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SOLUTIONS

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JP.076. Let ABC be an acute triangle. Prove that:

$$(a\cot A)^a(b\cot B)^b(c\cot C)^c \le (2r)^{a+b+c}$$

where $a = BC, b = CA, c = AB$, and r is the inradius.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Anh Tai Tran-Vietnam

$$\pi(a\cdot\cot a)^{\frac{a}{a+b+c}} \le 2\pi$$

$$LHS \le \frac{a^2\cdot\cot a}{\sum a} = \frac{\sum a^2[b^2+c^2-a^2]}{4\sum a\cdot S} \text{ (By Extended AM-GM)}$$

$$\sum a^2(b^2+c^2-a) = (a+b-c)(b+c-a)(a+c-b)\cdot\sum a$$

$$\Rightarrow LHS \le \frac{\pi(a+b-c)}{4S} = \frac{S}{a+b+c} = 2\pi \text{ (Shown)}$$

Solution 2 by Kevin Soto Palacios-Huarmey-Peru

Siendo ABC un triángulo acutángulo. Probar que

$$(a\cot A)^a(b\cot B)^b(c\cot C)^c \le (2r)^{a+b+c}$$

***Como ABC es un triàngulo acutàngulo** $\rightarrow \cot A, \cot B, \cot C > 0$*

Tener en cuenta lo siguiente en un triángulo ABC

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C = \frac{4abc}{8R^3} = \frac{16pRr}{8R^3} = \frac{2pr}{R^2}$$

$$\Leftrightarrow \frac{R^2(\sin 2A + \sin 2B + \sin 2C)}{2p} = r$$

Aplicando la desigualdad ponderada $MA \ge MG$

$$\frac{(a\cot A)a + (b\cot B)b + c(\cot C)c}{a+b+c} \ge \sqrt[a+b+c]{(a\cot A)^a(b\cot B)^b(c\cot C)^c}$$

$$\Leftrightarrow \frac{2R^2(\sin 2A + \sin 2B + \sin 2C)}{2p} \ge \sqrt[a+b+c]{(a\cot A)^a(b\cot B)^b(c\cot C)^c}$$

$$\Leftrightarrow (2r)^{a+b+c} \ge (a\cot A)^a(b\cot B)^b(c\cot C)^c \text{ (LQQD)}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$(a\cot A)^a(b\cot B)^b(c\cot C)^c \le \left[\frac{a(a\cot A) + b(b\cot B) + c(\cot C)}{a+b+c}\right]^{a+b+c}$$

***But** $a(a\cot A) + b(b\cot B) + c(\cot C)$*

$$= 2R^2(2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C)$$

$$= 2R^2(\sin 2A + \sin 2B + \sin 2C) = 2R^2(4\sin A\sin B\sin C)$$

$$= 8R^2\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) = \frac{abc}{R} = 4S \quad [S = area\ of\ \Delta\ ABC]$$

$$\therefore (a\cot A)^a(b\cot B)^b(c\cot C)^c \le \left(\frac{4S}{2s}\right)^{a+b+c} = (2r)^{a+b+c}$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\prod_{cyc}(a\cot A)^a \overset{AM\ge GM}{\le} \left(\frac{a^2\cot A + b^2\cot B + c^2\cot C}{a+b+c}\right)^{a+b+c}$$

$$= \left(\frac{\frac{R}{abc}\sum_{cyc} a^2(b^2+c^2-a^2)}{a+b+c}\right)^{a+b+c} = \left(\frac{R}{abc(a+b+c)}\sum_{cyc}(b^2c^2+c^2a^2-a^4)\right)^{a+b+c}$$

$$= \left(\frac{R}{abc(a+b+c)}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right)\right)^{a+b+c} = \left(\frac{R}{abc}\prod_{cyc}(a+b-c)\right)^{a+b+c}$$

$$= \left(\frac{R}{abc}\cdot 8\prod_{cyc}(p-a)\right)^{a+b+c} = \left(\frac{8R}{4Rrp}\cdot pr^2\right)^{a+b+c} = (2r)^{a+b+c}$$

JP.077. Let $a_1, a_2, \dots, a_9$ be non-negative real numbers such that $a_1 + a_2 + \cdots + a_9 = 1$. Prove that for all $\lambda \ge 4$, the following inequality holds

$$\sqrt{\sum_{1\le i\le 9} a_i^2} + \lambda\sqrt{\sum_{1\le i<j\le 9} a_i a_j} \le \frac{2\lambda+1}{3}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo $a_1, a_2, a_3 \dots a_9$ números reales no negativos de tal manera que

$$a_1 + a_2 + a_3 + \cdots + a_9 = 1.$$

Probar que para todo $\lambda \ge 4$, la siguiente desigualdad

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$$\sqrt{\sum_{1\le i\le 9} a_i^2}+\lambda\sqrt{\sum_{1\le i<j\le 9} a_ia_j}\le\frac{2\lambda+1}{3}$$

Utilizando la desigualdad ponderada Jensen para una funcíon concava

$$f(x)=\sqrt{x},x\ge 0$$

$$3\sqrt{\left(\sum_{1\le i\le 9} a_i^2\right)\cdot\frac{1}{9}}+6\lambda\sqrt{\left(\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{1}{36}}\le$$

$$\le(3+6\lambda)\sqrt{\left(\left(\sum_{1\le i\le 9} a_i^2\right)\cdot\frac{3}{9}+\left(\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{6\lambda}{36}\right)\cdot\frac{1}{3+6\lambda}}=$$

$$=(3+6\lambda)\sqrt{\left(\left(\left(\sum_{1\le i\le 9} a_i\right)^2-2\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{1}{3}+\left(\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{\lambda}{6}\right)\cdot\frac{1}{3+6\lambda}}=$$

$$=(3+6\lambda)\sqrt{\left(\left(1-2\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{1}{3}+\left(\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{\lambda}{6}\right)\cdot\frac{1}{3+6\lambda}}=$$

$$=(3+6\lambda)\sqrt{\left(\left(2-4\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{1}{6}+\left(\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{\lambda}{6}\right)\cdot\frac{1}{3+6\lambda}}$$

A lo cual es equivalente

$$\Rightarrow(3+6\lambda)\sqrt{\left(2+(\lambda-4)\sum_{1\le i<j\le 9} a_ia_j\right)\cdot\frac{1}{6(3+6\lambda)}}$$

Aplicando Maclaurin's inequality

$$\sum_{1\le i\le 9} a_i \ge 9\sqrt{\left(\sum_{1\le i<j\le 9} a_i a_j\right)\cdot\frac{1}{36}} \Leftrightarrow \frac{1}{9} \ge \sqrt{\left(\sum_{1\le i<j\le 9} a_i a_j\right)\cdot\frac{1}{36}} \Leftrightarrow$$

$$\Leftrightarrow \sum_{1\le i<j\le 9} a_i a_j \le \frac{36}{81} = \frac{4}{9}$$

Como $\rightarrow \lambda - 4 \ge 0$

$$(3+6\lambda)\sqrt{\left(2+(\lambda-4)\sum_{1\le i<j\le 9} a_i a_j\right)\cdot\frac{1}{6(3+6\lambda)}} \le (3+6\lambda)\sqrt{\left(2+(\lambda-4)\cdot\frac{4}{9}\right)\cdot\frac{1}{6(3+6\lambda)}} =$$

$$= (3+6\lambda)\sqrt{\frac{2(1+2\lambda)}{6\cdot 9\cdot 3(1+2\lambda)}}(3+6\lambda)\sqrt{\frac{1}{81}} = \frac{3+6\lambda}{9} = \frac{2\lambda+1}{3} \ \textit{(LQQD)}$$

La igualdad se alcanza cuando: $a_1 = a_2 = a_3 = \cdots = a_9 = \frac{1}{9}$

Solution 2 by Ravi Prakash-New Delhi-India

$$\sum_{i=1}^{a} a_i = 1, a_i \ge 0 \Rightarrow \sum_{i=1}^{a} a_i^2 + 2\sum_{i<j} a_i a_j = 1$$

Put

$$\sum_{i=1}^{a} a_i^2 = \sin^2\theta, 2\sum_{i<j} a_i a_j = \cos^2\theta$$

$\left(0 \le \theta \le \frac{\pi}{2}\right)$***. Also,***

$$\frac{1}{9}\sum_{i=1}^{a} a_i^2 \ge \left(\frac{1}{9}\sum_{i=1}^{a} a_i\right)^2 = \frac{1}{81} \Rightarrow \sum_{i=1}^{a} a_i^2 \ge \frac{1}{9}$$

$\Rightarrow \frac{1}{9} \le \sin^2\theta \le 1 \Rightarrow \frac{1}{3} \le \sin\theta \le 1$***. Now,***

$\sqrt{\sum_{i=1}^{a} a_i^2} + \lambda\sqrt{\sum_{i<j} a_i a_j} = \sin\theta + \frac{\lambda}{\sqrt{2}}\cos\theta = f(\theta)$ ***(say)***

For $\lambda = u, f(\theta) \le \sqrt{1+8} = 3 = \frac{2\lambda+1}{3}$***. So assume*** $\lambda > 4$

$$f'(\theta) = \cos\theta - \frac{\lambda}{\sqrt{2}}\sin\theta$$

$$f'(\theta)\geq 0\Leftrightarrow \cos\theta\geq\frac{\lambda}{\sqrt{2}}\sin\theta\Leftrightarrow 1-\sin^2\theta\geq\frac{\lambda^2}{2}\sin^2\theta\Leftrightarrow 1\geq\left(\frac{\lambda^2}{2}+1\right)\sin^2\theta$$

$\Leftrightarrow \sin^2\theta\leq\frac{2}{\lambda^2+2}<\frac{1}{9}$*. Not possible as* $\sin^2\theta\geq\frac{1}{9}\therefore f'(\theta)<0$

$\Rightarrow f(\theta)$ *decreases in* $\left[\sin^{-1}\left(\frac{1}{3}\right),\frac{\pi}{2}\right]\Rightarrow \max f(\theta)=f\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

$$=\frac{1}{3}+\frac{\lambda}{\sqrt{2}}\cdot\frac{2\sqrt{2}}{3}\Rightarrow f(\theta)\leq\frac{2\lambda+1}{3}$$

Solution 3 by Anas Adlany-El Zemamra-Morocco

First, we notice that the given condition can be reduced by squaring into

$$\sum_{i=1}^{9}a_i^2+2\sum_{1\leq i<j\leq 9}a_ia_j=1$$

or

$$\sum_{1\leq i<j\leq 9}a_ia_j=\frac{1-\sum_{i=1}^{9}a_i^2}{2}$$

and by setting $t=\sum_{i=1}^{9}a_i^2$ *the inequlity can be rewritten as follows*

$\sqrt{t}+\lambda\sqrt{\frac{1-t}{2}}\leq\frac{2\lambda+1}{3}$; *with* $t\in\left[\frac{1}{9},1\right)$ *because of* $\forall i, a_i\leq 1$ *and* $\sum_{i=1}^{9}a_i^2\geq\frac{\left(\sum_{i=1}^{9}a_i^2\right)^2}{9}=\frac{1}{9}$.

Now rearranging the inequality as write it as

$$\sqrt{t}-\frac{1}{3}-\frac{\lambda}{2}\left(\frac{t-\frac{1}{9}}{\sqrt{\frac{1-t}{2}}+\frac{2}{3}}\right)\leq 0\Leftrightarrow\left(\sqrt{t}-\frac{1}{3}\right)\left(1-\frac{\lambda}{2}\left(\frac{\sqrt{t}+\frac{1}{3}}{\sqrt{\frac{1-t}{2}}+\frac{2}{3}}\right)\right)\leq 0$$

hence, to finish the proof it suffices to show that

$$1-\frac{\lambda}{2}\left(\frac{\sqrt{t}+\frac{1}{3}}{\sqrt{\frac{1-t}{2}}+\frac{2}{3}}\right)\leq 0\Leftrightarrow 4\left(\sqrt{\frac{1-t}{8}}+\frac{1}{3}\right)\leq\lambda\left(\sqrt{t}+\frac{1}{3}\right)$$

which is obviously true since $\lambda\geq 4$ *and the range of* t *that gives*

$\frac{1-t}{8}\leq t\Leftrightarrow t\geq\frac{1}{9}$; *and we are done!*

JP.078. Let a, b, c be positive real numbers such that $a^2 = b^2 + c^2$. Prove that:

$$ab + bc + ca + (\sqrt{2} - 1)\frac{abc}{a + b + c} \leq 2a^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam

If a, b, c be positive real numbers such that $a^2 = b^2 + c^2$. Prove that

$$ab + bc + ca + (\sqrt{2} - 1)\cdot\frac{abc}{a+b+c} \leq 2a^2 \quad (1)$$

We have

$$(1) \Rightarrow \frac{ab+bc+ca+(\sqrt{2}-1)\cdot\frac{abc}{a+b+c}}{a^2} \leq 2 \Rightarrow \frac{b}{a} + \frac{c}{a} + \frac{b}{a}\cdot\frac{c}{a} + (\sqrt{2} - 1)\cdot\frac{\frac{b}{a}\cdot\frac{c}{a}}{1+\frac{b}{a}+\frac{c}{a}} \leq 2 \quad (2)$$

Put $\frac{b}{a} + \frac{c}{a} = x, \frac{b}{a}\cdot\frac{c}{a} = y$. *We have* $a^2 = b^2 + c^2 \Rightarrow x^2 - 2y = 1 \Rightarrow y = \frac{x^2-1}{2}$

We have (2) $\Rightarrow x + y + (\sqrt{2} - 1)\cdot\frac{y}{1+x} \leq 2 \Rightarrow x + \frac{x^2-1}{2} + (\sqrt{2} - 1)\cdot\frac{2}{1+x} \leq 2 \Rightarrow$

$$\Rightarrow \frac{x^2 + 2x - 1}{2} + \frac{\sqrt{2} - 1}{2}(x - 1) \leq 2 \Rightarrow \frac{x^2 + (1 + \sqrt{2})x - \sqrt{2}}{2} \leq 4$$

$$\Rightarrow x^2 + (1 + \sqrt{2})x - \sqrt{2} \leq 4 \Rightarrow x^2 + (1 + \sqrt{2})x - (4 + \sqrt{2}) \leq 0 \Rightarrow$$

$$\Rightarrow (x - \sqrt{2})[x + (1 + 2\sqrt{2})] \leq 0 \quad (3)$$

By BCS, we have $a^2 = b^2 + c^2 \geq \frac{(b+c)^2}{2} \Rightarrow \frac{b+c}{a} \leq \sqrt{2} \Rightarrow x \leq \sqrt{2} \Rightarrow$ *(3) true* $\Rightarrow$ *QED*

The equality occurs when $b = c = \frac{a\sqrt{2}}{2}$

Solution 2 by Ravi Prakash-New Delhi-India

Let $b = a\cos\theta, c = a\sin\theta$

$$\left(0 < \theta < \frac{\pi}{2}\right)$$

Now, $ab + bc + ca + (\sqrt{2} - 1)\frac{abc}{a+b+c} \leq 2a^2$

$$\Leftrightarrow \cos\theta + \sin\theta + \cos\theta\sin\theta + \frac{(\sqrt{2} - 1)\cos\theta\sin\theta}{1 + \cos\theta + \sin\theta} \leq 2$$

LHS

$$E = \cos\theta + \sin\theta + \cos\theta\sin\theta + \frac{(\sqrt{2} - 1)(\cos\theta\sin\theta)}{1 + \cos\theta + \sin\theta}$$

$$= (\cos\theta + \sin\theta) + \frac{(\cos\theta + \sin\theta)^2 - 1}{2} + \frac{(\sqrt{2}-1)}{2}\frac{\{(\cos\theta + \sin\theta)^2 - 1\}}{\cos\theta + \sin\theta + 1}$$

$$= t + \frac{1}{2}(t^2 - 1) + \frac{1}{2}(\sqrt{2}-1)(t-1) \text{ where } t = \cos\theta + \sin\theta \le \sqrt{2}$$

$$\therefore E \le \sqrt{2} + \frac{1}{2}(2-1) + \frac{1}{2}(\sqrt{2}-1)(\sqrt{2}-1) = 2$$

Solution 3 by Anas Adlany-El Zemamra-Morocco

We have $ab + bc + ca + \frac{(\sqrt{2}-1)abc}{a+b+c} \le 2a^2 \Leftrightarrow (\sqrt{2}-1)abc \le a^3 + b^3 + c^3 - 3abc$

$\Leftrightarrow (\sqrt{2}+2)abc \le a^3 + b^3 + c^3$*. By the AM-GM inequality, we have*

$2abc \le a(b^2 + c^2) = a^3$*. By Cauchy's inequality, we have* $b^3 + c^3 \ge \frac{a^4}{b+c}$

Hence we just still have to prove that $\sqrt{2abc} \le \frac{a^4}{b+c} \Leftrightarrow a^3 \ge \sqrt{2}bc(b+c)$

But $2bc \le a^2$*, hece it suffices to check that*

$a^3 \ge \frac{\sqrt{2}}{2}a^2(b+c) \Leftrightarrow \sqrt{2}a \ge b + c \Leftrightarrow (b-c)^2 \ge 0$*. Which is true. Therefore, as a result we have proved that* $(\sqrt{2}+2)abc \le a^3 + b^3 + c^3$*, and we are done.*

JP.079. Prove the inequality holds for all positive real numbers a, b, c

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{4}{2a+2b+3c} + \frac{4}{2b+2c+3a} + \frac{4}{2c+2a+3b}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar para todo los números R^+ *la siguiente desigualdad*

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{4}{3a+3b+2c} + \frac{4}{3b+3c+2a} + \frac{4}{3c+3a+2b}$$

Como $a, b, c > 0$*. Aplicando la desigualdad de Cauchy*

$\frac{1}{a+b} + \frac{1}{c+a} \ge \frac{4}{b+c+2a}$ *(A),*

$\frac{1}{b+c} + \frac{1}{a+b} \ge \frac{4}{c+a+2b}$ *(B),*

$\frac{1}{c+a} + \frac{1}{b+c} \ge \frac{4}{a+b+2c}$ *(C)*

Sumando (A)+(B)+(C)

$$\Rightarrow \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \geq \frac{4}{a+b+2c}+\frac{4}{b+c+2a}+\frac{4}{c+a+2b}$$

$$\Rightarrow \frac{1}{2(a+b)}+\frac{1}{2(b+c)}+\frac{1}{2(c+a)} \geq \frac{1}{a+b+2c}+\frac{1}{b+c+2a}+\frac{1}{c+a+2b}$$

$$\Rightarrow \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq$$

$$\geq \left(\frac{1}{a+b+2c}+\frac{1}{2(a+b)}\right)+\left(\frac{1}{b+c+2a}+\frac{1}{2(b+c)}\right)+\left(\frac{1}{c+a+2b}+\frac{1}{2(c+a)}\right)$$

$$\geq \frac{4}{a+b+2c+2(a+b)}+\frac{4}{b+c+2a+2(b+c)}+\frac{4}{c+a+2b+2(c+a)}=$$

$$=\frac{4}{3a+3b+2c}+\frac{4}{3b+3c+2a}+\frac{4}{3c+3a+2b}$$

Solution 2 by Nguyen Ngoc Tu-Ha Giang-Vietnam

We have: $\frac{3}{3a+3b+2c}=\frac{4}{2(a+b)+(a+c)+(b+c)} \leq \frac{4}{16}\left(\frac{2}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right)=\frac{1}{4}\left(\frac{2}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right)$

Similar $\frac{4}{3b+3c+2a} \leq \frac{1}{4}\left(\frac{2}{b+c}+\frac{1}{a+b}+\frac{1}{a+c}\right), \frac{4}{3c+3a+2b} \leq \frac{1}{4}\left(\frac{2}{a+c}+\frac{1}{a+b}+\frac{1}{b+c}\right)$

Hence $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{4}{3a+3b+2c}+\frac{4}{3b+3c+2a}+\frac{4}{3c+3a+2b}$

Solution 3 by Ravi Prakash-New Delhi-India

$$\frac{1}{a+b}+\frac{2}{b+c}+\frac{1}{c+a} \geq \frac{16}{a+b+2b+2c+c+a}$$

$\Rightarrow \frac{1}{a+b}+\frac{2}{b+c}+\frac{1}{c+a} \geq \frac{16}{2a+3b+3c}$. ***Similarly,*** $\frac{2}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{16}{3a+3b+2c}$

and $\frac{1}{a+b}+\frac{1}{b+c}+\frac{2}{c+a} \geq \frac{16}{3a+2b+3c}$. ***Adding, we get*** $4\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq$

$$\geq 16\left(\frac{1}{2a+3b+3c}+\frac{1}{3a+3b+2c}+\frac{1}{3a+2b+3c}\right)$$

$$\Rightarrow \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{4}{2a+3b+3c}+\frac{4}{3a+3b+2c}+\frac{4}{3a+2b+3c}$$

Solution 4 by Anas Adlany-El Zemamra-Morocco

Since the inequality is homogenious, we could suppose WLOG of the problem that $a+b+c=1$ ***and the inequality to prove can be reduce into*** $\sum \frac{3a-1}{(a-1)(a-3)} \geq 0$. ***For that, let f be, defined on $(0,1)$ by:***

$f(x)=\frac{3x-1}{(x-1)(x-3)}$; ***by differentiating we get*** $f'(x)=-\frac{3x^2-2x-5}{((x-1)(x-3))^2}$ ***on other time***

$f''(x)=\frac{2(3x^3-3x^2-15x+23)}{((x-1)(x-3))^3}>0$ ***hence f is convex, then by Jensen's inequality we have***

$$\sum\frac{3a-1}{(a-1)(a-3)}=\sum f(a)\geq 3f\left(\frac{1}{3}\right)=0 \text{ and we are done!}$$

JP.080. Prove that in any triangle ABC:

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\geq 2\sqrt{3}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Kevin Soto Palacios-Huarmey-Peru

Tener presente en un triángulo ABC, lo siguiente:

$$m_a^2=\frac{2b^2+2c^2-a^2}{4}\Leftrightarrow m_a^2+\frac{3a^2}{4}=\frac{2b^2+2c^2+2a^2}{4}. \text{ Por MA}\geq MG$$

$$\frac{2b^2+2c^2+2a^2}{4}=m_a^2+\frac{3a^2}{4}\geq\frac{2\sqrt{3}m_a a}{2}\Leftrightarrow a^2+b^2+c^2\geq 2\sqrt{3}m_a a$$

