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JP.091. Prove that the following inequalities hold for all positive real numbers:

a. $\frac{a^3}{ab+c^2}+\frac{b^3}{bc+a^2}+\frac{c^3}{ca+b^2}\geq\frac{3}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}$

b. $\frac{1}{a(b+c)}+\frac{1}{b(c+a)}+\frac{1}{c(a+b)}\geq\frac{3}{2}\cdot\frac{a+b+c}{a^3+b^3+c^3}$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Vadim Mitrofanov-Kiev-Ukraine

We have $C-S\sum_{cyc}\frac{1}{a(b+c)}=\frac{(a+b+c)^2}{\sum_{cyc}a^3(b+c)}\geq\frac{3}{2}\cdot\frac{(a+b+c)}{a^3+b^3+c^3}\Leftrightarrow$

$$\Leftrightarrow 2(a^4+b^4+c^4)\geq\sum_{cyc}a^3(b+c)$$

We have $C-S\sum_{cyc}\frac{a^3}{ab+c^2}\geq\frac{\left(a^2+b^2+c^2\right)^2}{\sum_{cyc}a\left(b^2+c^2\right)}\geq\frac{3}{2}\cdot\frac{\left(a^2+b^2+c^2\right)}{a+b+c}\Leftrightarrow$

$$\Leftrightarrow 2(a^3+b^3+c^3)\geq\sum_{cyc}a(b^2+c^2)$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a+b+c=p, ab+bc+ca=q$ and $r=abc$. We have

$$2p^3-7pq+9r\geq 0;\ \sum_{cyc}\frac{a^3}{ab+c^2}=\sum_{cyc}\frac{a^4}{a^2b+ac^2}=\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}ab(a+b)}$$

We need to prove, $\frac{\left(a^2+b^2+c^2\right)^2}{\sum_{cyc}ab(a+b)}\geq\frac{3}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}$

$$\Leftrightarrow 2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)\geq 3\sum_{cyc}ab(a+b)\Leftrightarrow 2(p^2-2q)p\geq 3(pq-3r)$$

$\Leftrightarrow 2p^3-7pq+9r\geq 0$, which is true $\sum_{cyc}\frac{a^3}{ab+c^2}\geq\frac{3}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}$ (proved)

b. $\sum_{cyc}\frac{(a^3+b^3+c^3)}{a(b+c)}=\sum_{cyc}\frac{a^2}{b+c}+\sum_{cyc}\frac{b^2-bc+c^2}{a}$

$$\overset{Bergstorm}{\geq}\frac{a+b+c}{2}+\frac{1}{4}\sum_{cyc}\frac{(b+c)^2}{a}\left[\begin{array}{c}\because a^2-ab+b^2\geq\frac{(a+b)^2}{4},\\ b^2-bc+c^2\geq\frac{(c+a)^2}{4}\ and\\ c^2-ca+a^2\geq\frac{(c+a)^2}{4}\end{array}\right]$$

$$\overset{Bergstorm}{\geq} \frac{a+b+c}{2}+a+b+c=\frac{3}{2}\cdot(a+b+c)\therefore\sum_{cyc}\frac{1}{a(b+c)}\geq\frac{3}{2}\cdot\frac{a+b+c}{a^3+b^3+c^3}$$ ***(proved)***

Solution 3 by Nguyen Ngoc Tu-Ha Giang-Vietnam

Using Hölder's inequality, we have: $a^3+b^3+c^3\geq\frac{1}{9}(a+b+c)^3$

$\Rightarrow\frac{3}{2}\cdot\frac{a+b+c}{a^3+b^3+c^3}\leq\frac{27}{2}\cdot\frac{1}{(a+b+c)^2}$. ***We will prove*** $\sum\frac{1}{a(b+c)}\geq\frac{27}{2(a+b+c)^2}$ ***is enough.***

We have $\sum\frac{1}{a(b+c)}\geq\frac{9}{2(ab+bc+ca)}\geq\frac{9}{2\cdot\frac{(a+b+c)^2}{3}}=\frac{27}{2(a+b+c)^2}$.

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c\in\mathbb{R}^+,\frac{1}{a(b+c)}+\frac{1}{b(c+a)}+\frac{1}{c(a+b)}\geq\frac{3}{2}\cdot\frac{a+b+c}{a^3+b^3+c^3}$$

$$LHS=\frac{\sum\{bc(a+b)(c+a)\}}{abc(a+b)(b+c)(c+a)}=\frac{\sum bc(\sum ab+a^2)}{abc(a+b)(b+c)(c+a)}\overset{(1)}{=}$$

$=\frac{(\sum ab)^2+abc(\sum a)}{abc(a+b)(b+c)(c+a)}$. ***Let*** $a+b=x,b+c=y,c+a=z,\therefore x+y>z,$

$y+z>x,z+x>y\Rightarrow x,y,z$ ***are 3 sides of a triangle with semiperimeter, circumradius, inradius*** $=s,R,r$ ***respectively. Now,*** $\sum a=\frac{\sum x}{2}=s,\therefore a=s-y,$

$$b=s-z,c=s-x;\sum ab=\sum(s-y)(s-z)=\sum\{s^2-s(y+z)+yz\}$$

$$=3s^2-s(4s)+s^2+4Rr+r^2\overset{(2)}{=}4Rr+r^2$$

$$\sum a^3=3abc+\left(\sum a\right)\left(\sum a^2-\sum ab\right)=$$

$$=\frac{3s(s-x)(s-y)(s-z)}{s}+s\left\{\left(\sum a\right)^2-3\sum ab\right\}=$$

$=\frac{3r^2s^2}{s}+s\{s^2-3(4R+r^2)\}=3r^2s+s(s^2-12Rr-3r^2)\overset{(3)}{=}s(s^2-12Rr)$; ***(1), (2), (3)*** $\Rightarrow$ ***given inequality*** $\Leftrightarrow\frac{r^2(4R+r)^2+r^2s^2}{r^2s\cdot 4Rrs}\geq\frac{3}{2}\cdot\frac{s}{s(s^2-12Rr)}\Leftrightarrow$

$$\Leftrightarrow s^4+s^2(16R^2-10Rr+r^2)\geq 192R^3r+96R^2r^2+12Rr^3$$

LHS of (4) $\overset{Gerretsen}{\geq} s^2(16Rr-5r^2)+s^2(16R^2-10Rr+r^2)$

$$=s^2(16R^2+6Rr-4r^2)\overset{Gerretsen}{\geq}(16Rr-5r^2)(16R^2+6Rr-4r^2)$$

$$\geq 192R^3r+96R^2r^2+12Rr^3 \Leftrightarrow (t-2)(32t^2+24t-5)\overset{?}{\geq} 0$$

$$\left(t=\frac{R}{r}\right)\to \textit{true (Euler) (proved)}$$

$$\forall a,b,c\in\mathbb{R}^+,\frac{a^3}{ab+c^2}+\frac{b^3}{bc+a^2}+\frac{c^3}{ca+b^2}\geq\frac{3}{2}\cdot\frac{\sum a^2}{\sum a}$$

$$LHS=\frac{a^4}{a^2b+c^2a}+\frac{b^4}{b^2c+a^2b}+\frac{c^4}{c^2a+b^2c}\overset{Bergstrom}{\geq}\frac{(\sum a^2)^2}{2\sum a^2b}\overset{?}{\geq}\frac{3\sum a^2}{2\sum a}$$

$$\Leftrightarrow\left(\sum a^2\right)\left(\sum a\right)\overset{?}{\geq}3\sum a^2b\Leftrightarrow\sum a^3+\sum a^2b+\sum ab^2\overset{?}{\geq}3\sum a^2b$$

$$\Leftrightarrow\sum a^3+\sum ab^2\overset{?}{\geq}2\sum a^2b$$ **(1). Now,** $a^3+ab^2\overset{A-G}{\geq}2a^2b,\ b^3+bc^2\overset{A-G}{\geq}2b^2c$

and, $c^3+ca^2\overset{A-G}{\geq}2c^2a$. ***Adding the last 3 inequalities, we find (1) is true (proved).***

