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SOLUTIONS

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PROBLEMS FOR JUNIORS

JP.511 Let $n \in \mathbb{N}^*$. Prove that among the numbers $\binom{2n}{1}, \binom{2n}{2}, \dots, \binom{2n}{n}$ exists at least a number which is not divisible with 16065.

Proposed by Mihaly Bencze-Romania

Solution 1 by proposer

$$\textit{WLOG, let's suppose that the numbers } \binom{2n}{1}, \binom{2n}{2}, \dots, \binom{2n}{n} \textit{ are divisible}$$

$$\textit{with } 16065, \textit{then from } \binom{2n}{2n-1} = \binom{2n}{1}, \dots, \binom{2n}{n+1} = \binom{2n}{n-1} \textit{ and the sum}$$

$$S_n = \sum_{k=1}^{2n-1} \binom{2n}{k} = 2^{2n} - 2 = 4^n - 2 \textit{ is divisible with } 16065.$$

$$S_{3k} = 4^{3k} - 2 = 64^k - 2 = (63+1)^k - 2 = m_{63} - 1,$$

$$S_{3k+1} = 4 \cdot 64^k - 2 = m_{63} + 2,$$

$$S_{3k+2} = 16 \cdot 64^k - 2 = m_{63} + 14, \textit{so } S_n \not\equiv 0 \ (mod\ 63)$$

$$S_{4k} = 250^k - 2 = (255+1)^k - 2 = m_{255} - 1,$$

$$S_{4k+1} = 4 \cdot 256^k - 2 = m_{255} + 2,$$

$$S_{4k+2} = 16 \cdot 256^k - 2 = m_{255} + 14,$$

$$S_{4k+3} = 64 \cdot 256^k - 2 = m_{255} + 62, \textit{so } S_n \not\equiv 0 \ (mod\ 255).$$

$$\textit{Therefore}, S_n \not\equiv 0 \ (mod\ 255 \cdot 63) \Leftrightarrow S_n \not\equiv 0 (\ mod\ 16065).$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

Lemma:

Let y be a natural number so that y is divisible by a Fermat prime $2^{2^m} + 1$. Then, for every natural numbers n, one of

$$\binom{2n}{1}, \binom{2n}{2}, \dots \dots, \binom{2n}{n}$$

is not divisible by y.

Proof:

Assume that the lemma is negative. Then $\binom{2n}{k} \equiv 0 \ mod\ y, \ \forall k = 1, 2, \dots, n$.

Consequently,

$$\sum_{k=1}^{2n-1}\binom{2n}{k}=2^{2n}-2\equiv 0\ (mod\ y)$$

It implies that $2^{2n}-2\equiv 0\left(mod\ 2^{2^m}+1\right)\ \Rightarrow\ 2^{2n-1}\equiv 1\left(mod\ 2^{2^m}+1\right)\ldots\ldots(*)$

By Fermat Little Theorem, we have $2^{2^{2^m}}\equiv 1\left(mod\ 2^{2^m}+1\right)\ldots\ldots(**)$

From $(*)$ and $(**)$ we obtain

$$2^{\gcd\left(2n-1,\ 2^{2^m}\right)}\equiv 1\left(mod\ 2^{2^m}+1\right)\Rightarrow\ 2\equiv 1\left(mod\ 2^{2^m}+1\right),$$

which is impossible. Thus, one of

$$\binom{2n}{1},\binom{2n}{2},\ldots\ldots,\binom{2n}{n}$$

should be indivisible by y.

Back to the problem:

Since 16065 is divisible by $17=2^{2^2}+1$, so the problem is proved.

JP.512 Solve the following equation:

$$\log_{a+1}(a^x+2a+1)=\log_a((a+1)^x-2a-1),a>1.$$

Proposed by Mihaly Bencze-Romania

Solution 1 by proposer

$$Let\ f(x)=\log_{a+1}(a^x+2a+1)\ bijective\ function\ and$$

$$f^{-1}(x)=\log_a((a+1)^x-2a-1).$$

$$So,the\ equation\ is\ equivalent\ to\ f(x)=f^{-1}(x)\Leftrightarrow f(x)=x.$$

$$\log_{a+1}(a^x+2a+1)=x\ or\ (a+1)^2-a^x=2a+1.$$

$$The\ function\ g(x)=(a+1)^x-a^x\ is\ strictly\ increasing\ for\ x>0.$$

$$If\ x\geq 0,then\ (a+1)^x\leq a^x\ which\ is\ an\ contradiction.$$

$$So,g\ a\ is\ injective\ function\ and\ the\ equation\ has\ an\ unique\ solution,x=2.$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

Let $y=\log_{a+1}(a^x+2a+1)=\log_a((a+1)^x-2a-1)$, then

$$a^x+2a+1=(a+1)^y\ldots\ldots(1)$$

$$(a+1)^x-2a-1=a^y\ldots\ldots(2)$$

By summing up (1) and (2), we obtain

$$(a+1)^x+a^x=(a+1)^y+a^y\ldots\ldots\ldots(3)$$

We consider the function $f(n)=(a+1)^n+a^n$ over $n\in\mathbb{R}$. Then

$$f'(n)=(a+1)^n\ln(a+1)+a^n\ln a>0,$$

consequently f is strictly increasing. Therefore, we should have $x=y$ by (3).

Setting $x=y$ to (2), we get $(a+1)^x-a^x-2a-1=0\ \ldots\ldots\ldots(4)$

From (4), we know that $x = 2$ satisfies for (4). Besides that, we also have that $(a+1)^x > a^x$. Since $a \in \mathbb{R}^+$, we should have $x > 0$. Consider another function $g(n) = (a+1)^n - a^n - 2a - 1$ over $n \in \mathbb{R}^+$, so

$$g'(n) = (a+1)^n \ln(a+1) - a^n \ln a > 0,$$

because clearly $(a+1)^n > a^n$ and $\ln(a+1) > \ln a$. Then, g is strictly increasing. It implies that $x = 2$ is the only solution satisfying (4) and finally we get $y = 2$ too.

Thus, the answer is $x = 2$.

Solution 3 by Marin Chirciu-Romania

We have $\log_{a=1}(a^x + 2a + 1) = \log_a((a+1)^x - 2a - 1) = t \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \log_{a+1}(a^x + 2a + 1) = t \\ \log_a((a+1)^x - 2a - 1) = t \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a^x + 2a + 1 = (a+1)^t \\ (a+1)^x - 2a - 1 = a^t \end{cases} \overset{+}{\Leftrightarrow} (a+1)^t + a^t = (a+1)^x + a^x$$

Because the function $x \to a^x, a > 1$ is strictly increasing it follows that the function

$x \to (a+1)^x + a^x$ is strictly increasing, so injective.

From $(a+1)^t + a^t = (a+1)^x + a^x \Rightarrow t = x$

We obtain $\log_{a+1}(a^x + 2a + 1) = x \Leftrightarrow a^x + 2a + 1 = (a+1)^x$, with the unique

solution $x = 2$, because:

$$a^x + 2a + 1 = (a+1)^x \Leftrightarrow \left(\frac{a}{a+1}\right)^x + (2a+1)\left(\frac{1}{a+1}\right)^x = 1$$

The equation has the form $f(x) = 1$, with f strictly decreasing function, so, injective.

Because $f(2) = 1$, it follows that $x = 2$ is unique solution.

JP.513 Solve for real numbers:

$$\begin{cases} 2\log_3(2^{x_1} + 5)\log_2(3^{x_1} - 5) = \log_3(2^{x_2} + 5) + \log_2^2(3^{x_3} - 5) \\ 2\log_3(2^{x_2} + 5)\log_2(3^{x_2} - 5) = \log_3(2^{x_3} + 5) + \log_2^2(3^{x_4} - 5) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 2\log_3(2^{x_n} + 5)\log_2(3^{x_n} - 5) = \log_3(2^{x_1} + 5) + \log_2^2(3^{x_2} - 5) \end{cases}$$

Proposed by Mihaly Bencze-Romania

Solution by proposer

$$By\ adding\ all\ equations, we\ get:$$

$$\sum_{k=1}^{n}(\log_3(2^{x_k}+5)-\log_2(3^{x_k}-5))^2=0\Rightarrow x_1=x_2=\cdots=x_n=x,$$

$$\log_3(2^x+5)=log_2(3^x-5)$$

$$\textit{The function } f:\mathbb{R}\to[\log_3 5,+\infty), f(x)=\log_3(2^x+5)\ \textit{is bijective and}$$

$$f^{-1}(x)=\log_2(3^x-5)\ \textit{is his reverse function.}$$

$$\textit{So, the equation can be written as}: f(x)=f^{-1}(x)\Leftrightarrow f(x)=x\Leftrightarrow$$

$$\log_3(2^x+5)=x\Leftrightarrow 3^x-2^x=5.$$

$$\textit{The function } g(x)=3^x-2^x \textit{ is strictly increasing for } x>0.$$

$$\textit{If } x\le 0, \textit{then } 3^x\le 2^x \textit{ which is an contradiction.}$$

$$\textit{Because } g \textit{ is a injective function, then the equation has an unique solution}$$

$$x=2. \textit{So, solution of the system is}: x_1=x_2=\cdots=x_n=2.$$

JP.514 On the set $M=\{2n+1|n\in\mathbb{N}^*\}$ define

$$a*b=a+(b-3)2^{[\log_2 a]-1}, (\forall)a,b\in M,$$

where $[\cdot]$ is the integer part. Prove that $(M,*)$ is a monoid.

Proposed by Mihaly Bencze-Romania

Solution 1 by proposer

$$1)\ (\forall)a,b\in M\Rightarrow a*b=\underbrace{a}_{odd}+\underbrace{(b-3)2^{[\log_2 a]-1}}_{even}=odd\ge 3$$

$$\Rightarrow a*b\in M \textit{ is stabile part.}$$

$$2)\ (\forall)a,b,c\in M\Rightarrow(a*b)*c=\left(a+(b-3)2^{[\log_2 a]-1}\right)*c=$$

$$=a+(b-3)2^{[\log_2 a]-1}+(c-3)2^{a+(b-3)2^{[\log_2 a]-1}}=$$

$$=a+(b-3)2^{[\log_2 a]-1}+(c-3)2^{[\log_2 a]+[\log_2 b]-2}.$$

$$\textit{Now, we prove that}:$$

$$[\log_2\left(a-(b-3)2^{[\log_2 a]-1}\right)=[\log_2 a]+[\log_2 b]-1.$$

$$\textit{Let } x=[\log_2 a]\in\mathbb{N}^* \textit{and } y=[\log_2 b]\in\mathbb{N}^*.$$

$$\textit{Because}: (x_iz_i+y_it_i)^2-(x_it_i+y_iz_i)^2=\left(x_i^2-y_i^2\right)\left(z_i^2-t_i^2\right)\ne 0, i=1,2,..,n$$

$$\det(x_1,y_2,x_2,y_2,\dots,x_n,y_n)=\left(x_1^2-y_1^2\right)\det(x_2,y_2,\dots,x_n,y_n))$$

$$and\ then\ \det(x_1, y_2, x_2, y_2, \dots, x_n, y_n) = \prod_{i=1}^{n}(x_i^2 - y_i^2) \neq 0.$$

$$So, (G,*)\ is\ the\ abelian\ group, where (\cdot)\ is\ product\ of\ matrix.$$

$$The\ function\ f\big(A(x_1, y_2, x_2, y_2, \dots, x_n, y_n)\big) = (x_1, y_2, x_2, y_2, \dots, x_n, y_n)\ is\ bijective.$$

$$Therefore, (G,\cdot) \cong (H,*).$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

Lemma: $\lfloor \log_2(a * b) \rfloor = \lfloor \log_2 a \rfloor + \lfloor \log_2 b \rfloor - 1$ for every $a, b \in M$.
Proof: Let $a = 2^{x_1} + y_1$, $b = 2^{x_2} + y_2$ with $x_1, x_2 \in \mathbb{Z}^+$; $y_i \in [0, 2^{x_i} - 1]$, $i = 1, 2$; and y_1, y_2 are odd. Then

$$a * b = 2^{x_1} + y_1 + (2^{x_2} + y_2 - 3)2^{x_1 - 1} \ \dots\dots (1)$$

Note that

$$2^{x_1} + y_1 + (2^{x_2} + y_2 - 3)2^{x_1-1} \leq 2^{x_1} + 2^{x_1} - 1 + (2^{x_2} + 2^{x_2} - 4)2^{x_1-1}$$
$$= 2^{x_1+x_2} - 1 \ \dots\dots (2)$$

and

$$2^{x_1} + y_1 + (2^{x_2} + y_2 - 3)2^{x_1-1} \geq 2^{x_1} + 1 + (2^{x_2} - 2)2^{x_1-1} = 2^{x_1+x_2-1} + 1 \ \dots\dots (3)$$

By $(1), (2), (3)$, we get $\lfloor \log_2(a * b) \rfloor = x_1 + x_2 - 1 = \lfloor \log_2 a \rfloor + \lfloor \log_2 b \rfloor - 1$, as desired. □

After that, we count

$$(a * b) * c = a + (b - 3)2^{\lfloor \log_2 a \rfloor - 1} + (c - 3)2^{\lfloor \log_2 a \rfloor + \lfloor \log_2 b \rfloor - 2} \ \dots\dots\dots (S.1)$$
$$a * (b * c) = a + \big(b + (c - 3)2^{\lfloor \log_2 b \rfloor - 1} - 3\big)2^{\lfloor \log_2 a \rfloor - 1} \ \dots\dots\dots (S.2)$$

where $a, b, c \in M$ and obtain $(S.1) = (S.2)$, thus $(M, *)$ is a semigroup.
Besides that, we can notice that 3 is the neutral element since

$$3 * a = a * 3 = a, \qquad \forall a \in M.$$

From these facts, we can deduce that $(M, *)$ is a monoid. ■

JP.515 Solve for real numbers:

$$\sqrt{\left(\frac{3}{5}\sin x + \frac{101}{15}\cos y\right)\left(\frac{5}{3}\sin x + \frac{17}{3}\cos y\right)} +$$
$$+\sqrt{\left(15 - \frac{52}{15}\sin x - \frac{388}{15}\cos y\right)\left(-7 + \frac{6}{5}\sin x + \frac{202}{15}\cos y\right)} = 4$$

Proposed by Mihaly Bencze-Romania

Solution by proposer

$$\text{Let be }\begin{cases} a=\frac{3}{5}\sin x+\frac{101}{15}\cos y\\ b=\frac{5}{3}\sin x+\frac{17}{3}\cos y\end{cases}\text{ and }\begin{cases} c=15-\frac{52}{15}\sin x-\frac{388}{15}\cos y\\ d=-7+\frac{6}{5}\sin x+\frac{202}{15}\cos y\end{cases}$$

From the conditions:

$$\begin{cases}\frac{3}{5}\sin x+\frac{101}{15}\cos y\ge 0\\ \frac{5}{3}\sin x+\frac{17}{3}\cos y\ge 0\end{cases}\text{ and }\begin{cases}15-\frac{52}{15}\sin x-\frac{388}{15}\cos y\ge 0\\ -7+\frac{6}{5}\sin x+\frac{202}{15}\cos y\ge 0\end{cases}$$

$$\text{we get: }\begin{cases} ab+cd=4\\ a^2+b^2+c^2+d^2=8\end{cases}\Rightarrow (a-b)^2+(c-d)^2=0\Leftrightarrow a=c\text{ and }b=d.$$

$$\sin x=\frac{1}{2}\text{ and }\cos x=\frac{1}{2}\Rightarrow x\in\left\{(-1)^k\cdot\frac{\pi}{6}+k\pi\middle|k\in\mathbb{Z}\right\}\text{ and }y\in\left\{\pm\frac{\pi}{3}+2k\pi\middle|k\in\mathbb{Z}\right\}$$

JP.516 In ΔABC the following relationship holds:

$$\left(\sum_{cyc} bc\right)^2+n\left(\sum_{cyc} a^2\right)^2\ge (n+1)(18Rr)^2, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

$$\text{We have: }LHS=(\sum bc)^2+n(\sum a^2)^2=(\sum bc)^2+(\sum a^2)^2+..+(\sum a^2)^2=$$

$$=x_1^2+x_2^2+\dots+x_n^2+x_{n+1}^2\ge\frac{(x_1+x_2+\dots+x_{n+1})^2}{n+1}=$$

$$=\frac{(\sum bc+\sum a^2+\sum a^2+\dots+\sum a^2)^2}{n+1}=\frac{\left(s^2+r^2+4Rr+2n(s^2-r^2-4Rr)\right)^2}{n+1}=$$

$$=\frac{\left((1+2n)s^2+(1-2n)(r^2+4Rr)\right)^2}{n+1}\overset{Gerretsen}{\ge}\frac{\left((1+2n)(16Rr-5r^2)+(1-2n)(r^2+4Rr)\right)^2}{n+1}$$

$$=\frac{\left(4r((6n+5)R-(3n+1)r)\right)^2}{n+1}=\frac{16r^2[(6n+5)R-(3n+1)r]^2}{n+1}\overset{Euler}{\ge}$$

$$\ge\frac{16r^2\left[(6n+5)R-(3n+1)\frac{R}{2}\right]^2}{n+1}=\frac{4r^2[9(n+1)R]^2}{n+1}=\frac{4r^2\cdot 81R^2(n+1)^2}{n+1}=$$

$$= 324(n+1)R^2r^2 = (n+1)(18Rr)^2 = RHS$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Tapas Das-India

In ΔABC

$$(ab+bc+ca)^2 + n(a^2+b^2+c^2) \ge (n+1)(18Rr)^2, n \in \mathbb{N}$$

$$(ab+bc+ca)^2 + n(a^2+b^2+c^2)^2 =$$

$$= (s^2+r^2+4Rr)^2 + n(2s^2-2r^2-8Rr)^2$$

$$\overset{Gerretsen}{\ge} (16Rr-5r^2+r^2+4Rr)^2 + n(32Rr-10r^2-2n^2-8Rr)^2$$

$$= (20Rr-4r^2) + n(24Rr-12r^2)^2$$

$$= (20Rr-4r^2)^2 + n(24Rr-8r^2-4r^2)^2$$

$$\overset{Euler}{\ge} (20Rr-4r^2)^2 + n\left(24Rr - 8\cdot\frac{R}{2}r - 4r^2\right)^2$$

$$= (20Rr-4r^2)^2 + n(20Rr-4r^2)^2 = (n+1)(20Rr+4r^2)^2$$

$$\overset{Euler}{\ge} (n+1)\left(20Rr - 4r\cdot\frac{R}{2}\right)^2 = (n+1)(18Rr)^2$$

JP.517 If $x, y, z \in [0; k]; k > 0$, then $y(x-z) - z(x-k) \le k^2$

Proposed by Laura Molea and Gheorghe Molea – Romania

Solution 1 by proposers

We construct the cube $ABCDA'B'C'D'$ with the side $k > 0$ and rectangular parallelepipeds $AB''C''D''MNPQ, PRSTP'R'C'T'$. If $AB'' = x \Rightarrow B''B = k-x$,

$$AD'' = y \Rightarrow D''D = k-y, A'M = z \Rightarrow AM = k-z$$

We have $V_1 = V_{AB''C''D''MNPQ} = xy(k-z)$

$$V_2 = V_{PRSTP'R'C'T'} = (k-x)(k-y)z$$

$$V = V_{ABCDA'B'C'D'} = k^3$$

But $V_1 + V_2 \le V \Rightarrow xy(k-z) + (k-x)(k-y)z \le k^3 \Leftrightarrow xyk - xyz + k^2z - kyz -$

$$-kxz + xyz \le k^3 \Leftrightarrow xy + kz - yz - xz \le k^2 \Leftrightarrow$$

$$\Leftrightarrow y(x-z) - z(x-k) \le k^2, \text{ Q.E.D.}$$

We have equality if: $x = y = k, z = 0$ or $x = y = 0, z = k$.

Solution 2 by Ivan Hadinata-Jember-Indonesia

Let $x = k\sin a$, $y = k\sin b$, $z = k\sin c$ where $a, b, c \in \left[0, \frac{\pi}{2}\right]$.

Thus, proving the inequality $y(x-z) - z(x-k) \le k^2$ is equivalent to proving

$$\sin a \cdot \sin b + \sin c \cdot (1 - \sin a - \sin b) \le 1$$

If $\sin a + \sin b \ge 1$, then

$$\sin a \cdot \sin b + \sin c \cdot (1 - \sin a - \sin b) \le \sin a \cdot \sin b \le 1.$$

(Equality holds iff $a = b = \frac{\pi}{2}$, $c = 0$)

If $\sin a + \sin b \le 1$, then

$$\sin a \cdot \sin b + \sin c \cdot (1 - \sin a - \sin b) \le \sin a \cdot \sin b - \sin a - \sin b + 1 =$$
$$= (1 - \sin a)(1 - \sin b) \le 1.$$

(Equality holds iff $a = b = 0$, $c = \frac{\pi}{2}$)

Solution 3 by Daniel Sitaru-Romania

$$f: [0,k]X[0,k]X[0,k] \to \mathbb{R}, [0,k] \text{ –compact}$$

$$f(x,y,z) = xy - yz - zx + zk$$

$$f'_x = y - z, f'_y = x - z, f'_z = -y - x + k$$

$$f''_{xx} = 0, f''_{yy} = 0, f''_{zz} = 0$$

$$f(0,0,0) = 0, f(k,0,0) = 0, f(0,k,0) = 0, f(0,0,k) = k^2$$

$$f(k,k,0) = k^2, f(0,k,k) = 0, f(k,0,k) = 0, f(k,k,k) = k^2$$

$$maxf(x,y,z) = max(0,k^2) = k^2$$

$$f(x,y,z) = xy - yz - zx + zk \le k^2$$

JP.518 Let $ABCD$ a convex quadrilateral, $\lambda \in \mathbb{R}$ and M, N be such that

$$\overrightarrow{AM} = \lambda \cdot \overrightarrow{AB}, \overrightarrow{DN} = \lambda \cdot \overrightarrow{DC}, \overrightarrow{AD} = 3\overrightarrow{BC}. \text{ Find } \lambda \in \mathbb{R} \text{ such that } \overrightarrow{MN} = 7\overrightarrow{BC}.$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$We\ have: \overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CN} \Rightarrow \lambda \cdot \overrightarrow{MN} = \lambda\left(\overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CN}\right) and$$

$$(1-\lambda)\overrightarrow{MN} = (1-\lambda)\left(\overrightarrow{MA} + \overrightarrow{AD} + \overrightarrow{DN}\right), hence$$

$$\overrightarrow{MN} = (\lambda + 1 - \lambda)\overrightarrow{MN} = \lambda\left(\overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CN}\right) + (1-\lambda)\left(\overrightarrow{MA} + \overrightarrow{AD} + \overrightarrow{DN}\right) =$$

$$= \lambda \cdot \overrightarrow{BC} + (1-\lambda)\overrightarrow{AD} + \left[\lambda\overrightarrow{MB} + (1-\lambda)\overrightarrow{MA}\right] + \left[\lambda \cdot \overrightarrow{CN} + (1-\lambda)\overrightarrow{DN}\right]$$

$$Let\ CE \parallel AB, CE = AB. From\ 3\overrightarrow{BC} = \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}, it\ follows:$$

$$2\overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{EC} + \overrightarrow{CD} = \overrightarrow{ED}, then\ DE \parallel BE, DE = 2BC\ and\ hence,$$

$$ABCE\ is\ an\ parallelogram, so\ AE \parallel BC, AE = BC.$$

$$Therefore, \quad \overrightarrow{MN} = \lambda \cdot \overrightarrow{BC} + (1-\lambda)\overrightarrow{AD}; \overrightarrow{MN} = (3-2\lambda)\overrightarrow{BC} \Rightarrow$$

$$\overrightarrow{MN} = 7\overrightarrow{BC} \Leftrightarrow 3 - 2\lambda = 7 \Leftrightarrow \lambda = -2.$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

Let O be the origin point. For all points X, define $\overline{X} = \overrightarrow{OX}$. From the given conditions, now we have the following equations:

$$\overline{M} - \overline{A} = \lambda(\overline{B} - \overline{A}) \quad \dots\dots\dots (1)$$
$$\overline{N} - \overline{D} = \lambda(\overline{C} - \overline{D}) \quad \dots\dots\dots (2)$$
$$\overline{D} - \overline{A} = 3(\overline{C} - \overline{B}) \quad \dots\dots\dots (3)$$
$$\overline{N} - \overline{M} = 7(\overline{C} - \overline{B}) \quad \dots\dots\dots (4)$$

Subtracting equation (1) from equation (2) gives us

$$(\overline{N} - \overline{M}) - (\overline{D} - \overline{A}) = \lambda(\overline{C} - \overline{B}) - \lambda(\overline{D} - \overline{A})$$
$$(\lambda - 1)(\overline{D} - \overline{A}) = \lambda(\overline{C} - \overline{B}) - (\overline{N} - \overline{M}) \quad \dots\dots\dots (5)$$

Substituting (3) and (4) to (5) yields

$$3(\lambda - 1)(\overline{C} - \overline{B}) = (\lambda - 7)(\overline{C} - \overline{B}).$$

Since $\overline{B} \neq \overline{C}$ then $3(\lambda - 1) = \lambda - 7$ and thus $\lambda = -2$ which is the only answer.

JP.519 In $\Delta ABC, AA', BB', CC'$ —internal bisectors, A'' —symmetric point of A to BC, $N \in (AB), M \in (AN)$ such that $\overrightarrow{CM} = x \cdot \overrightarrow{MN}, \overrightarrow{AB} = x \cdot \overrightarrow{AN}, x \in \mathbb{R}$. Prove that if $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 0$ then A, M, A'' are collinears.