Invirtiendo tenemos: $\frac{2\sqrt{3}a}{a^2+b^2+c^2}\leq\frac{1}{m_a}$ ***... (A)***

por lo tanto: $\frac{2\sqrt{3}b}{a^2+b^2+c^2}\leq\frac{1}{m_b}$ ***... (B) y*** $\frac{2\sqrt{3}c}{a^2+b^2+c^2}\leq\frac{1}{m_c}$ ***... (C)***

Sumando: (A)+(B)+(C)

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\geq\frac{2\sqrt{3}(a+b+c)}{a^2+b^2+c^2}\Leftrightarrow\left(\frac{a^2+b^2+c^2}{a+b+c}\right)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\geq 2\sqrt{3}\text{ (LQQD)}$$

JP.081. If $x,y,z>0$ then:

$$\sqrt{\frac{13x}{6x+7y}}+\sqrt{\frac{13y}{6y+7z}}+\sqrt{\frac{13z}{6z+7x}}\leq 3$$

Proposed by Marin Chirciu-Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo $x,y,z>0$. Probar que: $A=\sqrt{\frac{13x}{6x+7y}}+\sqrt{\frac{13y}{6y+7z}}+\sqrt{\frac{13z}{6z+7x}}\leq 3$. ***Aplicando la desigualdad de Cauchy***

$$A \le \sqrt{\left(\frac{13}{(6x+7y)(6y+7z)}+\frac{13}{(6y+7z)(6z+7x)}+\frac{13}{(6z+7x)(6x+7y)}\right)(x(6y+7z)+y(6z+7x)+z(6x+7y))}$$

$$\Leftrightarrow A \le \sqrt{\left(\frac{13(6y+7z)+13(6x+7y)+13(6z+7x)}{(6x+7y)(6y+7z)(6z+7x)}\right)\cdot(13xy+13yz+13zx)}$$

$$\Leftrightarrow A \le \sqrt{\frac{13(13x+13y+13z)}{(6x+7y)(6y+7z)(6z+7x)}\cdot(13xy+13yz+13zx)} =$$

$= \sqrt{\frac{13^3(x+y+z)(xy+yz+zx)}{(6x+7y)(6y+7z)(6z+7x)}}$. ***Es suficiente probar***

$13^3(x+y+z)(xy+yz+zx) \le 9(6x+7y)(6y+7z)(6z+7x)$. ***Ahora bien***

$$(x+y+z)(xy+yz+zx) = xy(x+y)+yz(y+z)+zx(z+x)+3xyz$$

$$\Rightarrow (x+y+z)(xy+yz+zx) = x^2y+y^2z+z^2x+y^2x+z^2y+x^2z+3xyz$$

$$(6x+7y)(6y+7z)(6z+7x) =$$

$$= 6x\cdot 7y(6x+7y)+6y\cdot 7z(6y+7z)+6z\cdot 7x(6z+7x)+(6^3+7^3)xyz$$

$$\Rightarrow (6x+7y)(6y+7z)(6z+7x) =$$

$= 252(x^2y+y^2z+z^2x)+294(y^2x+z^2y+x^2z)+559xyz$. ***Por lo tanto***

$$13^3(x^2y+y^2z+z^2x+y^2x+z^2y+x^2z+3xyz) \le$$

$$\le 9(252(x^2y+y^2z+z^2x)+294(y^2x+z^2y+x^2z)+559xyz)$$

$$\Leftrightarrow 2197(x^2y+y^2z+z^2x)+2197(y^2x+z^2y+x^2z)+6591xyz \le$$

$$\le 2268(x^2y+y^2z+z^2x)+2646(y^2x+z^2y+x^2z)+5031xyz$$

$$\Leftrightarrow 1560xyz \le 71(x^2y+y^2z+z^2x)+449(y^2x+z^2y+x^2z)$$

Aplicando MA ≥ MG

$$71(x^2y+y^2z+z^2x)+449(y^2x+z^2y+x^2z) \ge 213xyz+1347xyz = 1560xyz$$

(LQQD). Finalmente, concluimos que

$$A = \sqrt{\frac{13x}{6x+7y}}+\sqrt{\frac{13y}{6y+7z}}+\sqrt{\frac{13z}{6z+7x}} \le \sqrt{\frac{13^3(x+y+z)(xy+yz+zx)}{(6x+7y)(6y+7z)(6z+7x)}} \le \sqrt{9} = 3$$

Solution 2 by Anas Adlany-El Zemamra-Morocco

More generaly, we prove that $\sum\sqrt{\frac{x}{mx+ny}} \le \frac{3}{\sqrt{m+n}}$, ***where*** $(m,n) \in (\mathbb{N}^*)^2$.

$$\sum\sqrt{\frac{x}{mx+ny}}=\frac{\sum\sqrt{x(my+nz)(mz+nx)}}{\sqrt{\prod(mx+ny)}}\le\sqrt{\frac{(\sum x(my+nz))(\sum mz+nx)}{\prod(mx+ny)}}$$

$$=(m+n)\cdot\sqrt{\frac{(\sum xy)(\sum x)}{\prod(mx+ny)}}=(m+n)\cdot\sqrt{\frac{xyz+\prod(x+y)}{\prod(mx+ny)}}$$

$$=\frac{1}{\sqrt{m+n}}\cdot\sqrt{8\left(1+\frac{xyz}{\prod(x+y)}\right)}\le\frac{1}{\sqrt{m+n}}\cdot\sqrt{8\left(1+\frac{1}{8}\right)}=\frac{3}{\sqrt{m+n}}$$

where the last step come from Chebyshev's inequality, and we are done!

JP.082. If $a,b,c>0; a+b+c=3$ then:

$$\frac{a}{1+3b^4}+\frac{b}{1+3c^4}+\frac{c}{1+3a^4}\ge\frac{3}{4}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo $a,b,c>0$ de tal manera que $a+b+c=3$. Probar que

$$\frac{a}{1+3b^4}+\frac{b}{1+3c^4}+\frac{c}{1+3a^4}\ge\frac{3}{4}$$

$$\left(\frac{a}{1+3b^4}-a\right)+\left(\frac{b}{1+3c^4}-b\right)+\left(\frac{c}{1+3a^4}-c\right)\ge\frac{3}{4}-3$$

$$\Leftrightarrow-\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3a^4c}{1+3a^4}\ge-\frac{9}{4}\Leftrightarrow\frac{3ab^4}{1+3b^4}+\frac{3bc^4}{1+3c^4}+\frac{3a^4c}{1+3a^4}\le\frac{9}{4}$$

Como $a,b,c>0$. Aplicando MA$\ge MG$

$3a^4+1=a^4+a^4+a^4+1\ge4a^3, 3b^4+1\ge4b^3, 3c^4+1\ge4c^3$. ***Por lo tanto***

$$\Rightarrow\frac{3ab^4}{1+3b^4}+\frac{3bc^4}{1+3c^4}+\frac{3a^4c}{1+3a^4}\le\frac{3ab^4}{4b^3}+\frac{3bc^4}{4c^3}+\frac{3a^4c}{4a^3}=\frac{3}{4}(ab+bc+ca)\le$$

$$\le\frac{3}{4}\cdot\frac{(a+b+c)^2}{3}=\frac{3}{4}\cdot3=\frac{9}{4}$$

Solution 2 by Nguyen Minh Tri-Ho Chi Minh-Vietnam

If $a,b,c>0, a+b+c=3$ then

$$\frac{a}{1+3b^4}+\frac{b}{1+3c^4}+\frac{c}{1+3a^4}\ge\frac{3}{4}\Rightarrow ab+bc+ca\le3$$

$$\Leftrightarrow \frac{a(1+3b^4)-3ab^4}{1+3b^4}+\frac{b(1+3c^4)-3bc^4}{1+3c^4}+\frac{c(1+3a^4)-3ca^4}{1+3a^4}\ge\frac{3}{4}$$

$$\Leftrightarrow a-\frac{3ab^4}{1+3b^4}+b-\frac{3bc^4}{1+3c^4}+c-\frac{3ca^4}{1+3ab^4}\ge\frac{3}{4}$$

$$\Leftrightarrow (a+b+c)-\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\ge\frac{3}{4}$$

$$\Leftrightarrow 3-\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\ge\frac{3}{4}\ \textit{(1)}$$

Use Cauchy for 4 numbers, we have: $3b^4+1=b^4+b^4+b^4+1\ge 4\sqrt[4]{b^{12}}=4b^3$

$$\Leftrightarrow -\frac{3ab^4}{1+3b^4}\ge-\frac{3ab^4}{4b^3}=\frac{-3ab}{4}.\ \textit{Similarly}\ -\frac{3bc^4}{1+3c^4}\ge-\frac{3bc}{4};\frac{-3ca^4}{1+3a^4}\ge-\frac{3ac}{4}$$

$$\Rightarrow -\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\ge-\frac{3}{4}(ab+bc+ca)\ \textbf{but}\ ab+bc+ca\le 3$$

$$\Rightarrow -\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\ge-\frac{3}{4}\cdot 3=-\frac{9}{4}$$

$$\Rightarrow 3-\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\ge\frac{3}{4}\Rightarrow \textit{(1) true}\Rightarrow \textit{Q.E.D.}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Applying Tangent – Line Method,

Let $f(x)=\frac{1}{1+3x^4}$ ***for all*** $x>0$***, so the tangent to the curve*** $f(x)$ ***is***

$y=f(1)+f'(1)(x-1)=\frac{4-3x}{4}$***. So, we can write,*** $\frac{1}{1+3x^4}\ge\frac{4-3x}{4}$

$$\Leftrightarrow (x-1)^2(3x^2+2x+1)\ge 0\ \textit{for all}\ x>0$$

$$\therefore \sum_{cyc}\frac{a}{1+3b^4}\ge\sum_{cyc}\frac{a(4-3b)}{4}=\sum_{cyc}a-\frac{3}{4}\sum_{cyc}ab\ge\sum_{cyc}a-\frac{1}{4}\left(\sum_{cyc}a\right)^2=\frac{3}{4}\ \textit{(Proved)}$$

Solution 4 by Kunihiko Chikaya-Tokyo-Japan

$$\frac{a}{1+3b^4}+\frac{b}{1+3c^4}+\frac{c}{1+3a^4}\ge\frac{3}{4};\ a+b+c=3,a,b,c\in\mathbb{R}^+.$$

$$\frac{1}{1+3b^4}\ge-\frac{3}{4}+1\Leftrightarrow 3b(3b^2+2b+1)(b-1)^2\ge 0$$

$$\frac{a}{1+3b^4}\ge a\left(-\frac{3}{4}b+1\right)$$

$$\frac{b}{1+3c^4}\ge b\left(-\frac{3}{4}c+1\right)$$

$$\frac{c}{1+3a^4} \geq c\left(-\frac{3}{4}c+1\right)$$

$\therefore$ *LHS of (*)*$= a+b+c-\frac{3}{4}(ab+bc+ca) \geq a+b+c-\frac{1}{4}(a+b+c)^2 = \frac{3}{4}$

Equality: $a=b=c=1$

JP.083. In $\Delta\, ABC$ then following relationship holds:

$$(a^{2m}+b^{2m}+c^{2m})\left(\frac{1}{(a+b)^{2n}}+\frac{1}{(b+c)^{2n}}+\frac{1}{(c+a)^{2n}}\right) \geq$$

$$\geq 3^{m-n+2}\cdot 4^{m-2n}\cdot r^{2(m-n)};\ m,n \geq 1$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu - Romania

Solution by SK Rejuan-West Bengal-India

For $\Delta\, ABC$: With the help & m-th power theorem we get,

$$\frac{\sum a^{2m}}{3} \geq \left(\frac{\sum a^2}{3}\right)^m \Rightarrow \sum a^{2m} \geq 3^{1-m}\left(\sum a^2\right)^m \quad (1)$$

By Cauchy inequality we get, $(a+b)^2 \leq 2(a^2+b^2) \Rightarrow (a+b)^{2n} \leq 2^n(a^2+b^2)^n$

$$\Rightarrow \sum\frac{1}{(a+b)^{2n}} \geq \frac{1}{2^n}\sum\frac{1}{(a^2+b^2)^n} \quad (2)$$

Again by mth power theorem we get, $\frac{\sum\left(\frac{1}{a^2+b^2}\right)^n}{3} \geq \left\{\frac{\sum\left(\frac{1}{a^2+b^2}\right)}{3}\right\}^n$

$$\Rightarrow \sum\left(\frac{1}{a^2+b^2}\right)^n \geq 3^{1-n}\left(\sum\frac{1}{a^2+b^2}\right)^n \Rightarrow \frac{1}{2^n}\sum\frac{1}{(a^2+b^2)^n} \geq \frac{3^{1-n}}{2^n}\left(\sum\frac{1}{a^2+b^2}\right)^n \quad (3)$$

Again by AM≥HM we get,

$$\sum\frac{1}{a^2+b^2} \geq \frac{9}{\sum(a^2+b^2)} = \frac{9}{2\sum a^2} \Rightarrow \frac{3^{1-n}}{2^n} \times \left(\sum\frac{1}{a^2+b^2}\right)^n \geq \frac{3^{1-n}}{2^n}\left(\frac{9}{2\sum a^2}\right)^n \quad (4)$$

Combining (2), (3), (4) we get

$$\sum\frac{1}{(a+b)^{2n}} \geq \frac{3^{1-n}}{2^n}\left(\frac{9}{2\sum a^2}\right)^n \quad (5)$$

Multiplying (1) & (5) we get, $\sum a^{2m}\cdot\sum\frac{1}{(a+b)^{2n}} \geq 3^{1-m}\left(\sum a^2\right)^m\cdot\frac{3^{1-n}}{2^n}\left(\frac{9}{2\sum a^2}\right)^n$

$$= 3^{2-m-n+2n}\, 2^{-2n}\left(\sum a^2\right)^{m-n} = 3^{n-m+2}\cdot 4^{-n}\cdot\left(\sum a^2\right)^{(m-n)} \quad (6)$$

Also we know that $\sum a^2 \geq 36r^2 \Rightarrow \left(\sum a^2\right)^{m-n} \geq (36)^{m-n}\cdot r^{2(m-n)}$

$$\Rightarrow 3^{n-m+2}\cdot 4^{-n}\cdot\left(\sum a^2\right)^{m-n}\ge 3^{n-m+2}\cdot 4^{-n}\cdot 36^{(m-n)}\cdot r^{2(m-n)}$$

$$=3^{(n-m+2+2m-2n)}\cdot 4^{(-n+m-n)}\cdot r^{2(m-n)}$$

$$=3^{(m-n+2)}\cdot 4^{(m-2n)}\cdot r^{2(m-n)}\quad (7)$$

Combining (6) & (7) we get, $\sum a^{2m}\cdot\sum\frac{1}{(a+b)^{2n}}\ge 3^{(m-n+2)}\cdot 4^{(m-2n)}\cdot r^{2(m-n)}$

(proved)

JP.084. In ΔABC the following relationship holds:

$$\sum\left((a+b)\tan\frac{C}{2}\right)^m\cdot\sum\frac{1}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n}}\ge 3^{n+2}\cdot 4^{m-n}\cdot r^n;\ m\ge n\ge 1$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu - Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo $m\ge n\ge 1$. Probar en un triángulo la siguiente desigualdad

$$\left((a+b)\tan\frac{C}{2}+(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{C}{2}\right)^m\cdot$$

$$\left(\frac{1}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n}}+\frac{1}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^{2n}}+\frac{1}{\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)^{2n}}\right)\ge 3^{n+2}\cdot 4^{m-n}\cdot r^m$$

Recordar las siguientes identidades y desigualdades en un ΔABC

$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r}{p}, R\ge 2r, abc=4Rpr$. *Ahora bien, utilizando MA $\ge MG$*

$$(a+b)\tan\frac{C}{2}+(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{C}{2}\ge 3\sqrt[3]{(a+b)(b+c)(c+a)\cdot\frac{r}{p}}\ge 3\sqrt[3]{8abc\cdot\frac{r}{p}}=$$

$$=3\sqrt[3]{32Rr^2}\ge 3\sqrt[3]{64r^3}=12r$$

IRAN INEQUALITY

Siendo $x,y,z\ge 0$ se cumple la siguiente desigualdad

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge\frac{9}{4}\quad (A)$$

Realizamos los siguientes cambios de variables

$$x=\tan\frac{A}{2}>0, y=\tan\frac{B}{2}>0, z=\tan\frac{C}{2}>0 \Leftrightarrow xy+yz+zx=1$$

Por lo tanto tenemes en (A)

$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\geq\frac{9}{4}$. ***De la desigualdad ponderada de Cauchy***

$$\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)\cdot 3^{n-1}\geq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^n, \text{ donde } a,b,c>0, n\geq 1$$

Siendo $a=(x+y)^2, b=(y+z)^2, c=(z+x)^2 \Leftrightarrow x,y,z>0$

$$\Rightarrow \frac{1}{(x+y)^{2n}}+\frac{1}{(y+z)^{2n}}+\frac{1}{(z+x)^{2n}}\geq\frac{3}{3^n}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^n\geq$$

$$\geq\frac{3}{3^n}\cdot\frac{9^n}{4^n}=\frac{3^{n+1}}{4^n}$$

Utilizando $\rightarrow m-n\geq 0 \wedge m\geq 1$. ***Se concluye que***

$$LHS\geq(12r)^m\cdot\frac{3^{n+1}}{4^n}=4^m\cdot 3^m\cdot r^m\cdot\frac{3^{n+1}}{4^n}=4^{m-n}\cdot 3^{n+m+1}\cdot r^m\geq 4^{m-n}\cdot 3^{n+2}\cdot r^m$$

JP.085. Let ABC denote a triangle, I its incenter, R its circumradius, r its inradius and x, y and z the inradii of triangles $IBC, ICA,$ and IAB respectively. Prove that:

$$\frac{\sin A}{x}+\frac{\sin B}{y}+\frac{\sin C}{z}\leq\frac{4+3\sqrt{3}}{2r}+\frac{2}{R}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Lemma

Siendo I - Incentro. En un triángulo ABC, se cumple los siguiente

$$IA+IB+IC\leq 2(R+r)$$

Tener en cuenta las siguientes identidades y desigualdades en un ΔABC

$$IA=\frac{bc}{p}\cos\frac{A}{2}=\frac{bc}{p}\sqrt{\frac{p(p-a)}{bc}}=\sqrt{\frac{bc(p-a)}{p}},$$

$$IB=\sqrt{\frac{ca(p-b)}{p}}, IC=\sqrt{\frac{ab(p-c)}{p}}$$

$$a+b+c=2p,\ ab+bc+ca=p^2+r^2+4Rr,$$

$$p^2\le 4R^2+4Rr+3r^2$$ ***(Gerretsen Inequality)***

Aplicando la desigualdad de Cauchy

$$IA+IB+IC=\sqrt{\frac{bc(p-a)}{p}}+\sqrt{\frac{ca(p-b)}{p}}+\sqrt{\frac{ab(p-c)}{p}}\le$$

$$\le\sqrt{\left(\frac{p-a}{p}+\frac{p-b}{p}+\frac{p-c}{p}\right)(bc+ca+ab)}=\sqrt{ab+bc+ca}=$$

$$=\sqrt{p^2+r^2+4Rr}\le\sqrt{4R^2+4Rr+3r^2+r^2+4Rr}=$$

$$=\sqrt{4(R^2+2Rr+r^2)}=\sqrt{4(R+r)^2}=2(R+r)$$

(LQQD)

Siendo ABC un triángulo, con I - Incentro, R – circumradio, r - inradio, además x,y,z son los inradios de triángulos IBC, ICA, IAB. Probar que

$$\frac{\sin A}{x}+\frac{\sin B}{y}+\frac{\sin C}{z}\le\frac{4+3\sqrt{3}}{2r}+\frac{2}{r}$$

Recordar las siguientes desigualdades conocidas en un triángulo ABC

$\frac{p}{R}\le\frac{3\sqrt{3}}{2}, IA+IB+IC\le 2(R+r)$. ***De las condiciones, se verifica lo siguiente***

$$\frac{a}{x}=\frac{pa+ab\cos\frac{C}{2}+ca\cos\frac{B}{2}}{S}=\frac{pa+pIC+pIB}{S}=\frac{p(a+IC+IB)}{pr}=\frac{a+IC+IB}{r}$$

$$\frac{a}{x}=\frac{pa+ab\cos\frac{C}{2}+ca\cos\frac{B}{2}}{S}=\frac{pa+pIC+pIB}{S}=\frac{p(a+IC+IB)}{pr}=\frac{a+IC+IB}{r}$$

$$\frac{b}{y}=\frac{pb+bc\cos\frac{A}{2}+ab\cos\frac{C}{2}}{S}=\frac{pb+pIA+pIC}{S}=\frac{p(b+IA+IC)}{pr}=\frac{b+IA+IC}{r}$$

$$\frac{c}{z}=\frac{pc+ca\cos\frac{B}{2}+bc\cos\frac{A}{2}}{S}=\frac{pc+pIB+pIA}{S}=\frac{p(c+IB+IA)}{pr}=\frac{c+IB+IA}{r}$$

Lo cual implica $\Rightarrow\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{2p}{r}+\frac{2(IA+IB+IC)}{r}\Leftrightarrow$

$$\Leftrightarrow\frac{\sin A}{x}+\frac{\sin B}{y}+\frac{\sin C}{z}=\frac{p}{Rr}+\frac{IA+IB+IC}{Rr}\le\frac{3\sqrt{3}}{2r}+2\left(\frac{1}{R}+\frac{1}{r}\right)=\frac{4+3\sqrt{3}}{2r}+\frac{2}{R}$$

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JP.086. Let a, b, c be the side lengths of a triangle ABC with incentre I, circumradius R and inradius r. Prove that

$$\frac{\sqrt{AI}}{a}+\frac{\sqrt{BI}}{b}+\frac{\sqrt{CI}}{c}\le\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Lemma

Siendo I - incentro. En un triángulo ABC, se cumple lo siguiente

$$IA+IB+IC\le 2(R+r)$$

Solucion

Tener en cuenta las siguientes identidades y desigualdades en un $\Delta\ ABC$

$$IA=\frac{bc}{p}\cos\frac{A}{2}=\frac{bc}{p}\sqrt{\frac{p(p-a)}{bc}}=\sqrt{\frac{bc(p-a)}{p}},IB=\sqrt{\frac{ca(p-b)}{p}},IC=\sqrt{\frac{ab(p-c)}{p}}$$

$a+b+c=2p, ab+bc+ca=p^2+r^2+4Rr, p^2\le 4R^2+4Rr+3r^2$ ***(Gerretsen Inequality). Aplicando la desigualdad de Cauchy***

$$IA+IB+IC=\sqrt{\frac{bc(p-a)}{p}}+\sqrt{\frac{ca(p-b)}{p}}+\sqrt{\frac{ab(p-c)}{p}}\le$$

$$\le\sqrt{\left(\frac{p-a}{p}+\frac{p-b}{p}+\frac{p-c}{p}\right)(bc+ca+ab)}=\sqrt{ab+bc+ca}=$$

$$=\sqrt{p^2+r^2+4Rr}\le\sqrt{4R^2+4Rr+3r^2+r^2+4Rr}=\sqrt{4(R^2+2Rr+r^2)}=$$

$$=\sqrt{4(R+r)^2}=2(R+r)\ \textbf{(LQQD)}$$

Siendo a, b, c los lados de un triánguo ABC con Incentro I, circunradio R e inradio r.

