

$$= 9A^2+16B^2+25C^2+12(AB+BA)+20(BC+CB)+15(CA+AC)$$
$$= 9A^2+16B^2+25C^2.$$

$$\text{Then}: \ I_3+3A+4B+5C+9A^2+16B^2+25C^2 = I_3+3A+4B+5C+(3A+4B+5C)^2 =$$

$$= (3A+4B+5C-\omega I_3)(3A+4B+5C-\overline{\omega} I_3), \ \text{where } \omega = e^{\frac{2i\pi}{3}},$$

$$\text{and since } A,B,C \in M_3(R) \text{ then we have}:$$

$$det\left(I_3+3A+4B+5C+9A^2+16B^2+25C^2\right)$$
$$= det(3A+4B+5C-\omega I_3).det(3A+4B+5C-\overline{\omega} I_3) =$$

$$= det(3A+4B+5C-\omega I_3).\overline{det(3A+4B+5C-\omega I_3)} = |det(3A+4B+5C-\omega I_3)|^2 \geq 0.$$

SP.487 Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n^{2a}+\sqrt[3]{(k+1)^2(k^2+1)^2}}, a \in \mathbb{R}, a > 0$$

Proposed by Florică Anastase, Ionuț Bină-Romania

Solution 1 by proposers

$$k^3 < (k+1)(k^2+1) < (k+1)^3, (\forall)k = \overline{1,n}, n \geq 1$$

$$k^2 < \sqrt[3]{(k+1)^2(k^2+1)^2} < (k+1)^2, (\forall)k = \overline{1,n}, n \geq 1$$

$$\frac{1}{(k+1)^2+n^2} < \frac{1}{n^2+\sqrt[3]{(k+1)^2(k^2+1)^2}} < \frac{1}{k^2+n^2}, (\forall)k = \overline{1,n}, n \geq 1$$

$$\sum_{k=1}^{n} \frac{1}{(k+1)^2+n^2} \leq \sum_{k=1}^{n} \frac{1}{n^2+\sqrt[3]{(k+1)^2(k^2+1)^2}} < \sum_{k=1}^{n} \frac{1}{k^2+n^2}; (1)$$

$$n^{2a}+k^2 \geq 2kn^a, (\forall)k = \overline{1,n}, n \in \mathbb{N}^*$$

$$0 < \sum_{k=1}^{n} \frac{1}{k^2+n^{2a}} < \sum_{k=1}^{n} \frac{1}{2kn^a} = \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{2n^a}$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{2n^a} \overset{LCS}{=} \frac{1}{2} \cdot \lim_{n\to\infty} \frac{\frac{1}{n+1}}{(n+1)^a-n^a} =$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1}{(n+1)\left[\binom{a}{1}n^{a-1}+\binom{a}{2}n^{a-2}+\cdots+\binom{a}{a}\right]}=0$$

$$Therefore,\quad \Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n^{2a}+\sqrt[3]{(k+1)^2\left(k^2+1\right)^2}}=0, a\in\mathbb{R}, a>0$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$Note:\ n^{2n}\to a^{2n}\ or\ n^{2a}=\Psi(n,a)$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{\infty}I\begin{Bmatrix}1\ if\ k\le n\\ 0\ if\ k>n\\ k\le n\le 1\end{Bmatrix}\frac{1}{\Psi(n,a)+\sqrt[3]{(k+1)^2(k^2+1)^2}}\le$$

$$\le\frac{1}{\sqrt[3]{(k+1)^2(k^2+1)^2}}\le\frac{1}{\sqrt[3]{k^2k^4}}=\frac{1}{k^2}$$

$$The\ magnitade\ of\ the\ summand\ is\ upperbounded\ by\frac{1}{k^2}\ and\ \sum_{k=1}^{\infty}\frac{1}{k^2}<\infty$$

$$Then, by\ the\ dominated\ convergence\ theorem,$$

$$\Omega=\sum_{k=1}^{\infty}\lim_{n\to\infty}\frac{1}{\Psi(n,a)+\sqrt[3]{(k+1)^2(k^2+1)^2}}\cdot\lim_{n\to\infty}I\{k\le n=1\}$$

$$If\lim_{n\to\infty}\Psi(n,a)=\infty,(\forall)a>0, then\ the\ limit\ is\ 0.$$

$$Therefore,\quad \Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n^{2a}+\sqrt[3]{(k+1)^2\left(k^2+1\right)^2}}=0, a\in\mathbb{R}, a>0$$

SP.488 Let $A\in M_2(\mathbb{R})$ invertible such that $\det(A^2-2A+2I_2)=0$.

Find $Tr(A^1)$.

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\det(A^2-2A+2I_2)=\det(A^2-2A+I_2+I_2)=$$

$$=\det((A-I_2)^2+I_2)=\det\left((A-I_2)^2-i^2I_2^2\right)=$$

$$=\det\left((A-I_2+iI_2)(A-I_2-iI_2)\right)=\det(A-I_2+iI_2)\cdot\det(A-I_2-iI_2)=0;(1)$$

Let $f:\mathbb{R}\to\mathbb{R}, f(x)=\det(A-I_2+xI_2)=\det(A-I_2)+a_1x+x^2, a_1\in\mathbb{R}$

$$f(i)=\det(A-I_2+iI_2)=\det(A-I_2)-1+a_1;(2)$$

$$f(-i)=\det(A-I_2-iI_2)=\det(A-I_2)-1-a_1;(3)$$

$$(1)+(2)+(3)\Rightarrow a_1=0 \text{ and } \det(A-I_2)=1$$

$$\det(A-I_2)=\begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix}=(a-1)(d-1)-bc=ad-bc-a-d+1=1\Rightarrow$$

$$ad-bc-a-d=0\Rightarrow ad-bc=a+d\Rightarrow tr(A)=\det(A)=1$$

$$p_A(x)=x^2-tr(A)x+tr(A)\Rightarrow A^2-tr(A)\cdot A+tr(A)\cdot I_2=O_2\,\Big|\cdot A^{-1}\Rightarrow$$

$$A-tr(A)\cdot I_2+tr(A)\cdot A^{-1}=O_2\Rightarrow tr(A)\cdot A^{-1}=-A+tr(A)\cdot I_2\Rightarrow$$

$$tr(tr(A)\cdot A^{-1})=tr(-tr(A)\cdot I_2)\Rightarrow$$

$$tr(A)\cdot tr(A^{-1})=-tr(A)+2tr(A)\Rightarrow tr(A)\cdot tr(A^{-1})=tr(A)$$

$$tr(A)=\det(A)\neq 0\Rightarrow tr(A^{-1})=1.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$A^2-2A+2I_2=(A-(1+i)I_2)(A-(1-i)I_2)=(A-(1+i)I_2)\left(A-\overline{(1+i)}I_2\right).$$

$$\Rightarrow\ 0=det\left(A^2-2A+2I_2\right)=det(A-(1+i)I_2).det\left(A-\overline{(1+i)}I_2\right)=$$

$$=det(A-(1+i)I_2).\overline{det(A-(1+i)I_2)}=|det(A-(1+i)I_2)|^2.$$

$$Then:\ det(A-(1+i)I_2)=0.\ Similarly\ we\ have:\ det(A-(1-i)I_2)=0.$$

$$\Rightarrow 1+i\ and\ 1-i\ are\ eigenvalues\ of\ A.$$

$$Therefore,\qquad Tr\left(A^{-1}\right)=\frac{1}{1+i}+\frac{1}{1-i}=1.$$

SP.489 Let $p,q\in\mathbb{N}^*$ −odd numbers with $p\neq q$. Find all functions $f:\mathbb{R}\to\mathbb{R}$ continuous such that $f\left(\sqrt[p]{px+qy}\right)=f\left(\sqrt[q]{qx+py}\right),\forall x,y\in\mathbb{R}$.

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

If $x=y\Rightarrow f\left(\sqrt[p]{(p+q)x}\right)=f\left(\sqrt[q]{(q+p)x}\right)$. Let $(p+q)x=t$, then:

$$f\left(\sqrt[p]{t}\right)=f\left(\sqrt[q]{t}\right);\forall t\in\mathbb{R}$$

Suppose that $p<q$ (similarly for $p>q$), we have:

$$f\left(t^{\frac{1}{p}}\right)=f\left(t^{\frac{1}{q}}\right)$$

$$t\to t^p: f(t)=f\left(t^{\frac{p}{q}}\right); \frac{p}{q}<1, \frac{p}{q}=\alpha\in(0,1)$$

$$f(t)=f(t^\alpha)$$

$$f(t^\alpha)=f\left(t^{\alpha^2}\right)$$

$$\vdots$$

$$f\left(t^{\alpha^{n-1}}\right)=f\left(t^{\alpha^n}\right)$$

Hence,

$$f(t)=f\left(t^{\alpha^n}\right)\Rightarrow \lim_{n\to\infty} f(t)=\lim_{n\to\infty} f\left(t^{\alpha^n}\right)\Rightarrow$$

$$f(t)=f\left(\lim_{n\to\infty} t^{\alpha^n}\right)\Rightarrow f(t)=f(1)\Rightarrow f(t)=c, c\in\mathbb{R}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Replacing } y \textit{ by } -\frac{qx}{p} \textit{ we get}: f\left(\sqrt[p]{\frac{(p^2-q^2)x}{p}}\right)=f(0),\ \forall x\in R.$$

$$\textit{Replacing } x \textit{ by } \frac{p}{p^2-q^2}.x^p\ (\because p \textit{ is odd}), \textit{in the last equation we get}:$$

$$f(x)=f(0),\ \forall x\in R.$$

$$\textit{Hence, all constant functions satisfy our requirements.}$$

SP.490 If $x,y,z,t>0, x+y+z+t=4$ then:

$$xy+yz+zt+tx\le 4$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\text{Let be } f:\mathbb{R}\to\mathbb{R}, f(p)=(p-x-z)(p-y-t)$$

$$f(p)=p^2-(x+y+z+t)p+xy+yz+zt+tx$$

The equation $f(p)=0$ has two real roots: $p_1=x+z, p_2=y+t\Rightarrow\Delta\ge 0$

$$\Delta=(x+y+z+t)^2-4(xy+yz+zt+tx)\ge 0\Leftrightarrow$$

$$(x+y+z+t)^2\ge 4(xy+yz+zt+tx)\Leftrightarrow$$

$$16 \geq 4(xy + yz + zt + tx) \Leftrightarrow$$

$$xy + yz + zt + tx \leq 4$$

Equality holds for $x + z = y + t = 2$

Solution 2 by Marin Chirciu-Romania

Denote $x+z=a, y+t=b$ **.By** $x+y+z+t=4 \Leftrightarrow (x+z)+(y+t)=4$ **can be written** $a+b=4$ **, and** $xy+yz+zt+tx \leq 4 \Leftrightarrow y(x+z)+t(x+z) \leq 4 \Leftrightarrow$

$$\Leftrightarrow (x+z)(y+t) \leq 4 \text{ can be written } ab \leq 4.$$

Lemma

If $a,b>0$ **,** $a+b=4$ **then**

$$ab \leq 4.$$

Proof

$$ab \leq 4 \Leftrightarrow ab \leq \frac{(a+b)^2}{4} \Leftrightarrow (a-b)^2 \geq 0, \text{ equality for } a=b.$$

By Lemma for $(a,b)=(x+z,y+t)$ **follows** $xy+yz+zt+tx \leq 4$ **, equality for** $x+z=y+t=2$ **.**

SP.491 If $x \geq 0$ then:

$$(x+1)^{x+1} \cdot (x^2+1)^{x^2+1} \leq e^{x^2+x} \cdot \sqrt{e^{x^4+x^2}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f: [0,\infty) \to \mathbb{R}, f(x) = (x+1)\log(x+1) - x - \frac{x^2}{2}$, then:

$$f'(x) = \log(x+1) + 1 - 1 - x$$

$$f''(x) = \frac{1}{x+1} - 1 = \frac{-x}{x+1} < 0$$

$$\Rightarrow f' \text{ -decreasing} \Rightarrow f'(x) \leq f'(0) = 0 \Rightarrow f \text{ -decreasing} \Rightarrow f(x) \leq f(0) = 0$$

$$(x+1)\log(x+1) - x - \frac{x^2}{2} \leq 0$$

$$\log(x+1)^{x+1} \leq x + \frac{x^2}{2} \Leftrightarrow (x+1)^{x+1} \leq e^{x+\frac{x^2}{2}}; \quad (1)$$

Replace in (1), $x \to x^2$: $(x^2+1)^{x^2+1} \leq e^{x^2+\frac{x^4}{2}}$; (2)

Multiplying (1), (2), we get:

$$(x+1)^{x+1}\cdot(x^2+1)^{x^2+1}\le e^{x+x^2}\cdot e^{\frac{x^2}{2}+\frac{x^4}{2}}=e^{x+x^2}\sqrt{e^{x^4+x^2}}$$

Equality holds for $x=0$.

Solution 2 by Marin Chirciu-Romania

Lemma .

If $x\ge 0$ then

$$(x+1)^{x+1}\le e^{x^2+x}.$$

Proof.

$$(x+1)^{x+1}\le e^{x^2+x}\Leftrightarrow \ln(x+1)^{x+1}\le \ln e^{x^2+x}\Leftrightarrow (x+1)\ln(x+1)\le x^2+x.$$

Let be $f:[0,\infty)\to\mathbf{R}$, $f(x)=(x+1)\ln(x+1)-x^2-x$.

$$f'(x)=\ln(x+1)+(x+1)\frac{1}{x+1}-2x-1=\ln(x+1)-2x.$$

$$f''(x)=\frac{1}{x+1}-2=\frac{-2x-1}{x+1}<0\Rightarrow f' \text{ decreasing on } [0,\infty).$$

$f'(0)=0$ and f' decreasing on $[0,\infty)\Rightarrow f'(x)\le 0,\forall x\ge 0$.

