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PROBLEMS FOR JUNIORS

JP.496 In ΔABC the following relationship holds:

$$2(4R+r)^2 \geq \frac{r_a^7+r_b^7}{(r_a+r_b)^5}+\frac{r_b^7+r_c^7}{(r_b+r_c)^5}+\frac{r_c^7+r_a^7}{(r_c+r_a)^5}+\frac{95}{16}s^2 \geq 6s^2$$

Proposed by Alex Szoros-Romania

Solution 1 by proposer

$$Using: (x+y)^7 = x^7+y^7+7xy(x+y)(x^2+xy+y^2)^2, (\forall)x,y>0$$

$$we\ deduce\ that:$$

$$\frac{(x+y)^7-x^7-y^7}{7xy} = (x+y)(x^2+xy+y^2)^2; (1)$$

$$How\ x^2+xy+y^2 \geq \frac{3}{4}(x+y)^2, (\forall)x,y>0; \quad (2)$$

$$From\ (1)\ and\ (2), we\ get:$$

$$\frac{(x+y)^7-x^7-y^7}{7xy} \geq \frac{9}{16}(x+y)^5$$

$$\frac{(x+y)^7-x^7-y^7}{(x+y)^5} \geq \frac{63}{16}xy \Rightarrow (x+y)^2 \geq \frac{63}{16}xy+\frac{x^7+y^7}{(x+y)^5}$$

$$x^2+y^2 \geq \frac{31}{16}xy+\frac{x^7+y^7}{(x+y)^5}$$

$$2\sum_{cyc}x^2 \geq \frac{31}{16}\sum_{cyc}xy+\sum_{cyc}\frac{x^7+y^7}{(x+y)^5}; \quad (3)$$

$$On\ the\ other\ hand\ for\ all\ x,y>0\ holds: \frac{x^7+y^7}{2} \geq \left(\frac{x+y}{2}\right)^7$$

$$\frac{x^7+y^7}{(x+y)^5} \geq \frac{(x+y)^2}{64} \geq \frac{4xy}{64} = \frac{xy}{16}$$

$$\sum_{cyc}\frac{x^7+y^7}{(x+y)^5} \geq \frac{1}{16}\sum_{cyc}xy \Rightarrow \frac{31}{16}\sum_{cyc}xy+\sum_{cyc}\frac{x^7+y^7}{(x+y)^5} \geq 2\sum_{cyc}xy; \quad (4)$$

$$From\ (3)\ and\ (4), it\ follows:$$

$$2\sum_{cyc}x^2 \geq \frac{31}{16}\sum_{cyc}xy+\sum_{cyc}\frac{x^7+y^7}{(x+y)^5} \geq 2\sum_{cyc}xy$$

$$\sum_{cyc} x^2 \ge \frac{31}{32}\sum_{cyc} xy + \frac{1}{2}\sum_{cyc}\frac{x^7+y^7}{(x+y)^5} \ge \sum_{cyc} xy\,;(\forall)x,y>0;\ \ (5)$$

$$For\ x=r_a, y=r_b, z=r_c, from\ (5)\ we\ get:$$

$$\sum_{cyc} r_a^2 \ge \frac{31}{32}\sum_{cyc} r_ar_b + \frac{1}{2}\sum_{cyc}\frac{r_a^7+r_b^7}{(r_a+r_b)^5} \ge \sum_{cyc} r_ar_b\,;(6)$$

$$How\ in\ any\ \Delta ABC: \sum_{cyc} r_ar_b = s^2\ and$$

$$\sum_{cyc} r_a^2 = \left(\sum_{cyc} r_a\right)^2 - 2\sum_{cyc} r_ar_b = (4R+r)^2 - 2s^2; (6)$$

$$(4R+r)^2 - 2s^2 \ge \frac{31s^2}{32} + \frac{1}{2}\sum_{cyc}\frac{r_a^7+r_b^7}{(r_a+r_b)^7} \ge s^2$$

$$(4R+r)^2 \ge \frac{95}{32}s^2 + \frac{1}{2}\sum_{cyc}\frac{r_a^7+r_b^7}{(r_a+r_b)^5} \ge 3s^2$$

$$2(4R+r)^2 \ge \frac{95}{16}s^2 + \sum_{cyc}\frac{r_a^7+r_b^7}{(r_a+r_b)^5} \ge 6s^2$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{r_a^7+r_b^7}{(r_a+r_b)^5} + \frac{r_b^7+r_c^7}{(r_b+r_c)^5} + \frac{r_c^7+r_a^7}{(r_c+r_a)^5} + \frac{95}{16}s^2 \ge$$

$$\ge \frac{1}{32}\sum_{cyc}\frac{(r_a+r_b)^3(r_a^2+r_b^2)}{(r_a+r_b)^5} + \frac{95}{16}s^2 =$$

$$= \frac{1}{16}\left(\left(\sum_{cyc} r_a\right)^2 - 2\sum_{cyc} r_ar_b\right) + \frac{95}{16}s^2 =$$

$$= \frac{1}{16}((4R+r)^2 - 2s^2) + \frac{95}{16}s^2 \overset{DOUCET}{\gtrless} \frac{1}{16}(3s^2 - 2s^2) + \frac{95}{16}s^2 = 6s^2$$

$$2(4R+r)^2 = 2\sum_{cyc} r_a^2 + 2s^2 \ge \frac{r_a^7+r_b^7}{(r_a+r_b)^5} + \frac{r_b^7+r_c^7}{(r_b+r_c)^5} + \frac{r_c^7+r_a^7}{(r_c+r_a)^5} + \frac{95}{16}s^2 \Leftrightarrow$$

$$\sum_{cyc}\left(r_a^2 + r_b^2 - \frac{r_a^7+r_b^7}{(r_a+r_b)^5}\right) - \frac{31}{16}s^2 \ge 0 \Leftrightarrow$$

$$\sum_{cyc}\left(r_a^2+r_b^2-\frac{r_a^7+r_b^7}{(r_a+r_b)^5}\right)-\frac{31}{16}\sum_{cyc}r_ar_b\geq 0\Leftrightarrow$$

Denote $r_a=x, r_b=y, r_c=z$

$$\sum_{cyc}\frac{16(x^2+y^2)(x+y)^5-16(x^7+y^7)-31xy(x+y)^5}{16(x+y)^5}\geq 0\Leftrightarrow$$

$$\sum_{cyc}I(x,y)=\sum_{cyc}\frac{I_1(x,y)}{I_2(x,y)}\geq 0$$

$$I(x,y)=\frac{16(x^2+y^2)(x+y)^5-16(x^7+y^7)-31xy(x+y)^5}{16(x+y)^5}$$

$$I_1(x,y)=16(x^2+y^2)(x+y)^5-16(x^7+y^7)-31xy(x+y)^5$$

$$I_2(x,y)=16(x+y)^5>0$$

$$I_1(x,y)=7xy(x-y)^2(x+y)(7x^2-4xy+y^2)\geq 0$$

Equality holds for $a=b=c$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{r_a^7+r_b^7}{(r_a+r_b)^5}+\frac{r_b^7+r_c^7}{(r_b+r_c)^5}+\frac{r_c^7+r_a^7}{(r_c+r_a)^5}$$

$$=\sum_{cyc}\frac{(x+y)^7-7xy\left(x^5+y^5+3xy(x^3+y^3)+5x^2y^2(x+y)\right)}{(x+y)^5}$$

$$(x=r_a, y=r_b, z=r_c)=\sum_{cyc}(x+y)^2$$

$$-\sum_{cyc}\left(\frac{7xy}{(x+y)^4}\cdot(x^4-x^3y+x^2y^2-xy^3+y^4+3xy(x^2-xy+y^2)+5x^2y^2)\right)$$

$$=2\sum_{cyc}x^2+2\sum_{cyc}xy-\sum_{cyc}\left(\frac{7xy}{(x+y)^4}\cdot(x^4+2x^3y+2xy^3+3x^2y^2+y^4)\right)$$

$$=2\sum_{cyc}r_a^2+2\sum_{cyc}r_ar_b$$

$$-\sum_{cyc}\left(\frac{7xy}{2(x+y)^4}\cdot\left((x^4+4x^3y+4xy^3+6x^2y^2+y^4)+(x^4+y^4)\right)\right)$$

$$=2(4R+r)^2-4s^2+2s^2-\sum_{cyc}\left(\frac{7xy}{2(x+y)^4}\cdot(x+y)^4\right)-\frac{7xy(x^4+y^4)}{2(x+y)^4}$$

$$\overset{\text{Repeated Chebyshev}}{\leq} 2(4R+r)^2-2s^2-\frac{7}{2}\cdot\sum_{cyc} r_a r_b-\sum_{cyc}\left(\frac{7(x+y)^4}{16(x+y)^4}\cdot r_a r_b\right)$$

$$=2(4R+r)^2-2s^2-\frac{7s^2}{2}-\frac{7s^2}{16}\Rightarrow\frac{r_a^7+r_b^7}{(r_a+r_b)^5}+\frac{r_b^7+r_c^7}{(r_b+r_c)^5}+\frac{r_c^7+r_a^7}{(r_c+r_a)^5}+\frac{95}{16}s^2$$

$$=2(4R+r)^2-2s^2-\frac{56s^2}{16}-\frac{7s^2}{16}+\frac{95s^2}{16}=2(4R+r)^2-2s^2+\frac{32s^2}{16}$$

$$\therefore \boxed{2(4R+r)^2\geq\frac{r_a^7+r_b^7}{(r_a+r_b)^5}+\frac{r_b^7+r_c^7}{(r_b+r_c)^5}+\frac{r_c^7+r_a^7}{(r_c+r_a)^5}+\frac{95}{16}s^2}$$

$$\text{Again},\frac{r_a^7+r_b^7}{(r_a+r_b)^5}+\frac{r_b^7+r_c^7}{(r_b+r_c)^5}+\frac{r_c^7+r_a^7}{(r_c+r_a)^5}+\frac{95}{16}s^2$$

$$\overset{\text{Chebyshev}}{\geq}\frac{1}{2}\sum_{cyc}\frac{(r_b^2+r_c^2)(r_b^5+r_c^5)}{(r_b+r_c)^5}+\frac{95}{16}s^2$$

$$\overset{\text{Repeated Chebyshev}}{\geq}\frac{1}{32}\sum_{cyc}\frac{(r_b^2+r_c^2)(r_b+r_c)^5}{(r_b+r_c)^5}+\frac{95}{16}s^2=\frac{\sum_{cyc} r_a^2}{16}+\frac{95}{16}s^2$$

$$=\frac{(4R+r)^2-2s^2+95s^2}{16}\overset{\text{Trucht}}{\geq}\frac{3s^2-2s^2+95s^2}{16}=6s^2$$

$$\therefore \boxed{\frac{r_a^7+r_b^7}{(r_a+r_b)^5}+\frac{r_b^7+r_c^7}{(r_b+r_c)^5}+\frac{r_c^7+r_a^7}{(r_c+r_a)^5}+\frac{95}{16}s^2\geq 6s^2}$$

$\therefore$ combining, in any Δ ABC,

$$2(4R+r)^2\geq\frac{r_a^7+r_b^7}{(r_a+r_b)^5}+\frac{r_b^7+r_c^7}{(r_b+r_c)^5}+\frac{r_c^7+r_a^7}{(r_c+r_a)^5}+\frac{95}{16}s^2\geq 6s^2,$$

equalities iff Δ ABC is equilateral (QED)

JP.497 If $a,b,c>0$, then:

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\geq\frac{1}{2}\left(\sqrt[3]{7a^3+b^3}+\sqrt[3]{7b^3+c^3}+\sqrt[3]{7c^3+a^3}\right)$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

Lemma. If $a,b>0$, then:

$$\frac{7a^3}{b^2}+17b\geq 12\cdot\sqrt[3]{7a^3+b^3}$$

Proof of lemma. Using AM-GM inequality, we have:

$$\frac{7a^3}{b^2}+17b=\frac{7a^3+b^3}{b^2}+8b+8b\geq 3\cdot\sqrt[3]{\frac{7a^3+b^3}{b^2}\cdot 8b\cdot 8b}=12\cdot\sqrt[3]{7a^3+b^3}$$

Equality holds for $\frac{7a^3+b^3}{b^2}=8b \Leftrightarrow a=b$.

Now, using Lemma and summing, we get:

$$\sum_{cyc}\left(\frac{7a^3}{b^2}+17b\right)\ge\sum_{cyc}12\cdot\sqrt[3]{7a^3+b^3}\Leftrightarrow$$

$$\sum_{cyc}\frac{7a^3}{b^2}+17\sum_{cyc}a\ge 12\sum_{cyc}\sqrt[3]{7a^3+b^3};\quad (1)$$

Now, using Radon's inequality, we have:

$$\sum_{cyc}\frac{a^3}{b^2}\ge\frac{(\sum a)^3}{(\sum b)^2}=\sum_{cyc}a;\quad (2)$$

From (1) and (2), it follows:

$$\sum_{cyc}\frac{24a^3}{b^2}\overset{(2)}{\ge}\sum_{cyc}\frac{7a^3}{b^2}+17\sum_{cyc}a\overset{(1)}{\ge}12\sum_{cyc}\sqrt[3]{7a^3+b^3}$$

Hence,

$$\sum_{cyc}\frac{24a^3}{b^2}\ge 12\sum_{cyc}\sqrt[3]{7a^3+b^3}$$

$$\sum_{cyc}\frac{a^3}{b^2}\ge\frac{1}{2}\sum_{cyc}\sqrt[3]{7a^3+b^3}$$

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\ge\frac{1}{2}\left(\sqrt[3]{7a^3+b^3}+\sqrt[3]{7b^3+c^3}+\sqrt[3]{7c^3+a^3}\right)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM-GM\ inequality, we\ have:$$

$$\frac{1}{2}\sqrt[3]{7a^3+b^3}=\sqrt[3]{b.b.\left(\frac{7a^3}{8b^2}+\frac{b}{8}\right)}\le\frac{1}{3}\left(b+b+\left(\frac{7a^3}{8b^2}+\frac{b}{8}\right)\right).$$

$$Then:\ \frac{1}{2}\sqrt[3]{7a^3+b^3}\le\frac{7a^3}{24b^2}+\frac{17b}{24}\quad (and\ analogs)$$

$$Thus,\ \frac{1}{2}\sum_{cyc}\sqrt[3]{7a^3+b^3}\le\frac{7}{24}\sum_{cyc}\frac{a^3}{b^2}+\frac{17}{24}\sum_{cyc}a\quad (1)$$

By Radon's inequality, we have :

$$\sum_{cyc}\frac{a^3}{b^2}\geq\frac{(a+b+c)^3}{(b+c+a)^2}=\sum_{cyc}a\quad(2)$$

From (1) and (2), we have :

$$\frac{1}{2}\sum_{cyc}\sqrt[3]{7a^3+b^3}\leq\frac{7}{24}\sum_{cyc}\frac{a^3}{b^2}+\frac{17}{24}\sum_{cyc}\frac{a^3}{b^2}=\sum_{cyc}\frac{a^3}{b^2},\quad as\ desired.$$

Equality holds iff $a=b=c$.

JP.498 If $a,b,c>0$ then:

$$\sum_{cyc}\frac{(a+1)(b+1)}{a+b+2}\geq\frac{3}{2}+\sum_{cyc}\frac{ab}{a+b}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

First we prove that:

$$\frac{(a+1)(b+1)}{a+b+2}\geq\frac{1}{2}+\frac{ab}{a+b};\qquad(1)$$

$$(1)\Leftrightarrow 2(a+1)(b+1)(a+b)\geq(a+b)(a+b+2)+2ab(a+b+2)$$

$$(2ab+2a+2b+2)(a+b)\geq(a+b)(a+b+2)+2ab(a+b+2)$$

$$(a+b)(2ab+2a+2b+2-a-b-2)\geq 2ab(a+b+2)$$

$$(a+b)(2ab+a+b)\geq 2ab(a+b+2)$$

$$2a^2b+a^2+ab+2ab^2+ab+b^2\geq 2a^2b+2ab^2+4ab$$

$$a^2+b^2-2ab\geq 0\Leftrightarrow(a-b)^2\geq 0$$

By (1):

$$\sum_{cyc}\frac{(a+1)(b+1)}{a+b+2}\geq\frac{3}{2}+\sum_{cyc}\frac{ab}{a+b}$$

Equality holds for $a=b=c$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\frac{(a+1)(b+1)}{a+b+2}-\frac{ab}{a+b}=\frac{(ab+a+b+1)(a+b)-ab(a+b+2)}{(a+b+2)(a+b)}=$$

$$=\frac{a^2+b^2+a+b}{(a+b+2)(a+b)}\overset{CBS}{\geq}\frac{\frac{(a+b)^2}{2}+(a+b)}{(a+b+2)(a+b)}=\frac{1}{2}.$$

$$Then:\ \frac{(a+1)(b+1)}{a+b+2}\geq\frac{1}{2}+\frac{ab}{a+b}\ \ (and\ analogs)$$

$$Therefore,$$

$$\sum_{cyc}\frac{(a+1)(b+1)}{a+b+2}\geq\frac{3}{2}+\sum_{cyc}\frac{ab}{a+b}.$$

$$Equality\ holds\ iff\ a=b=c.$$

JP.499 Find $\lambda>0$ so that the double inequality

$$R^2\geq\frac{(a+b+c)^3-a^3-b^3-c^3}{\lambda(a+b+c)}\geq 2Rr$$

holds in any triangle ABC.

Proposed by Alex Szoros-Romania

Solution 1 by proposer

We assume the problem is solved. If ΔABC is equilateral, we have that:

$a=b=c=l$ and $R=2r$.

The relationship in the statement becomes:

$$R^2\geq\frac{27l^3-3l^3}{3l\lambda}\geq R^2\Rightarrow R^2\geq\frac{8l^2}{\lambda}\Rightarrow\lambda=24$$

For $\lambda=24$ we will show that:

$$R^2\geq\frac{(a+b+c)^3-a^3-b^3-c^3}{24(a+b+c)}\geq 2Rr$$

Using the identity:

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$

we get:

$$R^2\geq\frac{(a+b)(b+c)(c+a)}{16s}\geq 2Rr;\qquad (1)$$

But, $abc+(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)$

$$(a+b)(b+c)(c+a)=2s(s^2+r^2+2Rr)$$

Hence, (1) becomes:

$$R^2\geq\frac{s^2+r^2+2Rr}{8}\geq 2Rr\Leftrightarrow$$

$$8R^2-2Rr-r^2\geq 14Rr-r^2;\qquad (2)$$

From Gerretsen's inequality: $4R^2+4Rr+3r^2\geq s^2\geq 16Rr-5r^2$, we get:

$8R^2-2Rr-r^2\geq 4R^2+4Rr+3r^2\geq s^2\geq 16Rr-5r^2\geq 14Rr-r^2$ true.

So, (2) is true. In conclusion $\lambda=24$.

Solution 2 by Tapas Das-India

For an equilateral triangle ABC: $a=b=c=\frac{2s}{3}$ and $R=2r$.

So, the given inequality is true in equilateral triangle.

$$R^2\geq\frac{(a+b+c)^3-a^3-b^3-c^3}{\lambda(a+b+c)}\geq 2Rr$$

We put $R=2r, a=b=c=\frac{2s}{3}$ and we get:

$$\frac{27a^3-3a^3}{\lambda\cdot 3a}=4r^2\Rightarrow\frac{24a^3}{\lambda\cdot 3a}=4r^2\Rightarrow 8a^2=4r^2\lambda\Rightarrow$$

$$\lambda=\frac{2a^2}{r^2}=\frac{2}{r^2}\left(\frac{2s}{3}\right)^2=\frac{8}{9}\cdot\frac{s^2}{r^2}=\frac{8}{9}\cdot\frac{27r^2}{r^2}\Rightarrow\lambda=24.$$

So, it is suffices to prove for $\lambda=24$. We have:

$$\frac{(a+b+c)^3-a^3-b^3-c^3}{24(a+b+c)}=\frac{8s^3-2(s^3-3r^2s-6Rrs)}{24\cdot 2s}=\frac{s^2+r^2+2Rr}{8}$$

We need to show:

$$\frac{s^2+r^2+2Rr}{8}\geq 2Rr\Leftrightarrow s^2+r^2-14Rr\geq 0\Leftrightarrow$$

$$16Rr-5r^2+r^2-14Rr\geq 0\Leftrightarrow 2Rr-4r^2\geq 0\Leftrightarrow R\geq 2r\ (Euler);(1)$$

Again,

$$\frac{(a+b+c)^3-a^3-b^3-c^3}{24(a+b+c)}=\frac{s^2+r^2+2Rr}{8}$$

We need to show:

$$\frac{s^2+r^2+2Rr}{8} \le R^2 \Leftrightarrow s^2+r^2+2Rr \le 8R^2 \Leftrightarrow$$

$$2R^2-3Rr-2r^2 \ge 0 \Leftrightarrow (R-2r)(2R+r) \ge 0 \ (2)$$

From (1) and (2) is true for $\lambda = 24$.

JP.500 If $a,b,c>0, abc=1$ then:

$$\frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1} \ge 3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Lemma:

If $x>0$ then:

$$\frac{x^2}{x+1} \ge \frac{x}{x+1}+\log\sqrt{x}; \qquad (1)$$

Proof. **Let be $f:(0,\infty)\to\mathbb{R}; f(x)=\frac{x^2}{x+1}-\frac{x}{x+1}-\log\sqrt{x}$, then**

$$f'(x)=\left(\frac{x^2-x}{x^2+1}\right)'-\left(\frac{1}{2}\log x\right)'=\frac{x^2+2x+1}{(x+1)^2}-\frac{1}{2x}=$$

$$=\frac{(x-1)(2x^2+5x+1)}{2x(x+1)^2}$$

$$f'(x)=0 \Rightarrow x=1; f(x)\ge f(1)=0$$

$$\Rightarrow \frac{x^2}{x+1}-\frac{x}{x+1}-\log\sqrt{x}\ge 0; (\forall)x>0$$

Back to the problem:

Let be $x=a; x=b; x=c$ in (1). By adding:

$$\sum_{cyc}\frac{a^2}{a+1}\ge\sum_{cyc}\frac{a}{a+1}+\log\sqrt{a}+\log\sqrt{b}+\log\sqrt{c}\Leftrightarrow$$

$$\sum_{cyc}\frac{a^2}{a+1}\ge\sum_{cyc}\frac{a+1-a}{a+1}+\log\sqrt{abc}\Leftrightarrow\sum_{cyc}\frac{a^2}{a+1}\ge 3-\sum_{cyc}\frac{1}{a+1}+\log\sqrt{1}\Leftrightarrow$$

$$\sum_{cyc}\frac{a^2+1}{a+1}\ge 3$$

Equality holds for $a = b = c = 1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \frac{a^2+1}{a+1} = \frac{(a+1)^2+(a-1)^2}{2(a+1)} \geq \frac{a+1}{2}\ (and\ analogs)$$

$$Then: \frac{a^2+1}{a+1} + \frac{b^2+1}{b+1} + \frac{c^2+1}{c+1} \geq \frac{a+b+c+3}{2}.$$

$$By\ AM-GM\ inequality, we\ have:$$

$$a+b+c \geq 3\sqrt[3]{abc} = 3.$$

$$Therefore,$$

$$\frac{a^2+1}{a+1} + \frac{b^2+1}{b+1} + \frac{c^2+1}{c+1} \geq \frac{3+3}{2} = 3.$$

$$Equality\ holds\ iff\ a=b=c=1.$$

Solution 3 by Marin Chirciu-Romania

Using Holder's inequality:

$$\frac{a^2+1}{a+1} \geq \frac{a+1}{2} \Leftrightarrow (a-1)^2 \geq 0$$

Equality holds for $a = 1$.

$$LHS = \sum_{cyc} \frac{a^2+1}{a+1} \geq \sum_{cyc} \frac{a+1}{2} = \frac{\sum a + 3}{2} \overset{AM-GM}{\geq} \frac{3\sqrt[3]{abc}+3}{2} = \frac{3+\#}{2} = 3 = RHS.$$

Equality holds for $a = b = c = 1$.