Proposed by Florică Anastase-Romania

Solution by proposer

$$AA' - internal\ bisector \Rightarrow \frac{A'B}{A'C} = \frac{AB}{AC} = \frac{c}{b} \Rightarrow b \cdot \overrightarrow{A'B} = c \cdot \overrightarrow{A'C} \Rightarrow$$

$$b\left(\overrightarrow{AA'} + \overrightarrow{AB}\right) = c\left(\overrightarrow{A'A} + \overrightarrow{AC}\right) \Rightarrow (b+c)\overrightarrow{AA'} = b\overrightarrow{AB} + c\overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AA'} = \frac{b\overrightarrow{AB} + c\overrightarrow{AC}}{b+c}$$

$$Similarly, \overrightarrow{BB'} = \frac{c\overrightarrow{BC} + a\overrightarrow{BA}}{c+a}\ and\ \overrightarrow{CC'} = \frac{a\overrightarrow{CA} + b\overrightarrow{CB}}{b+a}.$$

$$\overrightarrow{AA'}+\overrightarrow{BB'}+\overrightarrow{CC'}=0\Leftrightarrow\frac{b\overrightarrow{AB}+c\overrightarrow{AC}}{b+c}+\frac{c\overrightarrow{BC}+a\overrightarrow{BA}}{c+a}+\frac{a\overrightarrow{CA}+b\overrightarrow{CB}}{b+a}=0\Leftrightarrow$$

$$\left(\frac{b}{b+c}-\frac{a}{a+c}\right)\overrightarrow{AB}+\left(\frac{c}{b+c}-\frac{b}{b+a}\right)\overrightarrow{AC}+\left(\frac{c}{a+c}-\frac{b}{b+a}\right)\overrightarrow{CB}=0$$

$$How\ \overrightarrow{AB},\overrightarrow{BC},\overrightarrow{CA}\ are\ not\ collinear\Rightarrow$$

$$\frac{b}{b+c}-\frac{a}{a+c}=0;\frac{c}{b+c}-\frac{b}{b+a}=0;\frac{c}{a+c}-\frac{b}{b+a}=0$$

$$\Rightarrow a=b=c\Rightarrow\Delta ABC-equilateral.$$

$$A''\ symmetric\ point\ of\ A, then\ ABA''C\ is\ an\ paralellogram, and\ hence$$

$$\overrightarrow{AA''}=\overrightarrow{AB}+\overrightarrow{AC};(1)$$

$$Now, from\ \overrightarrow{AB}=x\cdot\overrightarrow{AN},\overrightarrow{CM}=x\cdot\overrightarrow{MN}\Rightarrow\overrightarrow{CA}+\overrightarrow{AM}=x\left(\overrightarrow{MA}+\overrightarrow{AN}\right)$$

$$(1+x)\overrightarrow{AM}=\overrightarrow{AC}+x\cdot\overrightarrow{AN}=\overrightarrow{AB}+\overrightarrow{AC}\overset{(1)}{=}\overrightarrow{AA''}\Rightarrow A,M,A''-collinears.$$

JP.520 Prove, that in any ABC triangle, the following inequalities hold:

$$3\sqrt{\frac{2r}{R}}\le\sin\left(\frac{\hat{A}}{2}+\hat{B}\right)+\sin\left(\frac{\hat{B}}{2}+\hat{C}\right)+\sin\left(\frac{\hat{C}}{2}+\hat{A}\right)\le3$$

Proposed by Radu Diaconu – Romania

Solution 1 by proposer

We prove that:

$$\sin\left(\frac{\hat{A}}{2}+\hat{B}\right)+\sin\left(\frac{\hat{B}}{2}+\hat{C}\right)+\sin\left(\frac{\hat{C}}{2}+\hat{A}\right)=\sum_{cyc}\cos\frac{\hat{B}-\hat{C}}{2}$$

$$\sin\left(\frac{\hat{A}}{2}+\hat{B}\right)=\sin\left(\frac{\pi}{2}+\hat{B}-\frac{\hat{B}+\hat{C}}{2}\right)=\sin\left(\frac{\pi}{2}+\frac{\hat{B}-\hat{C}}{2}\right)=\cos\frac{\hat{B}-\hat{C}}{2}$$

$$\sin\left(\frac{\hat{B}}{2}+\hat{C}\right)=\sin\left(\frac{\pi}{2}+\hat{C}-\frac{\hat{C}+\hat{A}}{2}\right)=\sin\left(\frac{\pi}{2}+\frac{\hat{C}-\hat{A}}{2}\right)=\cos\frac{\hat{C}-\hat{A}}{2}$$

$$\sin\left(\frac{\hat{C}}{2}+\hat{A}\right)=\sin\left(\frac{\pi}{2}+\hat{A}-\frac{\hat{A}+\hat{B}}{2}\right)=\sin\left(\frac{\pi}{2}+\frac{\hat{A}-\hat{B}}{2}\right)=\cos\frac{\hat{A}-\hat{B}}{2}$$

$$\sin\left(\frac{\hat{A}}{2}+\hat{B}\right)+\sin\left(\frac{\hat{B}}{2}+\hat{C}\right)+\sin\left(\frac{\hat{C}}{2}+\hat{A}\right)=\cos\frac{\hat{B}-\hat{C}}{2}+\cos\frac{\hat{C}-\hat{A}}{2}+\cos\frac{\hat{A}-\hat{B}}{2}$$

Using the inequality $\cos\frac{\hat{A}-\hat{B}}{2}\geq\sqrt{\frac{2r}{R}}$ and the analogs, we obtain:

$$\cos\frac{\hat{B}-\hat{C}}{2}+\cos\frac{\hat{C}-\hat{A}}{2}+\cos\frac{\hat{A}-\hat{B}}{2}\geq 3\sqrt{\frac{2r}{R}}\quad (1)$$

$$\sum_{cyc}\sin\left(\frac{\hat{A}}{2}+\hat{B}\right)=\sum_{cyc}\cos\frac{\hat{B}-\hat{C}}{2}\leq 3\quad (2)$$

We have used the relationship:

$$\sum_{cyc}\cos\frac{\hat{B}-\hat{C}}{2}\leq 3$$

From (1) and (2), it follows the conclusion of the problem. Equality holds for an equilateral triangle. We prove that: $\cos\frac{\hat{A}-\hat{B}}{2}\geq\sqrt{\frac{2r}{R}}$. As:

$$\cos\frac{\hat{A}-\hat{B}}{2}=\cos\frac{\hat{A}}{2}\cos\frac{\hat{B}}{2}+\sin\frac{\hat{A}}{2}\sin\frac{\hat{B}}{2}=\frac{2p-c}{c}\sqrt{\frac{(p-a)(p-b)}{ab}};\frac{2r}{R}=\frac{8\cdot S^2}{p\cdot abc}$$

Inequality $\cos\frac{\hat{A}-\hat{B}}{2}\geq\sqrt{\frac{2r}{R}}$, is equivalent with:

$$\frac{2p-c}{c}\cdot\sqrt{\frac{(p-a)(p-b)}{ab}}\geq\sqrt{\frac{8p(p-a)(p-b)(p-c)}{pabc}}\Leftrightarrow\frac{2p-c}{c}\geq\sqrt{\frac{8(p-c)}{c}}\Leftrightarrow$$

$$2p-c\geq\sqrt{8c(p-c)}\Leftrightarrow(2p-c)^2\geq 8c(p-c)\Leftrightarrow 4p^2-12pc+9c^2\geq 0\Leftrightarrow$$

$$\Leftrightarrow(2p-3c)^2\geq 0\text{, obviously true.}$$

Solution 2 by Tapas Das-India

Prove that in any ΔABC

$$3\sqrt{\frac{2r}{R}}\leq\sin\left(\frac{A}{2}+B\right)+\sin\left(\frac{B}{2}+C\right)+\sin\left(\frac{C}{2}+A\right)\leq 3$$

Let $f(x)=\sin x, x\in(0,\pi)$

$\therefore f'(x)=\cos x, f''(x)=-\sin x<0$

$\therefore f$ is concave in $(0,\pi)$, using Jensen's we have

$$f\left(\frac{A}{2}+B\right)+f\left(\frac{B}{2}+C\right)+f\left(\frac{C}{2}+A\right)\leq 3f\left(\frac{\frac{A}{2}+B+\frac{B}{2}+C+\frac{C}{2}+A}{3}\right)$$

$$= 3f\left(\frac{\pi}{2}\right) = 3 \quad (\because A + B + C = \pi)$$

$$\frac{A}{2} + B = \frac{A}{2} + \frac{B}{2} + \frac{B}{2} = \frac{B}{2} + \frac{A+B}{2} = \frac{B}{2} + \frac{\pi - C}{2} = \frac{\pi}{2} + \left(\frac{B-C}{2}\right)$$

$$\therefore \sin\left(\frac{A}{2} + B\right) = \sin\left[\frac{\pi}{2} + \left(\frac{B-C}{2}\right)\right] = \cos\frac{B-C}{2}$$

Similarly, $\sin\left(\frac{B}{2} + C\right) = \cos\left(\frac{C-A}{2}\right)$

$$\sin\left(\frac{C}{2} + A\right) = \cos\left(\frac{A-C}{2}\right)$$

Now $\cos\frac{B-C}{2} = \cos\frac{B}{2} \cdot \cos\frac{C}{2} + \sin\frac{B}{2} \cdot \sin\frac{C}{2}$

$$= \sqrt{\frac{s(s-b)s(s-c)}{ac \cdot ab}} + \sqrt{\frac{(s-a)(s-c)}{ac} \cdot \frac{(s-a)(s-b)}{ab}}$$

Using Ravi's substitutions $\begin{cases} a = y + z \\ b = z + x \\ c = x + y \end{cases}$

and $a + b = c = 2(x + y + z)$

$$\therefore 2s = 2(x + y + z)$$

$$\therefore s = x + y + z$$

We need to show $\cos\frac{B-C}{2} \ge \sqrt{\frac{2r}{R}}$

Now the inequality equivalent to

$(2x + y + z)^2 \ge 8x(y + z)$. True according to AM-GM

$$\therefore \cos\frac{B-C}{2} \ge \sqrt{\frac{2r}{R}}$$

$$\therefore \sin\left(\frac{A}{2} + B\right) + \sin\left(\frac{B}{2} + C\right) + \sin\left(\frac{C}{2} + A\right) = \cos\frac{B-C}{2} + \cos\frac{C-A}{2} + \cos\frac{A-C}{2}$$

$$\ge 3\sqrt{\frac{2r}{R}}$$

JP.521 If $a,b,c,d>0$, such that $(a+b+c)(b+c+d)=1$, prove that:

$$\sqrt[3]{(a+b)(c+d)}+\sqrt[3]{(b+c)(d+a)}+\sqrt[3]{(c+d)(a+b)}+\sqrt[3]{(d+a)(b+c)}<$$

$$<\frac{1}{3}\left(\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}+4\right)$$

Proposed by Gheorghe Molea – Romania

Solution by proposer

$$\sqrt[3]{(a+b)(c+d)}=\sqrt[3]{\frac{a+b}{b+c}\cdot\frac{b+c}{a+b+c}\cdot\frac{c+d}{b+c+d}}\le$$

$$\le\frac{1}{3}\left(\frac{a+b}{b+c}+\frac{b+c}{a+b+c}+\frac{c+d}{b+c+d}\right)$$

$$\sqrt[3]{(b+c)(d+a)}=\sqrt[3]{\frac{b+c}{c+d}\cdot\frac{c+d}{a+b+c}\cdot\frac{d+a}{b+c+d}}\le$$

$$\le\frac{1}{3}\left(\frac{b+c}{c+d}+\frac{c+d}{a+b+c}+\frac{d+a}{b+c+d}\right)$$

$$\sqrt[3]{(c+d)(a+b)}=\sqrt[3]{\frac{c+d}{d+a}\cdot\frac{d+a}{a+b+c}\cdot\frac{a+b}{b+c+d}}\le$$

$$\le\frac{1}{3}\left(\frac{c+d}{d+a}+\frac{d+a}{a+b+c}+\frac{a+b}{b+c+d}\right)$$

$$\sqrt[3]{(d+a)(b+c)}\le\sqrt[3]{\frac{d+a}{a+b}\cdot\frac{a+b}{a+b+c}\cdot\frac{b+c}{b+c+d}}\le$$

$$\le\frac{1}{3}\left(\frac{d+a}{a+b}+\frac{a+b}{a+b+c}+\frac{b+c}{b+c+d}\right)$$

By adding the 4 inequalities we obtain:

$$\sqrt[3]{(a+b)(c+d)}+\sqrt[3]{(b+c)(d+a)}+\sqrt[3]{(c+d)(a+b)}+\sqrt[3]{(d+a)(b+c)}\le$$

$$\le\frac{1}{3}\left(\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}+4\right)\quad (*)$$

We will prove than we cannot have equality. $M_g=M_a\Leftrightarrow$ the numbers are equal, so:

$$\frac{a+b}{b+c}=\frac{b+c}{a+b+c}=\frac{c+d}{b+c+d}$$

$$\frac{b+c}{c+d}=\frac{c+d}{a+b+c}=\frac{d+a}{b+c+d}$$

$$\frac{c+d}{d+a}=\frac{d+a}{a+b+c}=\frac{a+b}{b+c+d}$$

$$\frac{d+a}{a+b}=\frac{a+b}{a+b+c}=\frac{b+c}{b+c+d}$$

Multiplying member with member the last equalities we obtain:

$$\frac{a+b}{b+c}\cdot\frac{b+c}{c+d}\cdot\frac{c+d}{d+a}\cdot\frac{d+a}{a+b}=$$

$$=\frac{(b+c)(c+d)(d+a)(a+b)}{(a+b+c)^4}=\frac{(c+d)(d+a)(a+b)(b+c)}{(b+c+d)^4}$$

$$\Rightarrow a+b+c=b+c+d\Rightarrow a=d,$$

$$(a+b+c)(b+c+d)=1\Rightarrow a+b+c=b+c+d=1$$

From $\frac{b+c}{a+b+c}=\frac{c+d}{b+c+d}\Rightarrow b=d$

From $\frac{c+d}{a+b+c}=\frac{d+a}{b+c+d}\Rightarrow a=c$, **so** $a=d=b=c$

From $\left.\begin{matrix}a+b+c=1\\ b+c+d=1\end{matrix}\right|\Rightarrow a=b=c=d=\frac{1}{3}$

Relationship $\frac{a+b}{b+c}=\frac{b+c}{a+b+c}$ **becomes** $1=\frac{2}{3}$ **(F), so we have strictly inequality in (*)**

JP.522 In acute triangle ABC the following relationship holds:

$$\left(\sum\frac{sin^2A}{cosA}\right)\left(\sum\frac{cosA}{sin^2A}\right)\geq 9+7\left(\frac{R-2r}{R+r}\right)$$

Proposed by Alexandru Szoros-Romania

Solution 1 by proposer

$$\sum\frac{sin^2A}{cosA}\geq\frac{\left(\sum sinA\right)^2}{\sum cosA}=\frac{\left(\frac{s}{R}\right)^2}{1+\frac{r}{R}}=\frac{s^2}{R(R+r)}\geq\frac{16Rr-5r^2}{R(R+r)}\Rightarrow$$

$$\sum\frac{sin^2A}{cosA}\geq\frac{r(16R-5r)}{R(R+r)}\qquad(1)$$

$$\sum\frac{cosA}{sin^2A}=\sum\left(\frac{b^2+c^2-a^2}{2bc}\right)\left(\frac{2R}{a}\right)^2=\frac{2R^2}{abc}\sum\left(\frac{b^2+c^2-a^2}{a}\right)=$$

$$=\frac{2R^2}{4sRr}\left(\sum\frac{b^2+c^2}{a}-\sum a\right)=\frac{R}{2sr}\left(\frac{\sum bc(b^2+c^2)}{abc}-2s\right)\geq$$

$$\geq\frac{R}{2sr}\left(\frac{2\sum b^2c^2}{abc}-2s\right)\geq\frac{R}{sr}\left(\sum a-s\right)=\frac{R}{r}\Rightarrow\sum\frac{cosA}{sin^2A}\geq\frac{R}{r} \qquad (2)$$

From (1) and (2) we get:

$$\left(\sum\frac{sin^2A}{cosA}\right)\left(\sum\frac{cosA}{sin^2A}\right)\geq\frac{16R-5r}{R+r}=9+7\left(\frac{R-2r}{R+r}\right)$$

Solution 2 by Marin Chirciu-Romania

Lemma 1. In ΔABC holds

$$\sum\frac{\cos A}{\sin^2 A}=\frac{s^2(s^2-8Rr)-r^2(4R+r)^2}{4s^2r^2}$$

$$\sum\frac{\cos A}{\sin^2 A}\geq\frac{7R-6r}{4r}$$

Solution:

$$\sum\frac{\cos A}{\sin^2 A}=\sum\frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{a^2}{4R^2}}=\frac{2R^2}{abc}\sum\frac{b^2+c^2-a^2}{a}$$

$$=\frac{2R^2}{4Rrs}\cdot\frac{s^2(s^2-8Rr)-r^2(4R+r)^2}{2Rrs}=\frac{s^2(s^2-8Rr)-r^2(4R+r)^2}{4s^2r^2}$$

We've used above:

$$\sum\frac{b^2+c^2-a^2}{a}=\frac{s^2(s^2-8Rr)-r^2(4R+r)^2}{2Rrs}$$

$$\sum\frac{\cos A}{\sin^2 A}=\frac{s^2(s^2-8Rr)-r^2(4R+r)^2}{4s^2r^2}=\frac{s^2-8Rr}{4r^2}-\frac{(4R+r)^2}{4s^2}\overset{Gerretsen}{\geq}$$

$$\geq\frac{16Rr-5r^2-2r^2-8Rr}{4r^2}-\frac{(4R+r)^2}{4\cdot\frac{r(4R+r)^2}{R+r}}=$$

$$=\frac{8Rr-7r^2}{4r^2}-\frac{R+r}{4r}=\frac{8R-7r}{4r}-\frac{R+r}{4r}=\frac{7R-6r}{4r}$$

Equality holds if and only if the triangle is equilateral.

Lemma 2. In ΔABC holds

$$\sum \frac{\sin^2 A}{\cos A} = \frac{r(8R^2 + 6Rr + r^2 - s^2)}{R[s^2 - (2R + r)^2]}$$

$$\sum \frac{\sin^2 A}{\cos A} \geq \frac{2R^2 + Rr - r^2}{Rr}$$

Solution:

$$\sum \frac{\sin^2 A}{\cos A} = \sum \frac{\frac{a^2}{4R^2}}{\frac{b^2 + c^2 - a^2}{2bc}} = \frac{abc}{2R} \sum \frac{a}{b^2 + c^2 - a^2} = \frac{4Rrs}{2R^2} \cdot \frac{8R^2 + 6Rr + r^2 - s^2}{2s[s^2 - (2R + r)^2]}$$
$$= \frac{r(8R^2 + 6Rr + r^2 - s^2)}{R[s^2 - (2R + r)^2]}$$

We've used above:

$$\sum \frac{a}{b^2 + c^2 - a^2} = \frac{8R^2 + 6Rr + r^2 - s^2}{2s[s^2 - (2R + r)^2]}$$

$$\sum \frac{\sin^2 A}{\cos A}$$
$$= \frac{r(8R^2 + 6Rr + r^2 - s^2)}{R[s^2 - (2R + r)^2]} \overset{Gerretsen}{\geq} \frac{r(8R^2 + 6Rr + r^2 - 4R^2 - 4Rr - 3r^2)}{R[4R^2 + 4Rr + 3r^2 - (4R^2 + 4Rr + r^2)]}$$
$$= \frac{r(4R^2 + 2Rr - 2r^2)}{R(2r^2)} = \frac{2R^2 + Rr - r^2}{Rr}$$

Equality holds if and only if the triangle is equilateral. Let's get back to the main problem

Using the lemmas above the inequality can be written:

$$\frac{7R - 6r}{4r} \cdot \frac{2R^2 + Rr - r^2}{Rr} \geq 9 + 7 \cdot \frac{R - 2r}{R + r} \Leftrightarrow$$

$$\Leftrightarrow \frac{7R - 6r}{4r} \cdot \frac{2R^2 + Rr - r^2}{Rr} \geq \frac{16R - 5r}{R + r} \Leftrightarrow$$

$$\Leftrightarrow \frac{14R^3 - 5R^2 r - 13Rr^2 + 6r^3}{4Rr^2} \geq \frac{16R - 5r}{R + r} \Leftrightarrow$$

$$\Leftrightarrow 14R^4 + 9R^3 r - 82R^2 r^2 + 13Rr^3 + 6r^4 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R - 2r)(14R^3 + 37R^2 r - 8Rr^2 - 3r^3) \geq 0,$$

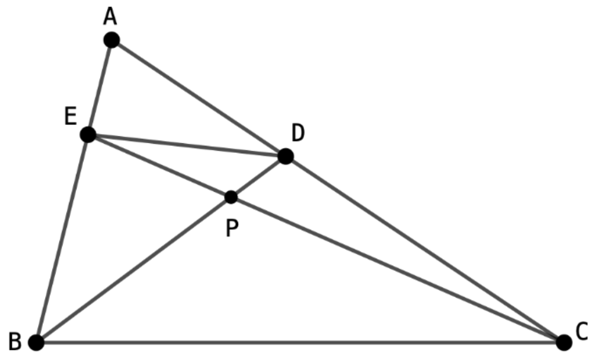
which follows from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

JP.523 On the sides AB and AC of a triangle ABC, consider the interior points E and D, respectively, such that $\left(\frac{AE}{EB}\right)^2+\left(\frac{AD^2}{DC}\right)^2=1$. The segments BD and CE intersect at point P. Find the ratio of the areas of quadrilateral $EBCD$ and triangle PBC.

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by proposer



Let $\frac{AE}{EB}=m$ and $\frac{AD}{DC}=n$, then $m^2+n^2=1$. Using the Menelaus' theorem in triangle ACE with intersecting line BPD, we have $\frac{AB}{BE}\cdot\frac{EP}{PC}\cdot\frac{CD}{DA}=1$. Also, we have $\frac{AB}{BE}=m+1, \frac{DC}{AD}=\frac{1}{n}$, so,

$(m+1)\cdot\frac{EP}{PC}\cdot\frac{1}{n}=1$, namely, $\frac{EP}{PC}=\frac{n}{m+1}\Leftrightarrow\frac{EC}{PC}=\frac{m+n+1}{m+1}$.

Again, using the Menelaus' theorem in triangle ABD with intersecting line CPE, we have $\frac{AC}{CD}\cdot\frac{DP}{PB}\cdot\frac{BE}{EA}=1$. So, $(n+1)\cdot\frac{DP}{PB}\cdot\frac{1}{m}=1\Leftrightarrow\frac{DP}{PB}=\frac{m}{n+1}\Leftrightarrow\frac{BD}{PB}=\frac{m+n+1}{n+1}$.

We have $\frac{EC}{PC}\cdot\frac{BD}{PB}=\frac{(m+n+1)^2}{(m+1)(n+1)}=\frac{m^2+n^2+2mn+2n+2m+1}{(m+1)(n+1)}=\frac{1+1+2mn+2m+2n}{(m+1)(n+1)}=\frac{2(1+mn+m+n)}{(m+1)(n+1)}=$ 2

Also, we have (Fig.) $\frac{[PBE]}{[PBC]}=\frac{PE}{PC},\frac{[PCD]}{[PBC]}=\frac{PD}{PB},\frac{[PDE]}{[PBC]}=\frac{PD\cdot PE}{PB\cdot PC}$. Now, we have

$$\frac{[EBCD]}{[PBC]}=\frac{[PBC]+[PBE]+[PED]+[PCD]}{[PBC]}=1+\frac{[PBE]}{[PBC]}+\frac{[PED]}{[PBC]}+\frac{[PCD]}{[PBC]}=$$

$$1+\frac{PE}{PC}+\frac{PD\cdot PE}{PB\cdot PC}+\frac{PD}{PB}=\frac{PB\cdot PC+PB\ \cdot PE+PD\cdot PE+PC\cdot PD}{PB\cdot PC}=$$

$$=\frac{(PB+PD)\cdot(PC+PE)}{PB\cdot PC}=\frac{BD\cdot EC}{PB\cdot PC}=2$$

So,

$$\frac{[EBCD]}{[PBC]}=2$$

JP.524 Prove that in any ΔABC the following inequality holds:

$$\frac{\cot\frac{A}{2}}{h_a}+\frac{\cot\frac{B}{2}}{h_b}+\frac{\cot\frac{C}{2}}{h_c}\ge\frac{4R+r}{F}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer

We have the inequality:

$$a(y+z)+b(z+x)+c(x+y)\ge 2\sqrt{(xy+xz+yz)(ab+ac+bc)},\forall x,y,z,a,b,c>0$$

true, because we can assume $x+y+z=1$, being homogeneous and then we must show that:

$$2\sqrt{(xy+xz+yz)(ab+ac+bc)}+ax+by+cz\le a+b+c,$$

which we prove applying twice Cauchy's inequality:

$$ax+by+cz+2\sqrt{(xy+yz+yz)(ab+ac+bc)}\le\sqrt{a^2+b^2+c^2}\sqrt{x^2+y^2+z^2}+$$
$$+\sqrt{2(xy+xz+yz)}\cdot\sqrt{2(ab+ac+bc)}\le$$
$$\le\sqrt{a^2+b^2+c^2+2(ab+ac+bc)}\cdot\sqrt{x^2+y^2+z^2+2(xy+xz+yz+1)}$$
$$=a+b+c$$

We take $a=\cot\frac{A}{2},b=\cot\frac{B}{2},c=\cot\frac{C}{2}\wedge x=\frac{1}{r_a},y=\frac{1}{r_b},z=\frac{1}{r_c}\Rightarrow$

$$\sum\cot\frac{A}{2}\left(\frac{1}{r_b}+\frac{r}{r_c}\right)\ge 2\sqrt{\left(\sum\cot\frac{A}{2}\cdot\cot\frac{A}{2}\right)\cdot\left(\sum\frac{1}{r_b}\cdot\frac{1}{r_c}\right)}\quad\textbf{(1)}$$

But $\frac{1}{r_b}+\frac{1}{r_c}=\frac{2}{h_a}$ **(2)**

From (1) + (2) $\Rightarrow\sum\frac{\cot\frac{A}{2}}{h_a}\ge\sqrt{\left(\sum\cot\frac{A}{2}\cdot\cot\frac{A}{2}\right)\left(\sum\frac{1}{r_br_c}\right)}$ **(3)**

But $\sum\frac{1}{r_br_c}=\frac{4R+r}{p^2r}\wedge\sum\cot\frac{A}{2}\cdot\cot\frac{B}{2}=\frac{4R+r}{r}$ **(4)**

From (3) + (4) $\Rightarrow \sum \frac{\cot\frac{A}{2}}{h_a} \geq \sqrt{\frac{(4R+r)^2}{p^2r^2}} = \frac{4R+r}{p^2r} = \frac{4R+r}{S}$

Solution 2 by Marin Chirciu-Romania

$$\sum \frac{\cot\frac{A}{2}}{h_a} \geq \frac{4R+r}{F}$$

Lemma. In ΔABC holds

$$\sum \frac{\cot\frac{A}{2}}{h_a} = \frac{4R+r}{F}.$$

Using $\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ and $h_a = \frac{2F}{a}$ we obtain:

$$\sum \frac{\cot\frac{A}{2}}{h_a} = \sum \frac{\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}}{\frac{2F}{a}} = \frac{1}{2F}\sum a\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} =$$

$$= \frac{1}{2F}\sum a\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} =$$

$$= \frac{1}{2F}\sum a\frac{s(s-a)}{F} = \frac{s}{2F^2}\sum a\,(s-a) = \frac{s}{2F^2}\cdot 2r(4R+r) = \frac{4R+r}{F}$$

We've used above $\sum a(s-a) = 2r(4R+r)$

Using the lemma we deduce that the relation from enunciation holds, with equality in any triangle.

Extensions by Marin Chirciu-Romania

Remark: In the same way : In ΔABC:

$$\frac{3r}{F} \leq \sum \frac{\tan\frac{A}{2}}{h_a} \leq \frac{2R-r}{F}$$

Marin Chirciu

Solution: Lemma: In ΔABC:

$$\sum \frac{\tan\frac{A}{2}}{h_a} = \frac{2R-r}{F}$$

Proof.

Using $\tan\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}$ and $h_a=\frac{2F}{a}$ we obtain:

$$\sum\frac{\tan\frac{A}{2}}{h_a}=\sum\frac{\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}}{\frac{2F}{a}}=\frac{1}{2F}\sum a\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}=$$

$$=\frac{1}{2F}\sum a\frac{(p-b)(p-c)}{\sqrt{p(p-a)(p-b)(p-c)}}=$$

$$=\frac{1}{2F}\sum a\frac{(p-b)(p-c)}{F}=\frac{1}{2F^2}\sum a(p-b)(p-c)=$$

$$=\frac{1}{2F^2}\cdot 2pr(2R-r)=\frac{2R-r}{F}$$

We used above $\sum a(p-b)(p-c)=2pr(2R-r)$.

