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SOLUTIONS

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DANIEL SITARU

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Proposed by

Daniel Sitaru – Romania

Nguyen Viet Hung – Hanoi – Vietnam

Marin Chirciu – Romania

George Apostolopoulos – Messolonghi – Greece

Martin Lukarevski – Stip – Macedonia

D.M. Bătinețu-Giurgiu – Romania

Neculai Stanciu – Romania

Mihály Bencze – Romania

Shivam Sharma – New Delhi – India

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Solutions by

Daniel Sitaru – Romania

Kevin Soto Palacios – Huarmey – Peru, Nguyen Ngoc Tu-Ha Giang-Vietnam, Boris Colakovic-Belgrade-Serbia, Truong Nguyen-Phan Rang-Vietnam, Nguyen Thanh Nho-Tra Vinh-Vietnam

Sanong Huayrerai-Nakon Pathom-Thailand

Sladjan Stankovik-Macedonia, Ravi Prakash-New Delhi-India

Soumava Chakraborty-Kolkata-India, Myagmarsuren Yadamsuren-Darkhan-Mongolia, Seyran Ibrahimov-Maasilli-Azerbaidian

SK Rejuan-West Bengal-India, Marin Chirciu – Romania

Nguyen Minh Tri-Vietnam, Nguyen Viet Hung – Hanoi – Vietnam

Mihály Bencze – Romania,Soumitra Mandal-Chandar Nagore-India

Shivam Sharma-New Delhi-India, Nguyen Phuc Tang-Hanoi-Vietnam

Tran Hong-Vietnam, Mohammed Hijazi-Amman-Jordania

Khalef Ahmad El Ruhemi-Jarash-Jordania

Atteiah Yahya Ahmed Atiya Al Zahrani-Jeddah-Kingdom Of Saudi Arabia

JP.061. Let a, b, c be positive real numbers such that $a+b+c=3$.

Prove that

$$\frac{1}{a^3+b^3+c^3}+\frac{8}{ab+bc+ca}\geq 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números R^+ de tal manera que $a+b+c=3$. Probar que

$\frac{1}{a^3+b^3+c^3}+\frac{8}{ab+bc+ca}\geq 3$. *Realizamos los siguientes cambios de variables*

$x=a^3+b^3+c^3>0, y=ab+bc+ca>0$. *La desigualdad propuesta es equivalente*

$\frac{1}{x}+\frac{8}{y}\geq 3$. *Aplicando la desigualdad de Cacuchy*

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}\right)(x+y+y+y+y+y+y+y+y)\geq 9^2=81$$

Es suficiente probar

$$x+8y\leq 27\Leftrightarrow a^3+b^3+c^3+8(ab+bc+ca)\leq (a+b+c)^3\Leftrightarrow$$

$\Leftrightarrow 8(ab+bc+ca)\leq 3(a+b)(b+c)(c+a)$. *Lo cual es cierto ya que*

$$3(a+b)(b+c)(c+a)\geq 3\cdot\frac{8}{9}(a+b+c)(ab+bc+ca)=8(ab+bc+ca)$$

Por lo tanto $\rightarrow \frac{1}{x}+\frac{8}{y}\geq\frac{81}{x+8y}\geq 3\Leftrightarrow\frac{1}{a^3+b^3+c^3}+\frac{8}{ab+bc+ca}\geq 3$

Solution 2 by Nguyen Ngoc Tu-Ha Giang-Vietnam

We have: $(a+b)(b+c)(c+a)\geq\frac{8}{9}(a+b+c)(ab+bc+ca)$

$\Rightarrow ab+bc+ca\leq\frac{3}{8}(a+b)(b+c)(c+a)$. *Hence*

$$\frac{1}{\sum a^3}+\frac{8}{\sum ab}\geq\frac{1}{\sum a^3}+\frac{64}{3(a+b)(b+c)(c+a)}\geq\frac{(1+8)^2}{\sum a^3+3\prod(a+b)}=\frac{81}{(a+b+c)^3}=3$$

Solution 3 by Boris Colakovic-Belgrade-Serbia

$$\frac{1}{a^3+b^3+c^3}+\frac{8}{ab+bc+ca}\geq 3|\cdot(a^3+b^3+c^3)\Leftrightarrow$$

$$\Leftrightarrow 1+\frac{8(a^3+b^3+c^3)}{ab+bc+ca}\geq 3(a^3+b^3+c^3)\qquad (1)$$

$$\frac{1}{ab+bc+ca} \geq \frac{3}{(a+b+c)^2} = \frac{1}{3}. \text{ From (1)} \Rightarrow$$

$$\Rightarrow 1 + \frac{8}{3}(a^3 + b^3 + c^3) \geq 3(a^3 + b^3 + c^3) \Leftrightarrow$$

$$\Leftrightarrow 3 + 8(a^3 + b^3 + c^3) \geq 9(a^3 + b^3 + c^3) \Leftrightarrow a^3 + b^3 + c^3 \leq 3 \quad (2)$$

$$a + b + c \overset{AM-GM}{\geq} 3\sqrt[3]{abc} \Rightarrow abc \leq 1$$

$$a^3 + b^3 + c^3 \overset{AM-GM}{\geq} 3\sqrt[3]{(abc)^3} \Rightarrow 3abc \leq a^3 + b^3 + c^3 \Rightarrow abc \leq \frac{a^3+b^3+c^3}{3} \leq 1 \Rightarrow \quad (2)$$

Equality holds for $a = b = c = 1$

Solution 4 by Truong Nguyen-Phan Rang-Vietnam

$$(a + b + c)(ab + bc + ca) \leq \frac{9}{8}(a + b)(b + c)(c + a)$$

$$\Leftrightarrow a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \geq 0$$

$$\text{So } \frac{1}{a^3+b^3+c^3} + \frac{8}{ab+bc+ca} = \frac{1}{a^3+b^3+c^3} + \frac{24}{(a+b+c)(ab+bc+ca)}$$

$$\geq \frac{1}{a^3 + b^3 + c^3} + \frac{8^2}{3(a + b)(b + c)(c + a)} \geq \frac{(1 + 8)^2}{a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a)} =$$

$$= \frac{9^2}{(a + b + c)^3} = \frac{9^2}{3^3} = 3$$

JP.062. Prove that if: $a, b, c, d \in [1, \infty)$ **then:**

$$3a + 3b + 2c + d \leq 6 + ab(1 + c + cd)$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

We prove by induction by n *that* $x_1, x_2, \dots, x_n \in [1, \infty)$; $n \in \mathbb{N}^*$ *implies:*

$$x_1 + x_2 + \cdots + x_n \leq n - 1 + x_1 x_2 \dots x_n. \text{ Checking:}$$

$$n = 1; \quad x_1 \leq 1 - 1 + x_1 \Leftrightarrow x_1 \leq x_1$$

$$n = 2; \quad x_1 + x_2 \leq 1 + x_1 x_2 \Leftrightarrow (x_1 - 1)(x_2 - 1) \geq 0 \text{ which it's true.}$$

$$P(k): x_1 + x_2 + \cdots + x_k \leq k - 1 + x_1 x_2 \dots x_k. \text{ Suppose that it's true.}$$

$$P(k + 1): x_1 + x_2 + \cdots + x_k + x_{k+1} \leq k + x_1 x_2 \dots + x_k x_{k+1}. \text{To proved.}$$

$$x_1 + x_2 + \cdots + x_k + x_{k+1} \leq k - 1 + x_1 x_2 \dots x_k + x_{k+1}$$

Remains to prove that: $k-1+x_1x_2\dots x_k+x_{k+1}\le k+x_1x_2\dots x_kx_{k+1}$

$$0\le x_1x_2\dots x_k(x_{k+1}-1)-(x_{k+1}-1)$$

$$(x_1x_2\dots x_k-1)(x_{k+1}-1)\ge 0$$

$$P(k)\to P(k+1)$$

For $x_1=a; x_2=b; x_3=c; x_4=d$

$$a+b\le 1+ab$$

$$a+b+c\le 2+abc$$

$$a+b+c+d\le 3+abcd$$

By adding: $3a+3b+2c+d\le 6+ab(1+c+cd)$

JP.063. Let x,y,z be non-negative real numbers satisfying $\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}=3$. Find the minimum possible value of

$$\sqrt[3]{x+2y+5z}+\sqrt[3]{y+2z+5x}+\sqrt[3]{z+2x+5y}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo x,y,z *números reales no negativos de tal manera que* $\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}=3$

Hallar el mínimo valor $E=\sqrt[3]{x+2y+5z}+\sqrt[3]{y+2z+5x}+\sqrt[3]{z+2x+5y}$

Como $x,y,z\ge 0$. *Aplicando la desigualdad de Holder*

$$\sqrt[3]{\frac{(x+y+y+z+z+z+z+z)(1+1+1+1+1+1+1+1)(1+1+1+1+1+1+1+1)}{64}}\ge\frac{\sqrt[3]{\left(\sqrt[3]{x}+2\sqrt[3]{y}+5\sqrt[3]{z}\right)^3}}{4}$$

$$\Rightarrow\sqrt[3]{x+2y+5z}\ge\frac{\sqrt[3]{x}+2\sqrt[3]{y}+5\sqrt[3]{z}}{4}\quad (A)$$

Análogamente para los siguientes términos se cumplirá

$$\sqrt[3]{y+2z+5x}\ge\frac{\sqrt[3]{y}+2\sqrt[3]{z}+5\sqrt[3]{x}}{4}\quad (B),$$

$$\sqrt[3]{z+2x+5y}\ge\frac{\sqrt[3]{z}+2\sqrt[3]{x}+5\sqrt[3]{y}}{4}\quad (C)$$

Sumando (A) + (B) + (C)

$$E=\sqrt[3]{x+2y+5z}+\sqrt[3]{y+2z+5x}+\sqrt[3]{z+2x+5y}\ge\frac{8\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)}{4}=\frac{8\cdot 3}{4}=6$$

La igualdad se alcanza cuando $x=y=z=1$

(LQQD)

Solution 2 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$\textbf{Let } a=\sqrt[3]{x}, b=\sqrt[3]{y}, c=\sqrt[3]{z}\Rightarrow a+b+c=3$$

$$P=\sqrt[3]{a^3+2b^3+5c^3}+\sqrt[3]{b^3+2c^3+5a^3}+\sqrt[3]{c^3+2a^3+5b^3}$$

$$\min(P)=?$$

$$\left(\frac{a^3+2b^3+5c^3}{1+2+5}\right)^{\frac{1}{3}}\ge\frac{a+2b+5c}{1+2+5}\quad \textit{(Radon)}$$

$$\Rightarrow\sqrt[3]{a^3+2b^3+5c^3}\ge\frac{1}{4}(a+2b+5c)\quad (1)$$

$$\textit{Simillary } \sqrt[3]{b^3+2c^3+5a^3}\ge\frac{1}{4}(b+2c+5a)\quad (2)$$

$$\sqrt[3]{c^3+2a^3+5b^3}\ge\frac{1}{4}(c+2a+5b)\quad (3)$$

$$(1)+(2)+(3)\Rightarrow P\ge 2(a+b+c)=6$$

$$\Rightarrow\min(P)=6\Leftrightarrow a=b=c=1\Leftrightarrow x=y=z=1$$

JP.064. Let a,b,c be non-negative real numbers. Prove that

$$\sqrt[3]{1+a^3}+\sqrt[3]{1+b^3}+\sqrt[3]{1+c^3}\ge\frac{\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}}{\sqrt[6]{2}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c números reales no negativos. Probar que

$$\sqrt[3]{1+a^3}+\sqrt[3]{1+b^3}+\sqrt[3]{1+c^3}\ge\frac{\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}}{\sqrt[6]{2}}$$

Como $a,b,c\ge 0$. Aplicando la desigualdad de MA ≥ MG y Holder

$$\sqrt[3]{a^3+1}+\sqrt[3]{1+b^3}\ge 2\sqrt[6]{\frac{(a^3+1)(1+b^3)(1+1)}{2}}=2\sqrt[6]{\frac{(a+b)^3}{2}}=\frac{2\sqrt{a+b}}{\sqrt[6]{2}}\quad (A)$$

Análogamente para los siguientes términos se cumplirá

$$\sqrt[3]{b^3+1}+\sqrt[3]{1+c^3}\ge\frac{2\sqrt{b+c}}{\sqrt[6]{2}}\quad (B),$$

$$\sqrt[3]{c^3+1}+\sqrt[3]{1+a^3}\ge\frac{2\sqrt{c+a}}{\sqrt[6]{2}}\quad (C)$$

Sumando (A) + (B) + (C)

$$2\sqrt[3]{1+a^3}+2\sqrt[3]{1+b^3}+2\sqrt[3]{1+c^3}\ge\frac{2\sqrt{a+b}+2\sqrt{b+c}+2\sqrt{c+a}}{\sqrt[6]{2}}$$

$$\Rightarrow\sqrt[3]{1+a^3}+\sqrt[3]{1+b^3}+\sqrt[3]{1+c^3}\ge\frac{\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}}{\sqrt[6]{2}}$$

Solution 2 by Sanong Hauerai-Nakonpathom-Thailand

$$\sqrt[3]{(1+a^3)}+\sqrt[3]{(1+b^3)}+\sqrt[3]{(1+c^3)}=(1+a^3)^{\frac{1}{3}}+(1+b^3)^{\frac{1}{3}}+(1+c^3)^{\frac{1}{3}}$$

$$=(1+a^3)^{\frac{2}{6}}+(1+b^3)^{\frac{2}{6}}+(1+c^3)^{\frac{2}{6}}=((1+a^3)^2)^{\frac{1}{6}}+((1+b^3)^2)^{\frac{1}{6}}+((1+c^3)^2)^{\frac{1}{6}}$$

$$\ge\left(((1+a^3)^2)^{\frac{1}{6}}((1+b^3)^2)^{\frac{1}{6}}\right)^{\frac{1}{2}}+\cdots+\left(((1+c^3)^2)^{\frac{1}{6}}((1+a^3)^2)^{\frac{1}{6}}\right)^{\frac{1}{2}}$$

$$=\left(((1+a^3)^2)^{\frac{1}{2}}((1+b^3)^2)^{\frac{1}{2}}\right)^{\frac{1}{6}}+\cdots+\left(((1+c^3)^2)^{\frac{1}{2}}((1+a^3)^2)^{\frac{1}{2}}\right)^{\frac{1}{6}}$$

$$=\left((1+a^3)(1+b^3)\right)^{\frac{1}{6}}+\cdots+\left((1+c^3)(1+a^3)\right)^{\frac{1}{6}}$$

$$=\left(\frac{2}{2}(1+a^3)(1+b^3)\right)^{\frac{1}{6}}+\cdots+\left(\frac{2}{2}(1+c^2)(1+a^3)\right)^{\frac{1}{6}}$$

$$\ge\frac{(a+b)^{\frac{1}{2}}+(b+c)^{\frac{1}{2}}+(c+a)^{\frac{1}{2}}}{2^{\frac{1}{6}}}.\ \textit{Because}\left(\frac{2}{2}(1+a^3)(1+b^3)\right)^{\frac{1}{6}}\ge\frac{(a+b)^{\frac{1}{2}}}{2^{\frac{1}{6}}}$$

$$\textit{If}\left(2(1+a^3)(1+b^3)\right)^{\frac{1}{6}}\ge(a+b)^{\frac{1}{2}}.\ \textit{If }2(1+a^3)(1+b^3)\ge(a+b)^3$$

$$\textit{If }2+a^3+b^3+a^3b^3\ge3a^2b+3b^2a.$$

$$\textit{If }(1+a^2+a^3b^3)+c+b^3+a^3b^3\ge3(a^2b+b^2a).\ \textit{Is to be true}$$

Solution 3 by Sladjan Stankovik-Macedonia

$a,b,c\ge0$. ***Prove that:*** $\sum_{cyc}\sqrt[3]{1+a^3}\ge\frac{1}{\sqrt[6]{2}}\cdot\sum_{cyc}\sqrt{a+b}\Leftrightarrow\sum_{cyc}\sqrt[3]{\frac{1+a^3}{2}}\ge\sum_{cyc}\sqrt{\frac{a+b}{2}}$

$$LHS=\sum_{cyc}\sqrt[3]{\frac{1+a^3}{2}}\ge\sum_{cyc}\frac{a+1}{2}=\frac{a+b+c+3}{2}\ge\sqrt{3(a+b+c)}\ge\sum_{cyc}\sqrt{\frac{a+b}{2}}=$$

$$\frac{1+a^3}{2} \geq \left(\frac{a+1}{2}\right)^3 \Leftrightarrow 3(a-1)^2 \cdot (a+1) \geq 0$$

$p+3 \geq 2\sqrt{3p} \Leftrightarrow \left(\sqrt{p}-\sqrt{3}\right)^2 \geq 0;\ 3(x^2+y^2+z^2) \geq (x+y+z)^2$ ***where:***

$$x = \sqrt{\frac{a+b}{2}};\ y = \sqrt{\frac{b+c}{2}};\ z = \sqrt{\frac{c+a}{2}}.$$

JP.065. Let ABC be an equilateral triangle inscribed in the circle (O) whose radius R. Prove that for an arbitrary point P lies on (O),

$$6\sqrt{2} < \frac{PA^3+PB^3+PC^3}{R^3} < 3\sqrt[4]{216}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Ravi Prakash-New Delhi-India

Let's take circle to be $x^2+y^2 = R^2$ ***(1)***

If $A(R\cos\theta, R\sin\theta)$ ***and*** $B(R\cos\phi, R\sin\phi)$***, are two points on (1), then***

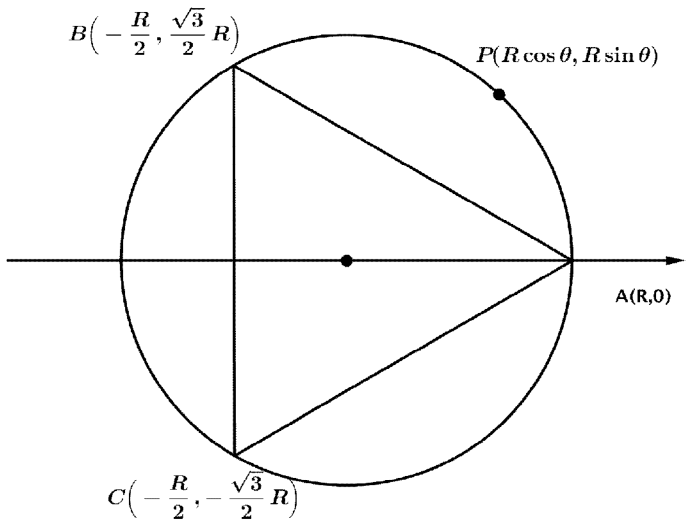
$$AB^2 = R^2(\cos\theta-\cos\phi)^2 + R^2(\sin\theta-\sin\phi)^2$$

$$= R^2[\cos^2\theta+\cos^2\phi-2\cos\theta\cos\phi+\sin^2\theta+\sin^2\phi-2\sin\theta\sin\phi]$$

$$= R^2[2-2\cos(\theta-\phi)] = 4R^2\sin^2\left(\frac{\theta-\phi}{2}\right) \Rightarrow AB = 2R\left|\sin\left(\frac{\theta-\phi}{2}\right)\right|$$

Let's take $\Delta\, ABC$ ***with vertices as*** $A(R,0), B\left(-\frac{R}{2}, \frac{\sqrt{3}}{2}R\right), C\left(-\frac{R}{2}, -\frac{\sqrt{3}}{2}R\right)$

In view of the symmetry we take point P ***as*** $(R\cos\theta, R\sin\theta)$ ***where*** $0 \leq \theta \leq \pi$



Now, $PA = 2R\left|\sin\frac{\theta}{2}\right|$ ***[For*** $A, \phi = 0$***]***

$$= 2R\sin\frac{\theta}{2} \ [\because 0 \le \phi \le \pi]$$

$$PB = 2R\left|\sin\left(\frac{\theta - \frac{2\pi}{3}}{2}\right)\right|$$

[For $B, \phi = \frac{2\pi}{3}$***]***

$$= 2R\left|\sin\left(\frac{\theta}{2} - \frac{\pi}{3}\right)\right|$$

Similarly, $PC = 2R\left|\sin\left(\frac{\theta}{2} + \frac{\pi}{3}\right)\right| = 2R\sin\left(\frac{\theta}{2} + \frac{\pi}{3}\right)$

$$E(\theta) = 6\sqrt{3}\cos\left(\frac{\theta}{2}\right) + 8\sin^3\left(\frac{\theta}{2}\right)$$

$$0 \le \theta \le \frac{2\pi}{3}$$

For $\frac{2\pi}{3} < \theta < \pi$

$$E(\theta) = 2\left[\begin{array}{c} 3\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right) - 3\sin\left(\frac{\pi}{3} - \frac{\theta}{2}\right) + \sin\left(\pi - \frac{3\theta}{2}\right) + \\ +3\sin\left(\frac{\pi}{3} + \frac{\theta}{2}\right) - \sin\left(\pi + \frac{3\theta}{2}\right) \end{array}\right]$$

$$= 2\left[3\sin\left(\frac{\theta}{2}\right) + 3(2)\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{3\theta}{2}\right)\right]$$

$$= 2\sin\left(\frac{3\theta}{2}\right) + 12\sin\left(\frac{\theta}{2}\right) = 18\sin\left(\frac{\theta}{2}\right) - 8\sin^3\frac{\theta}{2}$$

***Not that** $E(\theta)$ **is continuous at** $\theta = \frac{2\pi}{3}$ **and hence on** $[0,\pi]$*

Using** $4\sin^3 A = 3\sin A - \sin(3A)$**, we get

***For** $0 \le \theta \le \frac{2\pi}{3}$*

$$E(\theta) = 2\begin{bmatrix} 3\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right) + 3\sin\left(\frac{\pi}{3} - \frac{\theta}{2}\right) - \sin\left(\pi - \frac{3\theta}{2}\right) + \\ +3\sin\left(\frac{\pi}{3} + \frac{\theta}{2}\right) - \sin\left(\pi - \frac{3\theta}{2}\right) \end{bmatrix}$$

***if** $0 \le \theta \le \frac{2\pi}{3}$*

$$= 6\left[\sin\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\theta}{2}\right)\right] - 2\sin\left(\frac{3\theta}{2}\right)$$

$$= 6\sin\left(\frac{\theta}{2}\right) + 6\sqrt{3}\cos\left(\frac{\theta}{2}\right) - 6\sin\left(\frac{\theta}{2}\right) + 8\sin^3\left(\frac{\theta}{2}\right)$$

$$E'(\theta) = \begin{cases} -3\sqrt{3}\sin\left(\frac{\theta}{2}\right) + 12\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right); 0 < \theta < \frac{2\pi}{3} \\ 9\cos\left(\frac{\theta}{2}\right) - 12\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right); \frac{2\pi}{3} < \theta < \pi \end{cases}$$

$$= \begin{cases} 6\sin\left(\frac{\theta}{2}\right)\left[\sin\theta - \frac{\sqrt{3}}{2}\right]; 0 < \theta < \frac{2\pi}{3} \\ 12\cos\left(\frac{\theta}{2}\right)\left(\frac{\sqrt{3}}{2} - \sin\theta\right)\left(\frac{\sqrt{3}}{2} + \sin\theta\right); \frac{2\pi}{3} < \theta < \pi \end{cases}$$