$f(0)=0$ și $f'(x)\le 0,\forall x\ge 0\Rightarrow f(x)\le 0,\forall x\ge 0\Rightarrow f(x)=(x+1)\ln(x+1)-x^2-x\le 0$

$\Leftrightarrow (x+1)\ln(x+1)\le x^2+x$, equality for $x=0$.

By Lema $x\to x^2$ implies $(x^2+1)^{x^2+1}\le e^{x^4+x^2}$.

By multiplying $(x+1)^{x+1}\le e^{x^2+x}$ and $(x^2+1)^{x^2+1}\le e^{x^4+x^2}$ results conclusion.

Equality holds for $x=0$.

Solution 3 by Soumava Chakraborty-Kolkata-India

Taking log on both sides,

$$(x+1)^{x+1}.(x^2+1)^{x^2+1}\le e^{x^2+x}.\sqrt{e^{x^4+x^2}}$$

$$\Leftrightarrow (x+1)\ln(x+1)+(x^2+1)\ln(x^2+1)\overset{(*)}{\le} x^2+x+\frac{x^4+x^2}{2}$$

$$\text{Let } f(x)=x^2+x+\frac{x^4+x^2}{2}-(x+1)\ln(x+1)-(x^2+1)\ln(x^2+1)$$

$$\forall x\ge 0 \text{ and then}: f'(x)=2x^3+x-2x.\ln(x^2+1)-\ln(x+1)$$

$$\therefore f'(x)\overset{(\bullet)}{=}2x\left(x^2-\ln(x^2+1)\right)+(x-\ln(x+1))$$

$$\text{Now}, \forall m\ge 0, e^m\ge 1+m\Rightarrow \ln(m+1)\le m$$

and choosing $m\equiv x$ *and* $m\equiv x^2$ **separately, we arrive at:**

$$\ln(x+1)\le x \text{ and } \ln(x^2+1)\le x^2$$

$$\Rightarrow x^2-\ln(x^2+1), x-\ln(x+1)\ge 0$$

$$\Rightarrow 2x\left(x^2-\ln(x^2+1)\right)+\left(x-\ln(x+1)\right)\geq 0\ (\because x\geq 0)$$

$$\overset{\text{via }(\bullet)}{\Rightarrow}\ f'(x)\geq 0\Rightarrow f(x)\text{ is }\uparrow\text{ on }[0,\infty)$$

$$\Rightarrow \forall\, x\geq 0, f(x)\geq f(0)=0$$

$$\therefore x^2+x+\frac{x^4+x^2}{2}-(x+1)\ln(x+1)-(x^2+1)\ln(x^2+1)\geq 0$$

$$\forall\, x\geq 0$$

$$\Rightarrow x^2+x+\frac{x^4+x^2}{2}\geq (x+1)\ln(x+1)+(x^2+1)\ln(x^2+1)\ \forall\, x\geq 0$$

$$\Rightarrow (*)\text{ is true }\therefore (x+1)^{x+1}.(x^2+1)^{x^2+1}\leq e^{x^2+x}.\sqrt{e^{x^4+x^2}}\ \forall\, x\geq 0,$$

with equality iff $x=0$ (QED)

SP.492 If ABC, is a triangle, ω –Brocard's point and $x=\sin 2A+\sin 2B$, $y=\sin 2B+\sin 2C$, $z=\sin 2C+\sin 2A$, then prove that:

$$\frac{1}{3}\min\{x,y,z\}\leq \tan\omega\leq \frac{1}{3}\max\{x,y,z\}$$

Proposed by Cristian Miu-Romania

Solution 1 by proposer

Let us first prove that in any triangle ABC:

$$\sum_{cyc} a^3\cos(B-C)=3abc$$

We have:

$$\sum_{cyc} a^3\cos(B-C)=3abc\Leftrightarrow \sum_{cyc}\sin^3 A\cos(B-C)=3\prod_{cyc}\sin A$$

$$\sum_{cyc}\sin^2 A\,(\sin 2B+\sin 2C)=6\prod_{cyc}\sin A$$

$$\sum_{cyc}(1-\cos 2A)(\sin 2B+\sin 2C)=12\prod_{cyc}\sin A$$

$$2\sum_{cyc}\sin 2A-\sum_{cyc}\cos 2A\,(\sin 2B+\sin 2C)=12\prod_{cyc}\sin A$$

$$2\sum_{cyc}\sin 2A-\sum_{cyc}\sin 2(B+C)=12\prod_{cyc}\sin A$$

$$3\sum_{cyc}\sin 2A = 12\prod_{cyc}\sin A \text{ which is true, because}$$

$$\sum_{cyc}\sin 2A = 4\prod_{cyc}\sin A$$

$$\text{Now, } a^2 = b^2 + c^2 - 2bc\cos A\text{, so }\sum_{cyc}a^2 = \sum_{cyc}(b^2 + c^2 - 2bc\cos A) \Leftrightarrow$$

$$\sum_{cyc}a^2 = 4F\cot\omega \text{ because } F = \frac{1}{2}bc\sin A = \frac{ab}{2}\sin C = \frac{ac}{2}\sin B$$

$$\text{and }\sum_{cyc}\cot A = \cot\omega\text{, } F - \text{area of } ABC.$$

$$\Rightarrow \sum_{cyc}a^3\cos(B - C) = 3abc = 4s\cot\omega\text{. Hence,}$$

$$\min\{a\,cos(B - C), b\,cos(C - A), c\,cos(A - B)\} \le 3R\tan\omega \le$$

$$\le \max\{a\,cos(B - C), b\,cos(C - A), c\,cos(A - B)\}$$

because if t_1, t_2, t_3 are real numbers and u_1, u_2, u_3 are positive real numbers.

$$\min\left\{\frac{t_1}{u_1}, \frac{t_2}{u_2}, \frac{t_3}{u_3}\right\} \le \frac{t_1 + t_2 + t_3}{u_1 + u_2 + u_3} \le \max\left\{\frac{t_1}{u_1}, \frac{t_2}{u_2}, \frac{t_3}{u_3}\right\}$$

Therefore,

$$\frac{1}{3}\min\{x, y, z\} \le \tan\omega \le \frac{1}{3}\max\{x, y, z\}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$\sin 2A = 2\sin A\cos A = 2.\frac{2F}{bc}.\frac{b^2 + c^2 - a^2}{2bc} = \frac{2F(b^2 + c^2 - a^2)}{(bc)^2}\quad (and\ analogs)$$

$$Then:\ a^2y + b^2z + c^2x = \sum_{cyc}a^2y = 2F\sum_{cyc}a^2\left(\frac{c^2 + a^2 - b^2}{c^2a^2} + \frac{a^2 + b^2 - c^2}{a^2b^2}\right) =$$

$$= 2F\sum_{cyc}\left(1 + \frac{a^2}{c^2} - \frac{b^2}{c^2} + \frac{a^2}{b^2} + 1 - \frac{c^2}{b^2}\right) = 2F.6 = 12F.$$

$$Using\ this\ identity\ we\ get:$$

$$\tan\omega = \frac{4F}{a^2+b^2+c^2} = \frac{a^2y+b^2z+c^2x}{3(a^2+b^2+c^2)}.$$

$$Then: \ \tan\omega \ge \frac{(a^2+b^2+c^2).min\{x,y,z\}}{3(a^2+b^2+c^2)} = \frac{1}{3}min\{x,y,z\}.$$

$$Similarly\ we\ get: \tan\omega \le \frac{1}{3}max\{x,y,z\}.$$

$$Therefore, \ \frac{1}{3}min\{x,y,z\} \le \tan\omega \le \frac{1}{3}max\{x,y,z\}.$$

SP.493 If $a_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}; n \ge 1$ and

$$\lim_{n\to\infty}\frac{a_n}{\sqrt{n+1}}\cdot e^{2\sqrt{n}} = x > 0$$

then find:

$$\Omega(x) = \lim_{n\to\infty}(e^{a_{n+1}}-e^{a_n})\cdot x_n$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solutions by proposers

Denote:

$s_n = -2\sqrt{n}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$ **- Ioachimescu's sequence**

It is known that $\lim_{n\to\infty} s_n = s$ – Ioachimescu's constant

$$\Omega(x) = \lim_{n\to\infty}(e^{a_{n+1}}-e^{a_n})\cdot\frac{x_n}{\sqrt{n+1}}\cdot e^{2\sqrt{n}}\cdot e^{-2\sqrt{n}}\cdot\sqrt{n+1} =$$

$$= \lim_{n\to\infty}\frac{x_n}{\sqrt{n+1}}\cdot e^{2\sqrt{n}}\cdot\lim_{n\to\infty}\sqrt{n+1}\cdot e^{-2\sqrt{n}}\cdot(e^{a_{n+1}}-e^{a_n}) =$$

$$= x\lim_{n\to\infty}\sqrt{n+1}\cdot e^{a_n-2\sqrt{n}}\cdot(e^{a_{n+1}-a_n}-1) =$$

$$= x\lim_{n\to\infty}e^{s_n}\cdot\lim_{n\to\infty}\sqrt{n+1}\left(e^{\frac{1}{\sqrt{n+1}}}-1\right) = x\cdot e^s\cdot\lim_{n\to\infty}\frac{e^{\frac{1}{\sqrt{n+1}}}-1}{\frac{1}{\sqrt{n+1}}} = x\cdot e^s\cdot 1 = x\cdot e^s$$

SP.494 Let ABC be a triangle with inradius R and circumradius R. Prove that:

$$3\left(\frac{R}{2r}\right)^{-1} \le \sum_{cyc} \frac{(x+y)\sin A}{x\sin B + y\sin C} \le 3\left(\frac{R}{2r}\right), x, y > 0$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

Let $a = BC, b = CA, c = AB$ be the lengths of the side of triangle ABC. We'll prove that

$$\frac{1}{xb+yc} \le \frac{1}{(x+y)^2}\left(\frac{x}{b}+\frac{y}{c}\right); (*)$$

We have:

$$\frac{1}{xb+yc} - \frac{1}{(x+y)^2}\left(\frac{x}{b}+\frac{y}{c}\right) = \frac{1}{xb+yc} - \frac{xc+yb}{bc(x+y)^2} =$$

$$= \frac{bc(x+y)^2 - (xb+yc)(xc+yb)}{bc(x+y)^2(xb+yc)} = -\frac{xy(b-c)^2}{bc(x+y)^2(xb+yc)} \le 0$$

Equality holds for $b = c$. So, from (*)

$$\frac{a}{xb+yc} \le \frac{1}{(x+y)^2}\left(x\frac{a}{b}+y\frac{a}{c}\right)$$

Similarly,

$$\frac{b}{xc+ya} \le \frac{1}{(x+y)^2}\left(x\frac{b}{c}+y\frac{b}{a}\right) \text{ and } \frac{c}{xa+yb} \le \frac{1}{(x+y)^2}\left(x\frac{c}{a}+y\frac{c}{b}\right)$$

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb} \le \frac{1}{(x+y)^2}\left(x\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+y\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\right)$$

Now, we have:

$$(a^2+b^2+c^2)\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}\right) \ge \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \text{ or}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \le \sqrt{a^2+b^2+c^2}\cdot\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

It is well-known that:

$$a^2+b^2+c^2 \le 9R^2 \text{ and } \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \le \frac{1}{4r^2}. \text{So,}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le 3R\cdot\frac{1}{2r}\Rightarrow\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\le 3R\cdot\frac{1}{2r}.\text{ Namely,}$$

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}\le\frac{1}{(x+y)^2}\left(x\frac{3R}{2r}+y\frac{3R}{2r}\right)=\frac{1}{x+y}\cdot\frac{3R}{2r}$$

Using the law of the sines in ΔABC, we get:

$$\sum_{cyc}\frac{\sin A}{x\sin B+y\sin C}\le\frac{1}{x+y}\cdot\frac{3R}{2r}\Leftrightarrow\sum_{cyc}\frac{(x+y)\sin A}{x\sin B+y\sin C}\le 3\left(\frac{R}{2r}\right)$$

Also, we have:

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}\overset{AGM}{\ge}3\sqrt[3]{\frac{abc}{(xa+yb)(xb+yc)(xc+ya)}}\ge$$

$$\ge\frac{3\sqrt[3]{abc}}{\frac{xa+yb+xb+yc+xc+ya}{3}}=\frac{9\sqrt[3]{abc}}{(x+y)(a+b+c)}$$

We know that:

$R\ge 2r$ $(Euler)$, $abc=4Rrs$, where $2s=a+b+c$ and

$$\frac{3\sqrt{3}}{2}R\ge s\ge 3\sqrt{3}r\ (Mitrinovic)$$

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}\ge 9\sqrt[3]{\frac{4Rr(3\sqrt{3}r)}{(x+y)\cdot 2s}}\ge$$

$$\ge\frac{9\sqrt[3]{4\cdot 2r\cdot r\cdot 3\sqrt{3}r}}{(x+y)3\sqrt{3}R}=\frac{6r}{(x+y)R}$$

$$\sum_{cyc}\frac{a}{xb+yc}\ge\frac{3}{x+y}\left(\frac{R}{2r}\right)^{-1}$$

$$\sum_{cyc}\frac{(x+y)\sin A}{x\sin B+y\sin C}\ge 3\left(\frac{R}{2r}\right)^{-1}$$

Therefore,

$$3\left(\frac{R}{2r}\right)^{-1}\le\sum_{cyc}\frac{(x+y)\sin A}{x\sin B+y\sin C}\le 3\left(\frac{R}{2r}\right),x,y>0$$

Equality holds for ΔABC equilateral.