Using Lemma and Euler's inequality $R\geq 2r$ we deduce the conclusion.

Remark.

Between the sums $\sum\frac{\tan\frac{A}{2}}{h_a}$ and $\sum\frac{\cot\frac{A}{2}}{h_a}$ the following relationship exists:

In ΔABC:

$$\sum\frac{\cot\frac{A}{2}}{h_a}\leq 3\sum\frac{\tan\frac{A}{2}}{h_a}$$

Marin Chirciu

Solution.

Using the above Lemmas we have the sums $\sum\frac{\tan\frac{A}{2}}{h_a}=\frac{2R-r}{F}$ and $\sum\frac{\cot\frac{A}{2}}{h_a}=\frac{4R+r}{F}$.

Inequality can be written: $\frac{4R+r}{F}\leq 3\cdot\frac{2R-r}{F}\Leftrightarrow R\geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: In ΔABC:

$$\frac{3r}{2F}\leq\sum\frac{\tan\frac{A}{2}}{r_a}\leq\frac{3R}{4F}$$

Marin Chirciu

Solution: Lemma. In ΔABC:

$$\sum \frac{\tan\frac{A}{2}}{r_a} = \frac{3r}{2F}$$

Proof.

Using $\tan\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}}$ **and** $r_a = \frac{F}{p-a}$ **we obtain:**

$$\sum \frac{\tan\frac{A}{2}}{h_a} = \sum \frac{\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}}{\frac{F}{p-a}} = \frac{1}{2F}\sum(p-a)\sqrt{\frac{(p-b)(p-c)}{p(p-a)}} =$$

$$= \frac{1}{2F}\sum(p-a)\frac{(p-b)(p-c)}{\sqrt{p(p-a)(p-b)(p-c)}} = \frac{1}{2F}\sum(p-a)\frac{(p-b)(p-c)}{F}$$

$$= \frac{1}{2F^2}\sum(p-a)(p-b)(p-c) = \frac{1}{2F^2}\cdot 3pr^2 = \frac{3r}{2F}$$

We use above $(p-a)(p-b)(p-c) = pr^2$.

Using Lemma and Euler's inequality $R \geq 2r$ we deduce the conclusion.

Remark : In the same way : In ΔABC:

$$\frac{9}{2p} \leq \sum \frac{\cot\frac{A}{2}}{r_a} \leq \frac{1}{2p}\left(\frac{2R}{r} - 1\right)^2.$$

Marin Chirciu

Solution: Lemma.

In ΔABC:

$$\sum \frac{\cot\frac{A}{2}}{r_a} = \frac{p(p^2 - 2r^2 - 8Rr)}{2F^2}.$$

Proof.

Using $\cot\frac{A}{2} = \sqrt{\frac{p(p-a)}{(p-b)(p-c)}}$ **and** $r_a = \frac{F}{p-a}$ **we obtain:**

$$\sum \frac{\cot\frac{A}{2}}{r_a} = \sum \frac{\sqrt{\frac{p(p-a)}{(p-b)(p-c)}}}{\frac{F}{p-a}} = \frac{1}{2F}\sum (p-a)\sqrt{\frac{p(p-a)}{(p-b)(p-c)}} =$$

$$= \frac{1}{2F}\sum (p-a)\frac{p(p-a)}{\sqrt{p(p-a)(p-b)(p-c)}} =$$

$$= \frac{1}{2F}\sum (p-a)\frac{p(p-a)}{F} = \frac{p}{2F^2}\sum (p-a)^2 =$$

$$= \frac{p}{2F^2}\cdot (p^2-2r^2-8Rr) = \frac{p(p^2-2r^2-8Rr)}{2F^2}$$

We used above $\sum (p-a)^2 = p^2-2r^2-8Rr$.

Using the Lemma we obtain:

Right hand inequality:

$$\sum \frac{\cot\frac{A}{2}}{r_a} = \frac{p(p^2-2r^2-8Rr)}{2F^2} \overset{Gerretsen}{\leq} \frac{p(4R^2+4Rr+3r^2-2r^2-8Rr)}{2F^2} =$$

$$= \frac{p(4R^2-4Rr+r^2)}{2F^2} = \frac{p(2R-r)^2}{2F^2} = \frac{1}{2p}\left(\frac{2R}{r}-1\right)^2.$$

Equality holds if and only if the triangle is equilateral.

Left hand inequality:

$$\sum \frac{\cot\frac{A}{2}}{r_a} = \frac{p(p^2-2r^2-8Rr)}{2F^2} \overset{Gerretsen}{\leq} \frac{p(16Rr-5r^2-2r^2-8Rr)}{2F^2} =$$

$$= \frac{p(8Rr-7r^2)}{2F^2} = \frac{pr(8R-7r)}{2F^2} = \frac{8R-7r}{2F} \overset{Euler}{\geq} \frac{9r}{2pr} = \frac{9}{2p}$$

Equality holds if and only if the triangle is equilateral.

Remark: Between the sums $\sum \frac{\tan\frac{A}{2}}{r_a}$ and $\sum \frac{\cot\frac{A}{2}}{r_a}$ the following relationship exists:

In ΔABC:

$$\sum \frac{\cot\frac{A}{2}}{r_a} \geq 3\sum \frac{\tan\frac{A}{2}}{r_a}$$

Marin Chirciu

Solution: Using the above Lemmas we have the sums:

$$\sum\frac{\tan\frac{A}{2}}{r_a}=\frac{3r}{2F}\text{ and }\sum\frac{\cot\frac{A}{2}}{r_a}=\frac{p(p^2-2r^2-8Rr)}{2F^2}$$

Inequality can be written: $\frac{p(p^2-2r^2-8Rr)}{2F^2}\geq 3\cdot\frac{3r}{2F}\Leftrightarrow p^2-2r^2-8Rr\geq 9r^2\Leftrightarrow$

$\Leftrightarrow p^2\geq 8Rr+11r^2$, which follows from Gerretsen's inequality: $p^2\geq 16Rr-5r^2$.

It remains to prove that:

$16Rr-5r^2\geq 8Rr+11r^2\Leftrightarrow 8Rr\geq 16r^2\Leftrightarrow R\geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Solution 3 by Tapas Das-India

$$\sum\frac{\cot\frac{A}{2}}{h_a}=\frac{s}{r_a}\cdot\frac{1}{h_a}+\frac{s}{r_b}\cdot\frac{1}{h_b}+\frac{s}{r_c}\cdot\frac{1}{h_c}$$

$$=\frac{as}{r_a}\cdot\frac{1}{2F}+\frac{bs}{r_b}\cdot\frac{1}{2F}+\frac{cs}{r_c}\cdot\frac{1}{2F}=\frac{s}{2F}\left[\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right]$$

$$=\frac{s}{2F^2}[a(s-a)+b(s-b)+c(s-c)]=\frac{s}{2F^2}[2s^2-a^2-b^2-c^2]$$

$$=\frac{s}{2F^2}[2s^2-2s^2+2r^2+8Rr]=\frac{s}{2F^2}2r(4R+r)=\frac{F}{F^2}(4R+r)=\frac{4R+r}{F}$$

JP.525 Prove that in any ΔABC the following inequality holds:

$$\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c}{h_c}\leq\frac{(2R-r)^2}{r}$$

where n_a, n_b, n_c are Nagel's cevians.

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer

$$n_a^2=p(p-a)+\frac{(b-c)^2}{a}p\Rightarrow$$

$$\sum\frac{n_a^2}{h_a}=\sum\frac{p(p-a)}{h_a}+\frac{(b-c)^2}{ah_a}\cdot p=\sum\frac{p(p-a)}{\frac{2S}{a}}+\frac{(b-c)^2}{2S}\cdot p$$

$$=\frac{p}{2S}\sum(a(p-a)+(b-c)^2)=\frac{1}{2r}\left(\sum a(p-a)+\sum(b-c)^2\right)\quad\textbf{(1)}$$

$$\sum a(p-a)=2r(4R+r) \quad \textbf{(2)}$$

$$\sum(b-c)^2=2\left(a^2+b^2+c^2-ab-bc-ac\right)=2\left(2p^2-2r^2-8Rr-p^2-r^2-4Rr\right)=$$
$$2\left(p^2-12Rr-3r^2\right) \quad \textbf{(3)}$$

From (1) + (2) + (3) $\Rightarrow \sum \frac{n_a^2}{h_a}=\frac{1}{r}(4Rr+r^2+p^2-12Rr-3r^2)=$

$$=\frac{1}{r}(p^2-8Rr-2r^2) \overset{Gerretsen}{\leq} \frac{1}{r}(4R^2+4Rr+3r^2-8Rr-2r^2)$$

$$=\frac{1}{r}(4R^2-4Rr+r^2)=\frac{(2R-r)^2}{r}$$

Solution 2 and extensions by Marin Chirciu-Romania

Lemma: In ΔABC:

$$\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}=\frac{p^2-2r^2-8Rr}{r}$$

Proof.

Using $n_a^2=p(p-a)+\frac{p(b-c)^2}{a}$ **and** $h_a=\frac{2pr}{a}$ **we obtain:**

$$\sum\frac{n_a^2}{h_a}=\sum\frac{p(p-a)+\frac{p(b-c)^2}{a}}{\frac{2pr}{a}}=\frac{1}{2r}\sum[a(p-a)+(b-c)^2]=$$

$$=\frac{2r(4R+r)+2(p^2-3r^2-12Rr)}{2r}=\frac{2r(4R+r)+2(p^2-3r^2-12Rr)}{2r}=$$
$$=\frac{p^2-2r^2-8Rr}{r}$$

We've used above:

$\sum a(p-a)=2r(4R+r)$ **and** $\sum(b-c)^2=2(p^2-3r^2-12Rr)$

Let's get back to the main problem.

Using the Lemma we obtain:

$$LHS=\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}\overset{Lemma}{=}\frac{p^2-2r^2-8Rr}{r}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2-2r^2-8Rr}{r}=$$
$$=\frac{4R^2-4Rr+r^2}{r}=\frac{(2R-r)^2}{r}=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: Let's find an inequality with opposite sense.

$$\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}\geq 9r$$

where n_a, n_b, n_c are Nagel cevians.

Marin Chirciu

Solution: Lemma: In ΔABC

$$\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}=\frac{p^2-2r^2-8Rr}{r}$$

Proof.

Using $n_a^2=p(p-a)+\frac{p(b-c)^2}{a}$ **and** $h_a=\frac{2pr}{a}$ **we obtain:**

$$\sum\frac{n_a^2}{h_a}=\sum\frac{p(p-a)+\frac{p(b-c)^2}{a}}{\frac{2pr}{a}}=\frac{1}{2r}\sum[a(p-a)+(b-c)^2]=$$

$$=\frac{2r(4R+r)+2(p^2-3r^2-12Rr)}{2r}=\frac{2r(4R+r)+2(p^2-3r^2-12Rr)}{2r}=$$

$$=\frac{p^2-2r^2-8Rr}{r}$$

Let's get back to the main problem.

Using the Lemma we obtain:

$$LHS=\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}\overset{Lemma}{=}\frac{p^2-2r^2-8Rr}{r}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2-2r^2-8Rr}{r}=$$

$$=\frac{8Rr-7r^2}{r}=8R-7r\overset{Euler}{\geq}9r=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: The double inequality can be written:

In ΔABC**:**

$$9r\leq\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}\leq\frac{(2R-r)^2}{r},$$

where n_a,n_b,n_c **are Nagel's cevians.**

Remark: In the same way: In ΔABC**:**

$$\frac{9R}{2}\leq\frac{n_a^2}{r_a}+\frac{n_b^2}{r_b}+\frac{n_c^2}{r_c}\leq\frac{9R^4}{16r^3}$$

where n_a,n_b,n_c **are Nagel cevians.**

Marin Chirciu

Solution: Lemma: In ΔABC**:**

$$\frac{n_a^2}{r_a}+\frac{n_b^2}{r_b}+\frac{n_c^2}{r_c}=\frac{p^2(R-r)-r^2(4R+r)}{Rr}$$

Proof.

Using $n_a^2=p(p-a)+\frac{p(b-c)^2}{a}$ **and** $r_a=\frac{pr}{p-a}$ **we obtain:**

$$\sum\frac{n_a^2}{r_a}=\sum\frac{p(p-a)+\frac{p(b-c)^2}{a}}{\frac{pr}{p-a}}=\frac{1}{r}\sum\left[(p-a)^2+\frac{1}{a}(p-a)(b-c)^2\right]=$$

$$=\frac{1}{r}\left[\sum(p-a)^2+\sum\frac{1}{a}(p-a)(b-c)^2\right]=$$

$$=\frac{1}{r}\left[p^2-2r^2-8Rr+\frac{r(8R^2-2Rr-r^2-p^2)}{R}\right]=$$
$$=\frac{p^2(R-r)-r^2(4R+r)}{Rr}$$

We've used above:

$\sum(p-a)^2=p^2-2r^2-8Rr$ **and** $\sum\frac{1}{a}(p-a)(b-c)^2=\frac{r(8R^2-2Rr-r^2-p^2)}{R}$

Let's get back to the main problem. Using the Lemma we obtain:

Right hand inequality.

$$\frac{n_a^2}{r_a}+\frac{n_b^2}{r_b}+\frac{n_c^2}{r_c}\overset{Lemma}{=}\frac{p^2(R-r)-r^2(4R+r)}{Rr}\overset{Gerretsen}{\le}$$
$$\le\frac{(4R^2+4Rr+3r^2)(R-r)-r^2(4R+r)}{Rr}=$$
$$=\frac{4R^3-5Rr^2-4r^3}{Rr}\overset{Euler}{\le}\frac{9R^4}{16r^3}=RHS$$

We've use above $\frac{4R^3-5Rr^2-4r^3}{Rr}\overset{Euler}{\le}\frac{9R^4}{16r^3}\Leftrightarrow 9R^5-64R^3r^2+80Rr^4+64r^5\ge 0\Leftrightarrow$

$\Leftrightarrow(R-2r)(9R^4+18R^3r-28R^3r-28R^2r-32r^4)\ge 0$**, obviously from Euler's inequality** $R\ge 2r$**.**

Equality holds if and only if the triangle is equilateral.

Left hand inequality.

$$\frac{n_a^2}{r_a}+\frac{n_b^2}{r_b}+\frac{n_c^2}{r_c}\overset{Lemma}{=}\frac{p^2(R-r)-r^2(4R+r)}{Rr}\overset{Gerretsen}{\ge}$$
$$\ge\frac{(16Rr-5r^2)(R-r)-r^2(4R+r)}{Rr}=\frac{16R^2-25Rr+4r^2}{R}\overset{Euler}{\ge}\frac{9R}{2}$$

Equality holds if and only if the triangle is equilateral.

Remark.

Between the sums $\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}$ **and** $\frac{n_a^2}{r_a}+\frac{n_b^2}{r_b}+\frac{n_c^2}{r_c}$ **the following relationship exists:**

In ΔABC**:**

$$\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}\le\frac{n_a^2}{r_a}+\frac{n_b^2}{r_b}+\frac{n_c^2}{r_c}$$

where n_a,n_b,n_c **are Nagel's cevians.**

Marin Chirciu

Solution: Using the above Lemmas we have the sums:

$$\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}=\frac{p^2-2r^2-8Rr}{r}\text{ and }\frac{n_a^2}{r_a}+\frac{n_b^2}{r_b}+\frac{n_c^2}{r_c}=\frac{p^2(R-r)-r^2(4R+r)}{Rr}$$

Inequality can be written:

$$\frac{p^2-2r^2-8Rr}{r}\le\frac{p^2(R-r)-r^2(4R+r)}{Rr}\Leftrightarrow$$
$$\Leftrightarrow R(p^2-2r^2-8Rr)\le p^2(R-r)-r^2(4R+r)\Leftrightarrow$$

$\Leftrightarrow rp^2\le r(8R^2-2Rr-r^2)\Leftrightarrow p^2\le 8R^2-2Rr-r^2$,

which follows from Gerretsen's inequality $p^2\le 4R^2+4Rr+3r^2$**.**

It remains to prove:

$$4R^2+4Rr+3r^2\le 8R^2-2Rr-r^2\Leftrightarrow 2R^2-3Rr-2r^2\Leftrightarrow (R-2r)(2R+r)\ge 0,$$

obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Solution 3 by Tapas Das-India

$$n_a^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}$$

$$=s^2-\frac{4s(s-b)(s-c)}{a}=s^2-\frac{4s\cdot sr^2}{a(s-a)}\Rightarrow n_a^2=s^2-2r_ah_a$$

$$\frac{n_a^2}{h_a}=\frac{s^2-2r_ah_a}{h_a}=\frac{s^2}{h_a}-2r_a$$

$$\therefore\sum\frac{n_a^2}{h_a}=s^2\sum\frac{1}{h_a}-2\sum r_a=\frac{s^2}{r}-2(4R+r)$$

$$=\frac{s^2-8Rr+2r^2}{r}\overset{Gerretsen}{\le}\frac{4R^2-4Rr+r^2}{r}=\frac{(2R-r)^2}{r}$$

Equality holds if and only if the triangle is equilateral.

PROBLEMS FOR SENIORS

SP.511 Let triangle ABC with $\widehat{A}>90^\circ$ and let internal points M_1,M_2,M_3,M_4 on the side BC, such that $BM_1=M_1M_2=M_2M_3=M_3M_4=M_4C$. Also, R_1,R_2 denote the circumraius of triangles AM_1M_2,AM_3M_4, respectively.

Prove:

$$BC>\frac{20\sqrt{R_1R_2}}{3(\cot B+\cot C)}$$

Proposed by George Apostolopoulos-Greece

Solution by proposer

Let $a=BC,b=CA,c=AB$ be the lengths of the side of triangle ABC.

We use the Stewart's Theorem, we have:

$$BM_1\cdot AC^2+M_1C\cdot AB^2=BC\cdot AM_1^2+BM_1\cdot M_1C\cdot BC$$

$$So, \frac{a}{5}\cdot b^2+\frac{4a}{5}\cdot c^2=a\cdot AM_1^2+\frac{a}{5}\cdot\frac{4a}{5}\cdot a \text{ or } \frac{b^2+4c^2}{5}=AM_1^2+\frac{4a^2}{25}\Leftrightarrow$$

$$AM_1^2=\frac{b^2+4c^2}{5}-\frac{4a^2}{25}. Since\ \widehat{A}>90^\circ, so\ a^2>b^2+c^2.$$

$$Namely, AM_1^2<\frac{b^2+4c^2}{5}-\frac{4(b^2+c^2)}{25}=\frac{b^2+16c^2}{25}$$

$$Also, BM_2\cdot AC^2+M_2C\cdot AB^2=BC\cdot AM_2^2+BM_2\cdot M_2C\cdot BC, namely,$$

$$\frac{2a}{5}\cdot b^2+\frac{3a}{5}\cdot c^2=a\cdot AM_2^2+\frac{2a}{5}\cdot\frac{3a}{5}\cdot a, so\ AM_2^2<\frac{4b^2+9c^2}{25}$$

$$Similarly, AM_3^2<\frac{9b^2+4c^2}{25}, and\ AM_4^2<\frac{16b^2+c^2}{25}.$$

$$So, AM_1^2+AM_2^2+AM_3^2+AM_4^2<\frac{6b^2+6c^2}{5}<\frac{6}{5}\cdot BC^2$$

$$We\ know\ that: AM_1^2+AM_2^2+AM_3^2+AM_4^2>\frac{(AM_1+AM_2+AM_3+AM_4)^2}{4}$$

$$So, \frac{(AM_1+AM_2+AM_3+AM_4)^2}{4}<\frac{6}{5}\cdot BC^2 \text{ or}$$

$$AM_1+AM_2+AM_3+AM_4<2\sqrt{\frac{6}{5}}\cdot BC; (AM_1, AM_2, AM_3, AM_4 \text{ are different in pairs})$$

$$Also, we\ have: AM_1+AM_2+AM_3+AM_4>4\cdot\sqrt[4]{AM_1\cdot AM_2\cdot AM_3\cdot AM_4}, so$$

$$4\cdot\sqrt[4]{AM_1\cdot AM_2\cdot AM_3\cdot AM_4}<2\sqrt{\frac{6}{5}}\cdot BC \text{ or}$$

$$AM_1\cdot AM_2\cdot AM_3\cdot AM_4<\frac{1}{16}\cdot\frac{6^2}{5^2}\cdot BC^4;\ \ (1)$$

$$In\ triangles\ AM_1M_2\ and\ AM_3M_4, we\ have:$$

$$AM_1\cdot AM_2=2R_1h_a \text{ and } AM_3\cdot AM_4=2R_2h_a, so$$

$$AM_1\cdot AM_2\cdot AM_3\cdot AM_4=4R_1R_2\cdot h_a^2$$

$$Also, we\ know\ that: h_a=\frac{BC}{\cot B+\cot C}. Now, inequality\ (1), gives:$$

$$4R_1R_2\cdot\frac{BC^2}{(\cot B+\cot C)^2}<\frac{9}{100}BC^4\Leftrightarrow BC^2>\frac{400R_1R_2}{9(\cot B+\cot C)^2}$$

$$BC > \frac{20\sqrt{R_1R_2}}{3(\cot B + \cot C)}$$

SP.512 Find all functions $f: \mathbb{Z} \to \mathbb{Z}$ such that for all integers x, y the number $f^2(x) + 2xf(y) + y^2$ is a perfect square.

Proposed by Baris Koyuncu-Turkiye

Solution by proposer

We claim that the answer is any function f satisfying $|f(x)| = |cx|$ or $f(x) = cx, f(x) = -cx$. Putting $x = p$ where p is an odd prime number, in the original equation we obtain that $f^2(p) + y^2$ is a quadratic residue in $modp$ for $y \in \mathbb{Z}$. So all the numbers:

$$\left\{f^2(p) + 0^2, f^2(p) + 1^2, \dots, f^2(p) + \left(\frac{p-1}{2}\right)^2\right\} = \left\{0^2, 1^2, \dots, \left(\frac{p-1}{2}\right)^2\right\}$$

in $modp$. In particular, the sum of elements of these two sets must be congruent so

$$\left(\frac{p+1}{2}\right)f^2(p) \equiv 0 \Longrightarrow f(p) \equiv 0 (modp)$$

Now, putting $y = 1$ in the original equation tell us that:

$f^2(x) + 2xf(x) + 1$ is a perfect square. If $|f(x)| > |xf(1)|$ then:

$$(|f(x)| - 1)^2 < f^2(x) + 2xf(x) + 1 < (|f(x)| + 1)^2$$

Hence we need to have:

$$f^2(x) + 2xf(x) + 1 = (|f(x)|)^2 \Longrightarrow 2xf(1) = -1 \text{ −contradiction}$$

So: $|f(x)| \leq |xf(1)|$

In particular: $|f(p)| \leq |pf(1)|$ for all odd primes p. Also we proved that:

$p|f(p)$. Then: $0 \leq \left|\frac{f(p)}{p}\right| \leq |f(1)|$.

Thus there exists a positive integer c such that:

$|f(p)| = cp$ for infinitely many odd primes p.

Let a be an abitrary integer. Choose a sufficiently large odd prime q such that: $|f(q)| = cq$. Put $x = a, y = q$ in the original equation.

$f^2(a) \pm 2acq + q^2$ is a perfect square. But:

$$(q \pm ac - 1)^2 < f^2(a) \pm 2acq + q^2 < (q \pm ac + 1)^2$$

since q is large enough. Then we must have:

$$f^2(a) \pm 2acq + q^2 = (q \pm ac)^2 \Rightarrow f^2(a) = (ac)^2 \Rightarrow |f(a)| = |ac|$$

Thus we get $f(x) = cx$ or $f(x) = -cx$ for all integers x.

SP.513 Given $k \geq 4$. In any triangle ABC prove that:

$$\frac{3}{k} \leq \sum_{cyc} \frac{\sin^2 A}{2\sin^2 A + \sin^2 B + \sin^2 C} \leq \frac{9k+12}{64}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by proposer

We know that $\frac{1}{x+y} \leq \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right)$

Let $a = BC, b = CA, c = AB$ be the lengths of the sides of the triangle ABC. So,

$$\frac{1}{a^2+b^2} \leq \frac{1}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \frac{1}{b^2+c^2} \leq \frac{1}{4}\left(\frac{1}{b^2} + \frac{1}{c^2}\right)$$

and

$$\frac{1}{c^2+a^2} \leq \frac{1}{4}\left(\frac{1}{c^2} + \frac{1}{a^2}\right)$$

Also, we have

$$\frac{1}{2a^2+b^2+c^2} = \frac{1}{(a^2+b^2)+(a^2+c^2)} \leq \frac{1}{4}\left(\frac{1}{a^2+b^2} + \frac{1}{a^2+c^2}\right)$$

namely

$$\frac{1}{2a^2+b^2+c^2} \leq \frac{1}{4}\left(\frac{1}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \frac{1}{4}\left(\frac{1}{a^2} + \frac{1}{c^2}\right)\right) = \frac{1}{16}\left(\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{1}{a^2}\right)$$

Similarly,

$$\frac{1}{a^2+2b^2+c^2} \leq \frac{1}{16}\left(\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{1}{b^2}\right)$$

and

$$\frac{1}{a^2+b^2+2c^2} \leq \frac{1}{16}\left(\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{1}{c^2}\right)$$

Let r be the inradius of triangle ABC we'll prove that:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}$$

We have $(b-c)^2 \geq 0 \Leftrightarrow a^2-(b-c)^2 \leq a^2 \Leftrightarrow$

$$\frac{1}{a^2} \leq \frac{1}{a^2-(b-c)^2} = \frac{1}{(a+b-c)(a-b+c)}$$

Let $2s = a+b+c$

$$\frac{1}{a^2} \leq \frac{1}{2(s-c)\cdot 2(s-b)} = \frac{1}{4(s-b)(s-c)}$$

Similarly,

$$\frac{1}{b^2} \leq \frac{1}{4(s-c)(s-a)}, \text{ and } \frac{1}{c^2} \leq \frac{1}{4(s-a)(s-b)}$$

So,

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \leq \frac{1}{4}\left(\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}\right) =$$

$$\frac{1}{4}\cdot\frac{s-a+s-b+s-c}{(s-a)(s-b)(s-c)} = \frac{1}{4}\cdot\frac{s}{(s-a)(s-b)(s-c)}$$

We know that

$(r\cdot s)^2 = s(s-a)(s-b)(s-c)$ (Heron). So

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \leq \frac{1}{4}\cdot\frac{s}{r^2\cdot s} = \frac{1}{4r^2}$$

Now, we have

$\frac{1}{2a^2+b^2+c^2} \leq \frac{1}{16}\left(\frac{1}{4r^2}+\frac{1}{a^2}\right)$, and similarly

$\frac{1}{a^2+2b^2+c^2} \leq \frac{1}{16}\left(\frac{1}{4r^2}+\frac{1}{b^2}\right)$, and $\frac{1}{a^2+b^2+2c^2} \leq \frac{1}{16}\left(\frac{1}{4r^2}+\frac{1}{c^2}\right)$

Namely $\frac{a^2}{2a^2+b^2+c^2} \leq \frac{1}{16}\left(\frac{a^2}{4r^2}+1\right)$, $\frac{b^2}{a^2+2b^2+c^2} \leq \frac{1}{16}\left(\frac{b^2}{4r^2}+1\right)$,

$$\frac{c^2}{a^2+b^2+2c^2} \leq \frac{1}{16}\left(\frac{c^2}{4r^2}+1\right)$$

Adding up these inequalities, we have

$$\sum_{cyc}\frac{a^2}{2a^2+b^2+c^2} \leq \frac{1}{16}\left(\frac{a^2+b^2+c^2}{4r^2}+3\right)$$

We know that

$$a^2+b^2+c^2 \leq 9R^2$$

So

$$\sum_{cyc}\frac{a^2}{2a^2+b^2+c^2}\le\frac{1}{16}\left(\frac{9R^2}{4r^2}+3\right)$$

Now, by the Cauchy-Schwarz inequality, we have

$$\sum_{cyc}\frac{a^2}{2a^2+b^2+c^2}\ge\frac{(a+b+c)^2}{4(a^2+b^2+c^2)}=\frac{4s^2}{4(a^2+b^2+c^2)}=\frac{s^2}{a^2+b^2+c^2}$$

We know that $s\ge 3\sqrt{3}r$, namely

$$\sum_{cyc}\frac{a^2}{2a^2+b^2+c^2}\ge\frac{\left(3\sqrt{3}r\right)^2}{9R^2}=\frac{3r^2}{R^2}$$

So

$$\frac{3r^2}{R^2}\le\sum_{cyc}\frac{a^2}{2a^2+b^2+c^2}\le\frac{1}{16}\left(\frac{9R^2}{4r^2}+3\right)$$

Using the law of the sines, we get

$$\frac{3r^2}{R^2}\le\sum_{cyc}\frac{\sin^2 A}{2\sin^2 A+\sin^2 B+\sin^2 C}\le\frac{9R^2}{64r^2}+\frac{3}{16}$$

From $R\ge 2r$ (Euler), we have $\frac{R^2}{r^2}\ge 4$. If $k=\frac{R^2}{r^2}$, then we have:

$$\frac{3}{k}\le\sum_{cyc}\frac{\sin^2 A}{2\sin^2 A+\sin^2 B+\sin^2 C}\le\frac{9k+12}{64}$$

Equality holds if the triangle ABC is equilateral.