Probar que $\frac{\sqrt{AI}}{a}+\frac{\sqrt{BI}}{b}+\frac{\sqrt{CI}}{c}\le\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}$

Tener en cuenta las siguientes desigualdades conocidas en un triàngulo ABC

$AI+BI+CI\le 2(R+r), \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}$. ***Aplicando la desigualdad de Cauchy***

$$\frac{\sqrt{AI}}{a}+\frac{\sqrt{BI}}{b}+\frac{\sqrt{CI}}{c}\le\sqrt{(AI+BI+CI)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}\le\sqrt{2(R+r)\cdot\frac{1}{4r^2}}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS\overset{C-B-S}{\le}\sqrt{\sum AI}\cdot\sqrt{\sum\frac{1}{a^2}}=\frac{1}{4Rrs}\cdot\sqrt{\sum AI}\sqrt{\sum a^2b^2}$$

$$\overset{Goldstone}{\le}\frac{1}{4Rrs}\sqrt{\sum AI}\sqrt{4R^2s^2}=\frac{2Rs}{4Rrs}\sqrt{\sum AI}=\frac{1}{2r}\sqrt{\sum AI}\overset{?}{\le}\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}$$

$$\Leftrightarrow\sum AI\le 2(R+r)\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\le\frac{2(R+r)}{r}$$

$$\Leftrightarrow\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}}+\frac{\sqrt{ca}}{\sqrt{(s-c)(s-a)}}+\frac{\sqrt{ab}}{\sqrt{(s-a)(s-b)}}\le\frac{2(R+r)}{r}$$

$$\Leftrightarrow\frac{\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}}{\sqrt{(s-a)(s-b)(s-c)}}\le\frac{2(R+r)}{r}$$

$$\Leftrightarrow\frac{\sqrt{s}}{rs}\left\{\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\right\}\le\frac{2(R+r)}{r}$$

$$\Leftrightarrow\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\le 2\sqrt{s}(R+r)\quad\textit{(1)}$$

$$LHS\text{ of }(1)\overset{C-B-S}{\le}\sqrt{\sum ab}\sqrt{3s-2s}=\sqrt{\sum ab}\sqrt{s}\overset{?}{\le}2\sqrt{s}(R+r)$$

$$\Leftrightarrow\sum ab\le 4(R+r)^2\Leftrightarrow s^2+4Rr+r^2\le 4R^2+8Rr+4r^2$$

$$\Leftrightarrow s^2\le 4R^2+4Rr+3r^2\rightarrow\textit{true, by Gerretsen (proved)}$$

JP.087. Let ABC be an acute triangle. Prove that

$$\sqrt{\cos A\cdot\sin B\cdot\sin C}+\sqrt{\sin A\cdot\cos B\cdot\sin C}+\sqrt{\sin A\cdot\sin B\cdot\sin C}\le\frac{3}{2}\sqrt{\frac{3}{2}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo acutángulo ABC

$$\sum \sqrt{\cos A \sin B \sin C} \leq \frac{3}{2}\sqrt{\frac{3}{2}}$$

Dado que es triángulo acutángulo $\cos A, \cos B, \cos C > 0$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \Leftrightarrow 2\sin B \sin C \cos A = \sin^2 B + \sin^2 C - \sin^2 A$$

$$\Leftrightarrow 2\sin A \sin C \cos B = \sin^2 A + \sin^2 C - \sin^2 B \wedge 2\sin A \sin B \cos C =$$

$$= \sin^2 A + \sin^2 B - \sin^2 C$$

$$\Leftrightarrow 2\sum \sin B \sin C \cos A = \sin^2 A + \sin^2 B + \sin^2 C \Leftrightarrow$$

$$\Leftrightarrow \sum \sin B \sin C \cos A = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2}$$

Además $\frac{\sin^2 A + \sin^2 B + \sin^2 C}{2} = (1 + \cos A \cos B \cos C) \leq \frac{9}{8}$

Aplicando la desigualdad de Cauchy se tiene lo siguiente

$$\sum \sqrt{\cos A \sin B \sin C} = \sqrt{3\left(\sum \sin B \sin C \cos A\right)} = \sqrt{3\left(\frac{\sin^2 A + \sin^2 B + \sin^2 C}{2}\right)} \leq$$

$$\leq \sqrt{\frac{3 \cdot 9}{8}} = \frac{3}{2}\sqrt{\frac{3}{2}}$$

Solution 2 by Sanong Hauerai-Nakon Pathom-Thailand

$$-1 = \cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C -$$

$$-\sin A \cos B \sin C - \cos A \sin B \sin C$$

Hence $\sin A \sin B \cos C + \sin A \cos B \sin C + \cos A \sin B \sin C$

$$= 1 + \cos A \cos B \cos C \leq 1 + \frac{1}{8} = \frac{9}{8}$$

Because $\cos A + \cos B + \cos C \leq \frac{3}{2} \to \cos A \cos B \cos C \leq \frac{1}{8}$

Therefore $\sqrt{\sin A \sin B \cos C} + \sqrt{\sin A \cos B \sin C} + \sqrt{\cos A \sin B \sin C}$

$$\leq \sqrt{3(\sin A \sin B \cos C + \sin A \cos B \sin C + \cos A \sin B \sin C)}$$

$$\leq \sqrt{3\left(\frac{9}{8}\right)} = \frac{3}{2}\sqrt{\frac{3}{2}}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\sqrt{\sin A\cdot\sin B\cdot\cos C}=\sum_{cyc}\sqrt{\frac{a}{2R}\cdot\frac{b}{2R}\cdot\left(\frac{b^2+c^2-a^2}{2ab}\right)}$$

$$=\frac{1}{2\sqrt{2}R}\sum_{cyc}\sqrt{b^2+c^2-a^2}\le\frac{1}{2\sqrt{2}R}\sqrt{3\sum_{cyc}(b^2+c^2-a^2)}$$

[$\because\sqrt{x}$ is concave hence by Jensen's Inequality]

$$=\frac{1}{2\sqrt{2}R}\sqrt{3\sum_{cyc}a^2}\le\frac{1}{2\sqrt{2}R}\sqrt{3\cdot 9R^2}=\frac{3}{2}\sqrt{\frac{3}{2}}\text{ (Proved)}$$

JP.088. Let a,b,c be positive real numbers. Prove that

$$\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca}\ge\frac{9abc}{ab+bc+ca}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo a,b,c números R^+. Probar que $\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca}\ge\frac{9abc}{ab+bc+ca}$

Como $a,b,c>0$. Aplicando la desigualdad de Cauchy

$$\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca}\ge\frac{a^3+b^3}{\sqrt{(c^2+a^2)(c^2+b^2)}}+\frac{b^3+c^3}{\sqrt{(a^2+b^2)(a^2+c^2)}}+\frac{c^3+a^3}{\sqrt{(b^2+c^2)(b^2+a^2)}}.$$

Utilizando las siguientes desigualdades conocidas $\forall\ a,b,c>0$

$$\Rightarrow a^3+b^3\ge ab(a+b)\Leftrightarrow 2(a^3+b^3)\ge(a^2+b^2)(a+b)$$

Análogamente para los siguientes términos

$$2(b^3+c^3)\ge(b^2+c^2)(b+c),\ 2(c^3+a^3)\ge(c^2+a^2)(c+a)$$

Luego por MA≥MG

$$\frac{a^3+b^3}{\sqrt{(c^2+a^2)(c^2+b^2)}}+\frac{b^3+c^3}{\sqrt{(a^2+b^2)(a^2+c^2)}}+\frac{c^3+a^3}{\sqrt{(b^2+c^2)(b^2+a^2)}}\ge$$

$$\ge 3\sqrt[3]{\frac{(a^3+b^3)(b^3+c^3)(c^3+a^3)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}\ge$$

$$\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq 3\sqrt[3]{abc} = \frac{3\sqrt[3]{abc}\cdot(ab+bc+ca)}{ab+bc+ca} \geq$$

$$\geq \frac{3\sqrt[3]{abc}\cdot 3\sqrt[3]{a^2b^2c^2}}{ab+bc+ca} = \frac{9abc}{ab+bc+ca} \text{ (LQQD)}$$

Solution 2 by Sanong Hauerai-Nakon Pathom-Thailand

For $a,b,c>0$, Prove that: $\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca} \geq \frac{9abc}{ab+bc+ca}$***. We will show that***

$$\frac{a^3}{c^2+ab}+\frac{b^3}{a^2+bc}+\frac{c^3}{b^2+ca} \geq \frac{9abc}{2(ab+bc+ca)} \quad (1)$$

$$\frac{a^3}{b^2+ca}+\frac{b^3}{c^2+ab}+\frac{c^3}{a^2+bc} \geq \frac{9abc}{2(ab+bc+ca)} \quad (2)$$

$$\frac{a^3}{c^2+ab}+\frac{b^3}{a^2+bc}+\frac{c^3}{b^2+ca} \geq \frac{9abc}{2(ab+bc+ca)} \quad (3)$$

$$\text{Iff } \frac{a^4}{c^2a+a^2b}+\frac{b^4}{a^2b+b^2c}+\frac{c^4}{b^2c+c^2a} \geq \frac{9abc}{2(ab+bc+ca)}$$

$$\text{Iff } \frac{(a^2+b^2+c^2)^2}{2(a^2b+b^2c+c^2a)} \geq \frac{9(abc)}{2(ab+bc+ca)}$$

$$\text{Iff } \frac{27(a^2+b^2+c^2)^2}{b(a+b+c)^3} \geq \frac{9(abc)}{2(ab+bc+ca)}$$

$$\text{Iff } \frac{(a+b+c)}{9} \geq \frac{abc}{ab+bc+ca}$$

Iff $\frac{(ab+bc+ca)(a+b+c)}{9(ab+bc+ca)} \geq \frac{abc}{ab+bc+ca}$ ***and it is to be true. Similarly (2) is to be true***

Therefore this problem is to be true.

Solution 3 by Dinh Tien Dung-Hanoi-Vietnam

By Holder inequality we have: $\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca} \geq \frac{\left(\sqrt[3]{a^3+b^3}+\sqrt[3]{b^3+c^3}+\sqrt[3]{c^3+a^3}\right)^3}{3(a^2+b^2+c^2+ab+bc+ca)}$

Since $4(a^3+b^3) \geq (a+b)^3$***, we have:*** $\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca} \geq \frac{2(a+b+c)^3}{3(a^2+b^2+c^2+ab+bc+ca)}$

We need to show that: $\frac{2(a+b+c)^3}{3(a^2+b^2+c^2+ab+bc+ca)} \geq \frac{9abc}{ab+bc+ca}$***. The inequality is equivalent to***

$$2(a+b+c)^3(ab+bc+ca) \geq 27abc(a^2+b^2+c^2+ab+bc+ca)$$

We assume that $a=\min\{a,b,c\}$ ***and let*** $a=b-q, a=c-q \ (p,q\geq 0)$

The inequality is equivalente to

$$27a^3(p^2-pq+q^2)+3a^2(5p^3+6p^2q+6pq^2+5q^3)+$$
$$+a(4p^4+7p^3q+33p^2q^2+7pq^3+4q^4)+2pq(p^3+3p^2q+3pq^2+q^3)\geq 0$$

Equality if and only if $a=b=c$***. Q.E.D.***

JP.089. Let a,b,c be positive real numbers, take:

$$X=\frac{a}{b}+\frac{b}{a},Y=\frac{b}{c}+\frac{c}{b},Z=\frac{c}{a}+\frac{a}{c}.$$

Prove that: $X+Y+Z\geq 2\sqrt[4]{(X^2+Y^2+Z^2-3)(X+Y+Z+3)}$

Proposed by Nguyen Ngoc Tu – Ha Giang –Vietnam

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo a,b,c ***números*** R^+***, de tal manera que***

$X=\frac{a}{b}+\frac{b}{a}>0, Y=\frac{b}{c}+\frac{c}{b}>0, Z=\frac{c}{a}+\frac{a}{c}>0$. ***Probar que***

$X+Y+Z\geq 2\sqrt[4]{(X^2+Y^2+Z^2-3)(X+Y+Z+3)}$. ***Ahora bien***

$$X+Y+Z=\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}=\left(\frac{a+b}{c}+1\right)+\left(\frac{b+c}{a}+1\right)+\left(\frac{c+a}{b}+1\right)-3=$$

$$=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3\Rightarrow X+Y+Z+3=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\Rightarrow X^2+Y^2+Z^2=\left(\frac{a}{b}+\frac{b}{a}\right)^2+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2=$$

$$=\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}+2\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}+2\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}+2\right)$$

$$\Rightarrow X^2+Y^2+Z^2-3=\left(\frac{a^2+b^2}{c^2}+1\right)+\left(\frac{b^2+c^2}{a^2}+1\right)+\left(\frac{c^2+a^2}{b^2}+1\right)=$$

$=(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$. ***La desigualdad propuesta es equivalente***

$$\Leftrightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3\geq 2\sqrt[4]{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Aplicando la desigualdad de Cauchy

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3=\sqrt{(a+b+c)^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}-3=$$

$$=\sqrt{\left(\sum a^2+2\sum bc\right)\left(\sum\frac{1}{a^2}+2\sum\frac{1}{bc}\right)}-3\geq$$

$$\geq\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}+2\sqrt{\left(\sum bc\right)\left(\sum\frac{1}{bc}\right)}-3=\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}+2\sqrt{\left(\sum a\right)\left(\sum\frac{1}{a}\right)}-3$$

Utilizando MA≥ MG

$$\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}+\sqrt{\left(\sum a\right)\left(\sum\frac{1}{a}\right)}+\sqrt{\left(\sum a\right)\left(\sum\frac{1}{a}\right)}-3\geq$$

$$\geq\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}+\sqrt{\left(\sum a\right)\left(\sum\frac{1}{a}\right)}\geq$$

$$\geq 2\sqrt[4]{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\ \textit{(LQQD)}$$

Solution 2 by Richdad Phuc-Hanoi-Vietnam

We have $X=\frac{a}{b}+\frac{b}{a}\geq 2$ ***(AM-GM). Similar*** $Y\geq 2, Z\geq 2$

$$(X-2)(Y-2)\geq 0\Rightarrow XY\geq 2(X+Y-2)$$

similar $YZ\geq 2(Y+Z-2)$ ***and*** $ZX\geq 2(Z+X-2)$

Let $t=X+Y+Z$ $(t\geq 6)$***. We get***

$$X^2+Y^2+Z^2=(X+Y+Z)^2-2(XY+YZ+ZX)\leq t^2-8t+24$$

we need to prove that

$$t^4\geq 16(t^2-8t+21)(t+3)\Leftrightarrow (t-6)(t^3-10t^2+20t+168)\geq 0$$

$$(t-6)[t(t-6)^2+2(t-4)^2+136]\geq 0\ \textit{(true with } t\geq 6)$$

equality holds if $t=6\Leftrightarrow a=b=c$

JP.090. In $\Delta\, ABC$**:**

$$\frac{1+\cos A\cos B\cos B}{\sin A\sin B\sin A}\geq\frac{s}{3r}$$

Proposed by Martin Lukarevski-Skopje-Macedonia

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo ABC

$\frac{1+\cos A\cos B\cos B}{\sin A\sin B\sin A}\geq\frac{s}{3r}$. ***Recordar las siguientes identidades en un triángulo*** ABC

$$\sin^2 A+\sin^2 B+\sin^2 C=2(1+\cos A\cos B\cos C),\ S=2R^2\sin A\sin B\sin C$$

$$\Rightarrow \cot A+\cot B+\cot C=\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2\sin A\sin B\sin C}=\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}=$$

$=\frac{a^2+b^2+c^2}{4S}=\frac{a^2+b^2+c^2}{4sr}$. ***La desigualdad propuesta es equivalente***

$$\frac{a^2+b^2+c^2}{4sr}\geq\frac{s}{3r}\Leftrightarrow 3(a^2+b^2+c^2)\geq 4s^2=(a+b+c)^2$$

(Válido por desigualdad de Cauchy)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\cos A\cos B\cos C=\frac{b^2+c^2-a^2}{2bc}\cdot\frac{c^2+a^2-b^2}{2ca}\cdot\frac{a^2+b^2-c^2}{2ab}\quad \textit{(1)}$$

Numerator $=(\sum a^2-2a^2)(\sum a^2-2b^2)(\sum a^2-2c^2)$

$$=\left(\sum a^2\right)^3-2\left(\sum a^2\right)^2\left(\sum a^2\right)+4\left(\sum a^2\right)\left(\sum a^2b^2\right)-8a^2b^2c^2$$

$$=-\left(\sum a^2\right)^3+4\left(\sum a^2\right)\left\{\left(\sum ab\right)^2-2abc(2s)\right\}-128R^2r^2s^2$$

$$=\left(\sum a^2\right)\left\{4\left(\sum ab\right)^2-\left(\sum a^2\right)^2-16s\,abc\right\}-128R^2r^2s^2$$

$$=4\left(\sum a^2\right)\{(s^2+4Rr+r^2)^2-(s^2-4Rr-r^2)^2-16Rrs^2\}-128R^2r^2s^2$$

$$=4\left(\sum a^2\right)\{2s^2(8Rr+2r^2)-16Rrs^2\}-128R^2r^2s^2$$

$$=32r^2s^2(s^2-4Rr-r^2)-128R^2r^2s^2\overset{(2)}{=}32r^2s^2(s^2-4Rr-r^2-4R^2)$$

(1), (2) $\Rightarrow\cos A\cos B\cos C=\frac{32r^2s^2(s^2-4R^2-4Rr-r^2)}{128R^2r^2s^2}\overset{(3)}{=}\frac{s^2-4R^2-4Rr-r^2}{4R^2}$

Again, $\sin A\sin B\sin C=\frac{abc}{8R^3}=\frac{4Rrs}{8R^3}\overset{(4)}{=}\frac{rs}{2R^2}$

(3), (4) $\Rightarrow LHS=\frac{4R^2+s^2-4R^2-4Rr-r^2}{4R^2}\times\frac{2R^2}{rs}=\frac{s^2-4Rr-r^2}{2rs}\geq\frac{s}{3r}$

$$\Leftrightarrow 3s^2-12Rr-3r^2\geq 2s^2\Leftrightarrow s^2\geq 12Rr+3r^2\quad \textit{(i)}$$

Gerretsen $\Rightarrow s^2\geq 16Rr-5r^2$ $\therefore$ ***in order to prove (i), it suffices to prove:***

$16Rr-5r^2\geq 12Rr+3r^2\Leftrightarrow 4Rr\geq 8r^2\Leftrightarrow R\geq 2r\rightarrow$ ***true by Euler (Proved)***

SP.076. Let a, b, c be the side – lengths of an acute triangle with perimeter 1. Prove that

$$E_1 \geq a^a b^b c^c \geq E_2$$

where

$$E_1 = \frac{(b+c-a)(c+a-b)(a+b-c)}{(b^2+c^2-a^2)^a (c^2+a^2-b^2)^b (a^2+b^2-c^2)^c},$$

and

$$E_2 = \frac{(b^2+c^2-a^2)^{b+c}(c^2+a^2-b^2)^{c+a}(a^2+b^2-c^2)^{a+b}}{(b+c-a)(c+a-b)(a+b-c)}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c los lados de un triángulo acutángulo con perímetro 1.