Solution 2 by Marin Chirciu-Romania

First inequality:

By Bergström:

$$\sum\frac{(x+y)\sin A}{x\sin B+y\sin C}=\sum\frac{(x+y)\sin^2 A}{\sin A(x\sin B+y\sin C)}\overset{CS}{\geq}\frac{(x+y)\left(\sum\sin A\right)^2}{\sum\sin A(x\sin B+y\sin C)}=$$

$$=\frac{(x+y)\left(\sum\sin A\right)^2}{(x+y)\sum\sin B\sin C}=\frac{\left(\sum\sin A\right)^2}{\sum\sin B\sin C}=\frac{\left(\frac{p}{R}\right)^2}{\frac{p^2+r^2+4Rr}{4R^2}}=\frac{4p^2}{p^2+r^2+4Rr}\overset{(1)}{\geq}3\left(\frac{R}{2r}\right)^{-1},$$

(1) $\Leftrightarrow\frac{4p^2}{p^2+r^2+4Rr}\geq3\left(\frac{R}{2r}\right)^{-1}\Leftrightarrow\frac{4p^2}{p^2+r^2+4Rr}\geq\frac{6r}{R}\Leftrightarrow p^2(2R-3r)\geq3r^2(4R+r)$**,**

(By Gerretsen) $p^2\geq16Rr-5r^2\geq\frac{r(4R+r)^2}{R+r}$**.**

Reamains to prove:

$$\frac{r(4R+r)^2}{R+r}\cdot(2R-3r)\geq3r^2(4R+r)\Leftrightarrow(4R+r)(2R-3r)\geq3r(R+r)\Leftrightarrow$$

$$\Leftrightarrow 8R^2-13Rr-6r^2\geq0\Leftrightarrow(R-2r)(8R+3r)\geq0,$$ **(By Euler** $R\geq2r$**.)**

Equality holds for an equilateral triangle.

Second inequality:

$$\sum\frac{(x+y)\sin A}{x\sin B+y\sin C}=\sum\frac{(x+y)\frac{a}{2R}}{x\frac{b}{2R}+y\frac{c}{2R}}=\sum\frac{(x+y)a}{xb+yc}\overset{CBS}{\leq}(x+y)\sum a\cdot\frac{1}{4}\left(\frac{1}{xb}+\frac{1}{yc}\right)=$$

$$=(x+y)\frac{1}{4}\sum\left(\frac{a}{xb}+\frac{a}{yc}\right)=(x+y)\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)\sum\left(\frac{a}{b}+\frac{a}{c}\right)=\frac{x^2+y^2}{4xy}\sum\left(\frac{b}{c}+\frac{c}{b}\right)\overset{Bandila}{\leq}$$

$$\overset{Bandila}{\leq}\frac{x^2+y^2}{4xy}\sum\frac{R}{r}=\frac{x^2+y^2}{4xy}\cdot3\frac{R}{r}=3\left(\frac{R}{2r}\right)\cdot\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right).$$

Equality holds for an equilateral triangle.

SP.495 If $0\leq a\leq b<\frac{\pi}{2}$ then:

$$\int_a^b\frac{1+\tan^3 x}{\sqrt{1-\tan x+\tan^2 x}}dx\geq\tan b-\tan a$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

We will prove that if $p \geq 0$ then:

$$\frac{1+p^3}{\sqrt{1-p+p^2}} \geq 1+p^2;\ \ (1)$$

$$(1+p^3)^2 \geq (1+p^2)^2(1-p+p^2)$$

$$1+2p^3+p^6 \geq 1-p+p^2+2p^2-2p^3+2p^4+p^4-p^5+p^6$$

$$p^5+4p^3-3p^2+p-3p^4 \geq 0$$

$$p(p^4-3p^3+4p^2-3p+1) \geq 0$$

$$p(p-1)(p^3-2p^2+2p-1) \geq 0$$

$$p(p-1)\Big((p-1)(p^2+p+1)-2p(p-1)\Big) \geq 0$$

$$p(p-1)^2\left(\left(p-\frac{1}{2}\right)^2+\frac{3}{4}\right) \geq 0\ (true!)$$

For $p = \tan x$ in (1)

$$\frac{1+\tan^3 x}{\sqrt{1-\tan x+\tan^2 x}} \geq 1+\tan^2 x = 1+\frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Therefore,

$$\int_a^b \frac{1+\tan^3 x}{\sqrt{1-\tan x+\tan^2 x}}dx \geq \int_a^b \frac{1}{\cos^2 x}dx = \tan b - \tan a$$

Equality holds for $a = b$.

Solution 2 by Marin Chirciu-Romania

$$\int_a^b \frac{1+\tan^3 x}{\sqrt{1-\tan x+\tan^2 x}}dx \overset{\tan x=t}{=} \int_{\tan a}^{\tan b} \frac{1+t^3}{\sqrt{1-t+t^2}}\cdot\frac{1}{1+t^2}dt$$

Lemma:

If $t \geq 0$ then:

$$\frac{1+t^3}{\sqrt{1-t+t^2}}\cdot\frac{1}{1+t^2} \geq 1$$

Proof of lemma.

$$\frac{1+t^3}{\sqrt{1-t+t^2}}\cdot\frac{1}{1+t^2} = \frac{(1+t)(1-t+t^2)}{\sqrt{1-t+t^2}}\cdot\frac{1}{1+t^2} = \frac{(1+t)\sqrt{1-t+t^2}}{1+t^2} \overset{(1)}{\geq} 1,$$

$$(1)\Leftrightarrow\frac{(1+t)\sqrt{1-t+t^2}}{1+t^2}\ge 1\Leftrightarrow(1+t)\sqrt{1-t+t^2}\ge 1+t^2\Leftrightarrow$$

$$(1+t)^2(1-t+t^2)\ge(1+t^2)^2\Leftrightarrow(1+2t+t^2)(1-t+t^2)\ge 1+2t^2+t^4\Leftrightarrow$$

$$t^3+t\ge 0\Leftrightarrow t(t^2+t)\ge 0;(\forall)t\ge 0$$

By Lemma:

$$\int_{\tan a}^{\tan b}\frac{1+t^3}{\sqrt{1-t+t^2}}\cdot\frac{1}{1+t^2}dt\ge\int_{\tan a}^{\tan b}1\,dx=\tan b-\tan a$$

Equality holds for $a=b$.

Solution 3 by Angel Plaza-Spain

Since the subintegral function is positive and $(\tan x)'=1+\tan^2 x$, it is enough to prove that for $x\ge 0$, holds:

$$\frac{1+\tan^3 x}{\sqrt{1-\tan x+\tan^2 x}}\ge 1+\tan^2 x\Leftrightarrow$$

$$(1+\tan^3 x)^2-(1-\tan x+\tan^2 x)(1+\tan^2 x)^2\ge 0\Leftrightarrow$$

$$(1-\tan x+\tan^2 x)(\tan x-1)^2\tan x\ge 0\text{, which is true.}$$

Equality holds for $a=b$.

UNDERGRADUATE PROBLEMS

UP.481 Let $t\ge 0$ and $(a_n)_{n\ge 1}$ sequence of real numbers strictly positive such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{t+1}}=a>0.\text{ Find:}$$

$$\Omega(a,t)=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{\left(\sqrt[n]{(2n-1)!!}\right)^t}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$B_n=\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{\left(\sqrt[n]{(2n-1)!!}\right)^t}=\frac{\sqrt[n]{a_n}}{n^{t+1}}\left(\frac{n}{\sqrt[n+1]{(2n-1)!!}}\right)^t\cdot n(u_n-1)=$$

$$=\frac{\sqrt[n]{a_n}}{n^{t+1}}\left(\frac{n}{\sqrt[n+1]{(2n-1)!!}}\right)^t\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n\,;\forall n\geq 2$$

$$u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}=\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{t+1}}\cdot\frac{n^{t+1}}{\sqrt[n]{a_n}}\left(\frac{n+1}{n}\right)^{t+1}\,;\forall n\geq 1$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^{t+1}}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{n(t+1)}}}\overset{C-d'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{(n+1)(t+1)}}\cdot\frac{n^{n(t+1)}}{a_n}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{t+1}}\left(\frac{n}{n+1}\right)^{(n+1)(t+1)}=\frac{a}{e^{t+1}}$$

Similarly,

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\frac{e}{2},\qquad \lim_{n\to\infty}u_n=1,\qquad \lim_{n\to\infty}\frac{(u_n-1)}{\log u_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{t+1}}\left(\frac{n}{n+1}\right)^{t+1}=a\cdot\frac{e^{t+1}}{a}\cdot 1=e^{t+1}$$

Therefore,

$$\lim_{n\to\infty}B_n=\frac{a}{e^{t+1}}\cdot\left(\frac{2}{e}\right)^t\cdot 1\cdot\log e^{t+1}=\frac{2^t\cdot a}{e}(t+1)=\frac{2^t\cdot a(t+1)}{e}$$

Solution 2 by Angel Plaza-Spain

We will use the following result has been applied in [1] Proposition 2:

If $(a_n)_{n\geq1}$ is a sequence of real numbers strictly positive such that $\lim_{n\to\infty}\frac{a_{n+1}}{n^t\cdot a_n}=a, t>0$ then:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^t}=\frac{a}{e^t}$$

Note that for $b_{n+1}=(2n+1)!!$, then $\frac{b_{n+1}}{b_n}=\frac{(2n+1)!!}{(2n-1)!!}=2n+1$, so since

$$\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=2\Rightarrow$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{2}{e}\text{ and }\left(\sqrt[n]{(2n-1)!!}\right)^t\sim\left(\frac{2n}{e}\right)^t\text{ for }n\to\infty$$

$$\Omega(a,t)=\left(\frac{e}{2}\right)^t\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{n^t}=$$

$$=\left(\frac{e}{2}\right)^t\lim_{n\to\infty}\frac{(t+1)\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)}{(n+1)^{t+1}-n^{t+1}}\overset{LC-S}{=}\left(\frac{e}{2}\right)^t\lim_{n\to\infty}\frac{(t+1)\sqrt[n+1]{a_{n+1}}}{n^{t+1}}=$$

$$=(t+1)\left(\frac{e}{2}\right)^t\cdot\frac{a}{e^{t+1}}=\frac{(t+1)a}{2^te}$$

REFERENCES:

[1] D.M. Batineţu-Giurgiu, Angel Plaza, Daniel Sitaru, Florică Anastase, "New solutions for a few R.M.M. problems, available at:

https://www.ssmrmh.ro/2022/04/02/new-solutions-for-a-few-rmm-problems/.

UP.482 Let $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ an continuous function such that $\lim_{x\to\infty}\frac{f(x+1)}{x\cdot f(x)}=a>0$ and exist $\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x}$. Find:

$$\Omega(a)=\lim_{x\to\infty}\left((x+1)^2\cdot\left(f(x+1)\right)^{-\frac{1}{x+1}}-x^2\cdot\left(f(x)\right)^{-\frac{1}{x}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

We have:

$$\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{(f(n))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{f(n)}{n^n}}\overset{C-d'A}{=}\lim_{n\to\infty}\frac{f(n+1)}{(n+1)^{n+1}}\cdot\frac{n^n}{f(n)}=$$

$$=\lim_{n\to\infty}\frac{f(n+1)}{n\cdot f(n)}\left(\frac{n}{n+1}\right)^{n+1}=\frac{a}{e}$$

Let be $B:\mathbb{R}_+^*\to\mathbb{R}_+^*, B(x)=(x+1)^2\cdot\left(f(x+1)\right)^{-\frac{1}{x+1}}-x^2\cdot\left(f(x)\right)^{-\frac{1}{x}}=$

$$=\frac{x^2}{(f(x))^{\frac{1}{x}}}(u(x)-1)=\frac{x^2}{(f(x))^{\frac{1}{x}}}\cdot\frac{u(x)-1}{\log u(x)}\log u(x)=$$

$$=\frac{x}{(f(x))^{\frac{1}{x}}}\cdot\frac{u(x)-1}{\log u(x)}\log(u(x))^{x}$$

Let be $u:\mathbb{R}_+^*\to\mathbb{R}_+^*, u(x)=\left(\frac{x+1}{x}\right)^2\cdot\frac{(f(x))^{\frac{1}{x}}}{(f(x+1))^{\frac{1}{x+1}}}=$

$$=\left(\frac{x+1}{x}\right)^3\cdot\frac{(f(x))^{\frac{1}{x}}}{x}\cdot\frac{x+1}{(f(x+1))^{\frac{1}{x+1}}}\cdot\frac{x}{x+1}$$

So,

$$\lim_{x\to\infty}u(x)=1\cdot\frac{a}{e}\cdot\frac{e}{a}\cdot 1=1\Rightarrow\lim_{x\to\infty}\frac{u(x)-1}{\log u(x)}=1$$

$$\lim_{x\to\infty}(u(x))^x=\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{2x}\cdot\frac{f(x)}{f(x+1)}\cdot(f(x+1))^{\frac{1}{x+1}}=$$

$$=\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{2x+1}\cdot\frac{xf(x)}{f(x+1)}\cdot\frac{(f(x+1))^{\frac{1}{x+1}}}{x+1}=e^2\cdot\frac{1}{a}\cdot\frac{a}{e}=e$$

Therefore,

$$\lim_{x\to\infty}B(x)=\frac{e}{a}\cdot 1\cdot\log e=\frac{e}{a}$$

Solution 2 by Angel Plaza-Spain

We will use the following result [1, Proposition 2]:

If $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^t\cdot a_n}=a, t\geq 0$,

then: $\lim\limits_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^t}=\frac{a}{e^t}$.

Note in the conditions of the problem continuous variable x, may be changed by integer n, and according to the previous result, also

$$\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{(f(n))^{\frac{1}{n}}}{n}=\frac{a}{e}$$

Now,

$$\Omega(a)=\lim_{n\to\infty}\left((n+1)^2\cdot(f(n+1))^{-\frac{1}{n+1}}-n^2\cdot(f(n))^{-\frac{1}{n}}\right)=$$

$$=\lim_{n\to\infty}\frac{(n+1)^2\cdot\left(f(n+1)\right)^{-\frac{1}{n+1}}}{n+1}\overset{LCS}{=}\lim_{n\to\infty}(n+1)\cdot\left(f(n+1)\right)^{-\frac{1}{n+1}}=$$

$$=\lim_{n\to\infty}\frac{n+1}{\left(f(n+1)\right)^{\frac{1}{n+1}}}=\frac{e}{a}$$

REFERENCES

[1] D.M. Bătineţu-Giurgiu, Angel Plaza, Daniel Sitaru, Florică Anastase, "*New solutions for a few R.M.M. problems*", available at

https://www.ssmrmh.ro/2022/04/02/new-solutions-for-a-few-rmm-problems/.