***Note that** $E'(\theta) = 0$ **for** $\frac{\pi}{3}$**. Also, at** $\frac{2\pi}{3}$, $E'(\theta) = 0$**. Now,** $E(0) = 8\left[2\sin^3\left(\frac{\pi}{3}\right)\right] = 6\sqrt{3}$*

$$E\left(\frac{\pi}{3}\right) = 8\left[\sin^3\left(\frac{\pi}{6}\right) + \sin^3\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{2}\right)\right] = 8\left(\frac{1}{8} + \frac{1}{8} + 1\right) = 10$$

$E\left(\frac{2\pi}{3}\right) = 8\left[\frac{3\sqrt{3}}{8} + 0 + \frac{3\sqrt{3}}{8}\right] = 6\sqrt{3}$***. Also,** $E(\pi) = 10$**. As** $10 < 6\sqrt{3}$**,***

$\min E(\theta) = 10\sqrt{100} > \sqrt{72} = 6\sqrt{2}$ ***and*** $\max E(\theta) = 6\sqrt{3} = 3(2\sqrt{3}) = 3(2^4 \cdot 3^2)^{\frac{1}{4}}$

$$= 3(144)^{\frac{1}{4}} < 3\left(\frac{2}{6}\right)^{\frac{1}{4}}$$

JP.066. Prove that in any triangle

$$\sum \frac{m_a^2}{h_b h_c} \geq n \cdot \frac{R}{r} + (3-2n) \cdot \frac{s}{3r\sqrt{3}}, n \leq \frac{3}{2}$$

Proposed by Marin Chirciu – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC \sum \frac{m_a^2}{h_b h_c} \geq n \cdot \frac{R}{r} + (3-2n) \cdot \frac{s}{3\sqrt{3}r}$, *donde* $n \leq \frac{3}{2}$

Tener en cuenta lo siguiente

$$abc = 4RS, a + b + c = 2s = \frac{2S}{r} \Leftrightarrow (a+b+c)abc = 8S^2 \cdot \frac{R}{r}$$

Ahora bien

$\frac{R}{r} - \frac{2s}{3\sqrt{3}r} = \frac{r(3\sqrt{3}R - 2s)}{3\sqrt{3}r} \geq 0 \Leftrightarrow$ *Lo cual es cierto ya que en un*

$\Delta\ ABC \rightarrow 3\sqrt{3}R \geq 2s$. *Como* $n \leq \frac{3}{2}$

$$\rightarrow n \cdot \frac{R}{r} + (3-2n) \cdot \frac{s}{3\sqrt{3}r} = n \cdot \left(\frac{R}{r} - \frac{2s}{3\sqrt{3}r}\right) + \frac{3s}{3\sqrt{3}r} \leq \frac{3}{2}\left(\frac{R}{r} - \frac{2s}{3\sqrt{3}r}\right) + \frac{s}{\sqrt{3}r} = \frac{3R}{2r}$$

Es suficiente probar $\sum \frac{4m_a^2}{4h_b h_c} \geq \frac{3R}{2r}$

$$\Leftrightarrow \frac{(2b^2 + 2c^2 - a^2)bc}{16S^2} + \frac{(2c^2 + 2a^2 - b^2)ca}{16S^2} + \frac{(2a^2 + 2b^2 - c^2)ab}{16S^2} \geq \frac{3R}{2r}$$

$$\Leftrightarrow 2bc \sum (b^2 + c^2) - abc(a+b+c) \geq 16S^2 \cdot \frac{3R}{2r} = 3abc(a+b+c)$$

Aplicando MA ≥ MG

$$2bc \sum (b^2 + c^2) - abc(a+b+c) \geq 4 \sum b^2c^2 - abc(a+b+c) \geq$$

$$\geq 4abc(a+b+c) - abc(a+b+c) = 3abc(a+b+c)$$

JP.067. Prove that in any triangle

$$n \cdot \frac{s^2 + r^2}{Rr} + \sqrt[k]{\frac{2r}{R}} \geq 14n + 1, n \geq \frac{1}{2}, k \in N, k \geq 2.$$

Proposed by Marin Chirciu – Romania

Solution by Kevin Soto Palacios Peru – Huarmey – Peru

Probar en un triángulo ABC: $n\cdot\frac{s^2+r^2}{Rr}+\sqrt[k]{\frac{2r}{R}}\geq 14n+1, n\geq\frac{1}{2}, k\in N, k\geq 2$

Tener en cuenta lo siguiente

1) *Siendo a, b, c los lados de un triángulo se cumple la siguiente desigualdad*

$$(ab+bc+ca)\left(\frac{1}{ba}+\frac{1}{bc}+\frac{1}{ca}\right)\geq 9\Leftrightarrow\frac{(s^2+r^2+4Rr)}{2Rr}\geq 9\Leftrightarrow$$

$$\Leftrightarrow s^2+r^2-14Rr\geq 0$$

2) *Para todo $k\geq 2$ it checks* $\left(\frac{2r}{R}\right)^{k-1}\leq 1\Leftrightarrow\left(\frac{2r}{R}\right)^k\leq\frac{2r}{R}\Leftrightarrow\frac{2r}{R}\leq\sqrt[k]{\frac{2r}{R}}$

La desigualdad propuesta s equivalente

$$n\cdot\left(\frac{s^2+r^2}{Rr}-14\right)+\sqrt[k]{\frac{2r}{R}}\geq 1\Leftrightarrow n\left(\frac{s^2+r^2-14Rr}{Rr}\right)+\sqrt[k]{\frac{2r}{R}}\geq 1$$

Ahora bien $n\left(\frac{s^2+r^2-14Rr}{Rr}\right)+\sqrt[k]{\frac{2r}{R}}\geq\frac{1}{2}\left(\frac{s^2+r^2-14Rr}{Rr}\right)+\frac{2r}{R}$

Es suficiente probar

$$\frac{1}{2}\left(\frac{s^2+r^2-14Rr}{Rr}\right)+\frac{2r}{R}\geq 1\Leftrightarrow\frac{s^2+r^2}{2Rr}+\frac{4r^2}{2Rr}\geq 8\Leftrightarrow s^2\geq 16Rr-5r^2$$

(Válido por desigualdad de Gerretsen)

JP.068. Let a, b and c be the side lengths of a triangle ABC, with circumradius R and inradius r. Prove that

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a+b}+\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{b+c}+\frac{\tan\frac{C}{2}+\tan\frac{A}{2}}{c+a}\leq\frac{1}{r}-\frac{1}{R}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a+b}+\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{b+c}+\frac{\tan\frac{C}{2}+\tan\frac{A}{2}}{c+a}\leq\frac{1}{r}-\frac{1}{R}$$

Teniendo en cuenta las siguientes notaciones y desigualdades en un $\Delta\ ABC$

$$\tan\frac{A}{2}=\frac{(s-b)(s-c)}{S}, \tan\frac{B}{2}=\frac{(s-c)(s-a)}{S}, \tan\frac{C}{2}=\frac{(s-a)(s-b)}{S}$$

$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{a+b+c}{2}$ ***(Nesbit)***, $R\geq 2r$ ***(Euler)***

La desigualdad propuesta es equivalente

$$\frac{(s-b)(s-c)+(s-c)(s-a)}{S(a+b)}+\frac{(s-c)(s-a)+(s-a)(s-b)}{S(b+c)}+$$

$$+\frac{(s-a)(s-b)+(s-b)(s-c)}{S(c+a)}\leq\frac{1}{r}-\frac{1}{R}$$

$$\frac{(s-c)(s-b+s-a)}{S(a+b)}+\frac{(s-a)(s-c+s-b)}{S(b+c)}+\frac{(s-b)(s-c+s-a)}{S(c+a)}\leq\frac{1}{r}-\frac{1}{R}$$

$$\frac{(a+b-c)c}{2S(a+b)}+\frac{(b+c-a)a}{2S(b+c)}+\frac{(a+c-b)b}{2S(c+a)}\leq\frac{1}{r}-\frac{1}{R}$$

$$\sum\frac{(a+b)c}{2S(a+b)}-\sum\frac{c^2}{2S(a+b)}\leq\sum\frac{c}{2S}-\frac{a+b+c}{4S}=$$

$$=\frac{a+b+c}{2S}-\frac{a+b+c}{4S}=\frac{a+b+c}{4S}=\frac{2p}{4pr}=\frac{1}{2r}\leq\frac{1}{r}-\frac{1}{R}\ \textit{(LQQD)}$$

Lo cual es cierto ya que $\rightarrow\frac{1}{2r}\leq\frac{1}{r}-\frac{1}{R}\Leftrightarrow\frac{1}{R}\leq\frac{1}{2r}\Leftrightarrow R\geq 2r$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$s\tan\frac{A}{2}=s\cdot\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}=\frac{\Delta}{s-a}=\frac{rs}{s-a}$$

$\Rightarrow\tan\frac{A}{2}=\frac{r}{s-a}$. ***Similarly,*** $\tan\frac{B}{2}=\frac{r}{s-b}$ & $\tan\frac{C}{2}=\frac{r}{s-c}$. ***Now,*** $\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a+b}=\frac{r}{a+b}\left(\frac{1}{s-a}+\frac{1}{s-b}\right)$

$=\frac{r(s-b+s-a)}{(a+b)(s-a)(s-b)}\overset{(1)}{=}\frac{rc}{(s-a)(s-b)(a+b)}$. ***Similarly,*** $\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{b+c}\overset{(2)}{=}\frac{ra}{(s-b)(s-c)(b+c)}$,

and, $\frac{\tan\frac{C}{2}+\tan\frac{A}{2}}{c+a}\overset{(3)}{=}\frac{rb}{(s-c)(s-a)(c+a)}$

(1)+(2)+(3)⇒LHS

$$=r\left\{\frac{c}{(s-a)(s-b)(a+b)}+\frac{a}{(s-b)(s-c)(b+c)}+\frac{b}{(s-c)(s-a)(c+a)}\right\}$$

$\overset{(4)}{=}\frac{r\sum\{c(b+c)(c+a)(s-c)\}}{\{\prod(s-a)\}\{\prod(a+b)\}}$. ***Now,*** $\sum\{c(b+c)(c+a)(s-c)\}=\sum\{(cs-c^2)(\sum ab+c^2)\}$

$$=\sum(cs-c^2)\left(\sum ab\right)+s\sum a^3-\sum a^4$$

$$= \left(\sum ab\right)\cdot\left(s\sum a - \sum a^2\right) + s\left\{3abc + 2s\left(\sum a^2 - \sum ab\right)\right\} -$$

$$-\left\{\left(\sum a^2\right)^2 - 2\sum a^2b^2\right\}$$

$$= 2s^2(\sum ab) - (\sum ab)(\sum a^2) + 3sabc + 2s^2(\sum a^2) - 2s^2(\sum ab) -$$

$$-\left(\sum a^2\right)^2 + 2\left\{\left(\sum ab\right)^2 - 2abc(2s)\right\}$$

$$= \left(\sum a^2\right)\left(2s^2 - \sum ab\right) + 2\left(\sum ab\right)^2 - \left(\sum a^2\right)^2 - 5sabc$$

$$= \left(\sum a^2\right)(s^2 - 4Rr - r^2) - \left(\sum a^2\right)^2 + 2\left(\sum ab\right)^2 - 5sabc$$

$$= \frac{(\sum a^2)(\sum a^2)}{2} - \left(\sum a^2\right)^2 + 2\left(\sum ab\right)^2 - 5sabc$$

$$= \frac{4(\sum ab)^2 - (\sum a^2)^2 - 10s(4Rrs)}{2}$$

$$= \frac{(2\sum ab + \sum a^2)(2\sum ab - \sum a^2) - 40Rrs^2}{2}$$

$$= \frac{2(s^2 + 4Rr + r^2 + s^2 - 4Rr - r^2)(16Rr + 4r^2) - 40Rrs^2}{2}$$

$$= 2s^2(16Rr + 4r^2) - 20Rrs^2 = 2s^2(6Rr + 4r^2) \overset{(5)}{=} 4s^2r(3R + 2r)$$

Again, $\prod(a+b) = 2abc + \sum ab\,(2s - c)$

$$= 2abc + 2s\left(\sum ab\right) - 3abc = 2s\left(\sum ab\right) - 4Rrs \overset{(6)}{=} 2s(s^2 + 2Rr + r^2)$$

Also, $\prod(s-a) = \frac{s\cdot\{\prod(s-a)\}}{s} = \frac{r^2s^2}{s} \overset{(7)}{=} r^2s$

(4),(5),(6),(7)⇒LHS

$$= \frac{4s^2r^2(3R + 2r)}{(r^2s)\{2s(s^2 + 2Rr + r^2)\}} = \frac{2(3R + 2r)}{s^2 + 2Rr + r^2} \overset{?}{\le} \frac{1}{r} - \frac{1}{R} = \frac{R - r}{Rr}$$

$$\Leftrightarrow (s^2 + 2Rr + r^2)(R - r) \overset{?}{\ge} 2Rr(3r + 2r) \quad \textbf{(8)}$$

LHS of (8) $\ge (18Rr - 4r^2)(R - r)$ ***(Gerretsen)***

$$\overset{?}{\ge} 2Rr(3R + 2r)$$

$$\Leftrightarrow (9R-2r)(R-r) \overset{?}{\geq} 3R^2+2Rr$$

$$\Leftrightarrow 6R^2-13Rr+2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(6R-r) \overset{?}{\geq} 0 \rightarrow \textbf{true} \because R \geq 2r \text{ (Euler) (Proved)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\text{1) } \sum \frac{a}{b+c} \leq 2-\frac{r}{R} \text{ (ASSURE)}$$

$$\sum \frac{a}{b+c} = \frac{2\cdot(p^2-r^2-Rr)}{p^2+r^2+2Rr} \leq 2-\frac{r}{R} \Leftrightarrow$$

$$\Leftrightarrow \frac{2\cdot(p^2-r^2-Rr)}{p^2+r^2+2Rr}+\frac{r}{R} \leq 2 \Leftrightarrow \frac{2R\cdot(p^2-r^2-Rr)+r(p^2+r^2+2Rr)}{R\cdot(p^2+r^2+2Rr)} \leq 2$$

$$\Leftrightarrow p^2 \leq 6R^2+2Rr-r^2 \overset{Gerretsen}{\Rightarrow} 4R^2+4Rr+3r^2 \leq 6R^2+2Rr-r^2$$

$$\Leftrightarrow 2R^2-2Rr-4r^2 \geq 0 \Leftrightarrow (R-2r)\cdot(R+r) \geq 0 \text{ Euler}$$

True

$$\text{2) } -\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right) \overset{Schwarz}{\leq} -\frac{(a+b+c)^2}{2\cdot(a+b+c)} = -\frac{a+b+c}{2} = -p$$

$$\text{3) } \sum \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a+b} = \sum \frac{\sqrt{\frac{p-c}{p}}\cdot\left(\sqrt{\frac{p-b}{p-a}}+\sqrt{\frac{p-a}{p-b}}\right)}{a+b} =$$

$$= \sum \frac{\sqrt{\frac{p-c}{p}}\cdot\left(\frac{p-b+p-a}{\sqrt{(p-a)\cdot(p-b)}}\right)}{a+b} = \sum \frac{\sqrt{\frac{p-c}{p}}\cdot\left(\frac{c}{\sqrt{(p-a)(p-b)}}\right)}{a+b} =$$

$$= \sum \frac{(p-c)\cdot c}{\frac{S}{a+b}} = \frac{1}{S}\cdot\sum \frac{(p-c)\cdot c}{a+b} = \frac{1}{S}\cdot\left(p\cdot\sum\frac{c}{a+b}-\frac{c^2}{a+b}\right) \overset{(1)\,(2)}{\leq}$$

$$\leq \frac{1}{S}\cdot\left(p\cdot\left(2-\frac{r}{R}\right)-p\right) = \frac{1}{r}\cdot\left(2-\frac{r}{R}-1\right) = \frac{1}{r}-\frac{1}{R}$$

JP.069. Let a, b be positive real numbers such that $a^2+ab+b^2=9$. Find the maximal value of expression:

$$(a+b)^6+(ab)^5+2(ab)^3+(ab)^2-17$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c ***números*** R^+ ***de tal manera que*** $a^2 + ab + b^2 = 9$***. Hallar el máximo valor*** $E = (a+b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$***. Como*** $a, b, c > 0$***.***

Aplicando MA ≥ MG

$$9 = a^2 + ab + b^2 \geq 3\sqrt[3]{a^3b^3} = 3ab \Leftrightarrow 3 \geq ab$$

$$(a+b)^2 = (a^2 + ab + b^2) + ab = 9 + ab \leq 12. \textit{ Por lo tanto}$$

$$E = (a+b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17 \leq 12^3 + 3^5 + 2 \cdot 3^3 + 3^2 - 17 =$$

$$= 1728 + 243 + 54 + 9 - 17 = 2017;\ E \leq 2017$$

La igualdad se alcanza cuando $a = b = \sqrt{3}$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a, b, c > 0$$

$$a^2 + ab + b^2 = 9 \Rightarrow ab \leq 3 \textit{ from (AM-GM)}$$

$$(a+b)^6 = ((a+b)^2)^3 = (a^2 + ab + ab + b^2)^3 = (9 + ab)^3 \leq 1728$$

$$(ab)^5 \leq 24^3;\ 2(ab)^3 \leq 54;\ (ab)^2 \leq 9. \textit{ Problem: maximum values:}$$

$$(a+b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17 \leq 2017$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\textit{Let } E = (a+b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$= (a^2 + b^2 + ab + ab)^3 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$= (9 + ab)^3 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$= 9^3 + 243ab + 27(ab)^2 + (ab)^3 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$E = 712 + 243ab + 28(ab)^2 + 3(ab)^3 + (ab)^5. \textit{ As } a^2 + ab + b^2 = 9$$

$$4a^2 + 4b^2 + 4ab = 36 \Rightarrow 2\{(a+b)^2 + (a-b)^2\} + (a+b)^2 - (a-b)^2 = 36$$

$$\Rightarrow \frac{(a+b)^2}{12} + \frac{(a-b)^2}{36} = 1. \textit{ Let } a + b = 2\sqrt{3}\cos\theta$$

$$a - b = 6\sin\theta \Rightarrow a = \sqrt{3}\cos\theta + 3\sin\theta \textit{ and } b = \sqrt{3}\cos\theta - 3\sin\theta$$

$$\therefore ab = 3\cos^2\theta - 9\sin^2\theta = \frac{3}{2}(1 + \cos 2\theta) - \frac{9}{2}(1 - \cos 2\theta) = 6\cos 2\theta - 3 \leq 3$$

Thus, maximum possible value of E ***is*** $712 + (243)(3) + (28)(3^2) + 3(3^3) + 3^5$

$$= 712 + 729 + 252 + 81 + 243 = 2017$$

Solution 4 by SK Rejuan-West Bengal-India

$$a, b \in \mathbb{R}^+ \text{ and } a^2 + ab + b^2 = 9 \quad \textit{(i)}$$

By AM ≥ GM we get, $a^2 + ab + b^2 \geq 3ab \Rightarrow 9 \geq 3ab$ *[from (i)]*$\Rightarrow ab \leq 3$ *(ii)*

Again, $a^2 + ab + b^2 = 9 \Rightarrow (a + b)^2 - ab = 9$

$$\Rightarrow (a + b)^2 = 9 + ab \leq 9 + 3 \quad \textit{[from (ii)]} \Rightarrow (a + b)^2 \leq 12 \quad \textit{(iii)}$$

$$\Rightarrow LHS$$

$$= (a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$\leq (12)^3 + (3)^5 + 2 \cdot (3)^3 + (3)^2 - 17 \quad \textit{[from (iii) \& (ii)]}$$

$$= 1728 + 243 + 54 + 9 - 17$$

$$= 2017$$

$$\Rightarrow (a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17 \leq 2017$$

∴ ***Maximum value is*** 2017

JP.070. Let a, b be positive real numbers with $a^2 + ab + b^2 = k^2, k > 0$.