Probar que $E_1 \geq a^a b^b c^c \geq E_2$ ***donde*** $E_1 = \frac{(b+c-a)(c+a-b)(a+b-c)}{(b^2+c^2-a^2)^a (c^2+a^2-b^2)^b (a^2+b^2-c^2)^c}$,

$E_2 = \frac{(b^2+c^2-a^2)^{b+c}(c^2+a^2-b^2)^{c+a}(a^2+b^2-c^2)^{a+b}}{(b+c-a)(c+a-b)(a+b-c)}$. ***Como es un triángulo acutángulo***

$\to \cos A, \cos B, \cos C > 0$. ***Recordar las siguientes identidades en un triángulo*** ABC

$$b^2+c^2-a^2 = 2bc\cos A,\ c^2+a^2-b^2 = 2ca\cos B,\ a^2+b^2-c^2 = 2ab\cos C$$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C,\ abc = 4RS,$$

$$16S^2 = (a+b+c)(b+c-a)(c+a-b)(b+a-c)$$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$. ***Lo cual es equivalente***

$$\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C} \Leftrightarrow$$

$$\Leftrightarrow \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B = \sin A \sin B \sin C$$

Probaremos lo siguiente $(b+c-a)(a+c-b)(b+a-c) \geq$

$$\geq (b^2+c^2-a^2)^a \cdot a^a \cdot (c^2+a^2-b^2)^b b^b (a^2+b^2-c^2)^c c^c$$

$$\Leftrightarrow (b+c-a)(a+c-b)(b+a-c) \geq (2abc\cos A)^a (2abc\cos B)^b (2abc\cos C)^c$$

Recordar $\to 2p = a+b+c = 1$. ***Aplicando la desigualdad ponderada MA ≥ MG***

$$\frac{(2abc\cos A)a + (2abc\cos B)b + (2abc)\cos C}{a+b+c} \geq$$

$$\geq \sqrt[a+b+c]{(2abc\cos A)^a (2abc\cos B)^b (2abc\cos C)^c}$$

$$\Leftrightarrow 2abc(a\cos A + b\cos B + c\cos C) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$\Leftrightarrow 2abcR(\sin 2A + \sin 2B + 2\sin 2C) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$\Leftrightarrow 2abc\ R(4\sin A\sin B\sin C) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$\Leftrightarrow 2abcR\left(\frac{abc}{2R^3}\right) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$\Leftrightarrow \frac{a^2b^2c^2}{R^2} \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$(2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c \leq 16S^2 =$$

$$= (a+b+c)(b+c-a)(a+c-b)(b+a-c) = (b+c-a)(a+c-b)(b+a-c)$$

(LQQD). Por último, probaremos

$$a^ab^bc^c(b+c-a)(a+c-b)(b+a-c) \geq$$

$$\geq (b^2+c^2-a^2)^{b+c}(c^2+a^2-b^2)^{c+a}(a^2+b^2-c^2)^{a+b}$$

$$\Leftrightarrow a^ab^bc^c(b+c-a)(a+c-b)(b+a-c) \geq$$

$$\geq (2bc\cos A)^{b+c}(2ca\cos B)^{c+a}(2ab\cos C)^{a+b}$$

$$\Leftrightarrow (b+c-a)(a+c-b)(b+a-c) \geq$$

$$\left[\frac{(2ab\cos C)(2ca\cos B)}{a}\right]^a\left[\frac{(2bc\cos A)(2ab\cos C)}{b}\right]^b\left[\frac{(2ca\cos B)(2bc\cos A)}{c}\right]^c$$

$$\Leftrightarrow (b+c-a)(a+c-b)(b+a-c) \geq$$

$$\geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$

Utilizando la desigualdad ponderada $MA \geq MG$

$$\frac{(4abc\cos B\cos C)a + (4abc\cos C\cos A)b + (4abc\cos A\cos B)c}{a+b+c} \geq$$

$$\geq \sqrt[a+b+c]{(4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c}$$

$$\Leftrightarrow 4abc(a\cos B\cos C + b\cos C\cos A + c\cos A\cos B) \geq$$

$$\geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$

$$\Leftrightarrow 8Rabc\ (\sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B) \geq$$

$$\geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$

$$\Leftrightarrow Rabc\ (8\sin A\sin B\sin C) \geq$$

$$\geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$

$$\Leftrightarrow \frac{a^2b^2c^2}{R^2} \geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$

$$\Leftrightarrow (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c \leq 16S^2 =$$

$$= (a+b+c)(b+c-a)(a+c-b)(b+a-c) = (b+c-a)(a+c-b)(b+a-c)$$

SP.077. Prove that in any acute triangle ABC the following inequality holds

$$\frac{m_a}{h_a}\cos A + \frac{m_a}{h_b}\cos B + \frac{m_c}{h_c}\cos C \geq \frac{3}{2}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo acutángulo ABC la siguiente desigualdad

$\frac{m_a}{h_a}\cos A + \frac{m_b}{h_b}\cos B + \frac{m_c}{h_c}\cos C \geq \frac{3}{2}$. ***Como es un triángulo acutángulo***

$\cos A, \cos B, \cos C > 0$. ***Recordar la siguientes identidades y desigualdades en un***

$\Delta\ ABC\ h_a = \frac{bc}{2R}, h_b = \frac{ca}{2R}, h_c = \frac{ab}{2R}; m_a \geq \frac{b^2+c^2}{4R}, m_b \geq \frac{c^2+a^2}{4R}, m_c \geq \frac{a^2+b^2}{4R}$

Lo cual implica $\Rightarrow \frac{m_a}{h_a}\cos A + \frac{m_b}{h_b}\cos B + \frac{m_c}{h_c}\cos C \geq a\left(\frac{b^2+c^2}{2abc}\right)\cos A + b\left(\frac{c^2+a^2}{2abc}\right)\cos B +$

$+c\left(\frac{a^2+b^2}{2abc}\right)\cos C = \frac{3}{2}$. ***Lo cual es cierto ya que***

$$\Rightarrow a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2) = 3abc$$

$$\Leftrightarrow a^2(b^2+c^2)(2bc\cos A) + b^2(c^2+a^2)(2ca\cos B) + c^2(a^2+b^2)(2ab\cos C) = 6a^2b^2c^2$$

$$\Leftrightarrow a^2(b^2+c^2)(b^2+c^2-a^2) + b^2(c^2+a^2)(c^2+a^2-b^2) +$$

$$+c^2(a^2+b^2)(a^2+b^2-c^2) = a^2(b^2+c^2)^2 - a^4(b^2+c^2) +$$

$$+b^2(c^2+a^2)^2 - b^4(c^2+a^2) + c^2(a^2+b)^2 - c^4(a^2+b^2) =$$

$$= a^2(b^4+c^4) - a^4(b^2+c^2) + b^2(c^4+a^4) - b^4(c^2+a^2)$$

$$+c^2(a^4+b^4) - c^4(a^2+b^2) + 6a^2b^2c^2 = 6a^2b^2c^2$$

SP.078. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that:

$$a^{-a}b^{-b}c^{-c} + a^{-b}b^{-c}c^{-a} + a^{-c}b^{-a}c^{-b} \leq a^{-1} + b^{-1} + c^{-1}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

Applying the Weighted AM-GM inequality we obtain

$$a^{-a}b^{-b}c^{-c} = \left(\frac{1}{a}\right)^a\left(\frac{1}{b}\right)^b\left(\frac{1}{c}\right)^c \le a\cdot\frac{1}{a}+b\cdot\frac{1}{b}+c\cdot\frac{1}{c},$$

$$a^{-b}b^{-c}c^{-a} = \left(\frac{1}{a}\right)^b\left(\frac{1}{b}\right)^c\left(\frac{1}{c}\right)^a \le b\cdot\frac{1}{a}+c\cdot\frac{1}{b}+a\cdot\frac{1}{c},$$

$$a^{-c}b^{-a}c^{-b} = \left(\frac{1}{a}\right)^c\left(\frac{1}{b}\right)^a\left(\frac{1}{c}\right)^b \le c\cdot\frac{1}{a}+a\cdot\frac{1}{b}+b\cdot\frac{1}{c}.$$

Adding up these relations yields

$$a^{-a}b^{-b}c^{-c}+a^{-b}b^{-c}c^{-a}+a^{-c}b^{-a}c^{-b} \le (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = a^{-1}+b^{-1}+c^{-1}$$

as desired.

SP.079. Prove that for all positive real numbers a,b,c and integer $n\ge 3$, the following inequality holds

$$\frac{a^n+b^n+c^n}{9}\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right) \ge \left(\frac{b+c}{6a}+\frac{c+a}{6b}+\frac{a+b}{6c}\right)^n$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

$$\frac{a^n+b^n+c^n}{9}\left(\sum_{cyc}\frac{1}{a^n}\right) = \frac{1}{9}\sum_{cyc}\left(\frac{a^n+b^n}{2}\right)\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)$$

$$=\frac{1}{9}\sum_{cyc}\left(\frac{a+b}{2}\right)^n\left(\sum_{cyc}\frac{1}{a^n}\right) = \sum_{cyc}\left(\frac{a+b}{6}\right)^n\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)\cdot 3^{n-2}$$

$$=\sum_{cyc}\left(\frac{a+b}{6}\right)^n\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)\cdot(1+1+1)\cdot(1+1+1)\dots((n-2)times)$$

$$\overset{HOLDER}{\ge}\left(\sum_{cyc}\sqrt[n]{\left(\frac{a+b}{6}\right)^n\cdot\frac{1}{c^n}\cdot 1}\right)^n = \left(\sum_{cyc}\frac{a+b}{6c}\right)^n$$

SP.080. Prove that for all positive real numbers a, b, c the following inequality holds:

$$\frac{(a+b)^2}{a^2-ab+b^2}+\frac{(b+c)^2}{b^2-bc+c^2}+\frac{(c+a)^2}{c^2-ca+a^2}\geq\frac{9(a^2b+b^2c+c^2a+abc)}{a^3+b^3+c^3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

To solve this problem we must need the following results

Lemma 1. For any positive real numbers u, v, w, x, y, z then $\frac{u^3}{x}+\frac{v^3}{y}+\frac{w^3}{z}\geq\frac{(u+v+w)^3}{3(x+y+z)}$

Proof. By Hölder's inequality we have

$\left(\frac{u^3}{x}+\frac{v^3}{y}+\frac{w^3}{z}\right)(x+y+z)(1+1+1)\geq(u+v+w)^3$ *and the conclusion follows.*

Lemma 2. For all non-negative real numbers a, b, c then

$$a^2b+b^2c+c^2a+abc\leq\frac{4}{27}(a+b+c)^3.$$

Proof. Because the variables a, b, c are cyclic, without loss of generality we can suppose that b is between a and c. Then

$$a^2b+b^2c+c^2a+abc-(a^2b+c^2b+2abc)=c(b-a)(b-c)\leq 0$$

Consequently $a^2b+b^2c+c^2a+abc\leq(a^2b+c^2b+2abc)=b(a+c)^2$

On the other hand, applying the AM-GM inequality we get

$$b(a+c)^2=\frac{1}{2}(2b)(a+c)(a+c)\leq\frac{1}{2}\left(\frac{2b+(a+c)+(a+c)}{3}\right)^3=\frac{4}{27}(a+c+c)^3$$

Hence $a^2b+b^2c+c^2a+abc\leq\frac{4}{27}(a+b+c)^3$. *Come back to the main problem*

We use lemma 1 and lemma 2, respectively, to obtain $LHS=\frac{(b+c)^3}{b^3+c^3}+\frac{(c+a)^3}{c^3+a^3}+\frac{(a+b)^3}{a^3+b^3}\geq$

$\geq\frac{(2a+2b+2c)^3}{3(2a^3+2b^3+2c^2)}=\frac{4(a+b+c)^3}{3(a^3+b^3+c^3)}\geq\frac{9(a^2b+b^2c+c^2a+abc)}{a^3+b^3+c^3}$ *and we are done.*

SP.081. Let a, b, c be positive real numbers and $k\geq 2$. Prove that:

$$\sqrt{\frac{bc}{(b+ka)(c+ka)}}+\sqrt{\frac{ca}{(c+kb)(a+kb)}}+\sqrt{\frac{ab}{(a+kc)(b+kc)}}\geq\frac{3}{k+1}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

Using the AM-GM inequality we obtain:

$$\sum_{cyc}\sqrt{\frac{bc}{(b+ka)(c+ka)}}=\sum_{cyc}\frac{bc}{\sqrt{(bc+kca)(bc+kab)}}\ge\sum_{cyc}\frac{2bc}{2bc+k(ca+ab)}$$

After setting $bc=x, ca=y, ab=z$, the required inequality reduces to:

$$\frac{2x}{2x+k(y+z)}+\frac{2y}{2y+k(z+x)}+\frac{2z}{2z+k(x+y)}\ge\frac{3}{k+1}$$

Applying the Cauchy-Schwarz inequality yields

$$\sum_{cyc}\frac{2x}{2x+k(y+z)}=\sum_{cyc}\frac{2x^2}{2x^2+kx(y+z)}\ge\frac{2(x+y+z)^2}{2(x^2+y^2+z^2)+2k(xy+yz+zx)}=$$

$$=\frac{(x+y+z)^2}{(x+y+z)^2+(k-2)(xy+yz+zx)}\ge\frac{(x+y+z)^2}{(x+y+z)^2+\frac{k-2}{3}(x+y+z)^2}=$$

$=\frac{1}{1+\frac{k-2}{3}}=\frac{3}{k+1}$. *This completes the proof. The equality holds when $a=b=c$.*

SP.082. Let ABC be an equilateral triangle with side-length a and let M be any point inside the triangle. Prove that:

$$\frac{a^2}{2}\ge xMA+yMB+zMC\ge 2(xy+yz+zx)$$

where x,y,z denote the distances from M to the sides BC, CA, AB, respectively.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

We first see easily that

$$x\overrightarrow{MA}+y\overrightarrow{MB}+z\overrightarrow{MC}=\vec{0}\qquad (1)$$

Next, we have:

$$x\cdot PA\cdot MA+y\cdot PB\cdot MB+z\cdot PC\cdot MC\ge x\cdot\overrightarrow{PA}\cdot\overrightarrow{MA}+y\cdot\overrightarrow{PB}\cdot\overrightarrow{MB}+z\cdot\overrightarrow{PC}\cdot\overrightarrow{MC}=$$

$$=x\left(\overrightarrow{PM}+\overrightarrow{MA}\right)\overrightarrow{MA}+y\left(\overrightarrow{PM}+\overrightarrow{MB}\right)\overrightarrow{MB}+z\left(\overrightarrow{PM}+\overrightarrow{MC}\right)\overrightarrow{MC}=$$

$$=\overrightarrow{PM}\left(x\overrightarrow{MA}+y\overrightarrow{MB}+z\overrightarrow{MC}\right)+xMA^2+yMB^2+zMC^2=xMA^2+yMB^2+zMC^2.$$

Now we square both sides of (1) to obtain:

$$x^2MA^2+y^2MB^2+z^2MC^2+2xy\overrightarrow{MA}\cdot\overrightarrow{MB}+2yz\overrightarrow{MB}\cdot\overrightarrow{MC}+2zx\overrightarrow{MC}\cdot\overrightarrow{MA}=0$$

or $x^2MA^2+y^2MB^2+z^2MC^2+xy(MA^2+MB^2-AB^2)+yz(MB^2+MC^2-BC^2)+$

$+zx(MC^2+MA^2-CA^2)=0$ *or*

$(x+y+z)(xMA^2+yMB^2+zMC^2)=yza^2+zxb^2+xyc^2$ *or*

$xMA^2+yMB^2+zMC^2=\frac{(xy+yz+zx)a^2}{x+y+z}$ *(since $a=b=c$)*

Furthermore, it's not difficult to observe that: $x+y+z=h=\frac{a\sqrt{3}}{2}$*. Hence*

$xMA^2+yMB^2+zMC^2=\frac{2a}{\sqrt{3}}(xy+yz+zx)$*. Thus we have proved*

$$x\cdot PA\cdot MA+y\cdot PB\cdot MB+z\cdot PC\cdot MC\ge\frac{2a}{\sqrt{3}}(xy+yz+zx)\quad (2)$$

Also, using the AM-GM inequality we get:

$$xPA^2+xMA^2\ge 2xPA\cdot MA,$$

$$yPB^2+yMB^2\ge 2yPB\cdot MB,$$

$$zPC^2+zMC^2\ge 2zPC\cdot MC,$$

It follows that:

$(xPA^2+yPB^2+zPC^2)+(xMA^2+yMB^2+zMC^2)\ge 2x\cdot PA\cdot MA+2y\cdot PB\cdot MB+$

$+2z\cdot PC\cdot MC$*. On the other hand, according to the above proof, we have*

$$x\cdot PA\cdot MA+y\cdot PB\cdot MB+zc\cdot PC\cdot MC\ge xMA^2+yMB^2+zMC^2$$

Adding up two last results we obtain

$$xPA^2+yPB^2+zPC^2\ge x\cdot PA\cdot MA+y\cdot PB\cdot MB+z\cdot PC\cdot MC\quad (3)$$

Combining (2) and (3) gives us

$$xPA^2+yPB^2+zPC^2\ge x\cdot PA\cdot MA+y\cdot PB\cdot MB+z\cdot PC\cdot MC\ge\frac{2a}{\sqrt{3}}(xy+yz+zx)$$

The equalities occur if and only if $P\equiv M$. From here we take again $P\equiv O$ which is the center of the equilateral triangle ABC then to get

$$(x+y+z)R\ge xMA+yMB+zMC\ge\frac{2a}{R\sqrt{3}}(xy+yz+zx)$$

Also, $x+y+z=\frac{a\sqrt{3}}{2}$ *and* $R=\frac{a}{\sqrt{3}}$*. Therefore we find the desired result.*

SP.083.Let m_a, m_b, m_c be the lengths of the medians of a triangle with circumradius R. Prove that:

$$\left(1+\frac{1}{m_a}\right)\left(1+\frac{1}{m_b}\right)\left(1+\frac{1}{m_c}\right)\geq\left(1+\frac{2}{3R}\right)^3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

Applying the Cauchy – Schwarz Inequality, we have:

$$\frac{1}{m_a+x}+\frac{1}{m_b+x}+\frac{1}{m_c+x}\geq\frac{(1+1+1)^2}{m_a+m_b+m_c+3x}=\frac{9}{m_a+m_b+m_c+3x}, x\geq 0$$

It is well – known that $m_a+m_b+m_c\leq\frac{9R}{2}$***. So,*** $\frac{1}{m_a+x}+\frac{1}{m_b+x}+\frac{1}{m_c+x}\geq\frac{9}{\frac{9R}{2}+3x}=\frac{3}{\frac{3R}{2}+x}$.

Now, $\int_0^1\left(\frac{1}{m_a+x}+\frac{1}{m_b+x}+\frac{1}{m_c+x}\right)dx\geq\int_0^1\frac{3}{\frac{3R}{2}+x}dx$***. So,***

$$[\ln(m_a+x)+\ln(m_b+x)+\ln(m_c+x)]_0^1\geq 3\left[\ln\left(\frac{3R}{2}+x\right)\right]_0^1\Leftrightarrow$$

$$\Leftrightarrow\ln(m_a+1)+\ln(m_b+1)+\ln(m_c+1)-\ln m_a-\ln m_b-\ln m_c\geq$$

$$\geq 3\left(\ln\left(\frac{3R}{2}+1\right)-\ln\frac{3R}{2}\right)\Leftrightarrow\ln\frac{m_a+1}{m_a}+\ln\frac{m_b+1}{m_b}+\ln\frac{m_c+1}{m_c}\geq 3\ln\left(1+\frac{2}{3R}\right)\Leftrightarrow$$

$$\Leftrightarrow\ln\left(\frac{m_{a+1}}{m_a}\cdot\frac{m_b+1}{m_b}\cdot\frac{m_c+1}{m_c}\right)\geq\ln\left(1+\frac{2}{4R}\right)^3$$ ***. Namely***

$$\left(1+\frac{1}{m_a}\right)\left(1+\frac{1}{m_b}\right)\left(1+\frac{1}{m_c}\right)\geq\left(1+\frac{2}{3R}\right)^3$$ ***. Equality holds when triangle*** ABC ***is equilateral.***

SP.084. Prove that if $n\in\mathbb{N}^*$ then:

$$2\int_0^1\arctan(x^{n-1})\arctan(x^n)\,dx\leq\int_0^1\arctan^2(x^n)\,dx+\frac{1}{2n-1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$2\int_0^1\tan^{-1}x^{n-1}\tan^{-1}x^2\,dx\overset{AM\geq GM}{\lesssim}\int_0^1(\tan^{-1}x^{n-1})^2dx+\int_0^1(\tan^{-1}x^n)^2dx$$

$$\leq \int_0^1 x^{2n-2}\,dx + \int_0^1 (\tan^{-1} x^n)^2\,dx\ [\because \tan^{-1} x \leq x] = \frac{1}{2n-1} + \int_0^1 (\tan^{-1} x^n)^2 dx$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$2\int_0^1 a\tan(x^{n-1})\,a\tan(x^n)\,dx \overset{AM-GM}{\leq} \int_0^1 a\tan^2(x^{n-1})\,dx + \int_0^1 a\tan^2(x^n)\,dx$$

$$\overset{a\tan x\leq x}{\leq} \int_0^1 (x^{n-1})^2 dx + \int_0^1 a\tan^2(x^n)\,dx \leq \frac{1}{2n-1} + \int_0^1 a\tan^2(x^n)\,dx$$

Solution 3 by proposer

$$\int_0^1 (x^{n-1} - \arctan(x^n))^2 dx \geq 0$$

$$\int_0^1 x^{2n-2}\,dx - 2\int_0^1 x^{n-1}\arctan(x^n)\,dx + \int_0^1 \arctan^2(x^n)\,dx \geq 0$$

$$x \in [0,1] \Rightarrow x^{n-1} \in [0,1]$$

$$\tan x \geq x; \arctan x \leq x \Rightarrow \arctan(x^{n-1}) \leq x^{n-1}$$

$$\frac{1}{2n-1} + \int_0^1 \arctan^2(x^n)\,dx \geq 2\int_0^1 x^{n-1}\arctan(x^n)\,dx$$

$$\geq 2\int_0^1 \arctan(x^{n-1})\arctan(x^n)\,dx$$

SP.085. Prove that if $a, b \in (0,\infty)$; $n \in \mathbb{N}^*$ then:

$$\left(\frac{a}{b^n}+\frac{b}{a^n}\right)\left(\frac{a^n}{b}+\frac{b^n}{a}\right)\left(\frac{a^n}{b^n}+\frac{b}{a}\right)\left(\frac{b^n}{a^n}+\frac{a}{b}\right) \geq 8\left(\sqrt{\left(\frac{a}{b}\right)^{n-1}}+\sqrt{\left(\frac{b}{a}\right)^{n-1}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$h(x) = \left(x^{n+1} + \frac{1}{x^{n+1}}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right) = (x^2 - 1)\left(x^{n-1} - \frac{1}{x^{n+1}}\right)$$

$$\text{if } x \geq 1 \rightarrow x^2 - 1 \geq 0 \ \& \ x^{n-1} - \frac{1}{x^{n+1}} \geq 0$$

$$\text{if } x \leq 1 \rightarrow x^2 - 1 \leq 0 \ \& \ x^{n-1} - \frac{1}{x^{n+1}} \leq 0 \rightarrow \forall x > 0 \ \ h(x) \geq 0$$

$$f(t) = t^4 - 8t \qquad ; t \geq 2$$

$$f'(t) = 4t^3 - 8 \geq 0 \qquad \forall t \geq 2 \Rightarrow f(t) \geq f(2) = 0$$

$$g(z) = f\left(1 + \frac{1}{z}\right) \overset{z+\frac{1}{z}\geq 2}{\gtrless} 0 \Leftrightarrow \left(z + \frac{1}{z}\right)^4 - 8\left(z + \frac{1}{z}\right) \geq 0$$

$$\Rightarrow A = \left(\frac{a}{b^n} + \frac{b}{a^n}\right)\left(\frac{a^n}{b} + \frac{b^n}{a}\right)\left(\frac{a^n}{b^n} + \frac{b}{a}\right)\left(\frac{b^n}{a^n} + \frac{a}{b}\right)$$

$$= \left(\frac{(a^{n+1} + b^{n+1})^2}{a^{n+1}b^{n+1}}\right)^2 = \left(\frac{a^{\frac{n+1}{2}}}{b^{\frac{n+1}{2}}} + \frac{b^{\frac{n+1}{2}}}{a^{\frac{n+1}{2}}}\right)^4$$

$$\overset{\sqrt{\frac{a}{b}}=x}{=} \left(x^{n+1} + \frac{1}{x^{n+1}}\right)^4 \overset{h(x)\geq 0}{\geq} \left(x^{n-1} + \frac{1}{x^{n-1}}\right)^4$$

$$\overset{g(x^{n-1})\geq 0}{\geq} 8\left(x^{n-1} + \frac{1}{x^{n-1}}\right) = 8\left(\sqrt{\frac{a^{n-1}}{b^{n-1}}} + \sqrt{\frac{b^{n-1}}{a^{n-1}}}\right)$$