JP.092. Prove that the following inequalities holds for all positive real numbers a,b,c

a. $\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\geq\frac{3(a+b+c)}{a^2+b^2+c^2}$

b. $\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\geq\frac{3(a^2+b^2+c^2)}{a+b+c}$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

a. For $a,b,c>0$

$$\frac{a}{c^2}+\frac{c}{b^2}+\frac{b}{a^2}=\frac{a^2}{ac^2}+\frac{c^2}{cb^2}+\frac{b^2}{ba^2}\geq\frac{(a+b+c)^2}{ac^2+cb^2+ba^2}$$

$$\geq\frac{3(a+b+c)^2}{(a+b+c)(a^2+b^2+c^2)}=\frac{3}{a^2+b^2+c^2}$$

b. For $a,b,c>0$

$$\frac{a^3}{c^2}+\frac{c^3}{b^2}+\frac{b^2}{a^2}=\frac{a^4}{ac^2}+\frac{c^4}{cb^2}+\frac{b^4}{ba^2}\geq\frac{(a^2+b^2+c^2)^2}{ac^2+cb^2+ba^2}\geq\frac{3(a^2+b^2+c^2)^2}{(a+b+c)(a^2+b^2+c^2)}=\frac{3(a^2+b^2+c^2)}{(a+b+c)}.$$

Therefore it is true.

Solution 2 by Ravi Prakash-New Delhi-India

a. *Consider* $(a^2+b^2+c^2)\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right)=b+\frac{b^3}{a^2}+\frac{bc^2}{a^2}+c+\frac{c^3}{b^2}+\frac{a^2c}{b^2}+$

$$+a+\frac{a^3}{c^2}+\frac{ab^2}{c^2}=-2(a+b+c)+\left(2b+c+\frac{a^3}{c^2}+\frac{a^2c}{b^2}\right)+$$

$$+\left(2a+b+\frac{bc^2}{a^2}+\frac{c^3}{b^2}\right)+\left(2c+a+\frac{b^3}{a^2}+\frac{ab^2}{c^2}\right)$$

$$\geq -2(a+b+c)+5\left(b^2c\cdot\frac{a^3}{c^2}\cdot\frac{a^2c}{b^2}\right)^{\frac{1}{5}}+5\left(a^2b\cdot\frac{bc^2}{a^2}\cdot\frac{c^3}{b^2}\right)^{\frac{1}{5}}+$$

$$+5\left(c^2a\cdot\frac{b^3}{a^2}\cdot\frac{ab^2}{c^2}\right)^{\frac{1}{5}}=-2(a+b+c)+5(a+c+b)=3(a+b+c)$$

Solution 3 by Nguyen Ngoc Tu-Ha Giang-Vietnam

a. ***We have*** $\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\geq\frac{3(a+b+c)}{a^2+b^2+c^2}\Leftrightarrow\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right)\geq 3$

Use Cauchy – Schwarz and AM-GM inequality we have

$a^2+b^2+c^2\geq\frac{1}{3}(a+b+c)^2\Rightarrow\frac{a^2+b^2+c^2}{a+b+c}\geq\frac{a+b+c}{3}\geq\sqrt[3]{abc}$ ***and***

$\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\geq\frac{3}{\sqrt[3]{abc}}$. ***Hence*** $\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right)\geq 3$.

b. ***Use Lemma*** $(a+b+c)(a^2+b^2+c^2)\geq 3(a^2b+b^2c+c^2a)$ ***and Cauchy – Schwarz***

inequality we have $(a+b+c)(a^2+b^2+c^2)\left(\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\right)\geq$

$$\geq 3(a^2b+b^2c+c^2a)\left(\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\right)\geq 3(a^2+b^2+c^2)^2$$

$$\Rightarrow\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\geq\frac{3(a^2+b^2+c^2)}{a+b+c}$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

a. $\sum_{cyc}\frac{b}{a^2}=\sum_{cyc}\frac{\left(\frac{b}{a}\right)^2}{b}\geq\frac{\left(\frac{b}{a}+\frac{a}{c}+\frac{c}{b}\right)^2}{a+b+c}\overset{AM\geq GM}{\geq}\frac{9}{a+b+c}$. ***We need to prove***

$\frac{9}{a+b+c}\geq\frac{3(a+b+c)}{a^2+b^2+c^2}\Leftrightarrow 3\sum_{cyc}a^2\geq\left(\sum_{cyc}a\right)^2$ ***which is true.***

$\therefore\sum_{cyc}\frac{b}{a^2}\geq\frac{3(a+b+c)}{a^2+b^2+c^2}$ ***(proved)***

b. $\sum_{cyc}\frac{b^3}{a^2}=\sum_{cyc}\frac{b^4}{a^2b}\overset{Bergstrom}{\geq}\frac{\left(a^2+b^2+c^2\right)^2}{a^2b+b^2c+c^2a}$

we need to prove, $\frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a} \geq \frac{3(a^2+b^2+c^2)}{a+b+c}$

$\Leftrightarrow (\sum_{cyc} a^2)(\sum_{cyc} a) \geq 3\sum_{cyc} a^2b \Leftrightarrow \sum_{cyc} a^3 + \sum_{cyc} ab\,(a+b) \geq 3\sum_{cyc} a^2b$, *which is*

true $\begin{bmatrix} since, a^3 + a^2b + ab^2 \geq 3a^2b, \\ b^3 + b^2c + bc^2 \geq 3b^2c \text{ and} \\ c^3 + c^2a + ca^2 \geq 3c^2a \end{bmatrix}$; $\sum_{cyc} \frac{b^3}{a^2} \geq \frac{3(a^2+b^2+c^2)}{a+b+c}$ *(proved)*

JP.093. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that:

a. $\frac{1}{a+bc} + \frac{1}{b+ca} + \frac{1}{c+ab} \leq \frac{1}{4abc}$

b. $\frac{\sqrt{a}}{a+\sqrt{bc}} + \frac{\sqrt{b}}{b+\sqrt{ca}} + \frac{\sqrt{c}}{c+\sqrt{ab}} \leq \frac{1}{2\sqrt{abc}}$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Vadim Mitrofanov-Kiev-Ukraine

$$\sum_{cyc} \frac{1}{a+bc} = \sum_{cyc} \frac{1}{(a+b)(a+c)} = \frac{2}{(a+b)(b+c)(a+c)} \leq \frac{1}{4abc}$$

$$\sum_{cyc} \frac{\sqrt{a}}{a+\sqrt{bc}} \leq \sum_{cyc} \frac{\sqrt{a}}{2\sqrt{a\sqrt{bc}}} = \frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{2\sqrt[4]{abc}} \leq \frac{1}{2\sqrt{abc}} \Leftrightarrow \left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)^4 abc \leq 1$$

we have $\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)^4 \leq \left(3\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\right)^2 \leq 27 \Rightarrow 27abc \leq (a+b+c)^3 = 1$

Solution 2 by Ravi Prakash-New Delhi-India

$$\frac{1}{a+bc} + \frac{1}{b+ca} + \frac{1}{c+ab} \leq \frac{1}{4abc}$$

$$\Leftrightarrow \frac{1}{1-b-c+bc} + \frac{1}{1-c-a+ca} + \frac{1}{1-a-b+ab} \leq \frac{1}{4abc}$$

$$\Leftrightarrow \frac{1}{(1-b)(1-c)} + \frac{1}{(1-c)(1-a)} + \frac{1}{(1-a)(1-b)} \leq \frac{1}{4abc}$$

$$\Leftrightarrow \frac{(1-a)+(1-b)+(1-c)}{(1-a)(1-b)(1-c)} \leq \frac{1}{4ab} \Leftrightarrow 8abc \leq (1-a)(1-b)(1-c)$$

$$\Leftrightarrow 8abc \leq 1-(a+b+c)+ab+bc+ca-abc$$

$$\Leftrightarrow 9abc \leq ab+bc+ca \Leftrightarrow 9 \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \quad \textit{(1)}$$