UP.483 Prove that:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{8k^2+4k+1}=\frac{1}{2}\Im\left(\psi\left(\frac{1+i}{8}\right)-\psi\left(\frac{1+i}{4}\right)\right)$$

Proposed by Fao Ler-Iraq

Solution 1 by proposer

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{8k^2+4k+1}=2\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2+1}=-2\sum_{k=0}^{\infty}(-1)^k\Im\left(\frac{1}{4k+1+i}\right)=$$

$$=-2\Im\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{4k+1+i}\right)=-2\Im\left(\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{4k+i}\,dx\right)=$$

$$=-2\Im\left(\int_0^1 x^i\sum_{k=0}^{\infty}(-x^4)^k\,dx\right)=-2\Im\left(\int_0^1\frac{x^i}{x^4+1}dx\right)=$$

$$=-2\Im\left(\int_0^1\frac{x^{\frac{i}{4}}}{x+1}d\left(x^{\frac{1}{4}}\right)\right)=-\frac{1}{2}\Im\left(\int_0^1\frac{x^{\frac{i+1}{4}-1}}{x+1}dx\right)=$$

$$=-\frac{1}{4}\Im\left(\psi\left(\frac{i+1}{8}+\frac{1}{2}\right)-\psi\left(\frac{i+1}{8}\right)\right)=-\frac{1}{4}\Im\left(\psi\left(\frac{5+i}{8}\right)-\psi\left(\frac{1+i}{8}\right)\right)=$$

$$-\frac{1}{4}\Im\left(2\psi\left(\frac{i+1}{4}\right)-2\log 2-\psi\left(\frac{i+1}{8}\right)-\psi\left(\frac{1+i}{8}\right)\right)=$$

$$=\frac{1}{2}\Im\left(\psi\left(\frac{1+i}{8}\right)-\psi\left(\frac{1+i}{4}\right)\right)$$

Solution 2 by Le Thu-Vietnam

$$By\ partial\ fraction, one\ has$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{8k^2+4k+1}=\frac{i}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{k+z}-\frac{i}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{k+\bar{z}}, where\ z=\frac{1+i}{4}$$

$$=\frac{i}{4}\Phi(-1,1,z)-\frac{i}{4}\Phi(-1,1,\bar{z})=\frac{i}{4}[\Phi(-1,1,z)-\Phi(-1,1,\bar{z})]=$$

$$=\frac{i}{8}\left[\psi_0\left(\frac{z}{2}+\frac{1}{2}\right)-\psi_0\left(\frac{z}{2}\right)+\psi_0\left(\frac{\bar{z}}{2}\right)-\psi_0\left(\frac{\bar{z}}{2}+\frac{1}{2}\right)\right]=$$

$$=\frac{i}{8}\left[2\psi_0(z)-\psi_0\left(\frac{z}{2}\right)-\log(4)-\psi_0\left(\frac{z}{2}\right)+\psi_0\left(\frac{\bar{z}}{2}\right)-2\psi_0(\bar{z})+\psi_0\left(\frac{\bar{z}}{2}\right)+\log(4)\right]=$$

$$=\frac{i}{4}\left[\psi_0(z)-\psi_0(\bar{z})+\psi_0\left(\frac{\bar{z}}{2}\right)-\psi_0\left(\frac{z}{2}\right)\right]$$

$$Since\ \Im[f(z)]=\frac{f(z)-\overline{f(z)}}{2}=\frac{f(z)-f(\bar{z})}{2}\ \ if\ and\ only\ if\ f(z)\in\mathbb{R},(\forall)z\in\mathbb{D}\cap\mathbb{R}$$

$$\frac{1}{2}\Im\left(\psi\left(\frac{1+i}{8}\right)-\psi\left(\frac{1+i}{4}\right)\right)=\frac{1}{2}\Im\left[\psi_0\left(\frac{z}{2}\right)\right]-\frac{1}{2}\Im[\psi_0(z)]=$$

$$=\frac{i}{4}\left[\psi_0\left(\frac{\bar{z}}{2}\right)-\psi_0\left(\frac{z}{2}\right)\right]-\frac{i}{4}[\psi_0(\bar{z})-\psi_0(z)]=$$

$$=\frac{i}{4}\left[\psi_0(z)-\psi_0(\bar{z})+\psi_0\left(\frac{\bar{z}}{2}\right)-\psi_0\left(\frac{z}{2}\right)\right]=\sum_{k=0}^{\infty}\frac{(-1)^k}{8k^2+4k+1}, where\ z=\frac{1+i}{2}$$

UP.484 Find:

$$\Omega=\int_0^{\infty}\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by proposer

$$\text{Let us: }\Omega=\int_0^{\infty}\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx\,;A=\int_0^{1}\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx\,;B=\int_1^{\infty}\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx$$

We consider the integral A. We make the variable change: $x^2=y; x=\sqrt{y}$

We have, succesively: $A=\frac{1}{4}\int_0^1\frac{(1-y)y^{\frac{1}{4}}\log y}{1-y^3}dy=$

$$=\frac{1}{4}\left(\int_0^1\frac{y^{\frac{1}{4}}\log y}{1-y^3}dy-\int_0^1\frac{y^{\frac{5}{4}}\log y}{1-y^3}dy\right)=$$

$$=\frac{1}{4}\left(\int_0^1\sum_{n=0}^{\infty}y^{3n+\frac{1}{4}}\log y\,dy-\int_0^1\sum_{n=0}^{\infty}y^{3n+\frac{5}{4}}\log y\,dy\right)=$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\left(\int_0^1 y^{3n+\frac{1}{4}}\log y\,dy-\int_0^1 y^{3n+\frac{5}{4}}\log y\,dy\right)$$

We will use the following relationship: $\int_0^1 x^a\log x\,dx=-\frac{1}{(a+1)^2}$,

where $a\in\mathbb{R}, a\geq 0$.

We obtain: $A=\frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{1}{\left(3n+\frac{9}{4}\right)^2}-\frac{1}{\left(3n+\frac{5}{4}\right)^2}\right]=\frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{\frac{1}{9}}{\left(n+\frac{9}{12}\right)^2}-\frac{\frac{1}{9}}{\left(n+\frac{5}{12}\right)^2}\right]$

We now us the following relationship:

$\psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$, where $\psi_1(x)$ – is the trigamma function.

We obtained the value of the integral $A=\frac{1}{36}\left[\psi_1\left(\frac{9}{12}\right)-\psi_1\left(\frac{5}{12}\right)\right]$

We consider the integral B. We make the variable change: $x=\frac{1}{y}; y=\frac{1}{x}$.

We obtain: $B=-\int_0^1\frac{\sqrt{y}\log y}{y^4+y^2+1}dy$. By proceeding to the integral A, we obtain:

$B=\frac{1}{36}\left[\psi_1\left(\frac{3}{12}\right)-\psi_1\left(\frac{7}{12}\right)\right]$. Result:

$$\Omega=A+B=\frac{1}{36}\left[\psi_1\left(\frac{3}{4}\right)+\psi_1\left(\frac{1}{4}\right)-\psi_1\left(\frac{5}{12}\right)-\psi_1\left(\frac{7}{12}\right)\right]$$

We use the reflection formula: $\psi_1(x)+\psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$, we obtain:

$$\psi_1\left(\frac{5}{12}\right)+\psi_1\left(\frac{7}{12}\right)=4\pi^2(2-\sqrt{3})$$

The following special values is known: $\psi_1\left(\frac{1}{4}\right)=\pi^2+8G;\psi_1\left(\frac{3}{4}\right)$

$=\pi^2-8G$, where G is

Catalan's constant. Result: $\Omega=\frac{1}{18}\pi^2(2\sqrt{3}-3)$. Thus, the problem is solved.

Solution 2 by Probal Chakraborty-Kolkata-India

$$\Omega=\int_0^\infty\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx=\int_0^\infty\frac{(x^2-1)x\sqrt{x}\log x}{(x^2-1)(x^4+x^2+1)}dx=$$

$$=\int_0^\infty\frac{\left(x^{\frac{7}{2}}-x^{\frac{3}{2}}\right)\log x}{x^6-1}dx\overset{x^6=t}{=}\frac{1}{36}\int_0^\infty\frac{t^{-\frac{5}{6}}t^{\frac{7}{12}}-t^{\frac{3}{12}-\frac{5}{6}}}{t-1}\log t\,dt=$$

$$=\frac{1}{36}\int_0^\infty\frac{t^{-\frac{1}{4}}-t^{-\frac{7}{12}}}{t-1}\log t\,dt=\frac{1}{36}\int_0^\infty\frac{t^{-\frac{7}{12}}-t^{-\frac{1}{4}}}{1-t}\log t\,dt=$$

$$=\frac{1}{36}\int_0^1\frac{t^{-\frac{7}{12}}-t^{-\frac{1}{4}}}{1-t}\log t\,dt+\frac{1}{36}\int_1^\infty\frac{t^{-\frac{7}{12}}-t^{-\frac{1}{4}}}{1-t}\log t\,dt\overset{t=\frac{1}{z}}{=}$$

$$=\frac{1}{36}\left[\psi'\left(\frac{3}{4}\right)-\psi'\left(\frac{5}{12}\right)\right]-\frac{1}{36}\int_0^1\frac{-z^{\frac{7}{12}-1}+z^{\frac{1}{4}-1}}{1-z}\log z\,dz=$$

$$=\frac{1}{36}\left[\psi'\left(\frac{3}{4}\right)-\psi'\left(\frac{5}{12}\right)-\psi'\left(\frac{7}{12}\right)+\psi'\left(\frac{1}{4}\right)\right]=\frac{\pi^2}{36}\left[\csc^2\left(\frac{\pi}{4}\right)-\csc^2\left(\frac{\pi}{12}\right)\right]$$

$$\because -\int_0^1\frac{t^{z-1}}{1-t}\log t\,dt=\psi'(z)$$

$$\because -\psi'(1-z)-\psi'(z)=\pi\frac{d}{dz}[\cot(\pi z)]=-\pi^2\csc^2(\pi z)$$

Solution 3 by Le Thu-Vietnam

By the property of the improper integral, one has

$$\Omega=-\int_0^\infty\frac{\sqrt{x}\log x}{x^4+x^2+1}dx=\int_0^\infty\frac{\sqrt{x}(x^2-1)\log x}{1-x^6}dx\overset{u=x^6}{=}$$

$$=\frac{1}{36}\int_0^\infty\frac{\left(u^{-\frac{5}{12}}-u^{-\frac{3}{4}}\right)\log u}{1-u}du=\frac{1}{36}\int_0^\infty\frac{u^{\frac{7}{12}-1}}{1-u}du-\frac{1}{36}\int_0^\infty\frac{u^{\frac{1}{4}-1}}{1-u}du=$$

$$=\frac{1}{36}\left[\psi'\left(\frac{3}{4}\right)-\psi'\left(\frac{7}{12}\right)\right]+\frac{1}{36}\left[\psi'\left(\frac{1}{4}\right)-\psi'\left(\frac{5}{12}\right)\right]=$$

$$=\frac{1}{36}\left[\psi'\left(\frac{3}{4}\right)+\psi'\left(\frac{1}{4}\right)\right]-\frac{1}{36}\left[\psi'\left(\frac{7}{12}\right)+\psi'\left(\frac{5}{12}\right)\right]=$$

$$=\frac{1}{36}\cdot\frac{\pi^2}{\sin^2\left(\frac{\pi}{4}\right)}-\frac{1}{36}\cdot\frac{\pi^2}{\sin^2\left(\frac{\pi}{12}\right)}$$

$$where\ \psi'(z)=\int_0^1\frac{t^{z-1}\log t}{1-t}dt$$

$$Reflection\ formula: \psi'(1-z)+\psi'(z)=\frac{\pi^2}{\sin^2(\pi z)}$$

Solution 4 by Ankush Kumar Parcha-India

$$\Omega=\int_0^\infty\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx=\int_0^1\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx+\int_1^\infty\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx=$$

$$=\int_0^1\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx-\int_0^1\frac{\frac{1}{y\sqrt{y}}\log y}{\frac{y^4+y^2+1}{y^4}}\frac{dy}{y^2}=$$

$$=\int_0^1\frac{x\sqrt{x}(1-x^2)\log x}{1-x^6}dx-\int_0^1\frac{\sqrt{x}(1-x^2)\log x}{1-x^6}dx=$$

$$=\int_0^1\frac{x\sqrt{x}}{1-x^6}\log x\,dx-\int_0^1\frac{x^3\sqrt{x}}{1-x^6}\log x\,dx-\int_0^1\frac{\sqrt{x}}{1-x^6}\log x\,dx+\int_0^1\frac{x^2\sqrt{x}}{1-x^6}\log x\,dx=$$

$$=\sum_{n=0}^\infty\int_0^1x^{6n+\frac{3}{2}}\log x\,dx-\sum_{n=0}^\infty\int_0^1x^{6n+\frac{7}{2}}\log x\,dx-\sum_{n=0}^\infty\int_0^1x^{6n+\frac{1}{2}}\log x\,dx$$

$$+\sum_{n=0}^\infty\int_0^1x^{6n+\frac{5}{2}}\log x\,dx=$$

$$=\sum_{n=0}^\infty\frac{1}{\left(6n+\frac{3}{2}\right)^2}-\sum_{n=0}^\infty\frac{1}{\left(6n+\frac{5}{2}\right)^2}-\sum_{n=0}^\infty\frac{1}{\left(6n+\frac{7}{2}\right)^2}+\sum_{n=0}^\infty\frac{1}{\left(6n+\frac{9}{2}\right)^2}$$