Solution by proposer

From means inequality:

$$\frac{a}{b^n} + \frac{b}{a^n} \geq 2\sqrt{\frac{ab}{a^n b^n}} \qquad (1)$$

$$\frac{a^n}{b^n} + \frac{b}{a} \geq 2\sqrt{\frac{a^n b}{b^n a}} \qquad (2)$$

$$\frac{b^n}{a^n} + \frac{a}{b} \geq 2\sqrt{\frac{b^n a}{a^n b}} \qquad (3)$$

By multiplying the relationships (1); (2); (3):

$$\left(\frac{a}{b^n}+\frac{b}{a^n}\right)\left(\frac{a^n}{b^n}+\frac{b}{a}\right)\left(\frac{b^n}{a^n}+\frac{a}{b}\right)\geq\frac{8}{\sqrt{a^{n-1}\cdot b^{n-1}}}\quad (4)$$

We prove that: $\frac{a^n}{b}+\frac{b^n}{a}\geq a^{n-1}+b^{n-1}$ ***(5)***

$$a^{n+1}+b^{n+1}\geq a^n b+ab^n$$

$$a^n(a-b)-b^n(a-b)\geq 0$$

$$(a-b)(a^n-b^n)\geq 0$$

$$(a-b)^2(a^{n-1}+a^{n-2}b+\dots+b^{n-1})\geq 0\quad \textit{(true)}$$

We multiply the relationships (4); (5):

$$\left(\frac{a^n}{b}+\frac{b^n}{a}\right)\left(\frac{a}{b^n}+\frac{b}{a^n}\right)\left(\frac{a^n}{b^n}+\frac{b}{a}\right)\left(\frac{b^n}{a^n}+\frac{a}{b}\right)\geq\frac{8(a^{n-1}+b^{n-1})}{\sqrt{a^{n-1}\cdot b^{n-1}}}=$$

$$=8\left(\sqrt{\left(\frac{a}{b}\right)^{n-1}}+\sqrt{\left(\frac{b}{a}\right)^{n-1}}\right)$$

SP.086. Prove that if a, b, c are the length's sides in $\Delta\ ABC$ then:

$$\sin^2 a+\sin^2 b+\sin^2 c\geq 4\sin s\cdot\sin(s-a)\cdot\sin(s-b)\cdot\sin(s-c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ruanghaw Chaoka-Bangkok-Thailand

$$LHS=\frac{3-\cos 2a-\cos 2b-\cos 2c}{2}$$

$$RHS=(2\sin s\sin(s-a))(2\sin(s-b)\sin(s-c))$$

$$=(\cos a-\cos(2s-a))(\cos(-b+c)-\cos a)$$

$$=\cos a\cos(-b+c)-\cos^2 a-\cos(2s-a)\cos(-b+c)+\cos(2s-a)\cos a$$

$$=\frac{1}{2}(\cos(a-b+c)+\cos(a+b-c)-1-\cos 2a-\cos 2c-\cos 2b+\cos 2s+\cos(-a+b+c))$$

$LHS\geq RHS\Leftrightarrow 4\geq\cos(a-b+c)+\cos(a+b-c)+\cos 2s+\cos(-a+b+c)$ ***True***

$$\because 1\geq\cos x$$

Hold at $a-b+c=2k\pi, a+b-c=21\pi, 2s=2m\pi,$

$$-a+b+c=2n\pi\Leftrightarrow a=p\pi, b=q\pi, c=r\pi$$

Solution 2 by Geanina Tudose-Romania

$$2\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}=\cos\frac{a+b+c-b-c+a}{2}=$$

$$=\cos\frac{a+b+c+b+c-a}{2}=\cos a-\cos(b+c)$$

$2\sin\frac{a-b+c}{2}\sin\frac{a+b-c}{2}=\cos\frac{a-b+c-a+b+c}{2}-\cos\frac{a-b+c+a+b-c}{2}=\cos(c-b)-\cos a$. ***We have*** $\sin^2 a+\sin^2 b+\sin^2 c\ge(\cos a-\cos(b+c))(\cos(c-b)-\cos a)$

$$=2\cos a\cos b\cos c+\cos a\sin b\sin c-\cos^2 a-\cos^2 b\cos^2 c$$

$$-\cos b\sin b\cos c\sin c+\sin^2 b\sin^2 c-\sin b\cos a\sin c+\cos b\sin b\cos c\sin c$$

$$=2\cos a\cos b\cos c-1+\sin^2 a-(1-\sin^2 b)(1-\sin^2 c)+\sin^2 b\sin^2 c$$

$$\Leftrightarrow\sin^2 a+\sin^2 b+\sin^2 c\ge 2\cos a\cos b\cos c-2+\sin^2 a+\sin^2 b+\sin^2 c$$

$\Leftrightarrow 2\cos a\cos b\cos c\le 2$. ***which is obviously true***

Solution 3 by Kevin Soto Palacios-Huarmey-Peru

Siendo a,b,c ***los lados de un triángulo*** ABC***. Probar que***

$$2\sin^2 a+2\sin^2 b+2\sin^2 c\ge 8\sin s\sin(s-a)\sin(s-b)\sin(s-c)$$

$$1-\cos(2a)+1-\cos(2b)+1-\cos(2c)\ge 8\sin s\sin(s-a)\sin(s-b)\sin(s-c)$$

$$3-\cos(2a)-\cos(2b)-\cos(2c)\ge 8\sin s\sin(s-a)\sin(s-b)\sin(s-c)$$

Realizamos los siguientes cambios de variables

$$s-a=x>0,\ s-b=y>0,\ s-c=z>0,$$

$$x+y=c,\ y+z=a,\ z+x=b,\ x+y+z=s$$

$$\sin^2(x+y)+\sin^2(y+z)+\sin^2(z+x)\ge 4\sin(x+y+z)\sin x\sin y\sin z$$

Ahora bien, por transformaciones trigonométricas

$$E=4\sin(x+y+z)\sin x\sin y\sin z=$$

$$=(\cos(y+z)-\cos(2x+y+z))(\cos(y-z)-\cos(y+z))$$

$$2E=2\cos(y-z)\cos(y+z)-2\cos^2(y+z)-2\cos(2x+y+z)\cos(y-z)+$$

$$+2\cos(2x+y+z)\cos(y+z)$$

$$2E=2\cos(y+z)[\cos(y-z)+\cos(2z+y+z)]-1-\cos(2y+2z)-$$

$$-\cos(2z+2x)-\cos(2x+2y)$$

$$2E=4\cos(y+z)\cos(z+x)\cos(x+y)-1-\cos(2y+2z)-$$

$$-\cos(2z+2x)-\cos(2x+2y)$$

Por lo tanto $3-\cos(2y+2z)-\cos(2z+2x)-\cos(2x+2y)\geq$

$$\geq 4\cos(y+z)\cos(z+x)\cos(x+y)-1-\cos(2y+2z)-$$

$$-\cos(2z+2x)-\cos(2x+2y)$$

$$\leftrightarrow 4\geq 4\cos(y+z)\cos(z+x)\cos(x+y)\leftrightarrow 1\geq \cos a\cos b\cos c$$

(lo cual es evidente)

Solution 4 by proposer

$$\sin^2 a+\sin^2 b+\sin^2 c=1-\cos^2 a-\cos^2 b-\cos^2 c+2\geq$$

$$\geq 1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c=$$

$$=1-\sum\frac{1+\cos 2a}{2}+[\cos(a+b)+\cos(a-b)]\cos c=$$

$$=\frac{-1-\sum\cos 2a}{2}+\frac{\cos(a+b+c)+\sum\cos(a+b-c)}{2}=$$

$$=\frac{\cos(a+b+c)}{2}+\frac{1}{2}\sum(\cos(b+c-a)-\cos 2a)=$$

$$=-\sin^2\frac{a+b+c}{2}-\sum\sin\frac{a+b+c}{2}\sin\frac{b+c-3a}{2}=$$

$$=-\sin\frac{a+b+c}{2}\left[\left(\sin\frac{a+b+c}{2}+\sin\frac{b+c-3a}{2}\right)+\left(\sin\frac{a+b-3c}{2}+\frac{a-3b+c}{2}\right)\right]=$$

$$=4\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}\sin\frac{a+c-b}{2}\sin\frac{a+b-c}{2}=$$

$$=4\sin\frac{2s}{2}\sin\frac{2s-2a}{2}\sin\frac{2s-2b}{2}\sin\frac{2s-2c}{2}=4\sin s\sin(s-a)\sin(s-b)\sin(s-c)$$

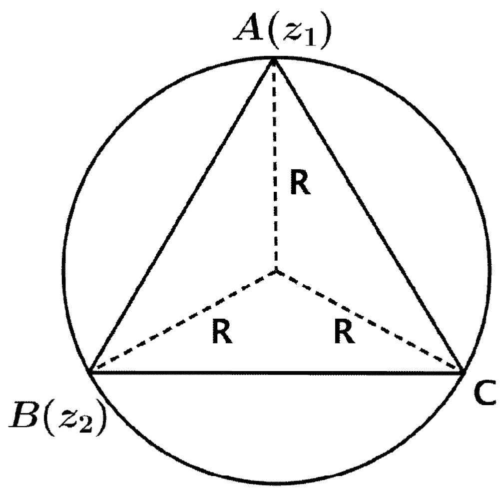
SP.087. Let z_1,z_2,z_3 be the affixes of A,B respectively C in acute-angled $\Delta\,ABC$.

Prove that:

$$\prod\left(\left|\frac{z_2-z_3}{z_2+z_3}\right|+\left|\frac{z_3-z_1}{z_3+z_1}\right|\right)\geq\frac{32sr^3}{(s^2-(2R+r)^2)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India



Let's take O the circumcentre of $\Delta\ ABC$.

Let's take $z_1 = R(\cos\alpha + i\sin\alpha)$; $z_2 = R(\cos\beta + i\sin\beta)$; $z_3 = R(\cos\gamma + i\sin\gamma)$

Now, $\frac{z_2-z_3}{z_2+z_3} = \frac{1-\frac{z_3}{z_2}}{1+\frac{z_3}{z_2}} = \frac{1-\cos(\gamma-\beta)-i\sin(\gamma-\beta)}{1+\cos(\gamma-\beta)+i\sin(\gamma-\beta)} \Rightarrow \left|\frac{z_2-z_3}{z_2+z_3}\right|^2 = \frac{(1-\cos(\gamma-\beta))^2+\sin^2(\gamma-\beta)}{(1+\cos(\gamma-\beta))^2+\sin^2(\gamma-\beta)}$

$= \frac{2[1-\cos(\gamma-\beta)]}{2[1+\cos(\gamma-\beta)]} = \frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A \therefore \left|\frac{z_2-z_3}{z_2+z_3}\right| = \tan A$. *Similalry for other expressions*

$$\therefore LHS = \prod(\tan A + \tan B) = \prod \frac{\sin(A+B)}{\cos A + \cos B} = \prod \frac{\sin C}{\cos A \cos B}$$

$$= \frac{\sin A \sin B \sin C}{\cos^2 A \cos^2 B \cos^2 C} = \frac{16R^4 \sin A \sin B \sin C}{(4R^2 \cos A \cos B \cos C)^2} \quad (1)$$

We now show that $4R^2 \cos A \cos B \cos C = s^2 - (2R+r)^2 = s^2 - (4R^2 + 4Rr + r^2)$

Now, $4Rr + r^2 = \frac{abc}{\Delta} \cdot \frac{\Delta}{s} + \frac{\Delta^2}{s^2}$ $[\Delta = area\ of\ \Delta ABC] = \frac{abc}{s} + \frac{1}{s}(s-a)(s-b)(s-c)$

$$= \frac{1}{s}[abc + (s-a)(s-b)(s-c)]$$

$$= \frac{1}{s}[abc + s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc] = -s^2 + ab + bc + ca$$

$$\therefore s^2 - (4R^2 + 4Rr + r^2) = s^2 - 4R^2 + s^2 - (ab+bc+ca)$$

$$= \frac{1}{2}(a+b+c)^2 - (ab+bc+ca) - (2R)^2 = \frac{1}{2}[a^2 + b^2 + c^2 - 2(2R)^2]$$

$$=\frac{1}{2}(2R)^2[\sin^2 A+\sin^2 B+\sin^2 C-2]=2R^2[-\cos^2 A-(\cos^2 B-\sin^2 C)]$$

$$=2R^2[-\cos^2 A-\cos(B+C)\cos(B-C)]=2R^2[-\cos^2 A+\cos A\cos(B-C)]$$

$$=2R^2(\cos A)[\cos(B-C)+\cos(B+C)]=4R^2\cos A\cos B\cos C\quad \textbf{(2)}$$

Also, $16R^4\sin A\sin B\sin C=4R^2(2R\sin A)(2R\sin B)\sin C$

$$=4R^2ab\sin C=4R^2(2\Delta)=8R^2\Delta=8R^2(sr)\geq 8(2r)^2(sr)$$

$$\therefore 16R^4\sin A\sin B\sin C\geq 32sr^3\quad \textbf{(3)}$$

From (1), (2), (3), we get $\prod\left(\frac{|z_2-z_3|}{|z_2+z_3|}+\frac{|z_1-z_3|}{|z_1+z_3|}\right)\geq\frac{32sr^3}{[s^2-(2R+r)^2]^2}$

SP.088. Let $a,b,c>0$ such that $ab+bc+ca+abc=4$.

Prove that:

$$(a+1)\sqrt{(b+1)(c+1)}+(b+1)\sqrt{(c+1)(a+1)}+(c+1)\sqrt{(a+1)(b+1)}\geq$$
$$\geq a+b+c+9$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Kevin Soto Palacios-Huarmey-Peru

Siendo $a,b,c>0$ *de tal manera que* $ab+bc+ca+abc=4$. *Probar que*

$$(a+1)\sqrt{(b+1)(c+1)}+(b+1)\sqrt{(c+1)(a+1)}+(c+1)\sqrt{(a+1)(b+1)}\geq$$
$$\geq a+b+c+9$$

De la condición, realizamos las siguientes sustituciones

$$a=\frac{2x}{y+z}>0,\ b=\frac{2y}{z+x}>0,\ c=\frac{2z}{x+y}>0,\ \textit{donde } x,y,z>0$$

Aplicando la desigualdad de Cauchy y MA ≥ MG

$$(a+1)\sqrt{(b+1)(c+1)}=\left(\frac{(x+y)+(x+z)}{y+z}\right)\sqrt{\frac{(y+z)+(y+x)}{z+x}\cdot\frac{(y+z)+(z+x)}{x+y}}\geq$$

$$\geq\left(\frac{(x+y)+(x+z)}{y+z}\right)\left(\frac{y+z}{\sqrt{(z+x)(x+y)}}+1\right)=$$

$$=\frac{(x+y)+(x+z)}{y+z}\cdot\frac{y+z}{\sqrt{(z+x)(x+y)}}+\frac{2x}{y+z}+1\geq\frac{2x}{y+z}+3\quad \textbf{(A)}$$

Analogamente para los siguientes términos

$$(b+1)\sqrt{(c+1)(a+1)}\geq\frac{2y}{z+x}+3\quad \textbf{(B)}$$

$$(c+1)\sqrt{(a+1)(b+1)}\geq\frac{2z}{x+y}+3\quad \textbf{(C)}$$

Sumando (A)+(B)+(C)

$$(a+1)\sqrt{(b+1)(c+1)}+(b+1)\sqrt{(c+1)(a+1)}+(c+1)\sqrt{(a+1)(b+1)}\geq$$

$$\geq \frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y} + 9 = a + b + c + 9$$

SP.089. Let r_a, r_b, r_c be the exradii of a triangle ABC, h_a, h_b, h_c the altitudes and let R, r, s denote the circumradius, inradius and semiperimeter respectively. Prove that

$$\frac{r_a^2}{h_a} + \frac{r_b^2}{h_b} + \frac{r_c^2}{h_c} \geq \frac{2s^2}{3}\left(\frac{1}{r} - \frac{1}{R}\right)$$

Proposed by Martin Lukarevski-Skopje

Solution by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $a \geq b \geq c$. *Then,* $r_a^2 \geq r_b^2 \geq r_c^2$ *and* $\frac{1}{h_a} \geq \frac{1}{h_b} \geq \frac{1}{h_c}$

$$\therefore LHS \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum r_a^2\right)\left(\sum \frac{1}{h_a}\right) = \frac{1}{3r}\{(4R+r)^2 - 2s^2\} \overset{?}{\geq} \frac{2s^2(R-r)}{3Rr}$$

$$\Leftrightarrow R(4R+r)^2 - 2Rs^2 \overset{?}{\geq} 2Rs^2 - 2rs^2 \Leftrightarrow (4R-2r)s^2 \overset{?}{\leq} R(4R+r)^2$$

Now, Rouche $\Rightarrow (4R-2r)s^2 \leq (2R^2 + 10Rr - r^2)(4R-2r) +$

$$+2(R-2r)(4R-2r)\sqrt{R^2-2Rr} \overset{?}{\leq} R(4R+r)^2$$

$$\Leftrightarrow (R-2r)(8R^2 - 12Rr + r^2) \overset{?}{\geq} 2(R-2r)(4R-2r)\sqrt{R^2-2Rr}$$

$\because R - 2r \geq 0$ *by Euler* $\therefore$ *it suffices to prove:* $8R^2 - 12Rr + r^2 > 4(2R-r)\sqrt{R^2-2Rr}$

$$[8R^2 - 12Rr + r^2 = (R-2r)(8r+4r) + 9r^2 > 0]$$

$$\Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 16(2R-r)^2(R^2-2Rr) > 0$$

$$\Leftrightarrow 16R^2r^2 + 8Rr^3 + r^4 > 0 \rightarrow true$$ ***(Proved)***

SP.090. If $u, v > 0$, with $2u - v > 0$ and α, β, γ are the measures of the angles of triangle ABC, then

$$\sum_{cyc} \frac{\sin\alpha}{u\sin\beta + v\sqrt{\sin\alpha\sin\beta}} \geq \frac{3}{u+v}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{\sin\alpha}{u\sin\beta+v\sqrt{\sin\alpha\sin\beta}}\overset{AM\ge GM}{\ge}2\sum_{cyc}\frac{\sin\alpha}{2u\sin\beta+v(\sin\alpha+\sin\beta)}$$

$$\ge 2\sum_{cyc}\frac{\sin^2\alpha}{v\sin^2\alpha+(2u+v)\sin\alpha\sin\beta}\overset{Bergstrom}{\ge}\frac{2(\sin\alpha+\sin\beta+\sin\gamma)^2}{v\sum_{cyc}\alpha+(2u+v)\sum_{cyc}\sin\alpha\sin\beta}$$

$$=\frac{2\left(\sum_{cyc}\sin\alpha\right)^2}{v\left(\sum_{cyc}\sin\alpha\right)^2+(2u-v)\sum_{cyc}\sin\alpha\sin\beta}$$

$$\ge 2\frac{\left(\sum_{cyc}\sin\alpha\right)^2}{v\left(\sum_{cyc}\sin\alpha\right)^2+\frac{2u-v}{3}\left(\sum_{cyc}\sin\alpha\right)^2}[\because 2u-v>0]=\frac{3}{u+v}$$

UP.076. Evaluate:

$$S=\sum_{n=1}^{\infty}\left(\frac{H_{2n}+1}{n^2}\right)$$

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Ali Shather-Nasyria-Iraq

$$S=\sum_{n=1}^{\infty}\frac{H_{2n+1}}{n^2}=\sum_{n=1}^{\infty}\frac{H_{2n}}{n^2}+\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}=S_1+S_2$$

$$S_1=\sum_{n=1}^{\infty}\frac{H_{2n}}{n^2}=4\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^2}=4\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\frac{1+(-1)^n}{2}\right)$$

$$=2\sum_{n=1}^{\infty}\frac{H_n}{n^2}+2\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^2}=4\xi(3)-\frac{5}{4}\xi(3)=\frac{11}{4}\xi(3)$$

$$S_2=\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}=\sum_{n=1}^{\infty}\frac{1}{n^2}+4\sum_{n=1}^{\infty}\frac{1}{2n+1}-2\sum_{n=1}^{\infty}\frac{1}{n}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}-2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}}\right)=\xi(2)-2H_{\frac{1}{2}}$$

$$=\xi(2)-2(2-2\ln 2)=\xi(2)-4+4\ln 2$$

$$\therefore S = S_1 + S_2 = \frac{11}{4}\xi(3) + \xi(2) + 4\ln 2 - 4$$

Solution 2 by Khalef Ruhemi-Jarash-Jordan

$$\textbf{Let } S := \sum_{n=1}^{\infty}\frac{H_{2n+1}}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\left(H_{2n} + \frac{1}{2n+1}\right)$$

$$= \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}}_{I_1} + \underbrace{\sum_{n=1}^{\infty}\frac{H_{2n}}{n^2}}_{I_2}$$

$$I_2 = \sum_{n=1}^{\infty}\frac{H_{2n}}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\int_0^1\left(\frac{1-x^{2n}}{1-x}\right)dx = \int_0^1\left(\frac{1}{1-x}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{x^{2n}}{n^2}\right)dx$$

$$= \int_0^1\left(\frac{1}{1-x}\right)\cdot\left(Li_2(1) - Li_2(x^2)\right)dx$$

$$= \left(Li_2(1) - Li_2(x^2)\right)\ln(1-x)\,|_1^0 + \int_0^1 \ln(1-x)\cdot\left(\ln\frac{(1-x^2)}{x^2}\cdot 2x\right)dx$$

$$= \int_0^1 2\ln\frac{(1-x)\ln(1-x^2)}{x}dx = 2\cdot\int_0^1 \ln\frac{(1-x)\cdot(\ln(1-x) + \ln(1+x))dx}{x}$$

$$= 2\cdot\int_0^1 \ln^2\frac{(1-x)}{x}dx + 2\int_0^1 \ln\frac{(1-x)\ln(1+x)}{x}dx$$

$$= 2\int_0^1 \ln^2\frac{(1-x)}{x}\cdot dx + \frac{1}{2}\int_0^1 \underbrace{\frac{\ln^2(1-x^2)}{x}}_{\substack{let\ x^2=y \\ \frac{dx}{x}=\frac{dy}{2y}}}dx - \frac{1}{2}\int_0^1 \underbrace{\frac{\ln^2\left(\frac{1-x}{1+x}\right)}{x}}_{\substack{let\frac{1-x}{1+x}=y \\ x=\frac{1-y}{1+y} \\ dx=-\frac{2dy}{(1+y)^2}}}dx$$

$$\therefore I_2 = 2\cdot\int_0^1\frac{\ln^2(1-x)}{x} + \frac{1}{4}\int_0^1\frac{\ln^2(1-x)}{x} - \int_0^1\frac{\ln^2 x\,dx}{(1-x)(1+x)}$$

$$= \frac{9}{4}\int_0^1\frac{\ln^2 x}{1-x} - \int_0^1\frac{\ln^2 x\,dx}{(1-x)(1+x)}$$

$$=\frac{9}{4}\int_0^1\frac{\ln^2 x\,dx}{1-x}-\frac{1}{2}\int_0^1\frac{\ln^2 x\,dx}{1-x}-\frac{1}{2}\int_0^1\frac{\ln^2 x\,dx}{1+x}$$

$$\therefore I_2=\left(\frac{7}{4}\right)\int_0^1\frac{\ln^2 x\,dx}{1-x}-\left(\frac{1}{2}\right)\int_0^1\frac{\ln^2 x\,dx}{1+x}$$

$$=\left(\frac{7}{4}\right)\cdot\left[\ln^2 x\ln(1-x)\,\Big|_1^0+\int_0^1 2\ln\frac{(1-x)}{x}\ln x\,dx\right]$$

$$-\left(\frac{1}{2}\right)\left[\ln^2 x\ln(1+x)\,\Big|_0^1-\int_0^1 2\ln\frac{(1+x)}{x}\ln x\,dx\right]$$

$$\therefore I_2=\left(\frac{7}{2}\right)\int_0^1\ln x\cdot\ln\frac{(1-x)}{x}dx+\int_0^1\ln x\frac{\ln(1+x)}{x}dx$$

$$=\left(\frac{7}{2}\right)\left[\ln x\cdot Li_2(x)\big|_1^0+\int_0^1\frac{Li_2(x)}{x}dx\right]+\ln(x)\,Li_2(-x)\big|_0^1+\int_0^1 Li_2\frac{(-x)}{x}dx$$

$$=\left(\frac{7}{2}\right)\int_0^1\frac{Li_2(x)}{x}dx+\int_0^1 Li_2\underbrace{\frac{(-x)}{x}}_{\substack{let-x=y\\dx=-dy}}dx=\left(\frac{7}{2}\right)\int_0^1\frac{Li_2(x)dx}{x}+\int_0^{-1}\frac{Li_2(x)}{x}dx$$

$$=\left(\frac{7}{2}\right)Li_3(1)+Li_3(-1)=\left(\frac{7}{2}\right)\varphi(3)-\frac{3}{4}\varphi(3)=\frac{11}{4}\varphi(3)\Rightarrow I_2=\frac{11}{4}\varphi(3)$$