Prove that

$$\sqrt{a + b} + \sqrt[4]{ab} \leq \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \sqrt{k}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c ***números*** R^+ ***de tal manera que*** $a^2 + ab + b^2 = k^2, k > 0$

$\sqrt{a + b} + \sqrt[4]{ab} \leq \frac{\sqrt{2}+1}{\sqrt[4]{3}} \cdot \sqrt{k}$. ***Ahora bien***

$$k^2 = a^2 + ab + b^2 \geq \frac{3}{4}(a + b)^2 + \frac{1}{4}(a - b)^2 \geq \frac{3}{4}(a + b)^2 \Leftrightarrow \sqrt{k} \geq \frac{\sqrt[4]{3} \cdot \sqrt{a + b}}{\sqrt{2}}$$

Aplicando MA ≥ MG

$$\Rightarrow \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \sqrt{k} \geq \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \frac{\sqrt[4]{3} \cdot \sqrt{a + b}}{\sqrt{2}} = \sqrt{a + b} + \sqrt{\frac{a + b}{2}} \geq \sqrt{a + b} + \sqrt[4]{ab}$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$k^2 \geq \frac{3}{4}(a + b)^2 \Rightarrow \textit{if} \sqrt{a + b} \Rightarrow a + b = x^2; \sqrt[4]{ab} \leq \sqrt{\frac{a+b}{2}}; x + \frac{x}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt[4]{3}} \cdot \frac{\sqrt[4]{3}}{\sqrt{2}} \cdot x$$

Solution 3 by Nguyen Ngoc Tu-Ha Giang-Vietnam

Fact

1) We have: $a^2+b^2\ge\frac{1}{2}(a+b)^2$

$$\Rightarrow a^2+b^2+2k^2\ge\frac{1}{2}(a+b)^2+\frac{2k^2}{3}+\frac{2k^2}{3}+\frac{2k^2}{3}\overset{AM-GM}{\ge}4\sqrt[4]{4\cdot(a+b)^2\cdot\frac{k^6}{27}}$$

$$=\frac{4\sqrt{2}}{\sqrt[4]{3}}\cdot k\cdot\sqrt{\frac{k}{3}}\cdot\sqrt{a+b}$$

Fact

$$\text{2) } ab+\frac{k^2}{3}+\frac{k^2}{3}+\frac{k^2}{3}\overset{AM-GM}{\ge}4\sqrt[4]{\frac{ab\cdot k^6}{27}}=\frac{4}{\sqrt[4]{3}}\cdot k\cdot\sqrt{\frac{k}{3}}\cdot\sqrt[4]{ab}$$

$$\Rightarrow a^2+b^2+ab+3k^2\ge\left(4\cdot k\cdot\sqrt{\frac{k}{3}}\cdot\frac{1}{\sqrt[4]{3}}\right)\left(\sqrt{2}\cdot\sqrt{a+b}+\sqrt[4]{ab}\right)$$

$$\Leftrightarrow 4k^2\ge\left(4k\cdot\sqrt{\frac{k}{3}}\cdot\frac{1}{\sqrt[4]{3}}\right)\left(\sqrt{2}\cdot\sqrt{a+b}+\sqrt[4]{ab}\right)\quad (*)$$

Fact

$$\text{3) } a+b\ge 2\sqrt{ab}\Rightarrow\sqrt{a+b}\ge\sqrt{2}\cdot\sqrt[4]{ab}$$

Fact

$$\text{4) } \sqrt{2}\cdot\sqrt{a+b}+\sqrt[4]{ab}=\left(\sqrt{2}-3+2\sqrt{2}\right)\left(\sqrt{a+b}\right)+\left(3-2\sqrt{2}\right)\sqrt{a+b}+\sqrt[4]{ab}$$

$$\overset{(3)}{\ge}3\left(\sqrt{2}-1\right)\cdot\sqrt{a+b}+\left(3-2\sqrt{2}\right)\sqrt{2}\cdot\sqrt[4]{ab}+\sqrt[4]{ab}$$

$$=3\left(\sqrt{2}-1\right)\left(\sqrt{a+b}+\sqrt[4]{ab}\right)$$

Hence, (*) becomes:

$$4k^2\ge\left(4k\cdot\sqrt{\frac{k}{3}}\cdot\frac{1}{\sqrt[4]{3}}\right)3\left(\sqrt{2}-1\right)\cdot\sqrt{a+b}+\sqrt[4]{ab}$$

$$\Rightarrow\sqrt{a+b}+\sqrt[4]{ab}\le\frac{4k^2}{4k\cdot\sqrt{\frac{k}{3}}\cdot\frac{1}{\sqrt[4]{3}}\cdot 3\left(\sqrt{2}-1\right)}=\frac{\sqrt{2}+1}{4\sqrt{3}}\sqrt{k}$$

Solution 4 by SK Rejuan-West Bengal-India

$a,b\in\mathbb{R}^+$ ***and*** $a^2+ab+b^2=k^2\quad$ ***(i)***

By AM ≥ GM we get, $a^2 + ab + b^2 \geq 3ab \Rightarrow k^2 \geq 3ab$ [as $k^2 = a^2 + ab + b^2$]

$$\Rightarrow ab \leq \frac{k^2}{3} \quad (ii)$$

$$\Rightarrow \sqrt[4]{ab} \leq \frac{\sqrt{k}}{\sqrt[4]{3}} \quad (iii)$$

Again, $a^2 + ab + b^2 = k^2 \Rightarrow (a+b)^2 - 2ab + ab = k^2$

$$\Rightarrow (a+b)^2 = k^2 + ab \leq k^2 + \frac{k^2}{3} \quad \text{[from (ii)]} \Rightarrow (a+b)^2 \leq \frac{4}{3}k^2$$

$$\Rightarrow \sqrt{a+b} \leq \frac{\sqrt{2}}{\sqrt[4]{3}}\sqrt{k} \quad (iv)$$

Adding (iv) & (iii) we get $\sqrt{a+b} + \sqrt[4]{ab} \leq \frac{\sqrt{2}}{\sqrt[4]{3}}\sqrt{k} + \frac{\sqrt{k}}{\sqrt[4]{3}} \Rightarrow \sqrt{a+b} + \sqrt[4]{ab} \leq \frac{\sqrt{2}+1}{\sqrt[4]{3}}\sqrt{k}$

JP.071. Let ABC be a triangle with circumradius R and inradius r, and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively.

Prove that

$$\left(\frac{w_a}{a}\right)^2 \cdot \tan\frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \cdot \tan\frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \cdot \tan\frac{C}{2} \leq \frac{3\sqrt{3}}{8} \cdot \frac{R}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo ABC un triángulo con circunradio R e inradio r, además w_a, w_b, w_c son los ángulos internos bisectors, opuestos a los lados a, b, c respectivamente.

Probar que $\left(\frac{w_a}{a}\right)^2 \cdot \tan\frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \cdot \tan\frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \cdot \tan\frac{C}{2} \leq \frac{3\sqrt{3}}{8} \cdot \frac{R}{r}$ (A)

Teniendo en cuento la siguientes identidades y desigualdades en un triángulo ABC

$$\tan\frac{A}{2} = \frac{(s-b)(s-c)}{S} = \frac{(a+c-b)(a+b-c)}{4S} = \frac{a^2-(b-c)^2}{4S} \leq \frac{a^2}{4S},$$

$$\tan\frac{B}{2} \leq \frac{b^2}{4S}, \tan\frac{C}{2} \leq \frac{c^2}{4S}$$

$$w_a \leq \sqrt{s(s-a)}, w_b \leq \sqrt{s(s-c)}, w_c \leq \sqrt{s(s-c)}, s \leq \frac{3\sqrt{3}R}{2}$$

Utilizando las desigualdades privias en (A)

$$\left(\frac{w_a}{a}\right)^2\tan\frac{A}{2}+\left(\frac{w_b}{b}\right)^2\tan\frac{B}{2}+\left(\frac{w_c}{c}\right)^2\tan\frac{C}{2}\le\frac{w_a^2}{4S}+\frac{w_b^2}{4S}+\frac{w_c^2}{4S}\le$$

$$\le\frac{s(s-a)+s(s-b)+s(s-c)}{4S}=\frac{s^2}{4sr}=\frac{s}{4r}\le\frac{3\sqrt{3}R}{8r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS\le\sum\left[\frac{s(s-a)}{a^2}\tan\frac{A}{2}\right]\quad\left(\because w_a\le\sqrt{s(s-a)}\right)=\sum\left(s\tan\frac{A}{2}\cdot\frac{s-a}{a^2}\right)$$

$$=\sum\left(r_a\cdot\frac{s-a}{a^2}\right)=\sum\left(\frac{\Delta}{s-a}\cdot\frac{s-a}{a^2}\right)$$

$$=\Delta\left(\sum\frac{1}{a^2}\right)=\Delta\cdot\frac{\sum a^2b^2}{16R^2r^2s^2}\overset{Goldstone}{\le}\frac{\Delta\cdot 4R^2S^2}{16R^2r^2s^2}$$

$$=\frac{rs\cdot 4R^2s^2}{16R^2r^2s^2}=\frac{s}{4r}\overset{Mitrinovic}{\le}\frac{3\sqrt{3}R}{2\cdot 4r}=\frac{3\sqrt{3}}{8}\cdot\frac{R}{r}$$

Solution 3 by Marin Chirciu – Romania

We prove the following helpful result: Lemma: In $\Delta\ ABC$

$$\left(\frac{w_a}{a}\right)^2\tan\frac{A}{2}+\left(\frac{w_b}{b}\right)^2\tan\frac{A}{2}+\left(\frac{w_c}{c}\right)^2\tan\frac{A}{2}\le\frac{p}{4r}.$$

Proof

We use $w_a^2\le p(p-a)$ and $\tan\frac{A}{2}=\frac{(p-b)(p-c)}{S}$

It follows $\left(\frac{w_a}{a}\right)^2\tan\frac{A}{2}\le\frac{p(p-a)}{a^2}\cdot\frac{(p-b)(p-c)}{S}=\frac{S}{a^2}$

We obtain $\sum\left(\frac{w_a}{a}\right)^2\tan\frac{A}{2}\le S\sum\frac{1}{a^2}\le rp\cdot\frac{1}{4r^2}=\frac{p}{4r}$.

The equality holds if and only if the triangle is equilateral.

Let's pass to solving the problem from the enuntiation.

Using the Lemma and Mitrinovic's inequality $p\le\frac{3R\sqrt{3}}{2}$ we obtain the conclusion.

The equality holds if and only if the triangle is equilateral.

JP.072. Let a, b and c denote, as usual, the lengths of the sides BC, CA, and AB, respectively, in $\Delta\ ABC$. Let R be the circumradius, r the inradius of $\Delta\ ABC$, and r_a, r_b and r_c the exradii to A, B and C, respectively.

Prove that

(a) $\frac{r_a}{a^3}+\frac{r_b}{b^3}+\frac{r_c}{c^3}\le\frac{\sqrt{3}}{8r^2}$, **(b)** $\frac{3R}{2r}\ge\sqrt{\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6}$

Proposed by George Apostolopoulos – Messolonghi - Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo** ABC: $\frac{r_a}{a^3}+\frac{r_b}{b^3}+\frac{r_c}{c^3}\le\frac{\sqrt{3}}{8r^2}$ **(A)

***Teniendo en cuenta las siguientes identidades y desigualdades en triángulo** ABC*

$$r_a=p\tan\frac{A}{2},r_b=p\tan\frac{B}{2},r_c=p\tan\frac{C}{2},S=pr,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{\sqrt{3}}{2r}$$

$$\tan\frac{A}{2}=\frac{(s-b)(s-c)}{S}=\frac{(a+c-b)(a+b-c)}{4S}=\frac{a^2-(b-c)^2}{4S}\le\frac{a^2}{4S},$$

$\tan\frac{B}{2}\le\frac{b^2}{4S},\tan\frac{C}{2}\le\frac{c^2}{4S}$. ***Utilizando las desigualdades previas en (A)***

$$\frac{r_a}{a^3}+\frac{r_b}{b^3}+\frac{r_c}{c^3}=\frac{p\tan\frac{A}{2}}{a^3}+\frac{p\tan\frac{B}{2}}{b^3}+\frac{p\tan\frac{C}{2}}{c^3}\le\frac{p}{4Sa}+\frac{p}{4Sb}+\frac{p}{4Sc}=$$

$$=\frac{1}{4r}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le\frac{1}{4r}\cdot\frac{\sqrt{3}}{2r}=\frac{\sqrt{3}}{8r^2}\text{ (LQQD)}$$

***Probar en un triángulo** ABC: $\left(\frac{3R}{2r}\right)^2\ge\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6$*

***1) Teniendo en cuenta las siguientes identidades en un triángulo** ABC*

$$r_a=\frac{S}{s-a},r_b=\frac{S}{s-b},r_c=\frac{S}{s-c},\frac{R}{2r}=\frac{abc}{8(s-a)(s-b)(s-c)}$$

Realizando los siguientes cambios de variables

$$x=s-a,y=s-b,z=s-c,x+y=b,y+z=a,z+x=c$$

2) Siendo** $x,y,z>0$**, se cumple la siguiente desigualdad

$$9(x+y)(y+z)(z+x)\ge 8(x+y+z)(xy+yzx+zx)$$

Solo es necesario demostrar lo siguiente

$$\frac{9R}{2r}\ge\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6\Leftrightarrow\frac{9(x+y)(y+z)(z+x)}{8xyz}\ge\frac{y}{x}+\frac{z}{y}+\frac{x}{z}+6$$

Es suficiente probar

$$\frac{(x+y+z)(xy+yz+zx)}{xyz}=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq\frac{y}{x}+\frac{z}{y}+\frac{x}{z}+6\Leftrightarrow$$

$$\Leftrightarrow 3+\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\geq\frac{y}{x}+\frac{z}{y}+\frac{x}{z}+6\ (MA\geq MG)$$

Se concluye que $\rightarrow\left(\frac{3R}{2r}\right)^2\geq\frac{9R}{2r}\geq\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6.$ ***(LQQD)***

JP.073. If $a,b,c>0;n\geq 1$ then:

$$\frac{3n(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq n+1$$

Proposed by Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c, de tal manera que $n\geq 1$. Probar que

$$\frac{3n(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq n+1$$

$$n\left(\frac{3(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}-1\right)+\frac{ab+bc+ca}{a^2+b^2+c^2}-1\geq 0$$

$$\Rightarrow n\left(\frac{2(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{(a^2+b^2+c^2)^2}\right)+\frac{(ab+bc+ca-a^2-b^2-c^2)(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}\geq 0$$

Como $n\geq 1$

Es suficiente demostrar

$$\frac{2(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{(a^2+b^2+c^2)^2}+\frac{(ab+bc+ca-a^2-b^2-c^2)(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}\geq 0$$

$$\Leftrightarrow 2(a^4+b^4+c^4)-2(a^2b^2+b^2c^2+c^2a^2)+(ab+bc+ca)(a^2+b^2+c^2)-(a^2+b^2+c^2)^2\geq 0$$

$$\Leftrightarrow a^4+b^4+c^4-4(a^2b^2+b^2c^2+c^2a^2)+ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\geq 0$$

$$\Leftrightarrow abc(a+b+c)+ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\geq$$

$$\geq 2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4$$

$$\Leftrightarrow abc(a+b+c)+ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\geq$$

$$\geq(a+b+c)(a+b-c)(b+c-a)(a+c-b)$$

$$\Leftrightarrow(a+b+c)\big(abc-(b+c-a)(a+c-b)(b+a-c)\big)+ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\geq 0$$

Es suficiente demostrar: $abc\geq(a+b-c)(b+c-a)(c+a-b)$

$$abc \geq (b^2 - (c-a)^2)(c+a-b)$$

$$abc \geq (b^2 - c^2 - a^2 + 2ac)(c+a-b)$$

$$abc \geq b^2c + b^2a - b^3 - c^3 - c^2a + c^2b - a^2c - a^3 + a^2b + 2c^2a + 2a^2c - 2abc$$

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + ca(c+a) + bc(b+c)$$

Válido por (Shur Inequality)

Solution 2 by Nguyen Minh Tri-Vietnam

$$\frac{3n(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq n+1$$

$$\Leftrightarrow 3n(a^4+b^4+c^4) + (a^2+b^2+c^2)(ab+bc+ac) \geq (n+1)(a^2+b^2+c^2)^2$$

$$\Leftrightarrow n[3(a^4+b^4+c^4) - (a^2+b^2+c^2)^2] + \sum ab(a^2+b^2) +$$

$$+abc(a+b+c) \geq (a^2+b^2+c^2)^2 \quad (**)$$

We have $n \geq 1; 3(a^4+b^4+c^4) \geq (a^2+b^2+c^2)^2$

$$\Rightarrow (n-1)[3(a^4+b^4+c^4) - (a^2+b^2+c^2)] \geq 0$$

$$\Leftrightarrow n[3(a^4+b^4+c^4) - (a^2+b^2+c^2)^2] \geq 3(a^4+b^4+c^4) - (a^2+b^2+c^2)^2$$

So we have to prove that: $3(a^4+b^4+c^4) - (a^2+b^2+c^2)^2 + \sum ab(a^2+b^2) +$

$$+abc(a+b+c) \geq (a^2+b^2+c^2) \quad (*)$$

$$\Leftrightarrow 3(a^4+b^4+c^4) + \sum ab(a^2+b^2) + abc(a+b+c) \geq 2(a^2+b^2+c^2)^2$$

$$\Leftrightarrow a^4+b^4+c^4 + abc(a+b+c) + \sum ab(a^2+b^2) \geq 4(a^2b^2+b^2c^2+a^2c^2)$$

Use Schur inequality exponent two and we have:

$a^4+b^4+c^4+abc(a+b+c) \geq \sum ab(a^2+b^2)$. ***So we need to prove:***

$$\sum ab(a^2+b^2) \geq 2\left(\sum a^2b^2\right) \quad (1)$$

Have: $a^2+b^2 \geq 2ab \Leftrightarrow ab(a^2+b^2) \geq 2a^2b^2$. ***Similarly:*** $bc(b^2+c^2) \geq 2b^2c^2$

$$ac(a^2+c^2) \geq 2a^2c^2 \Rightarrow \sum ab(a^2+b^2) \geq 2\left(\sum a^2b^2\right)$$

$\Rightarrow$ ***(1) true*** $\Rightarrow$ ***(*) true*** $\Rightarrow$ ***(**) true*** $\Rightarrow$ ***Q.E.D.***

JP.074. If $a,b,c,n > 0; n(ab+bc+ca) + 2abc = n^3$ **then:**

$$\frac{1}{a+b+2n}+\frac{1}{b+c+2n}+\frac{1}{c+a+2n}\le\frac{1}{n}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a,b,c>0$**, de tal manera que** $n(ab+bc+ca)+2abc=n^3$**. Probar que**

$$\frac{1}{a+b+2n}+\frac{1}{b+c+2n}+\frac{1}{c+a+2n}\le\frac{1}{n}$$

La desigualdad propuesta es equivalente

$$n\big((b+c+2n)(c+a+2n)+(c+a+2n)(a+b+2n)+(a+b+2n)(b+c+2n)\big)\le$$

$$\le(a+b+2n)(b+c+2n)(c+a+2n)$$

Ahora bien:

$$E=(b+c+2n)(c+a+2n)+(c+a+2n)(a+b+2n)+(a+b+2n)(b+c+2n)$$

$$E=12n^2+2n\big(2(b+c)+2(c+a)+2(a+b)\big)+(b+c)(c+a)+(c+a)(a+b)+$$

$$+(a+b)(b+c)$$

$$E=12n^2+8n(a+b+c)+a^2+b^2+c^2+3(ab+bc+ca)$$

$$F=(a+b+2n)(b+c+2n)(c+a+2n)$$

$$F=8n^3+4n^2\cdot 2(a+b+c)+2n\big((a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)\big)+$$

$$+(a+b)(b+c)(c+a)$$

$$F=8n^3+8n^2(a+b+c)+2n(a^2+b^2+c^2)+6n(ab+bc+ca)+$$

$$+(a+b)(b+c)(c+a)$$

$$\Leftrightarrow n\cdot E\le F\Leftrightarrow n\cdot E-F=4n^3-n(a^2+b^2+c^2)-3n(ab+bc+ca)-$$

$$-(a+b)(b+c)(c+a)\le 0\Leftrightarrow 4n(ab+bc+ca)+8abc\le$$

$$\le n(a^2+b^2+c^2+3ab+3bc+3ca)+(a+b)(b+c)(c+a)$$

$\Leftrightarrow$ **Lo cual es cierto ya que** $a^2+b^2+c^2\ge ab+bc+ca\wedge(a+b)(b+c)(c+a)\ge 8abc$

(Válido por MA $\ge$ MG)

Solution 2 by Ravi Prakash-New Delhi-India

$$n(ab+bc+ca)+2abc=n^3$$

$$\Leftrightarrow\left(\frac{a}{n}\right)\left(\frac{b}{n}\right)+\left(\frac{b}{n}\right)\left(\frac{c}{n}\right)+\left(\frac{c}{n}\right)\left(\frac{a}{n}\right)+2\left(\frac{a}{n}\right)\left(\frac{b}{n}\right)\left(\frac{c}{n}\right)=1\quad (1)$$

Let $\frac{a}{n}+1=x,\frac{b}{n}+1=y,\frac{c}{n}+1=z$

(1) now becomes

$$(x-1)(y-1)+(y-1)(z-1)+(z-1)(x-1)+$$
$$+2(x-1)(y-1)(z-1)=1$$
$$\Leftrightarrow xy-x-y+1+yz-y-z+1+zx-x-z+1+$$
$$+2(xyz-xy-yz-zx+x+y+z-1)=1$$

$\Leftrightarrow 2xyz=xy+yz+zx \Leftrightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$. *Let* $\frac{1}{x}=\alpha, \frac{1}{y}=\beta, \frac{1}{z}=\gamma$

So that $\alpha+\beta+\gamma=2$. *Now, to prove* $\frac{1}{a+b+2n}+\frac{1}{b+c+2n}+\frac{1}{c+a+2n}\le\frac{1}{n}$

$$\Leftrightarrow \frac{1}{\left(\frac{a}{n}+1\right)+\left(\frac{b}{n}+1\right)}+\frac{1}{\left(\frac{b}{n}+1\right)+\left(\frac{c}{n}+1\right)}+\frac{1}{\left(\frac{c}{n}+1\right)+\left(\frac{a}{n}+1\right)}\le 1$$

$$\Leftrightarrow \frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\le 1$$

$$\Leftrightarrow \frac{\alpha\beta}{\alpha+\beta}+\frac{\beta\gamma}{\beta+\gamma}+\frac{\gamma\alpha}{\alpha+\gamma}\le 1 \quad (2)$$

LHS of (2)

$$\frac{1}{2}\left\{\frac{2\alpha\beta}{\alpha+\beta}+\frac{2\beta\gamma}{\beta+\gamma}+\frac{2\gamma\alpha}{\gamma+\alpha}\right\}\le\frac{1}{2}\left\{\frac{\alpha+\beta}{2}+\frac{\beta+\gamma}{2}+\frac{\gamma+\alpha}{2}\right\}=\frac{\alpha+\beta+\gamma}{2}=\frac{2}{2}=1$$

∴ *(2) is true*

JP.075. Let R and r be the circumradius and the inradius of a triangle ABC respectively. Prove that

$$\csc A+\csc B+\csc C\ge 3\sqrt{3}\frac{R}{R+r}$$

Proposed by Martin Lukarevski – Stip – Macedonia

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo R y r circunradio e inradio de un triángulo ABC. Probar que

$$\csc A+\csc B+\csc C\ge\frac{3\sqrt{3}R}{R+r}$$

Tener en cuenta las siguientes identidades y desigualdades en un $\Delta\ ABC$

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}, R\ge 2r \quad \text{(Euler)}$$

La desigualdad propuesta es equivalente $2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq\frac{3\sqrt{3}R}{R+r}\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{3\sqrt{3}}{2(R+r)}$

Luego $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)}=\sqrt{\frac{3}{2Rr}}\geq\frac{3\sqrt{3}}{2(R+r)}$

Es necesario demostrar lo siguiente

$$\sqrt{\frac{3}{2Rr}}\geq\frac{3\sqrt{3}}{2(R+r)}\Leftrightarrow\sqrt{2}(R+r)\geq3\sqrt{Rr}\Leftrightarrow2(R+r)^2\geq9Rr\Leftrightarrow$$

$$\Leftrightarrow2(R+r)^2-9Rr=(R-2r)(2R-r)\geq0\quad(R\geq2r)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\csc A=2R\left(\sum\frac{1}{a}\right)=\frac{2R(\sum ab)}{4Rrs}=\frac{\sum ab}{2rs}$$

$\therefore$ ***it suffices to prove:*** $\frac{(\sum ab)^2}{4r^2s^2}\geq\frac{27R^2}{(R+r)^2}\Leftrightarrow(s^2+4Rr+r^2)^2(R+r)^2\geq108R^2r^2s^2$

$$\Leftrightarrow\left(s^4+(4Rr+r^2)^2+2s^2(4Rr+r^2)\right)(R+r)^2\geq108R^2r^2s^2\quad\textbf{(1)}$$

Now, Trucht $\Rightarrow4R+r\geq s\sqrt{3}\Rightarrow(4R+r)^2\geq3s^2$

$$\Leftrightarrow r^2(4R+r)^2\geq3s^2r^2\Leftrightarrow(4Rr+r^2)^2\geq3s^2r^2\quad\textbf{(2)}$$