Solution 2 and extensions by Marin Chirciu-Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\sum\frac{\sin^2 A}{2\sin^2 A+\sin^2 B+\sin^2 C}=$$

$$\sum\frac{\sin^2 A}{2\sin^2 A+\sin^2 B+\sin^2 C}=\sum\frac{\frac{a^2}{4R^2}}{2\cdot\frac{a^2}{4R^2}+\frac{b^2}{4R^2}+\frac{c^2}{4R^2}}=\sum\frac{a^2}{2a^2+b^2+c^2}$$

$$\sum\frac{a^2}{2a^2+b^2+c^2}=$$

$$=\frac{6p^6-2p^4(36Rr+5r^2)+2p^2r^2(156R^2+56Rr+5r^2)-6r^3(4R+r)^3}{9p^6-p^4(108Rr+23r^2)+p^2r^2(440R^2+200Rr+23r^2)-9r^3(4R+r)^3}$$

We've used above:

$$\sum\frac{a^2}{2a^2+b^2+c^2}=\frac{\sum a^2(2b^2+c^2+a^2)(2c^2+a^2+b^2)}{\prod(2a^2+b^2+c^2)},$$

$$\sum a^2(2b^2+c^2+a^2)(2c^2+a^2+b^2)=$$

$$=6p^6-2p^4(36Rr+5r^2)+2p^2r^2(156R^2+56Rr+5r^2)-6r^3(4R+r)^3,$$

$$\prod(2a^2+b^2+c^2)=$$

$$=9p^6-p^4(108Rr+23r^2)+p^2r^2(440R^2+200Rr+23r^2)-9r^3(4R+r)^3$$

Right hand inequality: We prove the stronger inequality:

$$\sum\frac{\sin^2 A}{2\sin^2 A+\sin^2 B+\sin^2 C}\le\frac{3}{4}$$

$$\frac{6p^6-2p^4(36Rr+5r^2)+2p^2r^2(156R^2+56Rr+5r^2)-6r^3(4R+r)^3}{9p^6-p^4(108Rr+23r^2)+p^2r^2(440R^2+200Rr+23r^2)-9r^3(4R+r)^3}\le\frac{3}{4}\Leftrightarrow$$

$$\Leftrightarrow 3p^6-p^4(36Rr+29r^2)+p^2r^2(72R^2+192Rr+29r^2)\ge 3r^3(4R+r)^3\Leftrightarrow$$

$$\Leftrightarrow p^2[p^2(3p^2-36Rr-29r^2)+r^2(72R^2+192Rr+29r^2)]\ge 3r^3(4R+r)^3,$$

which follows from Gerretsen's inequality $p^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(3(16Rr-5r^2)-36Rr-29r^2)$$
$$+r^2(72R^2+192Rr+29r^2)]\ge 3r^3(4R+r)^3\Leftrightarrow$$

$$\Leftrightarrow(16R-5r)(12R-44r)+(72R^2+192Rr+29r^2)\ge 3(R+r)(4R+r)\Leftrightarrow$$

$\Leftrightarrow 252R^2-627Rr+246r^2\ge 0\Leftrightarrow(R-2r)(252R-123r)\ge 0$, which follows from Euler's inequality $R\ge 2r$. It follows:

$$\sum\frac{\sin^2 A}{2\sin^2 A+\sin^2 B+\sin^2 C}\le\frac{3}{4}\overset{k\ge 4}{\le}\frac{9k+12}{64}$$

Equality holds if and only if the triangle is equilateral

Remark:

In ΔABC the following relationship holds:

$$\sum \frac{a^2}{2a^2+b^2+c^2} \le \frac{3}{4}$$

Marin Chirciu

Solution: Lemma:

$$\sum \frac{a^2}{2a^2+b^2+c^2} =$$
$$= \frac{6p^6 - 2p^4(36Rr+5r^2) + 2p^2r^2(156R^2+56Rr+5r^2) - 6r^3(4R+r)^3}{9p^6 - p^4(108Rr+23r^2) + p^2r^2(440R^2+200Rr+23r^2) - 9r^3(4R+r)^3}$$

Proof.

$$\sum \frac{a^2}{2a^2+b^2+c^2} = \frac{\sum a^2(2b^2+c^2+a^2)(2c^2+a^2+b^2)}{\prod(2a^2+b^2+c^2)}$$
$$\sum a^2(2b^2+c^2+a^2)(2c^2+a^2+b^2) =$$
$$= 6p^6 - 2p^4(36Rr+5r^2) + 2p^2r^2(156R^2+56Rr+5r^2) - 6r^3(4R+r)^3,$$
$$\prod(2a^2+b^2+c^2) =$$
$$= 9p^6 - p^4(108Rr+23r^2) + p^2r^2(440R^2+200Rr+2r^3) - 9r^3(4R+r)^3$$

Let's get back to the main problem. Using the lemma we obtain:

$$\sum \frac{a^2}{2a^2+b^2+c^2} \le \frac{3}{4} \Leftrightarrow$$
$$\Leftrightarrow \frac{6p^6 - 2p^4(36Rr+5r^2) + 2p^2r^2(156R^2+56Rr+5r^2) - 6r^3(4R+r)^3}{9p^6 - p^4(108Rr+23r^2) + p^2r^2(440R^2+200Rr+23r^2) - 9r^3(4R+r)^3} \le \frac{3}{4} \Leftrightarrow$$
$$\Leftrightarrow 3p^6 - p^4(36Rr+29r^2) + p^2r^2(72R^2+192Rr+29r^2) \ge 3r^3(4R+r)^3 \Leftrightarrow$$
$$\Leftrightarrow p^2[p^2(3p^2-36Rr-29r^2) + r^2(72R^2+192Rr+29r^2)] \ge 3r^3(4R+r)^3,$$

Which fllows from Gerretsen's inequality $p^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$.

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(3(16Rr-5r^2)-36Rr-29r^2)$$
$$+r^2(72R^2+192Rr+29r^2)] \ge 3r^3(4R+r)^3 \Leftrightarrow$$
$$\Leftrightarrow (16R-5r)(12R-44r) + (72R^2+192Rr+29r^2) \ge 3(R+r)(4R+r) \Leftrightarrow$$

$\Leftrightarrow 252R^2 - 627Rr + 246r^2 \geq 0 \Leftrightarrow (R-2r)(252R-123r) \geq 0$, which follows from Euler's inequality $R \geq 2r$.Equality holds if and only if the triangle is equilateral.

SP.514 Let be $P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ with $n \in \mathbb{N}, n \geq 2, a_i \in \mathbb{R}, (\forall)i = \overline{1,n}$. If the equation $P(x) = 0$ has all real roots then $(\forall)k > \max\{x_1, x_2, \dots, x_n\} + 1$, we have

$$(n-1) \cdot P(K) - P'(K) + 1 > 0$$

Proposed by Gheorghe Molea – Romania

Solution 1 by proposer

We denote $\max\{x_1, x_2, \dots, x_n\} = x_n$

We have $x_n \geq x_i, (\forall)i = \overline{1,n} \Rightarrow 1 + x_n \geq 1 + x_i, (\forall)i = \overline{1,n}$. Let be $K > x_n + 1$, we have $K > x_{n+1} \geq x_i + 1 \Rightarrow K > x_i + 1 \Rightarrow K - x_i > 1 > 0 \Rightarrow K - x_i > 0; (*)$

Also from $K > x_i + 1 \Rightarrow \frac{1}{K-x_i} < 1, (\forall)i = \overline{1,n}$; (**)

From (*) and (**) $\Rightarrow 0 < \frac{1}{k-x_i} < 1$

As $P(K) = \prod_{i=1}^n (K - x_i)$; $P(K) > 0$ from (*) and $\frac{P'(K)}{P(K)} = \sum_{i=1}^n \frac{1}{K-x_i}$ the inequality from enunciation becomes: $(n-1) - \frac{P'(K)}{P(K)} + \frac{1}{P((K)} > 0,\ (\forall)K > x_n + 1 \Leftrightarrow$

$$\Leftrightarrow (n-1) - \sum_{i=1}^n \frac{1}{k - x_i} + \prod \frac{1}{K - x_i} > 0$$

$$\Leftrightarrow \sum_{i=1}^n \frac{1}{K - x_i} - \prod_{i=1}^n \frac{1}{K - x_i} < n - 1, (\forall)0 < \frac{1}{k - x_i} < 1 \ (***)$$

We denote $\frac{1}{K-x_i} = a_i, (\forall)i = \overline{1,n} \Rightarrow 0 < a_i < 1, (\forall)i = \overline{1,n}$ and inequality (***) becomes:

$$\sum_i^n a_i - \prod_{i=1}^n a_i < n - 1 \Leftrightarrow 1 - \prod_{i=1}^n a_i < \sum_{i=1}^n (1 - a_i) \quad (1)$$

which must be proved for any a_i with $0 < a_i < 1, (\forall)i = \overline{1,n}$ with $n \geq 2$

For this we denote $1 - a_i = b_i, i = \overline{1,n}$ it is obviously that $0 < b_i < 1, (\forall) i = \overline{1,n}$ with $n \geq 2.$

Inequality (1) becomes:

$$1 - \prod_{i=1}^{n}(1 - b_i) < \sum_{i=1}^{n} b_i$$

which must be proved for $0 < b_i < 1; i = \overline{1,n}; n \geq 2.$

We use mathematical induction:

For $n = 2 \Rightarrow 1 - (1 - b_1)(1 - b_2) < b_1 + b \Leftrightarrow b_1 b_2 > 0$, obviously.

We suppose the inequality true for n and we will prove that:

$$1 - \prod_{i=1}^{n+1}(1 - b_i) < \sum_{i=1}^{n+1} b_i, (\forall) b_i \in (0;1)$$

Inequality $1 - \prod_{i=1}^{n}(1 - b_i) < \sum_{i=1}^{n} b_i$ we multiply with $(1 - b_{n+1}) \Rightarrow$

$$\Rightarrow 1 - b_{n+1} - \prod_{i=1}^{n}(1 - b_i) < \sum_{i=1}^{n} b_i - b_{n+1}\sum_{i=1}^{n} b_i$$

$$\Leftrightarrow 1 - \prod_{i=1}^{n+1}(1 - b_i) < \sum_{i=1}^{n+1} b_i - b_{n+1}\sum_{i=1}^{n} b_i < \sum_{i=1}^{n+1} b_i \Rightarrow 1 - \prod_{i=1}^{n+1}(1 - b_i) < \sum_{i=1}^{n} b_i$$

With this inequality (***) is proved and also the problem.

Solution 2 by Ivan Hadinata-Jember-Indonesia

Let $P(x) = (x - x_1)(x - x_2)\dots(x - x_n)$ where $x_i \in \mathbb{R}$, $\forall i = 1, 2, \dots, n$. Let $a_m = k - x_m - 1$ for all $m = 1, 2, \dots, n$. Then

$$P(k) = \prod_{m=1}^{n}(a_m + 1)$$

and

$$P'(k) = \sum_{i=1}^{n}\left(\prod_{\substack{m=1,\\ m\neq i}}^{n}(a_m + 1)\right)$$

Therefore,

$$(n-1)\cdot P(k) - P'(k) + 1 = z_1 - z_2 \quad \dots\dots\dots (1)$$

where

$$z_1=\sum_{i=1}^{n}\left(a_i\prod_{\substack{m=1,\\ m\neq i}}^{n}(a_m+1)\right)\ \dots\dots\dots(2)$$

and

$$z_2=a_1+\sum_{i=2}^{n}\left(a_i\prod_{m=1}^{i-1}(a_m+1)\right)\ \dots\dots\dots(3)$$

Since $n\geq 2$, clearly the following n inequalities hold:

$$a_i\prod_{\substack{m=1,\\ m\neq i}}^{n}(a_m+1)\geq a_i\prod_{m=1}^{i-1}(a_m+1),\qquad i=2,3,\dots,n$$

and $a_1(a_2+1)(a_3+1)\dots(a_n+1)>a_1$. Summing up these n inequalities yields that $z_1>z_2$. By (1), we will get $(n-1)\cdot P(k)-P'(k)+1>0$, as desired.

SP.515 If $a,b,c,t,k>0$ such that $(t+a)(t+b)(t+c)=2k$ and $k>\frac{t^3}{2}$, prove that:

$$\frac{1}{b(t+a)^2}+\frac{1}{c(t+b)^2}+\frac{1}{a(t+c)^2}\geq\frac{3t\sqrt[3]{4k^2}}{k^2}$$

Proposed by Gheorghe Molea – Romania

Solution 1 by proposer

From $(t+a)(t+b)(t+c)=2k\Rightarrow$

$$\Rightarrow(t+a)(t+b)=\frac{2k}{t+c}\Rightarrow t(t+a)+(tb+ab)=\frac{2k}{t+c}$$

$$MG\leq MA\Rightarrow\sqrt{(t^2+at)(tb+ab)}\leq\frac{t^2+at+tb+ab}{2}=\frac{k}{t+c}\Rightarrow$$

$$\Rightarrow tb(t+a)^2\leq\frac{k^2}{(t+c)^2}\Rightarrow\frac{1}{tb(t+a)^2}\geq\frac{(t+c)^2}{k^2}\text{ and the analogs.}$$

We obtain: $\sum\frac{1}{tb(t+a)^2}\geq\frac{1}{k^2}\cdot\sum(t+c)^2$ (*)

$$MP\geq MG\Rightarrow\sqrt{\frac{(t+a)^2+(t+b)^2+(t+c)^2}{3}}\geq$$

$$\geq\sqrt[3]{(t+a)(t+b)(t+c)}=\sqrt[3]{2k}\Rightarrow\frac{\sum(t+a)^2}{3}\geq\sqrt[3]{4k^2}\Rightarrow\sum(t+a)^2\geq 3\sqrt[3]{4k^2}$$

and inequality (*) becomes:

$$\sum\frac{1}{tb(t+a)^2}\geq\frac{3\sqrt[3]{4k^2}}{k^2}\Rightarrow\frac{1}{b(t+a)^2}+\frac{1}{c(t+b)^2}+\frac{1}{a(t+c)^2}\geq\frac{3t\sqrt[3]{4k^2}}{k^2}$$

We have equality $\Leftrightarrow a=b=c=\sqrt[3]{2k}-t$ which is bigger than 0, because $k>\frac{t^3}{2}$.

Solution 2 by Ivan Hadinata-Jember-Indonesia

Firstly, we use Holder inequality to get

$$2k=(t+a)(t+b)(t+c)\geq\left(t+\sqrt[3]{abc}\right)^3\quad\Longrightarrow\quad\sqrt[3]{abc}\leq\sqrt[3]{2k}-t$$

After that, we apply AM-GM inequality to get

$$\sum_{cyc}\frac{1}{b(t+a)^2}\geq 3\left(\frac{1}{abc\big((t+a)(t+b)(t+c)\big)^2}\right)^{\frac{1}{3}}\geq\frac{3}{\sqrt[3]{4k^2}\left(\sqrt[3]{2k}-t\right)}\quad\dots\dots(*)$$

Note that

$$t\left(\sqrt[3]{2k}-t\right)\cdot\sqrt[3]{16k^4}\leq\left(\frac{t+\sqrt[3]{2k}-t}{2}\right)^2\left(\sqrt[3]{16k^4}\right)=k^2$$

$$\Leftrightarrow$$

$$\frac{3}{\sqrt[3]{4k^2}\left(\sqrt[3]{2k}-t\right)}\geq\frac{3t\sqrt[3]{4k^2}}{k^2}\quad\dots\dots(**)$$

By $(*)$ and $(**)$, we get the desired inequality:

$$\frac{1}{b(t+a)^2}+\frac{1}{c(t+b)^2}+\frac{1}{a(t+c)^2}\geq\frac{3t\sqrt[3]{4k^2}}{k^2}.$$

Equality holds if and only if $a=b=c=t=\sqrt[3]{\frac{k}{4}}$.

Solution 3 by Tapas Das-India

If $a,b,c,t,k>0$ such that

$(t+a)(t+b)(t+c)=2k$ and $k>\frac{t^3}{2}$

Prove that:

$$\frac{1}{b(t+a)^2}+\frac{1}{c(t+b)^2}+\frac{1}{a(t+c)^2}\geq\frac{3t\sqrt[3]{4k^2}}{k^2}$$

$$(t+a)(t+b)(t+c)=2k$$

AM-GM, $2\sqrt{ta}\cdot 2\sqrt{tb}\cdot 2\sqrt{tc}\leq 2k$

$$4t\sqrt{t}\sqrt{abc} \le k \Rightarrow \sqrt{abc} \le \frac{k}{4t^{\frac{3}{2}}} \Rightarrow abc \le \frac{k^2}{16t^3}$$

$$\therefore \sum \frac{1}{b(t+a)^2} \overset{AM-GM}{\ge} 3\left[\frac{1}{abc(t+a)^2(t+b)^2(t+c)^2}\right]^{\frac{1}{3}} \ge 3\left[\frac{16t^3}{k^2 \cdot 4k^2}\right]^{\frac{1}{3}} = \frac{3t}{k}\sqrt[3]{\frac{4}{k}}$$

$$= \frac{3t}{k^2}\sqrt[3]{4k^2}$$

SP.516 Let be the acuteangled ΔABC and the points $B, A_1, A_2, \dots, A_{n-1}, C$ collinears in this order. Let $R, R_1, R_2, \dots, R_n$ be the circumradies of $\Delta ABC, \Delta ABA_1, \Delta A_1AA_2, \dots, A_{n-1}AC$. Prove that:

$$\max(R_1, R_2, \dots, R_n) \ge \frac{R\sin\widehat{A}}{n\cdot\sin\frac{\widehat{A}}{n}} \text{ and } \min(R_1, R_2, \dots, R_n) < \frac{\pi R\cdot\sin\widehat{A}}{2\widehat{A}}$$

Proposed by Radu Diaconu – Romania

Solution 1 by proposer

We apply sine theorem in the triangles $ABA_1, AA_1A_2, \dots, AA_{n-1}C$ and we have:

$$BA_1 = 2R_1\sin\left(\widehat{BAA_1}\right), A_1A_2 = 2R_2\sin\left(\widehat{A_1AA_2}\right), \dots A_{n-1}C = 2R_n\sin\left(\widehat{A_{n-1}AC}\right)$$

But $BC = BA_1 + A_1A_2 + \dots + A_{n-1}C$. It follows that:

$$R\sin\widehat{A} = R_1\sin\left(\widehat{BAA_1}\right) + R_2\sin\left(\widehat{A_1AA_2}\right) + \dots + R_n\sin\left(\widehat{A_{n-1}AC}\right)$$

$$R\sin\widehat{A} \le \max(R_1, R_2, \dots, R_n)\left(\sin\left(\widehat{BAA_1}\right) + \sin(A_1AA_2) + \dots \sin\left(\widehat{A_{n-1}AC}\right)\right)$$

We consider the function $f:(0,\pi) \to \mathbb{R}, f(x) = \sin x$, which is concave and from Jensen's inequality we have:

$$\sin\left(\widehat{BAA_1}\right) + \sin\left(\widehat{A_1AA_2}\right) + \dots + \sin\left(\widehat{A_{n-1}AC}\right) \le n\cdot\sin\frac{\widehat{A}}{n}$$

It follows that:

$$\max(R_1, R_2, \dots, R_n) \ge \frac{R\sin\widehat{A}}{\sin\left(\widehat{BAA_1}\right) + \sin\left(\widehat{A_1AA_2}\right) + \dots + \sin\left(\widehat{A_{n-1}AC}\right)} \ge$$

$$\ge \frac{R\sin\widehat{A}}{n\cdot\sin\frac{\widehat{A}}{n}} \Rightarrow \max(R_1, R_2, \dots, R_n) \ge \frac{R\sin\widehat{A}}{n\cdot\sin\frac{\widehat{A}}{n}}$$

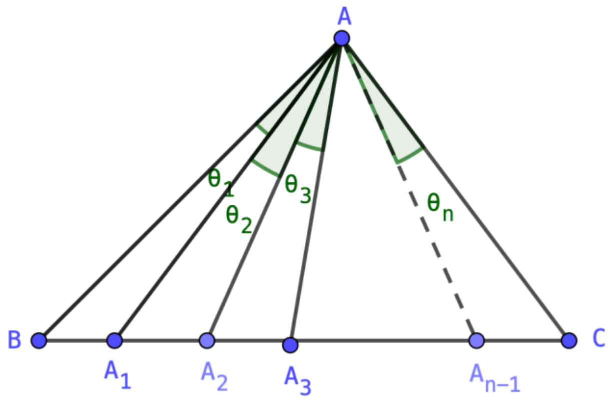
Then:

$$R\sin\widehat{A} \ge \min(R_1,R_2,\dots,R_n)\left(\sin\left(\widehat{BAA_1}\right)+\sin\left(\widehat{A_1AA_2}\right)+\cdots+\sin\left(\widehat{A_{n-1}AC}\right)\right) \quad (1)$$

Applying the inequality: $\sin x > \frac{2}{\pi}\cdot x$, where $x\in\left(0,\frac{\pi}{2}\right)$, relationship (1) becomes:

$$R\sin\widehat{A} > \min(R_1,R_2,\dots,R_n)\cdot\frac{2}{\pi}\cdot\widehat{A} \Rightarrow \min(R_1,R_2,\dots,R_n) < \frac{\pi R\cdot\sin\widehat{A}}{2\widehat{A}}$$

Solution 2 by Tapas Das-India



Let $\angle BAA_1=\theta_1, \angle A_1AA_2=\theta_2, \angle A_2AA_3=\theta_3,\dots,\angle A_{n-1}AC=\theta_n$

$$\therefore\ \theta_1+\theta_2+\theta_3+\cdots+\theta_n=A$$

Since ΔABC acute triangle so we have

$$\theta_1<\frac{\pi}{2},\theta_2<\frac{\pi}{2},\dots,\theta_n<\frac{\pi}{2} \quad (1)$$

R is the circumradius of the ΔABC

$$\therefore \frac{BC}{\sin A}=2R \qquad \therefore BC=2R\sin A$$

$R_1=$ circumradius of ΔABA_1

$$\therefore R_1=\frac{BA_1}{2\sin\theta_1}\Rightarrow BA_1=2R_1\sin\theta_1$$

Similarly $2R_2\sin\theta_2=A_1A_2, 2R_3\sin\theta_3=A_2A_3,\dots,2R_n\sin\theta_n=A_{n-1}C$

Now $BC=2R\sin A$

$$\Rightarrow BA_1+A_1A_2+A_2A_3+\cdots+A_{n-1}C=2R\sin A$$

$$\Rightarrow 2R_1\sin\theta_1+2R_2\sin\theta_2+\cdots+2R_n\sin\theta_n=2R\sin A$$

$$\Rightarrow R_1\sin\theta_1+R_2\sin\theta_2+\cdots+R_n\sin\theta_n=R\sin A$$

$$\Rightarrow \max(R_1,R_2,\dots,R_n)(\sin\theta_1+\sin\theta_2+\cdots+\sin\theta_n)\ge R\sin A$$

$$\Rightarrow \max(R_1, R_2, \dots, R_n)\, n \cdot \sin\left(\frac{\theta_1+\theta_2+\cdots+\theta_n}{n}\right) \overset{Jensen}{\geq} R\sin A$$

[Note: $\sin x$ is concave]

$$\Rightarrow \max(R_1, R_2, \dots, R_n)\, n \sin\left(\frac{A}{n}\right) \geq R\sin A$$

$$\therefore \max(R_1, R_2, \dots, R_n) \geq \frac{R\sin A}{n\sin\frac{A}{n}}$$

Jordan's inequality

$$\frac{\sin x}{x} > \frac{2}{\pi} \text{ for all } x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin x > \frac{2}{\pi}x$$

$$R\sin A = R_1\sin\theta_1 + R_2\sin\theta_2 + \cdots + R_n\sin\theta_n$$

$$\therefore R\sin A > \min(R_1, R_2, \dots, R_n)(\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_n)$$

$$\Rightarrow R\sin A \overset{Jordan}{>} \min(R_1, R_2, \dots, R_n)\frac{2}{\pi}(\theta_1+\theta_2+\cdots+\theta_n) = \min(R_1, R_2, \dots, R_n)\cdot\frac{2}{\pi}\cdot A$$

$$(\because A = \theta_1+\theta_2+\cdots+\theta_n)$$

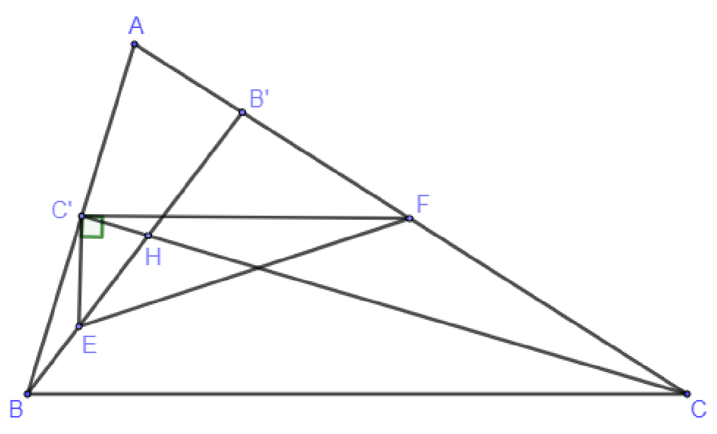
Or $\frac{\pi R\sin A}{2A} > \min(R_1, R_2, \dots, R_n)$

SP.517 In acute $\Delta ABC, BB', CC'$ –altitudes, $C' \in (AB), B' \in (AC)$, $\{H\} = BB' \cap CC'$, E, F middle points of $[BH], [AC]$ respectively. Prove that:

$$4EF^2 \geq (EC' + EB')^2 + (C'F + B'F)^2$$

Proposed by Floricǎ Anastase-Romania

Solution 1 by proposer



$$We\ have: \overrightarrow{C'F}\cdot\overrightarrow{C'E}=\frac{1}{2}\left(\overrightarrow{C'A}+\overrightarrow{C'C}\right)\cdot\frac{1}{2}\left(\overrightarrow{C'B}+\overrightarrow{C'H}\right)=$$

$$=\frac{1}{4}\left(\overrightarrow{C'A}\cdot\overrightarrow{C'B}+\overrightarrow{C'A}\cdot\overrightarrow{C'H}+\overrightarrow{C'C}\cdot\overrightarrow{C'B}+\overrightarrow{C'C}\cdot\overrightarrow{C'H}\right)=\frac{1}{4}\left(\overrightarrow{C'A}\cdot\overrightarrow{C'B}+0+0+\overrightarrow{C'C}\cdot\overrightarrow{C'H}\right)$$

$$=\frac{1}{4}\left(-\overrightarrow{C'A}\cdot\overrightarrow{C'B}+\overrightarrow{C'C}\cdot\overrightarrow{C'H}\right)=0, because$$

$$\Delta AC'C\sim\Delta HC'B\Leftrightarrow\frac{BC'}{C'C}=\frac{C'H}{C'A}\Leftrightarrow\overrightarrow{C'A}\cdot\overrightarrow{C'B}=\overrightarrow{C'C}\cdot\overrightarrow{C'H}.$$

$$Hence, \overrightarrow{C'F}\cdot\overrightarrow{C'E}=0\Leftrightarrow C'F\perp C'E. So, we\ have:$$

$$EF^2=(EC')^2+(C'F)^2\ and\ EF^2=(EB')^2+(B'F)^2$$

$$2EF=\sqrt{(EC')^2+(C'F)^2}+\sqrt{(EB')^2+(B'F)^2}\geq\sqrt{(EC'+EB')^2+(C'F+B'F)^2}$$

$$Therefore, 4EF^2\geq(EC'+EB')^2+(C'F+B'F)^2$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

It is well-known that if ΔPQR is a triangle with $\angle PQR=90°$ and S is the midpoint of side PR, then S is the center of (PQR). Therefore, we have $EC'+EB'=BE+EB'=BB'$ and $C'F+B'F=\frac{AC}{2}+B'F=\max\{AB',B'C\}$.