Let $I_1:=\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}=\sum_{n=1}^{\infty}\frac{1}{n^2}-2\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}}\right)$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}-2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}}\right)\ \therefore I_1=\varphi(2)-2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}}\right)$$

But $\int_0^1\underbrace{\left(\frac{1-x^{\frac{1}{2}}}{1-x}\right)}_{\substack{\downarrow\\ let\ x=y^2\\ dx=2ydy}}\cdot dx=\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+\frac{1}{2}}$

$$\therefore \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}}\right)=2\sum_{n=1}^{\infty}\frac{(1-x)}{(1-x^2)}\cdot x\,dx$$

$$=2\cdot\int_0^1\frac{xdx}{1+x}=2\int_0^1\left(1-\frac{1}{1+x}\right)dx=2\left(x-\ln(1+x)|_0^1\right)=2(1-\ln(2))$$

$$\therefore I_1=\varphi(2)-4(1-\ln(2))\therefore S=I_1+I_2=\frac{11}{4}\varphi(3)+\varphi(2)+4\ln(2)-4$$

$$\therefore \sum_{n=1}^{\infty}\frac{H_{2n+1}}{n^2}=\frac{11}{4}\varphi(3)+\frac{\pi^2}{6}+4\ln(2)-4$$

UP.077. Evaluate:

$$S=\prod_{n=1}^{\infty}\left(e\left(\frac{n}{n+1}\right)^n\sqrt{\frac{n}{n+1}}\right)$$

Proposed by Shivam Sharma-New Delhi-India

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$L=\prod_{n=1}^{\infty}\left(e\left(\frac{n}{n+1}\right)^n\sqrt{\frac{n}{n+1}}\right)=\lim_{p\to\infty}\prod_{n=1}^{p}\left(e\left(\frac{n}{n+1}\right)^n\sqrt{\frac{n}{n+1}}\right)$$

$$=\lim_{p\to\infty}e^p\prod_{n=1}^{p}\left(\frac{n^n}{(n+1)^{n+1}}\sqrt{n(n+1)}\right)=\lim_{p\to\infty}e^p\prod_{n=1}^{p}\left(\frac{n^n}{(n+1)^{n+1}}\right)\prod_{n=1}^{p}\left(\sqrt{n(n+1)}\right)$$

$$=\lim_{p\to\infty}e^p\frac{1^1}{(p+1)^{p+1}}\sqrt{p!\,(p+1)!}=\lim_{n\to\infty}\frac{e^p p!}{(p+1)^{p+\frac{1}{2}}}=\lim_{p\to\infty}\frac{e^p\left(\frac{p}{e}\right)^p\sqrt{2\pi p}}{(p+1)^{p+\frac{1}{2}}}$$

$$=\lim_{p\to\infty}\sqrt{2\pi\frac{p}{p+1}}\left(\frac{p}{p+1}\right)^p=\lim_{p\to\infty}\sqrt{2\pi\frac{p}{p+1}}\left(\frac{1}{\left(1+\frac{1}{p}\right)^p}\right)=\frac{\sqrt{2\pi}}{e}\because\lim_{p\to\infty}\left(1+\frac{1}{p}\right)^p=e$$

UP.078. Find:

$$\Omega=\lim_{n\to\infty}n\left(\sqrt[2n+2]{(2n+1)!!}-\sqrt[n]{(2n-1)!!}\right)\left(\sqrt[2n+2]{(n+1)!}-\sqrt[2n]{n!}\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Kays Tomy-Nador-Tunisia

$$\Omega = n\left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right)\left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!}\right)$$

$$\Omega \to \infty$$

First term of limit $\left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!}\right) \sim \left(\exp\frac{\ln(2n+1)!}{2n+2} - \exp\frac{\ln(n!)}{2n}\right)$

$$\sim\left[\exp\left(\frac{2n+2)\ln(2n+2)-(3n+2)}{2n+2}\right) - \exp\left(\frac{n\ln(n)-n}{2n}\right)\right]$$

$$\sim n\left[\exp\left(\frac{\ln(n+1)-1}{2}\right) - \exp\left(\frac{\ln(n)-1}{2}\right)\right]$$

$$\sim n\left(\sqrt{\frac{n+1}{e}} - \sqrt{\frac{n}{e}}\right) \sim \frac{1}{\sqrt{e}}\left(\frac{n}{\sqrt{n+1}+\sqrt{n}}\right) \sim \frac{1}{2}\sqrt{\frac{n}{e}} \to \text{(*)}$$

Now let us denote by $\theta_n = \frac{\ln(2n+1)!!}{2n+2} - \frac{\ln(2n-1)!!}{2n}$

$$\Rightarrow \theta_n \sim \frac{1}{2}\left\{\frac{\ln(2n+1)!!}{n} - \frac{\ln(2n-1)!!}{n}\right\} \Rightarrow \theta_n \sim \frac{1}{2}\ln\left(\sqrt[n]{\frac{(2n+1)!!}{(2n-1)!!}}\right)$$

And as we know that $m!\cdot n! \le (mn)!$***. Whenever*** $m,n \ge 2$***. Then***

$$(2n)! \le (2n+1)!\,(2n)! \le \left((2n+1)2n\right)!$$

$$\le \frac{\left((2n+1)2n\right)!\,(2n-1)!!}{(2n-1)!!} \le \frac{\left((2n+1)2n(2n-1)\right)!}{(2n-1)!!} \le \frac{(2n+1)!!}{(2n-1)!!}$$

Then it follows that $\frac{1}{2}\ln\sqrt[n]{(2n)!} \le \frac{1}{2}\ln\sqrt[n]{\frac{(2n+1)!!}{(2n-1)!!}} \sim \theta_n$

$$\Rightarrow \frac{1}{2}\frac{\ln(2n)!}{n} \le \theta_n \Rightarrow \frac{1}{2}\frac{2n\ln(2n)-2n}{n} \sim \frac{1}{2}\frac{\ln(2n)!}{n} \le \theta_n$$

$$\Rightarrow \underbrace{\ln(2n)-1}_{\to\infty} \le \theta_n \Rightarrow \theta_n \to \infty$$

Now lets look for the limit of $\sqrt[2n+2]{(2n+1)!!} - \sqrt[2n]{(2n-1)!!}$ ***(**)***

By combining (*) and (**) we get

$$\Omega = n\left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right)\left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!}\right) \to \infty$$

Solution 2 by Shivam Sharma-New Delhi-India

$$\Omega = \lim_{n\to\infty} n\left(((2n+1)!!)^{\frac{1}{2n+2}} - ((2n-1)!!)^{\frac{1}{2n}}\right)\left(((n+1)!)^{\frac{1}{2n+2}} - (n!)^{\frac{1}{2n}}\right)$$

As we know, $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$ & $(2n-1)!! = \frac{(2n)!}{2^n n!}$

Using this, we get,

$$\Rightarrow \lim_{n\to\infty} n\left(\left(\frac{(2n+1)!}{2^n n!}\right)^{\frac{1}{2n+2}} - \left(\frac{(2n)!}{2^n n!}\right)^{\frac{1}{2n}}\right)\left(((n+1)!)^{\frac{1}{2n+2}} - (n!)^{\frac{1}{2n}}\right)$$

As we know, the Stirling's formula, we get $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. *Using this, we get,*

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{\left(\left(\frac{2n+1}{e}\right)^{2n+1}\sqrt{2\pi(2n+1)}\right)^{\frac{1}{2n+2}}}{\left(2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)^{\frac{1}{2n+2}}} - \frac{\left(\left(\frac{2n}{e}\right)^{2n}\right)^{\frac{1}{2n}}}{\left(2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)}\right]$$

$$\left[\left(\left(\frac{n+1}{e}\right)^{n+1}\sqrt{2\pi(n+1)}\right)^{\frac{1}{2n+2}} - \left(\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)^{\frac{1}{2n}}\right]$$

As we can see many terms are cancelling, Now, applying ratio test, we get. Our limit,

$\Omega \to \infty$. *Hence* $\Omega = \infty$

Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco

$$\Omega = \lim_{+\infty} n\left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right)\left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!}\right)$$

$$= \lim_{+\infty} n\left(\sqrt[2n]{\frac{(2n+1)!}{2^n \times n!}} - \sqrt[2n]{\frac{(2n-1)!}{2^{n-1}(n-1)!}}\right)\left(\sqrt[2n]{(n+1)!} - \sqrt[2n]{n!}\right)$$

$$= \lim_{+\infty} n\left(\frac{(2n-1)!}{2^{n-1}(n-1)!}\right)^{\frac{1}{2n}}(n!)^{\frac{1}{2n}}\left(\left(\frac{(2n+1)!}{2^n\, n!}\right)\times\left(\frac{2^{n-1}(n-1)!}{(2n-1)!}\right)^{\frac{1}{2n}}\right)^{\frac{1}{2n}}\cdot$$

$$\cdot\left((n+1)^{\frac{1}{2n}}\times\frac{1}{(n!)^{\frac{1}{2n}}} - 1\right)$$

$$=\lim_{+\infty} n\left(\frac{(2n-1)!\times n}{2^{n-1}}\right)^{\frac{1}{2n}}\left(\left(\frac{2n\times(2n+1)}{2\times n}\right)^{\frac{1}{2n}}-1\right)\left((n+1)^{\frac{1}{2n}}-1\right)$$

$$=\lim_{+\infty} n\left(\frac{(2n)!}{2^n}\right)^{\frac{1}{2n}}\left(\sqrt[2n]{2n+1}-1\right)\left(\sqrt[2n]{n+1}-1\right)$$

$$=\lim_{+\infty}\frac{n}{\sqrt{2}}\cdot\left(\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}\right)^{\frac{1}{2n}}\left(\frac{e^{\frac{\ln(2n+1)}{2n}}-1}{\frac{\ln(2n+1)}{2n}}\right)\left(\frac{e^{\frac{\ln(n+1)}{2n}}-1}{\frac{\ln(n+1)}{2n}}\right)\times$$

$$\times\frac{\ln(2n+1)\ln(n+1)}{4n^2}$$

$$\sim\lim_{+\infty}\frac{n}{\sqrt{2}}\times\left(\frac{2n}{e}\right)\times\frac{\ln(2n+1)\ln(n+1)}{4n^2}=\lim_{+\infty}\frac{\sqrt{2}}{4e}\times\ln(2n+1)\ln(n+1)\to+\infty$$

UP.079. If $x,y,z>0$ and $b\ge a>0$ then:

$$\ln\frac{(x+b)(y+b)(z+b)}{(x+a)(x+b)(x+c)}\ge\frac{15}{8}\ln\frac{b}{a}+\frac{1}{16}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)(x^2+y^2+z^2)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

***If** $x,t>0$ **then** $\frac{t^3}{x+t}\ge\frac{5t^2-x^2}{8}\Leftrightarrow(x-t)^2(3t+x)\ge0;\frac{1}{x+t}\ge\frac{5}{8t}-\frac{x^2}{8t^3}$

$$\int_a^b\frac{dt}{x+t}\ge\int_a^b\left(\frac{5}{8t}-\frac{x^2}{8t^3}\right)dt\Leftrightarrow\ln\frac{b+x}{a+x}\ge\frac{5}{8}\ln\frac{b}{a}+\frac{x^2}{16}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\Rightarrow$$

$$\sum_{cyclic}\ln\frac{b+x}{a+x}\ge\sum_{cyclic}\left(\frac{5}{8}\ln\frac{b}{a}+\frac{x^2}{16}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\right)=\frac{15}{8}\ln\frac{b}{a}+\frac{1}{16}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)(x^2+y^2+z^2)$$

UP.080. Let be: $f:(0,\infty)\to(0,\infty)$ a function such that:

$\lim_{x\to\infty}\frac{f(x)}{x}=a\in(0,\infty)$ and $\lim_{x\to\infty}\left(\frac{f(x+1)}{f(x)}\right)^x=b\in(0,\infty)$. Find:

$$\Omega=\lim_{x\to\infty}\left(f(x+1)-f(x)\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Remus Florin Stanca-Romania

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)}=\lim_{x\to\infty}\frac{f(x+1)}{x+1}\cdot\frac{x}{f(x)}\cdot\frac{x+1}{x}=1$$

$$\Rightarrow\lim_{x\to\infty}\left(\frac{f(x+1)}{f(x)}\right)^x=\lim_{x\to\infty}\left(\frac{f(x+1)-f(x)}{f(x)}+1\right)^{\frac{f(x)}{f(x+1)-f(x)}\cdot x\cdot\frac{f(x+1)-f(x)}{f(x)}}=$$

$$=e^{\lim_{x\to\infty}\frac{x}{f(x)}\cdot(f(x+1)-f(x))}=b\Rightarrow e^{\frac{\Omega}{a}}=b\Rightarrow\frac{\Omega}{a}=\ln(b)\Rightarrow\Omega=a\ln(b).$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{x\to\infty}\frac{f(x)}{x}=a\Rightarrow\lim_{\substack{x\to\infty\\ n\in\mathbb{N}}}\frac{f(n)}{n}=a\text{ and }\lim_{x\to\infty}\left(\frac{f(x+1)}{f(x)}\right)^x=b\Rightarrow\lim_{\substack{x\to\infty\\ n\in\mathbb{N}}}\left(\frac{f(n+1)}{f(n)}\right)^n=b$$

$$\lim_{x\to\infty}(f(x+1)-f(x))=\lim_{\substack{x\to\infty\\ n\in\mathbb{N}}}(f(n+1)-f(n))=\lim_{x\to\infty}\left(\frac{f(n)}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)$$

$$\text{where } u_n=\frac{f(n+1)}{f(n)}\text{ for all } n\in\mathbb{N}$$

$$\left[\begin{array}{c}\lim_{x\to\infty}u_n=\lim_{x\to\infty}\left(\frac{f(n+1)}{n+1}\cdot\frac{n}{f(n)}\cdot\left(1+\frac{1}{n}\right)\right)=1\\ \text{hence, }\frac{u_n-1}{\ln u_n}\to 1\text{ as } n\to\infty\end{array}\right]$$

$$\therefore\lim_{x\to\infty}(f(x+1)-f(x))=a\ln b$$

UP.081. If

$$B_n(t)=n^{1-t}\left(\frac{(n+1)^{2t}}{\left(\sqrt[n+1]{(n+1)!}\right)^t}-\frac{n^{2t}}{\left(\sqrt[n]{n!}\right)^t}\right),\text{ with } t>0,\text{ then compute}$$

$$\lim_{n\to+\infty}B_n(t).$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$B_n(t)=n^{1-t}\left(\frac{(n+1)^{2t}}{\sqrt[n+1]{(n+1)!^t}}-\frac{n^{2t}}{\sqrt[n]{n!^t}}\right)$$

$$B_{n\to+\infty}(t)\sim n^{1-t}\left(\frac{(n+1)^{2t}}{\left(\left(\frac{n+1}{e}\sqrt[n+1]{2\pi(n+1)}\right)\right)^{\frac{t}{n+1}}}-\frac{n^{2t}}{\left(\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}\right)^{\frac{t}{n}}}\right)$$

$$\sim n^{1-t}\left(\frac{(n+1)^{2t}}{\left(\frac{n+1}{e}\right)^{t}}-\frac{n^{2t}}{\left(\frac{n}{e}\right)^{t}}\right)\sim n^{1-t}((n+1)^{t}e^{t}-n^{t}e^{t})$$

$$\sim n^{1-t}xn^{t}xe^{t}\left(\left(\frac{n+1}{n}\right)^{t}-1\right)\sim ne^{t}\ln\left(\left(\frac{n+1}{n}\right)^{t}\right)$$

$$\sim nte^{t}\ln\left(\frac{n+1}{n}\right)\sim nte^{t}\left(\frac{n+1}{n}-1\right)\sim te^{t}\Rightarrow\lim_{n\to+\infty}B_{n}(t)=te^{t}$$

Solution 2 by Kays Tomy-Nador-Tunisia

$$B_{n}(t)=n^{1-t}\left(\frac{(n+1)^{2t}}{\left(\sqrt[n+1]{(n+1)!}\right)^{t}}-\frac{n^{2t}}{\left(\sqrt[n]{n!}\right)^{t}}\right)$$

First of all, let us recall Stirling formula $\ln(n!)\sim n\ln(n)-n$. ***Then it follows***

$$\left(\sqrt[n]{n!}\right)^{t}=\left(\exp\left(\frac{\ln(n!)}{n}\right)\right)^{t}\sim\left(\exp\left(\frac{n\ln(n)-n}{n}\right)\right)^{t}$$

$$\sim(\exp(\ln(n)-1))^{t}\sim(\exp(\ln(n)))^{t}e^{-t}\sim n^{t}\cdot e^{-t}$$

likewise we get $\left(\sqrt[n+1]{(n+1)!}\right)^{t}\sim(n+1)^{t}\cdot e^{-t}$ ***then***

$$B_{n}(t)\sim n^{1-t}\left(\frac{(n+1)^{2t}}{(n+1)^{t}}-\frac{n^{2t}}{n^{t}}\right)\cdot e^{t}\sim n^{1-t}((n+1)^{t}-n^{t})e^{t}$$

$\sim n\left(\left(1+\frac{1}{n}\right)^{t}-1\right)e^{t}\sim n\left(1+\frac{1}{n}-1\right)\sim t\cdot e^{t}$. ***Finally we get*** $B_{n}(t)\to_{\infty}te^{t}=te^{t}$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$B_{n}(t)=n^{1-t}\left(\frac{(n+1)^{2t}}{\left(\sqrt[n+1]{(n+1)!}\right)^{t}}-\frac{n^{2t}}{\left(\sqrt[n]{n!}\right)^{t}}\right)$$

$=\left(\left(\frac{n}{\sqrt[n]{n!}}\right)^{t}\cdot\frac{u_{n}-1}{\ln u_{n}}\cdot\ln u_{n}^{n}\right)$ ***where*** $u_{n}=\left(\frac{(n+1)^{2}}{\sqrt[n+1]{(n+1)!}}\cdot\frac{\sqrt[n]{n!}}{n^{2}}\right)^{t}\ \forall n\in\mathbb{N}$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\overset{CAUCHY-D'ALEMBERT}{\cong}\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right)=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\left(1+\frac{1}{n}\right)\right)^t=1,\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\left(1+\frac{1}{n}\right)\cdot\left(1+\frac{1}{n}\right)^{2n}\right)^t=e^t$$

$$\therefore \lim_{n\to\infty}B_n(t)=\lim_{n\to\infty}\left(\left(\frac{n}{\sqrt[n]{n!}}\right)^t\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)=e^t\cdot\ln e^t=te^t\ \textit{(Ans :)}$$

UP.082. Let $n\in N$. Calculate

$$I_n=\int_0^{\frac{\pi}{2}}\sin^2 x\left(\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)+\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Kays Tomy-Nador-Tunisia

$$I_n=\int_0^{\frac{\pi}{2}}\sin^2 x\left(\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)+\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx$$

Let $J_n=\int_0^{\frac{\pi}{2}}\sin^2 x\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)dx.$

And $T_n=\int_0^{\frac{\pi}{2}}\sin^2 x\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)dx$

$\Rightarrow I_n=J_n+T_n$. *As* $\cos^2 x+\sin^2 x=1$. *Then we have*

$$(*)\begin{cases}J_n=\int_0^{\frac{\pi}{2}}\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)-dx-\int_0^{\frac{\pi}{2}}\cos^3 x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)dx\\ T_n=\int_0^{\frac{\pi}{2}}\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)dx-\int_0^{\frac{\pi}{2}}\cos^2 x\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)dx\end{cases}$$

Let us denote $K_n=\int_0^{\frac{\pi}{2}}\cos^3 x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)dx$