Solution 4 by Anas Chaabi-Morocco

$$\frac{a^2+1}{a+1} + \frac{b^2+1}{b+1} + \frac{c^2+1}{c+1} \geq 3 \Leftrightarrow \frac{(a+1)^2-2a}{a+1} + \frac{(b+1)^2-2b}{b+1} + \frac{(c+1)^2-2c}{c+1} \geq 3$$

$$\Leftrightarrow a+1+b+1+c+1 - \left(\frac{2a}{a+1} + \frac{2b}{b+1} + \frac{2c}{c+1}\right) \geq 3 \Leftrightarrow$$

$$\left(\frac{2a}{a+1} + \frac{2b}{b+1} + \frac{2c}{c+1}\right) \leq a+b+c\ which\ is\ true\ because:$$

$$\frac{2a}{a+1}+\frac{2b}{b+1}+\frac{2c}{c+1}\le\sqrt{a}+\sqrt{b}+\sqrt{c}\le\sqrt{a+b+c}=$$

$$=\frac{a+b+c}{\sqrt{a+b+c}}\cdot\sqrt{3}\le\frac{a+b+c}{\sqrt{3\sqrt[3]{abc}}}\cdot\sqrt{3}=a+b+c$$

Solution 5 by Nguyen Thai An-Vietnam

$$We\ have:\ \frac{a^2+1}{a+1}\ge\frac{a+1}{2}\Leftrightarrow(a-1)^2\ge 0\ true\,.$$

$$So,we\ have:$$

$$\frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1}\ge\frac{a+b+c+3}{2}$$

$$By\ AM-GM: a+b+c\ge 3\sqrt[3]{abc}=3\Rightarrow\frac{a+b+C+3}{3}\ge\frac{3+3}{2}=3$$

$$Therefore,\frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1}\ge 3.$$

$$Equality\ holds\ for\ a=b=c=1.$$

Solution 6 by Ivan Hadinata-Jember-Indonesia

$$a,b,c>0,abc=1\Longrightarrow\exists x,y,z>0,a=\frac{x}{z},b=\frac{y}{x},c=\frac{z}{y}$$

$$\sum_{cyc}\frac{a^2+1}{a+1}=\sum_{cyc}\frac{a^2+a-a+1}{a+1}=\sum_{cyc}a-\sum_{cyc}\frac{a-1}{a+1}=$$

$$=\sum_{cyc}\frac{x}{z}-\sum_{cyc}\frac{\frac{x}{z}-1}{\frac{x}{z}+1}=\sum_{cyc}\frac{x}{z}+\sum_{cyc}\frac{z-x}{z+x}=\sum_{cyc}\frac{x^2+z^2}{z(z+x)}\ge$$

$$\ge\sum_{cyc}\frac{(x+z)^2}{2z(z+x)}=\sum_{cyc}\frac{x+z}{2z}=\frac{1}{2}\sum_{cyc}\left(\frac{x}{z}+1\right)\overset{AM-GM}{\ge}\frac{1}{2}\sum_{cyc}2\sqrt{\frac{x}{z}}=$$

$$=\sum_{cyc}\sqrt{a}\overset{AM-GM}{\ge}3\sqrt[6]{abc}=3$$

$$Equality\ holds\ for\ a=b=c=1.$$

JP.501 If $a,b,c>0$, then:

$$\frac{a^{10}}{(b+c)^5(2a+b+c)^5}+\frac{b^{10}}{(c+a)^5(a+2b+c)^5}+\frac{c^{10}}{(a+b)^5(a+b+2c)^5}\ge\frac{3}{8^5}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{a^{10}}{(b+c)^5(2a+b+c)^5}=\sum_{cyc}\frac{a^{10}}{(s-a)^5(s+a)^5}=\sum_{cyc}\frac{(a^2)^5}{(s^2-a^2)^5}=$$

$$=\sum_{cyc}\left(\frac{a^2}{s^2-a^2}\right)^5\overset{Radon}{\geq}\frac{1}{3^4}\left(\sum_{cyc}\frac{a^2}{s^2-a^2}\right)^5=\frac{1}{3^4}\left(\sum_{cyc}\left(\frac{a^2}{s^2-a^2}+1\right)-3\right)^5=$$

$$=\frac{1}{3^4}\left(\sum_{cyc}\frac{s^2}{s^2-a^2}-3\right)^5\overset{Bergstrom}{\geq}\frac{1}{3^4}\left(s^2\cdot\frac{9}{\sum(s^2-a^2)}-3\right)^5=$$

$$=\frac{1}{3^4}\left(\frac{9s^2}{3s^2-(a^2+b^2+c^2)}-3\right)^5\geq\frac{1}{3^4}\left(\frac{9s^2}{3s^2-\frac{(a+b+c)^2}{3}}-3\right)^5=$$

$$=\frac{1}{3^4}\left(\frac{27s^2}{9s^2-s^2}-3\right)^5=\frac{1}{3^4}\left(\frac{27}{9-1}-3\right)^5=\frac{3^5}{3^4\cdot 8^5}=\frac{3}{8^5}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc}\frac{a^{10}}{(b+c)^5(2a+b+c)^5}=\sum_{cyc}\frac{a^{15}}{(ab+ca)^5(2a+b+c)^5}\geq$$

$$\overset{Hölder}{\geq}\frac{\left(\sum_{cyc}a\right)^{15}}{\left(\sum_{cyc}(ab+ca)\right)^5\left(\sum_{cyc}(2a+b+c)\right)^5\left(\sum_{cyc}1\right)^4}=$$

$$=\frac{3\left(\sum_{cyc}a\right)^{15}}{\left(2.3\sum_{cyc}bc\right)^5\left(4\sum_{cyc}a\right)^5}\geq\frac{3\left(\sum_{cyc}a\right)^{10}}{2^{15}.\left(\left(\sum_{cyc}a\right)^2\right)^5}=\frac{3}{2^{15}}=\frac{3}{8^5},\ \text{as desired.}$$

Equality holds iff $a=b=c$.

Solution 3 by Marin Chirciu-Romania

$$\text{If } a,b,c>0,\text{then }\sum_{cyc}\frac{a^{10}}{(b+c)^5(2a+b+c)^5}\geq\frac{3}{8^5}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

Solution.

$$Using\ Holder's\ inequality, we\ get:$$

$$LHS=\sum_{cyc}\frac{a^{10}}{(b+c)^5(2a+b+c)^5}=\sum_{cyc}\left(\frac{a^2}{(b+c)(2a+b+c)}\right)^5\overset{Holder}{\geq}$$

$$\geq\frac{1}{3^4}\left[\sum_{cyc}\frac{a^2}{(b+c)(2a+b+c)}\right]^5\overset{CBS}{\geq}\frac{1}{3^4}\cdot\left(\frac{3}{8}\right)^5\geq\frac{3}{8^5}, where$$

$$\sum_{cyc}\frac{a^2}{(b+c)(2a+b+c)}\overset{CBS}{\geq}\frac{(\sum a)^2}{\sum(b+c)(2a+b+c)}=\frac{\sum a^2+2\sum bc}{2\sum a^2+6\sum bc}\overset{(1)}{\geq}\frac{3}{8}, where$$

$$(1)\Leftrightarrow\frac{\sum a^2+2\sum bc}{2\sum a^2+6\sum bc}\geq\frac{3}{8}\Leftrightarrow\sum_{cyc}a^2\geq\sum_{cyc}bc\Leftrightarrow\sum_{cyc}(b-c)^2\geq 0$$

$$Equality\ holds\ for\ a=b=c.$$

$$Remark. The\ problem\ can\ be\ developed.$$

$$If\ a,b,c>0\text{ and }n\in\mathbb{N}, then\ \sum_{cyc}\frac{a^{2n}}{(b+c)^n(2a+b+c)^n}\geq\frac{3}{8^n}$$

Proposed by Marin Chirciu-Romania

Solution.

$$For\ n=0\ we\ get\ 3=3\ and\ for\ n=1, we\ get:\ \sum_{cyc}\frac{a^2}{(b+c)(2a+b+c)}\geq\frac{3}{8}.$$

$$For\ n\geq 2, we\ use\ Holdsr's\ inequality:$$

$$LHS=\sum_{cyc}\frac{a^{2n}}{(b+c)^n(2a+b+c)^n}=\sum_{cyc}\left(\frac{a^2}{(b+c)(2a+b+c)}\right)^n\overset{Holder}{\geq}$$

$$\geq\frac{1}{3^{n-1}}\left[\sum_{cyc}\frac{a^2}{(b+c)(2a+b+c)}\right]^n\overset{CBS}{\geq}\left(\frac{3}{8}\right)^n\cdot\frac{1}{3^{n-1}}\geq\frac{3}{8^n}=RHS, where$$

$$\sum_{cyc}\frac{a^2}{(b+c)(2a+b+c)}\overset{CBS}{\geq}\frac{(\sum a)^2}{\sum(b+c)(2a+b+c)}=\frac{\sum a^2+2\sum bc}{2\sum a^2+6\sum bc}\overset{(1)}{\geq}\frac{3}{8}, where$$

$$(1)\Leftrightarrow\frac{\sum a^2+2\sum bc}{2\sum a^2+6\sum bc}\geq\frac{3}{8}\Leftrightarrow\sum_{cyc}a^2\geq\sum_{cyc}bc\Leftrightarrow\sum_{cyc}(b-c)^2\geq 0$$

$$Equality\ holds\ for\ a=b=c.$$

$Note: For\ n=5\ we\ obtain\ the\ Proposed\ Problem\ JP.501\ from$
$RMM-34-Autumn\ 2024\ \ proposed\ by\ D.M.Batinetu-Giurgiu.$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\text{Via repeated use of Chebyshev's inequality}, (b+c)^5\leq 2^4\left(b^5+c^5\right)$$

$$\text{and } (2a+b+c)^5=(a+a+b+c)^5\le 4^4(a^5+a^5+b^5+c^5)$$

$$\therefore (b+c)^5(2a+b+c)^5\le 2^4.4^4.(b^5+c^5)(a^5+a^5+b^5+c^5)$$

$$\Rightarrow (b+c)^5(2a+b+c)^5\le 2^{12}(b^5+c^5)(2a^5+b^5+c^5) \text{ and analogs}$$

$$\therefore \frac{a^{10}}{(b+c)^5(2a+b+c)^5}+\frac{b^{10}}{(c+a)^5(a+2b+c)^5}+\frac{c^{10}}{(a+b)^5(a+b+2c)^5}$$

$$\ge \frac{1}{2^{12}}\sum_{cyc}\frac{a^{10}}{(b^5+c^5)(2a^5+b^5+c^5)}\overset{?}{\ge}\frac{3}{8^5}$$

$$\Leftrightarrow \sum_{cyc}\frac{x^2}{(y+z)(2x+y+z)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{8}\ (x=a^5,y=b^5,c=z^5)$$

$$\text{Now, }\sum_{cyc}\frac{x^2}{(y+z)(2x+y+z)}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}(2xy+y^2+yz+2xz+yz+z^2)}$$

$$=\frac{\sum_{cyc}x^2+2\sum_{cyc}xy}{2\sum_{cyc}x^2+6\sum_{cyc}xy}\overset{?}{\ge}\frac{3}{8}\Leftrightarrow 4\sum_{cyc}x^2+8\sum_{cyc}xy\overset{?}{\ge}3\sum_{cyc}x^2+9\sum_{cyc}xy$$

$$\Leftrightarrow \sum_{cyc}x^2\overset{?}{\ge}\sum_{cyc}xy\to \text{true}\Rightarrow (*)\text{ is true}$$

$$\therefore \frac{a^{10}}{(b+c)^5(2a+b+c)^5}+\frac{b^{10}}{(c+a)^5(a+2b+c)^5}+\frac{c^{10}}{(a+b)^5(a+b+2c)^5}\ge\frac{3}{8^5}$$

$$\forall\, a,b,c>0,''='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 5 by Ivan Hadinata-Jember-Indonesia

$$\sum_{cyc}\frac{a}{2a+3b+3c}=\sum_{cyc}\frac{a^2}{2a^2+3ab+3ac}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{(a+b+c)^2}{2(a+b+c)^2+2(ab+bc+ca)}\ge\frac{(a+b+c)^2}{2(a+b+c)^2+\frac{2}{3}(a+b+c)^2}=\frac{3}{8}$$

$$\sum_{cyc}\frac{a^{10}}{(b+c)^5(2a+b+c)^5}=\frac{1}{32}\sum_{cyc}\left(\frac{a^2}{(b+c)\left(a+\frac{b}{2}+\frac{c}{2}\right)}\right)^5\overset{AM-GM}{\ge}$$

$$\ge\frac{1}{32}\sum_{cyc}\left(\frac{4a^2}{\left(b+c+a+\frac{b}{2}+\frac{c}{2}\right)^2}\right)^5=\frac{1}{32}\sum_{cyc}\left(\frac{4a}{2a+3b+3c}\right)^{10}\ge$$

$$\ge\frac{3}{32}\cdot 4^{10}\left(\sum_{cyc}\frac{a}{3(2a+3b+3c)}\right)^{10}=\frac{3}{32}\cdot 4^{10}\cdot\frac{1}{3^{10}}\left(\sum_{cyc}\frac{a}{2a+3b+3c}\right)^{10}\ge$$

$$\geq \frac{3}{32}\cdot 4^{10}\cdot\frac{1}{3^{10}}\cdot\frac{3^{10}}{8^{10}}=\frac{3}{32}\cdot 4^{10}\cdot\frac{1}{8^{10}}=\frac{3}{2^{15}}=\frac{3}{8^5}$$

Equality holds iff $a=b=c$.

JP.502 If $m,n\geq 0, m+n=4$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{x\cdot a^m}{(y+z)h_a^n}+\frac{y\cdot b^n}{(z+x)h_b^n}+\frac{z\cdot c^n}{(x+y)h_c^n}\geq 2^{m-1}\cdot F^{2-n}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

Solution by proposer

$$\sum_{cyc}\frac{x\cdot a^m}{(y+z)h_a^n}=\sum_{cyc}\frac{x\cdot a^{m+n}}{(y+z)(a\cdot h_a)^n}=\frac{1}{(2F)^n}\sum_{cyc}\frac{x\cdot a^4}{y+z}=$$

$$=\frac{1}{(2F)^n}\sum_{cyc}\frac{x^2a^4}{xy+xz}\overset{Bergstrom}{\geq}\frac{1}{2^nF^n}\cdot\frac{(\sum xa^2)^2}{\sum(xy+xz)}=$$

$$=\frac{1}{2^nF^n}\cdot\frac{xa^2+yb^2+zc^2}{2(xy+yz+zx)}\overset{Oppenheim}{\geq}\frac{1}{2^nF^n}\cdot\frac{16(xy+yz+zx)F^2}{2(xy+yz+zx)}=$$

$$=2^{3-n}\cdot F^{2-n}=2^{3-4+m}\cdot F^{2-n}=2^{m-1}\cdot F^{2-n}$$

JP.503 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a+b}{ab}\cdot h_c\geq\frac{a+b+c}{R}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)h_c\geq\frac{a+b+c}{R};\qquad (1)$$

Using Pham Huu Duc's inequality:

$$x(a+b)+y(b+c)+z(c+a)\geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)};$$

$$(\forall)x,y,z,a,b,c>0$$

$$\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)h_c\geq 2\sqrt{(h_ah_b+h_bh_c+h_ch_a)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)};\qquad (2)$$

$$h_a h_b + h_b h_c + h_c h_a = \frac{2s^2 r}{R} \text{ and } \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}; \quad (3)$$

$$\textit{From } (2) \textit{ and } (3) \textit{ it follows that:}$$

$$\sum_{cyc} \left(\frac{1}{a} + \frac{1}{b}\right) h_c \geq 2\sqrt{\frac{s^2}{R^2}} = \frac{2s}{R} = \frac{a+b+c}{R}$$

Solution 2 by Marin Chirciu-Romania

$$\textit{Using } h_a = \frac{2F}{a}, \textit{we get:}$$

$$LHS = \sum_{cyc} \frac{b+c}{bc} \cdot h_a = \sum_{cyc} \frac{b+c}{bc} \cdot \frac{2F}{a} = \frac{2F}{abc} \sum_{cyc} (b+c) = \frac{2F}{4RF} \cdot 2 \sum_{cyc} a = \frac{a+b+c}{2}$$

$$= RHS$$

Solution 3 by Tapas Das-India

$$\sum_{cyc} \frac{a+b}{ab} \cdot h_c = \sum_{cyc} \frac{a+b}{ab} \cdot \frac{2F}{c} = \frac{2F}{abc} \sum_{cyc} (a+b) = \frac{2F}{abc} \cdot 2 \sum_{cyc} a =$$

$$= \frac{2F}{abc} \cdot 2 \cdot 2s = \frac{2F}{4RF} \cdot 4s = \frac{2s}{R} = \frac{a+b+c}{R}$$

JP.504 In ΔABC the following relationship holds:

$$\frac{A}{\pi A + BC + 12} + \frac{B}{\pi B + CA + 12} + \frac{C}{\pi C + AB + 12} \leq \frac{3+\pi}{32}$$

Proposed by Radu Diaconu-Romania

Solution 1 by proposer

We have:

$$\sum_{cyc} \frac{A}{\pi B + CA + 12} = \sum_{cyc} \frac{A}{(A+B+C)A + BC + 12} = \sum_{cyc} \frac{A}{(A+B)(A+C) + 4 + 8} \overset{(1)}{\leq}$$

$$\leq \sum_{cyc} \frac{A}{4\sqrt{(A+B)(A+C)} + 8} \overset{(2)}{\leq} \frac{1}{4} \sum_{cyc} \frac{A}{4} \left(\frac{1}{\sqrt{(A+B)(A+C)}} + \frac{1}{2}\right) =$$

$$= \frac{1}{16} \sum_{cyc} \sqrt{\frac{A}{A+B} \cdot \frac{A}{A+C}} + \sum_{cyc} \frac{A}{32} \overset{(3)}{\leq} \frac{1}{16} \sum_{cyc} \frac{1}{2} \left(\frac{A}{A+B} + \frac{A}{A+C}\right) + \frac{\pi}{32} =$$

$$=\frac{1}{16}\cdot\frac{3}{2}+\frac{\pi}{32}+\frac{3+\pi}{32}$$

$$(1)\Leftrightarrow(A+B)(A+C)+4\ge 4\sqrt{(A+B)(A+C)}\Leftrightarrow$$

$$\left(\sqrt{(A+B)(A+C)}\right)^2-4\sqrt{(A+B)(A+C)}+4\ge 0\Leftrightarrow$$

$$\left(\sqrt{(A+B)(A+C)}-2\right)^2\ge 0$$

Equality holds for $(A+B)(A+C)=4$.

$$(2)\Leftrightarrow\frac{1}{x+y}\le\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right),\text{where } x=\sqrt{(A+B)(A+C)}\text{ and } y=2.$$

Equality holds for $(A+B)(A+C)=4$.

$$(3)\Leftrightarrow\sqrt{\frac{A}{A+B}\cdot\frac{A}{A+C}}\le\frac{1}{2}\left(\frac{A}{A+B}\cdot\frac{A}{A+C}\right)$$

Equality holds for $\frac{A}{A+B}=\frac{A}{A+C}\Leftrightarrow A=B$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}:$$

$$\frac{A}{\pi A+BC+12}=\frac{A}{(A+B+C)A+BC+12}=\frac{A}{[(A+B)(A+C)+4]+8}\le$$

$$\overset{HM-AM}{\le}\frac{A}{4}\left(\frac{1}{(A+B)(A+C)+4}+\frac{1}{8}\right)\overset{AM-GM}{\le}\frac{A}{4.2\sqrt{4(A+B)(A+C)}}+\frac{A}{32}\le$$

$$\overset{AM-GM}{\le}\frac{1}{16}\cdot\frac{1}{2}\left(\frac{A}{A+B}+\frac{A}{A+C}\right)+\frac{A}{32}=\frac{1}{32}\left(\frac{A}{A+B}+\frac{A}{A+C}+A\right)\ \ (\textit{and analogs})$$

$$\textit{Therefore},$$

$$\sum_{cyc}\frac{A}{\pi A+BC+12}\le\sum_{cyc}\frac{1}{32}\left(\frac{A}{A+B}+\frac{A}{A+C}+A\right)=\sum_{cyc}\frac{1}{32}\left(\frac{A}{A+B}+\frac{B}{B+A}+A\right)=$$

$$=\sum_{cyc}\frac{1}{32}(1+A)=\frac{3+\pi}{32},\ \ \textit{as desired}.$$

JP.505 For $n \in \mathbb{N}^*, x \in \left(0, \frac{\pi}{2}\right)$ prove that:

$$\sum_{k=1}^{n}\left(\frac{4k+(k-1)^2\sin^2 2x}{(k-1)^2+4k\csc^2 2x}\right) \le (n+1)^2$$

Proposed by Florică Anastase, Flavius Pacionea-Romania

Solution by proposers

$$\frac{k+(k-1)^2\cdot\sin^2 x\cos^2 x}{(k-1)^2+k(\tan x+\cot x)^2} = \frac{k+(k-1)^2\cdot\sin^2 x\cos^2 x}{1+k^2+k(\tan x+\cot x)^2-2k} =$$

$$= \frac{k+(k-1)^2\cdot\sin^2 x\cos^2 x}{1+k^2+k(\tan^2 x+\cot^2 x)} =$$

$$= \frac{1+(k-1)\cos^2 x+(k-1)\sin^2 x+(k-1)^2\sin^2 x\cos^2 x}{k^2+k\cot^2 x+k\tan^2 x+1} =$$

$$= \frac{1+(k-1)\cos^2 x}{k+\tan^2 x}\cdot\frac{1+(k-1)\sin^2 x}{k+\cot^2 x}$$

$$\sqrt{\sum_{k=1}^{n}\left(\frac{k+(k-1)^2\cdot\sin^2 x\cos^2 x}{(k-1)^2+k(\tan x+\cot x)^2}\right)} = \sqrt{\sum_{k=1}^{n}\left(\frac{1+(k-1)\cos^2 x}{k+\tan^2 x}\cdot\frac{1+(k-1)\sin^2 x}{k+\cot^2 x}\right)} \le$$

$$\le \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1+(n-1)\cos^2 x}{n+\tan^2 x}+\frac{1+(n-1)\sin^2 x}{n+\cot^2 x}\right); \quad (1)$$

$$\text{Let: } S_n = \sum_{k=1}^{n}\left(\frac{1+(n-1)\cos^2 x}{n+\tan^2 x}+\frac{1+(n-1)\sin^2 x}{n+\cot^2 x}\right)$$

We use the mathematical induction for $n \in \mathbb{N}, n \ge 1$

$$n=1: S_1 = \frac{1}{1+\tan^2 x}+\frac{1}{1+\cot^2 x} = \frac{1}{\frac{1}{\cos^2 x}}+\frac{1}{\frac{1}{\sin^2 x}} = \sin^2 x+\cos^2 x = 1$$

$$n=2: S_2 = \frac{1}{1+\tan^2 x}+\frac{1}{1+\cot^2 x}+\frac{1+\sin^2 x}{2+\cot^2 x}+\frac{1+\cos^2 x}{2+\tan^2 x} =$$

$$= S_1+\frac{1+\sin^2 x}{\frac{1+\sin^2 x}{\sin^2 x}}+\frac{1+\cos^2 x}{\frac{1+\cos^2 x}{\cos^2 x}} = 1+\sin^2 x+\cos^2 x = 2$$

Let be $S_n = n, \forall n \in \mathbb{N}^*$ and then we prove that $S_{n+1} = n+1$.

$$S_{n+1}=\sum_{k=1}^{n+1}\left(\frac{1+(n-1)\cos^2 x}{n+\tan^2 x}+\frac{1+(n-1)\sin^2 x}{n+\cot^2 x}\right)=$$

$$=S_n+\frac{1+n\cos^2 x}{n+1+\tan^2 x}+\frac{1+n\sin^2 x}{n+1+\cot^2 x}=n+\frac{1+n\sin^2 x}{\frac{1+n\sin^2 x}{\sin^2 x}}+\frac{1+n\cos^2 x}{\frac{1+n\cos^2 x}{\cos^2 x}}=$$

$$=n+\sin^2 x+\cos^2 x=n+1;\ \ (2)$$

From (1) and (2), we get:

$$\sqrt{\sum_{k=1}^{n}\left(\frac{k+(k-1)^2\cdot\sin^2 x\cos^2 x}{(k-1)^2+k(\tan x+\cot x)^2}\right)}\le\frac{n+1}{2}\Leftrightarrow$$

$$\sum_{k=1}^{n}\left(\frac{4k+(k-1)^2\cdot\sin^2 2x}{(k-1)^2+k(\tan x+\cot x)^2}\right)\le(n+1)^2\Leftrightarrow$$

$$\sum_{k=1}^{n}\left(\frac{4k+(k-1)^2\sin^2 2x}{(k-1)^2+4k\csc^2 2x}\right)\le(n+1)^2$$

JP.506 For $a,b,c>0, a+b+c=1$ and $k\in\mathbb{N}$ prove:

$$\frac{a}{(a^2+abc)^{k+1}}+\frac{b}{(b^2+abc)^{k+1}}+\frac{c}{(c^2+abc)^{k+1}}\ge\left(\frac{27}{4}\right)^{k+1}$$

Proposed by Florică Anastase, Andreea Lixandru-Romania

Solution 1 by proposers

$$\frac{a}{(a^2+abc)^{k+1}}+\frac{b}{(b^2+abc)^{k+1}}+\frac{c}{(c^2+abc)^{k+1}}=$$

$$=\frac{1}{a^k}\cdot\frac{1}{(a+bc)^{k+1}}+\frac{1}{b^k}\cdot\frac{1}{(b+ca)^{k+1}}+\frac{1}{c^k}\cdot\frac{1}{(c+ab)^{k+1}}\overset{Radon}{\ge}$$

$$\ge\frac{\left(\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\right)^{k+1}}{(a+b+c)^k}=\left(\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\right)^{k+1};(1)$$

Now, from $a+b+c=1$, we have: $1-a=b+c>0, 1-b=c+a>0$,

$$1-c=a+b>0$$

$$\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}=\frac{1}{1-(b+c)+bc}+\frac{1}{1-(c+a)+ca}+\frac{1}{1-(a+b)+ab}$$

$$=$$

$$=\frac{1}{(1-b)(1-c)}+\frac{1}{(1-c)(1-a)}+\frac{1}{(1-a)(1-b)}=\frac{2}{(1-a)(1-b)(1-c)};\ (2)$$

$$(1-a)+(1-b)+(1-c)\overset{AM-GM}{\geq}3\sqrt[3]{(1-a)(1-b)(1-c)}\Leftrightarrow$$

$$2=3-(a+b+c)\geq 3\sqrt[3]{(1-a)(1-b)(1-c)}\Leftrightarrow$$

$$8\geq 27(1-a)(1-b)(1-c)\Leftrightarrow\frac{2}{(1-a)(1-b)(1-c)}\geq\frac{27}{4};(3)$$

From (2) and (3), we get:

$$\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}=\frac{2}{(1-a)(1-b)(1-c)}\geq\frac{27}{4};(4)$$

From (1) and (4), we get:

$$\frac{a}{(a^2+abc)^{k+1}}+\frac{b}{(b^2+abc)^{k+1}}+\frac{c}{(c^2+abc)^{k+1}}\geq\left(\frac{27}{4}\right)^{k+1}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ H\ddot{o}lder's\ inequality, we\ have:$$

$$\sum_{cyc}\frac{a}{(a^2+abc)^{k+1}}=\sum_{cyc}\frac{1}{a^k(a+bc)^{k+1}}\geq\frac{3^{2(k+1)}}{\left(\sum_{cyc}a\right)^k\left(\sum_{cyc}(a+bc)\right)^{k+1}}=$$

$$=\frac{3^{2(k+1)}}{1^k\left(1+\sum_{cyc}bc\right)^{k+1}}\geq\frac{3^{2(k+1)}}{\left(1+\frac{\left(\sum_{cyc}a\right)^2}{3}\right)^{k+1}}=\frac{3^{2(k+1)}}{\left(1+\frac{1}{3}\right)^{k+1}}=\left(\frac{27}{4}\right)^{k+1},\ \ as\ desired.$$

$$Equality\ holds\ iff\ a=b=c=\frac{1}{3}.$$

Solution 3 by Ivan Hadinata-Jember-Indonesia

$$\sum_{cyc}\frac{a}{(a^2+abc)^{k+1}}=\sum_{cyc}\frac{1}{a^k(a+bc)^{k+1}}=\sum_{cyc}\frac{1}{a^k(1-b-c+bc)^{k+1}}=$$

$$=\sum_{cyc}\frac{1}{a^k(1-b)^{k+1}(1-c)^{k+1}}=\sum_{cyc}\frac{1}{a^k(a+c)^{k+1}(a+b)^{k+1}}\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{1}{(abc)^k(a+b)^{2k+2}(b+c)^{2k+2}(c+a)^{2k+2}}}\overset{AM-GM}{\geq}$$

$$\geq \frac{3}{\left(\frac{a+b+c}{3}\right)^k\left(\frac{2a+2b+2c}{3}\right)^{2k+2}} = \frac{3}{\frac{1}{3^k}\cdot\frac{2^{2k+2}}{3^{2k+2}}} = \frac{3^{3k+3}}{2^{2k+2}} = \left(\frac{27}{4}\right)^{k+1}$$

$$Equality\ holds\ iff\ a=b=c=\frac{1}{3}.$$

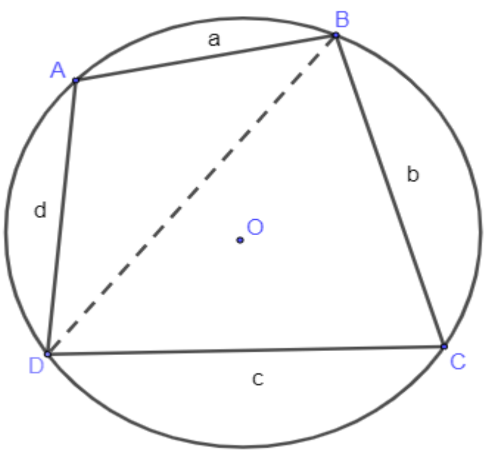
JP.507 Let $ABCD$ be a cyclic quadrilateral with circumradius R and area F. Prove:

$$\frac{\sum \sec^2\frac{A}{2}}{\sum \sec^{-2}\frac{A}{2}} \leq \frac{16R^4}{F^2}$$

where the sums are taken over all angles of the quadrilateral.