Consequently, by Pythagoras theorem,

$$(EC'+EB')^2+(C'F+B'F)^2=(BB')^2+(\max\{AB',B'C\})^2$$
$$=(\max\{AB,BC\})^2\ \dots\dots\dots(1)$$

Let Γ be the nine-point circle of ΔABC. Let X and Y be respectively the midpoints of AB and BC. Then $X,Y,E,F,B',C'\in\Gamma$. Since $\angle EB'F=90°$, so EF is diameter of Γ. It means that for every point $P_1,P_2\in\Gamma$, we have $P_1P_2\leq EF$. Then,

$$EF\geq\max\{B'X,B'Y\}=\frac{1}{2}\max\{AB,BC\}\ \dots\dots\dots(2)$$

By (1) and (2), we deduce

$$4EF^2\geq(EC'+EB')^2+(C'F+B'F)^2.$$

SP.518 Find:

$$\Omega=\lim_{x\to 0}\left(\frac{1}{x}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}3^{k-1}\sin^3\frac{x}{3^k}\right), a\in\mathbb{R}$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$Using\ the\ identity: \sin 3\alpha=3\sin\alpha-4\sin^3\alpha\ for\ \alpha=\left\{\frac{x}{3},\frac{x}{3^2},\dots,\frac{x}{3^{n-1}}\right\}, we\ get:$$

$$\begin{cases} \sin x = 3\sin\frac{x}{3} - 4\sin^3\frac{x}{3} \\ \sin\frac{x}{3} = 3\sin\frac{x}{3^2} - 4\sin^3\frac{x}{3^2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sin\frac{x}{3^{n-1}} = 3\sin\frac{x}{3^n} - 4\sin^3\frac{x}{3^n} \end{cases}$$

By multilying with $1, 3, 3^2, \dots, 3^{n-1}$ *and summing, it follows:*

$$\sum_{k=1}^{n} 3^{k-1}\sin^3\frac{x}{3^k} = \frac{1}{4}\left(3^n\sin\frac{x}{3^n} - \sin x\right) \textit{ and then}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} 3^{k-1}\sin^3\frac{x}{3^k} = \frac{1}{4}(x - \sin x)$$

Therefore,

$$\Omega = \lim_{x\to 0}\left(\frac{1}{x}\cdot\lim_{n\to\infty}\sum_{k=1}^{n} 3^{k-1}\sin^3\frac{x}{3^k}\right) = \lim_{x\to 0}\frac{1}{4x}(x - \sin x) = \lim_{x\to 0}\frac{1}{4}\left(1 - \frac{\sin x}{x}\right) = 0.$$

Solution 2 by Pham Duc Nam-Vietnam

$*$ Denote: $S = \sum_{k=1}^{n} 3^{k-1}\sin^3\left(\frac{x}{3^k}\right)$

From the trigonometric identity: $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta) \Rightarrow \sin^3(\theta)$

$$= \frac{1}{4}(3\sin(\theta) - \sin(3\theta))$$

Let: $\theta = \frac{x}{3^k} \Rightarrow \sin^3\left(\frac{x}{3^k}\right) = \frac{1}{4}\left(3\sin\left(\frac{x}{3^k}\right) - \sin\left(\frac{x}{3^{k-1}}\right)\right) \Rightarrow \sum_{k=1}^{n} 3^{k-1}\sin^3\left(\frac{x}{3^k}\right)$

$$= \frac{1}{4}\sum_{k=1}^{n}\left(3^k\sin\left(\frac{x}{3^k}\right) - 3^{k-1}\sin\left(\frac{x}{3^{k-1}}\right)\right)$$

Then S is a sum of a telescoping series, so it is easy to get: S

$$= \frac{1}{4}\left(-3^{1-1}\sin\left(\frac{x}{3^{1-1}}\right) + 3^n\sin\left(\frac{x}{3^n}\right)\right) = \frac{1}{4}\left(-\sin(x) + 3^n\sin\left(\frac{x}{3^n}\right)\right)$$

$$* \lim_{n\to\infty}\sum_{k=1}^{n} 3^{k-1}\sin^3\left(\frac{x}{3^k}\right) = \lim_{n\to\infty}\frac{1}{4}\left(-\sin(x) + 3^n\sin\left(\frac{x}{3^n}\right)\right) = -\frac{1}{4}\sin(x) + \frac{1}{4}\lim_{n\to\infty}3^n\sin\left(\frac{x}{3^n}\right)$$

$$= -\frac{1}{4}\sin(x) + \frac{1}{4}\lim_{n\to\infty}\frac{\sin\left(\frac{x}{3^n}\right)}{\frac{x}{3^n}}.x = -\frac{1}{4}\sin(x) + \frac{1}{4}x$$

$$* \Omega = \lim_{x\to 0}\left(\frac{1}{x}\lim_{n\to\infty}\sum_{k=1}^{n} 3^{k-1}\sin^3\left(\frac{x}{3^k}\right)\right) = \lim_{x\to 0}\left(\frac{1}{x}\left(-\frac{1}{4}\sin(x) + \frac{1}{4}x\right)\right) = 0$$

Solution 3 by Ivan Hadinata-Jember-Indonesia

- **Lemma:** $\sin x \le x$ **for every** $x \ge 0$.
- **Proof: When** $x > \frac{\pi}{2}$, **obviously** $x > 1 \ge \sin x$. **If** $x \in \left[0, \frac{\pi}{2}\right]$, **consider a unit circle** Γ **centered at the origin point** O **in Cartesian coordinate system. Let** Γ **intersects** x**-axis and** y**-axis at** A **and** B**, respectively. Let** C **be a point on the minor arc** AB **of** Γ **so that** $\angle AOC = x \in \left[0, \frac{\pi}{2}\right]$. **Geometrically, it is easy to see that the area of minor sector** ACO **of** Γ **is greater than or equal to the area of triangle** ΔACO**, that respectively their areas are** $\frac{x}{2}$ **and** $\frac{\sin x}{2}$. **Thus** $\frac{x}{2} \ge \frac{\sin x}{2}$ **and the result follows.** □

By lemma, we obtain

$$\frac{1}{x} \cdot \lim_{n\to\infty} \sum_{k=1}^{n} 3^{k-1} \sin^3 \frac{x}{3^k} \le \frac{1}{x} \cdot \lim_{n\to\infty} 3^{k-1} \left(\frac{x}{3^k}\right)^3 = \frac{x^2}{24} \;\dots\dots (1)$$

for every $x > 0$. **Observe that if** $x > 0$ **is approached as near as possible to** 0, **LHS of** (1) **is non-negative and RHS of** (1) **tends to** 0. **Thus,**

$$0 = \lim_{x\to 0+} \left(\frac{1}{x} \cdot \lim_{n\to\infty} \sum_{k=1}^{n} 3^{k-1} \sin^3 \frac{x}{3^k}\right) = \lim_{x\to 0+} \left(\frac{1}{-x} \cdot \lim_{n\to\infty} \sum_{k=1}^{n} 3^{k-1} \sin^3 \frac{-x}{3^k}\right)$$

$$= \lim_{x\to 0-} \left(\frac{1}{x} \cdot \lim_{n\to\infty} \sum_{k=1}^{n} 3^{k-1} \sin^3 \frac{x}{3^k}\right) \;\dots\dots (2)$$

By (2),

$$\Omega = \lim_{x\to 0} \left(\frac{1}{x} \cdot \lim_{n\to\infty} \sum_{k=1}^{n} 3^{k-1} \sin^3 \frac{x}{3^k}\right) = 0$$

SP.519 Let $x_i, i = 1, 2, \dots, n$ **be positive real numbers such that**

$$\prod_{i=1}^{n} x_i = 1$$

Prove:

$$\sum_{i=1}^{n} \left(\frac{x_i^6 + 1}{x_i + 1}\right)^2 \cdot x_{i+1} \ge n$$

where $x_1 = x_{n+1}$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by proposer

We'll prove that:

$\left(\frac{x_i^6+1}{x_i+1}\right)^2 \geq x_i^5$. We have

$$\left(x_i^6+1\right)^2 - x_i^5(x_i+1)^2 = x_i^7\left(x_i^5-1\right) - \left(x_i^5-1\right) =$$

$$\left(x_i^5-1\right)\cdot\left(x_i^7-1\right) = (x_i-1)\left(x_i^4+\dots+1\right)(x_i-1)\left(x_i^6+\dots+1\right) =$$

$$(x_i-1)^2\cdot\left(x_i^4+x_i^3+x_i^2+x_i+1\right)\left(x_i^6+x_i^5+x_i^4+x_i^3+x_i^2+x_i+1\right) \geq 0$$

Equality holds when $x_i = 1$. So

$$\left(\frac{x_i^6+1}{x_i+1}\right)^2 \geq x_i^5 \Leftrightarrow \left(\frac{x_i^6+1}{x_i+1}\right)^2 x_{i+1} \geq x_i^5\cdot x_{i+1}$$

Namely

$$\sum_{i=1}^{n}\left(\frac{x_i^6+1}{x_i+1}\right)^2 x_{i+1} \geq \sum_{i=1}^{n} x_i^5\cdot x_{i+1} \overset{AM-GM}{\geq}$$

$$n\sqrt[n]{(x_1x_2\dots x_n)^6} = n\cdot 1 = n$$

Equality holds when $x_i, i = 1,2,\dots,n$ equal to 1

Solution 2 by Ivan Hadinata-Jember-Indonesia

Observe that

$$\left(x_i^6+1\right)(1+1)^5 \underbrace{\geq}_{Holder} (x_i+1)^6 \quad\Rightarrow\quad \frac{x_i^6+1}{x_i+1} \geq \left(\frac{x_i+1}{2}\right)^5 \dots\dots(1)$$

Then

$$\sum_{i=1}^{n}\left(\frac{x_i^6+1}{x_i+1}\right)^2 x_{i+1} \underbrace{\geq}_{AM-GM} n\left(\prod_{i=1}^{n}\left(\frac{x_i^6+1}{x_i+1}\right)^2\cdot\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}} \underbrace{\geq}_{(1)} n\left(\prod_{i=1}^{n}\left(\frac{x_i+1}{2}\right)^{\frac{1}{n}}\right)^{10}$$

$$\underbrace{\geq}_{Holder} n\left(\frac{\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}+1}{2}\right)^{10} = n,$$

as desired.

Equality holds when $x_i, i = 1,2,\dots,n$ equal to 1

Solution 3 and extensions by Marin Chirciu-Romania

If $x_i > 0, i = \overline{1,n}, x_1x_2\dots x_n = 1$ then:

$$\left(\frac{x_1^6+1}{x_1+1}\right)^2 x_2 + \left(\frac{x_2^6+1}{x_2+1}\right)^2 x_3 + \dots + \left(\frac{x_n^6+1}{x_n+1}\right)^2 x_1 \geq n$$

Solution: Lemma: If $x > 0$ then:

$$\left(\frac{x^6+1}{x+1}\right)^2 \ge x^5$$

Proof.

$$\left(\frac{x^6+1}{x+1}\right)^2 \ge x^5 \Leftrightarrow \frac{x^{12}+2x^6+1}{x^2+2x+1} \ge x^5 \Leftrightarrow x^{12}+2x^6+1 \ge$$
$$\ge x^7+2x^6+x^5 \Leftrightarrow x^{12}+1 \ge x^7+x^5 \Leftrightarrow (x^7-1)(x^5-1) \ge 0,$$

because the factors (x^7-1) and (x^5-1) have the same sign. Let's get back to the main problem. Using the Lemma we obtain:

$$LHS = \left(\frac{x_1^6+1}{x_1+1}\right)^2 x_2 + \left(\frac{x_2^6+1}{x_2+1}\right)^2 x_3 + \dots + \left(\frac{x_n^6+1}{x_n+1}\right)^2 x_1 \overset{Lemma}{\ge}$$
$$\ge x_1^5x_2 + x_2^5x_3 + \dots + x_n^5x_1 \overset{AM-GM}{\ge}$$
$$\ge n\sqrt[n]{x_1^5x_2 \cdot x_2^5x_3 \cdot \dots \cdot x_n^5x_1} = n\sqrt[n]{x_1^6 \cdot x_2^6 \cdot \dots \cdot x_n^6} = n\sqrt[n]{(x_1x_1 \dots x_1)^6} = n$$

Equality holds if and only if $x_1 = x_2 = \dots = x_n = 1$.

Remark: The problem can be developed.

If $x_i > 0, i = \overline{1,n}, x_1x_2 \dots x_n = 1$ and $\lambda \in \mathbb{N}$ then:

$$\left(\frac{x_1^{\lambda+1}+1}{x_1+1}\right)^2 x_2 + \left(\frac{x_2^{\lambda}+1}{x_2+1}\right)^2 x_3 + \dots + \left(\frac{x_n^{\lambda+1}+1}{x_n+1}\right)^2 x_1 \ge n$$

Marin Chirciu

Solution: Lemma: If $x > 0$ and $\lambda \in \mathbb{N}$ then:

$$\left(\frac{x^{\lambda+1}+1}{x+1}\right)^2 \ge x^\lambda$$

Proof.

$$\left(\frac{x^{\lambda+1}+1}{x+1}\right)^2 \ge x^\lambda \Leftrightarrow \frac{x^{2\lambda+2}+2x^\lambda+1}{x^2+2x+1} \ge x^\lambda \Leftrightarrow$$
$$\Leftrightarrow x^{2\lambda+2}+2x^{\lambda+1}+1 \ge x^{\lambda+2}+2x^{\lambda+1}+x^\lambda \Leftrightarrow$$
$$\Leftrightarrow x^{2\lambda+2}+1 \ge x^{\lambda+2}+x^\lambda \Leftrightarrow (x^{\lambda+2}-1)(x^\lambda-1) \ge 0$$

because the factors $(x^{\lambda+2}-1)$ and $(x^\lambda-1)$ have the same sign. Let's get back to the main problem. Using the Lemma we obtain:

$$LHS = \left(\frac{x_1^{\lambda+1}+1}{x_1+1}\right)^2 x_2 + \left(\frac{x_2^{\lambda+1}+1}{x_2+1}\right)^2 x_3 + \dots + \left(\frac{x_n^{\lambda+1}+1}{x_n+1}\right)^2 x_1 \overset{Lemma}{\ge}$$
$$\ge x_1^\lambda x_2 + x_2^\lambda x_3 + \dots + x_n^\lambda x_1 \overset{AM-GM}{\ge} n\sqrt[n]{x_1^\lambda x_2 \cdot x_2^\lambda x_3 \cdot \dots \cdot x_n^\lambda x_1} =$$
$$= n\sqrt[n]{(x_1x_1 \dots x_1)^{\lambda+1}} = n.$$

Equality holds if and only if $x_1 = x_2 = \dots x_n = 1$

Note: For $\lambda = 5$ we obtain Problem SP.519 from RMM35 Winter Edition 2024, proposed by George Apostolopoulos.

SP.520 Prove that in any acute triangle ABC:

$$\sqrt{2}(13k^2-3) \le \sqrt{\pi(\cos^2 A+\cos^2 B)} \le \frac{\sqrt{2}}{2}k$$

where $k \in \left(0, \frac{1}{2}\right]$. The product is over all cyclic permutations of (A, B, C).

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by proposer

We have

$$\cos(A+B)\cdot\cos(A-B) = (\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B) =$$

$$= \cos^2 A\cos^2 B-\sin^2 A\sin^2 B = \cos^2 A\cos^2 B-(1-\cos^2 A)(1-\cos^2 B) =$$

$$= -1+\cos^2 A+\cos^2 B$$

So $\cos^2 A+\cos^2 B = 1+\cos(A+B)\cos(A-B)$

We know that $\cos(A+B) = -\cos C$, and $\cos(A-B) \le 1$

So $\cos^2 A+\cos^2 B \le 1-\cos C$. Similarly

$\cos^2 B+\cos^2 C \le 1-\cos A$, and $\cos^2 C+\cos^2 A \le 1-\cos B$

Multiplying up these inequalities, we have

$$(\cos^2 A+\cos^2 B)(\cos^2 B+\cos^2 C)(\cos^2 C+\cos^2 A)$$
$$\le (1-\cos A)(1-\cos B)(1-\cos C)$$

because $1-\cos A > 0,\ 1-\cos B > 0,\ 1-\cos C > 0$. It is well known that:

$(1-\cos A)(1-\cos B)(1-\cos C) = \frac{r^2}{2R^2}$, where r, R are the inradius, circumradius, respectively of ΔABC. Namely $\sqrt{\prod(\cos^2 A+\cos^2 B)} \le \frac{\sqrt{2}}{2}\cdot\frac{r}{R}$. Also, we know $R \ge 2r$ (Euler) so $\frac{r}{R} \le \frac{1}{2}$. Let $k = \frac{r}{R} \le \frac{1}{2}$ namely $\sqrt{\prod(\cos^2 A+\cos^2 B)} \le \frac{\sqrt{2}}{2}k, k \in \left(0, \frac{1}{2}\right]$

Now, we have

$$(\cos^2 A+\cos^2 B)(\cos^2 B+\cos^2 C)(\cos^2 C+\cos^2 A) \ge$$

$$(2\cos A\cos B)(2\cos B\cos C)(2\cos C\cos A) = 8(\cos A\cos B\cos C)^2$$

We know that $\cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}$, where s denotes the semiperimeter of ΔABC. So $\prod(\cos^2 A+\cos^2 B) \ge 8\left(\frac{s^2-(2R+r)^2}{4R^2}\right)^2 = \frac{(s^2-4R^2-4Rr-r^2)^2}{2R^4} \overset{(s\ge3\sqrt{3}r)}{\ge}$

$$\frac{(27r^2-4R^2-4Rr-r^2)^2}{2R^4}=\frac{(26r^2-4R^2-4Rr)^2}{2R^4}\overset{(R\ge 2r)}{\ge}\frac{(26r^2-4R^2-2R^2)^2}{2R^4}=$$

$$=2\left(\frac{13r^2-3R^2}{R^2}\right)^2. \text{ Namely}$$

$$\sqrt{\prod(\cos^2 A+\cos^2 B)}\ge\sqrt{2}\cdot\frac{13r^2-3R^2}{R^2}=\sqrt{2}\left(13\left(\frac{r}{R}\right)^2-3\right)$$

So

$$\sqrt{2}(13k^2-3)\le\sqrt{\prod(\cos^2 A+\cos^2 B)}\le\frac{\sqrt{2}}{2}k, k\in\left(0,\frac{1}{2}\right]$$

Equality holds if and only if the acute triangle ABC is an equilateral.

SP.521 If $F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n, \forall n\in\mathbb{N}$, i.e. $\{F_n\}_{n\ge 0}$ is Fibonacci's sequence, and $L_0=2, L_1=1, L_{n+2}=L_{n+1}+L_n, \forall n\in\mathbb{N}$, i.e. $\{L_n\}_{n\ge 0}$ is Lucas' sequence, then prove that:

$$\frac{F_nL_{n+2}^2}{F_{n+3}}+\frac{F_{n+1}L_{n+3}^2}{F_n+F_{n+2}}+(L_n+L_{n+2})^2-2\sqrt{6}\cdot\sqrt{L_nL_{n+1}}\cdot L_{n+2}\ge 0, \forall n\in\mathbb{N}^*$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by proposers

By an inequality of G. Tsintsifas we have: If $m,p,q\in\mathbb{R}_+^*$ and a,b,c are the side lengths of triangle ABC with the area S, then:

$$\frac{ma^2}{p+q}+\frac{pb^2}{q+m}+\frac{qc^2}{m+p}\ge 2\sqrt{3}S \quad (1)$$

We are taking into account that $\forall x,y,z\in\mathbb{R}_+^*$ we have a triangle ABC with:

$a=x+y, b=y+z, c=z+x$, and the semiperimeter $s=\frac{a+b+c}{2}=x+y+z$, so the area is $S=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{xyz(x+y+z)}$

Therefore by (1) it follows that:

$$\frac{m(x+y)^2}{p+q}+\frac{p(y+z)^2}{q+m}+\frac{q(z+x)^2}{m+p}\ge 2\sqrt{3}\cdot\sqrt{xyz(x+y+z)}, \forall x,y,z\in\mathbb{R}_+^* \quad (2)$$

If we take in (2): $x=L_n, y=L_{n+1}, z=L_{n+2}, m=F_n, p=F_{n+1}, q=F_{n+2}, n\in\mathbb{N}^*$, then we obtain that:

$$\frac{F_n(L_n+L_{n+1})^2}{F_{n+1}+F_{n+2}}+\frac{F_{n+1}(L_{n+1}+L_{n+2})^2}{F_{n+2}+F_n}+\frac{F_{n+2}(L_{n+2}+L_n)^2}{F_n+F_{n+1}}\geq$$

$$\geq 2\sqrt{3}\cdot\sqrt{L_nL_{n+1}L_{n+2}(L_n+L_{n+1}+L_{n+2})}=$$

$$=2\sqrt{3}\cdot\sqrt{2L_nL_{n+1}L_{n+2}^2}=2\sqrt{6}\cdot\sqrt{L_nL_{n+1}}\cdot L_{n+2}$$

$$\Leftrightarrow\frac{F_nL_{n+2}^2}{F_{n+3}}+\frac{F_{n+1}L_{n+3}^2}{F_n+F_{n+2}}+(L_n+L_{n+2})^2\geq 2\sqrt{6}\cdot\sqrt{L_nL_{n+1}}\cdot L_{n+2},\forall n\in\mathbb{N}^*$$

SP.522 If $\{\varphi\}_{n\geq 0}$ is the sequence of Fermat, i.e.

$$\varphi_{n+2}-3\varphi_{n+1}-2\varphi_n=0,\varphi_0=0,\varphi_1=1,\text{ then prove that:}$$

$$2(\varphi_n^2-\varphi_{n+1}\varphi_{n-1})=2(-2)^{n-1}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Angel Plaza – Spain

$$2(\varphi_n^2-\varphi_{n+1}\varphi_{n-1})=2(-2)^{n-1}$$

By solving the recurrence relation defining the sequence of Fermat, it is obtained $\varphi_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, where $\alpha=\frac{3+\sqrt{17}}{2}$, and $\beta=\frac{3-\sqrt{17}}{2}$ are the roots of the associated characteristic polynomial $x^2-3x-2=0$. Notice that $\alpha+\beta=3,\alpha-\beta=\sqrt{17}$, and $\alpha\cdot\beta=-2$.

After some algebra, the identity follows:

$$\varphi_n^2=\left(\frac{\varphi^n-\beta^n}{\alpha-\beta}\right)^2=\frac{\alpha^{2n}+\beta^{2n}-2(\alpha\beta)^n}{17}=\frac{\alpha^{2n}+\beta^{2n}+4(-2)^{n-1}}{17}$$

On the other hand, since $\alpha^2=3\alpha+2$, and $\beta^2=3\beta+2,\alpha^2+\beta^2=13$, and then

$$\varphi_{n+1}\varphi_{n-1}=\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha+\beta}\right)\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)=\frac{\alpha^{2n}+\beta^{2n}-(\alpha\beta)^{n-1}(\alpha^2+\beta^2)}{17}$$

$$=\frac{\alpha^{2n}+\beta^{2n}-13(-2)^{n-1}}{17}=\frac{\alpha^{2n}+\beta^{2n}-13(-2)^{n-1}}{17}$$

Therefore, $2(\varphi_n^2-\varphi_{n+1}\varphi_{n-1})=2(-2)^{n-1}$, and the problem is done.

Solution 2 by Ivan Hadinata-Jember-Indonesia

(by characteristic equation)

Rewrite the recurrence relation $\varphi_{n+2} - 3\varphi_{n+1} - 2\varphi_n = 0$ and its characteristic equation $x^2 - 3x - 2 = 0$ which has the real roots $\frac{3+\sqrt{17}}{2}$ and $\frac{3-\sqrt{17}}{2}$. Then $\varphi_n = A\left(\frac{3+\sqrt{17}}{2}\right)^n + B\left(\frac{3-\sqrt{17}}{2}\right)^n, \forall n \in \mathbb{Z}_{\geq 0}$. Since $\varphi_0 = 0,\ \varphi_1 = 1,$ we get $A = \sqrt{\frac{1}{17}}$ and $B = -\sqrt{\frac{1}{17}}$. Then, for every $n \in \mathbb{Z}^+$,

$$\varphi_n^2 = \frac{1}{17}\left(\left(\frac{3+\sqrt{17}}{2}\right)^{2n} + \left(\frac{3-\sqrt{17}}{2}\right)^{2n}\right) + \frac{4}{17}(-2)^{n-1} \ \dots\dots\dots (*)$$

and

$$\varphi_{n+1}\varphi_{n-1} = \frac{1}{17}\left(\left(\frac{3+\sqrt{17}}{2}\right)^{n+1} + \left(\frac{3-\sqrt{17}}{2}\right)^{n+1}\right)\left(\left(\frac{3+\sqrt{17}}{2}\right)^{n-1} + \left(\frac{3-\sqrt{17}}{2}\right)^{n-1}\right)$$

$$= \frac{1}{17}\left(\left(\frac{3+\sqrt{17}}{2}\right)^{2n} + \left(\frac{3-\sqrt{17}}{2}\right)^{2n}\right) - \frac{13}{17}(-2)^{n-1} \ \dots\dots\dots (**)$$

Subtracting $(*)$ by $(**)$ gives us $\varphi_n^2 - \varphi_{n+1}\varphi_{n-1} = \frac{4}{17}(-2)^{n-1} - \left(-\frac{13}{17}(-2)^{n-1}\right) = (-2)^{n-1}$ or $-2(\varphi_n^2 - \varphi_{n+1}\varphi_{n-1}) = (-2)^n$ for every $n \in \mathbb{Z}^+$.