By substitution variable $x = \frac{\pi}{2} - U \Rightarrow \sin x = \cos U$ and $\cos x = \sin u$ and $dx = -du$

$$\Rightarrow K_n = -\int_{\frac{\pi}{2}}^{0} \sin^3 u \cos^{2n}\left(\frac{\pi}{2}\cos u\right) du$$

$$\Rightarrow K_n = \int_0^{\frac{\pi}{2}} (1 - \cos^2 u) \sin(u) \cos^{2n+1}\left(\frac{\pi}{2}\cos(u)\right) du = T_n \text{ (**)}$$

Then combining (*) and (**) we get $I_n = \int_0^{\frac{\pi}{2}} \cos x \cos^{2n+1}\left(\frac{\pi}{2}\sin x\right) dx$

substituting $U = \frac{\pi}{2}\sin x \Rightarrow du = \frac{\pi}{2}\cos x\, dx \Rightarrow I_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1}(u)\, du$

Then $I_n = \frac{2}{\pi} W_{2n+1}$; with w_{2n+1} is the Wallis integral for der $2n + 1$

We know that $w_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}$. Finally we get $I_n = \frac{2^{2n+1}(n!)^2}{\pi\,(2n+1)!}$

Solution 2 by Shivam Sharma-New Delhi-India

As we know, the following Lemma,

If $f(x)$ is a continuos function defined on $[a, b]$, then,

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

Using the above Lemma, we get,

$$2I_n = \int_0^{\frac{\pi}{2}} \left[\begin{matrix} \sin^2(x)\left(\cos(x)\cos^{2n+1}\left(\frac{\pi}{2}\sin(x)\right) + \sin(x)\cos^{2n+1}\left(\frac{\pi}{2}\cos(x)\right)\right) + \\ + \cos^2(x)\left(\sin(x)\cos^{2n+1}\left(\frac{\pi}{2}\cos(x)\right) + \cos(x)\cos^{2n+1}\left(\frac{\pi}{2}\sin(x)\right)\right) \end{matrix}\right] dx$$

Applying the above Lemma again, we get,

$$2I_n = 2\int_0^{\frac{\pi}{2}} \sin(x)\cos^{2n+1}\left(\frac{\pi}{2}\cos(x)\right) dx \text{ (OR) } I_n = \int_0^{\frac{\pi}{2}} \sin(x)\cos^{2n+1}\left(\frac{\pi}{2}\cos(x)\right) dx$$

Let, $\frac{\pi}{2}\cos(x) = u$

$$\Rightarrow \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1}(u)\, du \Rightarrow \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1}(u)\, du \Rightarrow \frac{2}{\pi}\cdot\frac{2}{2}\int_0^{\frac{\pi}{2}} \cos^{2n+1}(u)\, du$$

$$\Rightarrow \frac{2}{\pi}\cdot\frac{1}{2}B\left(n + 1, \frac{1}{2}\right) \Rightarrow \frac{1}{\pi}B\left(n + 1, \frac{1}{2}\right)$$

(OR) $I_n = \frac{2^{2n+1}(n!)^2}{\pi\,(2n+1)!}$ $\;n > 0$ ***(Answer). For*** $n > 0$***, we get,*** $I_n = \frac{2^{2n+1}(n!)^2}{\pi(2n+1)!}$

(Answer). For $n = 0$***, we get,*** $I_n = \frac{2}{\pi}$ ***(Answer)***

Solution 3 by Ravi Prakash-New Delhi-India

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x\left(\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right) + \sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx \quad (1)$$

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$***, we get (2)***

$$I_n = \int_0^{\frac{\pi}{2}} \cos^2 x\left\{\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right) + \cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)\right\}dx \quad (3)$$

Adding (1) and (3), we get

$$2I_n = \int_0^{\frac{\pi}{2}} \left(\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx + \int_0^{\frac{\pi}{2}} \cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)dx$$

Using (2) on 2nd integral, we get

$$2I_n = \int_0^{\frac{\pi}{2}} \left(\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx + \int_0^{\frac{\pi}{2}} \left(\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx$$

$$\therefore I_n = \int_0^{\frac{\pi}{2}} \left(\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx$$

Put $\frac{\pi}{2}\cos x = \theta;\ -\frac{\pi}{2}\sin x\,dx = d\theta;\ x = 0, \theta = \frac{\pi}{2}; x = \frac{\pi}{2}, \theta = 0$

$$\therefore I_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta\,d\theta$$

$$= \frac{2}{\pi}\cdot\frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3} = \frac{\left(2^n(n!)\right)^2}{(2n+1)!}\cdot\frac{2}{\pi} = \frac{2^{2n+1}(n!)^2}{(2n+1)!\,\pi}$$

Solution 4 by Abdelhak Maoukouf-Casablanca-Morocco

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x\left(\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right) + \sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx$$

$$I_n=\int_0^{\frac{\pi}{2}}\sin^2 x\left(\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)\right)+\int_0^{\frac{\pi}{2}}\sin^2 x\left(\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx$$

$$I_n=\int_0^{\frac{\pi}{2}}\sin^2 x\left(\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)\right)+\int_0^{\frac{\pi}{2}}\cos^2 x\left(\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)\right)dx$$

$$I_n=\int_0^{\frac{\pi}{2}}\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)dx$$

$$\frac{\pi}{2}\sin x=t\Rightarrow\cos x\,dx=\frac{2}{\pi}dt\left(x=0\Leftrightarrow t=0\ \&x=\frac{\pi}{2}\Leftrightarrow t=\frac{\pi}{2}\right)$$

$$I_n=\int_0^{\frac{\pi}{2}}\frac{2}{\pi}\cos^{2n+1}t\,dt=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(1-\sin^2 t)\cos^{2n-1}t\,dt$$

$$I_n=I_{n-1}-\frac{2}{\pi}\left(\left[\frac{-1}{2n}\cos^{2n}t\sin t\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{-1}{2n}\cos^{2n+1}t\,dt\right)=I_{n-1}-\frac{1}{2n}I_n$$

$$\Rightarrow\frac{I_n}{I_n-1}=\frac{2n}{2n+1}\Rightarrow\prod_{n=1}^{p}\frac{I_n}{I_{n-1}}=\prod_{n=1}^{p}\frac{2n}{2n+1}\times\frac{2n}{2n}$$

$$\Rightarrow\frac{I_p}{I_0}=\frac{2^{2p}(p!)^2}{(2p+1)!}\ \&"\ I_0=\int_0^{\frac{\pi}{2}}\frac{2}{\pi}\cos x\,dx=\frac{2}{\pi}";\ I_n=\frac{2}{\pi}\cdot\frac{2^{2n}(n!)^2}{(2n+1)!}$$

UP.083. Prove that in any triangle ABC the following relationship holds:

$$R\sum(b+c-2a)^2\le 4(R-2r)\sum a^2$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum(b+c-2a)^2=b^2+c^2+4a^2+2bc-4ca-4ab+c^2+a^2+4b^2+$$

$$+2ca-4ab-4bc+a^2+b^2+4c^2+2ab-4bc-4ca$$

$$\overset{(1)}{=} 6\sum a^2 - 6\sum ab$$

(1)⇒ given inequality is $R(6\sum a^2 - 6\sum ab) \le 4(R-2r)\sum a^2$

$$\Leftrightarrow (R+4r)\sum a^2 \le 3R\left(\sum ab\right) \Leftrightarrow (2R+8r)(s^2-4Rr-r^2) \le 3R(s^2+4Rr+r^2)$$

$$\Leftrightarrow Rs^2 + (5R+8r)(4R+r)r \ge 8rs^2 \quad \textbf{(2)}$$

Now, LHS of (2) $\overset{Gerretsen}{\ge} Rr(16R-5r) + (5R+8r)(4R+r)r \overset{?}{\ge} 8rs^2$

$$\Leftrightarrow R(16R-5r) + (5R+8r)(4R+r) \overset{?}{\ge} 8s^2 \Leftrightarrow 8s^2 \overset{?}{\le} 36R^2 + 32Rr + 8r^2$$

$$\Leftrightarrow 2s^2 \overset{?}{\le} 9R^2 + 8Rr + 2r^2 \quad \textbf{(3)}$$

Now, LHS of (3) $\overset{Gerrestsen}{\le} 8R^2 + 8Rr + 6r^2 \overset{?}{\le} 9R^2 + 8Rr + 2r^2$

$\Leftrightarrow R^2 \overset{?}{\ge} 4r^2 \Leftrightarrow R \overset{?}{\ge} 2r \to$ ***true by Euler (Proved)***

UP.084. Evaluate

$$I = \int_0^1 \int_0^1 \frac{(\ln(x)\ln(y))^s}{1-xy} dx\, dy$$

Proposed by Shivam Sharma-New Delhi-India

Solution by Khalef Ruhemi-Jarash-Jordan

Evaluate $I := \int_0^1 \int_0^1 \frac{(\ln(x)\ln(y))^s}{1-xy} \cdot dx \cdot dy$ **(*)**

$$I = \int_0^1 \int_0^1 \frac{\left(\ln\left(\frac{1}{x}\right)\cdot \ln\left(\frac{1}{y}\right)\right)^s \cdot dx \cdot dy}{1-xy} = \int_0^1 \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s \cdot \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot \sum_{n=0}^{\infty} x^n \cdot y^n \cdot dx \cdot dy$$

$$= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s \cdot x^n \cdot \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot y^n \cdot dx \cdot dy$$

$$= \sum_{n=0}^{\infty} \left(\int_0^1 \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot y^n \cdot dy \cdot \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s x^n \cdot dx \right)$$

$$= \sum_{n=0}^{\infty} \left(\int_0^1 x^n \cdot \left(\ln\left(\frac{1}{x}\right)\right)^s dx \right)^2 = I \quad \textbf{(1)}$$

To find $\int_0^1 x^n\left(\ln\left(\frac{1}{x}\right)\right)^s dx := A$, let $\ln\left(\frac{1}{x}\right) = v \Rightarrow x = e^{-v} \Rightarrow dx = -e^{-v}dv$

$$\therefore A = \int_0^\infty v^s \cdot e^{-(1+n)v}dv = \frac{\Gamma(1+s)}{(1+n)^{1+s}} \therefore I = \sum_{n=0}^\infty \frac{\Gamma(1+s)}{(1+n)^{2(1+s)}}$$

$$\therefore I = \sum_{n=0}^\infty \frac{\Gamma^2(1+s)}{n^{2(1+s)}} = \Gamma^2(1+s)\cdot\sum_{n=0}^\infty \frac{1}{n^{2+2s}} = \Gamma^2(1+s)\mathcal{G}(2+2s), s > -\frac{1}{2}$$

$$\therefore \int_0^1\int_0^1 \frac{(\ln(x)\ln(y))^s}{1-xy}dx\,dy = \Gamma^2(1+s)\mathcal{G}(2+2s), s > -\frac{1}{2}$$

UP.085. Let k be positive integer. Calculate:

$$\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{k+1}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{k+1}{x}}\right)\left(\Gamma(x+1)\right)^{-\frac{k}{x}},$$

where $\Gamma(x)$ is the Gamma function (or Euler's second integral).

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{k+1}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{k+1}{x}}\right)\cdot\left(\Gamma(x+1)\right)^{-\frac{k}{x}}$$

$$= \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}}\left(\left(\Gamma(n+2)\right)^{\frac{k+1}{n+1}} - \left(\Gamma(n+1)\right)^{\frac{k+1}{n}}\right)\cdot\left(\Gamma(n+1)\right)^{-\frac{k}{n}}$$

$$= \lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{k+1} - \left(\sqrt[n]{n!}\right)^{k+1}\right)\cdot\left(\sqrt[n]{n!}\right)^{-k}$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) \text{ where } u_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{k+1} \quad \forall n\in\mathbb{N}$$

Now, $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}, \lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}\right)^{k+1} = 1$

$$\therefore \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1\cdot\lim_{n\to\infty}u_n^n = \lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{k+1} = e^{k+1}$$

$$\therefore \lim_{n\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{k+1}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{k+1}{x}}\right)\cdot\left(\Gamma(x+1)\right)^{-\frac{k}{x}}$$

$$= \frac{\ln e^{k+1}}{e} = \frac{k+1}{e}$$

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UP.086. Let $a>0, b, c>1$ and $f, g: R \to R$ be continuos and odd functions. Prove that:

$$\int_{-a}^{a} f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)dx=\left(\ln(bc)\right)\int_{0}^{a} f(x)\,g(x)dx$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Shivam Sharma-New Delhi-India

Let

$$I=\int_{-a}^{a} f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)dx$$

As we know, the following Lemma,

If $f(x)$ is a continuous function defined on $[-a,a]$, then,

$$\int_{-a}^{a} f(x)\,dx=\begin{cases}2\int_{0}^{a} f(x)dx\,, \text{if } f(x) \text{ is an even function}\\ 0, \text{if } f(x) \text{ is an odd function}\end{cases}$$

Using the above lemma, we get,

$$\Rightarrow \int_{-a}^{a} f(-x)\ln\left(b^{g(-x)}+c^{g(-x)}\right)dx \Rightarrow -\int_{-a}^{a} f(x)\ln\left(b^{-g(x)}+c^{-g(x)}\right)dx$$

$$\Rightarrow -\int_{-a}^{a} f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)+\int_{-a}^{a} f(x)\ln\left((bc)^{g(x)}\right)dx$$

$2I \Rightarrow \int_{-a}^{a} f(x)\,g(x)\ln(bc)\,dx$ ***(OR)*** $2I=2\ln(bc)\int_{0}^{a} f(x)\,g(x)dx$ ***(OR)***

$I=\ln(bc)\int_{0}^{a} f(x)\,g(x)dx$ ***(Proved)***

Solution 2 by Hasan Bostanlik-Sarkisla-Turkey

$$A=\int_{-a}^{a} f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)dx$$

$$x=-u \Rightarrow -\int_{a}^{-a} f(-u)\ln\left(\frac{1}{b^{g(u)}}+\frac{1}{c^{g(u)}}\right)du$$

$$A = -\int_{-a}^{a} f(u)\left(\ln\left(b^{g(u)} + c^{g(u)}\right) - g(x)\ln(b\cdot c)\right)du$$

$$2A = \int_{-a}^{a} f(u)g(u)\ln(bc)\,du \Rightarrow h(x) = g(x)f(x)$$

$$h(-x) = g(x)f(x) = h(x)$$

$$2A = 2\int_{0}^{a} f(u)g(u)\ln(bc)\,du$$

$$A = \int_{0}^{a} f(u)\,g(u)h(bc)du$$

Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco

$$I = \int_{-a}^{a} f(x)\ln\left(b^{g(x)} + c^{g(x)}\right)dx$$

$$= \int_{0}^{a} f(x)\ln\left(b^{g(x)} + c^{g(x)}\right)dx + \int_{a}^{0} f(-t)\ln\left(b^{g(-t)} + c^{g(-t)}\right)d(-t)$$

$$= \int_{0}^{a} f(x)\ln\left(b^{g(x)} + c^{g(x)}\right)dx + \int_{0}^{a} -f(t)\ln\left(b^{-g(t)} + c^{-g(t)}\right)dt$$

$$= \int_{0}^{a} f(x)\ln\left(b^{g(x)} + c^{g(x)}\right)dx - \int_{0}^{a} f(x)\ln\left(b^{-g(x)}c^{-g(x)}\left(c^{g(x)} + b^{g(x)}\right)\right)dx$$

$$= \int_{0}^{a} f(x)\ln\left(b^{g(x)} + c^{g(x)}\right)dx - \int_{0}^{a} f(x)\left(\ln\left(b^{-g(x)}c^{-g(x)}\right) + \left(c^{g(x)} + b^{g(x)}\right)\right)dx$$

$$= -\int_{0}^{a} f(x)\ln\left((bc)^{-g(x)}\right)dx = \ln(bc)\int_{0}^{a} f(x)\,g(x)dx$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$f(x) + f(-x) = 0,\ g(x) + g(-x) = 0$$

$$\Omega=\int_{-a}^{a}f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)dx=\int_{-a}^{a}f(a-a-x)\ln\left(b^{g(a-a-x)}+c^{g(a-a-x)}\right)dx$$

$$=\int_{-a}^{a}f(-x)\ln\left(b^{g(-x)}+c^{g(-x)}\right)dx=-\int_{-a}^{a}f(x)\ln\left(b^{-g(x)}+c^{-g(x)}\right)dx$$

$$=-\int_{-a}^{a}f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)+\ln(bc)\int_{-a}^{a}f(x)\,g(x)dx$$

$$\Rightarrow 2\Omega=\ln(bc)\int_{-a}^{a}f(x)\,g(x)dx=2\ln(bc)\int_{0}^{a}f(x)\,g(x)dx$$

$\Omega=\ln(bc)\int_0^a f(x)g(x)\,dx$ ***(Proved)***

UP.087. Let $a,b\in R$, $a<b$ and continuos functions

$f,g,h:R\to R$ such that $f(a+b-x)=-f(x)$, $g(a+b-x)=g(x)$,

$h(a+b-x)=-h(x),\forall x\in R$. Prove that

$$\int_{a}^{b}f(x)\,(\arctan(x))\ln\left(1+e^{h(x)}\right)dx=\frac{1}{2}\int_{a}^{b}f(x)\,h(x)\arctan(x)\,dx$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Shivam Sharma-New Delhi-India

$$\int_{a}^{b}f(x)\,(\tan^{-1}g(x))\ln\left(1+e^{h(x)}\right)dx=\frac{1}{2}\int_{a}^{b}f(x)\,h(x)\tan^{-1}(x)\,dx$$

As we know, the following Lemma,

If $f(x)$ is a continuos function defined on $[a,b]$, then,

$$\int_{a}^{b}f(x)\,dx=\int_{a}^{b}f(a+b-x)\,dx$$

Using the above Lemma, we get. Let,

$$I \Rightarrow \int_a^b f(x)\tan^{-1}(g(x))\ln(1+e^{h(x)})\,dx$$

$$\Rightarrow \int_a^b f(a+b-x)\tan^{-1}(g(a+b-x))\ln(1+e^{h(a+b-x)})\,dx$$

Given: $f(a+b-x) = -f(x);\ g(a+b-x) = g(x);\ h(a+b-x) = -h(x)$

Use the above given values, we get,

$$\Rightarrow \int_a^b (f(x))\tan^{-1}(g(x))\ln(1+e^{-h(x)})\,dx$$

$$\Rightarrow -\int_a^b f(x)\tan^{-1}(g(x))\ln(1+e^{h(x)})\,dx + \int_a^b f(x)\tan^{-1}(g(x))\ln(e^{h(x)})\,dx$$

$$2I = \int_a^b f(x)\tan^{-1}(g(x))\ln(e^{h(x)})\,dx;\ 2I = \int_a^b f(x)\,h(x)\tan^{-1}(g(x))\,dx$$

(OR) $I = \frac{1}{2}\int_a^b f(x)\,h(x)\tan^{-1}(g(x))\,dx$ ***(Proved)***

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$I = \int_a^b f(x)\arctan(g(x))\ln(1+e^{h(x)})\,dx =$$

$$= \int_a^b f(x)\arctan(g(x))\ln\left(e^{h(x)}(e^{-h(x)}+1)\right)dx$$

$$= \int_a^b f(x)\arctan(g(x))\left(h(x)+\ln(e^{-h(x)}+1)\right)dx$$

$$= \int_a^b f(x)\,h(x)\arctan(g(x))\,dx +$$

$$+ \int_a^b f(a+b-t)\arctan(g(a+b-t))\ln(e^{-h(a+b-t)}+1)\,d(a+b-t)$$

$$=\int_a^b f(x)\,h(x)\arctan\big(g(x)\big)\,dx-\int_a^b f(x)\arctan\big(g(x)\big)\ln\left(1+e^{h(x)}\right)dx$$

$$\Leftrightarrow I=\int_a^b f(x)\,h(x)\arctan\big(g(x)\big)\,dx-I\Leftrightarrow I=\frac{1}{2}\int_a^b f(x)\,h(x)\arctan\big(g(x)\big)\,dx$$

Solution 3 by Anas Adlany-El Zemamra-Morocco

$$B=\int_a^b f(x)\arctan\big(g(x)\big)\ln\left(1+e^{h(x)}\right)dx\ \ (u=a+b-x\Rightarrow du=-dx)$$

$$=-\int_a^b f(a+b-u)\arctan\big(g(a+b-x)\big)\ln\left(1+e^{h(a+b-u)}\right)du$$

$$=\int_a^b -f(x)\arctan\big(g(x)\big)\ln\left(\frac{1+e^{h(x)}}{e^{h(x)}}\right)$$

$$=-\int_a^b f(x)\arctan\big(g(x)\big)\left[\ln\left(1+e^{h(x)}\right)-h(x)\right]dx$$

$$=-B+\int_a^b f(x)\,h(x)\arctan\big(g(x)\big)\,dx\Rightarrow$$

$$\int_a^b f(x)\arctan\big(g(x)\big)\ln\left(1+e^{h(x)}\right)dx=\frac{1}{2}\int_a^b f(x)\,h(x)\arctan\big(g(x)\big)\,dx.!$$

UP.088. Let $f:R\to R$ be a continuous function such that

$f(x)=f(1-x),\forall x\in R$. Prove that

$$\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx=\sqrt{2}\cdot\int_0^1 f(x)\,dx$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Thanasis Xenos-Greece

$$I=\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx \quad \textbf{(1)}$$

$$t=1-x$$

$$I=\int_1^0\frac{\sqrt{t}+\sqrt{1-t}}{1+\sqrt{2(1-t)}}f(1-t)(-dt)$$

$$I=\int_0^1\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2(1-x)}}f(x)dx \quad \textbf{(2)}$$

$$\textbf{(1)+(2)}\Rightarrow 2I=\int_0^1\left(\sqrt{x}+\sqrt{1-x}\right)\cdot\left(\frac{1}{1+\sqrt{2x}}+\frac{1}{1+\sqrt{2(1-x)}}\right)f(x)dx$$

$$2I=\int_0^1\left(\sqrt{x}+\sqrt{1-x}\right)\cdot\frac{2+\sqrt{2x}+\sqrt{2(1-x)}}{\left(1+\sqrt{2x}\right)\cdot\left(1+\sqrt{2(1-x)}\right)}f(x)dx$$

$$2I=\sqrt{2}\int_0^1\frac{\left(\sqrt{x}+\sqrt{1-x}\right)\left(\sqrt{2}+\sqrt{x}+\sqrt{1-x}\right)}{\left(1+\sqrt{2x}\right)\cdot\left(1+\sqrt{2(1-x)}\right)}f(x)dx \quad \textbf{(3)}$$

$$\left(\sqrt{x}+\sqrt{1-x}\right)\cdot\left(\sqrt{2}+\sqrt{x}+\sqrt{1-x}\right)$$

$$=\sqrt{2x}+x+\sqrt{x(1-x)}+\sqrt{2(1-x)}+\sqrt{x(1+x)}+1-x$$

$$=1+\sqrt{2x}+\sqrt{2(1-x)}+2\sqrt{x(1-x)}$$

$$=\left(1+\sqrt{2x}\right)\left(1+\sqrt{2(1-x)}\right)$$

$$\textbf{(3)}\Rightarrow 2I=\sqrt{2}\cdot\int_0^1 f(x)\,dx\Rightarrow I=\frac{\sqrt{2}}{2}\cdot\int_0^1 f(x)\,dx$$