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer



$$We\ apply\ the\ Law\ of\ cosines\ in\ the\ triangle\ ABD\ and\ BDC\ to\ obtain:$$
$$BD^2 = a^2+d^2-2ad\cdot\cos A = b^2+c^2-2bc\cdot\cos C$$
$$So,\ because\ the\ quadrilateral\ is\ cyclic,\ we\ have\ \cos C = -\cos A,\ so,$$
$$2(ad+bc)\cos A = a^2-b^2-c^2-d^2,\ which\ means\ that$$
$$\cos A = \frac{a^2-b^2-c^2+d^2}{2(ad+bc)}$$
$$We\ have: F = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin A =$$
$$= \frac{1}{2}(ad+bc)\sin A,\ so,\ \sin A = \frac{2F}{ad+bc}.$$

Taking into account that $\cos^2\frac{A}{2}=\frac{1+\cos A}{2}$, *it follows that*

$$\cos^2\frac{A}{2}=\frac{1+\frac{a^2-b^2-c^2+d^2}{2(ad+bc)}}{2}=\frac{(a+d)^2-(b-c)^2}{2(ad+bc)}$$

$$=\frac{(a+d-b+c)(a+d+b-c)}{2(ad+bc)}=$$

$$=\frac{(s-b)(s-c)}{ad+bc}, \text{where } s=\frac{a+b+c+d}{2} \text{ denotes the semiperimeter of } ABCD.$$

Also, we have: $\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$, *so*, $\sin\frac{A}{2}=\frac{\sin A}{2\cos\frac{A}{2}}$, *namely*

$$\sin^2\frac{A}{2}=\frac{\sin^2 A}{4\cos^2\frac{A}{2}}=\frac{\left(\frac{2F}{ad+bc}\right)^2}{4\cdot\frac{(s-b)(s-c)}{ad+bc}}=\frac{F^2}{s-b(s-c)(ad+bc)}$$

It is well – known that $F=\sqrt{(s-a)(s-b)(s-c)(s-d)}$. *So*,

$$\sin^2\frac{A}{2}=\frac{(s-a)(s-d)}{ad+bc}, \text{we obtain the relations}$$

$$\tan^2\frac{A}{2}=\frac{(s-a)(s-d)}{(s-b)(s-c)}, \text{and similarly, we have}$$

$$\tan^2\frac{B}{2}=\frac{(s-b)(s-a)}{(s-c)(s-d)}, \tan^2\frac{C}{2}=\frac{(s-c)(s-b)}{(s-a)(s-d)} \text{ and } \tan^2\frac{D}{2}=\frac{(s-c)(s-b)}{(s-a)(s-d)}$$

Now, we have $\tan^2\frac{A}{2}=\frac{(s-a)(s-d)}{(s-b)(s-c)}\Leftrightarrow\frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}=\frac{(s-a)(s-d)}{(s-b)(s-c)}\Leftrightarrow$

$$\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}{\cos^2\frac{A}{2}}=\frac{(s-a)(s-d)+(s-b)(s-c)}{(s-b)(s-c)};\left(\because xy\le\frac{x^2+y^2}{4}\right)$$

$$\frac{1}{\cos^2\frac{A}{2}}=\frac{(s-a)(s-d)+(s-b)(s-c)}{(s-b)(s-c)}\le\frac{\frac{(s-a+s-d)^2}{4}+\frac{(s-b+s-c)^2}{4}}{(s-b)(s-c)}$$

$$=\frac{(b+c)^2+(d+a)^2}{4(s-b)(s-c)};\quad(*)$$

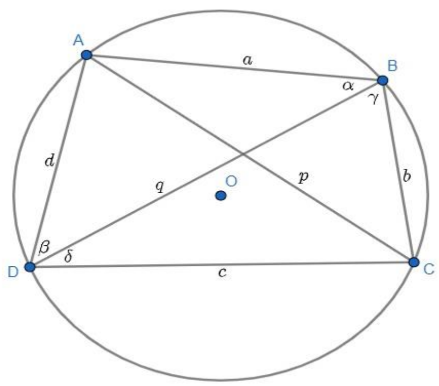
Now, we'll prove that:

Lemma: *In any triangle ABC the inequality* $b+c\le 4R\cos\frac{A}{2}$ *holds*:

Proof. *We easily deduce that* $b+c=2R(\sin B+\sin C)=4R\sin\frac{B+C}{2}\cos\frac{B-C}{2}$

$$\le 4R\sin\frac{\pi-A}{2}\cdot 1=4R\cos\frac{A}{2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India



$$\text{Via Ptolemy's theorem, } pq = ac + bd \text{ and via Ptolemy's second theorem,}$$

$$\frac{p}{q} = \frac{ad+bc}{ab+cd} \therefore p = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \text{ and } q = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}} \to (1)$$

$$\text{Now, } \tan^2\frac{A}{2} = \frac{(m-a)(m-d)}{m(m-q)} \left(m = \frac{a+d+q}{2}\right) = \frac{\left(\frac{d+q-a}{2}\right)\left(\frac{a+q-d}{2}\right)}{\left(\frac{a+d+q}{2}\right)\left(\frac{a+d-q}{2}\right)}$$

$$= \frac{q^2-(a-d)^2}{(a+d)^2-q^2} \overset{\text{via }(1)}{=} \frac{\frac{(ac+bd)(ab+cd)}{ad+bc}-(a-d)^2}{(a+d)^2-\frac{(ac+bd)(ab+cd)}{ad+bc}} = \frac{\frac{ad((b+c)^2-(a-d)^2)}{ad+bc}}{\frac{ad((a+d)^2-(b-c)^2)}{ad+bc}}$$

$$= \frac{4(s-d)(s-a)}{4(s-c)(s-b)} \therefore \tan^2\frac{A}{2} \overset{(i)}{=} \frac{(s-d)(s-a)}{(s-b)(s-c)}$$

$$\text{Again, } \tan^2\frac{B}{2} = \frac{(n-a)(n-b)}{n(n-p)} \left(n = \frac{a+b+p}{2}\right) = \frac{\left(\frac{b+p-a}{2}\right)\left(\frac{a+p-b}{2}\right)}{\left(\frac{a+b+p}{2}\right)\left(\frac{a+b-p}{2}\right)}$$

$$= \frac{p^2-(a-b)^2}{(a+b)^2-p^2} \overset{\text{via }(1)}{=} \frac{\frac{(ac+bd)(ad+bc)}{ab+cd}-(a-b)^2}{(a+b)^2-\frac{(ac+bd)(ad+bc)}{ab+cd}} = \frac{\frac{ab((c+d)^2-(a-b)^2)}{ab+cd}}{\frac{ab((a+b)^2-(c-d)^2)}{ab+cd}}$$

$$= \frac{4(s-b)(s-a)}{4(s-d)(s-c)} \therefore \tan^2\frac{B}{2} \overset{(ii)}{=} \frac{(s-a)(s-b)}{(s-c)(s-d)}$$

$$\text{We have, } \sum_{\text{cyc}} \sec^2\frac{A}{2} = \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} + \sec^2\frac{D}{2}$$

$$= \left(1+\tan^2\frac{A}{2}\right) + \left(1+\tan^2\frac{B}{2}\right) + \csc^2\frac{A}{2} + \csc^2\frac{B}{2} \left(\because \frac{A}{2}+\frac{C}{2} = \frac{B}{2}+\frac{D}{2} = \frac{\pi}{2}\right)$$

$$\overset{\text{via (i),(ii)}}{=} \left(1+\frac{(s-d)(s-a)}{(s-b)(s-c)}\right) + \left(1+\frac{(s-a)(s-b)}{(s-c)(s-d)}\right) + \left(1+\frac{(s-b)(s-c)}{(s-d)(s-a)}\right)$$

$$+\left(1+\frac{(s-c)(s-d)}{(s-a)(s-b)}\right)\left(\because \csc^2\frac{A}{2}=1+\frac{1}{\tan^2\frac{A}{2}}\text{ etc}\right)$$

$$=\frac{2s^2-s(b+c+d+a)+bc+ad}{(s-b)(s-c)}+\frac{2s^2-s(c+d+a+b)+cd+ab}{(s-c)(s-d)}$$

$$+\frac{2s^2-s(d+a+b+c)+ad+bc}{(s-d)(s-a)}+\frac{2s^2-s(a+b+c+d)+ab+cd}{(s-a)(s-b)}\to(2)$$

$$=\frac{(ab+cd)\big((s-a)(s-b)+(s-c)(s-d)\big)}{(s-a)(s-b)(s-c)(s-d)}$$

$$+\frac{(ad+bc)\big((s-d)(s-a)+(s-b)(s-c)\big)}{(s-a)(s-b)(s-c)(s-d)}=\frac{(ab+cd)^2+(ad+bc)^2}{(s-a)(s-b)(s-c)(s-d)}$$

$$\Rightarrow\sum_{cyc}\sec^2\frac{A}{2}\overset{(*)}{=}\frac{(ab+cd)^2+(ad+bc)^2}{(s-a)(s-b)(s-c)(s-d)}$$

$$\text{Also,}\sum_{cyc}\cos^2\frac{A}{2}\overset{\text{via }(2)}{=}\frac{(s-b)(s-c)}{2s^2-s(b+c+d+a)+bc+ad}$$

$$+\frac{(s-c)(s-d)}{2s^2-s(c+d+a+b)+cd+ab}+\frac{(s-d)(s-a)}{2s^2-s(d+a+b+c)+ad+bc}$$

$$+\frac{(s-a)(s-b)}{2s^2-s(a+b+c+d)+ab+cd}$$

$$=\frac{(s-b)(s-c)+(s-d)(s-a)}{ad+bc}+\frac{(s-c)(s-d)+(s-a)(s-b)}{ab+cd}$$

$$=\frac{2s^2-s(b+c+d+a)+bc+ad}{ad+bc}+\frac{2s^2-s(c+d+a+b)+cd+ab}{ab+cd}$$

$$\Rightarrow\sum_{cyc}\cos^2\frac{A}{2}\overset{(**)}{=}2\;\therefore\text{ via }(*),(**),\frac{\sum_{cyc}\sec^2\frac{A}{2}}{\sum_{cyc}\sec^{-2}\frac{A}{2}}\le\frac{16R^4}{F^2}$$

$$\Leftrightarrow\frac{(ab+cd)^2+(ad+bc)^2}{2(s-a)(s-b)(s-c)(s-d)}\le\frac{16R^4}{(s-a)(s-b)(s-c)(s-d)}$$

$$\Leftrightarrow(ab+cd)^2+(ad+bc)^2\overset{(\bullet)}{\le}32R^4$$

Now, via sine *law* and $\because$ circumradius of $\Delta\,ABD$ and $\Delta\,BCD$ is R,

$\therefore a=2R\sin\beta, b=2R\sin\delta, c=2R\sin\gamma, d=2R\sin\alpha$

$\therefore ab+cd=4R^2(\sin\beta\sin\delta+\sin\gamma\sin\alpha)$

$=2R^2\big(\cos(\beta-\delta)-\cos(\beta+\delta)+\cos(\gamma-\alpha)-\cos(\gamma+\alpha)\big)$

$=2R^2\big(\cos(\beta-\delta)-\cos(D)+\cos(\gamma-\alpha)-\cos(\pi-D)\big)$

$=2R^2\big(\cos(\beta-\delta)+\cos(\gamma-\alpha)\big)\le 2R^2(1+1)\Rightarrow ab+cd\overset{(?)}{\le}4R^2$

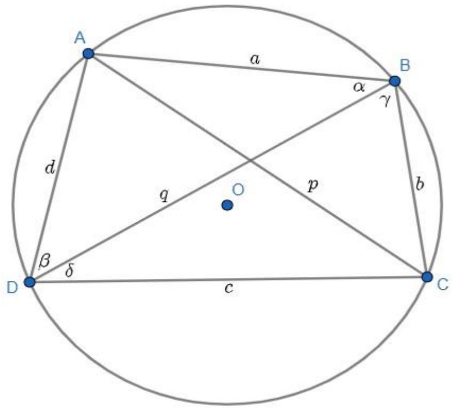
and, $ad+bc=4R^2(\sin\beta\sin\alpha+\sin\delta\sin\gamma)$

$=2R^2\big(\cos(\alpha-\beta)-\cos(\alpha+\beta)+\cos(\gamma-\delta)-\cos(\gamma+\delta)\big)$

$= 2R^2(\cos(\alpha-\beta) - \cos(\pi - A) + \cos(\gamma-\delta) - \cos(\pi - C))$
$= 2R^2(\cos(\alpha-\beta) + \cos A + \cos(\gamma-\delta) + \cos(\pi - A))$
$= 2R^2(\cos(\alpha-\beta) + \cos(\gamma-\delta)) \le 2R^2(1+1) \Rightarrow ad + bc \overset{(\square\square)}{\le} 4R^2$
$\therefore (\square), (\square\square) \Rightarrow (ab+cd)^2 + (ad+bc)^2 \le 16R^4 + 16R^4 = 32R^4 \Rightarrow (\bullet)$ is true
$\therefore \frac{\sum_{cyc}\sec^2\frac{A}{2}}{\sum_{cyc}\sec^{-2}\frac{A}{2}} \le \frac{16R^4}{F^2}$ in any cyclic quadrilateral ABCD (QED)

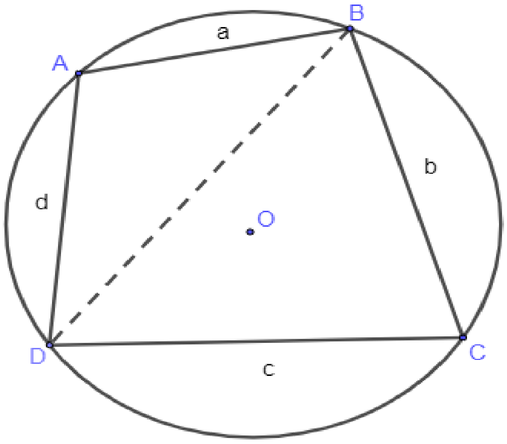
JP.508 Let $ABCD$ be a cyclic quadrilateral with circumradius R and area F. Prove that:

$$\csc A + \csc B + \csc C + \csc D \le \frac{8R^2}{F}$$



Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer



We apply the Law of cosines in the triangle ABD and BDC to obtain:

$$BD^2 = a^2 + d^2 - 2ad \cdot \cos A = b^2 + c^2 - 2bc \cdot \cos C$$

So, because the quadrilateral is cyclic, we have $\cos C = -\cos A$, *so,*

$2(ad+bc)\cos A = a^2 - b^2 - c^2 - d^2$, *which means that*

$$\cos A = \frac{a^2 - b^2 - c^2 + d^2}{2(ad+bc)}$$

$$\text{We have: } F = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin A =$$

$$= \frac{1}{2}(ad+bc)\sin A, \text{ so, } \sin A = \frac{2F}{ad+bc}.$$

Taking into account that $\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$, *it follows that*

$$\cos^2\frac{A}{2} = \frac{1 + \frac{a^2-b^2-c^2+d^2}{2(ad+bc)}}{2} = \frac{(a+d)^2-(b-c)^2}{2(ad+bc)}$$

$$= \frac{(a+d-b+c)(a+d+b-c)}{2(ad+bc)} =$$

$$= \frac{(s-b)(s-c)}{ad+bc}, \text{ where } s = \frac{a+b+c+d}{2} \text{ denotes the semiperimeter of } ABCD.$$

$$\text{So, } \sin^2\frac{A}{2} = \frac{(s-d)(s-a)}{\frac{2F}{\sin A}} = \frac{(s-d)(s-a)\sin A}{2F} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}(s-d)(s-a)}{2F}$$

$$\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{(s-d)(s-a)}{F}; \left(\because xy \le \frac{x^2+y^2}{4}\right)$$

$$\text{So, } \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} \le \frac{(s-a+s-d)^2}{4F} = \frac{(b+c)^2}{4F}; (*)$$

Now, we'll prove that: In any triangle ABC the inequality

$$b+c\le 4R\cos\frac{A}{2}\ holds.$$

We can easily deduce:

$$b+c=2R(\sin B+\sin C)=4R\sin\frac{B+C}{2}\cos\frac{B-C}{2}\le$$

$$\le 4R\sin\frac{\pi-A}{2}\cdot 1=4R\cos\frac{A}{2}$$

So, $b+c\le 4R\cos\frac{A}{2}$. *Applying the last inequality to triangles* ABC, BCD, CDA *and* DAB, *we obtain the following inequalities*:

$$a+b\le 4R\cos\frac{B}{2},\ b+c\le 4R\cos\frac{C}{2},\ c+d\le 4R\cos\frac{D}{2},\ d+a\le 4R\cos\frac{A}{2}$$

$$S,(*)\ gives\ \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\le\frac{16R^2\cos^2\frac{C}{2}}{4F}=\frac{4R^2\sin^2\frac{A}{2}}{F};\left(\frac{A}{2}+\frac{C}{2}=\frac{\pi}{2}\right),\ namely$$

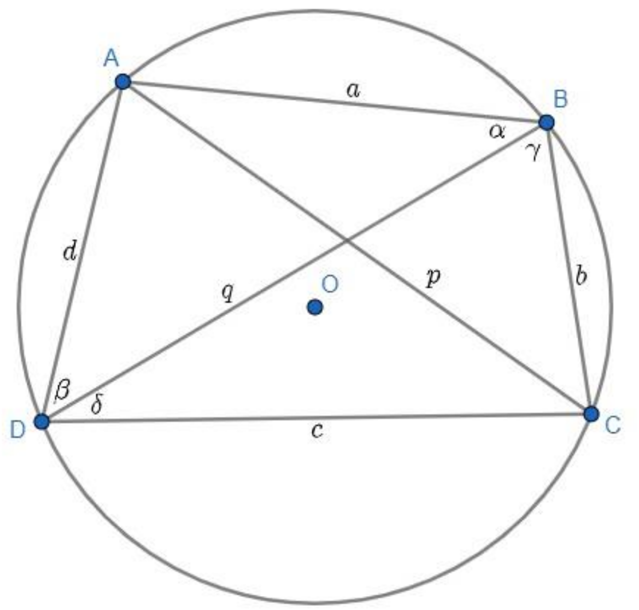
$$\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}}\le\frac{4R^2}{F}\Leftrightarrow\frac{1}{2\sin\frac{A}{2}\cos\frac{A}{2}}\le\frac{4R^2}{2F}\Leftrightarrow\frac{1}{\sin A}\le\frac{2R^2}{F}$$

$$Similarly,\ \frac{1}{\sin B}\le\frac{2R^2}{F},\ \frac{1}{\sin C}\le\frac{2R^2}{F}\ and\ \frac{1}{\sin D}\le\frac{2R^2}{F}$$

$$So,\ \sum_{cyc}\frac{1}{\sin A}\le\frac{8R^2}{F}\ or\ \sum_{cyc}\csc A\le\frac{8R^2}{F}$$

Equality holds if and only if the quadrilateral ABCD is square.

Solution 2 by Soumava Chakraborty-Kolkata-India



Via Ptolemy's theorem, $pq = ac + bd$ and via Ptolemy's second theorem,

$$\frac{p}{q} = \frac{ad+bc}{ab+cd} \therefore p = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \text{ and } q = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}} \to (1)$$

$$\csc A + \csc B + \csc C + \csc D = \csc A + \csc B + \csc(\pi - A) + \csc(\pi - B)$$

$$= 2\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right) \overset{\text{sine law}}{=} 4R\left(\frac{1}{q} + \frac{1}{p}\right)$$

$$\overset{\text{via (1)}}{=} \frac{4R}{pq}\left(\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} + \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}\right)$$

$$= \frac{4R}{pq} \cdot \frac{(ac+bd)(ad+bc+ab+cd)}{\sqrt{(ac+bd)(ad+bc)(ab+cd)}} \overset{\text{Ptolemy and Brahmagupta + Parameshwara}}{=} \frac{4R(a+c)(b+d)}{4FR}$$

$$\le \frac{8R^2}{F} \Leftrightarrow (a+c)(b+d) \overset{(*)}{\le} 8R^2$$

Now, via sine law and $\because$ circumradius of $\Delta\, ABD$ and $\Delta\, BCD$ is R, $\therefore a = 2R\sin\beta$, $b = 2R\sin\delta$, $c = 2R\sin\gamma$, $d = 2R\sin\alpha$

$$\therefore (a+c)(b+d) = 4R^2(\sin\beta + \sin\gamma)(\sin\delta + \sin\alpha)$$

$$= 4R^2(\sin\beta\sin\delta + \sin\gamma\sin\alpha + \sin\alpha\sin\beta + \sin\gamma\sin\delta)$$

$$= 2R^2\big(\cos(\beta-\delta) - \cos(\beta+\delta) + \cos(\gamma-\alpha) - \cos(\gamma+\alpha) + \cos(\alpha-\beta) - \cos(\alpha+\beta) + \cos(\gamma-\delta) - \cos(\gamma+\delta)\big)$$

$$= 2R^2\big(\cos(\beta-\delta) - \cos(D) + \cos(\gamma-\alpha) - \cos(B) + \cos(\alpha-\beta) - \cos(\pi - A) + \cos(\gamma-\delta) - \cos(\pi - C)\big)$$

$$= 2R^2\big(\cos(\beta-\delta) - \cos(D) + \cos(\gamma-\alpha) - \cos(\pi - D) + \cos(\alpha-\beta) + \cos A + \cos(\gamma-\delta) + \cos(\pi - A)\big)$$

$$= 2R^2\left(\cos(\beta-\delta)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)+\cos(\gamma-\delta)\right)$$
$$\le 2R^2(1+1+1+1) = 8R^2 \Rightarrow (*) \text{ is true}$$
$$\therefore \csc A + \csc B + \csc C + \csc D \le \frac{8R^2}{F} \text{ in any cyclic quadrilateral } ABCD \text{ (QED)}$$

JP.509 Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\left(1+\cot\frac{A}{2}\right)\left(1+\cot\frac{B}{2}\right)\left(1+\cot\frac{C}{2}\right) \ge \left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3$$

Proposed by George Apostolopoulos-Greece

Solution 1 by proposer

Use Huygens inequality:

$$\left(1+\cot\frac{A}{2}\right)\left(1+\cot\frac{B}{2}\right)\left(1+\cot\frac{C}{2}\right) \ge \left(1+\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}\right)^3 \ge$$

$$\ge \left(1+\sqrt{3}\right)^3 \ge \left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3, \text{ because}$$

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} \ge 3\cot\left(\frac{A+B+C}{6}\right) = 3\sqrt{3} \text{ and}$$

$$\sqrt{3} \ge \sqrt{3}\cdot\frac{2r}{R} \text{ true from } R \ge 2r \text{ (Euler).}$$

Equality holds if and only if the triangle ABC is equilateral.

Solution 2 by Marin Chirciu-Romania

$$\text{Using Huygen's inequality}: (1+x)(1+y)(1+z) \ge \left(1+\sqrt[3]{xyz}\right)^3; (\forall)x,y,z \ge 0,$$

we get:

$$LHS = \prod_{cyc}\left(1+\cot\frac{A}{2}\right) \overset{Huygens}{\ge} \left(1+\sqrt[3]{\prod_{cyc}\cot\frac{A}{2}}\right)^3 = \left(1+\sqrt[3]{\frac{s}{r}}\right)^3 \overset{(1)}{\ge}$$

$$\ge \left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3 = RHS,$$

$$\text{where } (1) \Leftrightarrow \left(1+\sqrt[3]{\frac{s}{r}}\right)^3 \ge \left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3 \Leftrightarrow \frac{s}{r} \ge 3\sqrt{3}\left(\frac{2r}{R}\right)^3$$

$$\text{which it is obtained from } s \ge 3\sqrt{3}r \text{ (Mitrinovic) and } R \ge 2r \text{ (Euler).}$$

Equality holds if and only if triangle is equilateral.