(2) $\Rightarrow$ ***LHS of (1)*** $\geq\{s^4+3s^2r^2+2s^2(4Rr+r^2)\}(R+r)^2\overset{?}{\geq}108R^2r^2$ ***(3)***

LHS of (3) $\overset{Gerretsen}{\geq}(16Rr-5r^2+3r^2+8Rr+2r^2)(R+r)^2$

$$=24Rr(R+r)^2\overset{?}{\geq}108R^2r^2\Leftrightarrow2(R+r)^2\overset{?}{\geq}9Rr\Leftrightarrow2R^2-5Rr+2r^2\overset{?}{\geq}0$$

$\Leftrightarrow(R-2r)(2R-r)\overset{?}{\geq}0\rightarrow$ ***true,*** $\because R\geq2r$ ***(Euler) (Proved)***

Solution 3 by Marin Chirciu – Romania

We have: $\sum\csc A=\sum\frac{1}{\sin A}=\sum\frac{2R}{a}=2R\sum\frac{1}{a}=2R\cdot\frac{\sum bc}{abc}=2R\cdot\frac{p^2+r^2+4Rr}{4pRr}=\frac{p^2+r^2+4Rr}{2rp}$

The inequality can be written:

$$\frac{p^2+r^2+4Rr}{2rp}\geq3\sqrt{3}\cdot\frac{R}{R+r}\Leftrightarrow\left(\frac{p^2+r^2+4Rr}{2rp}\right)^2\geq\frac{27R^2}{(R+r)^2}\Leftrightarrow$$

$$\Leftrightarrow(R+r)^2(p^2+r^2+4Rr)^2\geq108R^2r^2p^2\Leftrightarrow$$

$$\Leftrightarrow p^4(R+r)^2+2p^2[(R+r)^2(r^2+4Rr)-54R^2r^2]+r^2(4R+r)^2(R+r)^2\geq0\Leftrightarrow$$

$$\Leftrightarrow p^2[p^2(R+r)^2+2(R+r)^2(r^2+4Rr)-54R^2r^2]+r^2(4R+r)^2(R+r^2)\geq0$$

We distinguish the following cases:

1) ***If*** $p^2(R+r)^2+2(R+r)^2(r^2+4Rr)-54R^2r^2\geq 0$, ***the inequality is obvious.***
2) ***If*** $p^2(R+r)^2+2(R+r)^2(r^2+4Rr)-54R^2r^2<0$, ***we rewrite the inequality:***

$$p^2[54R^2r^2-2(R+r)^2(r^2+4Rr)-p^2(R+r)^2]\leq r^2(4Rr+r)^2(R+r)^2$$

which follows from Gerretsen's inequality: $16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$

It remains to prove that:

$$(4R^2+4Rr+3r^2)[54R^2r^2-2(R+r)^2(r^2+4Rr)-(16Rr-5r^2)(R+r)^2]\leq$$

$$\leq r^2(4R+r)^2(R+r)^2$$

$$\Leftrightarrow 24R^5-35R^4r-17R^3r^2-24R^2r^3+13Rr^3-2r^5\leq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(24R^4+13R^3r+9R^2r^2-6Rr^3+r^4)\geq 0$, ***obviously from Euler's inequality*** $R\geq 2r$. ***The equality holds if and only if the triangle is equilateral.***

SP.061. Let $x_1,x_2,\dots,x_n$ be non-negative real numbers satisfying

$$\frac{x_1}{1+x_1}+\frac{2x_2}{1+x_2}+\cdots+\frac{nx_n}{1+x_n}=1$$

Find the maximum possible value of

$P=x_1x_1^2\dots x_n^n$.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

From the given condition and by the AM-GM inequality, we obtain

$$\frac{1}{1+x_1}=\frac{2x_1}{1+x_2}+\frac{3x_3}{1+x_3}+\cdots+\frac{nx_n}{1+x_n}$$

$$\geq(2+3+\cdots+n)\sqrt[2+3+\cdots+n]{\left(\frac{x_2}{1+x_2}\right)^2\left(\frac{x_3}{1+x_3}\right)^3\dots\left(\frac{x_n}{1+x_n}\right)^n}$$

$$\frac{1}{1+x_2}=\frac{x_1}{1+x_1}+\frac{x_2}{1+x_2}+\frac{3x_3}{1+x_3}+\cdots+\frac{nx_n}{1+x_n}$$

$$\geq(2+3+\cdots+n)\sqrt[2+3+\cdots+n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)\left(\frac{x_3}{1+x_3}\right)^3\dots\left(\frac{x_n}{1+x_n}\right)^n}$$

$$\frac{1}{1+x_3}=\frac{x_1}{1+x_1}+\frac{2x_2}{1+x_2}+\frac{2x_3}{1+x_3}+\cdots+\frac{nx_n}{1+x_n}$$

$$\geq (2+3+\cdots+n)\sqrt[2+3+\cdots n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2\left(\frac{x_3}{1+x_3}\right)^2\cdots\left(\frac{x_n}{1+x_n}\right)^n}$$

… … …

$$\frac{1}{1+x_n}=\frac{x_1}{1+x_1}+\frac{2x_2}{1+x_2}+\cdots+\frac{(n-1)x_{n-1}}{1+x_{n-1}}+\frac{(n-1)x_n}{1+x_n}$$

$$\geq (2+3+\cdots+n)\sqrt[2+3+\cdots n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2\cdots\left(\frac{x_{n-1}}{1+x_{n-1}}\right)^{n-1}\left(\frac{x_n}{1+x_n}\right)^{n-1}}$$

From these relations above, we infer that

$$\frac{1}{1+x_1}\cdot\frac{1}{(1+x_2)^2}\cdot\frac{1}{(1+x_3)^3}\cdots\frac{1}{(1+x_n)^n}\geq$$

$$(2+3+\cdots+n)^{1+2+\cdots+n}\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2\left(\frac{x_3}{1+x_3}\right)^3\cdots\left(\frac{x_n}{1+x_n}\right)^n$$

Which implies that $x_1x_2^2\dots x_n^n\leq\frac{1}{(2+3+\cdots+n)^{1+2+\cdots+n}}$

The equality holds if and only if: $x_1=x_2=\cdots=x_n=\frac{1}{2+3+\cdots+n}$

Thus $\max P=\frac{1}{(2+3+\cdots+n)^{1+2+\cdots+n}}$

SP.062. If $a,b,c\in\mathbb{C}$ then:

$$\left|a^3+b^3+c^3-3abc\right|\leq|a+b+c|(|a|+|b|+|c|)^2$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\left|a^3+b^3+c^3-3abc\right|=\left|(a+b+c)(a+b\varepsilon+c\varepsilon^2)(a+b\varepsilon+c\varepsilon)\right|\leq$$

$$\leq|a+b+c|\left(|a|+|b\varepsilon|+\left|c\varepsilon^2\right|\right)(|a|+|b\varepsilon|+|c\varepsilon|)=$$

$$=|a+b+c|(|a|+|b|+|c|)(|a|+|b|+|c|)=|a+b+c|(|a|+|b|+|c|)^2$$ ***when:***

$$\varepsilon=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

SP.063. Prove that for any triangle ABC,

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$$\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\geq$$
$$\geq\sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\frac{1}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right).$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

$$\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\geq$$
$$\geq\sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\frac{1}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)$$

En un $\Delta\ ABC\rightarrow\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}>0$

Dividiendo $(\div)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ ***a la desigualdad propuesta***

$$\Leftrightarrow\sec\frac{C}{2}+\sec\frac{A}{2}+\sec\frac{B}{2}\geq\sqrt{3}+\frac{1}{2}\left(\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}\right)$$

$$\Leftrightarrow\sec\frac{C}{2}+\sec\frac{A}{2}+\sec\frac{B}{2}\geq\sqrt{3}+\frac{1}{2}\left(\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\left(\frac{C+A}{2}\right)}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\left(\frac{A+B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}\right)$$

$$\Leftrightarrow\sec\frac{C}{2}+\sec\frac{A}{2}+\sec\frac{B}{2}\geq\sqrt{3}+\frac{1}{2}\left(\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\right)$$

$$\Leftrightarrow\sec\frac{A}{2}-\tan\frac{A}{2}+\sec\frac{B}{2}-\tan\frac{B}{2}+\sec\frac{C}{2}-\tan\frac{C}{2}\geq\sqrt{3}$$

Calculamos la primera y segunda derivada

Sea $f(x)=\sec x-\tan x\,,x\in<0,\frac{\pi}{2}>,f'(x)=\sec x\tan x-\sec^2 x,$

$$f''(x)=\sec x\,(\sec x-\tan x)^2>0$$

Como $f''(x)>0$***, entonces*** $f(x)$ ***es estrictamente convexo en*** $<0,\frac{\pi}{2}>$

Dado que $\frac{A}{2},\frac{B}{2},\frac{C}{2}\in<0,\frac{\pi}{2}>$ ***de tal manera que*** $A+B+C=\pi$.

Aplicamos la desigualdad de Jensen

$$\sec\frac{A}{2}-\tan\frac{A}{2}+\sec\frac{B}{2}-\tan\frac{B}{2}+\sec\frac{C}{2}-\tan\frac{C}{2}=f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)\geq$$

$$\geq 3f\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right) = 3f\left(\frac{\pi}{6}\right)$$

$$\Leftrightarrow f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right) \geq 33f\left(\frac{\pi}{6}\right) = 3\left(\sec\frac{\pi}{6}-\tan\frac{\pi}{6}\right) = 3\left(\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right) = \sqrt{3}$$

$$\Rightarrow \sec\frac{A}{2}-\tan\frac{A}{2}+\sec\frac{B}{2}-\tan\frac{B}{2}+\sec\frac{C}{2}-\tan\frac{C}{2} \geq \sqrt{3}$$

(LQQD)

SP.064. If $x, y, z \in (0,1)$ then:

$$\left(\frac{yz}{1-x^2}\right)^{2n}+\left(\frac{zx}{1-y^2}\right)^{2n}+\left(\frac{xy}{1-z^2}\right)^{2n} \geq \frac{3^{3n+1}}{4^n}(xyz)^{2n}$$

for all $n \in \mathbb{N}$.

Proposed by Mihály Bencze – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = x - x^3$ *for all* $x \in (0,1), f'(x) = 1-3x^2, f''(x) = -6x$

$\because f'(x_0) = 0$ *where* $x_0 \in (0,1) \Rightarrow x_0 = \pm\frac{1}{\sqrt{3}}$ *choosing* $x_0 = \frac{1}{\sqrt{3}}$

$\therefore f''\left(\frac{1}{\sqrt{3}}\right) < 0$ *hence* f *attains maximum at* $x = \frac{1}{\sqrt{3}} \Rightarrow f(x) \leq f\left(\frac{1}{\sqrt{3}}\right)$

$$\therefore \sum_{cyc}\left(\frac{yz}{1-x^2}\right)^{2n} = (xyz)^{2n}\sum_{cyc}\frac{1}{(x-x^3)^{2n}} \geq (xyz)^{2n}\frac{3}{\left(\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}\right)^{2n}}$$

$$= \frac{3^{3n+1}}{4^n}(xyz)^{2n}$$ ***(proved)***

SP.065. If $a, b, c > 0$ and $n \in \mathbb{N}^*$ then:

$$2^n(a^n+b^n+c^n) \geq (a+b)^{n-1}(a+c)+(b+c)^{n-1}(b+a)+(c+a)^{n-1}(c+b)$$

Proposed by Mihály Bencze – Romania

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

Prove that $x^n+y^n+z^n \geq x^{n-1}z+z^{n-1}y+y^{n-1}x; x, y, z > 0, n \in \mathbb{N}$

Since $x^n+x^n+x^n+\cdots+x^n(n-1)term+2^n \geq nx^{n-1}z$

$$z^n+z^n+z^n+\cdots+z^n(n-1)term+y^n \geq nz^{n-1}y$$

$$y^n + y^n + y^n + \cdots + y^n (n-1) term + x^n \geq n y^{n-1} x$$

Hence $n(x^n + y^n + z^n) \geq n(x^{n-1}z + z^{n-1}y + y^{n-1}x)$

$$x^n + y^n + z^n \geq x^{n-1} + z^{n-1}y + y^{n-1}x$$

Solution

For $a, b, c > 0$ *and* $n \in \mathbb{N}$, *we have*

$$\frac{a^n + b^n}{2} \geq \left(\frac{a+b}{2}\right)^n, \frac{b^n + c^n}{2} \geq \left(\frac{b+c}{2}\right)^n, \frac{c^n + a^n}{2} \geq \left(\frac{c+a}{2}\right)^n$$

Hence $a^n + b^n + c^n \geq \left(\frac{a+b}{2}\right)^n + \left(\frac{b+c}{2}\right)^n + \left(\frac{c+a}{2}\right)^n \Rightarrow$

$$\Rightarrow 2^n(a^n + b^n + c^n) \geq (a+b)^n + (b+c)^n + (c+a)^n \geq$$

$$\geq (a+b)^{n-1}(c+a) + (b+c)^{n-1}(b+a) + (c+a)^{n-1}(c+b)$$

Therefore it is to be true.

SP.066. Let $t \in \mathbb{R}^+$ and $f: \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$\lim_{x\to\infty} \frac{f(x+1)}{x^t f(x)} \in \mathbb{R}_+$ and $\lim_{x\to\infty} (f(x))^{\frac{1}{x}} \cdot \frac{1}{x^t} \in \mathbb{R}_+^*$.

Prove that

$$\lim_{x\to\infty} (f(x))^{\frac{1}{x}} \cdot \frac{1}{x^t} = \frac{1}{e^t} \cdot \lim_{x\to\infty} \frac{f(x+1)}{x^t f(x)}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\lim_{x\to\infty} \frac{(f(x))^{\frac{1}{x}}}{x^t} = \lim_{\substack{x\to\infty \\ n\in\mathbb{N}}} \frac{\sqrt[n]{f(n)}}{n^t} = \lim_{x\to\infty} \sqrt[n]{\frac{f(n)}{n^{nt}}} \overset{CESARO-STOLZ}{=} \lim_{x\to\infty} \frac{\frac{f(n+1)}{(n+1)^{(n+1)t}}}{\frac{f(n)}{n^{nt}}}$$

$$= \lim_{x\to\infty} \left(\frac{f(n+1)}{n^t f(n)} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^{nt}} \cdot \left(\frac{n}{n+1}\right)^t \right) = \frac{1}{e^t} \cdot \lim_{x\to\infty} \frac{f(n+1)}{n^t f(n)} = \frac{1}{e^t} \cdot \lim_{x\to\infty} \frac{f(x+1)}{x^t f(x)}$$

SP.067. If $x, y, z > 0$ then:

$$\frac{1}{(x^2+yz)(3x^2+2y^2+z^2)}+\frac{1}{(y^2+zx)(3y^2+2z^2+x^2)}+$$
$$+\frac{1}{(z^2+xy)(3z^2+2x^2+y^2)}\le\frac{x^2+y^2+z^2+xy+yz+zx}{24x^2y^2z^2}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

For all $x,y,z>0$ **we have:** $\frac{x}{3x^2+2y^2+z^2}\ge\frac{1}{18}\left(\frac{2}{y}+\frac{1}{z}\right)\Leftrightarrow$

$$\Leftrightarrow\frac{3x^2y+6x^2z+2y^3+2z^3+4y^2z+yz^2}{18}\ge\sqrt[18]{(x^2y)^3(x^2z)(y^3)^2(z^3)^2(y^2z)^4yz^2}=$$

$$=xyz\text{ and }\frac{x}{x^2+yz}\le\frac{1}{4}\left(\frac{1}{y}+\frac{1}{z}\right)\Leftrightarrow y(x-z)^2+z(x-y)^2\ge 0\Rightarrow$$

$$\frac{x^2}{(x^2+yz)(3x^2+2y^2+z^2)}\le\frac{1}{72}\left(\frac{2}{y}+\frac{1}{z}\right)\left(\frac{1}{y}+\frac{1}{z}\right)=\frac{(y+z)(y+2z)}{72y^2z^2}\Rightarrow$$

$$\sum_{cyclic}\frac{1}{(x^2+yz)(3x^2+2y^2+z^2)}\le\sum_{cyclic}\frac{(y+z)(y+2z)}{72x^2y^2z^2}=$$

$$=\frac{3\sum x^2+3\sum xy}{72x^2y^2z^2}=\frac{\sum x^2+\sum xy}{24x^2y^2z^2}$$

SP.068. Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[mn+m]{(2n+1)!!}-\sqrt[mn]{(2n-1)!!}\right)\cdot n^{\frac{m-1}{m}}$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\Omega=\lim_{n\to\infty}\left(\sqrt[m(n+1)]{(2n+1)!!}-\sqrt[mn]{(2n-1)!!}\right)n^{\frac{m-1}{m}}$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[mn]{(2n-1)!!}}{\sqrt[m]{(n-1)}}\cdot\sqrt[m]{\left(1-\frac{1}{n}\right)}\frac{u_m-1}{\ln u_n}\cdot\ln u_n^n\right)\text{ where }u_n=\frac{\sqrt[m(n+1)]{(2n+1)!!}}{\sqrt[mn]{(2n-1)!!}}\text{ for all }n\in\mathbb{N}$$

Now, $\lim_{n\to\infty}\sqrt[m]{\sqrt[n]{\frac{(2n-1)!!}{(n-1)^m}}}\overset{D'ALEMBERT}{\cong}\lim_{n\to\infty}\sqrt[m]{\frac{(2n+1)!!}{n^{n+1}}\cdot\frac{(n-1)^n}{(2n-1)!!}}$

$$= \lim_{n\to\infty} \sqrt[m]{\frac{\frac{(2n+1)!}{2^m n!}}{\frac{(2n-1)!}{2^{m-1}(n-1)!}} \cdot \left(1-\frac{1}{n}\right)^n \cdot \frac{1}{n}} = \lim_{n\to\infty} \sqrt[m]{\frac{2n(2n-1)}{2n(n-1)} \cdot \left(1-\frac{1}{n}\right)^n} = \sqrt[m]{\frac{2}{e}}$$

$$\lim_{n\to\infty} u_m = \lim_{n\to\infty} \frac{\frac{\sqrt[m(n+1)]{(2n+1)!!}}{\sqrt[m]{n}}}{\frac{\sqrt[mn]{(2n-1)!!}}{\sqrt[m]{n-1}}} \cdot \sqrt[m]{\frac{n}{n-1}} = 1\text{ . Hence } \lim_{n\to\infty} \frac{u_m - 1}{\ln u_n} = 1$$

Now, $\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \sqrt[m]{\frac{(2n+1)!!}{(2n-1)!!}} \cdot \frac{1}{\sqrt[m(n+1)]{(2n+1)!!}}$

$$= \lim_{n\to\infty} \sqrt[m]{\frac{\frac{(2n+1)!!}{n^n}}{\frac{(2n-1)!!}{(n-1)^{n-1}}}} \cdot \frac{1}{\left(1-\frac{1}{n}\right)^m} \cdot \sqrt[m]{1-\frac{1}{n}} \cdot \frac{1}{\frac{\sqrt[m(n+1)]{(2n+1)!!}}{\sqrt[m]{n}}} = \sqrt[m]{e}$$

$$\therefore \Omega = \sqrt[m]{\frac{2}{e}} \cdot 1 \cdot \ln \sqrt[m]{e} = \frac{1}{m} \sqrt[m]{\frac{2}{e}} \text{ (proved)}$$

Solution 2 by Shivam Sharma-New Delhi-India

As we know, $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$ & $(2n-1)!! = \frac{(2n)!}{2^n n!}$. *Using this, we get,*

$$\Rightarrow \lim_{n\to\infty} \left[\left(\frac{(2n+1)!}{2^n n!}\right)^{\frac{1}{mn+m}} - \left(\frac{(2n)!}{2^n n!}\right)^{\frac{1}{mn}}\right] n^{\frac{m-1}{m}}$$

. As we know, the Stirling's formula,

$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. *Using this, we get,*

$$\Rightarrow \lim_{n\to\infty} \left[\frac{\left(\left(\frac{2n+1}{e}\right)^{2n+1} \sqrt{2\pi(2n+1)}\right)^{\frac{1}{mn+m}}}{\left(2^n \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{1}{mn+m}}} - \frac{\left(2\left(\frac{2n}{e}\right)^{2n} \sqrt{\pi n}\right)^{\frac{1}{mn}}}{\left(2^n \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{1}{mn}}}\right] \cdot n^{\frac{m-1}{m}}$$

As we can see many terms are cancelling now, applying ratio test,

$\frac{2n+2}{2n+1} \to 1$, *as* $n \to \infty$ & $\frac{2n+1}{2n} \to 1$, *as* $n \to \infty$*. so, our limit is now, equal to,*

$$\Omega = \frac{1}{m}\left(\frac{2}{e}\right)^{\frac{1}{m}}$$

SP.069. If $x, y, z > 0$ and $b \ge a > 0$ then:

$$\ln\frac{b^2+xy}{a^2+xy}+\ln\frac{b^2+yz}{a^2+yz}+\ln\frac{b^2+zx}{a^2+zx}\le (b-a)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

For all $y, z, t > 0$ *we have:* $\frac{t}{t^2+yz}\le\frac{1}{y}\left(\frac{1}{y}+\frac{1}{z}\right)\Leftrightarrow y(t-z)^2+z(y-t)^2\ge 0$

therefore $\int_a^b\frac{tdt}{t^2+yz}\le\int_a^b\frac{1}{4}\left(\frac{1}{y}+\frac{1}{z}\right)dt\Rightarrow\frac{1}{2}\ln\frac{b^2+yz}{a^2+yz}\le\frac{1}{4}\left(\frac{1}{y}+\frac{1}{z}\right)(b-a)\Rightarrow$

$$\sum_{cyclic}\ln\frac{b^2+yz}{a^2+yz}\le\frac{1}{2}\sum_{cyclic}\left(\frac{1}{y}+\frac{1}{z}\right)(b-a)=(b-a)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

SP.070. Prove that if $a, b, c \in \mathbb{R}$ then:

$$(2-a-b-c+abc)^2\le(a^2+2)(b^2+2)(c^2+2)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Phuc Tang-Hanoi-Vietnam

We have

$$(a^2+2)(b^2+2)=(a+b+1)^2+(a-1)^2+(b-1)^2+(ab-1)^2\ge$$

$\ge(1-a^2)+(1-b)^2+(ab-1)^2$. *By Cauchy – Schwarz*

$$(a^2+2)(b^2+2)(c^2+2)\ge[(1-a)^2+(1-b)^2+(ab-1)^2](1+1+c^2)\ge$$

$$\ge(2-a-b-c+abc)^2$$

Equality holds if $\begin{cases} a+b+1=0 \\ 1-a=1-b=\frac{ab-1}{c} \end{cases}\Leftrightarrow a=b=c=-\frac{1}{2}$