Solution 2 by Shivam Sharma-New Delhi-India

As we know the following Lemma,

If $f(x)$ is a continuous function defined on $[0,a]$, then,

$$\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx \quad \textbf{(I)}$$

Using the above lemma, we get $\Rightarrow \int_0^1\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2(1-x)}}f(1-x)dx$***. As it is given,***

$$f(x)=f(1-x) \ \textbf{(II)}$$

Using (II), we get, $\Rightarrow \int_0^1\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2(1-x)}}f(x)dx$***. Now,***

$$2I=\int_0^1 f(x)\left(\frac{1}{1+\sqrt{2x}}+\frac{1}{1+\sqrt{2(1-x)}}\right)(\sqrt{1-x}+\sqrt{x})dx$$

$$2I=\sqrt{2}\int_0^1 f(x)\left(\frac{1+\sqrt{2}\sqrt{1-x}+1+\sqrt{2}\sqrt{x}}{2x(1-x)+1+\sqrt{2}\sqrt{1-x}+\sqrt{2}\sqrt{x}}\right)(\sqrt{1-x}+\sqrt{x})dx$$

$$2I=\sqrt{2}\int_0^1 f(x)\left(\frac{1+2x(1-x)+\sqrt{2}\sqrt{1-x}+\sqrt{2}\sqrt{x}}{1+2x(1-x)+\sqrt{2}\sqrt{1-x}+\sqrt{2}\sqrt{x}}\right)dx$$

$2I=\sqrt{2}\int_0^1 f(x)\,dx$ ***(OR)*** $2I=\sqrt{2}\int_0^1 f(x)\,dx$ ***(OR)*** $I=\frac{1}{\sqrt{2}}\int_0^1 f(x)\,dx$ ***(Proved)***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\Omega=\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx=\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2(1-x)}}f(1-x)dx$$

Applying $g(a+b-x)=g(x)$

$$\therefore 2\Omega=\int_0^1(\sqrt{x}+\sqrt{1-x})\left(\frac{1}{1+\sqrt{2x}}+\frac{1}{1+\sqrt{2(1-x)}}\right)f(x)dx$$

$$=\int_0^1(\sqrt{x}+\sqrt{1-x})\cdot\frac{2+\sqrt{2x}+\sqrt{2(1-x)}}{(1+\sqrt{2x})(1+\sqrt{2(1-x)})}f(x)dx$$

$$=\int_0^1\frac{\sqrt{2}f(x)(\sqrt{x}+\sqrt{1-x})(\sqrt{2}+\sqrt{x}+\sqrt{1-x})}{\sqrt{2}(\sqrt{x}+\sqrt{1-x})+(\sqrt{x}+\sqrt{1-x})^2}dx=\sqrt{2}\int_0^1 f(x)\,dx$$

$$\therefore\int_0^1\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2x}}f(x)dx=\frac{1}{\sqrt{2}}\int_0^1 f(x)dx$$

Solution 4 by Hasan Bostanlik-Sarkisla-Turkey

$$I=\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2}\sqrt{x}}f(x)dx$$

$$x=1-u\Rightarrow-\int_1^0\frac{\sqrt{u}+\sqrt{1-u}}{1+\sqrt{2}\sqrt{1-u}}f(1-u)(-du)=\int_0^1\frac{\sqrt{u}+\sqrt{1-a}}{1+\sqrt{2}\sqrt{1-u}}f(u)du$$

$$2I=\int_0^1\left(\sqrt{u}+\sqrt{1-u}\right)f(u)\left(\frac{1}{1+\sqrt{2}\sqrt{1-u}}+\frac{1}{1+\sqrt{2}\sqrt{u}}\right)du$$

$$2I=\sqrt{2}\int\left(\sqrt{u}+\sqrt{1}-u\right)\left(f(u)\right)\left(\frac{\sqrt{2}+\sqrt{1-u}+\sqrt{u}}{\left(1+\sqrt{2}\sqrt{1-u}\right)\left(1+\sqrt{2}\sqrt{u}\right)}\right)du$$

$$2I=\sqrt{2}\int_0^1 f(u)\left[\frac{\sqrt{2}\sqrt{u}+\sqrt{2}\sqrt{1-u}+\sqrt{u}\sqrt{1-u}+1-u+u+\sqrt{u}\sqrt{1-u}}{1+\sqrt{2}\sqrt{u}+\sqrt{2}\sqrt{1-u}+2\sqrt{1-u}\sqrt{u}}\right]$$

$$2I=\sqrt{2}\int_0^1 f(u)\left[\frac{1+\sqrt{2}\sqrt{u}+\sqrt{2}\sqrt{1-u}+2\sqrt{u}\sqrt{1-u}}{1+2\sqrt{u}+\sqrt{2}\sqrt{1-u}+2\sqrt{u}\sqrt{u-1}}\right]du$$

$$2I=\sqrt{2}\int_0^1 f(u)\,du;\ I=\frac{1}{\sqrt{2}}\int_0^1 f(u)\,du$$

Solution 5 by Nirapada Pal-Jhargram-India

$$\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{2}\left(1+\sqrt{2x}\right)}f(x)dx=\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{2}+2\sqrt{x}}f(x)dx$$

$$=\frac{1}{2}\int_0^1\frac{2\sqrt{1-x}+2\sqrt{x}+\sqrt{2}-\sqrt{2}}{\sqrt{2}+2\sqrt{x}}f(x)dx=\frac{1}{2}\int_0^1 f(x)\,dx+\frac{1}{2}\int_0^1\frac{\sqrt{2(1-x)}-1}{1+\sqrt{2x}}f(x)dx$$

Let $I=\int_0^1\frac{\sqrt{2(1-x)}-1}{1+\sqrt{2x}}f(x)dx$**. Or,**

$$I=\int_0^1\frac{\sqrt{2x}-1}{1+\sqrt{2(1-x)}}f(1-x)dx=\int_0^1\frac{\sqrt{2x}-1}{1+\sqrt{2(1-x)}}f(x)dx$$

Or, $2I=\int_0^1\left[\frac{2(1-x)-1+2x-1}{\left(1+\sqrt{2x}\right)\left(1+\sqrt{2(1-x)}\right)}\right]f(x)dx=0$**. Or,** $I=0$

So, $\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{2}\left(1+\sqrt{2x}\right)}f(x)dx=\frac{1}{2}\int_0^1 f(x)\,dx$**. Or,** $\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx=\frac{1}{\sqrt{2}}\int_0^1 f(x)\,dx$

Solution 6 by Abdelhak Maoukouf-Casablanca-Morocco

$$I=\int_0^1\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx=\int_0^1\frac{1-2x}{1-2x}\cdot\frac{1-\sqrt{2x}}{\sqrt{1-x}-\sqrt{x}}f(x)dx$$

$$I=\int_0^1 \frac{1-\sqrt{2x}}{\sqrt{1-x}-\sqrt{x}}f(x)dx=\int_0^1 \frac{1-\sqrt{2(1-x)}}{\sqrt{x}-\sqrt{1-x}}f(x)dx$$

"$f(x)=f(1-x)$"

$$=\int_0^1 \frac{1-\sqrt{2x}+\sqrt{2x}-\sqrt{2(1-x)}}{\sqrt{x}-\sqrt{1-x}}f(x)dx=-I+\int_0^1 \frac{\sqrt{2x}-\sqrt{2(1-x)}}{\sqrt{x}-\sqrt{1-x}}f(x)dx$$

$$\rightarrow 2I=\sqrt{2}\int_0^1 f(x)\,dx \Leftrightarrow I=\frac{\sqrt{2}}{2}\int_0^1 f(x)\,dx$$

UP.089. Evalute:

$$\int_0^1 [\ln(x)\ln(1-x)+Li_2(x)]\left(\frac{Li_2(x)}{x(1-x)}-\frac{\zeta(2)}{1-x}\right)dx$$

Proposed by Shivam Sharma-New Delhi-India

Solution by Ali Shather-Iraq

$$I=\int_0^1 (\ln(x)\ln(1-x)+Li_2(x))\left(\frac{Li_2(x)}{x(1-x)}-\frac{\zeta(2)}{1-x}\right)dx$$

$$I=\int_0^1 \frac{\ln(x)\ln(1-x)\,Li_2(x)}{x}dx+\int_0^1 \frac{\ln(x)\ln(1-x)\,Li_2(x)}{1-x}-$$

$$-\zeta(2)\int_0^1 \frac{\ln(x)\ln(1-x)}{1-x}dx+\int_0^1 \frac{Li_2^2(x)}{x}dx+\int_0^1 \frac{Li_2^2(x)-\zeta(2)Li_2(x)}{1-x}dx$$

combining the second and the las term, we get

$$\int_0^1 \frac{Li_2^2(x)-Li_2(x)[\zeta(2)-\ln(x)\ln(1-x)]}{1-x}dx=\int_0^1 \frac{Li_2^2(x)-Li_2(x)[Li_2(x)+Li_2(1-x)]}{1-x}dx=$$

$$=-\int_0^1 \frac{Li_2(x)Li_2(1-x)}{1-x}dx=-\int_0^1 \frac{Li_2(1-x)Li_2(x)}{x}dx=$$

$$= -\int_0^1 \frac{Li_2(x)}{x}[\zeta(2) - \ln(x)\ln(1-x) - Li_2(x)]dx = -\zeta(2)\int_0^1 \frac{Li_2(x)}{x}dx +$$

$$+\int_0^1 \frac{\ln(x)\ln(1-x)\,Li_2(x)}{x}dx + \int_0^1 \frac{Li_2^2(x)}{x}dx$$

$$\therefore I = \int_0^1 \frac{\ln(x)\ln(1-x)\,Li_2(x)}{x}dx - \zeta(2)\int_0^1 \frac{\ln(x)\ln(1-x)}{1-x}dx + \int_0^1 \frac{Li_2^2(x)}{x}dx -$$

$$-\zeta(2)\int_0^1 \frac{Li_2(x)}{x}dx + \int_0^1 \frac{\ln(x)\ln(1-x)\,Li_2(x)}{x}dx + \int_0^1 \frac{Li_2^2(x)}{x}dx$$

$$I = 2\int_0^1 \frac{\ln(x)\ln(1-x)\,Li_2(x)}{x}dx - \zeta(2)\int_0^1 \frac{\ln(x)\ln(1-x)}{1-x}dx + 2\int_0^1 \frac{Li_2^2(x)}{x}dx - \zeta(2)\int_0^1 \frac{Li_2(x)}{x}dx$$

$$I = \left[-Li_2^2(x)\ln(x)\right]_0^1 + \int_0^1 \frac{Li_2^2(x)}{x} - \zeta(2)\int_0^1 \frac{\ln(1-x)\ln(x)}{x}dx + 2\int_0^1 \frac{Li_2^2(x)}{x}dx - \zeta(2)\zeta(3)$$

$$I = 3\int_0^1 \frac{Li_2^2(x)}{x}dx + \zeta(2)\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\ln(x)\,dx - \zeta(2)\zeta(3) =$$

$$= 3\int_0^1 \frac{Li_2^2(x)}{x}dx - \zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^3} - \zeta(2)\zeta(3) = 3\int_0^1 \frac{Li_2^2(x)}{x}dx - 2\zeta(2)\zeta(3)$$

$$\int_0^1 \frac{Li_2^2(x)}{x}dx = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(nk)^2}\int_0^1 x^{n+k-1}\,dx = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(nk)^2(n+k)} =$$

$$= \sum_{n=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{\infty}\frac{k}{n^2(n+k)} = \sum_{n=1}^{\infty}\frac{1}{k^3}\left[\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n(n+k)}\right]$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^3}\left[\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{H_k}{k}\right] = \zeta(3)\zeta(2) - \sum_{n=1}^{\infty}\frac{H_k}{k^4} = \zeta(3)\zeta(2) - \big(3\zeta(5) - \zeta(3)\zeta(2)\big) =$$

$$= 2\zeta(3)\zeta(2) - 3\zeta(5)$$

$$\therefore I = 3\big(2\zeta(2)\zeta(3) - 3\zeta(5)\big) - 2\zeta(2)\zeta(3) = 4\zeta(2)\zeta(3) - 9\zeta(5)$$

UP.090. Evaluate:

$$\int_0^1 (\ln(\Gamma(x)))(\sin(2k\pi x))\,dx, \qquad k \geq 1$$

Proposed by Shivam Sharma – New Delhi – India

Solution by Khalef Ruhemi-Jarash-Jordan

$$I = \int_0^1 \ln(\Gamma(x))\sin(2\pi kx)\,dx \,\ldots.\#$$

Notice that $\int_0^1 \frac{1-t^{x-1}}{1-t}\cdot dt = \frac{\Gamma'(x)}{\Gamma(x)} - \Gamma'(1)$

$$\therefore \int_0^1 \left(\frac{1}{1-t}\right)\left(\int_1^x (1-t^{v-1})dv\right)\cdot dt = \int_1^x \left(\frac{\Gamma'(v)}{\Gamma(v)} - \Gamma'(1)\right)dv$$

$$= \ln(\Gamma(v)) - v\Gamma(1)|_1^x = \ln(\Gamma(x)) - x\Gamma'(1) + \Gamma'(1)$$

$$= \int_0^1 \left(\frac{1}{1-t}\right)\cdot\left(v - \frac{t^{v-1}}{\ln(t)}\right)\cdot dt = \int_0^1 \left(\frac{1}{1-t}\right)\cdot\left(x - \frac{t^{x-1}}{\ln(t)} - 1 + \frac{1}{\ln(t)}\right)dt$$

$$= \int_0^1 \left(\frac{1-t^{x-1}-(1-x)\ln(t)}{(1-t)\ln(t)}\right)\cdot dt = \ln(\Gamma(x)) + \Gamma'(1)(1-x)$$

$$\therefore \ln(\Gamma(x)) = \Gamma'(1)(x-1) + \int_0^1 \frac{1-t^{x-1}-(1-x)\ln(t)}{(1-t)\ln(t)}\cdot dt \quad (1)$$

$$\therefore \ln(\Gamma(x))\sin(2\pi kx) = \Gamma'(1)(x-1)\sin(2\pi kx)$$

$$+ \int_0^1 \frac{\sin(2\pi kx) - \ln(t)\sin(2\pi kx) + \ln(t)x\sin(2\pi kx) - t^{x-1}\cdot\sin(2\pi kx)}{(1-t)\ln(t)}\cdot dt \quad (2)$$

$$\therefore \int_0^1 \ln(\Gamma(x))\sin(2\pi kx)\,dx = \Gamma'(1)\int_0^1 (x-1)\sin(2\pi kx)\,dx$$

$$+ \int_0^1 \frac{dt}{(1-t)\ln(t)}\cdot\left[\begin{array}{c}\int_0^1 \sin(2\pi kx)\,dx - \ln(t)\int_0^1 \sin(2\pi kx)\,dx \\ + \ln(t)\int_0^1 x\sin(2\pi kx)\,dx - \frac{1}{t}\int_0^1 t^x \sin(2\pi kx)\,dx\end{array}\right] \quad (3)$$

$$\int_0^1 \sin(2\pi kx)\,dx = \cos\frac{(2\pi kx)}{2\pi k}\cdot|_1^0 = \frac{1-1}{2\pi k} = 0$$

$$\int_0^1 x\sin(2\pi kx)\,dx = x\cos\frac{(2\pi kx)}{2\pi k}\Big|_1^0 + \frac{1}{2\pi k}\cdot\int_0^1 \cos(2\pi kx)\,dx$$

$$= -\frac{\cos(2\pi k)}{2\pi k} + \left(\frac{1}{4\pi^2k^2}\right)\sin(2\pi kx)\,\Big|_0^1 = -\frac{1}{2\pi k}$$

$$\int_0^1 t^x\sin(2\pi kx)\,dx = \int_0^1 e^{(\ln(t))x}\cdot\sin(2\pi kx)\,dx$$

$$= \frac{e^{\ln(t)x}}{\ln(t)}\cdot\sin(2\pi kx)\,\Big|_0^1 - \frac{2\pi k}{\ln(t)}\int_0^1 e^{\ln(t)x}\cdot\cos(2\pi kx)$$

$$= -\frac{2\pi k}{\ln(t)}\cdot\left(\frac{e^{\ln(t)x}}{\ln(t)}\cdot\cos(2\pi kx)\,\Big|_0^1 + \frac{2\pi k}{\ln(t)}\int_0^1 e^{\ln(t)x}\cdot\sin(2\pi kx)\,dx\right)$$

$$= -\frac{2\pi k}{\ln(t)}\cdot\left(\frac{(t-1)}{\ln(t)} + \frac{2\pi k}{\ln(t)}\int_0^1 e^{\ln(t)x}\sin(2\pi kx)\,dx\right)$$

$$\therefore \int_0^1 t^x\sin(2\pi kx)\,dx = \frac{2\pi k(1-t)}{\ln^2(t)} - \frac{4\pi^2k^2}{\ln^2(t)}\cdot\int_0^1 t^x\sin(2\pi kx)\,dx$$

$$\therefore \int_0^1 t^x\sin(2\pi kx)\,dx = \frac{\frac{2\pi k(1-t)}{\ln^2(t)}}{1+\frac{4\pi^2k^2}{\ln^2(t)}} = \frac{2\pi k(1-t)}{4\pi^2k^2+\ln^2(t)}$$

$$\therefore \int_0^1 t^x\sin(2\pi kx)\,dx = \frac{2\pi k(1-t)}{4\pi^2k^2+\ln^2(t)}$$

$$\therefore I = \int_0^1\left(-\frac{\ln(t)}{2\pi k} - \frac{2\pi k(1-t)}{t(4\pi^2k^2+\ln^2(t))}\right)\cdot\frac{dt}{(1-t)\ln(t)} - \frac{\Gamma'(1)}{2\pi k}\quad \textbf{(4)}$$

$$\therefore I = -\frac{\Gamma'(1)}{B} - \int_0^1\left(\frac{\ln(x)}{B} + \frac{B(1-x)}{x(B^2+\ln^2 x)}\right)\cdot\frac{dx}{(1-x)\ln(x)},\ B \coloneqq 2\pi k$$

$$\therefore I = -\frac{\Gamma'(1)}{B} - \int_0^1\left(\frac{1}{B(1-x)} + \frac{1}{B\ln(x)} + \frac{B}{x\ln x\,(B^2+\ln^2 x)} - \frac{1}{B\ln(x)}\right)dx$$

$$= -\frac{\Gamma'(1)}{B} - \frac{1}{B}\int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x}\right)dx + \int_0^1 \frac{1}{\ln(x)}\left(\frac{1}{B} - \frac{B}{x(B^2+\ln^2 x)}\right)dx$$

$$= -\frac{\Gamma'(1)}{B} + \frac{\Gamma'(1)}{B} + \int_0^1 \frac{1}{\ln(x)}\left(\frac{1}{B} - \frac{B}{x(B^2+\ln^2 x)}\right)dx$$

$$\therefore I = \int_0^1 \frac{1}{\ln(x)}\left(\frac{1}{B} - \frac{B}{x(B^2+\ln^2 x)}\right)dx \quad (6)$$

Let $\ln\left(\frac{1}{x}\right) = y \Rightarrow x = e^{-y} \Rightarrow dx = -e^{-y}\cdot dy$

$$\therefore I = \int_0^\infty -\frac{1}{y}\left(\frac{1}{B} - \frac{B}{e^{-y}(B^2+y^2)}\right)e^{-y}\cdot dy$$

$$= \int_0^\infty \left(\frac{-e^{-y}}{By} + \frac{B}{y(B^2+y^2)}\right)dy = \int_0^1 \left(\frac{1}{By} - \frac{e^{-y}}{By} - \frac{y}{B(B^2+y^2)}\right)dy$$

$$= \frac{1}{B}\int_0^\infty \left(\frac{1-e^{-x}}{x} - \frac{x}{B^2+x^2}\right)dx = I \quad (7)$$

Let $x = By \Rightarrow dx = Bdy$

$$\therefore I = \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1-e^{-Bx}}{x} - \frac{x}{1+x^2}\right)dx$$

$$\therefore I = \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1-e^{-BX}}{x} + \frac{1}{1+x} - \frac{x}{1+x^2} - \frac{1}{1+x}\right)dx$$

$$= \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1}{1+x} - \frac{x}{1+x^2}\right)dx + \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1-e^{-Bx}}{x} - \frac{1}{1+x}\right)dx$$

Since $\int_0^\infty \left(\frac{1}{1+x} - \frac{x}{1+x^2}\right)dx = \ln\left(\frac{1+x}{\sqrt{1+x^2}}\right)\Big|_0^\infty$

$$= \lim_{n\to\infty} \ln\left(\frac{\frac{1}{x}+1}{\sqrt{\frac{1}{x^2}+1}}\right) = \ln(1) = 0$$

$$\therefore I = \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1-e^{-Bx}}{x} - \frac{1}{1+x}\right)dx$$

$$\therefore I = \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1}{x} - \frac{1}{1+x} - \frac{e^{-x}}{x} + \frac{e^{-x}}{x} - \frac{e^{-Bx}}{x}\right) dx$$

$$= \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x}dx + \left(\frac{1}{B}\right)\int_0^\infty \frac{e^{-x}-e^{-Bx}}{x}dx$$

$$= -\frac{\Gamma'(1)}{B} + \left(\frac{1}{B}\right)\int_0^\infty \frac{e^{-x}-e^{-Bx}}{x}dx = I \quad (8)$$

Let $F(A) := \int_0^\infty \frac{e^{-x}-e^{-Ax}}{x}dx, A > 0$

$$\Rightarrow F(1) = 0 \Rightarrow F'(A) = \int_0^\infty e^{-Ax}dx = \frac{1}{A}$$

$$\therefore F(A) = \int_0^A \frac{dx}{x} = \ln(A) \Rightarrow I = \frac{\gamma}{B} + \frac{\ln(B)}{B}$$

$$\therefore I = \int_0^1 \ln(\Gamma(x))\sin(2\pi kx)\,dx = \frac{\gamma + \ln(2\pi k)}{2\pi k}$$