But $\frac{1}{3}=\frac{a+b+c}{3}\ge\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 9$***. Thus (1) is true. For*** $a,b,c>0, a+b+c=1$***,***

$$\frac{2\sqrt{abc}\sqrt{a}}{a+\sqrt{bc}}=\frac{2a\sqrt{bc}}{a+\sqrt{bc}}\le\frac{a+\sqrt{bc}}{2}$$

$$\therefore 2\sqrt{abc}\left[\frac{\sqrt{a}}{a+\sqrt{bc}}+\frac{\sqrt{b}}{b+\sqrt{ca}}+\frac{\sqrt{c}}{c+\sqrt{ab}}\right]\le\frac{1}{2}\left[a+b+c+\sqrt{bc}+\sqrt{ca}+\sqrt{ab}\right]\ \textbf{(1)}$$

But, $\sqrt{bc}+\sqrt{ca}+\sqrt{ab}=\sqrt{b}\sqrt{c}+\sqrt{c}\sqrt{a}+\sqrt{a}\sqrt{b}\le$

$$\le\left(\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2+\left(\sqrt{c}\right)^2=a+b+c\quad\textbf{(2)}$$

From (1), (2): $\sqrt[2]{abc}\left[\frac{\sqrt{a}}{a+\sqrt{bc}}+\frac{\sqrt{b}}{b+\sqrt{ca}}+\frac{\sqrt{c}}{c+\sqrt{ab}}\right]\le\frac{1}{2}[a+b+c+a+b+c]=1$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

a. $\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\le\frac{1}{4abc}$

$$a+b+c=1$$

$$\frac{\sum(a+bc)(b+ca)}{\prod(a+bc)}=\frac{\sum ab+\sum ab\,(a+b)+abc\sum a}{abc+a^2b^2c^2+abc\cdot\sum a^2+\sum a^2b^2}$$

$$=\frac{\sum ab+\sum ab\cdot\sum a-2abc\sum a}{abc+a^2b^2c^2+abc((\sum a)^2-2\sum ab)+((\sum ab)^2-2abc\sum a)}$$

$$=\frac{q+q\cdot p-2pr}{r+r^2+r(p^2-2q)+(q^2-2pr)}=$$

$$=\frac{2q-2r}{r+r^2+r(1-2q)+(q^2-2r)}=\frac{2q-2r}{2r-2r+r^2-2rq+q^2}=$$

$$=\frac{2(q-r)}{(q-r)^2}=\frac{q}{q-r}\overset{p=1}{=}\frac{2}{pq-r}\overset{pq\ge 9r}{\ge}\frac{2}{8r}=\frac{1}{4r}$$

$$a=b=c=\frac{1}{3}$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{1}{a+bc}=\sum_{cyc}\frac{1}{a(a+b+c)+bc}=\sum_{cyc}\frac{1}{(a+b)(a+c)}$$

$$=\frac{1}{(a+b)(b+c)(c+a)}\sum_{cyc}(a+b)=\frac{2(a+b+c)}{\prod_{cyc}(a+b)}\le\frac{2}{8abc}=\frac{1}{4abc}$$

(proved)

b. $\sum_{cyc}\frac{2a\sqrt{bc}}{a+\sqrt{bc}} = \sum_{cyc}\frac{2}{\frac{1}{a}+\frac{1}{\sqrt{bc}}} \overset{HM\le AM}{\le} \sum_{cyc}\frac{a+\sqrt{bc}}{2} = \frac{1}{2}\sum_{cyc} a + \frac{1}{2}\sum_{cyc}\sqrt{ab}$

$$\le \sum_{cyc} a = 1 \Rightarrow \sum_{cyc}\frac{\sqrt{a}}{a+\sqrt{bc}} \le \frac{1}{2\sqrt{abc}}$$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0, a + b + c = 1$, *we give:* $a = \frac{x}{x+y+z}, b = \frac{y}{x+y+z}, c = \frac{z}{x+y+z}$

Consider, since $4xyz\big((x+y)+(y+z)+(z+x)\big) \le (x+y+z)(x+y)(y+z)(z+x)$

Hence $\frac{(x+y)+(y+z)+(z+x)}{(x+y)(y+z)(z+x)} \le \frac{(x+y+z)}{4xyz}$

Hence $\frac{1}{(y+z)(z+x)} + \frac{1}{(x+y)(z+x)} + \frac{1}{(x+y)(y+z)} \le \frac{(x+y+z)}{4xyz}$

Hence $\frac{1}{x(x+y+z)+yz} + \frac{1}{y(x+y+z)+zx} + \frac{1}{z(x+y+z)+xy} \le \frac{(x+y+z)}{4(xyz)}$

Hence $\frac{(x+y+z)^3}{x(x+y+z)+yz} + \frac{(x+y+z)^2}{y(x+y+z)+zx} + \frac{(x+y+z)^2}{z(x+y+z)+xy} \le \frac{(x+y+z)^3}{4xyz}$

Hence $\frac{1}{\frac{x}{(x+y+z)}+\frac{yz}{(x+y+z)^2}} + \frac{1}{\frac{y}{(x+y+z)}+\frac{zx}{(x+y+z)^2}} + \frac{1}{\frac{z}{(x+y+z)}+\frac{xy}{(x+y+z)^2}} \le \frac{1}{\frac{4(xyz)}{(x+y+z)^3}}$

Therefore $\frac{1}{a+bc} + \frac{1}{b+ca} + \frac{1}{c+ab} \le \frac{1}{4abc}$ *is to be true.*

Solution 6 by Nguyen Ngoc Tu-Ha Giang-Vietnam

a. *We have* $1 = (a+b+c)^2 \ge 3(ab+bc+ca) \Rightarrow ab+bc+ca \le \frac{1}{3}$

$$\frac{1}{a+bc} + \frac{1}{b+ca} + \frac{1}{c+ab} \le \frac{1}{4abc} \Leftrightarrow \frac{abc}{a+bc} + \frac{abc}{b+ca} + \frac{abc}{c+ab} \le \frac{1}{4}$$

$\Leftrightarrow \sum\frac{a(a+bc)-a^2}{a+bc} \le \frac{1}{4} \Leftrightarrow \sum\frac{a^2}{a+bc} \ge \frac{3}{4}$ *with* $a + b + c = 1$

Using Cauchy – Schwarz we have: $\sum\frac{a^2}{a+bc} \ge \frac{(a+b+c)^2}{a+b+c+ab+bc+ca} \ge \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$

b. *We have*

$$\frac{1}{3} \ge ab+bc+ca \ge \frac{1}{3}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2 \Rightarrow \sqrt{ab}+\sqrt{bc}+\sqrt{ca} \le 1$$

$$\frac{\sqrt{a}}{a+\sqrt{bc}} + \frac{\sqrt{b}}{b+\sqrt{ca}} + \frac{\sqrt{c}}{c+\sqrt{ab}} \le \frac{1}{2\sqrt{abc}} \Leftrightarrow \sum\frac{a\sqrt{bc}}{a+\sqrt{bc}} \le \frac{1}{2}$$

$$\Leftrightarrow \sum \frac{a\left(a+\sqrt{bc}\right)-a^2}{a+\sqrt{bc}} \le \frac{1}{2} \Leftrightarrow \sum \frac{a^2}{a+\sqrt{bc}} \ge \frac{1}{2}$$

Using Cauchy – Schwarz: $\sum \frac{a^2}{a+\sqrt{bc}} \ge \frac{(a+b+c)^2}{a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \ge \frac{1^2}{1+1} = \frac{1}{2}$

JP.094. Let a,b,c be positive real numbers such that $ab+bc+ca=1$. Prove that:

$$bc\sqrt{a^2+2b}+ca\sqrt{b^2+2ca}+ab\sqrt{c^2+2ab} \ge 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

By Hölder's inequality we obtain:

$$\left(\sum_{cyc} bc\sqrt{a^2+2bc}\right)^2 \left(\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}\right) \ge (bc+ca+ab)^3 = 1$$