Solution 2 by Sladjan Stankovik-Macedonia

$$a, b, c \in \mathbb{R}$$

$$(2-a-b-c+abc)^2\le(a^2+2)(b^2+2)(c^2+2)$$

$$(2-p+c)^2\le c^2+2(q^2-2pc)+4(p^2-2q)+p$$

$$4+p^2+c^2-4p+4c-2pc\le c^2+2q^2-4pc+4p^2-8q+8$$

$$c\cdot(4+2p)+(-4-3p^2-4p-2q^2+8q)\le 0$$

If we fix p and $q \Rightarrow c$ has min or MAX $\Leftrightarrow$ two variables are equal.

$$\text{Let } b = c \Rightarrow \begin{cases} p = a + 2b \\ q = ab + b^2 \\ c = ab^2 \end{cases}$$

$$ab^2 \cdot (4 + 2a + 4b) \leq 4 + 3(a + 2b)^2 + 4(a + 2b) + 2(2ab + b^2)^2 - 8(2ab + b^2)$$

$$\Leftrightarrow a^2(-6b^2 - 3) + a \cdot (-4b^3 + 4b^2 + 4b - 4) - (2b^4 + 4b^2 + 8b + 4) \leq 0$$

$$D_a = (-4b^3 + 4b^2 + 4b - 4)^2 - 4(6b^2 + 3)(2b^4 + 4b^2 + 8b + 4) =$$

$$= -8\left(4b^6 + 4b^5 + 17b^4 + 16b^3 + 20b^2 + 16b + 4\right) \leq 0$$

$$D_a \leq 0$$

$$k_a = -6b^2 - 3 < 0 \Leftrightarrow -8(2b + 1)^2(b^2 + 2)^2 \leq 0;\ b = -\frac{1}{2} = a = c$$

SP.071. Prove that if $a, b, c \in \mathbb{R}$ then:

$$(3abc - a^3 - b^3 - c^3)^2 \leq (a^2 + b^2 + c^2)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$3abc - a^3 - b^3 - c^3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \Rightarrow (3abc - a^3 - b^3 - c^3)^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} x & y & y \\ y & x & y \\ y & y & x \end{vmatrix} = x^3 + 2y^3 - 3xy^2 \text{ where } x = a^2 + b^2 + c^2$$

$$y = bc + ca + ab = x^3 - y^2(2x - 2y) - xy^2$$

$$= (a^2 + b^2 + c^2)^3 - (ab + bc + ca)^2(a^2 + b^2 + c^2) -$$

$$-(ab + bc + ca)^2\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \leq (a^2 + b^2 + c^2)^3$$

Solution 2 by Nguyen Phuc Tang-Hanoi-Vietnam

$$\text{Let } \begin{cases} x = a^2 + b^2 + c^2 \\ y = ab + bc + ca \end{cases} \Rightarrow x \geq y. \text{ We have}$$

$$(a^3 + b^3 + c^3 - 3abc)^2 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

The inequality is equivalent to $(x + 2y)(x - y)^2 \leq x^3$ *(*)*

Case $x = 0 \Rightarrow a = b = c = 0 \Rightarrow$ *(*) is true*

Case $x > 0$

(*) $\Leftrightarrow (1+2t)(1-t)^2 \le 1$ ***(with*** $t = \frac{y}{x} \le 1$***)*** $\Leftrightarrow t^2(2t-3) \le 0$ ***which is true***

Equality holds if and only if $t = 0 \Leftrightarrow a = b = c = 0$

SP.072. If $a, b, c > 0; n \in \mathbb{N}^*$ **then:**

$$\left(\frac{2na}{b+(2n-1)c}\right)^{\frac{2}{3}} + \left(\frac{2nb}{c+(2n-1)a}\right)^{\frac{2}{3}} + \left(\frac{2nc}{a+(2n-1)b}\right)^{\frac{2}{3}} \ge 3$$

Proposed by Marin Chirciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\left(\frac{2na}{b+(2n-1)c}\right)^{\frac{2}{3}} = \sum_{cyc}\frac{2na}{\sqrt[3]{2na(b+(2n-1)c)^2}} \overset{AM\ge GM}{\ge} \sum_{cyc}\frac{6na}{2na+2b+2(2n-1)c}$$

$$= 3n\sum_{cyc}\frac{a}{na+b+(2n-1)c} = 3n\sum_{cyc}\frac{a^2}{na^2+ab+(2n-1)ca} \ge$$

$$\ge 3n\frac{(a+b+c)^2}{n\sum_{cyc}a^2+\sum_{cyc}ab+(2n-1)\sum_{cyc}ab} = 3 \textit{ (proved)}$$

SP.073. If $x, y, z > 0$ **then:**

$$log\left(1+\frac{1}{x}\right) + log\left(1+\frac{1}{y}\right) + log\left(1+\frac{1}{z}\right) \ge 3\,log\left(1+\frac{3}{x+y+z}\right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Tran Hong-Vietnam

Let $f(t) = \log\left(1+\frac{1}{t}\right)$ ***for*** $t > 0$

$$\Rightarrow f'(t) = \frac{\left(1+\frac{1}{t}\right)'}{\left(1+\frac{1}{t}\right)\ln 10} = -\frac{1}{t(t+1)\ln 10} \Rightarrow f''(t) = \frac{1}{\ln 10}\cdot\frac{2t+1}{[t(t+1)]^2} > 0\ \forall t > 0$$

$\Rightarrow$ ***using Jensen's inequality we have***

$$LHS = f(x)+f(y)+f(z) \ge 3f\left(\frac{x+y+z}{3}\right) = 3\log\left(1+\frac{3}{x+y+z}\right)$$

Proved. Equality $y \Leftrightarrow x = y = z$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0$***, we have***

$$\log\left(1+\frac{1}{x}\right)+\log\left(1+\frac{1}{y}\right)+\log\left(1+\frac{1}{z}\right)=\log\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)$$

$$\geq \log\left(1+\frac{1}{\sqrt[3]{xyz}}\right)^3=3\log\left(1+\frac{1}{\sqrt[3]{xyz}}\right)\geq 3\log\left(1+\frac{3}{x+y+z}\right)$$

Therefore it is to be true.

SP.074. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum value of:

$$P=\frac{x^4}{y^4\cdot\sqrt[3]{4z(x^5+1)}}+\frac{y^4}{z^4\cdot\sqrt[3]{4x(y^5+1)}}+\frac{z^4}{x^4\cdot\sqrt[3]{4y(z^5+1)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

**** By AM-GM inequality we have:***

$$\frac{x^4}{y^4\cdot\sqrt[3]{4z(x^5+1)}}=\frac{x^4}{y^4\cdot\sqrt[3]{4z(x+1)(x^4-x^3+x^2-x+1)}}=$$

$$=\frac{x^4}{y^4\cdot\sqrt[3]{2(zx+z)(2x^4-2x^3+2x^2-2x+2)}}\geq$$

$$\geq\frac{x^4}{y^4\left(\frac{2+zx+z+2x^4-2x^3+2x^2-2x+2}{3}\right)}=$$

$$=\frac{3x^4}{y^4(2x^4-2x^3+2x^2+zx-2x+z+4)}\Rightarrow$$

$$\Rightarrow\frac{x^4}{y^4\cdot\sqrt[3]{4z(x^5+1)}}\geq\frac{3x^4}{y^4(2x^4-2x^3+2x^2+zx-2x+z+4)}$$

$$+\ \textit{Similar: }\frac{y^4}{z^4\cdot\sqrt[3]{4x(y^5+1)}}\geq\frac{3y^4}{z^4(2y^4-2y^3+2y^2+xy-2y+x+4)}$$

$$\frac{z^4}{x^4\cdot\sqrt[3]{4y(z^5+1)}}\geq\frac{3z^4}{x^4(2z^4-2z^3+2z^2+yz-2z+y+4)}$$

$$-\textbf{\textit{ Hence: }}\Rightarrow\frac{P}{3}=\frac{1}{3}\left(\frac{x^4}{y^4.\sqrt[3]{4z(x^5+1)}}+\frac{y^4}{z^4.\sqrt[3]{4x(y^5+1)}}+\frac{z^4}{x^4.\sqrt[3]{4y(z^5+1)}}\right)\geq$$

$$\geq\frac{x^4}{y^4(2x^4-2x^3+2x^2+zx-2x+z+4)}+\frac{y^4}{z^4(2y^4-2y^3+2y^2+xy-2y+x+4)}+\frac{z^4}{x^4(2z^4-2z^3+2z^2+yz-2z+y+4)}\quad (1)$$

- Other, by Cauchy Schwarz inequality we have:

$$\frac{x^4}{y^4(2x^4-2x^3+2x^2+zx-2x+z+4)}+\frac{y^4}{z^4(2y^4-2y^3+xy-2y+x+4)}+$$

$$+\frac{z^4}{x^4(2z^4-2z^3+2z^2+yz-2z+y+4)}=$$

$$=\frac{\left(\frac{x^2}{y^2}\right)^2}{2x^4-2x^3+2x^2+zx-2x+z+4}+\frac{\left(\frac{y^2}{z^2}\right)^2}{2y^4-2y^3+2y^2+xy-2y+x+4}+$$

$$+\frac{\left(\frac{z^2}{x^2}\right)^2}{2z^4-2z^3+2z^2+yz-2z+y+4}\geq$$

$$\geq\frac{\left(\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)^2}{(2x^4-2x^3+2x^2+zx-2x+z+4)+(2y^4-2y^3+2y^2+xy-2y+x+4)+(2z^4-2z^3+2z^2+yz-2z+y+4)}\quad (2)$$

- Let (1), (2):

$$\Rightarrow\frac{P}{3}\geq\frac{\left(\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)^2}{(2x^4-2x^3+2x^2+zx-2x+z+4)+(2y^4-2y^3+2y^2+xy-2y+x+4)+(2z^4-2z^3+2z^2+yz-2z+y+4)}\quad (3)$$

- By AM-GM inequality and $x+y+z=3$. We have:

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}=\frac{\frac{x^2}{y^2}+\frac{x^2}{y^2}+\frac{y^2}{z^2}}{3}+\frac{\frac{y^2}{z^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}}{3}+\frac{\frac{z^2}{x^2}+\frac{z^2}{x^2}+\frac{x^2}{y^2}}{3}\geq$$

$$\geq\frac{3\cdot\sqrt[3]{\frac{x^2}{y^2}\cdot\frac{x^2}{y^2}\cdot\frac{y^2}{z^2}}}{3}+\frac{3\cdot\sqrt[3]{\frac{y^2}{z^2}\cdot\frac{y^2}{z^2}\cdot\frac{z^2}{x^2}}}{3}+\frac{3\cdot\sqrt[3]{\frac{z^2}{x^2}\cdot\frac{z^2}{x^2}\cdot\frac{x^2}{y^2}}}{3}=\sqrt[3]{\frac{x^4}{y^2z^2}}+\sqrt[3]{\frac{y^4}{z^2x^2}}+\sqrt[3]{\frac{z^4}{x^2y^2}}=\frac{x^2+y^2+z^2}{\sqrt[3]{x^2y^2z^2}}$$

$$\Rightarrow\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\frac{x^2+y^2+z^2}{\sqrt[3]{x^2y^2z^2}}\geq\frac{x^2+y^2+z^2}{\left(\frac{x+y+z}{3}\right)^2}=\frac{x^2+y^2+z^2}{\left(\frac{3}{3}\right)^2}=x^2+y^2+z^2\quad (4)$$

- Let (3), (4): $\Rightarrow \frac{P}{3} \ge \frac{(x^2+y^2+z^2)^2}{2(x^4+y^4+z^4)-2(x^3+y^3+z^3)+2(x^2+y^2+z^2)+(xy+yz+zx)-(x+y+z)+12}$ ***(5)***

We will prove:

$$\frac{(x^2+y^2+z^2)^2}{2(x^4+y^4+z^4)-2(x^3+y^3+z^3)+2(x^2+y^2+z^2)+(xy+yz+zx)-(x+y+z)+12} \ge \frac{1}{2} \quad (6)$$

$$\Leftrightarrow 2(x^2+y^2+z^2)^2 \ge 2(x^4+y^4+z^4)-2(x^3+y^3+z^3)+$$

$$+2(x^2+y^2+z^2)+2(x^2+y^2+z^2)+(xy+yz+zx)-(x+y+z)+12$$

$$\Leftrightarrow 2(x^3+y^3+z^3)+4(x^2y^2+y^2z^2+z^2x^2) \ge 2(x^2+y^2+z^2)+xy+yz+zx-3+12$$

$$(x+y+z=3)$$

$$\Leftrightarrow 2(x^3+y^3+z^3)+4(x^2y^2+y^2z^2+z^2x^2) \ge 2(x^2+y^2+z^2)+xy+yz+zx+9$$

$$\Leftrightarrow 18(x^3+y^3+z^3)+36(x^2y^2+y^2z^2+z^2x^2) \ge 18(x^2+y^2+z^2)+9(xy+yz+zx)+81$$

$$\Leftrightarrow 6(x+y+z)(x^3+y^3+z^3)+36(x^2y^2+y^2z^2+z^2x^2) \ge$$

$$\ge 2(x+y+z)^2(x^2+y^2+z^2)+(x+y+z)^2(xy+yz+zx)+(x+y+z)^4$$

(because $x+y+z=3$ ***then:*** $18=6(x+y+z); 18=2(x+y+z)^2; 81=(x+y+z)^4$***)***

$$\Leftrightarrow 6(x^4+y^4+z^4)+6\left(xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\right)+36(x^2y^2+y^2z^2+z^2x^2) \ge$$

$$\ge 3(x+y+z)^2(x^2+y^2+z^2+xy+yz+zx)$$

$$\Leftrightarrow 2(x^4+y^4+z^4)+2\left(xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\right)+12(x^2y^2+y^2z^2+z^2x^2) \ge$$

$$\ge (x^2+y^2+z^2+2xy+2yz+2zx)(x^2+y^2+z^2+xy+yz+zx)$$

$$\Leftrightarrow 2(x^4+y^4+z^4)+2\left(xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\right)+12(x^2y^2+y^2z^2+z^2x^2) \ge$$

$$\ge x^4+y^4+z^4+3\left(xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\right)+4(x^2y^2+y^2z^2+z^2x^2)+7xyz(x+y+z)$$

$$\Leftrightarrow x^4+y^4+z^4+8(x^2y^2+y^2z^2+z^2x^2) \ge xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)+7xyz(x+y+z) \quad (7)$$

- We have:

$$(x^2+y^2+z^2-xy-yz-zx)^2 \ge 0 \Leftrightarrow (x^2+y^2+z^2)+(xy+yz+zx)^2 \ge 2(x^2+y^2+z^2)(xy+yz+zx)$$

$$\Leftrightarrow x^4+y^4+z^4+3(x^2y^2+y^2z^2+z^2x^2) \ge 2\left(y(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\right) \quad (8)$$

- By AM-GM inequality:

$$xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2) \ge xy\cdot 2xy+yz\cdot 2yz+zx\cdot 2zx = 2(x^2y^2+y^2z^2+z^2x^2) \quad (9)$$

- Let (8), (9): $\Rightarrow x^4+y^4+z^4+3(x^2y^2+y^2z^2+z^2x^2) \ge xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)+2(x^2y^2+y^2z^2+z^2x^2)$

$$\Leftrightarrow x^4+y^4+z^4+(x^2y^2+y^2z^2+z^2x^2) \ge xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2) \quad (10)$$

+ Other: $x^2y^2+y^2z^2+z^2x^2 = \frac{x^2(y^2+z^2)}{2}+\frac{y^2(z^2+x^2)}{2}+\frac{z^2(x^2+y^2)}{2} \ge \frac{x^2\cdot 2yz}{2}+\frac{y^2\cdot 2zx}{2}+\frac{z^2\cdot 2xy}{2}$

$$\Rightarrow x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x + y + z) \Leftrightarrow xyz(x + y + z) \Leftrightarrow 7(x^2y^2 + y^2z^2 + z^2x^2) \geq 7xyz(x + y + z) \quad (11)$$

– Hence (10), (11):

$$\Rightarrow x^4 + y^4 + z^4 + 8(x^2y^2 + y^2z^2 + z^2x^2) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) + 7xyz(x + y + z)$$

$\Rightarrow$ Inequality (7) true $\Rightarrow$ (6) true.

– Let (5), (6): $\Rightarrow \frac{P}{3} \geq \frac{1}{2} \Rightarrow P \geq \frac{3}{2} \Rightarrow P_{Min} = \frac{3}{2}$*. Equality occurs if:*

$$\begin{cases} x, y, z > 0; x + y + z = 3 \\ x = y = z \end{cases} \Leftrightarrow x = y = z = 1$$

SP.075. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the minimum of expression:

$$P = 2(x + y + z) + \frac{x}{y^3 + z^3 + 1} + \frac{y}{z^3 + x^3 + 1} + \frac{z}{x^3 + y^3 + 1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo x, y, z números R^+ de tal manera que $xyz = 1$. Hallar el mínimo valor

$P = 2(x + y + z) + \frac{x}{y^3+z^3+1} + \frac{y}{z^3+x^3+1} + \frac{z}{x^3+y^3+1}$*. Aplicando la desigualdad de Cauchy*

$$\frac{x^2}{x(y^3 + z^3) + x} + \frac{y^2}{y(z^3 + x^3) + y} + \frac{z^2}{z(x^3 + y^3) + z} \geq$$

$$\geq \frac{(\sum x)^2}{\sum xy(x^2 + y^2) + \sum x} = \frac{(\sum x)^2}{\sum xy(x^2 + y^2) + (\sum x)xyz} = \frac{(\sum x)^2}{(\sum x^2)(\sum xy)}$$

Camo $x, y, z > 0$. Aplicando MA $\geq$ MG

$$\frac{(\sum x)^2(\sum xy)}{(\sum x^2)(\sum xy)(\sum xy)} \geq \frac{3(\sum x)^2}{\left(\frac{\sum x^2 + \sum xy + \sum xy}{3}\right)^3} = \frac{81(\sum x)^2}{(\sum x)^6} = \frac{81}{(x + y + z)^4}$$

Luego

$$P = 2(x + y + z) + \frac{x}{y^3 + z^3 + 1} + \frac{y}{z^3 + x^3 + 1} + \frac{z}{x^3 + y^3 + 1} \geq$$

$$\geq 2(x + y + z) + \frac{81}{(x + y + z)^4}$$

Nuevamente por MA $\geq$ MG

$$2(x+y+z)+\frac{81}{(x+y+z)^4}=\left(\frac{x+y+z}{3}+\frac{x+y+z}{3}+\frac{x+y+z}{3}+\frac{x+y+z}{3}+\frac{81}{(x+y+z)^4}\right)+$$

$$+\frac{2(x+y+z)}{3}\geq 5+2=7$$

Por transitividad $\rightarrow P=2(x+y+z)+\frac{x}{y^3+z^3+1}+\frac{y}{z^3+x^2+1}+\frac{z}{x^3+y^3+1}\geq 7$

La igualdad se alcanza cuando $x=y=z=1$

UP.061. Prove that in all triangle ABC with usual notations holds the following inequalities:

(a) $\frac{\tan^3\frac{A}{2}}{m\cdot\tan\frac{B}{2}+n\cdot\tan\frac{C}{2}}+\frac{\tan^3\frac{B}{2}}{m\cdot\tan\frac{C}{2}+n\cdot\tan\frac{A}{2}}+\frac{\tan^3\frac{C}{2}}{m\cdot\tan\frac{A}{2}+n\cdot\tan\frac{B}{2}}\geq\frac{\left((4R+r)^2-2s^2\right)^2}{(m+n)s^4}$;

(b) $\frac{\tan\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}+\frac{\tan\frac{B}{2}}{m+n\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}}+\frac{\tan\frac{C}{2}}{m+n\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\geq\frac{(4R+r)^2}{s(m(4R+r)+3nr)}$;

(c) $\frac{\tan^3\frac{A}{2}}{m\cdot\cot\frac{B}{2}+n\cdot\cot\frac{C}{2}}+\frac{\tan^3\frac{B}{2}}{m\cdot\cot\frac{C}{2}+n\cdot\cot\frac{A}{2}}+\frac{\tan^3\frac{C}{2}}{m\cdot\cot\frac{A}{2}+n\cdot\cot\frac{B}{2}}\geq\frac{(4R+r)r}{(m+n)s^2}$;

(d) $\frac{\tan\frac{A}{2}}{\left(x+y\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}\right)^m}+\frac{\tan\frac{B}{2}}{\left(x+y\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}\right)^m}+\frac{\tan\frac{C}{2}}{\left(x+y\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}\right)^m}\geq\frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$

for any positive real numbers m,n,x,y

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $m,n,x,y>0$***. Probar en un triángulo*** ABC ***las siguientes desigualdades***

a) $\frac{\tan^3\frac{A}{2}}{m\tan\frac{B}{2}+n\tan\frac{C}{2}}+\frac{\tan^3\frac{B}{2}}{m\tan\frac{C}{2}+n\tan\frac{A}{2}}+\frac{\tan^3\frac{C}{2}}{m\tan\frac{A}{2}+n\tan\frac{B}{2}}\geq\frac{\left((4R+r)^2-2s^2\right)^2}{(m+n)s^4}$

Tener en cuenta las siguientes identidades en un ΔABC

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1,\ \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\frac{s}{r}$$

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{4R+r}{s},$$

$$\cot\frac{A}{2}\cot\frac{B}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+\cot\frac{C}{2}\cot\frac{A}{2}=\prod\cot\frac{A}{2}\left(\sum\tan\frac{A}{2}\right)=\frac{4R+r}{r}$$

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2-2=\frac{(4R+r)^2-2s^2}{s^2}$$

Aplicando la desigualdad de Cauchy

$$\sum\frac{\tan^4\frac{A}{2}}{m\tan\frac{A}{2}\tan\frac{B}{2}+n\tan\frac{A}{2}\tan\frac{C}{2}}=\frac{\left(\sum\tan^2\frac{A}{2}\right)^2}{(m+n)\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)}=$$

$$=\frac{\left(\frac{(4R+r)^2-2s^2}{s^2}\right)^2}{(m+n)\cdot 1}=\frac{((4R+r)^2-2s^2)^2}{(m+n)s^4}$$

(LQQD)

b) $\frac{\tan\frac{A}{2}}{m+n\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\tan\frac{B}{2}}{m+n\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{\tan\frac{C}{2}}{m+n\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{(4R+r)^2}{s(m(4R+r)+3nr)}$

Aplicando la desigualdad de Cauchy

$$\sum\frac{\tan^2\frac{A}{2}}{m\tan\frac{A}{2}+n\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}\geq\frac{\left(\sum\tan\frac{A}{2}\right)^2}{m\left(\sum\tan\frac{A}{2}\right)+3\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}=$$

$$=\frac{\left(\frac{4R+r}{s}\right)^2}{m\left(\frac{4R+r}{s}\right)+3n\cdot\frac{r}{s}}=\frac{(4R+r)^2}{s(m(4R+r)+3nr)}$$

c) $\frac{\tan^3\frac{A}{2}}{m\cot\frac{B}{2}+n\cot\frac{C}{2}}+\frac{\tan^3\frac{B}{2}}{m\cot\frac{C}{2}+n\cot\frac{A}{2}}+\frac{\tan^3\frac{C}{2}}{m\cot\frac{A}{2}+n\cot\frac{B}{2}}\geq\frac{(4R+r)r}{(m+n)s^2}$