$$\because\int_0^1x^m\log^n x\,dx=\frac{(-1)^n n!}{(m+1)^{n+1}};m\neq 1,n>-1$$

$$36\Omega = \zeta\left(2,\frac{1}{4}\right) - \zeta\left(2,\frac{7}{12}\right) - \zeta\left(2,\frac{5}{12}\right) + \zeta\left(2,\frac{3}{4}\right)$$

$$= \psi'\left(\frac{1}{4}\right) - \psi'\left(\frac{5}{12}\right) - \psi'\left(\frac{7}{12}\right) + \psi'\left(\frac{3}{4}\right)$$

$$\psi^{(m)}(z) = (-1)^{m+1} m!\,\zeta(1+m,z)$$

$$\psi^{(n)}(1-z) = (-1)^n \psi^{(n)}(z) + (-1)^n \pi \frac{d^n}{dx^n}\cot(\pi z)$$

$$36\Omega = -\pi \frac{-\pi}{\sin^2\left(\frac{\pi}{4}\right)} - \left[-\pi \frac{-\pi}{\sin^2\left(\frac{5\pi}{12}\right)}\right] = 2\pi^2 - 2\pi^2\left(\sqrt{3}-1\right)^2$$

$$\Omega = \frac{\pi^2}{18}\left(2\sqrt{3}-3\right)$$

Solution 5 by Pham Duc Nam-Vietnam

$$Let\ I(a) = \int_0^\infty \frac{x^a}{x^2+x+1}dx\ ,where\ 0 < a < 1$$

$$I(a) = \int_0^\infty \frac{x^a(x-1)}{x^3-1}dx = \int_0^1 \frac{x^{a+1}-x^a}{x^3-1}dx + \int_0^1 \frac{x^{a+1}-x^a}{x^3-1}dx$$

$$Let\ t = \frac{1}{x}\ for\ latter\ integral \Rightarrow I(a) = \int_0^1 \frac{x^a + x^{-a}}{1-x^3}(1-x)dx$$

$$Now, using: \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}\ and\ change\ order\ of\ summation\ and\ integration:$$

$$I(a) = \sum_{k=0}^{\infty}\int_0^1 (x^a + x^{-a})(1-x)x^{3k}dx =$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{3k+a+1} + \frac{1}{3k-a+1} - \frac{1}{3k+a+2} - \frac{1}{3k-a+2}\right) =$$

$$= \frac{d}{da}\log\left(\prod_{k=0}^{\infty}\frac{(3k+a+1)(3k-a+2)}{(3k-a+1)(3k+a+2)}\right) =$$

$$= \frac{d}{da}\log\left(\frac{\sin\frac{\pi(a+1)}{3}}{\cos\frac{\pi\left(a+\frac{1}{2}\right)}{3}}\right) = \frac{2\pi}{\sqrt{3}}\cdot\frac{1}{2\cos\frac{2\pi a}{3}+1}$$

$2.$ ***Back to*** Ω***, let*** $t = x^2$***, we get:***

$$\Omega = \frac{1}{4}\int_0^{\infty}\frac{t^{\frac{1}{4}}\log t}{t^2+t+1}dt = \frac{1}{4}\cdot\frac{d}{da}I(a)\Big|_{a=\frac{1}{4}} = \frac{\pi}{2\sqrt{3}}\cdot\frac{d}{da}\left(\frac{1}{2\cos\frac{2\pi a}{3}+1}\right)_{a=\frac{1}{4}}$$

$$= \frac{\pi^2}{18}(2\sqrt{3}-3)$$

UP.485 Find:

$$\Omega = \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by proposer

Let $A = \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx$**, and** $B = \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\sin^{-1}x}{\sqrt{8x^4-6x^2+1}}dx$**. We have:**

$$A+B = \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\cos^{-1}x+\sin^{-1}x}{\sqrt{8x^4-6x^2+1}}dx = \frac{\pi}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\sqrt{8x^4-6x^2+1}}dx$$

But we have: $\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\sqrt{8x^4-6x^2+1}}dx$

$= 2\int_0^{\frac{1}{2}}\frac{1}{\sqrt{8x^4-6x^2+1}}dx$ **because the function under the intgral sign is even. We are going to calculate the integral:**

$$C = \int_0^{\frac{1}{2}}\frac{1}{\sqrt{8x^4-6x^2+1}}dx$$

We will show that the C **integral can be expresed using the complete elliptic integral of the first kind. The complete elliptic integral of the first kind is defined by**

$$K(k) = \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta\text{, with } -1 < k < 1$$

Substitute $t = \sin\theta$**, so** $dt = \cos\theta\, d\theta$**. We have:**

$$K(k)=\int_0^1 \frac{1}{\sqrt{1-k^2t^2}}\frac{1}{\sqrt{1-t^2}}\mathrm{dt}=\int_0^1 \frac{1}{\sqrt{k^2t^4-(1+k^2)t^2+1}}dt$$

Substitute $t=2x, dt=2dx$, we have:

$$K(k)=2\int_0^{\frac{1}{2}} \frac{1}{\sqrt{16k^2x^4-4(1+k^2)x^2+1}}dt\,.\text{ So, }K\left(\frac{1}{\sqrt{2}}\right)=2\int_0^{\frac{1}{2}}\frac{1}{\sqrt{8x^4-6x^2+1}}dx$$

$$\text{We have: } C=\frac{1}{2}K\left(\frac{1}{\sqrt{2}}\right)$$

The integral B is equal to zero, because the function under the integral sign is odd.

$$\text{So, we have: } A=\frac{\pi}{2}\cdot 2C=\frac{\pi}{2}K\left(\frac{1}{\sqrt{2}}\right).\text{ Therefore,}$$

$$\Omega=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx=\frac{\pi}{2}K\left(\frac{1}{\sqrt{2}}\right)$$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx=\int_{-\frac{1}{2}}^{0}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx+\int_0^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx$$
$$=I_1+I_2$$

$$I_1=\int_{-\frac{1}{2}}^{0}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx\overset{x\to -x}{=}\int_{\frac{1}{2}}^{0}-\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx=\int_0^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx=$$
$$=\int_0^{\frac{1}{2}}\frac{\pi-\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx=\int_0^{\frac{1}{2}}\frac{\pi}{\sqrt{8x^4-6x^2+1}}dx-\int_0^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx=$$
$$=\int_0^{\frac{1}{2}}\frac{\pi}{\sqrt{8x^4-6x^2+1}}dx-I_2$$

$$\Omega=I_1+I_2=\int_0^{\frac{1}{2}}\frac{\pi}{\sqrt{8x^4-6x^2+1}}dx-I_2+I_2=\int_0^{\frac{1}{2}}\frac{\pi}{\sqrt{8x^4-6x^2+1}}dx=$$
$$=\pi\int_0^{\frac{1}{2}}\frac{1}{\sqrt{8x^4-6x^2+1}}dx\overset{u=2x}{=}\frac{\pi}{2}\int_0^1\frac{1}{\sqrt{(1-u^2)\left(1-\frac{1}{2}u^2\right)}}du=\frac{\pi}{2}K\left(\frac{1}{\sqrt{2}}\right)$$

$$\Omega=\frac{\pi}{2}K\left(\frac{1}{\sqrt{2}}\right),\textit{where } K(k)-\textit{ complete eliptic integral of the first kind.}$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx=\int_{-\frac{1}{2}}^{0}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx+\int_{0}^{\frac{1}{2}}\frac{\cos^{-1}x}{\sqrt{8x^4-6x^2+1}}dx$$

$$2\Omega=2\pi\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{8x^4-6x^2+1}}\Rightarrow\Omega=\pi\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{8x^4-6x^2+1}}=$$

$$\overset{2x=y}{=}\frac{\pi}{2}\int_0^1\frac{\sqrt{2}}{\sqrt{y^4-3y^2+2}}dy=\frac{\pi\sqrt{2}}{2}\int_0^1\frac{dy}{\sqrt{(1-y^2)\big(1+(1-y^2)\big)}}\overset{1-y^2=k}{=}$$

$$=\frac{\pi\sqrt{2}}{4}\int_0^1\frac{dk}{\sqrt{(1-k)k(1+k)}}=\frac{\pi\sqrt{2}}{4}\int_0^1\frac{dk}{\sqrt{k(1-k^2)}}\overset{k=\sin y}{=}$$

$$=\frac{\pi\sqrt{2}}{4}\int_0^{\frac{\pi}{2}}\frac{\cos y}{\sqrt{\sin y}\cdot\sqrt{1-\sin^2 y}}dy=\frac{\pi\sqrt{2}}{4}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\sin y}}dy$$

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}x\cos^{2n-1}x\,dx\,;\Re(m)>0,\Re(n)>0$$

$$\frac{\pi}{\sin(n\pi)}=\Gamma(n)\Gamma(1-n),n\in(0,1)$$

$$\Omega=\frac{\pi\sqrt{2\pi}}{\Gamma\left(\frac{3}{4}\right)}=\frac{\sqrt{\pi}}{8}\Gamma^2\left(\frac{1}{4}\right)$$

UP.486 If $0<a\le b\le 1\le c\le d\le 2$ then:

$$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c+\tan^{-1}d\le 3\tan^{-1}\left(\frac{a+b+1}{3}\right)+\tan^{-1}(c+d-1)$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

Let be $f:(0,\infty)\to\mathbb{R}, f(x)=\tan^{-1}x$, then $f'(x)=\frac{1}{1+x^2}; f''(x)=\frac{-2x}{1+x^2}$

$\Rightarrow f$ –concave function. By Jensen's inequality and $a\le b\le 1$, we have:

$$f(a)+f(b)+f(1)\le 3f\left(\frac{a+b+1}{3}\right);(1)$$

$1\le c\le d; c+d=1+(c+d)-1$, by Karamata's inequality:

$$f(c)+f(d)\le f(1)+f(c+d-1);(2)$$

By adding (1) and (2):

$$f(a)+f(b)+f(1)+f(c)+f(d)\le f(1)+3f\left(\frac{a+b+1}{3}\right)+f(c+d-1)$$

$$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c+\tan^{-1}d\le 3\tan^{-1}\left(\frac{a+b+1}{3}\right)+\tan^{-1}(c+d-1)$$

Equality holds for $a=b=c=d=1$.

UP.487 Prove that:

$$\Omega=-\int_0^1\frac{t\log^3 t}{(1+t)^2}dt=\frac{7\pi^4}{120}-\frac{9}{2}\zeta(3)$$

Proposed by Said Attaoui-Algerie

Solution 1 by proposer

We have by making the change of the variable $t=e^{-x},dt=-e^{-x}dx$:

$$\Omega=\int_0^\infty\frac{x^3e^{-2x}}{(1+e^{-x})^2}dx$$

Since $e^{-x}<1$, we can use the geometric series:

$$\sum_{n=0}^{\infty}nx^n=\frac{x}{(1-x)^2}$$

We get:

$$\Omega=\int_0^\infty x^3e^{-x}\left(-\sum_{n=0}^{\infty}n(-1)^ne^{-nx}\right)dx=-\sum_{n=0}^{\infty}n(-1)^n\left(\int_0^\infty x^3e^{-n(n+1)x}\,dx\right)=$$

$$=-\sum_{n=0}^{\infty}n(-1)^n\left(\frac{\Gamma(4)}{(n+1)^4}\right)=$$

$$\left(\text{by the fact for all } a>0,b>0:\ \int_0^\infty t^ae^{-bx}dx=\Gamma(a+1).b^{a+1}\right)$$

$$=-6\sum_{n=0}^{\infty}n(-1)^n\left(\frac{1}{(n+1)^4}\right)=$$

$$=-6\sum_{n=1}^{\infty}(n-1)(-1)^n\left(\frac{1}{n^4}\right)=-6\left(-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right)$$

Using the property: $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}=-(1-2^{1-p})\zeta(p), p>1$, we can deduce:

$$\Omega=-6\left(\frac{3}{4}\zeta(3)-\frac{7}{8}\zeta(4)\right)=-\frac{9}{2}\zeta(3)+\frac{21}{4}\zeta(4)=-\frac{9}{2}\zeta(3)+\frac{7\pi^4}{120}$$

$$\text{Therefore, }\Omega=-\int_0^1\frac{t\log^3 t}{(1+t)^2}dt=\frac{7\pi^4}{120}-\frac{9}{2}\zeta(3)$$

Solution 2 by Le Thu-Vietnam

$$\Omega=\frac{x\log^3 x}{1+x}\bigg|_0^1-3\int_0^1\frac{\log^2 x}{1+x}dx-\int_0^1\frac{\log^3 x}{1+x}dx=$$

$$=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}=2\eta(3)=2(1-2^{1-3})\zeta(3)=\frac{3}{2}\zeta(3)$$

$$\textit{Similary}:\int_0^1\frac{\log^3 x}{1+x}dx=-6\sum_{n=0}^{\infty}\frac{(-1)^n}{(m+1)^4}=-6\eta(4)=$$

$$=-6(1-2^{1-4})\zeta(4)=-6\cdot\frac{7}{8}\cdot\frac{\pi^4}{90}=\frac{-\pi^4}{120}$$

$$\textit{Summing all of them, we obtain}:\ \Omega=\frac{7\pi^4}{120}-\frac{9}{2}\zeta(3)$$