Solution 3 by Ivan Hadinata-Jember-Indonesia

(by induction)

Let $P(n)$ be the statement that $\varphi_n^2 - \varphi_{n+1}\varphi_{n-1} = (-2)^{n-1}$. The problem wants us to show that $P(n)$ is true for all $n \in \mathbb{Z}^+$. Obviously $P(1)$ is true since $\varphi_0 = 0,\ \varphi_1 = 1,$ $\varphi_2 = 3$. Assume that $P(m)$ is true for some $m \in \mathbb{Z}^+$. We have

$$(-2)^{m-1} = \varphi_m^2 - \varphi_{m+1}\varphi_{m-1} = \varphi_m^2 - (3\varphi_m + 2\varphi_{m-1})\varphi_{m-1}$$
$$\Longrightarrow$$
$$-2\varphi_m^2 + 4\varphi_{m-1}^2 + 6\varphi_m\varphi_{m-1} = (-2)^m \ \dots\dots\dots (\#)$$

Besides that,

$$\varphi_{m+1}^2 = (3\varphi_m + 2\varphi_{m-1})^2$$
$$\Longrightarrow$$
$$\varphi_{m+1}^2 - 4\varphi_{m-1}^2 - 3\varphi_m(3\varphi_m + 4\varphi_{m-1}) = 0$$
$$\Longrightarrow$$
$$\varphi_{m+1}^2 - 4\varphi_{m-1}^2 - 3\varphi_m(\varphi_{m+1} + 2\varphi_{m-1}) = 0 \ \dots\dots\dots (\#\#)$$

Summing up $(\#)$ and $(\#\#)$ gives us

$$(-2)^m = \varphi_{m+1}^2 - \varphi_m(3\varphi_{m+1} + 2\varphi_m) = \varphi_{m+1}^2 - \varphi_{m+2}\varphi_m$$

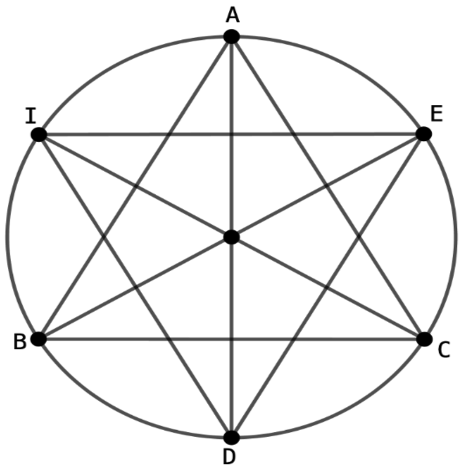
Thus, $P(m+1)$ is also true. By induction, $P(n)$ holds for all $n \in \mathbb{Z}^+$, and the result follows.

SP.523 Let be $\Delta ABC, D, E, F$ the points in which the interior bisectors intersect circumcenter. Prove that:

$$\frac{4}{3}R^2(4R+r)^2 \le DE^4+EF^4+FD^4 \le 4R^2(4R+r)(2R-r)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer



From sine theorem:

$$DE = 2R\sin\left(\frac{\pi-C}{2}\right) = 2R\sin\left(\frac{\pi}{2}-\frac{C}{2}\right) = 2R\cos\frac{C}{2} \text{ and the analogs} \Rightarrow$$

$$\left.\begin{array}{r} DE^4+EF^4+FD^4 = 16R^4\sum\cos^4\frac{A}{2} \\ \sum\cos^4\frac{A}{2} = \frac{(4R+r)^2-p^2}{8R^2} \end{array}\right\} \Rightarrow$$

$$DE^4+EF^4+FD^4 = 2R^2[(4R+r)^2-p^2] \quad (1)$$

$$p^2 \overset{Doucet}{\ge} 3r(4R+r) \Rightarrow -p^2 \le -3r(4R+r) \Rightarrow (4R+r)^2-p^2 \le$$

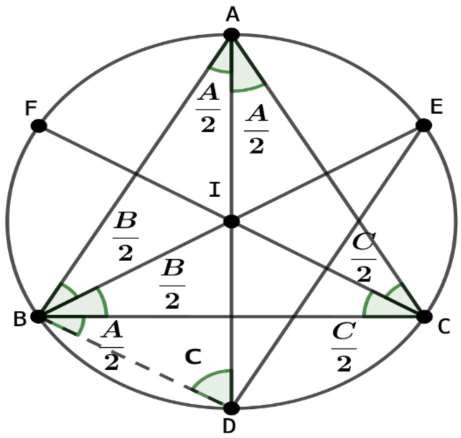
$$(4R+r)^2-3r(4R+r) = (4R+r)(4R+r-3r) = 2(4R+r)(2R-r) \quad (2)$$

$$\text{From } (1)+(2) \Rightarrow DE^4+EF^4+FD^4 \le 4R^2(4R+r)(2R-r)$$

$$p^2 \overset{Doucet}{\le} \frac{(4R+r)^2}{3} \Rightarrow -p^2 \ge -\frac{(4R+r)^2}{3} \Rightarrow (4R+r)^2-p^2 \ge \frac{2}{3}(4R+r)^2 \quad (3)$$

$$\text{From } (1)+(3) \Rightarrow DE^4+EF^4+FD^4 \ge \frac{4}{3}R^2(4R+r)^2$$

Solution 2 by Tapas Das-India



From ΔABD using sine law

$$\frac{BD}{\sin\frac{A}{2}} = \frac{C}{\sin C} \Rightarrow 2R\sin\frac{A}{2} = BD$$

$$\therefore BD = 2R\sin\frac{A}{2}$$

Again, $\angle BID = \angle IBA + \angle IAB = \angle DAC + \angle IBC = \angle CBD + \angle IBC = \angle IBD$

From $\Delta BID, ID = BD = 2R\sin\frac{A}{2}$ (analog)

Now from ΔIED we have

$$IE = 2R\sin\frac{B}{2}, ID = 2R\sin\frac{A}{2}$$

$$\angle EID = \pi - \left(\frac{B}{2} + \frac{A}{2}\right)$$

$$= \pi - \left(\frac{\pi}{2} - \frac{C}{2}\right) = \frac{\pi}{2} + \frac{C}{2}$$

Now from ΔIED we have

$$DE^2 = IE^2 + ID^2 - 2IE\cdot ID\cdot\cos\left(\frac{\pi}{2} + \frac{C}{2}\right)$$

$$\therefore DE^2 = 4R^2\sin^2\frac{B}{2} + 4R^2\sin^2\frac{A}{2} + 2\cdot 2R\sin\frac{B}{2}\cdot 2R\sin\frac{A}{2}\sin\frac{C}{2}$$

$$= 4R^2\left(\sin^2\frac{B}{2} + \sin^2\frac{A}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 4R^2\left[\sum \sin^2\frac{A}{2} + 2\prod \sin\frac{A}{2} - \sin^2\frac{C}{2}\right] = 4R^2\left[\left(1-\frac{r}{2R}\right) + 2\cdot\frac{r}{4R} - \sin^2\frac{C}{2}\right]$$

$$= 4R^2\left(1-\sin^2\frac{C}{2}\right) = 4R^2\cos^2\frac{C}{2}$$

$$\therefore DE = 2R\cos\frac{C}{2} \text{ (analog)}$$

$$\text{Similarly } DF = 2R\cos\frac{B}{2}$$

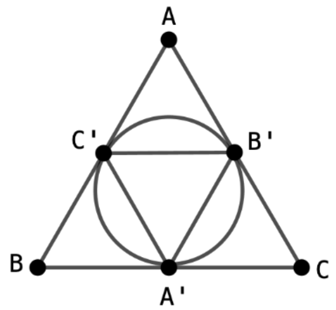
$$EF = 2R\cos\frac{A}{2}$$

SP.524 Let be ΔABC and A', B', C' the tangent points of circumcenter with the sides BC, AC, respectively AB. Prove that:

$$\frac{1}{A'B'\cdot A'C'} + \frac{1}{A'B'\cdot B'C'} + \frac{1}{A'C'\cdot B'C'} \le \left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\left(\frac{R}{r}+1\right)$$

Proposed by Marian Ursărescu – Romania

Solution by proposer



$$\angle C'A'B = \frac{\pi}{2} - \frac{B}{2} \wedge \angle B'A'C = \frac{\pi}{2} - \frac{C}{2} \Rightarrow$$

$$\angle B'A'C' = \frac{\pi}{2} - \frac{A}{2} \Rightarrow \text{from sine theorem}$$

$$\Rightarrow B'C' = 2r\sin\left(\frac{\pi}{2}-\frac{A}{2}\right) = 2r\cdot\cos\frac{A}{2} \text{ and the analogs} \Rightarrow$$

$$\sum\frac{1}{A'B'A'C'} = \frac{1}{4r^2}\sum\frac{1}{\cos\frac{A}{2}\cdot\cos\frac{B}{2}} \le \frac{1}{4r^2}\sum\frac{1}{\cos^2\frac{A}{2}} \quad (1)$$

$$\sum\frac{1}{\cos^2\frac{A}{2}} = 1 + \frac{(4R+r)^2}{p^2} \overset{Doucet}{\le} 1 + \frac{(4R+r)^2}{3r(4R+r)} = 1 + \frac{4R+r}{3r} = \frac{4(R+r)}{3r} \quad (2)$$

$$\text{From (1)}+\text{(2)} \Rightarrow \sum\frac{1}{A'B'\cdot A'C'} \le \frac{1}{3r^2}\cdot\frac{R+r}{r} = \frac{1}{3r^2}\left(\frac{R}{r}+1\right) \quad (3)$$

$$\frac{1}{3r^2}=\frac{1}{3}\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)^2\overset{Cauchy}{\leq}\frac{1}{3}\cdot 3\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)=\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\quad (4)$$

$$\text{From (3)}+(4)\Rightarrow\sum\frac{1}{A'B'\cdot A'C'}\leq\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\left(\frac{R}{r}+1\right)$$

SP.525 If $a,b,c\geq 0, a+b+c=3$ then:

$$343(ab+bc+ca)^3\leq 27(5+ab+c)(5+bc+a)(5+ca+b)$$

Proposed by Andrei Ștefan Mihalcea-Romania

Solution 1 by proposer

$$ab+bc+ca=c(a+b)+ab\underset{\sim}{\overset{AM-GM}{\leq}}c(a+b)+\frac{(a+b)^2}{4}=$$

$$=\frac{(a+b+4c)(a+b)}{4}=\frac{3(1+c)(a+b)}{4}$$

$$\frac{4}{3}(ab+bc+ca)\leq(1+c)(a+b)\underset{\sim}{\overset{GM-QM}{\leq}}\frac{c^2+2c+1+a^2+2ab+b^2}{2}$$

$$\frac{8}{3}(ab+bc+ca)\leq a^2+b^2+c^2+2ab+2c+1=$$

$$=(a+b+c)^2-2(ab+bc+ca)+2ab+2c+1$$

$$\frac{14}{3}(ab+bc+ca)\leq 9+\ 2ab+2c+1$$

$$\frac{7}{3}(ab+bc+ca)\leq 5+ab+c$$

$$7(ab+bc+ca)\leq 3(5+ab+c)$$

$$343(ab+bc+ca)^3\leq 27\prod_{cyc}(5+ab+c)$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

Let $X=27(5+ab+c)(5+bc+a)(5+ca+b)$ and $Y=ab+bc+ca$. (The problem wants us to show $X\geq(7Y)^3$). It is well-known that $a^2+b^2+c^2\geq Y$. Since $a+b+c=3$ then we can write

$$X=\frac{1}{27}\prod_{cyc}\left(5a^2+5b^2+5c^2+13ab+13bc+13ca+3(a^2+2bc)\right)$$

$$\underset{Holder\ inequality\ and\ a^2+b^2+c^2\geq Y}{\underbrace{\geq}}$$

$$\frac{1}{27}\left(18Y+3\left(\prod_{cyc}(a^2+2bc)\right)^{\frac{1}{3}}\right)^3 \dots\dots(1)$$

Note that

$$\left(\prod_{cyc}(a^2+2bc)\right)-(ab+bc+ca)^3=\beta_1+\beta_2\geq 0\dots\dots(2)$$

where

$$\beta_1=a^3b^3+b^3c^3+c^3a^3+3a^2b^2c^2-\sum_{sym}a^3b^2c\geq 0,$$

obtained from Schur's inequality, and

$$\beta_2=4abc(a^3+b^3+c^3)-2\sum_{sym}a^3b^2c\geq 0,$$

obtained by Muirhead inequality $\sum_{sym}a^4bc\geq\sum_{sym}a^3b^2c$ because $(4,1,1)\succ(3,2,1)$.

By (1) and (2), we get $X\geq\frac{1}{27}(18Y+3Y)^3=(7Y)^3$, as we expected. ■

Solution 3 and extensions by Marin Chirciu-Romania

Lemma: If $a,b,c\geq 0, a+b+c=3$ then:

$$ab+bc+ca\leq\frac{3}{7}(5+ab+c)$$

Proof: Homogenizing the inequality we can write:

$$ab+bc+ca\leq\frac{3}{7}\left(5\cdot\frac{(a+b+c)^2}{9}+ab+c\cdot\frac{a+b+c}{3}\right)\Leftrightarrow$$

$$\Leftrightarrow 21(ab+bc+ca)\leq 5a^2+5b^2+8c^2+19ab+13ac+13bc\Leftrightarrow$$

$$\Leftrightarrow 5a^2+5b^2+8c^2-2ab-8ac-8bc\geq 0\Leftrightarrow(a-b)^2+4(a-c)^2+4(b-c)^2\geq 0,$$

obviously with equality for $a=b=c=1$.

Let's get back to the main problem.

Using the Lemma and bypassing to the product in inequality $ab+bc+ca\leq\frac{3}{7}(5+ab+c)$ we obtain:

$$(ab+bc+ca)^3\overset{Lemma}{\leq}\prod\frac{3}{7}(5+ab+c)\Leftrightarrow 7^3(ab+bc+ca)^3\overset{Lemma}{\leq}$$

$$\leq 3^3\prod(5+ab+c)\Leftrightarrow$$

$$\Leftrightarrow 343(ab+bc+ca)^3\leq 27(5+ab+c)(5+bc+a)(5+ca+b).$$

Equality holds if and only if $a=b=c=1$.

Remark: The problem can be developed.

If $a,b,c\geq 0, a+b+c=3$ and $\lambda\geq 1$ then:

$$(\lambda+2)^3(ab+bc+ca)^3\leq 27(\lambda+ab+c)(\lambda+bc+a)(\lambda+ca+b)$$

Marin Chirciu

Solution: Lemma: If $a,b,c\geq 0, a+b+c=3$ and $\lambda\geq 1$ then:

$$ab+bc+ca\le\frac{3}{\lambda+2}(\lambda+ab+c)$$

Proof: Homogenizing the inequality we can write:

$$ab+bc+ca\le\frac{3}{\lambda+2}\left(\lambda\cdot\frac{(a+b+c)^2}{9}+ab+c\cdot\frac{a+b+c}{3}\right)\Leftrightarrow$$

$$\Leftrightarrow 3(\lambda+2)(ab+bc+ca)\le$$

$$\le\lambda a^2+\lambda b^2+(\lambda+3)c^2+(2\lambda+9)ab+(2\lambda+3)ac+(2\lambda+3)bc\Leftrightarrow$$

$$\Leftrightarrow\lambda a^2+\lambda b^2+(\lambda+3)c^2+(3-\lambda)ab-(\lambda+3)ac-(\lambda+3)bc\ge 0\Leftrightarrow$$

$$\Leftrightarrow(\lambda+3)c^2-(\lambda+3)c(a+b)+\lambda a^2+\lambda b^2+(3-\lambda)ab\ge 0$$

which follows from the fact that the trinome of 2^{nd} grade in variable c has $\Delta\le 0$.

Indeed: $\Delta=(\lambda+3)^2(a+b)^2-4(\lambda+3)(\lambda a^2+\lambda b^2+(3-\lambda)ab)=$

$$=(\lambda+3)[(3-3\lambda)a^2+(3-3\lambda)b^2+(6\lambda-6)a^2b]=$$

$$=3(\lambda+3)(1-\lambda)(a-b)^2\le 0,\ \text{for } \lambda\ge 1.$$

Let's get back to the main problem.Using the Lemma and bypassing to the product in inequality $ab+bc+ca\le\frac{3}{\lambda+2}(\lambda+ab+c)$ **we obtain:**

$$(ab+bc+ca)^3\overset{Lemma}{\le}\prod\frac{3}{\lambda+2}(\lambda+ab+c)\Leftrightarrow(\lambda+2)^3(ab+bc+ca)^3\le$$

$$\overset{Lemma}{\le}3^3\prod(\lambda+ab+c)\Leftrightarrow$$

$$\Leftrightarrow(\lambda+2)^3(ab+bc+ca)^3\le 27(\lambda+ab+c)(\lambda+bc+a)(\lambda+ca+b)$$

Equality holds if and only if $a=b=c=1$.

Note.

For $\lambda=5$ we obtain Problem SP.525 from RMM35 Winter Edition 2024, proposed by Andrei Ștefan Mihalcea.

UNDERGRADUATE PROBLEMS

UP.511 Prove that:

$$\int_0^\infty te^{2t}e^{-e^{-2t}}\,dt=-\frac{\gamma}{4}$$

where γ is the Euler-Mascheroni constant.

Proposed by Said Attaoui-Algerie

Solution by proposer

$$\int_0^\infty te^{2t}e^{-e^{-2t}}\,dt\overset{x=e^t}{=}\int_0^\infty\frac{\log x}{x^2}e^{-\frac{1}{x^2}}\frac{dx}{x}=\int_0^\infty\frac{1}{x^3}e^{-\frac{1}{x^2}}\log x\,dx\overset{\frac{1}{x}\to x}{=}$$

$$=\int_0^{\infty}\frac{1}{x^3}e^{-\frac{1}{x^2}}\log x\,dx=\int_0^{\infty}xe^{-x^2}\log x\,dx\overset{x^2\to x}{=}\frac{1}{4}\int_0^{\infty}e^{-x}\log x\,dx=$$

$$=\frac{1}{4}\lim_{a\to 0}\frac{d}{da}\left(\int_0^{\infty}x^a e^{-x}\,dx\right)=\frac{1}{4}\lim_{a\to 0}\frac{d}{da}\big(\Gamma(a+1)\big)=\frac{1}{4}\lim_{a\to 0}\big(\Gamma'(a+1)\big)=\frac{1}{4}\Gamma'(1)=$$

$$=\frac{1}{4}\psi(1)=-\frac{\gamma}{4}$$

where $\psi(z)$ is a first derivative of $\log(\Gamma(z))$ function defined as:

$$\psi(z)=\frac{d}{dz}\big(\log(\Gamma(z))\big)=\frac{\Gamma'(z)}{\Gamma(z)}=-\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n-1+z}\right)$$

UP.512 Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by proposers

$$We\ have:\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=e;$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D}{=}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{(n+1)e_n}=\frac{2}{e},where\ e_n=\left(1+\frac{1}{n}\right)^n,(\forall)n\in\mathbb{N}^*$$

$$Denote: u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}},(\forall)n\geq 2,\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}\right)=1$$

$$\lim_{n\to\infty}\frac{u_n-1}{\log u_n}=1;\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}=\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}=e$$

$$Denote: t_n=\frac{\pi}{4}u_n,n\geq 1,we\ have:$$

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)\right)=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n-1)!!}}{n}\cdot\lim_{n\to\infty}n\left(\tan t_n-\tan\frac{\pi}{4}\right)=$$

$$=\frac{2}{e}\cdot\lim_{n\to\infty}\frac{\sin\left(t_n-\frac{\pi}{4}\right)}{\cos t_n\cdot\cos\frac{\pi}{4}}\cdot n=\frac{2}{e}\cdot\lim_{n\to\infty}\left(\frac{\sin\left(t_n-\frac{\pi}{4}\right)}{t_n-\frac{\pi}{4}}\cdot n\left(t_n-\frac{\pi}{4}\right)\right)\cdot\frac{1}{\cos^2\frac{\pi}{4}}=$$

$$=\frac{4}{e}\cdot 1\cdot\lim_{n\to\infty}n\left(t_n-\frac{\pi}{4}\right)=\frac{4}{e}\cdot\lim_{n\to\infty}n\cdot\frac{\pi}{4}\cdot(u_n-1)=\frac{4}{e}\cdot\frac{\pi}{4}\cdot\lim_{n\to\infty}n\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1\right)=$$

$$=\frac{\pi}{e}\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{\pi}{e}\cdot e\cdot\lim_{n\to\infty}\sqrt[n]{n!}\,(u_n-1)=$$

$$=\pi\cdot\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\cdot\lim_{n\to\infty}n(u_n-1)=\frac{\pi}{e}\cdot\lim_{n\to\infty}\left(\frac{u_n-1}{\log u_n}\cdot\log u_n^n\right)=\frac{\pi}{e}\cdot 1\cdot\log e=\frac{\pi}{e}$$

Solution 2 by Angel Plaza-Spain

By Stirling approximation to $n!$, $\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}$ when $n\to\infty$, because

$$\sqrt[n]{(2n-1)!!}\sim\sqrt[n]{\frac{(2n)!}{(2n)!!}}\sim\sqrt[n]{\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{2^n n^n e^{-n}\sqrt{2\pi n}}}\sim\frac{4n^2e^{-2}}{2ne^{-1}}=\frac{2n}{e}.$$

$$\frac{\pi\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\sim\frac{\pi\sqrt[n+1]{(n+1)^{n+1}e^{-(n+1)}\sqrt{2\pi(n+1)}}}{\sqrt[n]{n^ne^{-n}\sqrt{2\pi n}}}\sim\frac{n+1}{n}.$$

Since

$$\lim_{x\to 0}\frac{\tan\left(\frac{\pi}{4}(1+x)\right)-1}{x}=\lim_{x\to 0}\frac{\pi}{4}\left(1+\tan^2\left(\frac{\pi}{4}(1+x)\right)\right)\text{ (by L'Hopital rule)}=\frac{\pi}{2},$$

the proposed limit follows $\Omega=\frac{2}{e}\cdot\frac{\pi}{2}=\frac{\pi}{e}$.

Solution 3 by Pham Duc Nam-Vietnam

$*$ As $n\to\infty$ we have Stirling's approximation for factorial and double factorial as follow:

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\Rightarrow(n+1)!\sim\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}\Rightarrow\sqrt[n+1]{(n+1)!}$$

$$\sim\left(\sqrt{2\pi(n+1)}\right)^{\frac{1}{n+1}}\left(\frac{n+1}{e}\right),\ \sqrt[n]{n!}\sim\left(\sqrt{2\pi n}\right)^{\frac{1}{n}}\left(\frac{n}{e}\right)$$

$$(2n-1)!!\sim\sqrt{2(2n-1)}\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}}\Rightarrow\sqrt[n]{(2n-1)!!}\sim\left(\sqrt{2(2n-1)}\right)^{\frac{1}{n}}\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2n}}$$

$$*\ \Omega = \lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\left(\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}\right)-1\right)\right)$$

$$= \lim_{n\to\infty}\left[\left(\sqrt{2(2n-1)}\right)^{\frac{1}{n}}\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2n}}\left(\tan\left(\frac{\pi}{4}\cdot\frac{\left(\sqrt{2\pi(n+1)}\right)^{\frac{1}{n+1}}}{\left(\sqrt{2\pi n}\right)^{\frac{1}{n}}}\left(\frac{n+1}{e}\right)\left(\frac{e}{n}\right)\right)-1\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2n}}\left(\tan\left(\frac{\pi(n+1)}{4n}\right)-1\right)$$

$$= \frac{1}{e}\lim_{n\to\infty}(2n-1)^{\frac{2n-1}{2n}}\left(\tan\left(\frac{\pi}{4}+\frac{\pi}{4n}\right)-1\right) = \frac{1}{e}\lim_{n\to\infty}(2n-1)^{\frac{2n-1}{2n}}\left(\frac{1+\tan\left(\frac{\pi}{4n}\right)}{1-\tan\left(\frac{\pi}{4n}\right)}-1\right)$$

$$= (-2)\frac{1}{e}\lim_{n\to\infty}(2n-1)^{\frac{2n-1}{2n}}\left(\frac{\tan\left(\frac{\pi}{4n}\right)}{\tan\left(\frac{\pi}{4n}\right)-1}\right)$$

$$= (-2)\frac{1}{e}\lim_{n\to\infty}(2n-1)^{\frac{2n-1}{2n}}\left(\frac{\tan\left(\frac{\pi}{4n}\right)}{\frac{\pi}{4n}}\cdot\frac{\pi}{4n}\cdot\frac{1}{\tan\left(\frac{\pi}{4n}\right)-1}\right)$$

$$= (-2)\frac{1}{e}\lim_{n\to\infty}(2n-1)^{\frac{2n-1}{2n}}\left(\frac{\pi}{4n}\cdot\frac{1}{\tan\left(\frac{\pi}{4n}\right)-1}\right)$$

$$= (-2)\frac{1}{e}\lim_{n\to\infty}\left(\frac{\pi}{4n}\cdot\frac{(2n-1)(2n-1)^{-\frac{1}{2n}}}{\tan\left(\frac{\pi}{4n}\right)-1}\right)$$

$$= (-2)\frac{1}{e}\lim_{n\to\infty}\left(\frac{\pi}{4n}\cdot\frac{(2n-1)}{\tan\left(\frac{\pi}{4n}\right)-1}\right) = (-2)\frac{1}{e}\lim_{n\to\infty}\left(\frac{\pi}{2\left(\tan\left(\frac{\pi}{4n}\right)-1\right)}-\frac{\pi}{4n\left(\tan\left(\frac{\pi}{4n}\right)-1\right)}\right)$$

$$= (-2)\frac{1}{e}\left(-\frac{\pi}{2}\right) = \frac{\pi}{e}$$

UP.513 Find:

$$\Omega = \lim_{x\to\infty}\left((x+a)\sin\frac{1}{x+a}\sqrt[x+1]{\Gamma(x+2)} - x\sin\frac{1}{x}\sqrt[x]{\Gamma(x+1)}\right); a > 0$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by proposers

$$Let\ f:\mathbb{R}_+^* \to \mathbb{R}, f(x) = (x+a)\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}\sin\frac{1}{x+a} - x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\frac{1}{x}$$

then we have to calculate $\lim_{x\to\infty} f(x)$ *. It is well – known that:*

$$\lim_{x\to\infty}\frac{1}{x}\left(\Gamma(x+1)\right)^{\frac{1}{x}}=\frac{1}{e}$$

$$\textit{Let } g:\mathbb{R}_+^*\to\mathbb{R},\ g(x)=\frac{x+a}{x}\cdot\frac{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}\cdot\frac{\sin\frac{1}{x+a}}{\sin\frac{1}{x}},\ \textit{and then}$$

$$f(x)=x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\frac{1}{x}\left(g(x)-1\right)=\frac{g(x)-1}{\log g(x)}\cdot x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\frac{1}{x}\cdot\log g(x)=$$

$$=\frac{g(x)-1}{\log g(x)}\cdot\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\frac{1}{x}\cdot\log\left(g(x)\right)^{x};\quad (1)$$

$$\lim_{x\to\infty}g(x)=\lim_{n\to\infty}\frac{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}{x+1}\cdot\frac{x}{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}\cdot\frac{x+1}{x}\cdot\frac{\sin\frac{1}{x+a}}{\frac{1}{x+a}}\cdot\frac{\frac{1}{x}}{\sin\frac{1}{x}}=\frac{1}{e}\cdot e\cdot 1\cdot 1\cdot 1$$

$$=1$$

$$\lim_{x\to\infty}\frac{g(x)-1}{\log\left(g(x)\right)}=1.\ \textit{By } (1), \textit{we have}:$$

$$\lim_{x\to\infty}f(x)=1\cdot\lim_{x\to\infty}\left(\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\frac{1}{x}\cdot\log\left(g(x)\right)^{x}\right)=$$

$$=\lim_{x\to\infty}\left(\frac{\Gamma(x+1)^{\frac{1}{x}}}{x}\cdot\frac{\sin\frac{1}{x}}{\frac{1}{x}}\cdot\log\left(g(x)\right)^{x}\right)=\frac{1}{e}\cdot 1\cdot\log\left(\lim_{x\to\infty}\left(g(x)\right)^{x}\right)=$$

$$=\frac{1}{e}\log\left(\lim_{x\to\infty}\left(\left(\frac{x+a}{x}\right)^{x}\left(\frac{\Gamma(x+2)^{\frac{1}{x+1}}}{\Gamma(x+1)^{\frac{1}{x}}}\right)^{x}\left(\frac{\sin\frac{1}{x+a}}{\sin\frac{1}{x}}\right)^{x}\right)\right)=$$

$$=\frac{1}{e}\log\left(e^{a}\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{\Gamma(x+2)^{\frac{1}{x+1}}}\right)\lim_{x\to\infty}\left(\frac{\sin\frac{1}{x+a}}{\sin\frac{1}{x}}\right)^{x}\right)=$$

$$=\frac{1}{e}\log\left(e^{a}\lim_{x\to\infty}\frac{x+1}{\Gamma(x+2)^{\frac{1}{x+1}}}\cdot\lim_{x\to\infty}\left(1+\frac{\sin\frac{1}{x+a}-\sin\frac{1}{x}}{\sin\frac{1}{x}}\right)^{x}\right)=$$

$$=\frac{1}{e}\log\left(e^{a}\cdot e\cdot\lim_{x\to\infty}\left(1+\frac{\sin\frac{1}{x+a}-\sin\frac{1}{x}}{\sin\frac{1}{x}}\right)^{x}\right)$$