Aplicando la desigualdad de Cauchy

$$\sum\frac{\tan^2\frac{A}{2}}{m\cot\frac{A}{2}\cot\frac{B}{2}+n\cot\frac{A}{2}\cot\frac{C}{2}}\geq\frac{\left(\sum\tan\frac{A}{2}\right)^2}{m\left(\sum\cot\frac{A}{2}\cot\frac{B}{2}\right)+n\left(\sum\cot\frac{A}{2}\cot\frac{B}{2}\right)}=$$

$$=\frac{\left(\sum\tan\frac{A}{2}\right)^2}{(m+n)\left(\sum\cot\frac{A}{2}\cot\frac{B}{2}\right)}=\frac{\left(\frac{4R+r}{s}\right)^2}{(m+n)\left(\frac{4R+r}{r}\right)}$$

$$\rightarrow\sum\frac{\tan^2\frac{A}{2}}{m\cot\frac{A}{2}\cot\frac{B}{2}+n\cot\frac{A}{2}\cot\frac{C}{2}}\geq\frac{(4R+r)r}{(m+n)s^2}$$

d) $\frac{\tan\frac{A}{2}}{\left(x+y\tan\frac{B}{2}\tan\frac{C}{2}\right)^m}+\frac{\tan\frac{B}{2}}{\left(x+y\tan\frac{C}{2}\tan\frac{A}{2}\right)^m}+\frac{\tan\frac{C}{2}}{\left(x+y\tan\frac{A}{2}\tan\frac{B}{2}\right)^m}\ge\frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$

La desigualdad propuesta es equivalente

$$\frac{\left(\tan\frac{A}{2}\right)^{m+1}}{\left(x\tan\frac{A}{2}+y\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^m}+\frac{\left(\tan\frac{B}{2}\right)^{m+1}}{\left(x\tan\frac{B}{2}+y\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{A}{2}\right)^m}+$$

$$+\frac{\left(\tan\frac{C}{2}\right)^{m+1}}{\left(x\tan\frac{C}{2}+y\tan\frac{C}{2}\tan\frac{A}{2}\tan\frac{B}{2}\right)^m}\ge\frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$$

Aplicando la desigualdad de Radon

$$\sum\frac{\left(\tan\frac{A}{2}\right)^{m+1}}{\left(x\tan\frac{A}{2}+y\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^m}\ge\frac{\left(\sum\tan\frac{A}{2}\right)^{m+1}}{\left(x\left(\sum\tan\frac{A}{2}\right)+3y\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^m}=$$

$$=\frac{\left(\frac{4R+r}{s}\right)^{m+1}}{\left(x\left(\frac{4R+r}{s}\right)+\frac{3yr}{s}\right)^m}=\frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\tan\frac{A}{2}=\frac{\Delta}{p(p-a)},\tan\frac{B}{2}=\frac{\Delta}{p(p-b)}\ \textit{and}\ \tan\frac{C}{2}=\frac{\Delta}{p(p-c)}$$

a) $\sum_{cyc}\frac{\tan^3\frac{A}{2}}{m\tan\frac{B}{2}+n\tan\frac{C}{2}}\overset{HOLDER}{\ge}\frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}{3(m+n)}$

$$=\frac{\Delta^2}{\left(p(p-a)(p-b)(p-c)\right)^2}\cdot\frac{\left(\sum(p-a)(p-b)\right)^2}{3(m+n)}=\frac{r^2(r+4R)^2}{\Delta^2\cdot3(m+n)}=\frac{(r+4R)^2}{3p^2(m+n)}$$

b) $\sum_{cyc}\frac{\tan\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}=\sum_{cyc}\frac{\tan^2\frac{A}{2}}{m\tan\frac{A}{2}+n\cdot\prod\tan\frac{A}{2}}$

$$\overset{BERGSTROM}{\ge}\frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}{m\sum\tan\frac{A}{2}+3n\prod\tan\frac{A}{2}}=\frac{\Delta^2}{\left(p(p-a)(p-b)(p-c)\right)^2}\cdot\frac{\left(\sum(p-a)(p-b)\right)^2}{\frac{\Delta}{p\prod(p-a)}m\left(\sum(p-a)(p-b)\right)+3n\frac{\Delta^3}{p^3\prod(p-a)}}$$

$$=\frac{1}{\Delta^2}\cdot\frac{r^2(r+4R)^2}{\frac{m}{\Delta}r(r+4R)+3n\frac{\Delta}{p^2}}=\frac{\Delta}{p^2}\cdot\frac{(r+4R)^2}{mr(r+4R)+3n\frac{\Delta^2}{p^2}}=\frac{(r+4R)^2}{p(m(r+4R)+3nr)}\ \textit{(proved)}$$

c) $\sum_{cyc}\frac{\tan^3\frac{A}{2}}{m\cot\frac{B}{2}+n\cot\frac{C}{2}}=\left(\prod_{cyc}\tan\frac{A}{2}\right)\sum_{cyc}\frac{\tan^2\frac{A}{2}}{m\tan\frac{C}{2}+n\tan\frac{B}{2}}\overset{BERGSTROM}{\geq}\left(\sum_{cyc}\tan\frac{A}{2}\right)\frac{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}{m+n}$

$$=\frac{\Delta^3}{p^3\prod(p-a)}\cdot\frac{\Delta}{p\prod(p-a)}\cdot\frac{1}{m+n}\left(\sum_{cyc}(p-a)(p-b)\right)$$

$$=\frac{\Delta}{p^2}\cdot\frac{1}{\Delta}\cdot\frac{r(r+4R)}{m+n}=\frac{r(r+4R)}{(m+n)p^2}\quad \textit{(proved)}$$

d) $\sum_{cyc}\frac{\tan\frac{A}{2}}{\left(x+y\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}\right)^m}=\sum_{cyc}\frac{\tan^{m+1}\frac{A}{2}}{\left(x\tan\frac{A}{2}+y\prod\tan\frac{A}{2}\right)^m}$

$$\overset{RADON}{\geq}\frac{\left(\sum\tan\frac{A}{2}\right)^{m+1}}{\left(x\sum_{cyc}\tan\frac{A}{2}+2y\prod\tan\frac{A}{2}\right)^m}=\frac{\left(\sum_{cyc}\frac{\Delta}{p(p-a)}\right)^{m+1}}{\left(x\sum_{cyc}\frac{\Delta}{p(p-a)}+3y\frac{\Delta^3}{p^3(p-a)}\right)^m}$$

$$=\frac{\Delta^{m+1}}{(p\prod(p-a))^{m+1}}\cdot\frac{\left(\sum_{cyc}(p-a)(p-b)\right)^{m+1}}{\left(\frac{\Delta}{p\prod(p-a)}x\sum_{cyc}(p-a)(p-b)+3y\frac{\Delta}{p^2}\right)^m}$$

$$=\frac{1}{\Delta^{m+1}}\cdot\frac{r^{m+1}(r+4R)^{m+1}}{\left(\frac{x}{\Delta}r(r+4R)+\frac{3y\Delta}{p^2}\right)^m}=\frac{1}{\Delta}\cdot\frac{r^{m+1}(r+4R)^{m+1}}{(xr(r+4R)+3yr^2)^m}=$$

$$=\frac{(r+4R)^{m+1}}{p(x(r+4R)+3yr)^m}$$

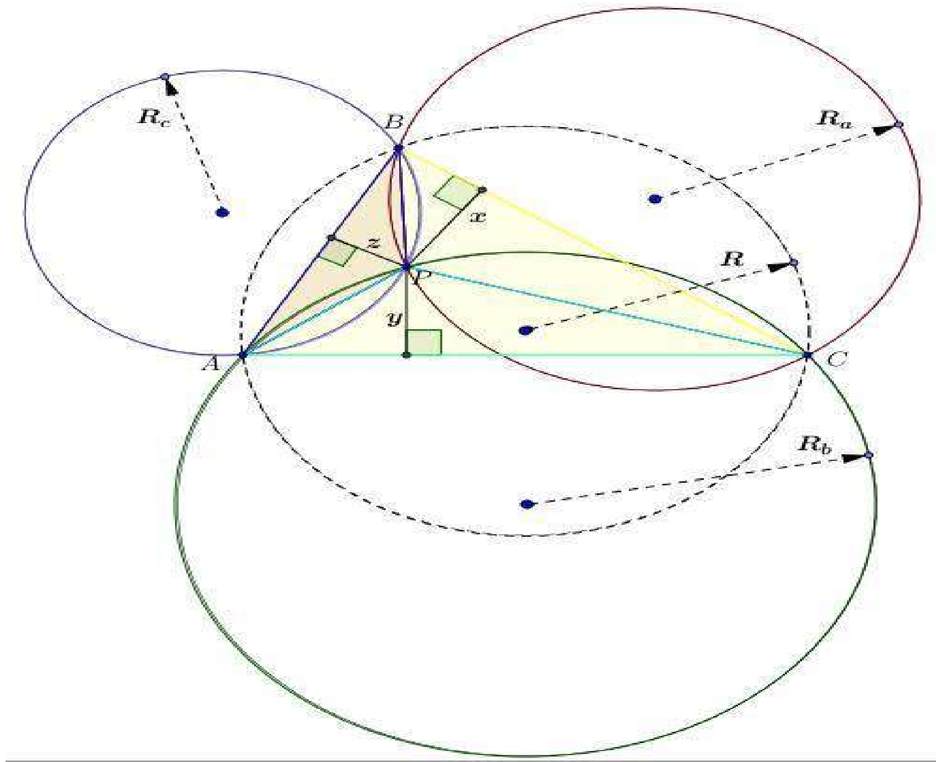
(proved)

UP.062. Given the equilateral triangle ABC and let P be any point in its plane. R, R_a, R_b, R_c denote respectively radii of the circumcircles of the triangles ABC, BPC, CPA, APB and x, y, z are respectively distances from P to the sides BC, CA, AB. Prove that

$$xR_a+yR_b+zR_c\geq\frac{3}{2}R^2.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru



Dado un triángulo equilátero ABC y sea P un punto en este plano. Siendo R, R_a, R_b, R_c, respectivamete los radios de las circunferencias circunscritas ABC, BPC, CPA, APB, además x, y, z son las distancias de P a los lados BC, CA, AB.

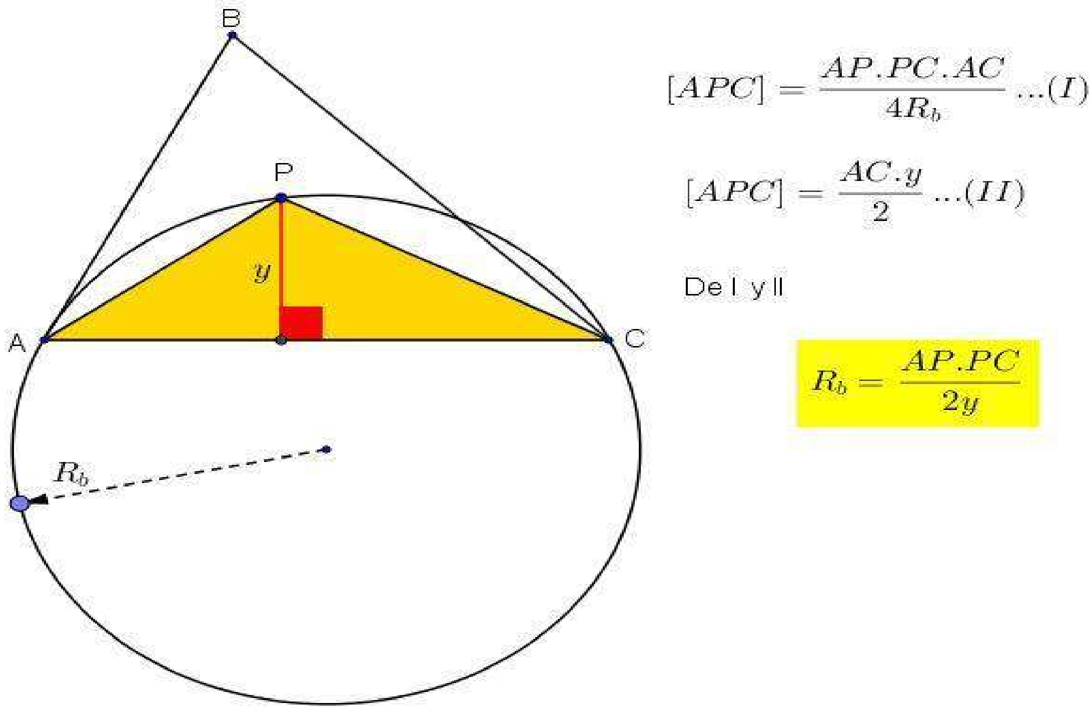
Probar que $xR_a + yR_b + zR_c \geq \frac{3}{2}R^2$

En un $\Delta\ ABC$ general, se cumple lo siguiente:

$$S_{\Delta BPC} = \frac{BC \cdot x}{2} = \frac{BP \cdot PC \cdot BC}{4R_a} \Leftrightarrow R_a = \frac{PB \cdot PC}{2x}$$

$$S_{\Delta CPA} = \frac{CA \cdot y}{2} = \frac{CP \cdot PA \cdot CA}{4R_b} \Leftrightarrow R_b = \frac{PC \cdot PA}{2y}$$

$$S_{\Delta CPA} = \frac{AB \cdot z}{2} = \frac{AP \cdot PB \cdot AB}{4R_b} \Leftrightarrow R_c = \frac{PA \cdot PB}{2z}$$



La desigualdad propuesta es equivalente: $PB \cdot PC + PC \cdot PA + PA \cdot PB \geq 3R^2$

Dado que es triángulo equilátero: $BC = CA = AB = l = 2R\sin 60° = \sqrt{3}R$

Es suficiente probar: $PB \cdot PC + BPC \cdot PA + PA \cdot PB \geq l^2$

Para todo, α, β, γ ***que satisface*** $\alpha + \beta + \gamma = 360°$ ***se cumple la siguiente desigualdad***$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma \geq -\frac{3}{2}$

En el triángulo APB***, por ley de cosenos tenemos***

$$l^2 = PA^2 + PB^2 - 2PA \cdot PB \cdot \cos\alpha \Leftrightarrow \cos\alpha = \frac{PA^2 + PB^2 - l^2}{2PA \cdot PB}$$

En el triángulo BPC***, por ley de cosenos tenemos***

$$l^2 = PB^2 + PC^2 - 2PB - 2PB \cdot PC \cdot \cos\beta \Leftrightarrow \cos\beta = \frac{PB^2 + PC^2 - l^2}{2PB \cdot PC}$$

En el triángulo CPA***, por ley de cosenos tenemos***

$$l^2 = PC^2 + PA^2 - 2PC \cdot PA \cdot \cos\gamma \Leftrightarrow \cos\gamma = \frac{PC^2 + PA^2 - l^2}{2PC \cdot PA}$$

Por lo tanto

$$\frac{PA^2+PB^2-l^2}{2PA\cdot PB}+\frac{PB^2+PC^2-l^2}{2PB\cdot PC}+\frac{PC^2+PA^2-l^2}{2PC\cdot PA}+\frac{3}{2}\geq 0$$

$$\Leftrightarrow (PA^2+PB^2-l^2)PC+(PB^2+PC^2-l^2)PA+(PC^2+PA^2-l^2)PB+$$

$$+3(PA\cdot PB\cdot PC)\geq 0$$

$$\Leftrightarrow PA\cdot PB(PA+PB)+PB\cdot PC(PB+PC)+PC\cdot PA(PC+PA)+$$

$$+3PA\cdot PB\cdot PC-l^2(PA+PB+PC)\geq 0$$

$$\Leftrightarrow (PA+PB+PC)(PA\cdot PB+PB\cdot PC+PC\cdot PA)-l^2(PA+PB+PC)\geq 0$$

$$\Leftrightarrow (PA+PB+PC)(PA\cdot PB+PB\cdot PC+PC\cdot PA-l^2)\geq 0$$

Donde se deduce $\rightarrow PA\cdot PB+PB\cdot PC+PC\cdot PA\geq l^2$

UP.063. If $a,b,c,d>0$ such that :

$$a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2=6.$$

then:

$$\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\geq 4.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean "a,b,c,d" números R^+ de tal manera que:

$$a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2=6$$

Probar que: $\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\geq 4$

Elevando al cuadrado la expresión, se tiene lo siguiente:

$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\left(\frac{bcd}{a}+\frac{cda}{b}\right)^2+\left(\frac{dab}{c}+\frac{abc}{d}\right)^2+2\left(\frac{bcd}{a}+\frac{cda}{b}\right)\left(\frac{dab}{c}+\frac{abc}{d}\right)$$

$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\left(\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+2c^2d^2\right)+\left(\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}+2a^2b^2\right)+$$

$$+2(b^2d^2+d^2a^2+b^2c^2+a^2c^2)$$

Ordenando la expresión convenienentemente:

$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}+$$

$$+2(a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2)$$

$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}+12 \quad (A)$$

Desde que: $a,b,c,d>0$***. Por: MA*** $\geq$ ***MG***

$$\frac{1}{a^4}+\frac{1}{b^4}\geq\frac{2}{a^2b^2} \quad (I),$$

$$\frac{1}{b^4}+\frac{1}{c^4}\geq\frac{2}{b^2c^2} \quad (II),$$

$$\frac{1}{c^4}+\frac{1}{d^4}\geq\frac{2}{c^2d^2} \quad (III)$$

$$\frac{1}{d^4}+\frac{1}{a^4}\geq\frac{2}{d^2a^2} \quad (IV)$$

$$\frac{1}{a^4}+\frac{1}{c^4}\geq\frac{2}{a^2c^2} \quad (V)$$

$$\frac{1}{b^4}+\frac{1}{d^4}\geq\frac{2}{b^2d^2} \quad (VI)$$

Sumando: (I) + (II) + (III) + (IV) + (V) + (VI):

$$\Rightarrow\frac{3}{a^4}+\frac{3}{b^4}+\frac{3}{c^4}+\frac{3}{d^4}\geq\frac{2}{a^2b^2}+\frac{2}{b^2c^2}+\frac{2}{c^2d^2}+\frac{2}{d^2a^2}+\frac{2}{a^2c^2}+\frac{2}{b^2d^2}$$

Multiplicando $\times\ (abcd)^2$***...***

$$\Rightarrow 3\left(\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}\right)\geq 2(c^2d^2+d^2a^2+a^2b^2+b^2c^2+b^2d^2+a^2c^2)$$

$$\Rightarrow\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}\geq\frac{2}{3}(a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2)=$$

$=\frac{2}{3}(6)=4$***. Finalmente tenemos en ... (A):***

$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}+12\geq 4+12=16$$

$$\Rightarrow\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\geq 4 \ \textit{(LQQD)}$$

UP.064. Let a,b,c,d be non-negative real numbers such that

$$a+b+c+d=4.$$

Prove that

$$ab(a+b)+cd(c+d)+4(a+b)(c+d)\leq$$
$$\leq\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}+16$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c, d números reales no negativos de tal manera que

$a + b + c + d = 4$. Probar que

$$ab(a+b) + cd(c+d) + 4(a+b)(c+d) \leq \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d} + 16$$

Tener en cuenta la siguiente identidad

$$(x+y)^3 = x^3 + y^3 + 3xy(x+y), \text{ donde } x = a+b, y = c+d$$

$$(a+b+c+d)^3 = (a+b)^3 + (c+d)^3 + 3(a+b)(c+d)(a+b+c+d)$$

$$(a+b+c+d)^3 = a^3 + b^3 + 3ab(a+b) + c^3 + d^3 + 3cd(c+d) + 12(a+b)(c+d)$$

La desigualdad propuesta es equivalente

$$\Leftrightarrow 3ab(a+b) + 3ab(a+b) + 12(a+b)(c+d) \leq 3\left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}\right) + 48$$

$$\Leftrightarrow (a+b+c+d)^3 \leq a^3 + b^3 + c^3 + d^3 + 3\left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}\right) + 48$$

$$\Leftrightarrow 64 \leq a^3 + b^3 + c^3 + d^3 + 3\left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}\right) + 48$$

$$\Leftrightarrow 16 \leq a^3 + b^3 + c^3 + d^3 + 3\left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}\right)$$

Como $a, b, c, d \geq 0$. Aplicando MA $\geq$ MG

$$a^3 + \sqrt[3]{a} + \sqrt[3]{a} + \sqrt[3]{a} \geq 4a,$$

$$b^3 + \sqrt[3]{b} + \sqrt[3]{b} + \sqrt[3]{b} \geq 4b,$$

$$c^3 + \sqrt[3]{c} + \sqrt[3]{c} + \sqrt[3]{c} \geq 4c,$$

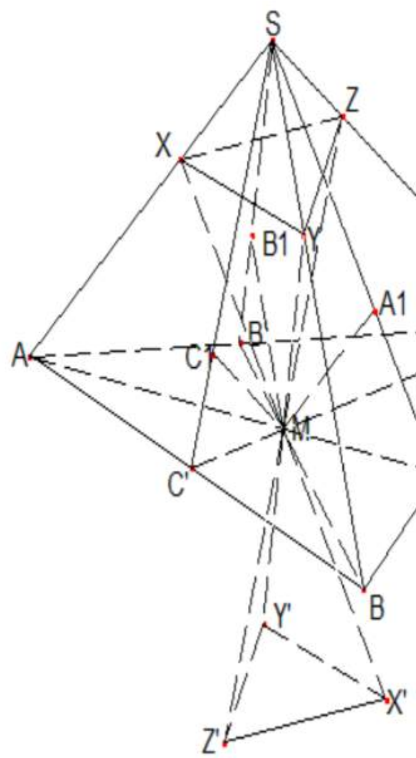
$$d^3 + \sqrt[3]{d} + \sqrt[3]{d} + \sqrt[3]{d} \geq 4d$$

$$\Rightarrow a^3 + b^3 + c^3 + d^3 + 3\left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}\right) \geq 4(a+b+c+d) = 16$$

UP.065. Let $SABC$ be a tetahedron and let M be any point inside the triangle ABC. The lines through M parallel with the planes SBC, SCA, SAB intersect SA, SB, SC at X, Y, Z respectively. Prove that

$$Vol(MXYZ) \leq \frac{2}{27} Vol(SABC)$$

Determine position of the point M such that the equality holds.



Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

Let X', Y', Z' be respectively images of X, Y, Z with the central symmetry M. Then

$$Vol(MXYZ) = Vol(MX'Y'Z').$$

Let A', B', C' be respectively intersections of the line AM with the sides BC, CA, AB. On the rays SA', SB', SC' we take points A_1, B_1, C_1, respectively such that $SA_1 = XM, SB_1 = YM, SC_1 = ZM$. Then the translation by vector $\overrightarrow{MS}$ transforms the tetahedron $MX'Y'Z'$ into the tetahedron $SA_1B_1C_1$. Therefore

$$Vol(MX'Y'Z') = Vol\,(SA_1B_1C_1)$$

Thus we have $Vol(MXYZ) = Vol(SA_1B_1C_1)$. Furthermore

$$\frac{Vol(SA_1B_1C_1)}{Vol(SA'B'C')} = \frac{SA_1}{SA'} \cdot \frac{SB_1}{SB'} \cdot \frac{SC_1}{SC'} = \frac{XM}{SA'} \cdot \frac{YM}{SB'} \cdot \frac{ZM}{SC'} = \frac{AM}{AA'} \cdot \frac{BM}{BB'} \cdot \frac{CM}{CC'}$$

From these above we deduce that: $\frac{Vol(MXYZ)}{Vol(SA'B'C')} = \left(1 - \frac{MA'}{AA'}\right)\left(1 - \frac{MB'}{BB'}\right)\left(1 - \frac{MC'}{CC'}\right)$

Using the AM-GM inequality and note that: $\frac{MA'}{AA'}+\frac{MB'}{BB'}+\frac{MC'}{CC'}=1$, *we obtain*

$$\left(1-\frac{MA'}{AA'}\right)\left(1-\frac{MB'}{BB'}\right)\left(1-\frac{MC'}{CC'}\right)\le\left(\frac{3-\frac{MA'}{AA'}-\frac{MB'}{BB'}-\frac{MC'}{CC'}}{3}\right)^3=\frac{8}{27}$$

Thus $\frac{Vol(MXYZ)}{Vol\ (SA'B'C')}\le\frac{8}{27}$. *On the other hand:* $\frac{Vol(SA'B'C')}{Vol\ (SABC)}=\frac{Area(A'B'C')}{Area(ABC)}\le\frac{1}{4}$

Multiplying up these two inequalities we get: $\frac{Vol(MXYZ)}{Vol(SABC)}\le\frac{2}{27}$, *which is the desired result. The equality holds when the point* M *is the centroid of the triangle* ABC. *Now we will prove a result that has just been used above as* $\frac{Area(A'B'C')}{Area(ABC)}\le\frac{1}{4}$. *Indeed, this is equivalent to*

$$\frac{Area(AB'C')}{Area(ABC)}+\frac{Area(BC'A')}{Area(ABC)}+\frac{Area(CA'B')}{Area(ABC)}\ge\frac{3}{4},\ or\ \frac{AB'}{AC}\cdot\frac{AC'}{AB}+\frac{BC'}{BA}\cdot\frac{BA'}{BC}+\frac{CA'}{CB}\cdot\frac{CB'}{CA}\ge\frac{3}{4}.$$

Setting $\frac{A'B}{A'C}=x,\frac{B'C}{B'A}=y,\frac{C'A}{C'B}=z$. *By the Ceva's theorem, we have* $xyz=1$. *Then our inequality becomes:* $\frac{1}{1+y}\cdot\frac{z}{1+z}+\frac{1}{1+z}\cdot\frac{x}{1+x}+\frac{1}{1+x}\cdot\frac{y}{1+y}\ge\frac{3}{4}$, *or* $\frac{x(1+y)+y(1+z)+z(1+x)}{(1+x)(1+y)(1+z)}\ge\frac{3}{4}$, *or*

$$4(x+y+z+xy+yz+zx)\ge3(1+x)(1+y)(1+z),\ or\ x+y+z+xy+yz+zx\ge6$$

The last inequality is true by $x+y+z\ge3\sqrt[3]{xyz}=3$, *and*

$xy+yz+zx\ge3\sqrt[3]{x^2y^2z^2}=3$. *The proof is complete and we are done.*

UP.066. Evaluate:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left(\frac{1}{n^3(2k-1)}\right)$$

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Mohammed Hijazi-Amman-Jordania

Find $\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{1}{k^3(2n-1)}$

$$=\sum_{k=1}^{\infty}\frac{1}{k^3}\left(\sum_{n=1}^{2k}\frac{1}{n}-\sum_{n=1}^{k}\frac{1}{2n}\right)=\sum_{k=1}^{\infty}\frac{1}{k^3}\left(H_{2k}-\frac{1}{2}H_k\right)$$

$$=8\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k)^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^3}=8\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k)^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^3}=8\sum_{k=1}^{\infty}\frac{H_k}{k^3}\cdot\frac{\left(1+(-1)^k\right)}{2}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^3}$$

$$=\frac{7}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^3}+4\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{k^3}$$

note that the blue one is well – known from (1995)

$$=\frac{7}{2}\cdot\frac{\pi^4}{72}+4\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{k^3} \quad \text{.........} \ (1)$$

now we want to find the red one:

$$\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{k^3}=\sum_{k=1}^{\infty}(-1)^k H_k\frac{1}{2}\int_0^1\ln^2 x\, x^{k-1}\,dx=\frac{1}{2}\int_0^1\ln^2(x)\left[\sum_{k=1}^{\infty}H_k(-x)^k\right]\frac{dx}{x}$$

$$=\frac{1}{2}\int_0^1\ln^2(x)\left[\frac{-\ln(1+x)}{1+x}\right]\frac{dx}{x}=-\frac{1}{2}\int_0^1\frac{\ln^2 x\ln(1+x)}{x(1+x)}dx$$

$$=\frac{1}{2}\int_0^1\frac{\ln^2 x\ln(1+x)}{(1+x)}dx-\frac{1}{2}\int_0^1\frac{\ln^2 x\ln(1+x)}{x}dx$$

$$=\frac{1}{6}\int_0^1\frac{3\ln^2(x)\ln(1+x)-3\ln(x)\ln^2(1+x)}{1+x}dx+\frac{1}{2}\int_0^1\frac{\ln x\ln^2(1+x)}{1+x}dx+\frac{1}{2}\int_0^{-1}Li_2'(-x)\ln^2(-x)\,dx$$

$$\frac{1}{6}\int_0^1\frac{\ln^3(x)}{1+x}dx-\frac{1}{6}\int_0^1\frac{\ln^3(1+x)}{1+x}dx-\frac{1}{6}\int_0^1\ln^3\left(\frac{x}{1+x}\right)\frac{dx}{1+x}-\frac{1}{6}\int_0^1\frac{\ln^3(1+x)}{x}dx-$$

$$-\int_0^{-1}Li_3'(x)\ln(-x)\,dx$$

$$=\frac{-1}{6}\int_0^{-1}\frac{\ln^3(-x)}{1-x}dx-\frac{1}{24}\ln^4(2)-\frac{1}{6}\int_0^{\frac{1}{2}}\frac{\ln^3(x)}{1-x}dx+\frac{1}{6}\int_1^2\frac{\ln^3(x)}{1-x}dx+\int_0^{-1}Li_4(x)\,dx\ldots\ (B)$$

Now using I.B.P 3 times for $\int\frac{\ln^3(\pm x)}{1-x}dx$ ***we will get***

$$\int\frac{\ln^3(\pm x)}{1-x}dx=-\ln(1-x)\ln^3(\pm x)-3Li_2(x)\ln^2(\pm x)+6Li_3(x)\ln(\pm x)-6Li_4(x)\ldots\ (A)$$

from (A) and (B) we will have

$$\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{k^3}=\frac{11}{360}\pi^4-\frac{\pi^2\ln^2(2)}{12}+\frac{\ln^4(2)}{12}+2Li_4\left(\frac{1}{2}\right)+\frac{7}{4}\ln(2)\zeta(3)\quad \text{.... (2)}$$

so from (1) and (2) we have:

$$\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{1}{k^3(2n-1)}=\frac{-53\pi^4}{720}-\frac{1}{3}\ln^2(2)\pi^2+\frac{1}{3}\ln^4(2)+\frac{1}{8}Li_4\left(\frac{1}{2}\right)+7\ln(2)\zeta(3)$$

Solution 2 by Khalef Ahmad El Ruhemi-Jarash-Jordania

$$I:=\sum_{n=1}^{\infty}\sum_{k=1}^{k=n}\frac{1}{n^3(2k-1)}$$

Define $I(x):=\sum_{n=1}^{\infty}\sum_{k=1}^{k=n}\frac{x^{2k-1}}{n^3(2k-1)}\Rightarrow I'(x)=\sum_{n=1}^{\infty}\sum_{k=1}^{k=n}\frac{x^{2k-2}}{n^3}=\sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\sum_{k=1}^{k=n}x^{2k-2}$

$$=\sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\left(\frac{1-x^{2n}}{1-x^2}\right)=\left(\frac{1}{1-x^2}\right)\cdot\sum_{n=1}^{\infty}\left(\frac{1-x^{2n}}{n^3}\right)$$

$$=\left(\frac{1}{1-x^2}\right)\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n^3}-\sum_{n=1}^{\infty}\frac{x^{2n}}{n^3}\right]=\frac{1}{(1-x^2)}[Li_3(1)-Li_3(x^2)]$$

Since $I(0)=0$*, then* $I=I(1)=\int_0^1 I'(x)\,dx$

$$\therefore I=\int_0^1\frac{\left(Li_3(1)-Li_3(x^2)\right)}{(1-x^2)}dx\quad \textit{integrate by parts}$$

$$=-\frac{1}{2}\left(Li_3(1)-Li_3(x^2)\right)\ln\left(\frac{1-x}{1+x}\right)\Big|_0^1+\frac{1}{2}\cdot\int_0^1\left(\ln\left(\frac{1-x}{1+x}\right)\right)x-\frac{Li_2(x^2)}{x^2}x\,2xdx$$

$$=\int_0^1\frac{\ln(1+x)}{x}\cdot Li_2(x^2)dx-\int_0^1\frac{\ln(1-x)}{x}\cdot Li_2(x^2)dx$$

$$=\int_0^1\frac{2\ln(1+x)}{x}\cdot\left(Li_2(x)+Li_2(-x)\right)dx-\int_0^1\frac{2\ln(1-x)}{x}\left(Li_2(x)+Li_2(-x)\right)dx$$

$$=2\cdot\int_0^1 Li_2(-x)\frac{\ln(1+x)}{x}dx-2\cdot\int_0^1\frac{\ln(1-x)\,x}{x}\cdot Li_2(x)dx+$$

$$+2\cdot\int_0^1\frac{\ln(1+x)}{x}Li_2(x)dx-2\cdot\int_0^1\frac{\ln(1-x)}{x}\cdot Li_2(-x)dx=I$$

$$\therefore I=-\left(Li_2(-x)\right)^2\Big|_0^1+\left(Li_2(x)\right)^2\Big|_0^1$$

$$+2\cdot\int_0^1\frac{\ln(1+x)}{x}\cdot Li_2(x)dx-2\int_0^1\frac{\ln(1-x)}{x}Li_2(x)dx$$ ***integrate by parts***

$$=\left(Li_2(1)\right)^2-\left(Li_2(-1)\right)^2+2\cdot\int_0^1\frac{\ln(1+x)}{x}\cdot Li_2(x)dx$$

$$+2\left(Li_2(x)Li_2(-x)|_0^1-\int_0^1 Li_2(x)\times\frac{\ln(1+x)}{-x}dx\right)$$

$$=(Li_2(1))^2-\left(Li_2(-1)\right)^2+2Li_2(1)Li_2(-1)+4\int_0^1\frac{\ln(1+x)Li_2(x)}{x}\cdot dx=I\quad (*)$$

But $\frac{\ln(1+x)}{x}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cdot x^{k-1}}{k}$

$$\Rightarrow\int_0^1\frac{\ln(1+x)Li_2(x)}{x}\cdot dx=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cdot\left(\int_0^1 x^{k-1}\cdot Li_2(x)dx\right)$$ ***integrate by parts***

$$\int_0^1 x^{k-1}\cdot Li_2(x)\,dx=\frac{x^k}{k}Li_2(x)\Big|_0^1-\int_0^1\frac{x^k}{k}\times-\frac{\ln(1-x)}{x}dx$$

$$=\frac{Li_2(1)}{k}+\frac{1}{k}\cdot\int_0^1 x^{k-1}\ln(1-x)\,dx=\frac{Li_2(1)}{k}-\frac{H_k}{k^2}$$

$$\therefore\int_0^1\frac{\ln(1+x)\,Li_2(x)}{x}\cdot dx=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cdot\left(\frac{Li_2(1)}{k}-\frac{H_k}{k^2}\right)$$

$$\therefore\int_0^1\frac{\ln(1+x)\,Li_2(x)}{x}dx=Li_2(1)\cdot\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cdot H_k}{k^3}$$

$$=\varphi(2)\eta(2)-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_k}{k^3}$$

$$=\left(\frac{\pi^2}{6}\right)\left(1-\frac{1}{2}\right)\left(\frac{\pi^2}{6}\right)+\left(-\frac{11\pi^4}{360}+\frac{1}{12}\ln^4(2)-\frac{\pi^2\ln^2(2)}{12}+2Li_4\left(\frac{1}{2}\right)+\frac{7}{4}\ln(2)\,\varphi(3)\right)$$

$$=\frac{\pi^4}{72}-\frac{11\pi^4}{360}+\frac{1}{12}\ln^4(2)-\frac{\pi^2\cdot\ln^2(2)}{12}+2Li_4\left(\frac{1}{2}\right)+\frac{7}{4}\ln(2)\,\varphi(3)$$

going to (*), and using $Li_2(-1)=-\frac{1}{2}\varphi(2)=-\frac{\pi^2}{12}$

$$I=\frac{\pi^4}{36}-\frac{\pi^4}{144}-\frac{\pi^4}{36}+\frac{\pi^4}{18}-\frac{11\pi^4}{90}+\frac{1}{3}\ln^4(2)-\frac{\pi^2\cdot\ln^2(2)}{3}$$

$$+8Li_4\left(\frac{1}{2}\right)+7\ln(2)\,\varphi(3)$$

$$=-\frac{53\pi^4}{720}-\frac{1}{3}\pi^2\cdot\ln^2(2)+\frac{1}{3}\ln^4(2)+7\ln(2)\,\varphi(3)+8Li_4\left(\frac{1}{2}\right)$$

$$\therefore\sum_{n=1}^{\infty}\sum_{k=1}^{k=n}\frac{1}{n^3(2k-1)}=-\frac{53}{720}\pi^4-\frac{1}{3}\pi^2\cdot\ln^2(2)+\frac{1}{3}\ln^4(2)$$

$$+7\ln(2)\,\varphi(3)+9Li_4\left(\frac{1}{2}\right)$$

UP.067. Evaluate:

$$S=\sum_{n=1}^{\infty}H_n\left[\zeta(8)-\frac{1}{1^8}-\frac{1}{2^8}-\frac{1}{3^8}\cdot\ldots\cdot\frac{1}{n^8}\right]$$

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Mohammed Hijazi-Amman-Iordania

Using summation by parts for

$$\sum_{n=1}^{N}a_nb_n=A_Nb_N-\sum_{n=1}^{N-1}(b_{n+1}-b_n)$$

where $A_n=a_1+a_2+\cdots+a_n$ ***in this problem with*** $a_n=H_n$ ***and*** $b_n=\zeta(8)-H_n^{(8)}$

$$S=\sum_{n=1}^{\infty}H_n\left(\zeta(8)-H_n^{(8)}\right)=\sum_{n=1}^{\infty}[(n+1)H_{n+1}-(n+1)]\frac{1}{(n+1)^8}$$

$$\text{so }S=-\zeta(7)+\sum_{n=1}^{\infty}\frac{H_n}{n^7}=\frac{\pi^8}{4200}-\zeta(3)\zeta(5)-\zeta(7)$$

Solution 2 by Shivam Sharma – New Delhi – India

Applying Abel's summation, with $a_n=H_n, b_n=\left(\zeta(8)-\frac{1}{1^8}-\frac{1}{2^8}-\frac{1}{3^8}-\cdots-\frac{1}{n^8}\right)$

$$S=\lim_{n\to\infty}H_n\left[\zeta(8)-\frac{1}{1^8}-\frac{1}{2^8}-\frac{1}{3^8}-\cdots-\frac{1}{(n+1)^8}\right]+\sum_{n=1}^{\infty}\left[\frac{H_{n+1}}{(n+1)^7}-\frac{1}{(n+1)^7}\right]$$

$$\Rightarrow 0+\sum_{n=1}^{\infty}\left[\frac{H_{n+1}}{(n+1)^7}-\frac{1}{(n+1)^7}\right]\Rightarrow\sum_{n=1}^{\infty}\left[\frac{H_{n+1}}{(n+1)^7}-\frac{1}{(n+1)^7}\right]\Rightarrow\sum_{k=2}^{\infty}\left[\frac{H_k}{k^7}-\frac{1}{k^7}\right]$$

$$\Rightarrow\sum_{k=1}^{\infty}\frac{H_k}{k^7}-\sum_{k=1}^{\infty}\frac{1}{k^7}\dots\quad(1)$$

As we know,

$$\sum_{k=1}^{\infty}\frac{H_k}{k^m}=\frac{1}{2}\left[(m+2)\zeta(m+1)-\sum_{n=1}^{m-2}\{(\zeta(m-n))(\zeta(n+1))\}\right]$$

if $m=7$***, we get***

$$\sum_{n=1}^{\infty}\frac{H_k}{k^7}=\frac{\pi^8}{4200}-\zeta(3)\zeta(5)$$

Now put this result in equation (1), we get, $S=\frac{\pi^8}{4200}-\zeta(3)\zeta(5)-\zeta(7)$

UP.068. Prove that if $a,b,c\in\mathbb{R}^*$ then:

$$(abc-ab-bc-ca)^2\leq 4(1+a^2)(1+b^2)(1+c^2)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow 4\sum a^2+4+3\sum a^2b^2+3a^2b^2c^2+2abc(\sum ab)\overset{(1)}{\geq}2abc(\sum a)$

Now, $\forall m,n,p\in\mathbb{R}^*,\sum m^2-\sum mn=\frac{1}{2}[(m-n)^2+(n-p)^2+(p-m)^2]\geq 0$

$$\therefore 2\sum a^2b^2\overset{(2)}{\geq}2abc\left(\sum a\right)$$

(1), (2) $\Rightarrow$ ***it suffices to prove:*** $4\sum a^2+4+\sum a^2b^2+3a^2b^2c^2+2abc(\sum ab)\geq 0\Leftrightarrow$

$$\Leftrightarrow 3x^2+\left(2\sum ab\right)x+\left(\sum a^2b^2+4\sum a^2+4\right)\geq 0\ \ (x=abc)$$

Let $f(x)=3x^2+(2\sum ab)x+(\sum a^2b^2+4\sum a^2+4)$***, which is a quadratic in*** x ***as***

$$x=abc\neq 0\ \ (\because a,b,c\in\mathbb{R}^*)$$

Discriminant Δ ***of*** $f(x)=4(\sum ab)^2-4\cdot 3(\sum a^2b^2+4\sum a^2+4)=$

$$=4\left\{\sum a^2b^2+2abc\left(\sum a\right)-3\sum a^2b^2-12\left(\sum a^2+1\right)\right\}=$$

$$= 4[2\{abc(\sum a) - \sum a^2b^2\} - 12(\sum a^2 + 1)] < 0 \ \textit{(using (2) \&} \because 12(\sum a^2 + 1) > 0\textit{)}$$

$\therefore f(x) > 0$ ***($\because f(x)$ never touches x – axis as roots of $f(x) = 0$ are imaginary)***

$\therefore f(x) \geq 0$ ***(Done)***

UP.069. Prove that if $n \in \mathbb{N}^*$; $a > 1$ then:

$$(n + a - 1)(a - 1)^{n-1} \leq a^n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

As $a > 1, 0 < \frac{1}{a} < 1 \Rightarrow 0 < 1 - \frac{1}{a} < 1$. ***Now,*** $\frac{1-\left(1-\frac{1}{a}\right)^n}{\frac{1}{a}} = \frac{1-\left(1-\frac{1}{a}\right)^n}{1-\left(1-\frac{1}{a}\right)}$

$$= 1 + \left(1 - \frac{1}{a}\right) + \left(1 - \frac{1}{a}\right)^2 + \cdots + \left(1 - \frac{1}{a}\right)^{n-1}$$

$$\geq \underbrace{\left(1 - \frac{1}{a}\right)^{n-1} + \left(1 - \frac{1}{a}\right)^{n-1} + \cdots + \left(1 - \frac{1}{a}\right)^{n-1}}_{"n"\text{ times}}$$

$$\Rightarrow 1 - \left(1 - \frac{1}{a}\right)^n \geq \frac{n}{a}\left(1 - \frac{1}{a}\right)^{n-1} \Rightarrow \left(1 - \frac{1}{a}\right)^n + \frac{n}{a}\left(1 - \frac{1}{a}\right)^{n-1} \leq 1$$

$$\Rightarrow \left(1 - \frac{1}{a} + \frac{n}{a}\right)\left(1 - \frac{1}{a}\right)^{n-1} \leq 1 \Rightarrow (a - 1 + n)(a - 1)^{n-1} \leq a^n$$

Solution 2 by SK Rejuan-West Bengal-India

$$n \in \mathbb{N}^*, a > 1,$$

Let us consider two positive terms $(a - 1)$ ***and*** $(n + a - 1)$ ***with the associated weights*** $(n - 1)$ ***and*** 1 ***respectively. Applying weighted AM ≥ weighted GM***

$$\frac{(a - 1)(n - 1) + (n + a - 1) \cdot 1}{(n - 1) + 1} \geq \{(a - 1)^{n-1} \cdot (n + a - 1)^1\}^{\frac{1}{n-1+1}}$$

$$\Rightarrow \left(\frac{an - a - n + 1 + n + a - 1}{n}\right) \geq \{(a - 1)^{n-1}(n + a - 1)\}^{\frac{1}{n}}$$

$$\Rightarrow \left(\frac{an}{n}\right)^n \geq (n + a - 1)(a - 1)^{(n-1)} \Rightarrow a^n \geq (n + a - 1)(a - 1)^{n-1}$$

$$\Rightarrow (n + a - 1)(a - 1)^{n-1} \leq a^n$$

[Proved]

Solution 3 by Nguyen Phuc Tang-Hanoi-Vietnam

If $n = 1$ we have equality hold. If $n > 1$, we have

$$RHS - LHS = [(a-1)+1]^n - n(a-1)^{n-1} - (a-1)^n = \sum_{k=2}^{n} C_n^k (a-1)^{n-k} > 0$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = x^n$ for all $x \in [a-1, a]$ where $n \in \mathbb{N}^*$