The proof will be completed if we show that $\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab} \le 1$. ***Indeed, we will use Cauchy – Schwarz inequality by the following way***

$$\sum_{cyc} \frac{bc}{a^2+2bc} = \sum_{cyc} \frac{(a^2+2bc)-a^2}{2(a^2+2bc)} =$$

$= \frac{3}{2} - \sum_{cyc} \frac{a^2}{2(a^2+2bc)} \le \frac{3}{2} - \frac{(a+b+c)^2}{2(a^2+2bc+b^2+2ca+c^2+2ab)} = 1$ ***and we are done.***

JP.095. Prove that for all positive real numbers a,b,c :

$$\frac{a(b^2+c^2)}{2a^2+bc}+\frac{b(c^2+a^2)}{2b^2+ca}+\frac{c(a^2+b^2)}{2c^2+ab} \ge \frac{6abc}{ab+bc+ca}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{a(b^2+c^2)}{2a^2+bc} = \sum \frac{abc(b^2+c^2)}{bc(2a^2+bc)} = abc\sum \frac{b^2+c^2}{bc(2a^2+bc)} \ge$$

$$\overset{BERGSTROM}{\ge} abc \cdot \frac{2(\sum a)^2}{\sum b^2c^2+2abc\sum a} = abc \cdot \frac{2(\sum a)^2}{(\sum ab)^2} \ge abc \cdot \frac{2\cdot 3\sum ab}{(\sum ab)^2} = \frac{6abc}{ab+bc+ca}$$

JP.096. Let a,b,c positive numbers such that $a^4+b^4+c^4=3$. Prove that:

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\geq 9$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\overset{AM-GM}{\geq}3\sqrt[3]{\frac{1}{a^2b^2c^2}}\cdot 3\sqrt[3]{\frac{1}{a^2b^2c^2}}=$$

$$=\frac{9}{\sqrt[3]{a^4b^4c^4}}\overset{AM-GM}{\geq}\frac{9}{\frac{a^4+b^4+c^4}{3}}=9$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\left(\sum\frac{a^3}{b^5}\right)\left(\sum\frac{b^3}{a^5}\right)=\left(\sum\frac{\left(\frac{a}{b}\right)^5}{a^2}\right)\left(\sum\frac{\left(\frac{b}{a}\right)^5}{b^2}\right)\geq$$

$$\overset{BERGSTROM}{\geq}\frac{\left(\sum\frac{a^{\frac{5}{2}}}{b^{\frac{5}{2}}}\right)^2\cdot\left(\sum\frac{b^{\frac{5}{2}}}{a^{\frac{5}{2}}}\right)^2}{(a^2+b^2+c^2)^2}\overset{AM-GM}{\geq}\frac{3^2\cdot 3^2}{(a^2+b^2+c^2)^2}\geq\frac{81}{3\sum a^4}=\frac{81}{9}=9$$

Solution 3 by Rozeta Atanasova-Skopje

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\overset{AM-GM}{\geq}3\sqrt[3]{\frac{1}{a^2b^2c^2}}\cdot 3\sqrt[3]{\frac{1}{a^2b^2c^2}}=$$

$$=\frac{9}{\sqrt[3]{a^4b^4c^4}}\overset{AM-GM}{\geq}\frac{9}{\frac{a^4+b^4+c^4}{3}}=\frac{9}{\frac{3}{3}}=9$$

JP.097. Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=8$. Prove that:

$$\frac{a}{a+1}+\sqrt{\frac{2b}{b+1}}+2\sqrt[4]{\frac{2c}{c+1}}\leq\frac{7}{2}$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

By AM-GM:

$$\frac{a}{a+1}+\sqrt{\frac{2b}{b+1}\cdot 1}+2\cdot\sqrt[4]{\frac{2c}{c+1}\cdot 1\cdot 1\cdot 1}\le\frac{a}{a+1}+\frac{\frac{2b}{b+1}+1}{2}+\frac{2\left(\frac{2c}{c+1}+1+1+1\right)}{4}$$

$$=\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+2\quad (1)$$

We prove that: $\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\le\frac{3}{2}$

$$\Leftrightarrow\frac{a(b+1)(c+1)+b(c+1)(a+1)+c(a+1)(b+1)}{(a+1)(b+1)(c+1)}\le\frac{3}{2}$$

$$\Leftrightarrow 2(3abc+2(ab+bc+ca)+a+b+c)\le 3(abc+ab+bc+ca+a+b+c+1)$$

$$\Leftrightarrow 3abc+ab+bc+ca\le a+b+c+3\quad (2)$$

Other: $8=(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca)$

$$\Leftrightarrow (a+b+c)(ab+bc+ca)\le 9$$

$$\Rightarrow 9\ge 3\sqrt[3]{abc}\cdot 3\sqrt[3]{(abc)^2}=9abc\Leftrightarrow abc\le 1\quad (3)$$

$$\begin{cases}9\ge(a+b+c)(ab+bc+ca)\ge\sqrt{3(ab+bc+ca)}\cdot(ab+bc+ca)\\ \Rightarrow ab+bc+ac\le 3\quad (4)\end{cases}$$

(3), (4) $\Rightarrow 3abc+ab+bc+ca\le 6$ **(5)**

$$8=(a+b)(b+c)(c+a)\le\frac{\left((a+b)+(b+c)+(c+a)\right)^3}{27}=\frac{8(a+b+c)^3}{27}$$

$$\Rightarrow (a+b+c)^3\ge 27\Rightarrow a+b+c+3\ge 6\quad (6)$$

(5), (6) $\Rightarrow 3abc+ab+bc+ca\le a+b+c+3$

$\Rightarrow$ **(2) true** $\Rightarrow \frac{a}{a+1}+\sqrt{\frac{2b}{b+1}}+2\cdot\sqrt[4]{\frac{2c}{c+1}}\le\frac{7}{2}$

JP.098. Let a, b and c be the side lengths of a triangle ABC with incenter I. Prove that:

$$\frac{1}{IA^2}+\frac{1}{IB^2}+\frac{1}{IC^2}\ge 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$IA = \frac{r}{\sin\frac{A}{2}} \text{ etc}$$

$$\therefore \sum \frac{1}{IA^2} = \frac{1}{r^2}\sum \sin^2\frac{A}{2} \quad (1)$$

$$\text{Also, } 3\sum\frac{1}{a^2} = \frac{3\sum a^2b^2}{a^2b^2c^2} \overset{Goldstone}{\le} \frac{12R^2s^2}{16R^2r^2s^2} = \frac{3}{4r^2} \quad (2)$$

$$(1),(2) \Rightarrow \text{it suffices to prove: } \sum\sin^2\frac{A}{2} \ge \frac{3}{4} \Leftrightarrow \sum\left(2\sin^2\frac{A}{2}\right) \ge \frac{3}{2} \Leftrightarrow \sum(1-\cos A) \ge \frac{3}{2}$$

$$\Leftrightarrow 3 - 1 - \frac{r}{R} \ge \frac{3}{2} \Leftrightarrow \frac{2R-r}{R} \ge \frac{3}{2} \Leftrightarrow R \ge 2r \to \text{true (Euler) (proved)}$$