Solution 3 by Tapas Das-India

$$\cot\frac{A}{2}=\frac{s}{r_a},\sum_{cyc}r_a=4R+r,\sum_{cyc}r_ar_b=s^2,\prod_{cyc}r_a=s^2r,\sum_{cyc}\frac{1}{r_a}=\frac{1}{r}$$

$$\prod_{cyc}\left(1+\cot\frac{A}{2}\right)=\prod_{cyc}\left(1+\frac{s}{r_a}\right)=1+\sum_{cyc}\frac{s}{r_a}+\sum_{cyc}\frac{s^2}{r_ar_b}+\frac{s^3}{r_ar_br_c}=$$

$$=1+s\sum_{cyc}\frac{1}{r_a}+s^2\sum_{cyc}\frac{1}{r_ar_b}+\frac{s^3}{s^2r}=1+\frac{2s}{r}+\frac{4R+r}{r}\overset{\substack{Mitrinovic\\Euler}}{\geq}$$

$$\geq 2+2\cdot\frac{3\sqrt{3}r}{r}+4\cdot\frac{2r}{r}=10+6\sqrt{3};(1)$$

$$Now,\left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3\leq\left(1+\sqrt{3}\cdot\frac{2r}{2r}\right)^3=\left(1+\sqrt{3}\right)^3=10+6\sqrt{3};(2)$$

$$From\ (1)\ and\ (2), it\ follows:$$

$$\prod_{cyc}\left(1+\cot\frac{A}{2}\right)\geq\left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3$$

JP.510 In ΔABC the following relationship holds:

$$9\sqrt{2}r\leq\sum_{cyc}\sqrt{r_a^2+r_b^2}\leq(4R+r)\sqrt{6\left(\frac{R-r}{R+r}\right)}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

$$\sum_{cyc}\sqrt{r_a^2+r_b^2}\overset{CBS}{\leq}\sqrt{3\sum_{cyc}(r_a^2+r_b^2)}=\sqrt{6\sum_{cyc}r_a^2}\leq\sqrt{6(4R+r)^2\left(\frac{R-r}{R+r}\right)}=$$

$$=(4R+r)\sqrt{6\left(\frac{R-r}{R+r}\right)}, where\ we\ used$$

$$\sum_{cyc}r_a^2=(4R+r)^2-2s^2\overset{Gerretsen}{\leq}(4R+r)^2-2\frac{r(4R+r)^2}{R+r}=$$

$$=(4R+r)^2\left(1-\frac{2r}{R+r}\right)=(4R+r)^2\left(\frac{R-r}{R+r}\right)$$

$$\sum_{cyc}\sqrt{r_a^2+r_b^2}\overset{AGM}{\geq}\sum_{cyc}\sqrt{\frac{(r_a+r_b)^2}{2}}=\frac{1}{\sqrt{2}}\sum_{cyc}(r_a+r_b)=\sqrt{2}\sum_{cyc}r_a=$$

$$=\sqrt{2}(4R+r)\overset{Euler}{\geq}\sqrt{2}\cdot 9r=9\sqrt{2}r$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Tapas Das-India

$$\sum_{cyc}\sqrt{r_a^2+r_b^2}\overset{AGM}{\geq}\sum_{cyc}\sqrt{2r_ar_b}=\sqrt{2}\sum_{cyc}\sqrt{r_ar_b}\geq\sqrt{2}\sum_{cyc}h_c\geq$$

$$\geq 3\sqrt{2}\sqrt[3]{h_ah_bh_c}\geq 3\sqrt{2}\sqrt[3]{27r^3};\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\right)$$

$$\sum_{cyc}\sqrt{r_a^2+r_b^2}\overset{CBS}{\leq}\sqrt{6\left(\sum_{cyc}r_a^2\right)}=\sqrt{6\left(\left(\sum_{cyc}r_a\right)^2-2\sum_{cyc}r_ar_b\right)}=$$

$$=\sqrt{6[(4R+r)^2-2s^2]}$$

We need to prove:

$$\sqrt{6[(4R+r)^2-2s^2]}\leq(4R+r)\sqrt{6\left(\frac{R-r}{R+r}\right)}\Leftrightarrow$$

$$[(4R+r)^2-2s^2](R+r)\leq(4R+r)^2(R-r)\Leftrightarrow$$

$$[(4R+r)^2-2(16Rr-5r^2)](R+r)\leq(4R+r)^2(R-r)\Leftrightarrow$$

$$6Rr^2-12r^3\geq 0\Leftrightarrow R\geq 2r\ (Euler).$$

PROBLEMS FOR SENIORS

SP.496 If $f:\mathbb{R}\to\mathbb{R}$ derivable, $f(0)=0, f'(0)=1$, then find:

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to 0}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+\cdots+f\left(\frac{x}{n}\right)\right)-\log n\right)$$

Proposed by Marin Chirciu-Romania

Solution by proposer

$$Using\ \lim_{h\to 0}\frac{f(h)-f(0)}{h}=f'(0), we\ get:$$

$$\lim_{x\to 0}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+\cdots+f\left(\frac{x}{n}\right)\right)=\sum_{k=1}^{n}\frac{1}{k}\lim_{x\to 0}\frac{f\left(\frac{x}{k}\right)-f(0)}{\frac{x}{k}-0}=$$

$$=\sum_{k=1}^{n}\frac{1}{k}f'(0)=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\cdot 1=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to 0}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+\cdots+f\left(\frac{x}{n}\right)\right)-\log n\right)=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)=C=05772\ldots$$

SP.497 Let $D\ ABC$ be a triangle with circumradius R. Let r_a, r_b, r_c be the exradii. Prove that:

$$\frac{r_a^4}{\sin(2A)}+\frac{r_b^4}{\sin(2B)}+\frac{r_c^4}{\sin(2C)}\geq\frac{81\sqrt{3}}{8}R^4$$

Proposed by George Apostolopoulos-Greece

Solution 1 by proposer

It is well known that:

$$r_a=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},\ r_b=4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}\text{ and}$$

$$r_c=4R\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2},\text{ then using CBS:}$$

$$\frac{r_a^4}{\sin(2A)}+\frac{r_b^4}{\sin(2B)}+\frac{r_c^4}{\sin(2C)}\geq$$

$$\geq\frac{\left(r_a^2+r_b^2+r_c^2\right)^2}{\sin(2A)+\sin(2B)+\sin(2C)};\qquad (*)$$

We'll prove that:

$$r_a^2+r_b^2+r_c^2\geq\frac{27}{4}R^2$$

$$\text{We have: } r_a^2+r_b^2+r_c^2=16R^2\sum_{cyc}\sin^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}=$$

$$=2R^2\sum_{cyc}(1-\cos A)(1+\cos B)(1+\cos C)=$$

$$=2R^2\left(3+\sum_{cyc}\cos A-\sum_{cyc}\cos A\cos B-3\cos A\cos B\cos C\right)$$

We know that:

$$\cos A+\cos B+\cos C=\frac{R+r}{r}$$

$$\sum_{cyc}\cos A\cos B=\frac{s^2+r^2-4R^2}{4R^2}$$

$$\cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2}$$

So: $r_a^2+r_b^2+r_c^2=2R^2\left(3+\frac{R+r}{R}-\frac{s^2+r^2-4R^2}{4R^2}-3\cdot\frac{s^2-(2R+r)^2}{4R^2}\right)=$

$$=(4R+r)^2-2s^2$$

Using Gerretsen's inequality, we get:

$$r_a^2+r_b^2+r_c^2\geq 8R^2-5r^2$$

But $R\geq 2r(Euler)\Leftrightarrow r\leq\frac{R}{2}$, namely $r_a^2+r_b^2+r_c^2\geq\frac{27R^2}{4}$

Now, we'll prove that:

$\sin(2A)+\sin(2B)+\sin(2C)\leq\sin A+\sin B+\sin C$. We have:

$$\sin(2A)+\sin(2B)=2\sin(A+B)\cos(A-B)\leq 2\sin(A+B)=2\sin C$$

Similarly,

$\sin(2B)+\sin(2C)\leq 2\sin A$ and $\sin(2C)+\sin(2A)\leq 2\sin B$

So, $2(\sin(2A)+\sin(2B)+\sin(2C))\leq 2(\sin A+\sin B+\sin C)\Leftrightarrow$

$$\sin(2A)+\sin(2B)+\sin(2C)\leq\sin A+\sin B+\sin C$$

Also, we know that: $\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}$

The inequality $(*)$, gives:

$$\frac{r_a^4}{\sin(2A)}+\frac{r_b^4}{\sin(2B)}+\frac{r_c^4}{\sin(2C)}\geq\frac{\left(\frac{27R^2}{4}\right)^2}{\sin A+\sin B+\sin C}\geq\frac{\left(\frac{27R^2}{4}\right)^2}{\frac{3\sqrt{3}}{2}}=\frac{81\sqrt{3}}{8}R^4$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Marin Chirciu-Romania

Lemma: In acute ΔABC holds: $\sum_{cyc}\frac{r_a^4}{\sin 2A}\ge\frac{R^2(4R+r)^4}{18sr}$

Proof. $LHS=\sum_{cyc}\frac{r_a^4}{\sin 2A}\overset{Holder}{\ge}\frac{(\sum r_a)^4}{9\sum\sin 2A}=\frac{(4R+r)^4}{9\cdot\frac{2r}{R^2}}=\frac{R^2(4R+r)^4}{18sr}\overset{(1)}{\ge}\frac{81\sqrt{3}}{8}R^4$
$=RHS$

$$\text{where }(1)\Leftrightarrow\frac{R^2(4R+r)^4}{18sr}\ge\frac{81\sqrt{3}}{8}R^4\Leftrightarrow 4(4R+r)^4\ge 729\sqrt{3}R^2sr$$

$$\text{which follows from } s\le\frac{3\sqrt{3}R}{2}\ (Mitrinovic)$$

$$\text{Remains to prove}: 4(4R+r)^4\ge 729\sqrt{3}R^2\cdot\frac{3\sqrt{3}R}{2}\cdot r\Leftrightarrow$$

$$8(4R+r)^4\ge 6561R^3r\Leftrightarrow 8(256R^4+256R^3r+96R^2r^2+16Rr^3+r^4)\ge 6561R^3r\Leftrightarrow$$

$$2048R^4+2048R^3r+768R^2r^2+128Rr^3+8r^4\ge 6561R^3r\Leftrightarrow$$

$$(R-2r)(2048R^3-417R^2r-66Rr^2-4r^3)\ge 0 \text{ which is true from } R\ge 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral. Remark. The problem can be developed:

If $n\in\mathbb{N}$, in acute ΔABC holds:

$$\frac{r_a^n}{\sin 2A}+\frac{r_b^n}{\sin 2B}+\frac{r_c^n}{\sin 2C}\ge\left(\frac{4R+r}{3}\right)^{n-2}\cdot\frac{9\sqrt{3}}{2}R^2$$

Proposed by Marin Chirciu – Romania

Solution.

For $n=0, n=1$ the problem it is solved using Bergstrom's inequality.

For $n\ge 2$, it is use Holder's inequality.

Lemma: If $n\in\mathbb{N}$, in acute ΔABC holds: $\sum_{cyc}\frac{r_a^n}{\sin 2A}\ge\frac{R^2(4R+r)^n}{3^{n-2}\cdot 2sr}$

Proof. $LHS=\sum_{cyc}\frac{r_a^n}{\sin 2A}\overset{Holder}{\ge}\frac{(\sum r_a)^n}{3^{n-2}\sum\sin 2A}=\frac{(4R+r)^n}{3^{n-2}\cdot\frac{2sr}{R^2}}=\frac{R^2(4R+r)^n}{3^{n-2}\cdot 2sr}=$

$$=\frac{R^2(4R+r)^{n-2}}{2\cdot 3^{n-2}}\cdot\frac{(4R+r)^2}{sr}\overset{(1)}{\ge}\frac{R^2(4R+r)^{n-2}}{2\cdot 3^{n-2}}\cdot 9\sqrt{3}=\left(\frac{4R+r}{3}\right)^{n-2}\cdot\frac{9\sqrt{3}}{2}R^2$$

$$(1)\Leftrightarrow\frac{(4R+r)^2}{sr}\ge 9\sqrt{3}\text{ which follows from } s\le\frac{3\sqrt{3}R}{2}\ (Mitrinovic)$$

$$\text{Remains to prove}: (4R+r)^2\ge 9\sqrt{3}\cdot\frac{3\sqrt{3}R}{2}\cdot r\Leftrightarrow 2(4R+r)^2\ge 81Rr$$

$$32R\sqrt{2}-65Rr+2r^2\geq 0 \Leftrightarrow (R-2r)(32R-r)\geq 0 \textit{ which is true from } R \geq 2r(\textit{Euler}).$$

Equality holds if and only if triangle is equilateral.

SP.498 Let $A,B\in M_3(\mathbb{R})$ such that $AB=BA$. Prove that

$\det(A^2+B^2)=0$ if and only if $\det(A+B)=2(\det A+\det B)$ and

$\det(A-B)=2(\det A-\det B)$.

Proposed by Florentin Vișescu-Romania

Solution by proposer

$$\det(A^2+B^2)=\det(A^2-(iB)^2)=\det[(A-iB)(A+iB)]=$$
$$=\det(A-iB)\cdot\det(A+iB)=\det(A-iB)\cdot\det(A+\varepsilon B)=$$
$$=\det(A+iB)\cdot\det(A+\bar{\imath}B)=\det(A+iB)\cdot\overline{\det(A+\imath B)}=|\det(A+iB)|^2=0$$
$$\det(A-iB)=0$$

It is known that $(\forall)A,B\in M_3(\mathbb{C})$ and $(\forall)x\in\mathbb{C}$:

$$\det(A+xB)=x^3\det(B)+\frac{\det(A+B)+\det(A-B)-2\det A}{2}x^2+\frac{\det(A+B)-\det(A-B)-2\det B}{2}x+\det A$$

Then

$$\det(A-iB)=0\Leftrightarrow$$
$$-i\det(B)-\frac{\det(A+B)+\det(A-B)-2\det A}{2}+\frac{\det(A+B)-\det(A-B)-2\det B}{2}i+\det A=0$$

$$\begin{cases}-\det B+\dfrac{\det(A+B)-\det(A-B)-2\det B}{2}=0\\ \det A-\dfrac{\det(A+B)+\det(A-B)-2\det A}{4}=0\end{cases}$$

$$\begin{cases}\det(A+B)-\det(A-B)=4\det B\\ \det(A+B)+\det(A-B)=4\det A\end{cases}$$

$$\begin{cases}\det(A+B)=2(\det A+\det B)\\ \det(A-B)=2(\det A-\det B)\end{cases}$$

SP.499 Let $A,B\in M_3(\mathbb{R})$ such that $AB=BA$. Prove that:

$\det(A^2+AB+B^2)=0$ if and only if $\det(A+B)=\det A+\det B$ and

$\det(A-B)=3(\det A-\det B)$.

Proposed by Florentin Vișescu-Romania

Solution by proposer

$$\det(A^2+AB+B^2)=\det\left(A^2+2A\cdot\frac{1}{2}B+\frac{1}{4}B^2+\frac{3}{4}B^2\right)=$$

$$=\det\left[\left(A+\frac{1}{2}B\right)^2-\left(\frac{i\sqrt{3}}{2}B\right)^2\right]=\det\left[\left(A+\frac{1+i\sqrt{3}}{2}B\right)\left(A+\frac{1-i\sqrt{3}}{2}B\right)\right]=$$

$$=\det\left(A+\frac{1+i\sqrt{3}}{2}B\right)\cdot\det\left(A+\frac{1-i\sqrt{3}}{2}B\right)=\det(A-\varepsilon B)\cdot\det(A+\varepsilon B)=$$

$$=\det(A-\varepsilon B)\cdot\det(A-\bar{\varepsilon}B)=\det(A-\varepsilon B)\cdot\overline{\det(A+\varepsilon B)}=|\det(A-\varepsilon B)|^2=0$$

$$\det(A-\varepsilon B)=0$$

It is known that $(\forall)A,B\in M_3(\mathbb{C})$ and $(\forall)x\in\mathbb{C}$:

$$\det(A+xB)=x^3\det(B)+\frac{\det(A+B)+\det(A-B)-2\det A}{2}x^2$$

$$+\frac{\det(A+B)-\det(A-B)-2\det B}{2}x+\det A$$

Then, $\det(A-\varepsilon B)=0\Leftrightarrow$

$$-\det(B)+\frac{\det(A+B)+\det(A-B)-2\det A}{2}\bar{\varepsilon}$$

$$-\frac{\det(A+B)-\det(A-B)-2\det B}{2}\varepsilon+\det A=0$$

$$\begin{cases}-\det B-\dfrac{\det(A+B)+\det(A-B)-2\det A}{4}+\dfrac{\det(A+B)-\det(A-B)-2\det B}{4}+\det A=0\\ \dfrac{\sqrt{3}}{2}\dfrac{\det(A+B)+\det(A-B)-2\det A}{4}+\dfrac{\sqrt{3}}{2}\dfrac{\det(A+B)-\det(A-B)-2\det B}{4}=0\end{cases}$$

$$\begin{cases}-2\det(A-B)-6\det B+6\det A=0\\ 2\det(A+B)-2\det A-2\det B=0\end{cases}$$

$$\begin{cases}\det(A+B)=\det A+\det B\\ \det(A-B)=3\det A-3\det B\end{cases}$$

SP.500 Find:

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\cos x-4\sin x}{12\sin 2x+7\cos^2 x+4\cos x+3\sin x+9}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$12\sin 2x+7\cos^2 x+4\cos x+3\sin x+9=$$
$$=24\sin x\cos x+7\cos^2 x+4\cos x+3\sin x+9(\sin^2 x+\cos^2 x)=$$
$$=24\sin x\cos x+16\cos^2 x+9\sin^2 x+4\cos x+3\sin x=$$
$$=3\sin x(3\sin x+4\cos x)+4\cos x(3\sin x+4\cos x)+3\sin x+4\cos \quad =$$
$$=(3\sin x+4\cos x)(3\sin x+4\cos x+1)$$

Denote: $y=3\sin x+4\cos x\Rightarrow dy=(3\cos x-4\sin x)dx$

If $x=0\Rightarrow y=4$ and if $x=\frac{\pi}{2}\Rightarrow y=3$

$$\Omega=\int_4^3\frac{dt}{t(t+1)}=\int_4^3\left(\frac{1}{t}-\frac{1}{t+1}\right)dt=\log 3-\log 4-(\log 4-\log 5)=\log\left(\frac{15}{16}\right)$$

Solution 2 by Pham Duc Nam-Vietnam

$$*\ 12\sin 2x+7\cos^2 x+4\cos x+3\sin x+9$$
$$=4\cos x+3\sin x+(16-9)\cos^2 x+24\sin x\cos x+9$$
$$=4\cos x+3\sin x+16\cos^2 x+9(1-\cos^2 x)+24\sin x\cos x$$
$$=4\cos x+3\sin x+(4\cos x)^2+(3\sin x)^2+2(3\sin x)(4\cos x)$$
$$=4\cos x+3\sin x+(4\cos x+3\sin x)^2$$
$$\Rightarrow\int_0^{\frac{\pi}{2}}\frac{3\cos x-4\sin x}{12\sin 2x+7\cos^2 x+4\cos x+3\sin x+9}dx=$$
$$=\int_0^{\frac{\pi}{2}}\frac{d(4\cos x+3\sin x)}{4\cos x+3\sin x+(4\cos x+3\sin x)^2}$$
$$=\log\left|\frac{4\cos x+3\sin x}{4\cos x+3\sin x+1}\right|\Bigg|_0^{\frac{\pi}{2}}=\log\frac{3}{4}-\log\frac{4}{5}=\log\frac{15}{16}$$

Solution 3 by Marin Chirciu-Romania

$$\textit{Let } t=3\sin x+4\cos x \textit{ and with}$$
$$12\sin 2x+7\cos^2 x+4\cos x+3\sin x+9$$
$$=(3\sin x+4\cos x)^2+(3\sin x+4\cos x)=t^2+t$$
$$\textit{we get}:$$
$$\int_0^{\frac{\pi}{2}}\frac{3\cos x-4\sin x}{12\sin 2x+7\cos^2 x+4\cos x+3\sin x+9}dx=\int_4^3\frac{dt}{t^2+t}=$$

$$= \int_4^3 \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = (\log t - \log(t+1))|_4^3 = \log \frac{t}{t+1}\Big|_4^3 =$$

$$= \log \frac{3}{4} - \log \frac{4}{5} = \log \frac{15}{16}$$

Solution 4 by Kartick Chandra Betal-India

$$\int_0^{\frac{\pi}{2}} \frac{3\cos x - 4\sin x}{12\sin 2x + 7\cos^2 x + 4\cos x + 3\sin x + 9} dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{3\cos x - 4\sin x}{(4\cos x)^2 + 2\cdot 4\cos x \cdot 3\sin x + (3\sin x)^2 + (4\cos x + 3\sin x)} dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{d(3\sin x + 4\cos x)}{(3\sin x + 4\cos x)(3\sin x + 4\cos x + 1)} dx =$$

$$= \int_4^3 \frac{dz}{z(z+1)} = \left[\log\left(\frac{z}{z+1}\right)\right]_4^3 = \log\left(\frac{3}{4}\cdot\frac{5}{4}\right) = \log\left(\frac{15}{16}\right)$$

SP.501 If $M \in Int(\Delta ABC)$ such that $x = MA, y = MB, z = MC$, then:

$$\frac{x}{h_a h_b \sqrt{yz}} + \frac{y}{h_b h_c \sqrt{zx}} + \frac{z}{h_c h_a \sqrt{xy}} \ge \frac{\sqrt{3}}{F}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

$$\sum_{cyc} \frac{x}{h_a h_b \sqrt{yz}} = \sum_{cyc} \frac{abx}{ah_a \cdot bh_b \sqrt{yz}} = \frac{1}{4F^2} \sum_{cyc} \frac{abx}{\sqrt{yz}} \ge$$

$$\ge \frac{1}{4F^2} \cdot 3 \cdot \sqrt[3]{\prod_{cyc} \frac{abx}{\sqrt{yz}}} = \frac{1}{4F^2} \cdot 3 \cdot \sqrt[3]{a^2b^2c^2} \overset{Carliz}{\ge} \frac{1}{4F^2} \cdot 4\sqrt{3}F = \frac{\sqrt{3}}{F}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM - GM\ inequality, we\ have:$$

$$\frac{x}{h_ah_b\sqrt{yz}}+\frac{y}{h_bh_c\sqrt{zx}}+\frac{z}{h_ch_a\sqrt{xy}}\geq\frac{3}{\sqrt[3]{(h_ah_bh_c)^2}},\ \ with:\ h_ah_bh_c=\frac{2F^2}{R}.$$

Therefore,

$$\frac{x}{h_ah_b\sqrt{yz}}+\frac{y}{h_bh_c\sqrt{zx}}+\frac{z}{h_ch_a\sqrt{xy}}\geq 3\sqrt[3]{\frac{R.R}{4F^4}}\overset{Euler\ \&\ Mitrinovic}{\geq}3\sqrt[3]{\frac{2r.2s}{4F^4.3\sqrt{3}}}\overset{sr=F}{=}\frac{\sqrt{3}}{F}.$$

Equality holds iff M is the center of the equilateral ΔABC.

SP.502 If $x,y,z>0$, then:

$$(2x^2+1)(2y^2+1)(2z^2+1)\geq\frac{9}{2}(xy+yz+zx)$$

Proposed by D.M. Bătineţu-Giurgiu, Claudia Nănuţi-Romania

Solution 1 by proposers

$$(u^2+1)(v^2+1)\geq\frac{3}{4}((u+v)^2+1);(\forall)u,v>0; \qquad (1)$$

$$4u^2v^2+4(u^2+v^2)+4\geq 3(u^2+v^2)+6uv+3$$

$$4u^2v^2-4uv+1+u^2+v^2+2uv\geq 0$$

$$(2uv-1)^2+(u-v)^2\geq 0$$

Equality holds for $2uv=1$ and $u=v\Leftrightarrow u=v=\frac{1}{\sqrt{2}}$.

$$(t^2+1)(w^2+1)\geq(t+w)^2;(\forall)t,w>0; \qquad (2)$$

$$t^2w^2+t^2+w^2+1\geq t^2+w^2+2tw\Leftrightarrow t^2w^2-2tw+1\geq 0$$

$(tw-1)^2\geq 0$. Equality holds for $tw=1$.

$$(u^2+1)(v^2+1)(w^2+1)\overset{(1)}{\geq}\frac{3}{4}((u+v)^2+1)(w^2+1)\overset{(2)}{\geq}$$

$$\geq\frac{3}{4}\big((u+v)+w\big)^2;(\forall)u,v,w>0; \qquad (3)$$

In (3) we take: $u=\sqrt{2}x;v=\sqrt{2}y;w=\sqrt{2}z$ then:

$$(2x^2+1)(2y^2+1)(2z^2+1)\geq\frac{3}{4}\left(\sqrt{2}(x+y+z)\right)^2-\frac{3}{2}(x+y+z)^2\geq$$

$$\geq\frac{9}{2}(xy+yz+zx)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\boldsymbol{We\ have:}$$

$$(2x^2+1)(2y^2+1)=\left[\left(2x^2+\frac{1}{2}\right)+\frac{1}{2}\right]\left[\left(\frac{1}{2}+2y^2\right)+\frac{1}{2}\right]=$$

$$=\left(2x^2+\frac{1}{2}\right)\left(\frac{1}{2}+2y^2\right)+x^2+y^2+\frac{3}{4}\overset{CBS}{\geq}(x+y)^2+\frac{(x+y)^2}{2}+\frac{3}{4}=\frac{3}{2}\left((x+y)^2+\frac{1}{2}\right).$$

$$\boldsymbol{Therefore,}$$

$$(2x^2+1)(2y^2+1)(2z^2+1)\geq\frac{3}{2}\left((x+y)^2+\frac{1}{2}\right)(1+2z^2)\geq$$

$$\overset{CBS}{\geq}\frac{3}{2}(x+y+z)^2\geq\frac{3}{2}.3(xy+yz+zx)=\frac{9}{2}(xy+yz+zx).$$

$$\boldsymbol{Equality\ holds\ iff\ x=y=z=\frac{1}{2}}.$$

Solution 3 by Ivan Hadinata-Jember-Indonesia

***Lemma 1:** If $x,y,z>0$ then:*

$$\left(\sum_{cyc}x\right)^2\geq 3\sum_{cyc}xy\quad(1)$$

Proof:

$$\left(\sum_{cyc}x\right)^2=2\sum_{cyc}xy+\sum_{cyc}\frac{x^2+y^2}{2}\overset{AM-GM}{\geq}2\sum_{cyc}xy+\sum_{cyc}xy=3\sum_{cyc}xy$$

***Lemma 2:** If $x,y>0$ then:*

$$(2x^2+1)(2y^2+1)\geq 2(x+y)^2\quad(2)$$

Proof:

$$(2x^2+1)(2y^2+1)=4x^2y^2+1+2(x^2+y^2)\overset{AM-GM}{\geq}$$
$$\geq 2\sqrt{4x^2y^2\cdot 1}+2(x^2+y^2)=2(x+y)^2$$

***Lemma 3:** If $x,y,z>0$ then:*

$$(2x^2+1)(2y^2+1) \geq \frac{3}{4}(2(x+y)^2+1) \quad (3)$$

Proof:

$$4(2x^2+1)(2y^2+1) = 16x^2y^2+1+6(x^2+y^2)+2(x^2+y^2)+3 \overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{16x^2y^2\cdot 1}+6(x^2+y^2)+2\cdot 2xy+3 = 6(x+y)^2+3$$

Back to the problem:

$$(2x^2+1)(2y^2+1)(2z^2+1) \overset{(3)}{\geq} \frac{3}{4}(2(x+y)^2+1)(2z^2+1) \overset{(2)}{\geq}$$

$$\geq \frac{3}{2}(x+y+z)^2 \overset{(1)}{\geq} \frac{9}{2}(xy+yz+zx)$$

Equality holds for $x=y=z=\frac{1}{2}$.