Applying Lagrange's Mean Value Theorem,

$$\frac{a^n-(a-1)^n}{a-(a-1)} = n\xi^{n-1} \text{ where } \xi \in [a-1, a] \text{ i.e. } a-1 \le \xi \le a$$

$$\Rightarrow a^n - (a-1)^n \ge n(a-1)^{n-1} \Rightarrow a^n \ge (a-1)^n + n(a-1)^{n-1}$$

$$\Rightarrow a^n \ge (a-1)^{n-1}(a-1+n) \quad \text{(proved)}$$

UP.070. If $a, b \in \mathbb{R}; a < b; f, g: [a, b] \to \mathbb{R}$ are continuous functions such that: $f(a+b-x)=f(x)$, $g(a+b-x)g(x)=1$, $\forall x\in[a,b]$ then:

$$\int_a^b \frac{f(x)\cdot g(x)}{1+g(x)}dx = \frac{1}{2}\int_a^b f(x)\,dx$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution1 by Ravi Prakash-New Delhi-India

Let

$$I = \int_a^b \frac{f(x)g(x)}{1+g(x)}dx \qquad (1)$$

$$= \int_a^b \frac{f(a+b-x)g(a+b-x)}{1+g(a+b-x)}dx = \int_a^b \frac{f(x)\left(\frac{1}{g(x)}\right)}{1+\frac{1}{g(x)}}dx$$

$$I = \int_a^b \frac{f(x)}{g(x)+1}dx \qquad (2)$$

Adding (1) and (2) we get

$$2I = \int_a^b \frac{f(x)(g(x)+1)}{g(x)+1}dx = \int_a^b f(x)\,dx \Rightarrow I = \frac{1}{2}\int_a^b f(x)dx$$

Solution 2 by SK Rejuan-West Bengal-India

$$a, b \in \mathbb{R} \text{ and } a < b$$

$$f, g: [a, b] \to \mathbb{R} \text{ are continuous functions}$$

$$f(a+b-n) = f(n), g(a+b-n)g(n) = 1 \quad \forall n \in [a, b] \Rightarrow g(n) \neq 0, \ \ \forall n \in [a, b]$$

Let,

$$I = \int_a^b \frac{f(n)\cdot g(n)}{1+g(n)}dn = \int_a^b \frac{f(a+b-n)g(a+b-n)}{1+g(a+b-n)}dn$$

$$= \int_a^b \frac{f(n)g(a+b-n)}{1+g(a+b-n)}dn \quad \text{[as, } f(a+b-x) = f(n)\text{]}$$

$$= \int_a^b \frac{\{f(n)g(a+b-n)\}\cdot g(n)}{\{1+g(a+b-n)\}g(n)}dn = \int_a^b \frac{f(n)\{g(a+b-n)\cdot g(n)\}}{g(n)+g(a+b-n)g(n)}dn$$

$$= \int_a^b \frac{f(n)}{g(n)+1}dn \quad (1)$$

$$I = \int_a^b \frac{f(n)}{1+g(n)}dn \quad \text{[from (1)]}$$

$$= \int_a^b \frac{f(n)+f(n)f(n)-f(n)g(n)}{1+g(n)}dx = \int_a^b \frac{f(n)\{1+g(n)\}-f(n)g(n)}{1+g(n)}dn$$

$$= \int_a^b f(n)dn - \int_a^b \frac{f(n)g(n)}{1+g(n)}dn = \int_a^b f(n)\,dn - I \quad \text{[assem...]}$$

$$\Rightarrow 2I = \int_a^b f(n)\,dn \Rightarrow I = \frac{1}{2}\int_a^b f(n)\,dn \Rightarrow \int_a^b \frac{f(n)g(n)}{1+g(n)}dn = \frac{1}{2}\int_a^b f(n)\,dn$$

[proved]

Solution 3 by Shivam Sharma-New Delhi-India

As we know, the following lemma,

If $f(x)$ is a continuous functions defined on $[a, b]$; then,

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Using this, we get,

$$I \Rightarrow \int_a^b \frac{f(a+b-x)g(a+b-x)}{1+g(a+b-x)}dx$$

Given $\Rightarrow f(a+b-x) = f(x)$

$$g(a+b-x) = \frac{1}{g(x)}$$

Using this and putting these values, we get,

$$\Rightarrow \int_a^b \frac{f(x)\cdot\left(\frac{1}{g(x)}\right)}{1+\left(\frac{1}{g(x)}\right)}dx \Rightarrow \int_a^b \frac{f(x)}{1+g(x)}dx$$

Then, $2I = \int_a^b \frac{f(x)}{1+g(x)}[1+g(x)]dx$

$$2I = \int_a^b f(x)\,dx$$

Then, $I = \frac{1}{2}\int_a^b f(x)\,dx$ ***(Proved)***

UP.071. Evaluate:

$$\int_0^1 \ln\left[\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\right]\frac{x\,dx}{1-x^2}$$

Proposed by Shivam Sharma-New Delhi-India

Solution 1 by Khalef Ruhemi-Jarash-Iordania

$$I := \int_0^1 \ln\left[\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\right]\cdot\frac{x\,dx}{1-x^2}$$

Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$

$$\therefore I = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \ln\left(\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}\right)^2\cdot\frac{\sin\theta\cos\theta}{\cos^2\theta}d\theta = \int_0^{\frac{\pi}{2}} \ln\left(\frac{\tan\theta+1}{\tan\theta-1}\right)^2\cdot\tan(\theta)\,d\theta$$

let $\tan\theta = x; \theta = \tan^{-1}(x)$; $d\theta = \frac{dx}{1+x^2}$

$$\therefore I=\int_0^\infty \frac{x}{1+x^2}\cdot\ln\left(\frac{1+x}{1-x}\right)^2\cdot dx=\int_0^\infty \frac{2x}{1+x^2}\cdot\ln\left|\frac{1+x}{1-x}\right|\cdot dx,\ \textit{let } x=\frac{1}{y}, dx=-\frac{dy}{y^2}$$

$$\therefore I=\int_0^\infty \frac{\frac{2}{x}}{\left(1+\frac{1}{x^2}\right)x^2}\ln\left|\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right|\cdot dx=\int_0^\infty \frac{2}{x(1+x^2)},\ln\left|\frac{1+x}{1-x}\right|dx$$

$$=\int_0^\infty 2\ln\left|\frac{1+x}{1-x}\right|\cdot\left(\frac{1}{x}-\frac{x}{1+x^2}\right)\cdot dx$$

$$=\int_0^\infty \frac{2\ln\left|\frac{1+x}{x}\right|}{x}\cdot dx-\int_0^\infty \frac{2x}{1+x^2}\ln\left|\frac{1+x}{1-x}\right|dx$$

$$\therefore I=2\int_0^\infty \frac{\ln\left|\frac{1+x}{1-x}\right|dx}{x}-I\Rightarrow I=\int_0^\infty \underbrace{\frac{\ln\left|\frac{1+x}{1-x}\right|dx}{x}}_{\textit{integrating by parts}}\qquad (*)$$

$$I=\lim_{a\to 0^+}\ln\left|\frac{1+x}{1-x}\right|\cdot\ln x\Big|_a^{\frac{1}{a}}-\int_0^\infty(\ln x)\cdot\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$=\lim_{a\to 0^+}\left(\ln\left(\frac{\left(1+\frac{1}{a}\right)}{\left(1-\frac{1}{a}\right)}\right)\right)\ln\left(\frac{1}{a}\right)-\ln\left(\left|\frac{1+a}{1-a}\right|\ln(a)\right)$$

$$-\int_0^\infty \ln x\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx.$$

$$=\lim_{a\to 0^+}-2\ln\left|\frac{1+a}{1-a}\right|\cdot\ln(\theta)-\int_0^\infty \ln x\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

But $\lim_{a\to 0^+}-2\ln(a)\ln\left|\frac{1+a}{1-a}\right|=-2\lim_{a\to 0^+}(\ln(a)\ln(1+a)-\ln(a)\ln(1-a))$

$$=-2(0-0)=0$$

$$\therefore I=-\int_0^\infty \ln(x)\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$\therefore -I = \int_0^1 (\ln x)\cdot\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx + \int_1^\infty \ln(x)\underbrace{\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx}_{\substack{\downarrow \\ let\ x=\frac{1}{y},dx=-\frac{dy}{yz}}}$$

$$\therefore -I = \int_0^1 \frac{\ln x}{1+x}dx + \int_0^1 \frac{\ln x}{1-x}dx + \int_0^1 -\ln(x)\cdot\left(\frac{1}{1+x}-\frac{1}{1-x}\right)\cdot\frac{dx}{x}$$

$$-I = \int_0^1 \frac{\ln x\,dx}{1+x} + \int_0^1 \frac{\ln x\,dx}{1-x} + \int_0^1 \ln x\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$-I = \int_0^1 \frac{\ln x\,dx}{1+x} + \int_0^1 \frac{\ln x\,dx}{1-x} + \int_0^1 \frac{\ln x}{1+x}dx + \int_0^1 \frac{\ln x\,dx}{1-x}$$

$$\therefore -I = 2\cdot\int_0^1 \frac{\ln x dx}{1+x} + 2\int_0^1 \frac{\ln x dx}{1-x} \quad \textit{integrating by parts}$$

$$\Rightarrow -\frac{1}{2}I = \ln x\ln(1+x)\Big|_0^1 - \int_0^1 \frac{\ln(1+x)}{x}dx$$

$$-\ln x\ln(1-x)\Big|_0^1 + \int_0^1 \frac{\ln(1-x)}{x}dx$$

$$\therefore -\frac{1}{2}I = -Li_2(x)\Big|_0^1 + Li_2(-x)\Big|_0^1$$

$$= -Li_2(1) + Li_2(-1) = \varphi(2) - \eta(2)$$

$$= -\frac{\pi^2}{6} - \frac{1}{2}\left(\frac{\pi^2}{6}\right) = -\frac{\pi^2}{6} - \frac{\pi^2}{12} = -\frac{3\pi^2}{12} = -\frac{\pi^2}{4}$$

$$\therefore -\frac{1}{2}I = -\frac{\pi^2}{4} \Rightarrow I = \frac{\pi^2}{2},\ \textit{(Note: } \eta(2) = \frac{1}{2}\varphi(2) = \frac{\pi^2}{12}\textit{)}$$

$$\therefore \int_0^1 \ln\left[\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\right]\cdot\frac{x\cdot dx}{1-x^2} = \frac{\pi^2}{2}$$

Solution 2 by Mohammed Hijazi-Amman-Iordania

Let $\tanh u = x$. ***So***

$$I=\int_0^\infty \ln\left[\left(\frac{\tanh u+\operatorname{sech} u}{\tanh u-\operatorname{sech} u}\right)^2\right]\tanh u\,du$$

$$I=2\int_0^\infty \ln\left[\left(\frac{\sinh u+1}{\sinh u-1}\right)\right]\frac{\sinh u}{\cosh u}du$$

Let $\sinh u=t$

$$I=2\int_0^\infty \ln\left|\frac{1+t}{t-1}\right|\frac{t}{1+t^2}du \quad \text{let } t=\frac{1}{y}$$

$$2I=2\int_0^\infty \frac{\ln\left|\frac{1+y}{1-y}\right|}{y}dy$$

hence $I=\int_0^1\frac{\ln\left|\frac{1+y}{1-y}\right|}{y}dy+\int_0^\infty\frac{\ln\left|\frac{1+y}{1-y}\right|}{y}dy$ *for the red one let* $y=\frac{1}{m}$

so $I=\int_0^1\frac{\ln\left|\frac{1+y}{1-y}\right|}{y}dy+\int_0^1\frac{\ln\left|\frac{1+\frac{1}{m}}{1-\frac{1}{m}}\right|}{\frac{1}{m}}\frac{1}{m^2}dm$

$I=\int_0^1\frac{\ln\left|\frac{1+y}{1-y}\right|}{y}dy+\int_0^1\frac{\ln\left|\left(\frac{m+1}{m-1}\right)\right|}{m}dm$ *the integral in the red is same as the blue one using the power series:*

$$I=2\int_0^1 2\sum_{n=0}^{n=\infty}\frac{y^{2n}}{(2n+1)}dy=4\sum_{n=0}^{n=\infty}\frac{1}{(2n+1)^2}=4\frac{3}{4}\xi(2)=\frac{\pi^2}{2}$$

Solution 3 by Atteiah Yahya Ahmed Atiya Al Zahrani-Jeddah-Kingdom Of Saudi Arabia

$$I=\int_0^1 \ln\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\cdot\frac{x}{1-x^2}dx=$$

$$=-\frac{1}{2}\cdot\ln(1-x^2)\cdot\ln\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\Bigg|_{x=0}^{x\to\infty}-2\cdot PV\int_0^1\frac{\ln(1-x^2)}{(2\cdot x^2-1)\sqrt{1-x^2}}dx$$

$$I=-2\cdot PV\int_0^1\frac{\ln(1-x^2)}{(2x^2-1)\sqrt{1-x^2}}dx$$

$$x=\frac{y}{\sqrt{y^2+1}}\quad dx=\frac{1}{(y^2+1)\sqrt{y^2+1}}\quad y=\frac{x}{\sqrt{1-x^2}}\quad x:(0,1)\quad y:(0,\infty)$$

$$I=2\cdot PV\int_0^{\infty}\frac{\ln(y^2+1)}{y^2-1}dy=-2\cdot PV\int_0^1\int_0^{\infty}\frac{\partial}{\partial\alpha}\frac{\ln(\alpha(y^2-1)+2)}{y^2-1}dy\,d\alpha$$

$$I=\int_0^1\int_0^{\infty}\frac{2}{\alpha(y^2-1)+2}dy\,d\alpha=\int_0^1\frac{1}{\alpha}\int_0^{\infty}\frac{2}{y^2+\left(\sqrt{\frac{2}{\alpha}-1}\right)^2}dy\,d\alpha$$

$$I=\pi\cdot\int_0^1\frac{1}{\sqrt{\alpha\cdot(2-\alpha)}}d\alpha=2\cdot\pi\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-\beta^2}}d\beta=2\cdot\pi\cdot\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi^2}{2}$$

UP.072. If $x,y,z>0$ and $b\ge a>0$ then:

$$\frac{x}{z}\ln\frac{x^2+bz}{x^2+az}+\frac{y}{x}\ln\frac{y^2+bx}{y^2+ax}+\frac{z}{y}\ln\frac{z^2+by}{z^2+ay}\le\frac{3}{4}\ln\frac{b}{a}+\frac{b-a}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

For $x,z,t>0$ we have: $\frac{x}{x^2+tz}\le\frac{1}{4}\left(\frac{1}{t}+\frac{1}{z}\right)\Leftrightarrow t(x-z)^2+z(x-t)^2\ge 0$

therefore $\int_a^b\frac{xdt}{x^2+tz}\le\frac{1}{4}\int_a^b\left(\frac{1}{t}+\frac{1}{z}\right)dt\Rightarrow\frac{x}{z}\ln\frac{x^2+bz}{x^2+az}\le\frac{1}{4}\ln\frac{b}{a}+\frac{b-a}{4z}\Rightarrow$

$$\Rightarrow\sum_{cyclic}\frac{x}{z}\ln\frac{x^2+bz}{x^2+az}\le\sum_{cyclic}\left(\frac{1}{4}\ln\frac{b}{a}+\frac{b-a}{4z}\right)=\frac{3}{4}\ln\frac{b}{a}+\frac{b-a}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

UP.073. Let be $a,b,c\in\mathbb{C}$. Solve the following equation:

$$x^3-(a+b+c)x^2+(ab+bc+ca-1)x+b-abc=0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$x^3-(a+b+c)x+(ab+bc+ca-1)x+b-abc=0\quad \textbf{(1)}$$

$$\Rightarrow x^3-(a+b+c)x+(ab+bc+ca)x-abc=x-b$$

$$\Rightarrow (x-a)(x-b)(x-c) = x-b \Rightarrow x-b = 0 \text{ or } (x-a)(x-c) = 1$$

$$\Rightarrow x = b \text{ or } x^2 - (a+c)x + ac - 1 = 0. \text{ Now, } x^2 - (a+c)x + ac - 1 = 0$$

$$D = (a+c)^2 - 4(ac-1) = (a-c)^2 + 4 \therefore x = \frac{(a+c) \pm \sqrt{(a-c)^2+4}}{2}$$

Thus, roots of (1) has $b, \frac{1}{2}\left[(a+c) \pm \sqrt{(a-c)^2+4}\right]$

Solution 2 by SK Rejuan-West Bengal-India

$a, b, c \in C$. *The given equation is,*

$$x^3 - (a+b+c)x^2 + (ab+bc+ca-1)x + b - abc = 0$$

$$\Rightarrow x^3 - bx^2 - (a+c)^2x^2 + b(a+c)x + (ac-1)x + b - abc = 0$$

$$\Rightarrow x^2(x-b) - (a+c)x(x-b) + (ac-1)(x-b) = 0$$

$$\Rightarrow (x-b)\{x^2 - (a+c)x + (ac-1)\} = 0$$

Either $x = b$

Or, $x = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-1)}}{2} = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4ac + 4}}{2} = \frac{(a+c)^2 \pm \sqrt{(a-c)^2+4}}{2}$

$\therefore$ *Solutions of the given equation* $x = b, \frac{(a+c) \pm \sqrt{(a-c)^2+4}}{2}$

UP.074. Let $(a_n)_{n\geq 1}$ be positive real sequences with $\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in R_+^*$ and $f \in R[x], f(x) \in R_+^*, \forall x \in R_+^*$; $u, v \in R$ such that $u + v = 1$. Find the following limit

$$\lim_{n\to\infty}\left((n+1)^u \sqrt[n+1]{\left(a_{n+1}f(n+1)\right)^v} - n^u \sqrt[n]{\left(a_n f(n)\right)^v}\right).$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+, u + v = 1$ *then*

$$\lim_{n\to\infty}\left((n+1)^u \sqrt[n+1]{\left(a_{n+1}f(n+1)\right)^v} - n^u \sqrt[n]{\left(a_n f(n)\right)^v}\right)$$

$$= \lim_{n\to\infty}\left(\sqrt[n]{\left(\frac{a_n f(n)}{n^n}\right)^v} \cdot \frac{w_n - 1}{\ln w_n} \cdot \ln w_n^n\right) \text{ where } w_n = \left(1 + \frac{1}{n}\right)^u \frac{\sqrt[n+1]{\left(a_{n+1}f(n+1)\right)^v}}{\sqrt[n]{\left(a_n f(n)\right)^v}}$$

$$\textit{now, } \lim_{n\to\infty} \sqrt[n]{\left(\frac{a_n f(n)}{n^n}\right)^v} \overset{D'ALEMBERT}{=} \lim_{n\to\infty}\left(\frac{a_{n+1}}{na_n}\cdot\frac{\frac{f(n+1)}{f(n)}}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{n}{n+1}\right)^v = \left(\frac{a}{e}\right)^v$$

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{\left(\frac{a_{n+1}f(n+1)}{(n+1)^{n+1}}\right)^v}}{\sqrt[n]{\left(\frac{a_n f(n)}{n^n}\right)^v}} = 1, \textit{ so, } \lim_{n\to\infty}\frac{w_n-1}{\ln w_n} = 1$$

$$\lim_{n\to\infty} w_n^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nu}\cdot\left(\frac{a_{n+1}}{na_n}\cdot\frac{f(n+1)}{f(n)}\cdot\frac{1}{\sqrt[n+1]{\left(\frac{a_{n+1}f(n+1)}{(n+1)^{n+1}}\right)^v}}\cdot\frac{n}{n+1}\right)^v = e$$

$$\therefore \lim_{n\to\infty}\left((n+1)^u\sqrt[n+1]{\left(a_{n+1}f(n+1)\right)^v} - n^u\sqrt[n]{\left(a_n f(n)\right)^v}\right)$$

$$= \left(\frac{a}{e}\right)^v\cdot 1\cdot\ln e = \left(\frac{a}{e}\right)^v \textit{ (Proved)}$$

UP.075. Let $(a_n)_{n\ge1}, (b_n)_{n\ge1}$, be real positive sequences with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a \in R_+^*$. If $\lim_{n\to\infty}(n(a_n - a)) = b \in R$ and $\lim_{n\to\infty}(n(b_n - a)) = c \in R$, evaluate

$$\lim_{n\to\infty}\left(a_{n+1}\sqrt[n+1]{(n+1)!} - b_n\sqrt[n]{n!}\right).$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{D'ALEMBERT}{=} \lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\right) = \frac{1}{e}$$

$$\textit{we know, } \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \frac{1}{e}$$

$$\lim_{n\to\infty}\left(a_{n+1}\sqrt[n+1]{(n+1)!} - b_n\sqrt[n]{n!}\right)$$

$$= \lim_{n\to\infty}\left((a_{n+1}-a)\sqrt[n+1]{(n+1)!} - (b_n - a)\sqrt[n]{n!} + a\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)\right)$$

$$= \lim_{n\to\infty}\left(n(a_{n+1}-a_n)\frac{\sqrt[n+1]{(n+1)!}}{n+1}\left(1+\frac{1}{n}\right)-n(b_n-a)\frac{\sqrt[n]{n!}}{n}+\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\right)$$

$$= \frac{a+b-c}{e} \text{ (Proved)}$$

Solution 2 by Shivam Sharma-New Delhi-India

Applying Stolz – Cesaro theorem, we get, $\frac{a_{n+1}-a_n}{n+1-n} = a_{n+1}-a_n$. *Similarly,* $b_n = b_n - a$

&, $\frac{(a_{n+1})^{\frac{1}{n+1}}-(a_n)^{\frac{1}{n}}}{n+1-n} = (a_{n+1})^{\frac{1}{n+1}}-(a_n)^{\frac{1}{n}}$. *Using these all, we get,*

$$\Rightarrow \lim_{n\to\infty}\left[\frac{[n(a_{n+1}-a_n)]}{n+1}\cdot\frac{(n+1)}{n}\sqrt[n+1]{(n+1)!}-\frac{[n(b_n-a)]}{n}\sqrt[n]{n!}+\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right]$$

As we know, the Stirling's formula, $n! = \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$. *Using this, we get,*

$$\Rightarrow \lim_{n\to\infty}\left[\begin{array}{c}\frac{[n(a_{n+1}-a_n)]}{n+1}\cdot\frac{(n+1)}{n}\left(\left(\frac{n+1}{e}\right)^{n+1}\sqrt{2\pi(n+1)}\right)^{n+1}-\\ -\frac{[n(b_n-a)]}{n}\left(\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)^{\frac{1}{n}}+\left(\left(\frac{n+1}{e}\right)^{n+1}\sqrt{2\pi(n+1)}\right)^{\frac{1}{n+1}}-\left(\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)^{\frac{1}{n}}\end{array}\right]$$

As we can see many terms are cancelling, so now, applying Ratoin test,

$\frac{n+1}{n}\to 1$, *as* $n\to\infty$ *so, our limit equals. As we use the given data,*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a \quad \text{(I)}$$

&

$$\lim_{n\to\infty}\left(n(a_n-a)\right) = b \quad \text{(II)}$$

&

$$\lim_{n\to\infty}\left(n(b_n-a)\right) = c \quad \text{(III)}$$

Using (I), (II), (III), we get,

$$L = \frac{1}{e}[a+b-c]$$

(Answer)