JP.099. If $x, y, z > 0$ and $b \ge a > 0$ then:

$$\int_a^b \frac{x\,dy}{3x^2+2y^2+z^2} + \int_a^b \frac{y\,dz}{3y^2+2z^2+x^2} + \int_a^b \frac{z\,dx}{3z^2+2x^2+y^2}$$

$$\le \frac{1}{3}\ln\frac{b}{a} + \frac{b-a}{18}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\text{We have for } x,t,z > 0; \frac{x}{3x^2+2t^2+z^2} \le \frac{1}{18}\left(\frac{2}{t}+\frac{1}{z}\right) \Leftrightarrow$$

$$\Leftrightarrow 3x^2t + 6x^2z + 2t^3 + 2z^3 + 4t^2z + tx^2 \ge 18xtz \Leftrightarrow$$

$$\Leftrightarrow \frac{3x^2t + 6x^2z + 2t^3 + 2z^3 + 4t^2z + tz^2}{18} \ge \sqrt[18]{(x^2t)^3(x^2t)^6(t^3)^2(z^3)^2(t^2z)^4t} = xtz \Rightarrow$$

$$\int_a^b \frac{x\,dt}{3x^2+2t^2+z^2} \le \frac{1}{18}\int_a^b\left(\frac{2}{t}+\frac{1}{z}\right)dt \Rightarrow \int_a^b \frac{x\,dt}{3x^2+2t^2+z^2} \le \frac{1}{9}\ln\frac{b}{a} + \frac{b-a}{18z} \Rightarrow$$

$$\sum_{cyclic}\int_a^b \frac{xdy}{3x^2+2y^2+z^2} \le \sum_{cyclic}\left(\frac{1}{9}\ln\frac{b}{a} + \frac{b-a}{18z}\right) = \frac{1}{3}\ln\frac{b}{a} + \frac{b-a}{18}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

JP.100. Let in triangle w_a, w_b, w_c be the angle bisectors and R, r the circumradius and inradius respectively. Prove the inequality:

$$\frac{3}{R+r} \le \frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \le \frac{1}{r}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \overset{AM \ge GM}{\ge} \frac{3}{\sqrt[3]{w_a w_b w_c}} \to \textbf{(1)}$$

Now, $w_a w_b w_c = \left(\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}\right)\left(\frac{2\sqrt{ca}}{c+a}\sqrt{s(s-b)}\right)\left(\frac{2\sqrt{ab}}{a+b}\sqrt{s(s-c)}\right)$

$$= \frac{8abcs \cdot rs}{\prod(a+b)} = \frac{32Rr^2s^3}{\prod(a+b)} \to \textbf{(2)}$$

Again, $\prod(a+b) = 2abc + \sum ab(2s-c) = 2s(s^2+4Rr+r^2) - 4Rrs$

$$= 2s(s^2+2Rr+r^2) \to \textbf{(3)}$$

$$\textbf{(2), (3)} \Rightarrow w_a w_b w_c = \frac{16Rr^2s^2}{s^2+2Rr+r^2} \to \textbf{(4)}$$

$$\textbf{(4), (1)} \Rightarrow \frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \ge 3\sqrt[3]{\frac{s^2+2Rr+r^2}{16Rr^2s^2}} \ge \frac{3}{R+r}$$

$$\Leftrightarrow (R+r)^3(s^2+2Rr+r^2) \ge 16Rr^2s^2 \to \textbf{(a)}$$

Now, LHS of (a) $\overset{Gerretsen}{\ge} (R+r)^3(18Rr-4r^2)$ and

RHS $\overset{Gerretsen}{\le} 16Rr^2(4R^2+4Rr+3r^2)$

$\therefore$ *in order to prove (a), it suffices to prove:*

$$(R+r)^3(18Rr-4r^2) \ge 16Rr^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 9t^4 - 7t^3 - 11t^2 - 21t - 2 \ge 0 \ \ (\text{where } t = \tfrac{R}{r})$$

$$\Leftrightarrow (t-2)(9t^3+11t^2+11t+1) \ge 0 \to \text{true} \because t \ge 2 \ (\text{Euler})$$

$\Rightarrow$ (a) is true $\Rightarrow \frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \ge \frac{3}{R+r}$ is proved. Now, $\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \le \frac{1}{r} \Leftrightarrow \frac{\sum w_a w_b}{w_a w_b w_c} \le \frac{1}{r}$

$$\sum w_a w_b = \sum\left(\left(\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}\right)\left(\frac{2\sqrt{ca}}{c+a}\sqrt{s(s-b)}\right)\right)$$

$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sum\left[\left((a+b)\sqrt{c}\right)\left(\sqrt{(s-a)(s-b)}\right)\right]$$

$$\overset{C-B-S}{\leq}\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{\sum c(a+b)^2}\sqrt{\sum(s-a)(s-b)}$$

$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{\sum c\,(a^2+2ab+b^2)}\sqrt{\sum(s^2-s(a+b)+ab)}$$

$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{\sum ab\,(2s-c)+6abc}\sqrt{3s^2-4s^2+s^2+4Rr+r^2}$$

$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{2s(s^2+4Rr+r^2)+12Rrs}\sqrt{4Rr+r^2}$$

$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}$$

$$=\frac{4s\sqrt{4Rrs}}{2s(s^2+2Rr+r^2)}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}\quad \textit{(by (3))}$$

$$\therefore\sum w_aw_b\leq\frac{4s\sqrt{4Rrs}}{2s(s^2+2Rr+r^2)}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}\rightarrow \textit{(5)}$$

$$\therefore\frac{\sum w_aw_b}{w_aw_bw_c}\overset{by\,(5),(4)}{\leq}\frac{4s\sqrt{4Rrs}}{2s(s^2+2Rr+r^2)}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}\cdot\frac{s^2+2Rr+r^2}{16Rr^2s^2}$$

$$=\frac{\sqrt{4Rrs}}{8Rr^2s^2}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}$$

$$=\frac{\sqrt{R(4R+r)(s^2+10Rr+r^2)}}{2\sqrt{2}Rrs}\overset{?}{\leq}\frac{1}{r}\Leftrightarrow 8R^2s^2\overset{?}{\geq}R(4R+r)(s^2+10Rr+r^2)$$

$$\Leftrightarrow(4R-r)s^2\overset{?}{\geq}(4R+r)(10Rr+r^2)\rightarrow \textit{(b)}$$

$$\Leftrightarrow 8R^2s^2\overset{?}{\geq}R(4R+r)(s^2+10Rr+r^2)$$

$$\Leftrightarrow(4R-r)s^2\overset{?}{\geq}(4R+r)(10Rr+r^2)\rightarrow \textit{(b)}$$

Now, LHS of (b) $\geq(4R-r)(16Rr-5r^2)\overset{?}{\geq}(4R+r)(10Rr+r^2)$

$$\Leftrightarrow 12R^2-25Rr+2r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(12R-r)\overset{?}{\geq}0$$

$\rightarrow$ ***true*** $\because R\geq 2r$ ***(Euler)*** $\Rightarrow$ ***(b) is true*** $\Rightarrow\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\leq\frac{1}{r}$ ***is proved.***

JP.101. Let x, y, z be positive real numbers with $xyz = 1$. Prove that:

$$\frac{\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1}}{x^2+y^2+z^2} \leq \sqrt{2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

We have $(x-1)^4 \geq 0 \Leftrightarrow x^4-4x^3+6x^2-4x+1 \geq 0 \Leftrightarrow$

$$\Leftrightarrow 2x^4-4x^3+6x^2-4x+2 \geq x^4+1 \Leftrightarrow x^4-2x^3+3x^2-2x+1 \geq \frac{x^4+1}{2} \Leftrightarrow$$

$\Leftrightarrow (x^2-x+1)^2 \geq \frac{x^4+1}{2} \Leftrightarrow \frac{\sqrt{x^4+1}}{\sqrt{2}} \leq x^2-x+1$. *Similarly* $\frac{\sqrt{y^4+1}}{\sqrt{2}} \leq y^2-y+1$, *and* $\frac{\sqrt{z^4+1}}{\sqrt{2}} \leq z^2-z+1$. *Adding up these inequalities, we get:*

$$\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1} \leq \sqrt{2}(x^2+y^2+z^2)+\sqrt{2}\big(3-(x+y+z)\big) \quad (1)$$

By AM-GM inequlity we have $x+y+z \geq 3\sqrt{xyz} = 3$, *so* $3-(x+y+z) \leq 0$. *Now (1) gives* $\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1} \leq \sqrt{2}(x^2+y^2+z^2)$, *namely* $\frac{\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1}}{x^2+y^2+z^2} \leq \sqrt{2}$. *Equality holds when* $x=y=z=1$.