Solution 3 by Pham Duc Nam-Vietnam

$$\textit{We known}:\frac{1}{1+x}=\sum_{k=0}^{\infty}(-1)^k x^k\Rightarrow-\frac{1}{(1+x)^2}=\sum_{k=0}^{\infty}(-1)^k kx^{k-1}$$

$$\Omega=-\int_0^1\frac{x\log^3 x}{(x+1)^2}dx=\int_0^1 x\log^3 x\sum_{k=0}^{\infty}(-1)^k kx^{k-1}\,dx=$$

$$=\sum_{k=0}^{\infty}(-1)^k k\int_0^1 x^k\log^3 x\,dx=\sum_{k=0}^{\infty}(-1)^k k\cdot\frac{-6}{(k+1)^4}=$$

$$=-6\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{(k+1)^3}-\frac{1}{(k+1)^4}\right)=$$

$$=-6\left((1-2^{1-3})\zeta(3)-(1-2^{1-4})\zeta(4)\right)=$$

$$= -6\left(\frac{3}{4}\zeta(3) - \frac{7}{8}\zeta(4)\right) = \frac{21}{4}\zeta(4) - \frac{9}{2}\zeta(3) = \frac{7\pi^4}{120} - \frac{9}{2}\zeta(3)$$

Solution 4 by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^1\int_0^1 Li_2(x-y)\,dxdy \overset{x-y=z}{=} \int_0^1\int_{x-1}^x Li_2(z)\,dz$$

$$\begin{cases} u = \int_{x-1}^x Li_2(z)\,dz \\ dv = dx \end{cases};\ \begin{cases} du = [Li_2(x) - Li_2(x-1)]dx \\ v = x \end{cases}$$

$$\Omega = x\int_{x-1}^x Li_2(z)\,dz\Big|_0^1 - \int_0^1 xLi_2(x)\,dx + \int_0^1 xLi_2(x-1)\,dx =$$

$$= \int_0^1 Li_2(x)\,dx - \int_0^1 xLi_2(x)\,dx + \int_0^1 xLi_2(x-1)\,dx =$$

$$= \int_0^1 (1-x)Li_2(x)\,dx + \int_0^1 xLi_2(x-1)\,dx =$$

$$= \int_0^1 (1-x)Li_2(x)\,dx + \int_0^1 (1-x)Li_2(-x)\,dx =$$

$$= \int_0^1 (1-x)[Li_2(x) + Li_2(-x)]\,dx = \frac{1}{2}\int_0^1 (1-x)Li_2(x^2)\,dx =$$

$$= \frac{1}{2}\int_0^1 Li_2(x^2)\,dx - \frac{1}{2}\int_0^1 xLi_2(x^2)\,dx = \frac{1}{2}\int_0^1 Li_2(x^2)\,dx - \frac{1}{4}\int_0^1 Li_2(x)\,dx =$$

$$= \frac{1}{2}[\zeta(2) + 4\log 2 - 4] - \frac{1}{4}[\zeta(2) - 1] = \frac{1}{4}\zeta(2) + 2\log 2 - \frac{7}{4}$$

Solution 5 by Amin Hajiyev-Azerbaijan

$$\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n x^n\,\frac{d}{dt}\left(\frac{1}{1+t}\right) = \sum_{n=0}^{\infty}\frac{n(-1)^n x^n}{x}$$

$$-\frac{x}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\Omega = -\int_0^1 \frac{t\log^3 t}{(1+t)^2}dt = \sum_{n=0}^{\infty}(-1)^n n\int_0^1 t^n\log^3 t\,dt \overset{IBP}{=} -3\sum_{n=0}^{\infty}\frac{n(-1)^n}{n+1}\int_0^1 t^n\log^2 t\,dt \overset{IBP}{=}$$

$$= 6\sum_{n=0}^{\infty}\frac{n(-1)^n}{(n+1)^2}\int_0^1 t^n \log t\, dt \overset{IBP}{=} -6\sum_{n=0}^{\infty}\frac{n(-1)^n}{(n+1)^3}\int_0^1 t^n\, dt = -6\sum_{n=0}^{\infty}\frac{n(-1)^n}{(n+1)^4} =$$

$$= 6\sum_{n=0}^{\infty}\frac{(n-1)(-1)^n}{n^4} = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{n^3} - \sum_{n=0}^{\infty}\frac{(-1)^n}{n^4} =$$

$$= 6\big(\eta(4)-\eta(3)\big) = 6\big((1-2^{1-4})\zeta(4) - (1-1^{1-3})\zeta(3) =$$

$$= \frac{21}{4}\zeta(4) - \frac{9}{2}\zeta(3) = \frac{21}{4}\cdot\frac{\pi^4}{90} - \frac{9}{2}\zeta(3) = \frac{7\pi^4}{120} - \frac{9}{2}\zeta(3)$$

$$-\int_0^1 \frac{t\log^3 t}{(1+t)^2}dt = \frac{7\pi^4}{120} - \frac{9}{2}\zeta(3)$$

UP.488 If $0 < a \le b$ then:

$$\left(\int_a^b \frac{x^2+1}{x^3+1}dx\right)\left(\int_a^b \frac{\sqrt{x}}{x^3+1}dx\right) \le 2\big(\sqrt{b}-\sqrt{a}\big)(\tan^{-1} b - \tan^{-1} a)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

We will prove that for $x \ge 0$:

$$\frac{\sqrt{x}}{x^3+1} \le \frac{1}{x^2+1};(1)$$

$$(1) \Leftrightarrow x(x^2+1)^2 \le (x^3+1)^2 \Leftrightarrow x(x^4+2x^2+1) \le x^6+2x^3+1 \Leftrightarrow$$

$$x^6 - x^5 + 1 - x \ge 0 \Leftrightarrow x^5(x-1) - (x-1) \ge 0 \Leftrightarrow$$

$(x-1)^2(x^4+x^3+x^2+x+1) \ge 0$ **wich is clearly true.**

By (1), it follows:

$$\int_a^b \frac{\sqrt{x}}{x^3+1}dx \le \int_a^b \frac{1}{x^2+1}dx = \tan^{-1} b - \tan^{-1} a;(2)$$

By (1), we have:

$$\frac{x^2+1}{x^3+1} \le \frac{1}{\sqrt{x}}$$

$$\int_a^b \frac{x^2+1}{x^3+1}dx \le \int_a^b \frac{1}{\sqrt{x}}dx = 2(\sqrt{b}-\sqrt{a});\ (3)$$

By multiplying (2), (3):

$$\left(\int_a^b \frac{x^2+1}{x^3+1}dx\right)\left(\int_a^b \frac{\sqrt{x}}{x^3+1}dx\right) \le 2(\sqrt{b}-\sqrt{a})(\tan^{-1} b - \tan^{-1} a)$$

Equality holds for $a = b$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{By Chebyshev's inequality we have}:\ x^3+1 = x^3+1^3 \ge \frac{1}{2}(x^2+1^2)(x+1).$$

$$\textit{Also by AM}-\textit{GM inequality we have}:\ x+1 \ge 2\sqrt{x}.$$

$$\textit{Then}:\ x^3+1 \ge (x^2+1)\sqrt{x},\ \ \forall x > 0.$$

$$\textit{Therefore},\ \left(\int_a^b \frac{x^2+1}{x^3+1}dx\right)\left(\int_a^b \frac{\sqrt{x}}{x^3+1}dx\right) \le \left(\int_a^b \frac{1}{\sqrt{x}}dx\right)\left(\int_a^b \frac{1}{x^2+1}dx\right) =$$

$$= \left[2\sqrt{x}\right]_a^b \cdot \left[\tan^{-1} x\right]_a^b = 2(\sqrt{b}-\sqrt{a})(\tan^{-1} b - \tan^{-1} a).$$

UP.489 Find:

$$\Omega = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\left[\sum_{k=1}^{n} \sqrt[2k+1]{\frac{2k+1}{2k-1}}\right] + \sqrt[3]{(k+1)^2(k^2+1)^2}},\ a \in \mathbb{R}, a > 0, [*] - \mathbf{GIF}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$S_n = \sum_{k=1}^{n} \sqrt[2k+1]{\frac{2k+1}{2k-1}},\ n \in \mathbb{N}^*$$

$$\sqrt[2k+1]{\frac{2k+1}{2k-1}} = \sqrt[2k+1]{1+\frac{2}{2k-1}} > 1, \forall k = \overline{1,n} \Rightarrow S_n = \sum_{k=1}^{n} \sqrt[2k+1]{\frac{2k+1}{2k-1}} > n;\ (1)$$

$$\sqrt[2k+1]{\frac{2k+1}{2k-1}} = \sqrt[2k+1]{1+\frac{2}{2k-1}} = \sqrt[2k+1]{\underbrace{1\cdot 1\cdot \ldots \cdot 1}_{2k-terms}\left(1+\frac{2}{2k-1}\right)} \overset{AGM}{<}$$

$$<\frac{2k\cdot 1+1+\frac{2}{2k-1}}{2k+1}=1+\frac{2}{(2k-1)(2k+1)}=1+\frac{1}{2k-1}-\frac{1}{2k+1}$$

$$\Rightarrow S_n=\sum_{k=1}^{n}\sqrt[2k+1]{\frac{2k+1}{2k-1}}<n+1-\frac{1}{2n+1}<n+1;\ \ (2)$$

From (1) and (2), we obtain that: $n<S_n<n+1\Rightarrow [S_n]=n$

Therefore,

$$i^3<(i+1)(i^2+1)<(i+1)^3,(\forall)i=\overline{1,n},n\geq 1$$

$$i^2<\sqrt[3]{(i+1)^2(i^2+1)^2}<(i+1)^2,(\forall)i=\overline{1,n},n\geq 1$$

$$\frac{1}{(i+1)^2+n}<\frac{1}{n+\sqrt[3]{(i+1)^2(i^2+1)^2}}<\frac{1}{i^2+n},(\forall)i=\overline{1,n},n\geq 1$$

$$\sum_{i=1}^{n}\frac{1}{(i+1)^2+n}\leq\sum_{i=1}^{n}\frac{1}{n+\sqrt[3]{(i+1)^2(i^2+1)^2}}<\sum_{k=1}^{n}\frac{1}{i^2+n};(1)$$

$$n+i^2\geq 2i\sqrt{n},(\forall)i=\overline{1,n},n\in\mathbb{N}^*$$

$$0<\sum_{i=1}^{n}\frac{1}{i^2+n}<\sum_{i=1}^{n}\frac{1}{2i\sqrt{n}}=\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{2\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{2\sqrt{n}}\overset{LCS}{=}\frac{1}{2}\cdot\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\sqrt{n+1}-\sqrt{n}}=0$$

So, we obtain that:

$$\mathrm{L}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\left[\sum_{k=1}^{n}\sqrt[2k+1]{\frac{2k+1}{2k-1}}\right]+\sqrt[3]{(i+1)^2(i^2+1)^2}}=0$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ k\in N.\ We\ have:$$

$$1\leq\sqrt[2k+1]{\frac{2k+1}{2k-1}}=\sqrt[2k+1]{\underbrace{1.1\dots 1}_{2k\ times}\cdot\frac{2k+1}{2k-1}}\overset{AM-GM}{\leq}\frac{\underbrace{1+\cdots+1}_{2k\ times}+\frac{2k+1}{2k-1}}{2k+1}=$$

$$=\frac{2k+1+\frac{2}{2k-1}}{2k+1}=1+\frac{2}{(2k+1)(2k-1)}=1+\frac{1}{2k-1}-\frac{1}{2k+1}.$$

$$\Rightarrow\ n\le\sum_{k=1}^{n}\sqrt[2k+1]{\frac{2k+1}{2k-1}}\le\sum_{k=1}^{n}\left(1+\frac{1}{2k-1}-\frac{1}{2k+1}\right)=n+1-\frac{1}{2n+1}<n+1.$$

$$Then:\ \left[\sum_{k=1}^{n}\sqrt[2k+1]{\frac{2k+1}{2k-1}}\right]=n.$$

$$Now,we\ have:\ \sqrt[3]{(i+1)^2(i^2+1)^2}>\sqrt[3]{i^2.(i^2)^2}=i^2,\ \ \forall i\in N,$$

$$\Rightarrow\ 0<\frac{1}{n+\sqrt[3]{(i+1)^2(i^2+1)^2}}<\frac{1}{n+i^2}\overset{AM-GM}{\le}\frac{1}{2i\sqrt{n}},\ \ \forall i=\overline{1,n}.$$

$$Then:\ 0<\sum_{i=1}^{n}\frac{1}{n+\sqrt[3]{(i+1)^2(i^2+1)^2}}<\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\frac{1}{i}\le\frac{1}{2\sqrt{n}}\left(1+\sum_{i=2}^{n}\int_{i-1}^{i}\frac{1}{t}dt\right)=$$

$$=\frac{1}{2\sqrt{n}}\left(1+\int_{1}^{n}\frac{1}{t}dt\right)=\frac{1+ln(n)}{2\sqrt{n}}\underset{n\to\infty}{\to}0.$$

$$Therefore,\ \Omega=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\left[\sum_{k=1}^{n}\sqrt[2k+1]{\frac{2k+1}{2k-1}}\right]+\sqrt[3]{(i+1)^2(i^2+1)^2}}=0.$$

UP.490 Find:

$$\Omega=\lim_{n\to\infty}\left(\prod_{i=1}^{2n}e^{(-1)^k k!(2n-k)!}\right)^{\frac{1}{(2n)!}}$$

Proposed by Florică Anastase, Ionuț Bină-Romania

Solution 1 by proposer

$$\text{Let: } S_n=\sum_{k=0}^{2n}(-1)^k k!\,(2n-k)!\,,\text{ then:}$$

$$\frac{S_n}{(2n)!}=\sum_{k=0}^{2n}\frac{(-1)^k k!\,(2n-k)!}{(2n)!}=\sum_{k=0}^{2n}\frac{(-1)^k}{\binom{2n}{k}}$$