SP.503 Solve for real positive numbers:

$$e\cdot(x^x+(1+\log x)^2)^e = 1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}, f(x)=x^x+(1+\log x)^2$**, then:**

$$f'(x) = x^x\log x + x\cdot x^{x-1}+2(1+\log x)\cdot\frac{1}{x} =$$

$$= x^x(1+\log x)+\frac{2}{x}(1+\log x) = (1+\log x)\left(x^x+\frac{2}{x}\right)$$

$$sgn(f'(x)) = sgn(1+\log x)$$

$$f'(x)=0 \Rightarrow 1+\log x = 0 \Rightarrow \log x = -1 \Rightarrow x = -\frac{1}{e}$$

$$\min_{x>0} f(x) = f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}} + \left(1+\log\left(\frac{1}{e}\right)\right)^2 = e^{-\frac{1}{e}}$$

$$\Rightarrow f(x) \geq e^{-\frac{1}{e}} \Rightarrow x^x+(1+\log x)^2 \geq e^{-\frac{1}{e}} \Rightarrow (x^x+(1+\log x)^2)^e \geq \frac{1}{e}$$

$$\Rightarrow e\cdot(x^x+(1+\log x)^2)^e \geq 1$$

Equality holds for $x=\frac{1}{e}$.

Solution 2 by Pham Duc Nam-Vietnam

$$e(x^x+(1+log\,x)^2)^e = 1(*)$$

$*$ Equation holds: $x>0$

$$*\,(*) \Leftrightarrow (x^x + (1+\log x)^2)^e = \frac{1}{e} \Leftrightarrow x^x + (1+\log x)^2 = \sqrt[e]{\frac{1}{e}}$$

Let: $f(x) = x^x + (1+\log x)^2 \Rightarrow f'(x) = x^x(\log x + 1) + \frac{2}{x}(\log x + 1)$

$$= \frac{(\log x + 1)(x^{x+1} + 2)}{x}$$

$$\Rightarrow f'(x) = 0 \Leftrightarrow \log x + 1 = 0 \Rightarrow x = \frac{1}{e} \therefore (x^{x+1} + 2 > 0 \forall x > 0)$$

Table

x	0		$\frac{1}{e}$		∞
$f'(x)$		$-$	0	$+$	
$f(x)$	∞	$\searrow$	$\sqrt[e]{\frac{1}{e}}$	$\nearrow$	∞

$$\Rightarrow f(x) \text{ is strictly decreasing in } \left(0, \frac{1}{e}\right) \text{,and strictly increasing in } \left(\frac{1}{e}, \infty\right)$$

and: $f(x) \geq \sqrt[e]{\frac{1}{e}}, \exists x_0 = \frac{1}{e} \Rightarrow f\left(\frac{1}{e}\right) = \sqrt[e]{\frac{1}{e}} \Rightarrow f\left(\frac{1}{e}\right)$ is minimum $\Rightarrow f(x) = \sqrt[e]{\frac{1}{e}} \Leftrightarrow x = \frac{1}{e}$

$$\Rightarrow \text{Equation has root is: } x = \frac{1}{e}$$

SP.504 If $x, y, z > 0, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$, then:

$$4\sum_{cyc} x^2 + 9 \geq \sum_{cyc}\left(5x + \frac{4}{x+1}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$4\sum_{cyc} x^2 + 9 \geq \sum_{cyc}\left(5x + \frac{4}{x+1}\right) \Leftrightarrow 4\sum_{cyc} x^2 + 12 \geq 3 + 5\sum_{cyc} x + 4\sum_{cyc}\frac{1}{x+1} \Leftrightarrow$$

$$\sum_{cyc} x^2 + 3 \ge \frac{3}{4} + \frac{5}{4}\sum_{cyc} x + \sum_{cyc}\frac{1}{x+1} \Leftrightarrow$$

$$\sum_{cyc} x^2 + \sum_{cyc}\frac{1}{x} - \frac{5}{4}\sum_{cyc} x - \sum_{cyc}\frac{1}{4} - \sum_{cyc}\frac{1}{x+1} \ge 0$$

We will prove that if $x > 0$, then:

$$x^2 + \frac{1}{x} - \frac{5}{4}x - \frac{1}{4} - \frac{1}{x+1} \ge 0 \Leftrightarrow$$

$$4(x^3(x+1) - x + x + 1) - (5x+1)(x^2+x) \ge 0 \Leftrightarrow$$

$$4x^4 - x^3 - 6x^2 - x + 4 \ge 0 \Leftrightarrow (x-1)^2(4x^2+7x+4) \ge 0$$

Equality holds for $x = y = z = 1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$3 = \sum_{\text{cyc}}\frac{1}{x} \overset{\text{Bergstrom}}{\ge} \frac{9}{\sum_{\text{cyc}} x} \Rightarrow \sum_{\text{cyc}} x \overset{(*)}{\ge} 3$$

$$\text{Now, } \sum_{\text{cyc}}\left(5x + \frac{4}{x+1}\right) = 5\sum_{\text{cyc}} x + 4\sum_{\text{cyc}}\frac{1}{x+1} \overset{\text{A-G}}{\le} 5\sum_{\text{cyc}} x + 2\sum_{\text{cyc}}\frac{1}{\sqrt{x}}$$

$$\overset{\text{CBS}}{\le} 5\sum_{\text{cyc}} x + 2\sqrt{3}.\sqrt{\sum_{\text{cyc}}\frac{1}{x}} \overset{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3}{=} 5\sum_{\text{cyc}} x + 2\sqrt{3}.\sqrt{3} = 5\sum_{\text{cyc}} x + 6 \overset{?}{\le} 4\sum_{\text{cyc}} x^2 + 9$$

$$\Leftrightarrow 4\sum_{\text{cyc}} x^2 + 3 - 5\sum_{\text{cyc}} x \underbrace{\overset{?}{\ge}}_{(**)} 0$$

$$\text{Again, LHS of } (**) \ge \frac{4}{3}\left(\sum_{\text{cyc}} x\right)^2 + 3 - 5\sum_{\text{cyc}} x \overset{?}{\ge} 0$$

$$\Leftrightarrow 4t^2 - 15t + 9 \overset{?}{\ge} 0 \left(t = \sum_{\text{cyc}} x\right) \Leftrightarrow (4(t-3)+9)(t-3) \overset{?}{\ge} 0 \to \text{true}$$

$$\because t = \sum_{\text{cyc}} x \overset{\text{via } (*)}{\ge} 3 \Rightarrow (**) \text{ is true} \therefore 4\sum_{\text{cyc}} x^2 + 9 \ge \sum_{\text{cyc}}\left(5x + \frac{4}{x+1}\right) \ \forall x,y,z > 0$$

$$\text{such that}: \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3,'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{By AM-GM inequality, we have}: \ x^2 + x^2 + \frac{1}{x} \ge 3x \text{ and } x + \frac{1}{x} \ge 2.$$

$$Then: \ 4x^2 \ge 2\left(3x - \frac{1}{x}\right) = 5x + \left(x + \frac{1}{x}\right) - \frac{3}{x} \ge 5x + 2 - \frac{3}{x} \ (and\ analogs)$$

$$Thus, \quad 4\sum_{cyc} x^2 + 9 \ge \sum_{cyc}\left(5x + 2 - \frac{3}{x}\right) + 9 = \sum_{cyc} 5x + 6 \ (1)$$

$$By\ AM - HM\ inequality, we\ have: \frac{4}{x+1} \le \frac{1}{x} + 1 \ (and\ analogs)$$

$$Then: \sum_{cyc} \frac{4}{x+1} \le \sum_{cyc}\left(\frac{1}{x} + 1\right) = 3 + 3 = 6 \ (2)$$

$$From\ (1)\ and\ (2), we\ have:$$

$$\sum_{cyc} x^2 + 9 \ge \sum_{cyc}\left(5x + \frac{4}{x+1}\right), \ as\ desired.$$

$$Equality\ holds\ iff\ x = y = z = 1.$$

SP.505 If $x, y, z > 0, \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} = 3$, then:

$$x^2\sqrt{x} + y^2\sqrt{y} + z^2\sqrt{z} \ge x^2 + y^2 + z^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$x^2\sqrt{x} + y^2\sqrt{y} + z^2\sqrt{z} \ge x^2 + y^2 + z^2 \Leftrightarrow \sum_{cyc} x^2\sqrt{x} + 3 \ge \sum_{cyc} x^2 + 3 \Leftrightarrow$$

$$\sum_{cyc} x^2\sqrt{x} + \sum_{cyc} \frac{1}{\sqrt{x}} \ge \sum_{cyc}(x^2 + 1) \Leftrightarrow \sum_{cyc}\left(x^2\sqrt{x} + \frac{1}{\sqrt{x}} - x^2 - 1\right) \ge 0 \Leftrightarrow$$

$$\sum_{cyc}\left(\frac{x^3 + 1}{\sqrt{x}} - x^2 - 1\right) \ge 0$$

$$We\ will\ prove\ that:$$

$$\frac{x^3 + 1}{\sqrt{x}} - x^2 - 1 \ge 0; (\forall)x > 0$$

$$x^3 + 1 \ge \sqrt{x}(x^2 + 1) \Leftrightarrow x^6 - x^5 - x + 1 \ge 0 \Leftrightarrow x^5(x - 1) - (x - 1) \ge 0$$

$$(x - 1)^2(x^4 + x^3 + x^2 + x + 1) \ge 0$$

$$Equality\ holds\ for\ x = y = z = 1.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$WLOG, we\ assume\ that:\ x \ge y \ge z,\ then:$$

$$x^2\sqrt{x} \ge y^2\sqrt{y} \ge z^2\sqrt{z}\ and\ \frac{1}{\sqrt{x}} \le \frac{1}{\sqrt{y}} \le \frac{1}{\sqrt{z}}.$$

$$By\ Chebyshev's\ inequality, we\ have:$$

$$x^2+y^2+z^2 = x^2\sqrt{x}.\frac{1}{\sqrt{x}} + y^2\sqrt{y}.\frac{1}{\sqrt{y}} + z^2\sqrt{z}.\frac{1}{\sqrt{z}}$$

$$\le \frac{1}{3}\left(x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z}\right)\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}\right)$$

$$= x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z}.$$

$$Equality\ holds\ iff\ x=y=z=1.$$

Solution 3 by Marin Chirciu-Romania

$$With\ the\ substitution\ \left(\sqrt{x},\sqrt{y},\sqrt{z}\right)=(a,b,c), then\ problem\ can\ be\ write:$$

$$If\ a,b,c>0, then\ a^5+b^5+c^5 \ge a^4+b^4+c^4$$

$$Use\ AM-GM\ inequality, we\ have: 3 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 3\sqrt[3]{\frac{1}{abc}} \Rightarrow abc \ge 1$$

$$\sum_{cyc} a^5 = \sum_{cyc} a\cdot a^4 \overset{Chebyshev}{\ge} \frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^4\right) \overset{AM-GM}{\ge} \sqrt[3]{abc}\cdot\sum_{cyc} a^4 \overset{abc\ge1}{\ge} \sum_{cyc} a^4$$

$$Equality\ holds\ for\ a=b=c=1 \Leftrightarrow x=y=z=1.$$

$$Remark. The\ problem\ can\ be\ developed.$$

$$If\ x,y,z>0, \frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}} = 3\ and\ n\in\mathbb{N}, then:$$

$$x^n\sqrt{x}+y^n\sqrt{y}+z^n\sqrt{z} \ge x^n+y^n+z^n$$

Proposed by Marin Chirciu – Romania

Solution.

$$For\ n=0, inequality\ can\ be\ write: \sqrt{x}+\sqrt{y}+\sqrt{z} \ge 3, which\ as\ follows\ from$$

$$\left(\sum_{cyc}\sqrt{x}\right)\left(\sum_{cyc}\frac{1}{\sqrt{x}}\right) \ge 0\ and\ from\ \sum_{cyc}\frac{1}{\sqrt{x}} = 3$$

$$Let\ be\ n\ge1. With\ the\ substitution\ \left(\sqrt{x},\sqrt{y},\sqrt{z}\right)$$

$$= (a,b,c), the\ problem\ can\ be\ written\ as:$$

$$If\ a,b,c>0,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3,then\ a^{2n+1}+b^{2n+1}+c^{2n+1}\ge a^{2n}+b^{2n}+c^{2n}$$

$$With\ AM-GM\ inequality,we\ have: 3=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3\sqrt[3]{\frac{1}{abc}}\Rightarrow abc\ge 1$$

$$\sum_{cyc}a^{2n+1}=\sum_{cyc}a\cdot a^{2n}\overset{Chebyshev}{\ge}\frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^{2n}\right)\overset{AM-GM}{\ge}$$

$$\ge\sqrt[3]{abc}\cdot\sum_{cyc}a^{2n}\overset{abc\ge 1}{\ge}\sum_{cyc}a^{2n}$$

$$Equality\ holds\ for\ a=b=c=1\Leftrightarrow x=y=z=1.$$

Note: For $n=2$ we onbtain the Proposed Problem SP.505 from RMM − 34 Autumn Edition, proposed by Daniel Sitaru − Romania.

SP.506 Determine the sequence $(a_n)_{n\ge 1}, a_n\in\mathbb{R}^*$ such that:

$$\binom{n-1}{0}\cdot\binom{n-1}{1}a_1+\binom{n-1}{1}\cdot\binom{n}{2}a_2+\cdots+\binom{n-1}{n-1}\cdot\binom{n}{n}a_n$$

$$=\left[\binom{2n}{n}-1\right]a_n;(\forall)n\ge 1$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$\text{For } n=1\Rightarrow\binom{0}{0}\cdot\binom{1}{1}a_1=\left(\binom{2}{1}-1\right)a_1\Rightarrow a_1=a_1$$

$$\text{For } n=2\Rightarrow\binom{1}{0}\cdot\binom{2}{1}a_1+\binom{1}{1}\cdot\binom{2}{2}a_2=\left(\binom{4}{2}-1\right)a_2\Rightarrow 2a_1+a_2=5a_2$$

$$4a_2=2a_1\Rightarrow a_2=\frac{a_1}{2}$$

$$\text{For } n=3\Rightarrow a_3=\frac{a_1}{3}$$

Using mathematical induction after $n\in\mathbb{N}, n\ge 1$, we prove that:

$$a_n=\frac{a_1}{n};(\forall)n\in\mathbb{N}^*$$

$$P(1): a_1=a_1;(\forall)\ n\ge 1$$

Suppose that $P(1),P(2),\dots,P(n)$ are true and we will to prove:

$$P(n+1): a_{n+1} = \frac{a_1}{n+1}$$

Using the hypothesis of the induction, replacing n with n + 1 and using the identity:

$$\sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k} = \frac{1}{n}\left(\binom{2n}{n}-1\right)$$

$$\sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}\cdot\binom{n}{k} = \sum_{k=1}^{n}\left(\frac{1}{k}\cdot\binom{n-1}{k-1}\right)\cdot\binom{n}{k} = \sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\cdot\binom{n}{k} =$$

$$= \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}^2 = \frac{1}{n}\left(\binom{2n}{n}-1\right)$$

SP.507 In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a+b}{ab}\cdot r_c \geq \frac{a+b+c}{R}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right) r_c \geq \frac{a+b+c}{R}; \qquad (1)$$

Using Pham Huu Duc's inequality:

$$x(a+b)+y(b+c)+z(c+a) \geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)};$$
$$(\forall) x,y,z,a,b,c > 0$$

$$\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right) r_c \geq 2\sqrt{(r_a r_b + r_b r_c + r_c r_a)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)}; \qquad (2)$$

$$r_a r_b + r_b r_c + r_c r_a = s^2 \text{ and } \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} = \frac{1}{2Rr}; \qquad (3)$$

From (2) and (3) it follows that:

$$\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right) r_c \geq 2\sqrt{\frac{s^2}{R^2}} = \frac{2s}{R} = \frac{a+b+c}{R}$$

Solution 2 by Marin Chirciu-Romania

$$Using\ r_a=\frac{F}{s-a},we\ get:$$

$$LHS=\sum_{cyc}\frac{b+c}{bc}r_a=\sum_{cyc}\frac{b+c}{bc}\cdot\frac{F}{s-a}=\frac{F}{abc}\sum_{cyc}\frac{a(b+c)(s-b)(s-c)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{F}{4RF}\cdot\frac{4Rrs^2}{r^2s}=\frac{s}{r}\overset{Euler}{\geq}\frac{2s}{R}=\frac{a+b+c}{R}=RHS,where$$

$$\sum_{cyc}a(b+c)(s-b)(s-c)=4Rrs^2$$

Equality holds if and only if triangle is equilateral.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a+b}{ab}.r_c=\sum_{cyc}\frac{a(b+c)}{abc}.\frac{rs}{s-a}=\frac{rs}{4Rrs}.\sum_{cyc}\frac{a(s+s-a)}{s-a}$$

$$=\frac{1}{4R}\left(s.\sum_{cyc}\frac{a-s+s}{s-a}+\sum_{cyc}a\right)=\frac{1}{4R}\left(-3s+\frac{s^2}{r^2s}\sum_{cyc}(s-b)(s-c)+2s\right)$$

$$=\frac{s}{4R}\left(\frac{4R+r}{r}-1\right)=\frac{s}{r}\overset{Euler}{\geq}\frac{2s}{R}=\frac{a+b+c}{R}$$

$$\therefore \text{ in } any\ \Delta\ ABC,\sum_{cyc}\frac{a+b}{ab}.r_c\geq\frac{a+b+c}{R},\text{"=" iff }\Delta\text{ ABC is equil}ateral\text{ (QED)}$$

Solution 4 by Tapas Das-India

$$\sum_{cyc}\frac{a+b}{ab}r_c=\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)r_c=\sum_{cyc}\frac{1}{a}(r_b+r_c)\overset{AM-GM}{\geq}\sum_{cyc}\frac{2\sqrt{r_br_c}}{a}\geq$$

$$\overset{h_a\leq\sqrt{r_br_c}}{\geq}4F\sum_{cyc}\frac{1}{a^2}\geq 4F\cdot\sum_{cyc}\frac{1}{ab}=4F\cdot\frac{a+b+c}{abc}=4F\cdot\frac{a+b+c}{4RF}=\frac{a+b+c}{R}$$

SP.508 Let $A_1A_2\dots A_n$ a convex polygon, $n\in\mathbb{N},n\geq 3$. Prove that:

$$\sum_{i=1}^{n}\left(\frac{A_i^{p+1}+1}{A_i^p+1}\right)^k\geq\frac{[(n-2)\pi+n]^k}{2^k\cdot n^{k-1}},p,k\in\mathbb{N}$$

Proposed by Radu Diaconu-Romania

Solution by proposer

Lemma: If $p\in\mathbb{N}$, then:

$$\frac{A^{p+1}+1}{A^p+1} \geq \frac{A+1}{2}$$

Proof. **We have:**

$$\frac{A^{p+1}+1}{A^p+1} \geq \frac{A+1}{2} \Leftrightarrow 2A^{p+1}+2 \geq A^{p+1}+A^p+A+1 \Leftrightarrow$$

$$A^{p+1}-A^p-(A-1) \geq 0 \Leftrightarrow A^p(A-1)-(A-1) \geq 0 \Leftrightarrow$$

$$(A-1)(A^p-1) \geq 0 \Leftrightarrow (A-1)^2(A^{p-1}+A^{p-2}+\cdots+A+1) \geq 0$$

Equality holds for $A=1$.

Using lemma and Holder's inequality, we get:

$$\sum_{i=1}^{n}\left(\frac{A_i^{p+1}+1}{A_i^p+1}\right)^k \geq \sum_{k=1}^{n}\left(\frac{A_i+1}{2}\right)^k \geq \frac{1}{n^2}\left(\sum_{k=1}^{n}\frac{A_i+1}{2}\right)^k =$$

$$= \frac{1}{n^{k-1}}\left(\frac{1}{2}\sum_{i=1}^{n}A_i+\frac{1}{2}\sum_{i=1}^{n}1\right)^k = \frac{[(n-2)\pi+n]^k}{2^k \cdot n^{k-1}}, p,k \in \mathbb{N}$$

For $k=1$, inequality is obviously true from lemma, and for $k=0$ we obtain equality $n = n$.

Equality holds if and only of $k=0$.

SP.509 Let ABC be an acute triangle. Prove that:

$$\left(9\sqrt{3}\cot^3 A+2\right)\left(9\sqrt{3}\cot^3 B+2\right)\left(9\sqrt{3}\cot^3 C+2\right) \geq 125$$

Proposed by George Apostolopoulos-Greece

Solution 1 by proposer

$$We\ know\ that:\ \cot A\cot B+\cot B\cot C+\cot C\cot A=1.$$

$$Setting: x=\sqrt{3}\cot A, y=\sqrt{3}\cot B, z=\sqrt{3}\cot C, we\ have\ xy+yz+zx=3.$$

$$Using\ the\ Holder's\ inequality, we\ have:$$

$$(9x^3+6)(9y^3+6)(9z^3+6) =$$

$$= \left(4x^3+4x^3+4+x^3+1+1\right)\left(4y^3+4+4y^3+1y^3+1\right)\left(4+4z^3+4z^3+1+1+z^3\right) \geq$$

$$\geq (4xy+4zx+4yz+x+y+z)^3 = (12+x+y+z)^3 \geq$$

$$\geq\left(12+\sqrt{3(xy+yz+zx)}\right)^3=15^3$$

$$So, 3(3x^3+2)\cdot 3(3y^3+2)\cdot 3(3z^3+2)\geq 15^3$$

Namely,

$$(9\sqrt{3}\cot^3 A+2)(9\sqrt{3}\cot^3 B+2)(9\sqrt{3}\cot^3 C+2)\geq 125$$

Equality holds if and only if ΔABC is equilateral.

Solution 2 by Tapas Das-India

$$\prod_{cyc}(9\sqrt{3}\cot^3 A+2)=8\prod_{cyc}\left(\frac{9\sqrt{3}\cot^3 A}{2}+1\right)=$$

$$=8\left[1+\frac{9\sqrt{3}}{2}\left(\sum_{cyc}\cot^3 A\right)+\frac{243}{4}\left(\sum_{cyc}\cot^3 A\cot^3 B\right)+\frac{729\cdot 3\sqrt{3}}{8}\prod_{cyc}\cot^3 A\right]=$$

$$=8\left[1+\frac{9\sqrt{3}}{2}\left(\sum_{cyc}\frac{s^3}{r_a^3}\right)+\frac{243}{4}\cdot s^6\left(\sum_{cyc}\frac{1}{r_a^3r_b^3}\right)+\frac{2187\sqrt{3}}{8}\cdot\frac{s^9}{(r_ar_br_c)^3}\right]\overset{AGM}{\geq}$$

$$\geq 8\left[1+\frac{9\sqrt{3}}{2}\cdot 3\cdot\frac{s^3}{r_ar_br_c}+\frac{243}{4}\cdot s^6\cdot\frac{3}{r_a^2r_b^2r_c^2}\right]=$$

$$=8\left[1+\frac{9\sqrt{3}}{2}\cdot\frac{3s^3}{s^2r}+\frac{243}{4}\cdot\frac{3s^6}{s^4r^2}+\frac{2187\sqrt{3}}{8}\cdot\frac{s^9}{s^6r^3}\right]=$$

$$=8\left[1+\frac{9\sqrt{3}}{2}\cdot\frac{2s}{r}+\frac{243}{4}\cdot\frac{3s^2}{r^2}+\frac{2187\sqrt{3}}{8}\cdot\frac{s^3}{r^3}\right]\geq$$

$$\geq 8\left[1+\frac{9\sqrt{3}}{2}\cdot 3\cdot 3\sqrt{3}+\frac{243}{4}\cdot 3(3\sqrt{3})^2+\frac{2187\sqrt{3}}{8}\cdot 27\cdot 3\sqrt{3}\right]=$$

$$=8\left[1+\frac{81\cdot 3}{2}+\frac{243}{4}\cdot 3\cdot 9\cdot 3+\frac{2187}{8}\cdot 3\cdot 3\cdot 27\right]\geq 571887>125$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$(9\sqrt{3}\cot^3 \mathrm{A}+2)(9\sqrt{3}\cot^3 \mathrm{B}+2)(9\sqrt{3}\cot^3 \mathrm{C}+2)\geq 125$$

$$\Leftrightarrow ln\prod_{\text{cyc}}(9\sqrt{3}\cot^3 \mathrm{A}+2)\geq ln125\Leftrightarrow\sum_{\text{cyc}}ln(9\sqrt{3}\cot^3 \mathrm{A}+2)\overset{(*)}{\geq}3ln5$$

$$\text{Let } \mathrm{f}(x)=ln(9\sqrt{3}\cot^3 x+2)\ \forall\, x\in\left(0,\frac{\pi}{2}\right)$$

$$\therefore f''(x) = -\frac{\cot x.\operatorname{cosec}^2 x}{\left(9\sqrt{3}\cot^3 x+2\right)^2}(729\cot^3 x.(1+\cot^2 x) - 108\sqrt{3}(1+\cot^2 x) - 1458\cot^5 x - 108\sqrt{3}\cot^2 x)$$

$$= -\frac{27\cot x.\operatorname{cosec}^2 x}{\left(9\sqrt{3}\cot^3 x+2\right)^2}\left(27m^3 - 27m^5 - 4\sqrt{3}(1+2m^2)\right)\ (m = \cot x)$$

$$= \frac{27\cot x.\operatorname{cosec}^2 x}{\left(9\sqrt{3}\cot^3 x+2\right)^2}\left(27m^5 - 27m^3 + 4\sqrt{3}(1+2m^2)\right)$$

$$\Rightarrow f''(x) \overset{(**)}{=} \frac{27\cot x.\operatorname{cosec}^2 x}{\left(9\sqrt{3}\cot^3 x+2\right)^2}\left(4\sqrt{3} + m^2\left(27m^3 + 8\sqrt{3} - 27m\right)\right)$$

$$\text{Now, } 27m^3 + 8\sqrt{3} = 27m^3 + 4\sqrt{3} + 4\sqrt{3} \overset{A-G}{\geq} 3\sqrt[3]{27m^3.4\sqrt{3}.4\sqrt{3}}$$

$$= 18\sqrt[3]{6}m \overset{?}{>} 27m \Leftrightarrow 2.\sqrt[3]{6} \overset{?}{>} 3 \Leftrightarrow 8.6 \overset{?}{>} 27 \to \text{true} \Rightarrow 27m^3 + 8\sqrt{3} - 27m > 0$$

$$\overset{\text{via }(**)}{\Rightarrow} f''(x) > 0 \Rightarrow f(x) = ln\left(9\sqrt{3}\cot^3 x + 2\right) \text{ is convex on } \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \sum_{cyc} ln\left(9\sqrt{3}\cot^3 A + 2\right) \overset{Jensen}{\geq} 3ln\left(9\sqrt{3}\cot^3\frac{\pi}{3} + 2\right) = 3ln\left(\frac{9\sqrt{3}}{3\sqrt{3}} + 2\right) = 3ln5$$

$\Rightarrow (*)$ is true $\therefore$ in acute Δ ABC, $\left(9\sqrt{3}\cot^3 A + 2\right)\left(9\sqrt{3}\cot^3 B + 2\right)\left(9\sqrt{3}\cot^3 C + 2\right) \geq 125$, " = " iff Δ ABC is equilateral (QED)

Solution 4 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since ABC is an acute triangle, then we have : $\cot A, \cot B, \cot C > 0$.