JP.102. Let $x, y, z > 0$ be positive real numbers. Then:

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \geq \frac{4\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

Solution by Soumitra Mandal-Chandar Nagore-India

We know, $\left(\sum_{cyc} xy\right)^2 \geq 3xyz(x+y+z)$

$\frac{4\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)} \leq \frac{4(xy+yz+zx)}{(x+y)(y+z)(z+x)}$, *we need to prove,*

$$\sum_{cyc}\frac{1}{x+y} \geq \frac{4(xy+yz+zx)}{(x+y)(y+z)(z+x)} \Leftrightarrow \sum_{cyc}(x+y)(x+z) \geq 4(xy+yz+zx)$$

$\Leftrightarrow x^2+y^2+z^2 \geq xy+yz+zx$, *which is true.*

$$\therefore \sum_{cyc} \frac{1}{x+y} \geq \frac{4\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)}$$

(proved)

JP.103. Let $x, y, z > 0$ be positive real numbers. Then in triangle ABC with semiperimeter s and inradius r.

$$\frac{x}{y+z}cot^2\frac{A}{2} + \frac{y}{z+x}cot^2\frac{B}{2} + \frac{z}{x+y}cot^2\frac{C}{2} \geq 18 - \frac{s^2}{2r^2}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

Solution by Soumitra Mandal-Chandar Nagore-India

$$\cot\frac{A}{2} = \frac{p(p-a)}{\Delta}, \cot\frac{B}{2} = \frac{p(p-b)}{\Delta} \textit{ and } \cot\frac{C}{2} = \frac{p(p-c)}{\Delta}$$

$$\sum_{cyc} \frac{x}{y+z}\cot^2\frac{A}{2} = (x+y+z)\sum_{cyc}\frac{\cot^2\frac{A}{2}}{y+z} - \sum_{cyc}\cot^2\frac{A}{2}$$

$$\overset{Bergström}{\geq} \frac{1}{2}\left(\sum_{cyc}\cot\frac{A}{2}\right)^2 - \sum_{cyc}\cot^2\frac{A}{2}$$

$$= \frac{1}{2}\left(\sum_{cyc}\frac{p(p-a)}{\Delta}\right)^2 - \sum_{cyc}\frac{p^2(p-a)^2}{\Delta^2} = \frac{p^2}{2r^2} - \frac{p^2\left\{\left(\sum_{cyc}(p-a)\right)^2 - 2\sum_{cyc}(p-a)(p-b)\right\}}{\Delta^2}$$

$$= \frac{p^2}{2r^2} - \frac{p^2 - 2r(r+4R)}{r^2} = \frac{2(r+4R)}{r} - \frac{p^2}{2r^2} \geq \frac{2(r+8r)}{r} - \frac{p^2}{2r^2} = 18 - \frac{p^2}{2r^2}$$

JP.104. Let r_a, r_b, r_c be the exradii, h_a, h_b, h_c the altitudes and m_a, m_b, m_c the medians of a triangle ABC with semiperimeter s, circumradius R and inradius r. Then

$$\frac{r_a^2}{h_b m_c} + \frac{r_b^2}{h_c m_a} + \frac{r_c^2}{h_a m_b} \geq \frac{54r^2}{s^2 - r^2 - 4Rr}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{ra^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\ge\frac{54r^2}{p^2-r^2-4Rr}$$

$$\textbf{1)}\ r_a+r_b+r_c=4R+r$$

$$\textbf{2)}\ h_b m_c+h_c m_a+h_a m_b\overset{\substack{h_a\le m_a\\ h_b\le m_b\\ h_c\le m_c}}{\le} m_b m_c+m_c m_a+m_a m_b\le m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$$

$$LHS:\sum_{\Delta}\frac{r_a^2}{h_b m_c}\overset{Bergström}{\ge}\frac{(\sum r_a)^2}{h_b m_c+h_c m_a+h_a m_b}\overset{(2);(1)}{\ge}$$

$$\ge\frac{(4R+r)^2}{\frac{3}{4}(a^2+b^2+c^2)}\overset{Euler}{\ge}\frac{81r^2}{\frac{3}{4}\cdot 2(p^2-4Rr-r^2)}=\frac{54r^2}{p^2-4Rr-r^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\because h_a\le m_a\ \textit{etc,}$$

$$\therefore LHS\overset{h_a\le m_a,etc}{\ge}\sum\frac{r_a^2}{m_b m_c}\overset{Bergström}{\ge}\frac{(\sum r_a)^2}{\sum m_b m_c}\overset{m_b m_c\le\frac{2a^2+bc}{4}}{\ge}\frac{(4R+r)^2}{\sum_{cyc}\left(\frac{2a^2+bc}{4}\right)}$$

$$=\frac{4(4R+r)^2}{2\sum a^2+\sum ab}=\frac{4(4R+r)^2}{4(s^2-4Rr-r^2)+s^2+4Rr+r^2}$$

$$=\frac{4(4R+r)^2}{5s^2-12Rr-3r^2}\overset{?}{\ge}\frac{54r^2}{s^2-4Rr-r^2}$$

$$\Leftrightarrow\frac{2(4R+r)^2}{27r^2}\overset{?}{\ge}\frac{5(s^2-4Rr-r^2)+8Rr+2r^2}{s^2-4Rr-r^2}$$

$$\Leftrightarrow\frac{2(4R+r)^2-135r^2}{27r^2}\overset{?}{\ge}\frac{8Rr+2r^2}{s^2-4Rr-r^2}$$

$$\Leftrightarrow(32R^2+16Rr-133r^2)(s^2-4Rr-r^2)\overset{?}{\ge}27r^3(8R+2r)\quad\textbf{(1)}$$

LHS of (1) $\overset{Gerretsen}{\ge} 6r(2R-r)(32R^2+16Rr-133r^2)\overset{?}{\ge}27r^3(8R+2r)$

$\Leftrightarrow 32t^3-159t+62\overset{?}{\ge}0$ ***(where*** $t=\frac{R}{r}$***)*** $\Leftrightarrow(t-2)(32t^2+64t-31)\overset{?}{\ge}0\to$ ***true***

$\because t\ge 2$ ***(Euler)*** $\Rightarrow$ ***(1) is true (Proved)***

JP.105. Let $m>0$ and F be the area of the triangle ABC. Then:

$$\frac{a^{m+2}}{b^m+c^m}+\frac{b^{m+2}}{c^m+a^m}+\frac{c^{m+2}}{a^m+b^m}\geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

Solution by Soumava Chakraborty-Kolkata-India

$$LHS=\sum_{cyc}\left(a^2\cdot\frac{x}{y+z}\right)(x=a^m,y=b^m,z=c^m)$$

$$\geq 4F\sqrt{\frac{x}{y+z}\cdot\frac{y}{z+x}+\frac{y}{z+x}\cdot\frac{z}{x+y}+\frac{z}{x+y}\cdot\frac{x}{y+z}}$$

$$\left(\begin{array}{c}\because a^2m'+b^2n'+c^2p'\geq 4R\sqrt{m'n'+n'p'+p'm'}\\ \forall m',n',p'\in\mathbb{R}^+ \text{ and as}, \frac{x}{y+z},\frac{y}{z+x},\frac{z}{x+y}>0\\ \because x,y,z>0\end{array}\right)$$

$$\overset{?}{\geq} 2\sqrt{3}F$$

$$\Leftrightarrow \frac{xy}{(y+z)(z+x)}+\frac{yz}{(z+x)(x+y)}+\frac{zx}{(x+y)(y+z)}\overset{?}{\geq}\frac{3}{4}$$

$$\Leftrightarrow \frac{\sum\{xy(x+y)\}}{2xyz+\sum x^2y+\sum xy^2}\overset{?}{\geq}\frac{3}{4}$$

$$\Leftrightarrow 4\sum x^2y+4\sum xy^2\overset{?}{\geq}6xyz+3\sum x^2y+3\sum xy^2$$

$$\Leftrightarrow \sum x^2y+\sum xy^2\overset{?}{\geq}6xyz\rightarrow \textit{true by AM-GM}$$