Let be the function $h:\mathbb{R}_+^*\to\mathbb{R}$,

$$h(x)=\left(\sin\frac{1}{x+a}-\sin\frac{1}{x}\right)\cdot\frac{1}{\sin\frac{1}{x}},\ \textit{and we obtain that}:$$

$$\lim_{x\to\infty}f(x)=\frac{1}{e}\cdot\log\left(e^{1+e}e^{\lim\limits_{x\to\infty}xh(x)}\right)=\frac{1}{e}\left(1+a+\lim_{x\to\infty}xh(x)\right);\ \ (2)$$

$$\lim_{x\to\infty}xh(x)=\lim_{x\to\infty}\left(x\cdot\frac{\sin\frac{1}{x+a}-\sin\frac{1}{x}}{\sin\frac{1}{x}}\right)=\lim_{x\to\infty}\frac{\frac{1}{x}}{\sin\frac{1}{x}}\cdot\lim_{x\to\infty}\left(x^2\left(\sin\frac{1}{x+a}-\sin\frac{1}{x}\right)\right)=$$

$$=1\cdot\lim_{x\to\infty}\left(x^2\cdot 2\sin\frac{\frac{1}{x+a}-\frac{1}{x}}{2}\cos\frac{\frac{1}{x+a}+\frac{1}{x}}{2}\right)$$

$$=\lim_{x\to\infty}\left(2x^2\sin\frac{-a}{2x(x+a)}\cos\frac{2x+a}{2x(x+a)}\right)=$$

$$=-2\lim_{x\to\infty}\left(x^2\sin\frac{a}{2x(x+a)}\right)=-2\lim_{x\to\infty}\left(x^2\cdot\frac{\sin\frac{a}{2x(x+a)}}{\frac{a}{2x(x+a)}}\cdot\frac{a}{2x(x+a)}\right)=$$

$$=-2a\lim_{x\to\infty}\frac{x^2}{2x(x+a)}=-2a\cdot\frac{1}{2}=-a;\ \ (3)$$

By (2) *and* (3), *we obtain that*: $\lim\limits_{x\to\infty}f(x)=\frac{1}{e}(1+a-a)=\frac{1}{e}$ *and we are done.*

UP.514 If $f:(0,\infty)\to(0,\infty)$ is a convex function, $0<a\le b$ then:

$$\frac{1}{4a}\int_0^{4a}f(x)\,dx-\frac{1}{3a+b}\int_0^{3a+b}f(y)\,dy\ge\frac{1}{a+3b}\int_0^{a+3b}f(z)\,dz-\frac{1}{4b}\int_0^{4b}f(t)\,dt$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\int_0^1 f(4ax)\,dx\overset{4ax=y}{=}\int_0^{4a}f(y)\cdot\frac{1}{4a}dy=\frac{1}{4a}\int_0^1 f(x)\,dx$$

$$\int_0^1 f(4ax)\,dx=\frac{1}{4a}\int_0^1 f(x)\,dx;\ \ (1)$$

$$\textit{Analogous}:\int_0^1 f(4bx)\,dx=\frac{1}{4b}\int_0^{4b}f(t)\,dt;\ \ (2)$$

$$\int_0^1 f((3a+b)x)\,dx = \frac{1}{3a+b}\int_0^{3a+b} f(y)\,dy; \quad (3)$$

$$\int_0^1 f((a+3b)x)\,dx = \frac{1}{a+3b}\int_0^{a+3b} f(z)\,dz; \quad (4)$$

$$f - convex\ function. By\ Jensen's\ inequality:$$

$$\frac{3}{4}f(4ax) + \frac{1}{4}f(4bx) \ge ((3a+b)x)$$

$$\frac{1}{4}f(4ax) + \frac{3}{4}f(4bx) \ge f((a+3b)x)$$

$$By\ adding:$$

$$f(4ax) + f(4bx) \ge f((3a+b)x) + f((a+3b)x)$$

$$\int_0^1 f(4ax)\,dx + \int_0^1 f(4bx)\,dx \ge \int_0^1 f((3a+b)x)\,dx + \int_0^1 f((a+3b)x)\,dx$$

$$By\ (1),(2),(3)\ and\ (4):$$

$$\frac{1}{4a}\int_0^{4a} f(x)\,dx + \frac{1}{4b}\int_0^{4b} f(t)\,dt \ge \frac{1}{3a+b}\int_0^{3a+b} f(y)\,dy + \frac{1}{a+3b}\int_0^{a+3b} f(z)\,dz$$

$$\frac{1}{4a}\int_0^{4a} f(x)\,dx - \frac{1}{3a+b}\int_0^{3a+b} f(y)\,dy \ge \frac{1}{a+3b}\int_0^{a+3b} f(z)\,dz - \frac{1}{4b}\int_0^{4b} f(t)\,dt$$

$$Equality\ holds\ for\ a = b.$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

Lemma: Function $F(x) \overset{\text{def}}{=} \frac{1}{x}\int_0^x f(x)\,dx$ is convex over $x \in \mathbb{R}^+$.

Proof: Consider the function $F(x) \overset{\text{def}}{=} \frac{1}{x}\int_0^x f(t)\,dt, x \in \mathbb{R}^+$. We shall prove that F is a convex function. By substituting $y = \frac{t}{x}$, we get

$$\frac{1}{x}\int_0^x f(t)\,dt = \int_0^1 f(xy)\,dy$$

Let $x, y_1, y_2 \in \mathbb{R}^+$ with $x \in (0,1)$. Since f is convex, then

$$F((1-x)y_1 + xy_2) = \int_0^1 f((1-x)y_1y + xy_2y)\,dy$$

$$\le \int_0^1 (1-x)f(y_1y)\,dy + \int_0^1 xf(y_2y)\,dy = (1-x)F(y_1) + xF(y_2)$$

Consequently, F is convex. □

Observe that the wanted inequality in the problem is equivalent with

$$\frac{1}{4a}\int_0^{4a} f(x)\,dx+\frac{1}{4b}\int_0^{4b} f(x)\,dx\geq\frac{1}{a+3b}\int_0^{a+3b} f(x)\,dx+\frac{1}{3a+b}\int_0^{3a+b} f(x)\,dx$$

or

$$F(4a)+F(4b)\geq F(a+3b)+F(3a+b).$$

Since F is convex and $(4b,4a)\succ(a+3b,3a+b)$; by Karamata inequality we obtain

$$F(4b)+F(4a)\geq F(a+3b)+F(3a+b),$$

as desired.

UP.515 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(k+1)x\right)\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by proposer

$$Let\ be:\begin{cases}S_1=\sum_{k=0}^{n}\binom{n}{k}\cos(k+1)x\\ S_2=\sum_{k=0}^{n}\binom{n}{k}\sin(k+1)x\end{cases}$$

$$Using\ the\ identity:\sum_{k=1}^{n}\binom{n}{k}z^{k+1}=z(1+z)^n\ for\ z=\cos x+i\sin x,we\ get:$$

$$S_1+iS_2=(\cos x+i\sin x)(\cos x+i\sin x+1)^n;\ \ (1)$$

$$But:\cos x+i\sin x+1=2\cos^2\frac{x}{2}+2i\sin\frac{x}{2}\cos\frac{x}{2}=2\cos\frac{x}{2}\left(\cos\frac{x}{2}+i\sin\frac{x}{2}\right),$$

$$then\ (1)\ becomes:$$

$$S_1+iS_2=2^n\cos^n\frac{x}{2}\left(\cos\frac{n+2}{2}x+i\sin\frac{n+2}{2}x\right);\ \ (2)$$

$$By\ develop\ and\ identifying\ in\ (2),it\ follows:$$

$$\begin{cases}S_1=\sum_{k=0}^{n}\binom{n}{k}\cos(k+1)x=2^n\cos^n\frac{x}{2}\cos\frac{n+2}{2}x\\ S_2=\sum_{k=0}^{n}\binom{n}{k}\sin(k+1)x=2^n\cos^n\frac{x}{2}\sin\frac{n+2}{2}x\end{cases}$$

$$Therefore,$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(k+1)x\right)\right)=\lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{x\to\frac{\pi}{n}}\left(2^n\cos^n\frac{x}{2}\sin\frac{n+2}{2}x\right)\right)=$$

$$=\lim_{n\to\infty}\cos^n\frac{\pi}{2n}\sin\frac{n+2}{n}\cdot\frac{\pi}{2}=1.$$

Solution 2 by Pham Duc Nam-Vietnam

$*$ Denote: $S=\sum_{k=0}^{n}\binom{n}{k}\sin(x(k+1))=\sum_{k=0}^{n}\binom{n}{k}(\sin(kx)\cos(x)+\cos(kx)\sin(x))$

$=\cos(x)\sum_{k=0}^{n}\binom{n}{k}\sin(kx)+\sin(x)\sum_{k=0}^{n}\binom{n}{k}\cos(kx)$

$=\cos(x)\,\Im\sum_{k=0}^{n}\binom{n}{k}e^{kix}+\sin(x)\,\Re\sum_{k=0}^{n}\binom{n}{k}e^{kix}$

$=\cos(x)\,\Im\left(1+e^{ix}\right)^n+\sin(x)\,\Re\left(1+e^{ix}\right)^n$

$*\left(1+e^{ix}\right)^n=(1+\cos(x)+i\sin(x))^n=\left(\sqrt{\sin^2(x)+(1+\cos(x))^2}e^{i\arctan\left(\frac{\sin(x)}{1+\cos(x)}\right)}\right)^n$

$=\left(2\cos\left(\frac{x}{2}\right)e^{i\arctan\left(\tan\left(\frac{x}{2}\right)\right)}\right)^n$

$=2^n\cos^n\left(\frac{x}{2}\right)e^{in\left(\frac{x}{2}\right)}\Rightarrow\Im\left(1+e^{ix}\right)^n=2^n\cos^n\left(\frac{x}{2}\right)\sin\left(\frac{nx}{2}\right),\Re\left(1+e^{ix}\right)^n$

$=2^n\cos^n\left(\frac{x}{2}\right)\cos\left(\frac{nx}{2}\right)$

$\Rightarrow S=2^n\cos^n\left(\frac{x}{2}\right)\left(\cos(x)\sin\left(\frac{nx}{2}\right)+\sin(x)\cos\left(\frac{nx}{2}\right)\right)=2^n\cos^n\left(\frac{x}{2}\right)\sin\left(\frac{nx}{2}+x\right)$

$*\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(x(k+1))\right)=\lim_{x\to\frac{\pi}{n}}\left[2^n\cos^n\left(\frac{x}{2}\right)\sin\left(\frac{nx}{2}+x\right)\right]=2^n\cos^n\left(\frac{\pi}{2n}\right)\sin\left(\frac{\pi}{2}+\frac{\pi}{n}\right)$

$=2^n\cos^n\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{n}\right)$

$*\,\Omega=\lim_{n\to\infty}\left(\frac{1}{2^n}\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(x(k+1))\right)\right)=\lim_{n\to\infty}\left(\frac{1}{2^n}.2^n\cos^n\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{n}\right)\right)$

$=\lim_{n\to\infty}\left[\cos^n\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{n}\right)\right]=1$

UP.516 Prove that:

$$\int_0^1\frac{1}{(1-x(1-x))}dx=2\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}$$

Deduce the value of the serie:

$$\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}$$

Proposed by Said Attaoui – Algeria

Solution 1 by proposer

Recall that $\binom{2n}{n}=\frac{(2n)!}{n!(2n-n)!}=\frac{(2n)!}{(n!)^2}$.

We have

$$\int_0^1\frac{1}{(1-x(1-x))}dx=\int_0^1\frac{1-x+x}{(1-x(1-x))}dx$$

$$=\int_0^1\frac{1-x}{(1-x(1-x))}dx+\int_0^1\frac{x}{(1-x(1-x))}dx$$

$$=\int_0^1\frac{x}{(1-x(1-x))}dx+\int_0^1\frac{x}{(1-x(1-x))}dx=2\int_0^1\frac{x}{(1-x(1-x))}dx$$

$=2\int_0^1 x\left(\sum_{n=0}^{\infty}\left(x(1-x)\right)^n\right)dx$, since $x(1-x)<1$

$$=2\sum_{n=1}^{\infty}\left(\int_0^1 x^{n-1}(1-x)^n dx\right)=2\sum_{n=1}^{\infty}\beta(n,n+1)=2\sum_{n=1}^{\infty}\frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)}$$

$=2\sum_{n=1}^{\infty}\frac{(n!)^2}{n(2n)!}$, since $n\Gamma(n)=\Gamma(n+1)=n!,\forall n$

$$=2\sum_{n=1}^{\infty}\frac{1}{n\frac{(2n)!}{(n!)^2}}=2\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}$$

Now

$$\int_0^1\frac{1}{(1-x(1-x))}dx=\int_0^1\frac{1}{x^2-x+1}dx=\int_0^1\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$=\frac{2}{\sqrt{3}}\int_0^1\frac{\frac{\partial}{\partial x}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)}{\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)^2+1}=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)\right]_0^1$$

$$=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{1}{\sqrt{3}}\right)-\arctan\left(-\frac{1}{\sqrt{3}}\right)\right]=\frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{4}{\sqrt{3}}\frac{\pi}{6}=\frac{2\pi}{3\sqrt{3}}$$

It yields

$$\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}=\frac{\pi}{3\sqrt{3}}=\int_0^1\frac{x}{(1-x(1-x))}dx$$

Solution 2 by Pham Duc Nam-Vietnam

$$=\int_0^1 \frac{1}{(1-x(1-x))}dx, \text{ we have } \forall x\in(0,1)\Rightarrow x(1-x)\in(0,1)$$

$$\Rightarrow \text{We can use the series expansion of } \frac{1}{(1-x(1-x))}$$

$$\text{And: } \frac{1}{(1-x(1-x))}=\sum_{n=0}^{\infty}\big(x(1-x)\big)^n=\sum_{n=0}^{\infty}x^n(1-x)^n$$

$$\Rightarrow \Omega=\int_0^1 \frac{1}{(1-x(1-x))}dx$$

$$=\sum_{n=0}^{\infty}\int_0^1 x^n(1-x)^n dx$$

$$=\sum_{n=0}^{\infty}B(n+1,n+1)\text{, where B}(u,v)\text{ is the Beta function and B}(u,v)$$

$$=\int_0^1 x^{u-1}(1-x)^{v-1}dx, \Re(u),\Re(v)>0$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma^2(n+1)}{\Gamma(2n+2)}$$

$$=\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)(2n)!}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)\frac{(2n)!}{n!\,n!}}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)\binom{2n}{n}}$$

$$=\sum_{n=1}^{\infty}\frac{1}{(2n-1)\binom{2n-2}{n-1}}$$

$$*\text{ Now we have: }\frac{1}{(2n-1)\binom{2n-2}{n-1}}=\frac{1}{(2n-1)\frac{(2n-2)!}{(n-1)!\,(n-1)!}}=\frac{1}{(2n-1)\frac{\frac{(2n)!}{(2n-1)2n}}{\frac{(n)!}{n}\cdot\frac{(n)!}{n}}}$$

$$=\frac{2n}{(2n-1)\frac{n^2(2n)!}{(n)!\,(n)!\,(2n-1)}}=\frac{2}{\frac{n(2n)!}{(n)!\,(n)!}}=\frac{2}{n\binom{2n}{n}}$$

$$\Rightarrow \Omega=\sum_{n=1}^{\infty}\frac{1}{(2n-1)\binom{2n-2}{n-1}}=2\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}\text{ (Q.E.D)}$$

$$\Omega=\int_0^1\frac{1}{(1-x(1-x))}dx=\int_0^1\frac{1}{x^2-x+1}dx$$

$$=\frac{2\pi}{3\sqrt{3}}(\textit{basic integral}),\text{ apply the above proof: }\Omega=2\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}\Leftrightarrow\frac{2\pi}{3\sqrt{3}}$$

$$=2\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}=\frac{\pi}{3\sqrt{3}}$$

UP.517 Prove the equality:

$$\int_0^{\infty}\frac{\ln x}{x^3-x\sqrt{x}+1}dx=\frac{8\pi^2}{81}\left(5\sin\frac{\pi}{18}-\sqrt{3}\cos\frac{\pi}{18}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I=\int_0^{\infty}\frac{\ln x}{x^3-x\sqrt{x}+1}dx;\ A=\int_0^1\frac{\ln x}{x^3-x\sqrt{x}+1}dx;\ B=\int_1^{\infty}\frac{\ln x}{x^3-x\sqrt{x}+1}dx$$

We consider the intergral A. We make the variable change: $x=y^{\frac{2}{3}}$

We have, successively:

$$A=\frac{4}{9}\int_0^1\frac{(1+y)y^{-\frac{1}{3}}\ln y}{1+y^3}dy=\frac{4}{9}\left(\int_0^1\frac{y^{-\frac{1}{3}}\ln y}{1+y^3}dy+\int_0^1\frac{y^{\frac{2}{3}}\ln y}{1+y^3}dy\right)$$

$$A=\frac{4}{9}\left(\begin{array}{c}\int_0^1\sum_{k=0}^{\infty}y^{6k-\frac{1}{3}}\ln y\,dy-\int_0^1\sum_{k=0}^{\infty}y^{6k+\frac{8}{3}}\ln y\,dy+\\+\int_0^1\sum_{k=0}^{\infty}y^{6k+\frac{2}{3}}\ln y\,dy-\int_0^1\sum_{k=0}^{\infty}y^{6k+\frac{11}{3}}\ln y\,dy\end{array}\right)$$

$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left(\begin{array}{c}\int_0^1 y^{6k-\frac{1}{3}}\ln y\,dy-\int_0^1 y^{6k+\frac{8}{3}}\ln y\,dy+\\+\int_0^1 y^{6k+\frac{2}{3}}\ln y\,dy-\int_0^1 y^{6k+\frac{11}{3}}\ln y\,dy\end{array}\right)$$

We will use the following relationship:

$\int_0^1 x^a\ln x\,dx=-\frac{1}{(a+1)^2}$, **where $a\in\mathbb{R},a\geq 0$**

We obtain:

$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left[-\frac{1}{\left(6k+\frac{2}{3}\right)^2}+\frac{1}{\left(6k+\frac{11}{3}\right)^2}-\frac{1}{\left(6k+\frac{5}{3}\right)^2}+\frac{1}{\left(6k+\frac{14}{3}\right)^2}\right]$$

$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left[-\frac{\frac{1}{36}}{\left(k+\frac{2}{18}\right)^2}+\frac{\frac{1}{36}}{\left(k+\frac{11}{8}\right)^2}-\frac{\frac{1}{36}}{\left(k+\frac{5}{18}\right)^2}+\frac{\frac{1}{36}}{\left(k+\frac{14}{18}\right)^2}\right]$$

We now use the following relationship: $\Psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$

where $\Psi_1(x)$ is the trigamma function. We have obtained the value of the integral A:

$$A=\frac{1}{81}\left[-\Psi_1\left(\frac{1}{9}\right)+\Psi_1\left(\frac{11}{18}\right)-\Psi_1\left(\frac{5}{18}\right)+\Psi_1\left(\frac{7}{9}\right)\right]$$

We consider the integral B. We make the variable change: $x=\frac{1}{y}$; $y=\frac{1}{x}$.

We obtain:

$$B=-\int_0^1\frac{y\ln y}{y^3-y\sqrt{y}+1}dy$$

By proceeding similarly to the integral A, we obtain:

$$B=\frac{1}{81}\left[\Psi_1\left(\frac{2}{9}\right)-\Psi_1\left(\frac{13}{18}\right)+\Psi_1\left(\frac{7}{18}\right)-\Psi_1\left(\frac{8}{9}\right)\right]$$

Result: $I=A+B$

$$I=\frac{1}{81}\left[\begin{array}{c}-\Psi_1\left(\frac{1}{9}\right)+\Psi_1\left(\frac{11}{18}\right)-\Psi_1\left(\frac{5}{18}\right)+\Psi_1\left(\frac{7}{9}\right)+\Psi_1\left(\frac{2}{9}\right)-\\-\Psi_1\left(\frac{13}{18}\right)+\Psi_1\left(\frac{7}{18}\right)-\Psi_1\left(\frac{8}{9}\right)\end{array}\right]$$

We use the reflection formula:

$$\Psi_1(x)+\Psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\Psi_1\left(\frac{1}{9}\right)+\Psi_1\left(\frac{8}{9}\right)=\frac{\pi^2}{\sin^2\frac{\pi}{9}};\ \Psi_1\left(\frac{2}{9}\right)+\Psi_1\left(\frac{7}{9}\right)=\frac{\pi^2}{\sin^2\frac{2\pi}{9}};$$

$$\Psi_1\left(\frac{5}{18}\right)+\Psi_1\left(\frac{13}{18}\right)=\frac{\pi^2}{\sin^2\frac{5\pi}{18}};\ \Psi_1\left(\frac{7}{18}\right)+\Psi_1\left(\frac{11}{18}\right)=\frac{\pi^2}{\sin^2\frac{7\pi}{18}}$$

Result:

$$I=\frac{\pi^2}{81}\left(-\frac{1}{\sin^2\frac{\pi}{9}}+\frac{1}{\sin^2\frac{2\pi}{9}}-\frac{1}{\sin^2\frac{5\pi}{18}}+\frac{1}{\sin^2\frac{7\pi}{18}}\right)$$

We have:

$$-\frac{1}{\sin^2\frac{\pi}{9}}+\frac{1}{\sin^2\frac{2\pi}{9}}=4\sin\frac{\pi}{18}-4\sqrt{3}\cos\frac{\pi}{18}$$

We will prove this equality. We use the relationship: $\sin 3a=\sin a\,(1+2\cos 2a)$

We consider:

$$E=\frac{1}{\sin^2\frac{\pi}{9}}-\frac{1}{\sin^2\frac{2\pi}{9}}=\frac{\left(1+2\cos\frac{2\pi}{9}\right)^2}{\sin^2\frac{\pi}{3}}-\frac{\left(1+2\cos\frac{4\pi}{9}\right)^2}{\sin^2\frac{2\pi}{3}}$$

$$E=\frac{16}{3}\left(1+\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}\right)\left(\cos\frac{2\pi}{9}-\cos\frac{4\pi}{9}\right)=$$

$$=\frac{16}{3}\left(\cos\frac{2\pi}{9}+2\cos^2\frac{2\pi}{9}\right)\cdot 2\sin\frac{\pi}{9}\sin\frac{\pi}{3}$$

$$E=\frac{16\sqrt{3}}{3}\cos\frac{2\pi}{9}\left(1+2\cos\frac{2\pi}{9}\right)\sin\frac{\pi}{9}=\frac{16\sqrt{3}}{3}\cos\frac{2\pi}{9}\cdot\frac{\sin\frac{\pi}{3}}{\sin\frac{\pi}{9}}\cdot\sin\frac{\pi}{9}$$

$E=8\cos\frac{2\pi}{9}$. But: $\cos\frac{2\pi}{9}=\cos\frac{4\pi}{18}=\cos\frac{3\pi+\pi}{18}=\cos\left(\frac{\pi}{6}+\frac{\pi}{18}\right)=\frac{\sqrt{3}}{2}\cos\frac{\pi}{18}-\frac{1}{2}\sin\frac{\pi}{18}$

So:

$$E=-4\sin\frac{\pi}{18}+4\sqrt{3}\cos\frac{\pi}{18}$$

and the equality is proved.