By Hölder's inequality, we have :

$$\left(\frac{9\sqrt{3}}{2}\cot^3 A + \frac{9\sqrt{3}}{2}\cot^3 A + \frac{3}{2} + \frac{1}{2}\right)\left(\frac{9\sqrt{3}}{2}\cot^3 B + \frac{3}{2} + \frac{9\sqrt{3}}{2}\cot^3 B + \frac{1}{2}\right)\left(\frac{3}{2} + \frac{9\sqrt{3}}{2}\cot^3 C + \frac{9\sqrt{3}}{2}\cot^3 C + \frac{1}{2}\right) \geq \left(\frac{9}{2}\cot A\cot B + \frac{9}{2}\cot A\cot C + \frac{9}{2}\cot B\cot C + \frac{1}{2}\right)^3 =$$

$$= \left[\frac{9}{2}(\cot A\cot B + \cot B\cot C + \cot C\cot A) + \frac{1}{2}\right]^3 = \left(\frac{9}{2}.1 + \frac{1}{2}\right)^3 = 125.$$

Therefore,

$$\left(9\sqrt{3}\cot^3 A + 2\right)\left(9\sqrt{3}\cot^3 B + 2\right)\left(9\sqrt{3}\cot^3 C + 2\right) \geq 125.$$

Equality holds iff ΔABC is equilateral.

SP.510 Let ABC be an arbitrary triangle having the sides a, b, c. Denote by m_a, s_a the lengths of the median and the symmedian corresponding to the side a, and the analogs. Let ω Brocard's angle and M be the set

$$M = \left\{\frac{m_a}{s_a}, \frac{m_b}{s_b}, \frac{m_c}{s_c}\right\}. \text{ Prove that:}$$

$$\frac{2R}{r}\max M \geq \frac{1}{\sin^2\omega} \geq \frac{2R}{r}\min M$$

Proposed by Vasile Jiglău-Romania

Solution 1 by proposer

$$Since: \frac{m_a}{s_a} = \frac{b^2+c^2}{2bc} \text{ and analogs,}$$

$$\text{the inequality from enuciacion is equivalent to:}$$

$$\frac{R}{r}\max M' \geq \frac{1}{\sin^2\omega} \geq \frac{R}{r}\min M', \text{where } M' = \left\{\frac{a}{b}+\frac{b}{a}, \frac{b}{c}+\frac{c}{b}, \frac{c}{a}+\frac{a}{c}\right\}$$

$$a) \text{ Suppose thet the sides of the triangle verify: } c \geq b \geq a; (1). \text{ We have:}$$

$$\frac{c}{a}+\frac{a}{c} \geq \frac{a}{b}+\frac{b}{a} \Leftrightarrow bc^2+a^2b \geq a^2c+cb^2 \Leftrightarrow$$

$$(bc-a^2)(c-b) \geq 0, \text{ and in a similar way:}$$

$$\frac{c}{a}+\frac{a}{c} \geq \frac{b}{c}+\frac{c}{b}. \text{ We conclude that, under hypothesis (1), } \max M = \frac{a}{c}+\frac{c}{a}.$$

$$\text{We have to prove that: } \frac{R}{r}\left(\frac{a}{c}+\frac{c}{a}\right) \geq \frac{1}{\sin^2\omega}. \text{ With } \frac{R}{r} = \frac{abcs}{4F^2}, \frac{1}{\sin\omega} = \frac{\sqrt{\sum a^2b^2}}{2F},$$

$$\text{thus inequality becomes equivalent to:}$$

$$\frac{abcs}{4F^2}\cdot\frac{a^2+c^2}{ac} \geq \frac{a^2b^2+b^2c^2+c^2a^2}{4F^2} \Leftrightarrow$$

$$b(a+b+c)(a^2+c^2) \geq 2(a^2b^2+b^2c^2+c^2a^2) \Leftrightarrow$$

$$(b-a)(a+c-b)[c(c-a)+a(c-b)]+a(c-b)[b(b-a)+c(c-a)] \geq 0,$$

$$\text{which is true from (1)}$$

$$b) \text{ As above, } \frac{1}{\sin^2\omega} \geq \frac{R}{r}\left(\frac{a}{b}+\frac{b}{a}\right) \Leftrightarrow$$

$$f(a,b,c) = 2(a^2b^2+b^2c^2+c^2a^2) - c(a+b+c)(a^2+b^2) \geq 0; \quad (2)$$

$$and\ \frac{1}{\sin^2\omega}\ge\frac{R}{r}\left(\frac{c}{b}+\frac{b}{c}\right)\Leftrightarrow$$

$$g(a,b,c)=4(a^2b^2+b^2c^2+c^2a^2)-(a+b+c)[a(b^2+c^2)+c(a^2+b^2)]\ge 0;\ \ (3)$$

We prove that:

$$f(a,b,c)+g(a,b,c)\ge 0\Leftrightarrow$$

$$4(a^2b^2+b^2c^2+c^2a^2)-(a+b+c)[a(b^2+c^2)+c(a^2+b^2)]\ge 0$$

Since in any triangle holds:

$$8(a^2b^2+b^2c^2+c^2a^2)\ge(a+b+c)(a+b)(b+c)(c+a)$$

it is enough to prove:

$$(a+b)(b+c)(c+a)\ge 2[a(b^2+c^2)+c(a^2+b^2)]\Leftrightarrow$$

$$(c-a)(b-a)(a+c)\ge 0\ which\ is\ true\ from\ (1).$$

It results that at least one of $f(a,b,c)$ *and* $g(a,b,c)$ *is positive, therefore at least one of the inequalities* (2) *and* (3) *holds, and from here it results the conclusion.*

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ m_1=maxM\ \ and\ \ m_2=minM.$$

$$We\ have:\ \frac{m_a}{s_a}=\frac{b^2+c^2}{2bc}\ \ (and\ analogs)$$

$$Then:\ \ m_1\ge\frac{b^2+c^2}{2bc}\ge m_2\ \ or\ \ 2bcm_1\ge b^2+c^2\ge 2bcm_2\ \ (and\ analogs)$$

$$And\ since:\ \frac{1}{\sin^2\omega}=\frac{a^2b^2+b^2c^2+c^2a^2}{4F^2}=\frac{1}{8F^2}\sum_{cyc}a^2(b^2+c^2),\ \ then:$$

$$\frac{1}{8F^2}\sum_{cyc}a^2.2bcm_1\ge\frac{1}{\sin^2\omega}\ge\frac{1}{8F^2}\sum_{cyc}a^2.2bcm_2.$$

$$And\ since:\ \sum_{cyc}a^2bc=abc(a+b+c)=4RF.2s=\frac{8RF^2}{r},then:$$

$$\frac{2R}{r}maxM\ge\frac{1}{\sin^2\omega}\ge\frac{2R}{r}minM,\ \ as\ desired.$$

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UNDERGRADUATE PROBLEMS

UP.496 Prove that:

$$\int_0^1\int_0^1 \frac{\log(1+x^2)\log(1+y)}{xy^{\frac{3}{2}}}dxdy = \frac{\zeta(2)}{4}(\pi - 2\log(2))$$

Proposed by Said Attaoui – Oran – Algeria

Solution 1 by proposer

We have

$$\int_0^1\int_0^1 \frac{\log(1+x^2)\log(1+y)}{xy^{\frac{3}{2}}}dxdy = \left(\int_0^1 \frac{\log(1+x^2)}{x}dx\right)\left(\frac{\log(1+y)}{y^{\frac{3}{2}}}dy\right)$$

$$= 2\left(\int_0^1 \frac{\log(1+x^2)}{x}dx\right)\left(\frac{\log(1+y^2)}{y^2}dy\right)$$

{by replacing y by y^2}

Now, applying the geometric series $-\log(1-t) = \sum_{n=1}^{\infty}\frac{t^n}{n}$ to have

$$\int_0^1 \frac{\log(1+x^2)}{x} = \int_0^1 \frac{1}{x}\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{n}\right)dx$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\int_0^1 x^{2n-1}\,dx\right) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\zeta(2)}{4}$$

By the property $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p} = -(1-2^{2-p})\zeta(p), p > 1$

Similarly

$$\int_0^1 \frac{\log(1+y^2)}{y^2}dx = \int_0^1 \frac{1}{y^2}\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{y^{2n}}{n}\right)dy$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\int_0^1 y^{2n-2}\,dy\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(2n-1)}$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 2\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)}}_{=\frac{\pi}{4}} - \underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}}_{\log(2)} = \frac{\pi}{2} - \log(2)$$

Thereby

$$\left(\int_0^1 \frac{\log(1+x^2)}{x}dx\right)\left(\int_0^1 \frac{\log(1+y^2)}{y^2}dy\right) = \frac{\zeta(2)}{4}\left(\frac{\pi}{2} - \log(2)\right)$$

Finally

$$\int_0^1 \int_0^1 \frac{\log(1+x^2)\log(1+y)}{xy^{\frac{3}{2}}}dxdy = \frac{\zeta(2)}{4}(\pi - 2\log(2))$$

Proof of

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} = \frac{\pi}{4}$.

We have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 x^{2n-2}\,dx = \int_0^1 \sum_{n=1}^{\infty} (-1)^{n-1} x^{2(n-1)} dx$$

$$= \int_0^1 \sum_{n=1}^{\infty} (-x^2)^n\,dx = \int_0^1 \frac{1}{1+x^2}dx = \arctan(1) = \frac{\pi}{4}$$

Solution 2 by Vincent Nguyen-USA

$$\int_0^1 \int_0^1 \frac{\log(1+x^2)\log(1+y)}{xy^{\frac{3}{2}}}dxdy = \int_0^1 \int_0^1 \frac{1}{xy^{\frac{3}{2}}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{y^m}{m} dxdy$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^1 \frac{x^{2n}}{nx}dx \sum_{m=1}^{\infty} (-1)^{m+1} \int_0^1 \frac{y^m}{my^{\frac{3}{2}}}dy =$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^1 \frac{x^{2n-1}}{n}dx \sum_{m=1}^{\infty} (-1)^{m+1} \int_0^1 \frac{y^{m-\frac{3}{2}}}{m}dy =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cdot 2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m(2m-1)} = \frac{1}{2}\eta(2) \cdot 2\left(2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1} - \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\right) =$$

$$= \frac{1}{4}\zeta(2) \cdot 2(2arctan1 - log2) = \frac{1}{4}\zeta(2)(\pi - 2log2)$$

UP.497 Prove that:

$$F(x) = \int \frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx = \frac{1}{2}Li_2\left(-\frac{1}{x^2}\right) + \frac{\log\left(1+\frac{1}{x^2}\right)}{x} - \frac{2}{x} + 2\arctan\left(\frac{1}{x}\right)$$

Deduce

$$\int_1^{\infty} \frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx$$

Proposed by Said Attaoui – Oran – Algeria

Solution 1 by proposer

It is easy to verify that $\frac{\partial}{\partial x}\left(\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)\right)=\frac{\log\left(1+\frac{1}{x^2}\right)}{x}$

Then, we have by integrating by parts

$$\int\frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx=\int\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)dx-\int\frac{1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)-\left[-\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)+\int\frac{1}{x}\left(\frac{-\frac{2}{x^3}}{1+\frac{1}{x^2}}\right)dx\right]$$

$$=\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)+\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)+2\int\frac{1}{x^2(1+x^2)}dx$$

$$=\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)+\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)+2\int\frac{1}{x^2}dx-2\int\frac{1}{1+x^2}dx$$

$$=\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)+\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)+2\int\frac{1}{x^2}dx+2\int\frac{-\frac{1}{x^2}}{1+\left(\frac{1}{x}\right)^2}dx$$

$$=\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)+\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)-\frac{2}{x}+2\arctan\left(\frac{1}{x}\right)$$

So, we deduce that

$$\int_1^\infty\frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx=F(\infty)-F(1)=-\frac{1}{2}Li_2(-1)-\log(2)+2-2\arctan(1)$$

$$=\frac{\zeta(2)}{4}+2-\frac{\pi}{2}-\log(2)$$

Since

$$\arctan(1)=\frac{\pi}{4},\quad Li_2(-1)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\zeta(2)}{2}$$

By the property:

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}=-(1-2^{1-p})\zeta(p),p>1$$

Solution 2 by Vincent Nguyen-USA

$$F(x)=\int\frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx=$$

$$=\int\frac{1}{x}\log\left(1+\frac{1}{x^2}\right)dx-\int\frac{1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx=I_1-I_2$$

$$I_1\overset{u=-\frac{1}{x^2}}{\cong}\int\frac{1}{x}\log(1-u)\cdot\frac{x^3}{2}du=\frac{1}{2}\int\frac{-\log(1-u)}{u}du=\frac{1}{2}Li_2(u)=\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)+C$$

$$I_2=\int\frac{1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx\overset{IPB}{\cong}-\frac{\log\left(1+\frac{1}{x^2}\right)}{x}-2\int\frac{1}{x^2(x^2+1)}dx=$$

$$=-\frac{\log\left(1+\frac{1}{x^2}\right)}{x}-2\int\left(\frac{1}{x^2}-\frac{1}{x^2+1}\right)dx=-\frac{\log\left(1+\frac{1}{x^2}\right)}{x}+\frac{2}{x}-2\arctan x+C=$$

$$=-\frac{\log\left(1+\frac{1}{x^2}\right)}{x}+\frac{2}{x}-2\left(\frac{\pi}{2}-\arctan x\right)+C=-\frac{\log\left(1+\frac{1}{x^2}\right)}{x}+\frac{2}{x}-2\arctan\frac{1}{x}+C$$

$$F(x)=\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)-\left(-\frac{\log\left(1+\frac{1}{x^2}\right)}{x}+\frac{2}{x}-2\arctan\frac{1}{x}\right)=$$

$$=\frac{1}{2}Li_2\left(-\frac{1}{x^2}\right)+\frac{\log\left(1+\frac{1}{x^2}\right)}{x}-\frac{2}{x}+2\arctan\frac{1}{x}+C$$

$$\int\frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx=\lim_{R\to\infty}\int_0^R\frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx=$$

$$=\lim_{R\to\infty}\left(\frac{1}{2}Li_2\left(-\frac{1}{R^2}\right)-\left(-\frac{\log\left(1+\frac{1}{R^2}\right)}{R}+\frac{2}{R}-2\arctan\frac{1}{R}\right)\right)=$$

$$=-\left(\frac{1}{2}\left(-\eta(2)+\log 2-2+\frac{\pi}{2}\right)\right)=2-\log 2-\frac{\pi}{2}+\frac{\pi^2}{24}$$

UP.498 Prove that:

$$\int_0^1\int_0^1\frac{(1+2x-x^2)(1-2y+y^2)}{\sqrt{x^2y^2-2yx^2-2xy^2+4xy}}dxdy=\frac{9}{8}\zeta(2)$$

Proposed by Said Attaoui-Oran-Algerie

Solution 1 by proposer

Remark that: $x^2y^2-2yx^2-2xy^2+4xy=(4-2x-2y+xy)xy=$

$=(2-x)(2-y)xy$**, we have:**

$$\int_0^1\int_0^1\frac{(1+2x-x^2)(1-2y+y^2)}{\sqrt{x^2y^2-2yx^2-2xy^2+4xy}}dxdy=\int_0^1\int_0^1\frac{\left(1+x(2-x)\right)(1-y)^2}{\sqrt{(2-x)(2-y)xy}}dxdy$$

$$\text{Let: } J=\int_0^1\int_0^1\frac{(1+2x-x^2)(1-2y+y^2)}{\sqrt{x^2y^2-2yx^2-2xy^2+4xy}}dxdy$$

So, by replacing x by $(1-x)$ and y by $(1-y)$, we obtain:

$$J=\int_0^1\int_0^1\frac{\left(1+x(2-x)\right)(1-y)^2}{\sqrt{(2-x)(2-y)xy}}dxdy=\int_0^1\int_0^1\frac{(2-x^2)y^2}{\sqrt{(1-x^2)(1-y^2)}}dxdy=$$

$$=\left[\int_0^1\frac{2-x^2}{\sqrt{1-x^2}}dx\right]\left[\int_0^1\frac{y^2}{\sqrt{1-y^2}}dy\right]=\left[\int_0^1\frac{2-x^2}{\sqrt{1-x^2}}dx\right]\left[\int_0^1\frac{x^2}{\sqrt{1-x^2}}dx\right]=$$

$$=\left[\int_0^1\left(\frac{1}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right)dx\right]\left[\int_0^1\left(\frac{1}{\sqrt{1-x^2}}-\sqrt{1-x^2}\right)dx\right]=$$

$$=\left[\int_0^1\frac{1}{\sqrt{1-x^2}}dx+\int_0^1\sqrt{1-x^2}dx\right]\left[\int_0^1\frac{1}{\sqrt{1-x^2}}dx-\int_0^1\sqrt{1-x^2}dx\right]=$$

$$=\left(\int_0^1\frac{1}{\sqrt{1-x^2}}dx\right)^2-\left(\int_0^1\sqrt{1-x^2}dx\right)^2=$$

$$=\left([\sin^{-1}x]_0^1\right)^2-\left(\frac{1}{2}\left[x\sqrt{1-x^2}+\sin^{-1}x\right]_0^1\right)^2=$$

$$=(\sin^{-1}x)^2-\left(\frac{1}{2}\sin^{-1}x\right)^2=\frac{\pi^2}{4}-\frac{\pi^2}{16}=\frac{18}{16}\cdot\frac{\pi^2}{6}=\frac{9}{8}\zeta(2)$$

Solution 2 by Vincent Nguyen-USA

$$\Omega=\int_0^1\int_0^1\frac{(1+2x-x^2)(1-2y+y^2)}{\sqrt{x^2y^2-2yx^2-2xy^2+4xy}}dxdy=$$

$$=-\int_0^1\int_0^1\frac{((x-1)^2-2)(y-1)^2}{\sqrt{x(2-x)\cdot y(2-y)}}dxdy=$$

$$= -\int_0^1 \frac{(y-1)^2}{\sqrt{y(2-y)}}dy \cdot \int_0^1 \frac{(x-1)^2-2}{\sqrt{x(2-x)}}dx = -I_1 \cdot I_2$$

$$I_1 = \int_0^1 \frac{(y-1)^2}{\sqrt{y(2-y)}}dy \overset{y=sint}{\cong} \int_0^{\frac{\pi}{2}} \frac{sin^2t \cdot costdt}{\sqrt{1-sin^2t}} = \int_0^{\frac{\pi}{2}} sin^2tdt = \frac{\pi}{4}$$

$$I_2 = \int_0^1 \frac{(x-1)^2-2}{\sqrt{x(2-x)}}dx \overset{x-1=t}{\cong} \int_0^1 \frac{t^2-2}{\sqrt{1-t^2}}dt = I_1 - 2\int_0^1 \frac{1}{\sqrt{1-t^2}}dt = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$$

$$\Omega = -I_1 \cdot I_2 = \frac{\pi}{4} \cdot \frac{3\pi}{4} = \frac{9}{8} \cdot \frac{\pi^2}{6} = \frac{9}{8}\zeta(2)$$

UP.499 If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{1+xy} \leq \frac{2(b-a)^2}{(1+a)(1+b)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\frac{1}{1+xy} = \frac{x+y}{(x+y)(1+xy)} = \frac{x}{(x+y)(1+xy)} + \frac{y}{(x+y)(1+xy)} =$$

$$= \frac{1}{\left(1+\frac{y}{x}\right)(1+xy)} + \frac{1}{\left(1+\frac{x}{y}\right)(1+xy)} =$$

$$= \frac{1}{\left(1^2+\left(\sqrt{\frac{y}{x}}\right)^2\right)\left(1^2+\left(\sqrt{xy}\right)^2\right)} + \frac{1}{\left(1^2+\left(\sqrt{\frac{x}{y}}\right)^2\right)\left(1^2+\left(\sqrt{xy}\right)^2\right)} \overset{CBS}{\leq}$$

$$\leq \frac{1}{\left(1\cdot 1+\sqrt{\frac{y}{x}}\cdot\sqrt{xy}\right)^2} + \frac{1}{\left(1\cdot 1+\sqrt{\frac{x}{y}}\cdot\sqrt{xy}\right)^2} =$$

$$= \frac{1}{(1+y)^2} + \frac{1}{(1+x)^2}$$

$$\int_a^b \int_a^b \frac{dxdy}{1+xy} \leq \int_a^b \int_a^b \frac{1}{(1+y)^2}dxdy + \int_a^b \int_a^b \frac{1}{(1+x)^2}dxdy =$$

$$= 2\int_a^b dx \int_a^b \frac{dy}{(1+y)^2} = 2(b-a)\left(\frac{-1}{1+b} — 11 + a..=\right.$$

$$=-2(b-a)\cdot\frac{1+a-1-b}{(1+a)(1+b)}=\frac{2(b-a)^2}{(1+a)(1+b)}$$

$$\text{Equality holds for } a=b.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } x,y>0. \text{ We have :}$$

$$\frac{1}{1+xy}-\left(\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}\right)=\frac{(1+x)^2(1+y)^2-(1+xy)[(1+x)^2+(y+1)^2]}{(1+xy)(1+x)^2(1+y)^2}=$$

$$=\frac{x^2y^2+2xy-x^3y-xy^3-1}{(1+xy)(1+x)^2(1+y)^2}=-\frac{xy(x-y)^2+(xy-1)^2}{(1+xy)(1+x)^2(1+y)^2}\le 0.$$

$$\text{Then : }\frac{1}{xy+1}\le\frac{1}{(x+1)^2}+\frac{1}{(y+1)^2},\quad \forall x,y>0,$$

$$\text{Therefore,}$$

$$\int_a^b\int_a^b\frac{dxdy}{1+xy}\le\int_a^b\int_a^b\left(\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}\right)dxdy=2(b-a)\left[-\frac{1}{1+x}\right]_a^b=$$

$$=2(b-a)\left(\frac{1}{1+a}-\frac{1}{1+b}\right)=\frac{2(b-a)^2}{(1+a)(1+b)}.$$

$$\text{Equality holds iff } a=b.$$

Solution 3 by Tapas Das-India

$$\text{Lemma: For all } x,y>0 \text{ holds: }\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}\ge\frac{1}{1+xy}$$

$$\text{Proof. We have: }\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}-\frac{1}{1+xy}=\frac{xy(x^2+y^2)-x^2y^2-2xy+1}{(1+x)^2(1+y)^2(1+xy)}=$$

$$=\frac{xy(x-y)^2+(xy-1)^2}{(1+x)^2(1+y)^2(1+xy)}\ge 0\Rightarrow\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}-\frac{1}{1+xy}\ge 0 \text{ and hence,}$$

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}\ge\frac{1}{1+xy};(\forall)x,y>0$$

$$\text{Now, we get: }\int_a^b\int_a^b\frac{dxdy}{1+xy}\le\int_a^b\int_a^b\frac{dxdy}{(1+x)^2}+\int_a^b\int_a^b\frac{dxdy}{(1+y)^2}=$$

$$=2\left(-\frac{1}{1+x}\right)\Big|_a^b y|_a^b=2\left(\frac{1}{1+a}-\frac{1}{1+b}\right)(b-a)=\frac{2(b-a)^2}{(1+a)(1+b)}$$

UP.500 Let $0 < a < b$ and $f: [a,b] \to \mathbb{R}$. If f −differentiable on $[a,b]$ and $f(a) = f(b)$, then $(\exists)\ c_1, c_2 \in (a,b)$ such that $af(c_1) + bf(c_2) = 0$.