SP.091. Prove that for all positive real numbers a,b,c,d:

$$\frac{a^2}{a+b+c}+\frac{b^2}{b+c+d}+\frac{c^2}{c+d+a}+\frac{d^2}{d+a+b}\geq\frac{a+b+c+d}{3}+\frac{4(2a+b-2c-d)^2}{27(a+b+c+d)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

We have: $\frac{a^2}{a+b+c}=\frac{5a-b-c}{9}+\frac{(b+c-2a)^2}{9(a+b+c)},\frac{b^2}{b+c+d}=\frac{5b-c-d}{9}+\frac{(c+d-2b)^2}{9(b+c+d)},$

$$\frac{c^2}{c+d+a}=\frac{5c-d-a}{9}+\frac{(d+a-2c)^2}{9(c+d+a)},\frac{d^2}{d+a+b}=\frac{5d-a-b}{9}+\frac{(a+b-2d)^2}{9(d+a+b)}$$

Adding up these relations we obtain: $\sum_{cyc}\frac{a^2}{a+b+c}=\frac{a+b+c+d}{3}+\sum_{cyc}\frac{(b+c-2a)^2}{9(a+b+c)}$.

Now we use Cauchy – Schwarz inequaity (or Titu's lemma) to get

$$\sum_{cyc}\frac{(b+c-2a)^2}{9(a+b+c)}=\frac{(b+c-2a)^2}{9(a+b+c)}+\frac{(c+d-2b)^2}{9(b+c+d)}+\frac{(-d-a+2c)^2}{9(c+d+a)}+$$

$$+\frac{(-a-b+2d)^2}{9(d+a+b)}\geq\frac{4(2a+b-2c-d)^2}{27(a+b+c+d)}$$

Therefore $\frac{a^2}{a+b+c}+\frac{b^2}{b+c+d}+\frac{c^2}{c+d+a}+\frac{d^2}{d+a+b}\geq\frac{a+b+c+d}{3}+\frac{4(2a+b-2c-d)^2}{27(a+b+c+d)}$ *as desired.*

SP.092. Prove that for all positive real numbers a,b,c:

a. $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}+\frac{(b-c)^2}{2(a+b+c)}$

b. $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{a+b+c}{2}+\frac{(a+b-2c)^2}{2(a+b+c)}$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}+\frac{(b-c)^2}{2(a+b+c)}$$

Given inequality $\Leftrightarrow\frac{\sum a^2(\sum ab+a^2)}{(a+b)(b+c)(c+a)}\geq\frac{(a+b+c)^2+(a+b-2c)^2}{2(a+b+c)}$

$$\Leftrightarrow 2\left(\sum a\right)\left\{\sum a^4+\left(\sum ab\right)\left(\sum a^2\right)\right\}\geq$$

$$\geq(a+b)(b+c)(c+a)\{(a+b+c)^2+(a+b-2c)^2\}$$

$$\Leftrightarrow 2(a^5+b^5+c^5)+2a^4b+2a^4c+2a^3c^2+2ab^4+2b^4c+2b^3c^2\geq$$

$$\geq 4a^3b^2+4a^2b^3+4a^2b^2c+a^2bc^2+a^2c^3+ab^2c^2+ac^4+b^2c^3+bc^4 \quad \textit{(1)}$$

Now, $2(a^5+ab^4)\overset{A-G}{\geq}4a^3b^2$ *(a)*

$$2(b^5+a^4b)\overset{A-G}{\geq}4a^2b^3 \quad \textit{(b)}$$

$$2(a^4c+b^4c)\overset{A-G}{\geq}4a^2b^2c \quad \textit{(c)}$$

$$c^2(a^3+b^3)\ge c^2ab(a+b)=a^2bc^2+ab^2c^2 \quad \textbf{(d)}$$

$$c^2(a^3+c^3)\ge c^2ac(a+c)=a^2c^3+ac^4 \quad \textbf{(e)}$$

$$c^2(b^3+c^3)\ge c^2bc(b+c)=b^2c^3+bc^4 \quad \textbf{(f)}$$

(a)+(b)+(c)+(d)+(e)+(f)⇒ (1) is true

(Proved)

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge\frac{a+b+c}{2}+\frac{(a+b-2c)^2}{2(a+b+c)}$$

Given inequality $\Leftrightarrow \frac{\sum a^2(\sum ab+a^2)}{(a+b)(b+c)(c+a)}\ge\frac{(a+b+c)^2+(a+b-2c)^2}{2(a+b+c)}$

$$\Leftrightarrow 2\left(\sum a\right)\left\{\sum a^4+\left(\sum ab\right)\left(\sum a^2\right)\right\}\ge$$

$$\ge(a+b)(b+c)(c+a)\{(a+b+c)^2+(a+b-2c)^2\}$$

$$\Leftrightarrow 2\left(a^5+b^5+c^5\right)+2a^4b+2a^4c+2a^3c^2+2ab^4+2b^4c+2b^3c^2\ge$$

$$\ge 4a^3b^2+4a^2b^3+4a^2b^2c+a^2bc^2+a^2c^3+ab^2c^2+ac^4+b^2c^3+bc^4 \textbf{(1)}$$

Now, $2\left(a^5+ab^4\right)\overset{A-G}{\ge}4a^3b^2$ ***(a)***

$$2\left(b^5+a^4b\right)\overset{A-G}{\ge}4a^2b^3 \quad \textbf{(b)}$$

$$2(a^4c+b^4c)\overset{A-G}{\ge}4a^2b^2c \quad \textbf{(c)}$$

$$c^2(a^3+b^3)\ge c^2ab(a+b)=a^2bc^2+ab^2c^2 \quad \textbf{(d)}$$

$$c^2(a^3+c^3)\ge c^2ac(a+c)=a^2c^3+ac^4 \quad \textbf{(e)}$$

$$c^2(b^3+c^3)\ge c^2bc(b+c)=b^2c^3+bc^4 \quad \textbf{(f)}$$

(a)+(b)+(c)+(d)+(e)+(f)⇒(1) is true (Proved)

SP.093. Prove that in any triangle ABC the following inequality holds

$$\frac{(b+c)a}{m_a^2}+\frac{(c+a)b}{m_b^2}+\frac{(a+b)c}{m_c^2}\ge 8$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $s-a=x, s-b=y, s-c=z$***. Then*** $x,y,z>0$ ***and*** $s=x+y+z$

$\therefore a=y+z, b=z+x, c=x+y$***. Now, given inequality*** $\Leftrightarrow$

$$\Leftrightarrow \frac{(b+c)a}{2b^2+2c^2-a^2}+\frac{(c+a)b}{2c^2+2a^2-b^2}+\frac{(a+b)c}{2a^2+2b^2-c^2}\overset{(1)}{\geq} 2$$

Now, $2b^2+2c^2-a^2=2(z+x)^2+2(x+y)^2-(y+z)^2$

$$=2z^2+2x^2+4zx+2x^2+2y^2+4xy-y^2-z^2-2yz$$

$$=z^2+y^2+4x^2+2yz+4xy+4zx-4yz\overset{(a)}{=}(y+z+2z)^2-4yz$$

$$(a)\Rightarrow \frac{(b+c)a}{2b^2+2c^2-a^2}=\frac{(y+z)(y+z+2x)}{(y+z+2x)^2-4yz}\ \ (i)$$

Similalry, $\frac{(c+a)b}{2c^2+2a^2-b^2}\overset{(ii)}{=}\frac{(z+x)(z+x+2y)}{(z+x+2y)^2-4zx}$ **&** $\frac{(a+b)c}{2a^2+2b^2-c^2}\overset{(iii)}{=}\frac{(x+y)(x+y+2z)}{(x+y+2z)^2-4xy}$