The following relationship is proved similarly.

$$-\frac{1}{\sin^2\frac{5\pi}{18}}+\frac{1}{\sin^2\frac{7\pi}{18}}=36\sin\frac{\pi}{18}-4\sqrt{3}\cos\frac{\pi}{18}$$

Result: $I=\frac{8\pi^2}{81}\left(5\sin\frac{\pi}{18}-\sqrt{3}\cos\frac{\pi}{18}\right)$

UP.518 Let $F,f,g:[0,1]\to\mathbb{R}$ such as $g'(x)>0$ for every $x\in[0,1]$ and $F'(x),\frac{f'(x)}{g'(x)}$ are Riemann integrable. Find:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(F\left(\frac{k}{n}\right)-F\left(\frac{k-1}{n}\right)\right)\frac{f'\left(\frac{k}{n}\right)+f'\left(\frac{k-1}{n}\right)}{g'\left(\frac{k}{n}\right)+g'\left(\frac{k-1}{n}\right)}$$

Proposed by Cristian Miu – Romania

Solution by proposer

From Lagrange theorem we obtain that:

$$F\left(\frac{k}{n}\right)-F\left(\frac{k-1}{n}\right)=\frac{1}{n}F'(x_{kn}),x_{kn}\in\left[\frac{k-1}{n},\frac{k}{n}\right]\text{ so}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(F\left(\frac{k}{n}\right)-F\left(\frac{k-1}{n}\right)\right)\frac{f'\left(\frac{k}{n}\right)+f'\left(\frac{k-1}{n}\right)}{g'\left(\frac{k}{n}\right)+g'\left(\frac{k-1}{n}\right)}=$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}F'(x_{kn})\cdot\frac{f'\left(\frac{k}{n}\right)+f'\left(\frac{k-1}{n}\right)}{g'\left(\frac{k}{n}\right)+g'\left(\frac{k-1}{n}\right)}$$

Now let us notice that

$$\min\left(\frac{f'\left(\frac{k}{n}\right)}{g'\left(\frac{k}{n}\right)},\frac{f'\left(\frac{k-1}{n}\right)}{g'\left(\frac{k-1}{n}\right)}\right)\le\frac{f'\left(\frac{k}{n}\right)+f'\left(\frac{k-1}{n}\right)}{g'\left(\frac{k}{n}\right)+g'\left(\frac{k-1}{n}\right)}\le\max\left(\frac{f'\left(\frac{k}{n}\right)}{g'\left(\frac{k}{n}\right)},\frac{f'\left(\frac{k-1}{n}\right)}{g'\left(\frac{k-1}{n}\right)}\right)$$

because

$g'(x)>0$ **for every** $x\in[0,1]$

Let us recall Jarnik theorem

If $h_1,h_2:[a,b]\to\mathbb{R}$ are two derivable function and $h_2'(x)\neq0$ for every $x\in[a,b]$ then $\frac{h_1'(x)}{h_2(x)}$ has Darboux property. So we can write that

$$\frac{f'\left(\frac{k}{n}\right)+f'\left(\frac{k-1}{n}\right)}{g'\left(\frac{k}{n}\right)+g'\left(\frac{k-1}{n}\right)}=\frac{f'(y_{kn})}{g'(y_{kn})}$$

$$y_{kn} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

Now we need to find

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f'(x_{kn}) \cdot \frac{f'(y_{kn})}{g'(y_{kn})}$$

where

$$x_{kn} \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \text{ and } y_{kn} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

$F'(x)$ and $\frac{f'(x)}{g'(x)}$ are Riemann integrable so there exists

$$\int_0^1 f'(x)' \cdot \frac{f'(x)}{g'(x)} dx$$

Let $t_n = \frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \frac{f'(y_{kn})}{g'(y_{kn})}$

Then $\left|t_n - \int_0^1 F'(x) \cdot \frac{f'(x)}{g'(x)} dx\right| =$

$$= \left|\begin{array}{c} \frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \cdot \frac{f'(y_{kn})}{g'(y_{kn})} - \frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \cdot \frac{f'(x_{kn})}{g'(x_{kn})} + \\ + \frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \frac{f'(x_{kn})}{g'(x_{kn})} - \int_0^1 F'(x) \frac{f'(x)}{g'(x)} dx \end{array}\right| \le$$

$$\le \left|\frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \left(\frac{f'(y_{kn})}{g'(y_{kn})} - \frac{f'(x_{kn})}{g'(x_{kn})}\right)\right| + \left|\frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \frac{f'(x_{kn})}{g'(x_{kn})} - \int_0^1 F'(x) \cdot \frac{f'(x)}{g'(x)}\right|$$

But $F'(x) \cdot \frac{f'(x)}{g'(x)}$ is Riemann integrable so

$$\lim_{n\to\infty} \left|\frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \frac{f'(x_{kn})}{g'(x_{kn})} - \int_0^1 F'(x) \frac{f'(x)}{g'(x)}\right| = 0$$

Let us prove that

$$\lim_{n\to\infty} \left|\frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \left(\frac{f'(y_{kn})}{g'(y_{kn})}\right) - \frac{f'(x_{kn})}{g'(x_{kn})}\right| = 0$$

$$\left|\frac{1}{n}\sum_{k=1}^{n} F'(x_{kn}) \left(\frac{f'(y_{kn})}{g'(y_{kn})}\right) - \frac{f'(x_{kn})}{g'(x_{kn})}\right| \le \frac{1}{n}\sum_{k=1}^{n} |F'(x_{kn})| \cdot \left|\frac{f'(y_{kn})}{g'(y_{kn})} - \frac{f'(x_{kn})}{g'(x_{kn})}\right| \le$$

$$\leq \frac{M}{n}\sum_{k=1}^{n}\left|\frac{f'(y_{kn})}{g'(y_{kn})}-\frac{f'(x_{kn})}{g'(x_{kn})}\right|$$

because $F'(x)$ is Riemann integrable so is bounded. Now

$$\frac{1}{n}\sum_{k=1}^{n}\left|\frac{f'(y_{kn})}{g'(y_{kn})}-\frac{f'(x_{kn})}{g'(x_{kn})}\right| \leq \frac{1}{n}\sum_{k=1}^{n}(M_{kn}-m_{kn})$$

where M_{kn} and m_{kn} are upper bound and the lower bound of $\frac{f'(x)}{g'(x)}$ on $\left[\frac{k-1}{n},\frac{k}{n}\right]$. But $\frac{f'(x)}{g'(x)}$ is Riemann integrable so by Darboux theorem

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(M_{kn}-m_{kn}) = 0$$

In conclusion

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(F\left(\frac{k}{n}\right)\cdot F\left(\frac{k-1}{n}\right)\right)\cdot\frac{f'\left(\frac{k}{n}\right)+f'\left(\frac{k-1}{n}\right)}{g'\left(\frac{k}{n}\right)+g'\left(\frac{k-1}{n}\right)} = \int_0^1 F'(x)\cdot\frac{f'(x)}{g'(x)}dx$$

UP.519 In triangle ABC_Δ we note H the orthocentre and O the circumcenter of the triangle. Let D, E, F be the midpoints of $[BC], [AC]$ and $[AB]$ and let A_1, B_1, C_1 be the points symmetric to H with respect to D, E and F, and let H_1 be the orthocentre of the triangle $A_1B_1C_1$. Prove that $HH_1 = 2OH$.

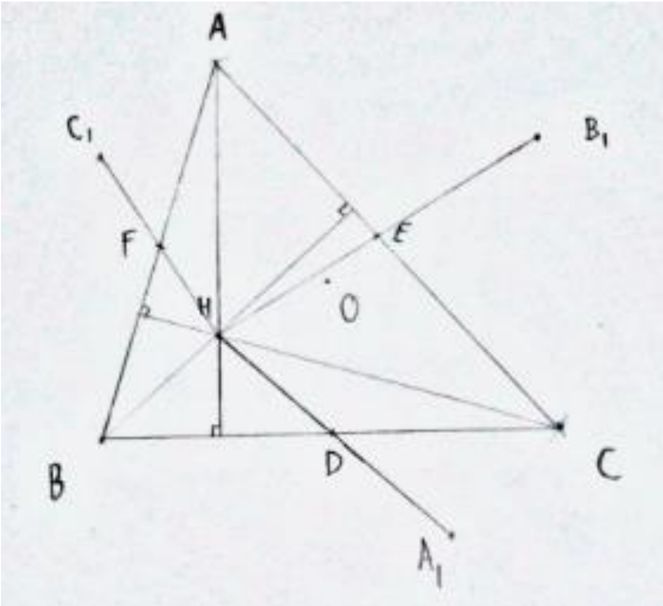
Proposed by Pal Orban – Romania

Solution 1 by proposer

Let O be the origin of the complex plane and let $A, B, C, H, A_1, B_1, H_1$ have the affixes $a, b, c, h, a_1, b_1, c_1, h_1 \in \mathbb{C}$. From Sylvester's theorem: $h = a + b + c$, and based on the symmetry given $h + a_1 = b + c \Rightarrow a + b + c + a_1 = b + c \Rightarrow a_1 = -a$, analogously: $b_1 = -b, c_1 = -c$. Hence, A_1, B_1 and C_1 lie on the circumcircle of ABC_Δ, opposite to points A, B, C. We can then apply Sylvester's theorem again, to get: $h_1 = a_1 + b_1 + c_1 = -(a + b + c) = -h$, implying that O is the midpoint of HH_1, hence $HH_1 = 2OH$.

Solution 2 by Ivan Hadinata-Jember-Indonesia

Since AH and OD are perpendicular to BC, so $AH \parallel OD$. Let G be the centroid of ΔABC. Then, $\frac{AG}{GD} = 2$ and O, G, H are collinear on the Euler line. Therefore, $\Delta AHG \sim \Delta DOG$ and $AH = 2OD$. Because D is the midpoint of A_1H, so there exists a homothety, with center A_1 and factor 2, mapping OD to AH. Thus, A_1 is the reflection of A with respect to O. By using similar way, B_1 and C_1 are the reflection of B and C, respectively. Therefore, $\Delta A_1B_1C_1$ is the reflection of ΔABC with respect to point O. Consequently, H_1 is the reflection of point H with respect to point O and it implies that $HH_1 = 2OH$. ■



UP.520 If $a_n > 0; n \in \mathbb{N}^*$ is such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a > 0$$

then find:

$$\Omega(a) = \lim_{n\to\infty} \left(H_n - \log \sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

$$H_n - \log \sqrt[n]{a_n} = H_n - \log n + \log n - \log \sqrt[n]{a_n} = \gamma_n + \log \frac{n}{\sqrt[n]{a_n}}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{a_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{a_n}} \overset{CAUCHY-D'ALEMBERT}{=} \lim_{n\to\infty} \frac{(n+1)^{n+1}}{a_{n+1}} \cdot \frac{a_n}{n^n} =$$

$$= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^{n+1}}\cdot\frac{na_n}{a_n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}\cdot\lim_{n\to\infty}\frac{1}{\frac{a_{n+1}}{na_n}} = e\cdot\frac{1}{a} = \frac{e}{a}$$

$$\Omega(a) = \lim_{n\to\infty}\left(\gamma_n + \log\frac{n}{\sqrt[n]{a_n}}\right) = \gamma + \log\left(\frac{e}{a}\right) = \gamma + 1 - \log a$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

We decompose the expression as follows.

$$\Omega(a) = \lim_{n\to\infty}\left(H_n - \log\sqrt[n]{a_n}\right) = \lim_{n\to\infty}(H_n - \log n) - \lim_{n\to\infty}\log\frac{\sqrt[n]{a_n}}{n}$$

$$= \gamma - \lim_{n\to\infty}\log\frac{\sqrt[n]{a_n}}{n} \quad \dots\dots(1)$$

Setting $c_n = \frac{a_n}{n}$ for all $n \in \mathbb{Z}^+$. We will obtain

$$\lim_{n\to\infty}\frac{c_{n+1}}{c_n} = \lim_{n\to\infty}\frac{a_{n+1}}{na_n}\cdot\left(\frac{n}{n+1}\right)^{n+1} = \lim_{n\to\infty}\frac{a_{n+1}}{na_n}\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1} = \frac{a}{e}.$$

By Cauchy Second Limit Theorem,

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty}\sqrt[n]{c_n} = \frac{a}{e}.$$

Then, from (1), it deduces

$$\Omega(a) = \gamma - \lim_{n\to\infty}\log\frac{\sqrt[n]{a_n}}{n} = \gamma - \log\frac{a}{e} = \gamma + 1 - \log a.$$

(Note: γ =Euler-Mascheroni constant, e =Euler number.)

UP.521 If $a_1 = 1; a_{n+1} = a_n + e^{H_n}\cdot\sin\frac{\pi}{n}; n \in \mathbb{N}^*$ then find:

$$\Omega = \lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solutions 1 by proposers

$$\lim_{n\to\infty}(a_{n+1} - a_n) = \lim_{n\to\infty}e^{H_n}\cdot\sin\frac{\pi}{n} =$$

$$= \lim_{n\to\infty}e^{H_n - \ln n}\cdot e^{\ln n}\cdot\sin\frac{\pi}{n} = e^{\gamma}\cdot\lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cdot\pi = \pi\cdot e^{\gamma}$$

$$\Omega = \lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}} = \lim_{n\to\infty}\frac{a_n}{n}\cdot\frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty}\frac{a_n}{n}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} \overset{CESARO-STOLZ}{=}$$

$$=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n+1-n}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}\overset{CAUCHY-D'ALEMBERT}{=}\pi\cdot e^{\gamma}\cdot\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=$$

$$=\pi\cdot e^{\gamma}\cdot\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\pi\cdot e^{\gamma}\cdot e=\pi\cdot e^{\gamma+1}$$

Solution 2 by Ivan Hadinata-Jember-Indonesia

Let $(b_n)_{k=1}^{\infty}$ be the sequence where $b_1=1$ and $b_n=e^{H_{n-1}}\sin\frac{\pi}{n-1}$ for all $n\geq 2$.
Then, $a_n=b_1+b_2+\cdots+b_{n-1}+b_n,\forall n\geq 2$ and $a_1=b_1$.
Remind that $e=\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}$, so we have

$$\Omega=\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}}=\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right)\left(\lim_{n\to\infty}\frac{a_n}{n}\right)=e\left(\lim_{n\to\infty}\frac{a_n}{n}\right)\quad\dots\dots\dots\dots(1)$$

Observe that

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}e^{H_{n-1}}\sin\frac{\pi}{n-1}=\lim_{n\to\infty}e^{H_n}\sin\frac{\pi}{n}=\pi\left(\lim_{n\to\infty}e^{H_n-\log n}\right)\left(\lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)=e^{\gamma}\pi.$$

By Cauchy first limit theorem,

$$\lim_{n\to\infty}\frac{a_n}{n}=\lim_{n\to\infty}\frac{b_1+b_2+\cdots+b_n}{n}=\lim_{n\to\infty}b_n=e^{\gamma}\pi\quad\dots\dots\dots\dots(2)$$

Thus, (1) and (2) give us

$$\Omega=e^{\gamma+1}\pi.$$

UP.522 Find $x,y>0$ such that:

$$81x^2+y+\frac{1}{2x+y}=16x+1$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$81x^2-16x-1+y+\frac{1}{2x+y}=0$$

$$81x^2-18x+1+2x-2+y+\frac{1}{2x+y}=0$$

$$81\left(x^2-\frac{2x}{9}+\frac{1}{81}\right)+2x+y+\frac{1}{2x+y}-2=0$$

$$81\left(x-\frac{1}{9}\right)^2+\left(\sqrt{2x+y}\right)^2+\left(\frac{1}{\sqrt{2x+y}}\right)^2-2=0$$

$$81\left(x-\frac{1}{9}\right)^2+\left(\sqrt{2x+y}-\frac{1}{\sqrt{2x+y}}\right)^2=0$$

$$\begin{cases} x-\frac{1}{9}=0 \\ \sqrt{2x+y}-\frac{1}{\sqrt{2x+y}} \end{cases} \Rightarrow x=\frac{1}{9}$$

$$2x+y-1=0 \Rightarrow \frac{2}{9}+y-1=0 \Rightarrow y=\frac{7}{9}$$

$$\text{Solution: } \begin{cases} x=\frac{1}{9} \\ y=\frac{7}{9} \end{cases}$$

UP.523 If $x,y,z>0, xyz=x+y+z+2$, then:

$$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}} \geq \frac{6}{\sqrt{xyz}}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

The inequality can be equivalently written:

$$\sum \frac{1}{\sqrt{x}} \geq \frac{6}{\sqrt{xyz}} \Leftrightarrow \sum \sqrt{yz} \geq 6$$

Deconditioning the relationship $xyz=x+y+z=2$ by the substitution

$$(x,y,z)=\left(\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}\right)$$

We obtain:

$$\sum \sqrt{yz} \geq 6 \Leftrightarrow \sum \sqrt{\frac{c+a}{b}\cdot\frac{a+b}{c}} \geq 6 \Leftrightarrow \sum \sqrt{\frac{(a+b)(a+c)}{bc}} \geq 6$$

which follows from means inequality:

$$\sum \sqrt{\frac{(a+b)(a+c)}{bc}} \overset{AGM}{\geq} 3\sqrt[3]{\prod \frac{(a+b)(a+c)}{bc}}=3\sqrt[3]{\sqrt{\prod \frac{(a+b)(a+c)}{bc}}}=$$

$$= 3\sqrt[3]{\sqrt{\frac{(a+b)^2(b+c)^2(c+a)^2}{a^2b^2c^2}}} = 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}} \overset{Cesaro}{\geq} 3\sqrt[3]{8} = 6$$

Equality holds if and only if $a = b = c \Leftrightarrow x = y = z = 2$.

Solution 2 by Ivan Hadinata-Jember-Indonesia

Let $a = \sqrt{\frac{1}{xy}},\ b = \sqrt{\frac{1}{xz}},\ c = \sqrt{\frac{1}{yz}}$. Then,

$$xyz = x + y + z + 2 \quad \Leftrightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$$

Therefore, there exists a non-degenerate triangle ΔABC in such a way that $a = \sin\frac{A}{2}$, $b = \sin\frac{B}{2},\ c = \sin\frac{C}{2}$. Thus,

$$\sum_{cyc}\sqrt{xy} = \sum_{cyc}\csc\frac{A}{2}$$

Since $y = \sin x,\ x \in (0,\pi)$ is concave; by Jensen inequality we have

$$\sum_{cyc}\sin\frac{A}{2} \leq 3\sin\left(\frac{1}{3}\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right)\right) = 3\sin\frac{\pi}{6} = \frac{3}{2}$$

CS-inequality helps us to get

$$9 \leq \left(\sum_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\csc\frac{A}{2}\right) \leq \frac{3}{2}\left(\sum_{cyc}\csc\frac{A}{2}\right) \quad \Rightarrow \quad 6 \leq \sum_{cyc}\csc\frac{A}{2} = \sum_{cyc}\sqrt{xy}$$

$$\Rightarrow \quad \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} \geq \frac{6}{\sqrt{xyz}}$$

Solution 3 by David Chatarashvili-Georgia

$$x, y, z > 0; xyz = x + y + z + 2$$

Then: $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} \geq \frac{6}{\sqrt{xyz}}$

1) $xyz = x + y + z + 2 \geq 3\sqrt[3]{xyz} + 2$ (Cauchy Inequality)

$$\sqrt[3]{xyz} = t, t^3 = xyz, t > 0$$

$$t^3 \geq 3t + 2 \Rightarrow t^3 - 3t - 2 \geq 0 \Rightarrow (t+1)^2(t-2) \geq 0 \Rightarrow$$

$$\Rightarrow t \geq 2 \Rightarrow \sqrt[3]{xyz} \geq 2$$

2) $S = \sqrt{yz} + \sqrt{xz} + \sqrt{xy} \geq 3\sqrt[3]{\sqrt{x^2y^2z^2}} = 3\sqrt[3]{xyz} \geq 6$

$$\sqrt{yz} + \sqrt{xz} + \sqrt{xy} \geq 6$$

$$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}\ge\frac{6}{\sqrt{xyz}}$$

UP.524 In ΔABC holds:

$$\sum\frac{(m_b+m_c)^{n+1}}{\left(m_a+\sqrt{m_bm_c}\right)^n}\ge\frac{12r}{R}(2R-r),n\in\mathbb{N}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Using the inequality $\sum x^2\ge\sum yz$ for $(x,y,z)=\left(\sqrt{m_a},\sqrt{m_b},\sqrt{m_c}\right)$ we have

$$\sum m_a\ge\sum\sqrt{m_bm_c}$$

With Radon's inequality we obtain:

$$\sum\frac{(m_b+m_c)^{n+1}}{\left(m_a+\sqrt{m_bm_c}\right)^n}\overset{Radon}{\ge}\frac{[\sum(m_b+m_c)]^{n+1}}{\left[\sum\left(m_a+\sqrt{m_bm_c}\right)\right]^n}=\frac{(2\sum m_a)^{n+1}}{(2\sum m_a)^n}=$$

$$=2\sum m_a\ge2\cdot\frac{6R}{r}(2R-r)=\frac{12r}{R}(2R-r)$$

Above, we've used:

$$\sum m_a\overset{Tereshin}{\ge}\sum\frac{b^2+c^2}{4R}=\sum\frac{a^2}{2R}=\frac{2(s^2-r^2-4Rr)}{2R}\overset{Gerretsen}{\ge}$$

$$\ge\frac{16Rr-5r^2-r^2-4Rr}{R}=\frac{12Rr-6r^2}{R}=\frac{6r(2R-r)}{R}=\frac{6r}{R}(2R-r)$$

Equality holds if and only if the triangle is equilateral.

Solution 2 by Tapas Das-India

$$\sum\frac{(m_b+m_c)^{n+1}}{\left(m_a+\sqrt{m_bm_c}\right)^n}$$

$$\overset{Radon}{\ge}\frac{2^{n+1}(m_a+m_b+m_c)^{n+1}}{\left[m_a+m_b+m_c+\sqrt{m_bm_c}+\sqrt{m_cm_a}+\sqrt{m_am_b}\right]^n}$$

$$\overset{AM-GM}{\ge}\frac{2^{n+1}(m_a+m_b+m_c)^{n+1}}{\left[(m_a+m_b+m_c+\frac{m_b+m_c}{2}+\frac{m_c+m_a}{2}+\frac{m_a+m_b}{2}\right]^n}$$

$$=\frac{2^{n+1}(m_a+m_b+m_c)^{n+1}}{2^n(m_a+m_b+m_c)^n}=2(m_a+m_b+m_c)$$

$$\overset{Tereshin}{\geq} 2\left(\frac{b^2+c^2}{4R}+\frac{c^2+a^2}{4R}+\frac{a^2+b^2}{4R}\right)=\frac{4}{4R}\left(\sum a^2\right)=\frac{1}{R}2(s^2-r^2-4Rr)$$

$$\overset{Gerretsen's}{\geq}\frac{1}{R}\times 2(16Rr-5r^2-4Rr)=\frac{1}{R}\cdot 2(12Rr-6r^2)=\frac{12r}{R}(2R-r)$$

UP.525 In acute ΔABC holds:

$$2s\left(2+\frac{3R}{r}-\frac{R^2}{r^2}\right)\leq\sum\frac{b+c}{\cos A}\leq\frac{4s}{3}\sum\sec A$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Lemma.

In ΔABC:

$$\sum\frac{b+c}{\cos A}=\frac{2s(s^2+r^2-2Rr-4R^2)}{s^2-(2R+r)^2}$$

Proof.

$$\sum\frac{b+c}{\cos A}=\frac{\sum(b+c)\cos B\cos C}{\prod\cos A}=\frac{2s(s^2+r^2-2Rr-4R^2)}{s^2-(2R+r)^2}$$

We've use above:

$$\sum(b+c)\cos B\cos C=\frac{s(s^2+r^2-2Rr-4R^2)}{2R^2}$$

and

$$\prod\cos A=\frac{s^2-(2R+r)^2}{4R^2}$$

$$\sum bc(b+c)\,(a^2+b^2-c^2)(a^2+c^2-b^2)=32s^3r^2(s^2+r^2-2Rr-4R^2)$$

Let's get back to the main problem. Using the Lemma we obtain:

RHS $\sum\frac{b+c}{\cos A}=\frac{2s(s^2+r^2-2Rr-4R^2)}{s^2-(2R+r)^2}\overset{(1)}{\leq}\frac{4s}{3}\sum\sec A$

Where (1) $\Leftrightarrow\frac{2s(s^2+r^2-2Rr-4R^2)}{s^2-(2R+r)^2}\leq\frac{4s}{3}\sum\sec A\Leftrightarrow$

$$\Leftrightarrow \frac{2s(s^2+r^2-2Rr-4R^2)}{s^2-(2R+r)^2} \le \frac{4s}{3}\cdot\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2} \Leftrightarrow$$

$$\Leftrightarrow 3(s^2+r^2-2Rr-4R^2) \le 2(s^2+r^2-4R^2) \Leftrightarrow$$

$$\Leftrightarrow 3(s^2+r^2-2Rr-4R^2) \le 2(s^2+r^2-4R^2) \Leftrightarrow s^2 \le 4R^2+6Rr-r^2$$

which follows from Gerretsen's inequality: $s^2 \le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$4R^2+4Rr+3r^2 \le 4R^2+6Rr-r^2 \Leftrightarrow R \ge 2r \quad \text{(Euler)}$$

We've used above:

$$\sum \sec A = \sum \frac{1}{\cos A} = \frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}$$

Equality holds if and only if the triangle is equilateral.

LHS $\sum \frac{b+c}{\cos A} = \frac{2s(s^2+r^2-2Rr-4R^2)}{s^2-(2R+r)^2} \overset{Walker}{\underset{Gerretsen}{\ge}}$

$$\ge \frac{2s(2R^2+8Rr+3r^2+r^2-2Rr-4R^2)}{4R^2+4Rr+3r^2-(2R+r)^2} =$$

$$= \frac{2s(4r^2+6Rr-2R^2)}{2r^2} = \frac{2s(2r^2+3Rr-R^2)}{r^2} = 2s\left(2+\frac{3R}{r}-\frac{R^2}{r^2}\right)$$

Equality holds if and only if the triangle is equilateral.

Solution 2 by Tapas Das-India

$$\sum \frac{b+c}{\cos A} = \sum \frac{2s-a}{\cos A} = \sum \frac{2s}{\cos A} - \sum \frac{a}{\cos A}$$

$$= 2s\frac{\sum \cos A\cos B}{\prod \cos A} - 2R\sum \tan A$$

$$= 2s\cdot\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2} - 2R\cdot\frac{2sr}{s^2-(2R+r)^2} = 2s\left[\frac{s^2+r^2-4R^2-2Rr}{s^2-(2R+r)^2}\right] \ge$$

$$\overset{Walker's\ and\ Gerretsen's}{\ge} 2s\left[\frac{(2R^2+8Rr+3r^2)+r^2-4R^2-2Rr}{(4R^2+4Rr+3r^2)-(2R+r)^2}\right]$$

$$= 2s\left[\frac{4r^2+6Rr-2R^2}{2r^2}\right] = 2s\left[2+3\frac{R}{r}-\frac{R^2}{r^2}\right]$$

$$\therefore \sum \frac{b+c}{\cos A} \ge 2s\left[2+3\frac{R}{r}-\frac{R^2}{r^2}\right]$$

Note: Using following identities:

1) $\sum \cos A \cdot \cos B = \frac{s^2+r^2-4R^2}{4R^2}$

2) $\prod \cos A = \frac{s^2-(2R+r)^2}{4R^2}$

3) $\sum \tan A = \frac{2sr}{s^2-(2R+r)^2}$

4) $a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$

$$\sum \frac{b+c}{\cos A} \overset{Chebyshev}{\leq} \frac{1}{3}\left(\sum (b+c)\right) \cdot \sum \frac{1}{\cos A}$$

WLOG: $a \geq b \geq c \therefore A \geq B \geq C$

$\therefore a+b \geq c+a \geq b+c$

and $\cos A \leq \cos B \leq \cos C$

$$\therefore \sum \frac{b+c}{\cos A} \leq \frac{1}{3}\left(\sum (b+c)\right) \cdot \sum \frac{1}{\cos A} = \frac{1}{3}(2s) \cdot 2 \cdot \sum \sec A = \frac{4s}{3}\sum \sec A$$