Using: $\frac{1}{\binom{2n}{k}}=\frac{2n+1}{2(n+1)}\left(\frac{1}{\binom{2n+1}{k}}+\frac{1}{\binom{2n+1}{k+1}}\right)$, **we get:**

$$\frac{S_n}{(2n)!}=\sum_{k=0}^{2n}\frac{2n+1}{2(n+1)}\cdot(-1)^k\left(\frac{1}{\binom{2n+1}{k}}+\frac{1}{\binom{2n+1}{k+1}}\right)=$$

$$=\frac{2n+1}{2(n+1)}\sum_{k=0}^{2n}(-1)^k\left(\frac{1}{\binom{2n+1}{k}}+\frac{1}{\binom{2n+1}{k+1}}\right)=$$

$$=\frac{2n+1}{2(n+1)}\left(\frac{1}{\binom{2n+1}{0}}+\frac{1}{\binom{2n+1}{2n+1}}\right)=\frac{2n+1}{n+1}$$

Hence,

$$\sum_{k=1}^{2n}\frac{(-1)^k}{\binom{2n}{k}}=\sum_{k=0}^{2n}\frac{(-1)^k}{\binom{2n}{k}}-1=\frac{2n+1}{n+1}-1=\frac{n}{n+1}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\prod_{i=1}^{2n}e^{(-1)^k k!(2n-k)!}\right)^{\frac{1}{(2n)!}}=e^{\lim_{n\to\infty}\frac{n}{n+1}}=e$$

Solution 2 by Angel Plaza-Spain

$$\Omega=\lim_{n\to\infty}\left(\prod_{i=1}^{2n}e^{(-1)^k k!(2n-k)!}\right)^{\frac{1}{(2n)!}}=\lim_{n\to\infty}e^{\sum_{i=1}^{2n}\frac{(-1)^k}{\binom{2n}{k}}},\sum_{i=0}^{n}\frac{(-1)^k}{\binom{2n}{k}}=\frac{(n+1)(1+(-1)^n)}{n+2}$$

From where, changing n by $2n$,

$$\sum_{i=0}^{2n}\frac{(-1)^k}{\binom{2n}{k}}=\frac{(2n+1)(1+(-1)^{2n})}{2n+2}=\frac{2n+1}{n+1}$$

$$\sum_{i=1}^{2n}\frac{(-1)^k}{\binom{2n}{k}}=\frac{2n+1}{n+1}-1=\frac{n}{n+1}$$

$$\Omega=\lim_{n\to\infty}\left(\prod_{i=1}^{2n}e^{(-1)^k k!(2n-k)!}\right)^{\frac{1}{(2n)!}}=e^{\lim_{n\to\infty}\frac{n}{n+1}}=e$$

UP.491 In $\Delta ABC, a,b,c-$ sides and h_a,h_b,h_c- altitudes.

$$\text{If } \Omega(x,\alpha)=\int\frac{\cos\alpha\,dx}{\sin x-\sin\alpha}+\int\frac{\sin\alpha\,dx}{\cos x-\cos\alpha}\text{ then prove:}$$

$$e^{\Omega(h_a^2,a)}+e^{\Omega(h_b^2,b)}+e^{\Omega(h_c^2,c)}\ge s\left(\frac{F}{R}+1\right)$$

Proposed by Floricǎ Anastase-Romania

Solution 1 by proposer

$$\text{Denote: } m=\frac{x-\alpha}{2}, n=\frac{x+\alpha}{2}$$

Let $\tan\frac{x}{2}=t$ and $\tan\frac{\alpha}{2}=\beta\Rightarrow\sin\alpha=\frac{2\beta}{1+\beta^2}$, then:

$$I(x,\alpha)=\int\frac{dx}{\sin x-\sin\alpha}=\int\frac{1}{\frac{2t}{1+t^2}-\frac{2\beta}{1+\beta^2}}\frac{2dt}{1+t^2}$$

$$=(1+\beta^2)\int\frac{dt}{(1+\beta^2)t-(1+t^2)\beta}=$$

$$=(1+\beta^2)\int\frac{dt}{\beta t^2-(1+\beta^2)t+\beta}=-(1+\beta^2)\int\frac{dt}{(\beta t-1)(t-\beta)}=$$

$$=\frac{1+\beta^2}{\beta^2-1}\int\frac{\beta(t-\beta)-(\beta t-1)}{(\beta t-1)(t-\beta)}dt=\frac{\beta^2+1}{\beta^2-1}\left(\int\frac{\beta}{\beta t-1}dt-\int\frac{dt}{t-\beta}\right)=$$

$$=\frac{\beta^2+1}{\beta^2-1}\log\left|\frac{\beta t-1}{t-\beta}\right|+C$$

But $\beta=\tan\frac{\alpha}{2}$ and $\frac{\beta^2+1}{\beta^2-1}=-\frac{1}{\cos\alpha}$, hence:

$$I(x,\alpha)=\int\frac{dx}{\sin x-\sin\alpha}=\frac{1}{\cos\alpha}\log\left|\frac{\sin\frac{x-\alpha}{2}}{\cos\frac{x+\alpha}{2}}\right|+C=\frac{1}{\cos\alpha}\log\left|\frac{\sin m}{\cos n}\right|;(1)$$

Now, for $\frac{\pi}{2}-x=t$ and $\frac{\pi}{2}-\alpha=\gamma$, we have:

$$J(x,\alpha)=\int\frac{dx}{\cos x-\cos\alpha}=\int\frac{dx}{\sin\left(\frac{\pi}{2}-x\right)-\sin\left(\frac{\pi}{2}-\alpha\right)}=-\int\frac{dt}{\sin t-\sin\alpha}=$$

$$=I(t,\gamma)=-\frac{1}{\cos\gamma}\log\left|\frac{\sin\frac{t-\beta}{2}}{\cos\frac{t+\beta}{2}}\right|=\frac{1}{\sin\alpha}\log\left|\frac{\sin\frac{x+\alpha}{2}}{\sin\frac{x-\alpha}{2}}\right|=\frac{1}{\sin\alpha}\log\left|\frac{\sin n}{\sin m}\right|;(2)$$

From (1) and (2) we have:

$$\Omega(x,\alpha)=\cos\alpha\cdot I(x,\alpha)+\sin\alpha\cdot J(x,\alpha)=$$

$$=\cos\alpha\cdot\frac{1}{\cos\alpha}\log\left|\frac{\sin m}{\cos n}\right|+\sin\alpha\cdot\frac{1}{\sin\alpha}\log\left|\frac{\sin n}{\sin m}\right|=\log|\tan n|=\log\left|\tan\frac{x+\alpha}{2}\right|$$

$$e^{\Omega(h_a^2,a)}=\tan\frac{h_a^2+a}{2}\geq\frac{h_a^2+a}{2}$$

Analogous,

$$e^{\Omega(h_b^2,b)}\geq\frac{h_b^2+b}{2}\text{ and }e^{\Omega(h_c^2,c)}\geq\frac{h_c^2+c}{2}$$

By adding, it follows:

$$e^{\Omega(h_a^2,a)}+e^{\Omega(h_b^2,b)}+e^{\Omega(h_c^2,c)}\geq\frac{\left(h_a^2+h_b^2+h_c^2\right)+(a+b+c)}{2};(3)$$

Now, WLOG, we can assume that $a\geq b\geq c$. On account of rearrangement inequality:

$$\frac{1}{a}\leq\frac{1}{b}\leq\frac{1}{c}\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{1}{a}\cdot\frac{1}{b}+\frac{1}{b}\cdot\frac{1}{c}+\frac{1}{c}\cdot\frac{1}{a}=\frac{a+b+c}{abc}$$

$$h_a=\frac{2F}{a}=\frac{2rs}{a}\Rightarrow\frac{1}{a^2}=\frac{h_a^2}{4r^2s^2}$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{h_a^2+h_b^2+h_c^2}{4r^2s^2}$$

$$\text{But }[ABC]=\frac{abc}{4R}=\frac{(a+b+c)r}{2}\Rightarrow a+b+c=\frac{2F}{r}\text{ and }abc=4RF$$

$$\Rightarrow\frac{a+b+c}{abc}=\frac{1}{2Rr}\Rightarrow h_a^2+h_b^2+h_c^2\geq\frac{r}{2R}(a+b+c)^2;(4)$$

From (3) and (4) it follows that:

$$e^{\Omega(h_a^2,a)}+e^{\Omega(h_b^2,b)}+e^{\Omega(h_c^2,c)}\geq\frac{r(a+b+c)^2}{4R}+s=s\left(\frac{F}{R}+1\right)$$

Solution 2 by Tapas Das-India

$$\int\frac{\cos\alpha}{\sin x-\sin\alpha}dx+\int\frac{\sin\alpha}{\cos x-\cos\alpha}dx=\int\left[\frac{\cos\alpha}{\sin x-\sin\alpha}+\frac{\sin\alpha}{\cos x-\cos\alpha}\right]dx=$$

$$=\int\frac{(\cos\alpha\cos x+\sin\alpha\sin x)-(\cos^2\alpha+\sin^2\alpha)}{(\sin x-\sin\alpha)(\cos x-\cos\alpha)}dx=$$

$$=\int\frac{\cos(x-\alpha)-1}{2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}\left(-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}\right)}dx=$$

$$=\int\frac{-2\sin^2\frac{x-\alpha}{2}}{-2\cos\frac{x+\alpha}{2}\sin^2\frac{x-\alpha}{2}\cdot 2\sin\frac{x+\alpha}{2}}dx=\int\frac{dx}{2\cos\frac{x+\alpha}{2}\sin\frac{x+\alpha}{2}}=\int\frac{dx}{\sin(x+\alpha)}$$

$$\Omega(x,\alpha)=\log\left|\tan\frac{x+\alpha}{2}\right|$$

$$e^{\Omega(h_a^2,a)}+e^{\Omega(h_b^2,b)}+e^{\Omega(h_c^2,c)}=\tan\frac{h_a^2+a}{2}+\tan\frac{h_b^2+b}{2}+\tan\frac{h_c^2+c}{2}\geq$$

$$\geq\frac{h_a^2+a}{2}+\frac{h_b^2+b}{2}+\frac{h_c^2+c}{2}=\frac{1}{2}(h_a^2+h_b^2+h_c^2)+\frac{1}{2}(a+b+c)\geq$$

$$\geq\frac{1}{2}(h_ah_b+h_bh_c+h_ch_a)+\frac{1}{2}\cdot 2s=\frac{1}{2}\left(\frac{4F^2}{ab}+\frac{4F^2}{bc}+\frac{4F^2}{ca}\right)+s=$$

$$=2F^2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+s=2F^2\cdot\frac{a+b+c}{abc}+s=$$

$$=2F\cdot F\cdot\frac{2s}{4RF}+s=\frac{Fs}{R}+s=s\left(\frac{F}{R}+1\right)$$

$$Let\ f(x)=\tan x-x\ then\ f'(x)=\sec^2 x-1>0$$

$$f-is\ increasing\ function, then\ f(x)>f(0)=0$$

$$\tan x>x\ and\ thus, \tan\frac{h_a^2+a}{2}\geq\frac{h_a^2+a}{2}$$

Solution 3 by Adrian Popa-Romania

$$\int\frac{\cos\alpha}{\sin x-\sin\alpha}dx+\int\frac{\sin\alpha}{\cos x-\cos\alpha}dx=\int\left[\frac{\cos\alpha}{\sin x-\sin\alpha}+\frac{\sin\alpha}{\cos x-\cos\alpha}\right]dx=$$

$$=\int\frac{(\cos\alpha\cos x+\sin\alpha\sin x)-(\cos^2\alpha+\sin^2\alpha)}{(\sin x-\sin\alpha)(\cos x-\cos\alpha)}dx=$$

$$=\int\frac{\cos(x-\alpha)-1}{2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}\left(-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}\right)}dx=$$

$$=\int\frac{-2\sin^2\frac{x-\alpha}{2}}{-2\cos\frac{x+\alpha}{2}\sin^2\frac{x-\alpha}{2}\cdot 2\sin\frac{x+\alpha}{2}}dx=$$

$$=\int\frac{dx}{2\cos\frac{x+\alpha}{2}\sin\frac{x+\alpha}{2}}=\int\frac{dx}{\sin(x+\alpha)}=\int\frac{\sin(x+\alpha)}{\sin^2(x+\alpha)}dx=$$

$$=\int\frac{\sin(x+\alpha)}{1-\cos^2(x+\alpha)}dx\overset{\cos(x+\alpha)=t}{=}\int\frac{dt}{t^2-1}=\frac{1}{2}\log\left|\frac{1-t}{1+t}\right|=\frac{1}{2}\log\left|\frac{1-\cos(x+\alpha)}{1+\cos(x+\alpha)}\right|$$

$$\Rightarrow\Omega(x,\alpha)=\log\left|\tan\frac{x+\alpha}{2}\right|$$

$$e^{\Omega(x,\alpha)}=e^{\log\left|\tan\frac{x+\alpha}{2}\right|}\ge\frac{x+\alpha}{2}$$

$$e^{\Omega(h_a^2,a)}+e^{\Omega(h_b^2,b)}+e^{\Omega(h_c^2,c)}=\tan\frac{h_a^2+a}{2}+\tan\frac{h_b^2+b}{2}+\tan\frac{h_c^2+c}{2}\ge$$

$$\ge\frac{h_a^2+a}{2}+\frac{h_b^2+b}{2}+\frac{h_c^2+c}{2}=\frac{1}{2}\left(h_a^2+h_b^2+h_c^2\right)+\frac{1}{2}(a+b+c)\overset{?}{\ge}\frac{Fs}{R}+s\Leftrightarrow$$

$$\frac{1}{2}\left(h_a^2+h_b^2+h_c^2\right)\ge\frac{Fs}{R}$$

$$\frac{ah_a}{2}=F\Rightarrow h_a=\frac{2F}{a}\Rightarrow 2F^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge\frac{Fs}{R}$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{a+b+c}{abc}=\frac{1}{2Rr}(true).$$

UP.492 Let $(a_n)_{n\ge1}$ be sequence of real numbers strictly positive such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^2a_n}=a>0.\ \textbf{Find:}$$