Proposed by Marian Ursărescu-Romania

Solution by proposer

We apply Lagrange's Theorem on the interval $\left[a, \frac{2ab}{a+b}\right]$:

$$(\exists)\ c_1 \in \left(a, \frac{2ab}{a+b}\right) \text{ such that } \frac{f\left(\frac{2ab}{a+b}\right) - f(a)}{\frac{2ab}{a+b} - a} = f'(c_1), \text{then}$$

$$\frac{f\left(\frac{2ab}{a+b}\right) - f(a)}{\frac{a(b-a)}{a+b}} = f'(c_1) \Rightarrow f\left(\frac{2ab}{a+b}\right) - f(a) = \frac{a(b-a)}{a+b} \cdot f'(c_1); \quad (1)$$

We apply Lagrange's Theorem on the interval $\left[\frac{2ab}{a+b}, b\right]$:

$$(\exists)\ c_2 \in \left(\frac{2ab}{a+b}, b\right) \text{ such that } \frac{f(b) - f\left(\frac{2ab}{a+b}\right)}{b - \frac{2ab}{a+b}} = f'(c_2), \text{then}$$

$$f(b) - f\left(\frac{2ab}{a+b}\right) = \frac{b(b-a)}{a+b} f'(c_2); \quad (2)$$

$$\text{From (1) and (2)}: f(b) - f(a) = \frac{b-a}{a+b}\left(af'(c_1) + bf'(c_2)\right)$$

$$\frac{b-a}{a+b}\left(f'(c_1) + bf'(c_2)\right) = 0 \Rightarrow af'(c_1) + bf'(c_2) = 0$$

UP.501 Determine all functions $f: \mathbb{R} \to (0, \infty)$ such that

$$f(x) \cdot f(3x) \cdot f(9x) \cdot f(27x) = 3^x, (\forall)\ x \in \mathbb{R}$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$\log_3 f(x) + \log_3 f(3x) + \log_3 f(9x) + \log_3 f(27x) = \log_3 3^x$$

$$\text{Let}: g(x) = \log_3 f(x), \text{then}: g(x) + g(3x) + g(9x) + g(27x) = x$$

$$x \to 3x: \ g(3x) + g(9x) + g(27x) + g(81x) = 3x$$

$$g(81x) - g(x) = 2x \Rightarrow$$

$$g(x)-g\left(\frac{x}{81}\right)=\frac{2}{81}x$$

$$g\left(\frac{x}{81}\right)-g\left(\frac{x}{81^2}\right)=\frac{2}{81^2}x$$

… … … … … … … … … … … …

$$g\left(\frac{x}{81^{n-1}}\right)-g\left(\frac{x}{81^n}\right)=\frac{2}{81^n}x$$

By adding:

$$g(x)-g\left(\frac{x}{81^n}\right)=\frac{2}{81}x\left(1+\frac{1}{81}+\cdots+\frac{1}{81^{n-1}}\right)$$

$$\lim_{n\to\infty}\left(g(x)-g\left(\frac{x}{81^n}\right)\right)=\lim_{n\to\infty}\frac{2}{81}x\cdot\frac{\frac{1}{81^n}-1}{\frac{1}{81}-1}$$

$$g(x)-g(0)=\frac{2}{81}x\cdot\frac{1}{\frac{80}{81}}=\frac{1}{40}x,\text{but } g(0)=0,\text{hence}$$

$$g(x)=\frac{x}{40}\Rightarrow\log_3 f(x)=\frac{x}{40}\Rightarrow f(x)=3^{\frac{x}{40}}$$

UP.502 Let $A_1A_2\dots A_n$ a convex polygon, $n\ge 3, n\in\mathbb{N}$. Prove that:

$$\begin{vmatrix}-1 & a & a & \dots & a\\ a & -1 & a & \dots & a\\ a & a & -1 & \cdots & a\\ \dots & \dots & \dots & \cdots & \dots\\ a & a & a & \cdots & -1\end{vmatrix}\cdot\begin{vmatrix}-1 & b & b & \dots & b\\ b & -1 & b & \dots & b\\ b & b & -1 & \cdots & b\\ \dots & \dots & \dots & \cdots & \dots\\ b & b & b & \cdots & -2\end{vmatrix}=$$

$$=(2p+1)^{n-1}[(n-1)2p-1][(n-2)\pi+1]^{n-1}[(n-1)(n-2)\pi-1].$$

where $a=a_1+a_2+\cdots+a_n=2p, (n-1)2p-1>0, b=\widehat{A_1}+\widehat{A_2}+\cdots+\widehat{A_n}$ and the order of the determinants is n.

Proposed by Radu Diaconu-Romania

Solution by proposer

$$Let\ us\ denote: d_1=\begin{vmatrix}-1 & a & a & \dots & a\\ a & -1 & a & \dots & a\\ a & a & -1 & \cdots & a\\ \dots & \dots & \dots & \cdots & \dots\\ a & a & a & \cdots & -1\end{vmatrix}\ and\ d_2=\begin{vmatrix}-1 & b & b & \dots & b\\ b & -1 & b & \dots & b\\ b & b & -1 & \cdots & b\\ \dots & \dots & \dots & \cdots & \dots\\ b & b & b & \cdots & -2\end{vmatrix}$$

$$d_1 \overset{l_1+(l_2,l_3,\dots,l_n)}{=} [(n-1)a-1]\cdot\begin{vmatrix}1 & 1 & 1 & \dots & 1\\ a & -1 & a & \dots & a\\ a & a & -1 & \cdots & a\\ \dots & \dots & \dots & \cdots & \dots\\ a & a & a & \cdots & -1\end{vmatrix}=$$

$$=[(n-1)a-1]\cdot\begin{vmatrix}1 & 1 & 1 & \dots & 1\\ 0 & -1-a & 0 & \dots & 0\\ 0 & 0 & -1-a & \cdots & 0\\ \dots & \dots & \dots & \cdots & \dots\\ 0 & 0 & 0 & \cdots & -1-a\end{vmatrix}=$$

$$=[(n-1)a-1)]\cdot\begin{vmatrix}-1-a & 0 & 0 & \dots & 0\\ 0 & -1-a & 0 & \dots & 0\\ 0 & 0 & -1-a & \cdots & 0\\ \dots & \dots & \dots & \cdots & \dots\\ 0 & 0 & 0 & \cdots & -1-a\end{vmatrix}=$$

$$=[(n-1)a-1](-1-a)^{n-1}=(-1)^{n-1}(1+a)^{n-1}[(n-1)a-1]$$

Therefore,

$$d_1=\begin{vmatrix}-1 & a & a & \dots & a\\ a & -1 & a & \dots & a\\ a & a & -1 & \cdots & a\\ \dots & \dots & \dots & \cdots & \dots\\ a & a & a & \cdots & -1\end{vmatrix}=(-1)^{n-1}(1+a)^{n-1}[(n-1)a-1];(1)$$

In a similar way, we get:

$$d_2=\begin{vmatrix}-1 & b & b & \dots & b\\ b & -1 & b & \dots & b\\ b & b & -1 & \cdots & b\\ \dots & \dots & \dots & \cdots & \dots\\ b & b & b & \cdots & -2\end{vmatrix}=(-1)^{n-1}(1+b)^{n-1}[(n-1)b-1];\quad(2)$$

By multipliyng (1) and (2), it follows:

$$\begin{vmatrix}-1 & a & a & \dots & a\\ a & -1 & a & \dots & a\\ a & a & -1 & \cdots & a\\ \dots & \dots & \dots & \cdots & \dots\\ a & a & a & \cdots & -1\end{vmatrix}\cdot\begin{vmatrix}-1 & b & b & \dots & b\\ b & -1 & b & \dots & b\\ b & b & -1 & \cdots & b\\ \dots & \dots & \dots & \cdots & \dots\\ b & b & b & \cdots & -2\end{vmatrix}=$$

$$=(2p+1)^{n-1}[(n-1)2p-1][(n-2)\pi+1]^{n-1}[(n-1)(n-2)\pi-1].$$

UP.503 Let $f:[n-1,n]\to[n,n+1]$ continuous function such that

$$\int_{n-1}^{n}\left(1+xf'(x)\right)dx\le nf(n)-(n-1)f(n-1),\text{ then prove:}$$

$$\int_{n-1}^{n}\frac{dx}{f(x)}\le\frac{2}{n+1},n\in\mathbb{N}^*$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$\text{We have: } \int_{n-1}^{n} \left(1 + xf'(x)\right) dx \le nf(n) - (n-1)f(n-1) \Leftrightarrow$$

$$nf(n) - (n-1)f(n-1) - \int_{n-1}^{n} xf'(x)\, dx \ge 1$$

$$\text{But: } \int_{n-1}^{n} xf'(x)\, dx \stackrel{IBP}{=} xf(x)|_{n-1}^{n} - \int_{n-1}^{n} f(x)\, dx$$

$$\int_{n-1}^{n} xf'(x)\, dx = nf(n) - (n-1)f(n-1) - \int_{n-1}^{n} f(x)\, dx$$

$$\int_{n-1}^{n} f(x)\, dx = nf(n) - (n-1)f(n-1) - \int_{n-1}^{n} xf'(x)\, dx \ge 1$$

We have:

$$\frac{(f(x)-n)(f(x)-n-1)}{f(x)} \le 0, (\forall)x \in [n-1, n], n \in \mathbb{N}^* \Leftrightarrow$$

$$\frac{f^2(x) - nf(x) - f(x) - nf(x) + n^2 + n}{f(x)} \le 0 \Leftrightarrow f(x) - (2n+1) + \frac{n(n+1)}{f(x)} \le 0 \Leftrightarrow$$

$$\frac{n(n+1)}{f(x)} \le (2n+1) - f(x) \Leftrightarrow \frac{1}{f(x)} \le \frac{2n+1}{n(n+1)} - \frac{1}{n(n+1)} f(x) \Leftrightarrow$$

$$\int_{n-1}^{n} \frac{1}{f(x)} dx \le \frac{2n+1}{n(n+1)} \int_{n-1}^{n} dx - \frac{1}{n(n+1)} \int_{n-1}^{n} f(x)\, dx \Leftrightarrow$$

$$\int_{n-1}^{n} \frac{1}{f(x)} dx \le \frac{2n+1}{n(n+1)} \cdot \left(n - (n-1)\right) - \frac{1}{n(n+1)} \Leftrightarrow$$

$$\int_{n-1}^{n} \frac{1}{f(x)} dx \le \frac{2n+1}{n(n+1)} - \frac{1}{n(n+1)} \Leftrightarrow \int_{n-1}^{n} \frac{1}{f(x)} dx \le \frac{2}{n+1}$$

UP.504 Let $a, b, c, d > 1$ and $f: [a, b] \to [c, d]$ continuous function for which

$$(\exists)\lambda \in (a, b) \text{ such that } a\int_{a}^{\lambda} f(x)\, dx + b\int_{\lambda}^{b} f(x)\, dx \ge a + c, \text{ then prove:}$$

$$\int_{a}^{b} \frac{x}{f(x)} dx \le \left(\frac{1}{a} + \frac{1}{c}\right) \cdot \frac{b^2 - a^2 - 2}{2}$$

Proposed by Florică Anastase-Romania

Solution by proposer

Let $F:[a,b]\to\mathbb{R}, F(t)=\int_a^t f(x)\,dx$.

Because f –continuous function, the function F is derivable and

$F'(t)=f(t),(\forall)t\in[a,b]$. We have:

$$\int_a^b xf(x)\,dx \overset{IBP}{=} xF(x)|_a^b-\int_a^b F(x)\,dx=b\int_a^b f(x)\,dx-\int_a^b F(x)\,dx$$

Using M.V.T. $(\exists)\lambda\in(a,b)$, such that:

$$\int_a^b F(x)\,dx=(b-a)F(\lambda)$$

$$\int_a^b xf(x)\,dx=b\int_a^b f(x)\,dx-(b-a)F(\lambda)=$$

$$=b\left(\int_a^\lambda f(x)\,dx+\int_\lambda^b f(x)\,dx\right)-(b-a)\int_a^\lambda f(x)\,dx=a\int_a^\lambda f(x)\,dx+b\int_\lambda^b f(x)\,dx$$

So, we have:

$$a\int_a^\lambda f(x)\,dx+b\int_\lambda^b f(x)\,dx=\int_a^b xf(x)\,dx\ge a+c;\quad (1)$$

On the other hand, we have:

$$\frac{\big(c-f(x)\big)\big(ax-xf(x)\big)}{f(x)}\le 0,(\forall)x\in[a,b]\Leftrightarrow\frac{acx-cxf(x)-axf(x)+xf^2(x)}{f(x)}\le 0$$

$$\frac{acx}{f(x)}-(cx+ax)+xf(x)\le 0\Leftrightarrow\frac{acx}{f(x)}\le(a+c)x-xf(x)\Leftrightarrow$$

$$\frac{x}{f(x)}\le\frac{a+c}{ac}x-\frac{1}{ac}xf(x)\Leftrightarrow\frac{x}{f(x)}\le\left(\frac{1}{a}+\frac{1}{c}\right)x-\frac{1}{ac}xf(x)\Leftrightarrow$$

$$\int_a^b\frac{x}{f(x)}dx\le\left(\frac{1}{a}+\frac{1}{c}\right)\int_a^b x\,dx-\frac{1}{ac}\int_a^b xf(x)\,dx\overset{(1)}{\Leftrightarrow}$$

$$\int_a^b\frac{x}{f(x)}dx\le\left(\frac{1}{a}+\frac{1}{c}\right)\cdot\frac{b^2-a^2}{2}-\left(\frac{1}{a}+\frac{1}{c}\right)\Leftrightarrow$$

$$\int_a^b\frac{x}{f(x)}dx\le\left(\frac{1}{a}+\frac{1}{c}\right)\cdot\frac{b^2-a^2-2}{2}$$

UP.505 Find:

$$\Omega = \lim_{x\to\infty}\left([2x](2x)^{\frac{1}{[2x]}} - [x]x^{\frac{1}{[x]}} - \left[x+\frac{1}{2}\right]\right)$$

where $[a]$ is the greatest integer less than a.

Proposed by Cristian Miu-Romania

Solution by proposer

$$\Omega = \lim_{x\to\infty}\left([2x](2x)^{\frac{1}{[2x]}} - [x]x^{\frac{1}{[x]}} - \left[x+\frac{1}{2}\right]\right) =$$

$$= \lim_{x\to\infty}\left([2x]\left((2x)^{\frac{1}{[2x]}} - 1\right) - [x]\left(x^{\frac{1}{[x]}} - 1\right)\right)$$

$$\textit{because } [x] + \left[x+\frac{1}{2}\right] = [2x], \textit{it is easy to prove that.}$$

$$\textit{Now, we prove that: } \lim_{x\to\infty}([x]\left(x^{\frac{1}{[x]}} - \log x\right) = 0.$$

$$\textit{For } x > 1, \textit{using Lagrange's theorem for } f(x) = \log x \textit{ on } \left[x, x^{\frac{1}{[x]}}\right]$$

$$\textit{we can write } \frac{\log x^{\frac{1}{[x]}}}{x^{\frac{1}{[x]}} - 1} = \frac{1}{c_x}, \textit{where } 1 \le c_x \le x^{\frac{1}{[x]}}$$

$$\textit{So, } \frac{[x]\left(x^{\frac{1}{[x]}} - 1\right)}{\log x} = c_x \Leftrightarrow [x]\left(x^{\frac{1}{[x]}} - 1\right) - \log x = (c_x - 1)\log x$$

$$\textit{Let us write } F(x) = [x]\left(x^{\frac{1}{[x]}} - 1\right) - \log x. \textit{We obtain:}$$

$$(c_x - 1)\log x = F(x).$$

$$\textit{But } 0 \le c_x - 1 \le x^{\frac{1}{[x]}} - 1 \Leftrightarrow 0 \le (c_x - 1)\log x \le \left(x^{\frac{1}{[x]}} - 1\right)\log x$$

$$0 \le F(x) \le \left(x^{\frac{1}{[x]}} - 1\right)\log x \Leftrightarrow 0 \le F(x) \le \frac{x^{\frac{1}{[x]}} - 1}{\log x}\cdot\frac{\log^2 x}{[x]}\cdot[x] \Leftrightarrow$$

$$0 \le F(x) \le \frac{x^{\frac{1}{[x]}} - 1}{\log x^{\frac{1}{[x]}}}\cdot\frac{\log^2 x}{x}\cdot\frac{x}{[x]}$$

$$Now, \lim_{x\to\infty}\frac{x^{\frac{1}{[x]}}-1}{\log x^{\frac{1}{[x]}}}=1, because \lim_{x\to\infty}x^{\frac{1}{[x]}}=1 \ and \lim_{x\to 1}\frac{x-1}{\log x}=1$$

$$\lim_{x\to\infty}\frac{\log^2 x}{x}=0 \ and \lim_{x\to\infty}\frac{x}{[x]}=1$$

$$So, we\ obtain: \lim_{x\to\infty}\left(\frac{x^{\frac{1}{[x]}}-1}{\log x^{\frac{1}{[x]}}}\cdot\frac{\log^2 x}{x}\cdot\frac{x}{[x]}\right)=0, that\ means:$$

$$\lim_{x\to\infty}F(x)=0, so, \lim_{x\to\infty}[2x]\left((2x)^{\frac{1}{[2x]}}-1\right)-\log 2x=0$$

$$Now, we\ obtain\ the\ result:$$

$$\lim_{x\to\infty}\left([2x]\left((2x)^{\frac{1}{[2x]}}-1\right)-[x]\left(x^{\frac{1}{[x]}}-1\right)\right)=\log 2$$

$$Therefore,$$

$$\Omega=\lim_{x\to\infty}\left([2x](2x)^{\frac{1}{[2x]}}-[x]x^{\frac{1}{[x]}}-\left[x+\frac{1}{2}\right]\right)=\log 2.$$

UP.506 Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E and F be chosen on sides BC, CA and AB so that AD, BE, and CF bisect the angle of ABC.

Prove that: $\left(\frac{EF}{BC}\right)^4+\left(\frac{FD}{CA}\right)^4+\left(\frac{DE}{AB}\right)^4+\frac{3}{16}\le\frac{3}{8}\left(\frac{R}{2r}\right)^2.$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer

$$Let\ a=BC, b=CA, c=AB\ be\ the\ side\ lengths\ of\ \Delta ABC.$$

$$We\ know\ that\ AE=\frac{bc}{a+c}\ and\ AF=\frac{bc}{a+b}.$$

$$By\ the\ Law\ of\ cosines\ in\ \Delta AEF:$$

$$EF^2=AE^2+AF^2-2AE\cdot AF\cdot\cos A=$$

$$=\left(\frac{bc}{a+c}\right)^2+\left(\frac{bc}{a+b}\right)^2-2\left(\frac{bc}{a+c}\right)\cdot\left(\frac{bc}{a+b}\right)\cdot\frac{b^2+c^2-a^2}{2bc}=$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc[2bc+(b-c)^2-a^2]}{(a+b)(a+c)}=$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{a^2bc}{(a+b)(a+c)}-\frac{bc(b-c)^2[(a+b)(a+c)-bc]}{(a+b)^2(a+c)^2}\le$$

$$\le\frac{a^2bc(b+c)}{(a+b)(b+c)(c+a)}\overset{Cesaro}{\le}\frac{a^2bc(b+c)}{8abc}=\frac{a^2}{8}\left(\frac{b}{a}+\frac{c}{a}\right)$$

Then $EF^2\le\frac{a^2}{8}\left(\frac{b}{a}+\frac{c}{a}\right)$ *and analogs. Therefore,* $\frac{EF^2}{BC^2}\le\frac{1}{8}\left(\frac{b}{a}+\frac{c}{a}\right)$

Hence, $\left(\frac{EF}{BC}\right)^4\le\frac{1}{64}\left(\frac{b}{a}+\frac{c}{a}\right)^2$ *and analogs.*

It is well – known that $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}$ *(Bandila). So,* $\frac{a^2}{b^2}+\frac{b^2}{a^2}\le\frac{R^2}{r^2}-2.$

Similarly, $\frac{b^2}{c^2}+\frac{c^2}{b^2}\le\frac{R^2}{r^2}-2$ *and* $\frac{c^2}{a^2}+\frac{a^2}{c^2}\le\frac{R^2}{r^2}-2.$

Now, we have:

$$\left(\frac{EF}{BC}\right)^4+\left(\frac{FD}{CA}\right)^4+\left(\frac{DE}{AB}\right)^4\le\frac{1}{64}\left(\left(\frac{b}{a}+\frac{c}{a}\right)^2+\left(\frac{c}{b}+\frac{a}{b}\right)^2+\left(\frac{a}{c}+\frac{b}{c}\right)^2\right)=$$

$$=\frac{1}{64}\left(\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\frac{2bc}{a^2}+\frac{2ca}{b^2}+\frac{2ab}{c^2}\right)\le$$

$$\le\frac{1}{62}\left(\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\frac{b^2+c^2}{a^2}+\frac{c^2+a^2}{b^2}+\frac{a^2+b^2}{c^2}\right)=$$

$$=\frac{1}{64}\cdot2\left(\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)\right)\le\frac{1}{32}\left(\left(\frac{R^2}{r^2}-2\right)+\left(\frac{R^2}{r^2}-2\right)+\left(\frac{R^2}{r^2}-2\right)\right)=$$

$$=\frac{3}{32}\cdot\frac{R^2}{r^2}-\frac{3}{16}=\frac{3}{8}\left(\frac{R}{2r}\right)^2-\frac{3}{16}$$

Therefore,

$$\left(\frac{EF}{BC}\right)^4+\left(\frac{FD}{CA}\right)^4+\left(\frac{DE}{AB}\right)^4+\frac{3}{16}\le\frac{3}{8}\left(\frac{R}{2r}\right)^2$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ a=BC,b=CA,c=AB\ be\ the\ side\ lengths\ of\ \Delta ABC. We\ know\ that:$$

$$AE=\frac{bc}{a+c}\ and\ AF=\frac{bc}{a+b}. By\ the\ Law\ of\ cosines\ in\ \Delta AEF:$$

$$EF^2=AE^2+AF^2-2.AE.AF.\cos A=\left(\frac{bc}{a+c}\right)^2+\left(\frac{bc}{a+b}\right)^2-2\left(\frac{bc}{a+c}\right)\left(\frac{bc}{a+b}\right).\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc\left[(b-c)^2+2bc-a^2\right]}{(a+b)(a+c)}=$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{a^2bc}{(a+b)(a+c)}+\frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2}-\frac{bc(b-c)^2}{(a+b)(a+c)}=$$

$$=\frac{a^2bc}{(a+b)(a+c)}-\frac{abc(a+b+c)(b-c)^2}{(a+b)^2(a+c)^2}\le\frac{a^2bc}{(a+b)(a+c)}\overset{AM-GM}{\le}\frac{a^2bc}{2\sqrt{ab}.2\sqrt{ac}}=\frac{a\sqrt{bc}}{4}.$$

$$Then:\left(\frac{EF}{BC}\right)^4\le\frac{bc}{16a^2}\overset{AM-GM}{\le}\frac{b^2+c^2}{32a^2}\quad(And\ analogs)$$

$$\left(\frac{EF}{BC}\right)^4+\left(\frac{FD}{CA}\right)^4+\left(\frac{DE}{AB}\right)^4+\frac{3}{16}\le\sum_{cyc}\frac{b^2+c^2}{32a^2}+\frac{3}{16}=\frac{1}{32}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)^2\overset{Bandila}{\le}$$

$$\le\frac{1}{32}\sum_{cyc}\left(\frac{R}{r}\right)^2=\frac{3}{8}.\left(\frac{R}{2r}\right)^2. Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

Solution 3 by Soumava Chakraborty-Kolkata-India

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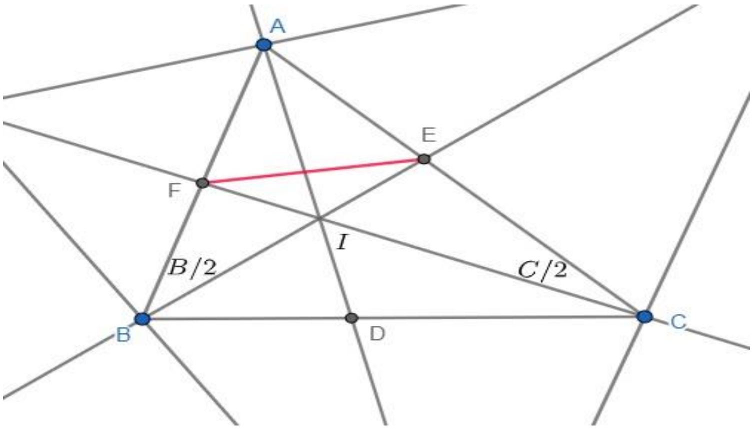

$$\text{Angle}-\text{bisector theorem} \Rightarrow \frac{AF}{BF}=\frac{b}{a} \Rightarrow \frac{AF+BF}{BF}=\frac{b+a}{a} \Rightarrow BF \overset{(i)}{=} \frac{ca}{a+b}$$

$$\textit{and also}, \text{angle}-\text{bisector theorem} \Rightarrow \frac{AE}{CE}=\frac{c}{a} \Rightarrow \frac{AE+CE}{CE}=\frac{c+a}{a} \Rightarrow CE \overset{(ii)}{=} \frac{ab}{c+a}$$

$$\text{Cosine } \textit{law} \Rightarrow \left(BF^2+w_b^2-2BF.w_b.\cos\frac{B}{2}\right)+\left(CE^2+w_c^2-2CE.w_c.\cos\frac{C}{2}\right)=2FE^2$$

$$\overset{\text{via (i),(ii)}}{\Rightarrow} \frac{a^2b^2}{(c+a)^2}+\frac{4ca}{(c+a)^2}.s(s-b)-2.\frac{ca}{a+b}.\frac{2ca}{c+a}.\frac{s(s-b)}{ca}+\frac{c^2a^2}{(a+b)^2}$$

$$+\frac{4ab}{(a+b)^2}.s(s-c)-2.\frac{ab}{c+a}.\frac{2ab}{a+b}.\frac{s(s-c)}{ab}=2FE^2$$

$$\Rightarrow 2FE^2=\frac{a^2b^2}{(c+a)^2}+\frac{c^2a^2}{(a+b)^2}+\frac{4ca}{c+a}.s(s-b)\left(\frac{1}{c+a}-\frac{1}{a+b}\right)$$

$$+\frac{4ab}{a+b}.s(s-c)\left(\frac{1}{a+b}-\frac{1}{c+a}\right)$$

$$\overset{A-G}{\le} \frac{a^2b^2}{4ca}+\frac{c^2a^2}{4ab}+4sa\left(\frac{1}{a+b}-\frac{1}{c+a}\right)\left(\frac{b(s-c)}{a+b}-\frac{c(s-b)}{c+a}\right)=\frac{a^2(b^3+c^3)}{4abc}$$

$$+\frac{4sa}{(a+b)^2(c+a)^2}.(c+a-a-b)\big(b(s-c)(2s-b)-c(s-b)(2s-c)\big)$$

$$\Rightarrow \frac{FE^2}{a^2} \le \frac{b^3+c^3}{8abc}$$

$$-\frac{2s(b-c)}{a(a+b)^2(c+a)^2}.\big(b(2s^2-bs-2cs+bc)-c(2s^2-cs-2bs+bc)\big)$$

$$=\frac{b^3+c^3}{8abc}-\frac{2s(b-c)}{a(a+b)^2(c+a)^2}.\big(2s^2(b-c)+bc(b-c)-s(b-c)(b+c)\big)$$

$$=\frac{b^3+c^3}{8abc}-\frac{2s(b-c)^2}{a(a+b)^2(c+a)^2}.\big(2s^2+bc-s(2s-a)\big)$$

$$=\frac{b^3+c^3}{8abc}-\frac{2s(b-c)^2(bc+as)}{a(a+b)^2(c+a)^2} \le \frac{b^3+c^3}{8abc} \Rightarrow \frac{FE^4}{a^4} \le \frac{(b^3+c^3)^2}{64a^2b^2c^2} \textit{ and analogs}$$

$$\Rightarrow \sum_{cyc} \frac{FE^4}{a^4} \le \frac{\sum_{cyc} a^6 + \sum_{cyc} a^3b^3}{32a^2b^2c^2} = \frac{\left(\sum_{cyc} a^3\right)^2 - \sum_{cyc} a^3b^3}{32a^2b^2c^2} \overset{?}{\le} \frac{3(R^2-2r^2)}{32r^2}$$

$$\Leftrightarrow 4s^2(s^2-6Rr-3r^2)^2 - \left(\sum_{cyc} ab\right)^3 + 3abc\prod_{cyc}(b+c) \overset{?}{\le} 48R^2s^2(R^2-2r^2)$$

$$\Leftrightarrow 4s^2(s^2-6Rr-3r^2)^2 - (s^2+4Rr+r^2)^3 + 24Rrs^2(s^2+2Rr+r^2)$$
$$\overset{?}{\le} 48R^2s^2(R^2-2r^2)$$

$$\Leftrightarrow 3s^6 - (36Rr+27r^2)s^4 - (48R^4-240R^2r^2-144Rr^3-33r^4)s^2$$
$$-r^3(4R+r)^3 \underset{(*)}{\overset{?}{\le}} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\le} (12R^2-24Rr-18r^2)s^4$$
$$-(48R^4-240R^2r^2-144Rr^3-33r^4)s^2 - r^3(4R+r)^3$$
$$\overset{\text{Gerretsen}}{\le} (12R^2-24Rr)s^4 - 18r^2(16Rr-5r^2)s^2$$
$$-(48R^4-240R^2r^2-144Rr^3-33r^4)s^2 - r^3(4R+r)^3 \overset{?}{\le} 0$$

$$\Leftrightarrow (12R^2-24Rr)s^4 - (48R^4-240R^2r^2+144Rr^3-123r^4)s^2 - r^3(4R+r)^3 \underset{(**)}{\overset{?}{\le}} 0$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\le} (12R^2-24Rr)(4R^2+4Rr+3r^2)s^2$$
$$-(48R^4-240R^2r^2+144Rr^3-123r^4)s^2 - r^3(4R+r)^3 \overset{?}{\le} 0$$
$$\Leftrightarrow (48R^3-180R^2r+216Rr^2-123r^3)s^2 + r^2(4R+r)^3 \underset{(***)}{\overset{?}{\ge}} 0$$