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}-\sqrt[n]{\frac{a_n}{n!}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$B_n=\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}-\sqrt[n]{\frac{a_n}{n!}}=\sqrt[n]{\frac{a_n}{n!}}(u_n-1)=\sqrt[n]{\frac{a_n}{n!}}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n=$$

$$=\frac{1}{n}\sqrt[n]{\frac{a_n}{n!}}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n=\sqrt[n]{\frac{a_n}{n^n\cdot n!}}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n,n\ge 2$$

$$u_n=\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}\cdot\sqrt[n]{\frac{n!}{a_n}}=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}=$$

$$=\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2}\cdot\frac{n^2}{\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{(n+1)}{\sqrt[n+1]{(n+1)!}}\cdot\frac{n}{n+1},\forall n\ge 2$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^2}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{2n}}}\overset{C-D}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{2(n+1)}}\cdot\frac{n^{2n}}{a_n}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}\left(\frac{n}{n+1}\right)^{2n+2}=\frac{a}{e^2}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e},\quad\lim_{n\to\infty}u_n=\frac{a}{e^2}\cdot\frac{e^2}{e}\cdot\frac{1}{e}\cdot e\cdot 1\Rightarrow\lim_{n\to\infty}\frac{u_n-1}{\log u_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)!}\cdot\frac{n!}{a_n}\cdot\sqrt[n+1]{\frac{(n+1)!}{a_{n+1}}}=\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}\cdot\frac{n^2}{n+1}\cdot\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n+1]{a_{n+1}}}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}\cdot\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}\left(\frac{n}{n+1}\right)^2=a\cdot\frac{1}{e}\cdot\frac{e^2}{a}\cdot 1=e$$

$$\lim_{n\to\infty}B_n=1\cdot\log e\cdot\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n!\cdot n^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^2}\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=\frac{a}{e^2}\cdot e=\frac{a}{e}$$

Solution 2 by Angel Plaza-Spain

By the Stolz-Cesaro Lemma,

$$\Omega=\lim_{n\to\infty}\frac{1}{n+1}\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}=\lim_{n\to\infty}\frac{e\cdot\sqrt[n+1]{a_{n+1}}}{(n+1)^2}\overset{Stirling}{=}\frac{a}{e},$$

where, in the last step, the following result has been applied in [1] Proposition 2:

If $(a_n)_{n\ge1}$ is a sequence of real numbers strictly positive such that $\lim_{n\to\infty}\frac{a_{n+1}}{n^t\cdot a_n}=a, t>0$ then:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^t}=\frac{a}{e^t}$$

REFERENCES:

[1] D.M. Batineţu-Giurgiu, Angel Plaza, Daniel Sitaru, Florică Anastase, "New solutions for a few R.M.M. problems, available at

https://www.ssmrmh.ro/2022/04/02/new-solutions-for-a-few-rmm-problems/.

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{a_n}{n!}}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n n!}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}(n+1)!}\cdot\frac{n!\,n^n}{a_n}=$$

$$=\lim_{n\to\infty}\frac{1}{(n+1)^2}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{a_{n+1}}{a_n}\cdot\frac{n^2}{(n+1)^2}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{a}{e}$$

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}-\sqrt[n]{\frac{a_n}{n!}}\right)=\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{a_n}{n!}}\cdot n\cdot\frac{\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}}{\sqrt[n]{\frac{a_n}{n!}}}=$$

$$=\frac{a}{e}\lim_{n\to\infty}n\left(e^{\log\left(\frac{\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}}{\sqrt[n]{\frac{a_n}{n!}}}\right)}-1\right)=\frac{a}{e}\lim_{n\to\infty}n\log\left(\frac{\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}}{\sqrt[n]{\frac{a_n}{n!}}}\right)=$$

$$=\frac{a}{e}\lim_{n\to\infty}\log\left(\frac{\left(\frac{a_{n+1}}{(n+1)!}\right)^{\frac{n+1}{n+1}-\frac{1}{n+1}}}{\frac{a_n}{n!}}\right)=\frac{a}{e}\lim_{n\to\infty}\log\left(\frac{a_{n+1}}{n^2a_n}\cdot\frac{n^2}{(n+1)^2}\cdot\frac{1}{\frac{1}{n+1}\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}}\right)$$

$$=\frac{a}{e}\log\left(a\cdot 1\cdot\frac{e}{a}\right)=\frac{a}{e}\log e=\frac{a}{e}.$$

UP.493 Let $t\geq 0$ and $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ sequences of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{t+1}}=a>0$, $\lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^t}=b>0$. Find:

$$\Omega(a,b,t)=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{\sqrt[n]{b_n}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

We have:

$$B_n=\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{\sqrt[n]{b_n}}=\frac{\sqrt[n]{a_n}}{\sqrt[n]{b_n}}(u_n-1)=\frac{\sqrt[n]{a_n}}{\sqrt[n]{b_n}}\cdot\frac{1}{n}\cdot n\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n=$$

$$=\frac{\sqrt[n]{a_n}}{\sqrt[n]{b_n}}\cdot\frac{1}{n}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n=\frac{\sqrt[n]{a_n}}{n^{tn}}\cdot\frac{n^t}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n,n\geq 2,\text{where}$$

$$u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}=\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{t+1}}\cdot\frac{n^{t+1}}{\sqrt[n]{a_n}}\left(\frac{n+1}{n}\right)^{t+1},n\geq 2$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^{t+1}}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{n(t+1)}}}\overset{C-D}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{(n+1)(t+1)}}\cdot\frac{n^{nt}}{a_n}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{t+1}}\left(\frac{n}{n+1}\right)^{(t+1)n}=\frac{a}{e^{t+1}}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^t}=\frac{a}{e^t}$$

$$\lim_{n\to\infty}u_n=\frac{a}{e^{t+1}}\cdot\frac{e^{t+1}}{a}\cdot 1=1\Rightarrow\lim_{n\to\infty}\frac{u_n-1}{\log u_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{t+1}}\cdot\frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}}\left(\frac{n}{n+1}\right)^{t+1}=$$

$$=a\cdot\frac{e^{t+1}}{a}\cdot 1=e^{t+1}$$

$$\Rightarrow\lim_{n\to\infty}B_n=\frac{a}{e^{t+1}}\cdot\frac{e^t}{b}\cdot 1\cdot\log e^{t+1}=\frac{a(t+1)}{be}$$

Solution 2 by Angel Plaza-Spain

We will use the following result has been applied in [1] Proposition 2:

If $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^t\cdot a_n}=a,t>0$ then:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^t}=\frac{a}{e^t}$$

$$\Omega(a,b,t)=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{n^t}\cdot\frac{n^t}{\sqrt[n]{b_n}}=$$

$$=\lim_{n\to\infty}\frac{(t+1)\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)}{(n+1)^{t+1}-n^{t+1}}\cdot\frac{n^t}{\sqrt[n]{b_n}}\overset{LC-S}{=}\lim_{n\to\infty}\frac{(t+1)\sqrt[n+1]{a_{n+1}}}{n^{t+1}}\cdot\frac{n^t}{\sqrt[n]{b_n}}=$$

$$=\frac{(t+1)a}{e^{t+1}}\cdot\frac{e^t}{b}=\frac{(t+1)a}{eb}$$

REFERENCES:

[1] D.M. Batineţu-Giurgiu, Angel Plaza, Daniel Sitaru, Florică Anastase, "New solutions for a few R.M.M. problems, available at:

https://www.ssmrmh.ro/2022/04/02/new-solutions-for-a-few-rmm-problems/.

UP.494 If $f,g:\mathbb{R}\to(0,\infty)$, f,g –derivable, f',g' –continuous, $0<a\le b$ then:

$$16\int_a^b (f'(x)+g'(x))\left(f^3(x)+g^3(x)\right)dx\ge\left(f(b)+g(b)\right)^4-\left(f(a)+g(a)\right)^4$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Lemma. If $p,q>0$ then $4(p^3+q^3)\ge(p+q)^3$

Proof. $4(p^3+q^3)\ge p^3+3p^2q+3pq^2+q^3\Leftrightarrow$

$$4p^3+4q^3\ge p^3+3p^2q+3pq^2+q^3\Leftrightarrow 3p^3+3q^3-3p^2q-3pq^2\ge 0\Leftrightarrow$$

$$p^3-p^2q+q^3-3pq^2\ge 0\Leftrightarrow p^2(p-q)-q^2(p-q)\ge 0\Leftrightarrow$$

$$(p-q)(p^2-q^2)\ge 0\Leftrightarrow(p-q)(p-q)(p+q)\ge 0\Leftrightarrow$$

$$(p-q)^2(p+q)\ge 0\ (true!)$$

Let be $p=f(x),q=g(x)\Rightarrow 4\left(f^3(x)+g^3(x)\right)\ge\left(f(x)+g(x)\right)^3$

$$4(f'(x)+g'(x))\left(f^3(x)+g^3(x)\right)\ge\left(f(x)+g(x)\right)'\left(f(x)+g(x)\right)^3$$

$$4\int_a^b (f'(x)+g'(x))\left(f^3(x)+g^3(x)\right)dx\ge\int_a^b\left(f(x)+g(x)\right)'\left(f(x)+g(x)\right)^3dx$$

$$4\int_a^b (f'(x)+g'(x))\left(f^3(x)+g^3(x)\right)dx\ge\frac{\left(f(b)+g(b)\right)^4}{4}-\frac{\left(f(a)+g(a)\right)^4}{4}$$

$$16\int_a^b (f'(x)+g'(x))\left(f^3(x)+g^3(x)\right)dx\ge\left(f(b)+g(b)\right)^4-\left(f(a)+g(a)\right)^4$$

Equality holds for $a=b$ or $f\equiv g$.

Solution 2 by Marin Chirciu-Romania

Using $x^3+y^3\ge\frac{(x+y)^3}{4}\Leftrightarrow 4(x^3+y^3)\ge(x+y)^3\Leftrightarrow(x+y)(x-y)^2\ge 0$,

We get $f^3(x)+g^3(x)\ge\frac{(f(x)+g(x))^3}{4}$.

$$16\int_a^b(f'(x)+g'(x))(f^3(x)+g^3(x))\ge 16\int_a^b(f'(x)+g'(x))\frac{(f(x)+g(x))^3}{4}dx\Leftrightarrow$$

$$\Leftrightarrow 16\int_a^b(f'(x)+g'(x))(f^3(x)+g^3(x))\ge 4\int_a^b(f'(x)+g'(x))(f(x)+g(x))^3dx.$$

$$4\int_a^b(f'(x)+g'(x))(f(x)+g(x))^3dx=(f(b)+g(b))^4-(f(a)+g(a))^4.$$

Denote $f(x)+g(x)=h(x)$:

$$4\int_a^b h^3(x)h'(x)dx=h^4(b)-h^4(a):$$

$$\int_a^b h^3(x)h'(x)dx=\frac{h^4(x)}{4}\Big|_a^b=\frac{h^4(b)}{4}-\frac{h^4(a)}{4}.$$

UP.495 Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\frac{7^n}{7^{2n}+40\cdot 7^n+175}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\frac{7^n}{7^{2n}+40\cdot 7^n+175}=\frac{1}{6}\cdot\frac{6\cdot 7^n}{7^{2n}+35\cdot 7^n+5\cdot 7^n+175}=$$

$$=\frac{1}{6}\cdot\frac{7^n(7-1)}{\frac{1}{7}(7^{2n-1}+5\cdot 7^n+5\cdot 7^{n-1}+25)}=\frac{1}{6}\cdot\frac{7^n\left(\frac{7-1}{7}\right)}{7^n(7^{n-1}+5)+5(7^{n-1}+5)}=$$

$$=\frac{1}{6}\cdot\frac{7^n\left(1-\frac{1}{7}\right)}{(7^{n-1}+5)(7^n+5)}=\frac{1}{6}\cdot\frac{7^n-7^{n-1}}{(7^{n-1}+5)(7^n+5)}=$$

$$=\frac{1}{6}\cdot\frac{7^n+5-(7^{n-1}+5)}{(7^n+5)(7^{n-1}+5)}=\frac{1}{6}\left(\frac{1}{7^{n-1}+5}-\frac{1}{7^n+5}\right)$$

Therefore,

$$\Omega=\sum_{n=1}^{\infty}\frac{7^n}{7^{2n}+40\cdot 7^n+175}=\sum_{n=1}^{\infty}\frac{1}{6}\left(\frac{1}{7^{n-1}+5}-\frac{1}{7^n+5}\right)=$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{7^{k-1}+5}-\frac{1}{7^k+5}\right)=\frac{1}{6}\cdot\lim_{n\to\infty}\left(\frac{1}{6}-\frac{1}{7^n+5}\right)=\frac{1}{36}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\frac{7^n}{7^{2n}+40.7^n+175}=\frac{7^n}{(7^n+5)(7^n+35)}=\frac{7^{n-1}}{(7^n+5)(7^{n-1}+5)}=$$

$$\frac{(7^n+5)-(7^{n-1}+5)}{6(7^n+5)(7^{n-1}+5)}=\frac{1}{6(7^{n-1}+5)}-\frac{1}{6(7^n+5)}.$$

$$\textit{Therefore},\ \Omega=\sum_{n=1}^{\infty}\left(\frac{1}{6(7^{n-1}+5)}-\frac{1}{6(7^n+5)}\right)=\frac{1}{6(7^{1-1}+5)}=\frac{1}{36}.$$

Solution 3 by Angel Plaza-Spain

Since: $\frac{7^n}{7^{2n}+40\cdot 7^n+175}=\frac{1}{6(7^{n-1}+5)}-\frac{1}{6(7^n+5)}$, *the proposed series*

telescopes: $\sum_{n=1}^{k}\frac{7^n}{7^{2n}+40\cdot 7^n+175}=\frac{1}{36}-\frac{1}{6(7^k+5)}$ *and then*

$$\Omega=\sum_{n=1}^{\infty}\frac{7^n}{7^{2n}+40\cdot 7^n+175}=\lim_{n\to\infty}\left(\frac{1}{36}-\frac{1}{6(7^k+5)}\right)=\frac{1}{36}$$