$\boxed{Case\ 1}$ $48R^3-180R^2r+216Rr^2-123r^3 \ge 0$ *and* **then,**
LHS of $(***) \ge r^2(4R+r)^3 > 0 \Rightarrow (***)$ **is true (strict inequality)**
$\boxed{Case\ 2}$ $48R^3-180R^2r+216Rr^2-123r^3 < 0$ *and* **then, LHS of** $(***)$

$$= -\left(-(48R^3-180R^2r+216Rr^2-123r^3)\right)s^2 + r^2(4R+r)^3 \overset{\text{Gerretsen}}{\ge}$$
$$-\left(-(48R^3-180R^2r+216Rr^2-123r^3)\right)(4R^2+4Rr+3r^2) + r^2(4R+r)^3 \overset{?}{\ge} 0$$
$$\Leftrightarrow 192t^5 - 528t^4 + 352t^3 - 120t^2 + 168t - 368 \overset{?}{\ge} 0 \quad \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(120t^4 + 72(t-2)t^3 + 64t^2 + 8t + 184) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$\Rightarrow (***)$ **is true** $\therefore$ **combining** *cases* **1, 2,** $(***) \Rightarrow (**) \Rightarrow (*)$ **is true for** *all* **triangles**

$$\therefore \left(\frac{EF}{BC}\right)^4 + \left(\frac{FD}{CA}\right)^4 + \left(\frac{DE}{AB}\right)^4 \le \frac{3(R^2-2r^2)}{32r^2} \ \textit{or},$$

$$\left(\frac{EF}{BC}\right)^4 + \left(\frac{FD}{CA}\right)^4 + \left(\frac{DE}{AB}\right)^4 + \frac{3}{16} \le \frac{3}{8}\left(\frac{R}{2r}\right)^2, \text{"="} \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

UP.507 Prove that:

$$\int_0^1\int_0^1 \frac{\log x}{(x^2-1)(1+x^2y)^2}dxdy = \frac{1}{2}G + \frac{3}{8}\zeta(2)$$

where $G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ design the Catalan's constant and

$$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k} \text{ is zeta function.}$$

$$\text{Find the value of the series: } \Omega = \sum_{n=0}^{\infty}\frac{1}{(4n+1)^2}.$$

Proposed by Said Attaoui-Oran-Algerie

Solution 1 by proposer

$$\textit{Remark that} \int \frac{1}{(1+x^2y)^2}dy = -\frac{1}{x^2(1+x^2y)}. \textit{So,}$$

$$\int_0^1 \frac{1}{(1+x^2y)^2}dy = -\frac{1}{x^2(1+x^2)} + \frac{1}{x^2} = \frac{1}{1+x^2}. \textit{Then,}$$

$$\int_0^1\int_0^1 \frac{\log x}{(x^2-1)(1+x^2y)^2}dxdy = -\int_0^1 \frac{\log x}{(1-x^2)(1+x^2)}dx \overset{not.}{=} -J$$

By decomposition into simple elements, we can write:

$$\frac{1}{(1-x^2)(1+x^2)} = \frac{ax+b}{1+x^2} + \frac{c}{1-x} + \frac{d}{1+x}$$

$$\begin{cases} 1 = b+c+d \\ 0 = a-c+d \\ \frac{1}{4} = c = d \end{cases} \Rightarrow \begin{cases} b = \frac{1}{2} \\ a = 0 \\ \frac{1}{4} = c = d \end{cases}. \textit{So,}$$

$$\frac{1}{(1-x^2)(1+x^2)} = \frac{1}{2(1+x^2)} + \frac{1}{4(1-x)} + \frac{1}{4(1+x)}$$

$$\textit{Hence, } J = \frac{1}{2}\underbrace{\int_0^1 \frac{\log x}{1+x^2}dx}_{J_1=-G} + \frac{1}{4}\underbrace{\int_0^1 \frac{\log x}{1-x}dx}_{J_2=-\zeta(2)} + \frac{1}{4}\underbrace{\int_0^1 \frac{\log x}{1+x}dx}_{J_3=-\frac{1}{2}\zeta(2)}$$

$$\textit{Proof of } J_1. \textit{ Integrating by parts, we get since: } \arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}.$$

$$J_1=\underbrace{[\arctan x\log x]_0^1}_{=0}-\int_0^1\frac{\arctan x}{x}dx=-\int_0^1\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2n+1}\right)dx=$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\int_0^1 x^{2n}\,dx\right)=-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{2n+1}x^{2n+1}\Big|_0^1\right)=$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-G$$

Proof of J_2. *Using the expansion series of* $\log(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$, *we get*:

$$J_2=\int_0^1\frac{\log(1-x)}{x}dx=-\int_0^1\left(\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}\right)dx=$$

$$=-\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^1 x^{n-1}\,dx\right)=-\sum_{n=1}^{\infty}\frac{1}{n^2}=-\zeta(2)$$

Proof of J_3. *Dirtectly we have*:

$$-\frac{1}{2}\zeta(2)=\frac{1}{2}J_2=\frac{1}{2}\int_0^1\frac{\log x}{1-x}dx=2\int_0^1\frac{x\log x}{1-x^2}dx\overset{x\to x^2}{=}$$

$$=-2\int_{061}\frac{\log x}{1+x}dx+2\int_0^1\frac{\log x}{1-x^2}dx=$$

$$=-2\int_0^1\frac{\log x}{1+x}dx+\int_0^1\frac{\log x}{1-x}dx+\int_0^1\frac{\log x}{1+x}dx=-J_3+J_2$$

$$Thus, J_3=\frac{1}{2}J_2=-\frac{1}{2}\zeta(2).$$

$$Finally, -J=\frac{1}{2}G+\frac{3}{8}\zeta(2)=\int_0^1\int_0^1\frac{\log x}{(x^2-1)(1+x^2y)^2}dxdy.$$

Now, the double integrals can be expressed by applying the geometric series

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}, |x|<1, as:$$

$$\int_0^1\int_0^1\frac{\log x}{(x^2-1)(1+x^2y)^2}dxdy=-\int_0^1\frac{\log x}{1-x^4}dx=$$

$$=-\sum_{n=0}^{\infty}\left(\int_0^1 x^{4n}\log x\,dx\right)=\sum_{n=0}^{\infty}\left(\int_0^{\infty}te^{-(4n+1)t}\,dt\right)\overset{x=e^{-t}}{=}\sum_{n=0}^{\infty}\frac{1}{(4n+1)^2}$$

By the fact that $\int_0^{\infty}t^a e^{-bt}\,dt=\frac{\Gamma(a+1)}{b^{a+1}}, a>0, b>0.$

We deduce so: $\sum_{n=0}^{\infty}\frac{1}{(4n+1)^2}=\frac{1}{2}G+\frac{3}{8}\zeta(2).$

Solution 2 by Pham Duc Nam-Vietnam

$$*\int_0^1\int_0^1 \frac{\log(x)}{(x^2-1)(1+x^2y)^2}dxdy = \int_0^1 \frac{\log(x)}{(x^2-1)}dx\int_0^1 \frac{1}{(1+x^2y)^2}dy$$

$$= \int_0^1 \frac{\log(x)}{(x^2-1)}dx\left(\frac{1}{x^2}\int_0^1 \frac{d(1+x^2y)}{(1+x^2y)^2}\right)$$

$$= \int_0^1 \frac{\log(x)}{(x^2-1)}dx\left(-\frac{1}{x^2}\cdot\frac{1}{1+x^2y}\Big|_0^1\right) = \int_0^1 \frac{\log(x)}{(x^2-1)}dx\left(\frac{1}{x^2}-\frac{1}{x^2}\cdot\frac{1}{1+x^2}\right) = \int_0^1 \frac{\log(x)}{(x^4-1)}dx$$

$$= \int_0^1 \frac{\log(x)}{(x^4-1)}dx = \frac{1}{2}\int_0^1 \log(x)\left(\frac{1}{x^2-1}-\frac{1}{x^2+1}\right)dx$$

$$= \frac{1}{2}\left(-\sum_{k=0}^{\infty}\int_0^1 x^{2k}\log(x)\,dx - \sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{2k}\log(x)\,dx\right)$$

$$= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}+\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\right) = \frac{1}{2}G + \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{1}{2}G + \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k\ is\ even}^{\infty}\frac{1}{k^2}\right)$$

$$= \frac{1}{2}G + \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{j=1}^{\infty}\frac{1}{4j^2}\right)$$

$$= \frac{1}{2}G + \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{1}{4}\sum_{j=1}^{\infty}\frac{1}{j^2}\right) = \frac{1}{2}G + \frac{1}{2}\left(\zeta(2)-\frac{1}{4}\zeta(2)\right) = \frac{1}{2}G + \frac{3}{8}\zeta(2)$$

$$*\int_0^1 \frac{\log(x)}{(x^4-1)}dx = -\sum_{k=0}^{\infty}\int_0^1 x^{4k}\log(x)\,dx = \sum_{k=0}^{\infty}\frac{1}{(4k+1)^2} = \Omega \Rightarrow LHS = \frac{1}{2}G + \frac{3}{8}\zeta(2)$$

Solution 3 by Vincent Nguyen-USA

$$\Omega = \int_0^1\int_0^1 \frac{\log x}{(x^2-1)(1+x^2y)^2}dxdy = \int_0^1 \frac{\log x}{x^2-1}\int_0^1 \frac{1}{(1+x^2y)^2}dydx =$$

$$= \int_0^1 \frac{\log x}{x^2-1}\left(\frac{1}{x^2(1+x^2)}-\frac{1}{x^2}\right)dx = -\int_0^1 \frac{\log x}{x^2-1}\cdot\frac{1}{x^2+1}dx = -\int_0^1 \frac{\log x}{x^4-1}dx =$$

$$= -\int_0^1 \log x\sum_{n=0}^{\infty}x^{4n}dx \overset{x=e^{-t}}{\cong} \sum_{n=0}^{\infty}\int_0^{\infty} te^{-4nt}\cdot e^{-t}dt = \sum_{n=0}^{\infty}\int_0^{\infty} te^{-(4n+1)t}dt =$$

$$= \sum_{n=0}^{\infty}\mathcal{L}\{t\}(4n+1) = \sum_{n=0}^{\infty}\frac{1}{(2(2n)+1)^2} = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^2}\right) =$$

$$=\frac{1}{2}(S_1+S_2)$$

$$S_1=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\right)=$$

$$=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}+\sum_{n=-\infty}^{0}\frac{1}{(-2n+1)^2}\right)=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}+\sum_{n=-\infty}^{-1}\frac{1}{(2n+1)^2}\right)=$$

$$=\frac{1}{2}\sum_{-\infty}^{\infty}\frac{1}{(2n+1)^2},\qquad S_2=G$$

Let be $f(z)=\frac{1}{(2z+1)^2}$ with a second order pole $z=-\frac{1}{2}$.

$$\sum_{-\infty}^{\infty}\frac{1}{(2n+1)^2}=-\underbrace{Res}_{z=-\frac{1}{2}}\left(\frac{\pi\cot(\pi z)}{(2z+1)^2}\right)=-\lim_{z\to-\frac{1}{2}}\frac{d}{dz}\left[\left(z+\frac{1}{2}\right)^2\cdot\frac{\pi\cot(\pi z)}{(2z+1)^2}\right]=$$

$$=-\lim_{z\to-\frac{1}{2}}\frac{d}{dz}\left[\frac{\pi\cot(\pi z)}{4}\right]=\lim_{z\to-\frac{1}{2}}\left[\frac{\pi^2\csc^2(\pi z)}{4}\right]=\frac{\pi^2}{4}$$

$$S_1=\frac{1}{2}\cdot\frac{\pi^2}{4}=\frac{3}{4}\cdot\frac{\pi^2}{6}=\frac{3}{4}\zeta(2)$$

$$\Omega=\frac{1}{2}(S_1+S_2)=\frac{1}{2}\left(\frac{3}{4}\zeta(2)+G\right)=\frac{G}{2}+\frac{3}{8}\zeta(2)$$

UP.508 Find:

$$\Omega=\lim_{n\to\infty}e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}\cdot\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\Omega=\lim_{n\to\infty}e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}\cdot\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)=$$

$$=\lim_{n\to\infty}e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n}\cdot e^{\log n}\cdot\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)=$$

$$=e^{\gamma}\cdot\lim_{n\to\infty}n\left(\sqrt[n]{\pi}-\sqrt[n]{e}\right)\left(\sqrt[n]{\pi^2}+\sqrt[n]{\pi e}+\sqrt[n]{e^2}\right)=$$

$$= e^{\gamma}\cdot\lim_{n\to\infty} n\cdot\sqrt[n]{e}\cdot\left(\sqrt[n]{\frac{\pi}{e}}-1\right)\left(\sqrt[n]{\pi^2}+\sqrt[n]{\pi e}+\sqrt[n]{e^2}\right)=$$

$$= e^{\gamma}\cdot\lim_{n\to\infty}\sqrt[n]{e}\cdot\frac{\sqrt[n]{\frac{\pi}{e}}-1}{\frac{1}{n}}\cdot\lim_{n\to\infty}\left(\sqrt[n]{\pi^2}+\sqrt[n]{\pi e}+\sqrt[n]{e^2}\right)=$$

$$= e^{\gamma}\cdot\lim_{n\to\infty}\sqrt[n]{e}\cdot\lim_{n\to\infty}\frac{e^{\frac{1}{n}\log\left(\frac{\pi}{e}\right)}-1}{\frac{1}{n}\log\left(\frac{\pi}{e}\right)}\cdot\log\left(\frac{\pi}{e}\right)(1+1+1)=$$

$$= e^{\gamma}\cdot 1\cdot\log\left(\frac{\pi}{e}\right)\cdot 3 = 3e^{\gamma}\log\left(\frac{\pi}{e}\right).$$

Solution 2 by Toubal Fethi-Algeria

$$\Omega=\lim_{n\to\infty} e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)=\lim_{n\to\infty} e^{(H_n-\log n)+\log n}\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)=$$

$$= e^{\gamma}\lim_{n\to\infty} n\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)=e^{\gamma}\lim_{n\to\infty}\left(\frac{\sqrt[n]{\pi^3}-1}{\frac{1}{n}}-\frac{\sqrt[n]{e^3}-1}{\frac{1}{n}}\right)=$$

$$= e^{\gamma}(\log(\pi^3)-\log(e^3)) = 3e^{\gamma}\log\left(\frac{\pi}{e}\right)$$

Solution 3 by Adrian Popa-Romania

$$\Omega=\lim_{n\to\infty} e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)=e^{\gamma}\lim_{n\to\infty} n\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)=$$

$$= e^{\gamma}\lim_{n\to\infty}\left(\frac{\sqrt[n]{\pi^3}-1}{\frac{1}{n}}-\frac{\sqrt[n]{e^3}-1}{\frac{1}{n}}\right)=e^{\gamma}\lim_{x\to 0}\frac{(\pi^3)^x-(e^3)^x}{x}=$$

$$= e^{\gamma}\lim_{x\to 0}\frac{[(\pi^3)^x-1]-[(e^3)^x-1]}{x}=\log(\pi^3)-\log(e^3)=$$

$$= e^{\gamma}\log\left(\frac{\pi}{e}\right)^3 = 3e^{\gamma}(\log\pi-1)$$

UP.509 Find:

$$\Omega=\lim_{n\to\infty}\left(e^{3H_{n+1}}-e^{3H_n}\right)\left(\sqrt[n]{\pi}-1\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\Omega=\lim_{n\to\infty}\left(e^{3H_{n+1}}-e^{3H_n}\right)\left(\sqrt[n]{\pi}-1\right)^2=\lim_{n\to\infty}\left(e^{3H_n+\frac{3}{n+1}}-e^{3H_n}\right)\left(\sqrt[n]{\pi}-1\right)^2=$$

$$=\lim_{n\to\infty}e^{3H_n}\left(e^{\frac{3}{n+1}}-1\right)\left(\frac{e^{\frac{1}{n}\log\pi}-1}{\frac{1}{n}\log\pi}\right)^2\left(\frac{1}{n}\log\pi\right)^2=$$

$$=\lim_{n\to\infty}e^{3H_n-3\log n}\cdot e^{3\log n}\cdot\frac{e^{\frac{3}{n+1}}-1}{\frac{3}{n+1}}\cdot\frac{3}{n+1}\cdot\frac{1}{n^2}\cdot\log^2\pi=$$

$$=\lim_{n\to\infty}e^{3\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)}\cdot n^3\cdot\frac{3}{n+1}\cdot\frac{1}{n^2}\cdot\log^2\pi=$$

$$=e^{3\gamma}\cdot\lim_{n\to\infty}\frac{3n}{n+1}\cdot\log^2\pi=e^{3\gamma}\cdot 3\cdot\log^2\pi=3e^{3\gamma}\log^2\pi$$

Solution 2 by Vincent Nguyen-USA

$$\Omega=\lim_{n\to\infty}\left(e^{3H_{n+1}}-e^{3H_n}\right)\left(\sqrt[n]{\pi}-1\right)^2=\lim_{n\to\infty}\left(e^{3\left(H_n+\frac{1}{n+1}\right)}-e^{3H_n}\right)\left(\sqrt[n]{\pi}-1\right)^2=$$

$$=\lim_{n\to\infty}e^{3H_n}\left(e^{\frac{3}{n+1}}-1\right)\left(\sqrt[n]{\pi}-1\right)^2=\lim_{n\to\infty}e^{3H_n-3logn}e^{logn}\left(e^{\frac{3}{n+1}}-1\right)\left(\sqrt[n]{\pi}-1\right)^2=$$

$$=e^{3\gamma}\lim_{n\to\infty}n^3\left(e^{\frac{3}{n+1}}-1\right)\left(e^{\frac{log\pi}{n}}-1\right)^2=$$

$$=e^{3\gamma}\lim_{n\to\infty}n^3\left(\left(1+\frac{3}{n+1}\right)-1\right)\left(\left(1+\frac{log\pi}{n}\right)-1\right)^2=$$

$$=e^{3\gamma}\lim_{n\to\infty}n^3\left(\frac{3}{n+1}\right)\left(\frac{log\pi}{n}\right)^2=e^{3\gamma}\,log^2\pi\lim_{n\to\infty}\left(\frac{3n}{n+1}\right)=3e^{3\gamma}log^2\pi$$

UP.510 Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{n^2+k}\right)\cdot\frac{1}{\log n}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by proposers

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{n^2+k}\right)\cdot\frac{1}{\log n} = \lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{k}-\sum_{k=1}^{n^2}\frac{1}{k}\right)\cdot\frac{1}{\log n} =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{k}-\log(n^4)-\left(\sum_{k=1}^{n^2}\frac{1}{k}-\log(n^2)\right)+\log(n^4)-\log(n^2)\right)\cdot\frac{1}{\log n} =$$

$$= \lim_{n\to\infty}\frac{1}{\log n}\left(\sum_{k=1}^{n^4}\frac{1}{k}-\log(n^4)\right)-\lim_{n\to\infty}\frac{1}{\log n}\left(\sum_{k=1}^{n^2}\frac{1}{k}-\log(n^2)\right)$$

$$+\lim_{n\to\infty}\frac{\log(n^4)-\log(n^2)}{\log n} = 0-0+2=2.$$

Solution 2 by Arnab Debnath-India

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{n^2+k}\right)\cdot\frac{1}{\log n} = \lim_{n\to\infty}\left(\sum_{k=1}^{n^4+n^2}\frac{1}{k}-\sum_{k=1}^{n^2}\frac{1}{k}\right)\cdot\frac{1}{\log n} =$$

$$= \lim_{n\to\infty}\frac{\psi(n^4+n^2+1)-\psi(n^2+1)}{\log n} =$$

$$= \lim_{n\to\infty}\frac{\log(n^4+n^2+1)-\frac{1}{2(n^4+n^2+1)}-\log(n^2+1)+\frac{1}{2(n^2+1)}}{\log n} =$$

$$= \lim_{n\to\infty}\frac{\log(n^4+n^2+1)}{\log n}-\lim_{n\to\infty}\frac{\log(n^2+1)}{\log n} =$$

$$= \lim_{n\to\infty}\frac{8n^2+2}{2n^2+1}-\lim_{n\to\infty}\frac{2n^2}{n^2+1} = 4-2=2$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\cong \log n$$

$$\sum_{k=1}^{n^4}\frac{1}{n^2+k}\cong \log(n^4+n^2)-\log(n^2)=\log\left(\frac{n^4+n^2}{n^2}\right)=\log(n^2+1)$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{n^2+k}\right)\cdot\frac{1}{\log n}\cong\lim_{n\to\infty}\frac{\log(n^2+1)}{\log n}=\lim_{n\to\infty}\frac{\frac{2n}{n^2+1}}{\frac{1}{n}}=2$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\sum_{k=1}^{n^4}\frac{1}{n^2+k}=\sum_{k=1}^{\infty}\left(\frac{1}{n^2+k}-\frac{1}{k+n^4+n^2}\right)\overset{m=k-1}{=}$$

$$=-\gamma+\sum_{m=0}^{\infty}\left(\frac{1}{m+1}-\frac{1}{m+1+n^2+n^4}\right)+\gamma-\sum_{m=0}^{\infty}\left(\frac{1}{m+1}-\frac{1}{m+1+n^2}\right)=$$

$$=\Psi(n^4+n^2+1)-\Psi(n^2+1)\overset{\Psi(z)=\log z-\frac{1}{2z}}{\Rightarrow}$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{n^2+k}\right)\cdot\frac{1}{\log n}=$$

$$=\lim_{n\to\infty}\frac{1}{\log n}\left(\log\left(\frac{n^4+n^2+1}{n^2+1}\right)-\frac{1}{2(n^4+n^2+1)}+\frac{1}{2(n^2+1)}\right)=2$$

Solution 5 by Le Thu-Vietnam

$$By\ Euler-Maclaurin\ formula, one\ has$$

$$\sum_{k=1}^{n^4}\frac{1}{n^2+k}\sim\int_1^{n^4}\frac{1}{n^2+x}dx=\log(n^2+x)\Big|_1^{n^4}=$$

$$=2\log n+\log(1+n^2)-\log(1+n^2)=2\log n$$

$$Hence,$$

$$\Omega=\lim_{n\to\infty}\frac{2\log n}{\log n}=2$$

Solution 6 by Samir Zaakouni -Morocco

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{n^2+k}\right)\cdot\frac{1}{\log n}=\lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\int_0^1 x^{n^2+k-1}\,dx\right)\cdot\frac{1}{\log n}=$$

$$=\lim_{n\to\infty}\left(\int_0^1 x^{n^2}\sum_{k=1}^{n^4}x^{k-1}\,dx\right)\cdot\frac{1}{\log n}=\lim_{n\to\infty}\left(\int_0^1\frac{x^{n^2}-x^{n^4+n^2}}{1-x}dx\right)\cdot\frac{1}{\log n}=$$

$$=\lim_{n\to\infty}\left(-\gamma+\int_0^1\frac{1-x^{n^4+n^2}}{1-x}dx+\gamma-\int_0^1\frac{1-x^{n^2}}{1-x}dx\right)\cdot\frac{1}{\log n}=$$

$$=\lim_{n\to\infty}\left(H_{n^4+n^2}-H_{n^2}\right)\cdot\frac{1}{\log n}=\lim_{n\to\infty}\frac{H_{n^4+n^2}}{\log n}-\lim_{n\to\infty}\frac{H_{n^2}}{\log n}$$

$$H_{n^4+n^2} \sim \log(n^2+n^4) - \log(n^4) = 4\log n$$

$$H_{n^2} \sim \log(n^2) = 2\log n$$

$$H_{n^4+n^2} \cdot \frac{1}{\log n} \sim 4 \ and \ H_{n^2} \cdot \frac{1}{\log n} \sim 2$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n^4} \frac{1}{n^2+k}\right) \cdot \frac{1}{\log n} = 2$$