(i)+(ii)+(iii)⇒ given inequality ⇔

$$\Leftrightarrow (y+z)(y+z+2x)\{(z+x+2y)^2-4zx\}\{(x+y+2z)^2-4xy\}+$$
$$+(z+x)(z+x+2y)\{(x+y+2z)^2-4xy\}\{(y+z+2x)^2-4yz\}+$$
$$+(x+y)(x+y+2z)\{(y+z+2x)^2-4yz\}\{(z+x+2y)^2-4zx\}\geq$$
$$\geq 2\{(x+y+2z)^2-4xy\}\{(z+x+2y)^2-4zx\}\{(y+z+2x)^2-4yz\}$$

$$\Leftrightarrow 10\sum x^5y+10\sum xy^5+77\sum x^4y^2+77\sum x^2y^4+$$
$$+150\sum x^3y^3\overset{(2)}{\geq}118xyz\left(\sum x^3\right)+90xyz\left(\sum x^2y+\sum xy^2\right)++78x^2y^2z^2$$

Now, $59\sum x^4y^2+59\sum x^2y^4=$

$$=59\{x^4(y^2+z^2)+y^4(z^2+x^2)+z^4(x^2+y^2)\}\underset{(iv)}{\overset{A-G}{\geq}}118xyz\left(\sum x^3\right)$$

Now, $\forall u,v,w\in\mathbb{R}^+,\sum u^3+3uvw\overset{Shur}{\geq}\sum u^2v+\sum uv^2$ ***and*** $\sum u^3\overset{A-G}{\geq}3uvw$

Adding the last 2, $2\sum u^3\geq\sum u^2v+\sum uv^2$ ***(b)***

(b) $\Rightarrow 150\sum x^3y^3\geq 75xyz(\sum x^2y+\sum xy^2)$ ***(v)***

Again, $15\sum x^4y^2+15\sum x^2y^4\overset{A-G}{\geq}30\sum x^3y^3$

(vi) $\geq 15xyz(\sum x^2y+\sum xy^2)$ ***(by (b))***

Also, $3\sum x^4y^2+3\sum x^2y^4\overset{A-G}{\geq}18x^2y^2z^2$ ***(vii)***

$10\sum x^5y+10\sum xy^5\overset{A-G}{\geq}60x^2y^2z^2$ ***(viii)***

(iv)+(v)+(vi)+(vii)+(viii)⇒ (2) is true (proved)

SP.094. Prove that in any acute triangle ABC the following inequality holds

$$\frac{\cos B\cos C}{\sin A}+\frac{\cos C\cos A}{\sin B}+\frac{\cos A\cos B}{\sin C}\le\frac{\sqrt{3}}{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

$$\sum\frac{\cos A\cos B}{\sin C}\le\frac{\sqrt{3}}{2}$$

Since $\Delta\ ABC$ is acute then $\sin A, \sin B, \sin C>0$. So, the inequality is equivalent to:

$$\sum\cos A\cos B\sin A\sin B\le\frac{\sqrt{3}}{2}\sin A\sin B\sin C\Leftrightarrow$$

$$\Leftrightarrow\sum\sin 2A\sin 2B\le 2\sqrt{3}\sin A\sin B\sin C$$

We have: $\sum\sin 2A\sin 2B\le\frac{(\sum\sin 2A)^2}{3}=\frac{[4\sin A\sin B\sin C]^2}{3}\le 2\sqrt{3}\sin A\sin B\sin C$

$\Leftrightarrow\sin A\sin B\sin C\le\frac{3\sqrt{3}}{8}$, *this is true by AM-GM since:*

$$\sin A\sin B\sin C\le\frac{(\sin A+\sin B+\sin C)^3}{27}\le\frac{\left(\frac{3\sqrt{3}}{2}\right)^3}{27}=\frac{3\sqrt{3}}{8}\Rightarrow \textit{Q.E.D.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{1}{\prod\sin A}\sum\cos B\cos C\sin B\sin C=\frac{1}{4\prod\sin A}\sum(2\sin B\cos B)(2\sin C\cos C)$$

$$=\frac{1}{12\prod\sin A}\cdot 3\sum\sin 2B\sin 2C$$

$$\le\frac{1}{12\prod\sin A}\left(\sum\sin 2A\right)^2\ \left(\because 3\sum xy\le\left(\sum x\right)^2, \forall x,y,z\right)$$

$$=\frac{1}{12}\frac{(4\prod\sin A)^2}{(\prod\sin A)}=\frac{4}{3}(\sin A\sin B\sin C)$$

$$=\frac{4}{3}\cdot\frac{abc}{8R^3}=\frac{16Rrs}{24R^3}=\frac{2rs}{3R^2}\overset{Euler}{\le}\frac{RS}{3R^2}=\frac{s}{3R}\overset{Mitrinovic}{\le}\frac{3\sqrt{3}R}{2\cdot 3R}=\frac{\sqrt{3}}{2}$$

(proved)

SP.095. Let a, b, c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that:

$$(b^4+c^4)\sin^2 A+(c^4+a^4)\sin^2 B+(a^4+b^4)\sin^2 C\le\frac{81}{4}(3R^4-16r^4)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{1}{4R^2}\{(b^2+c^2)^2-2b^2c^2\}a^2+$$

$$+\frac{1}{4R^2}\{(c^2+a^2)^2-2c^2a^2\}b^2+\frac{1}{4R^2}\{(a^2+b^2)^2-2a^2b^2\}c^2\le\frac{81}{4}(3R^4-16r^4)$$

$$\Leftrightarrow(b^2+c^2)^2a^2+(c^2+a^2)^2b^2+(a^2+b^2)^2c^2\le$$

$$\le 81R^2(3R^4-16r^4)+6a^2b^2c^2\ \ (1)$$

WLOG, we may assume $a\ge b\ge c$***. Then,*** $a^2(b^2+c^2)\ge b^2(c^2+a^2)\ge c^2(a^2+b^2)$

$$b^2+c^2\le c^2+a^2\le a^2+b^2$$

$$\therefore \text{LHS of (1)}\overset{\text{Chebyshev}}{\le}\frac{1}{3}\{\sum a^2(b^2+c^2)\}\{\sum(b^2+c^2)\}$$

$$=\frac{4}{3}\left(\sum a^2b^2\right)\left(\sum a^2\right)\overset{\text{Goldstone}}{\le}\frac{4}{3}(4R^2s^2)\left(\sum a^2\right)$$

$$\overset{\text{Leibnitz}}{\le}\frac{4}{3}(4R^2s^2)(9R^2)=48R^4s^2\overset{?}{\le}81R^2(3R^4-16r^4)+96R^2r^2s^2$$

$$\Leftrightarrow 16R^2s^2\overset{?}{\le}27(3R^4-16r^4)+32r^2s^2$$

$$\Leftrightarrow s^2(16R^2-32r^2)\overset{?}{\le}81R^4-432r^4\ \ (2)$$

$$\text{Now, LHS of (2)}\overset{\text{Gerretsen}}{\le}(4R^2+4Rr+3r^2)(16R^2-32r^2)\overset{?}{\le}81R^4-432r^4$$

$$\Leftrightarrow 17t^4-64t^3+80t^2+128t-336\ge 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{(t-2)(17t^2+4t+28)+224\}\ge 0\rightarrow \text{true}\because t=\frac{R}{r}\ge 2\ \text{(Euler)}$$

(Proved)

SP.096. Let ABC be a triangle and w_a, w_b, w_c are bisectors of ABC. Prove that:

$$\frac{1}{aw_a^2}+\frac{1}{bw_b^2}+\frac{1}{cw_c^2}\ge\frac{1}{R\Delta}$$

where R is the circumradius of ABC, Δ is area of ABC.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution 1 by Soumava Chakraborty-Kolkata-India

$$w_a^2=\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=\frac{4bcs(s-a)}{(b+c)^2}$$

$$\Rightarrow\frac{1}{aw_a^2}=\frac{(b+c)^2}{4abcs(s-a)}\quad (1)$$

$$\text{Similarly, }\frac{1}{bw_b^2}\overset{(2)}{=}\frac{(c+a)^2}{4abcs(s-b)}\ \&\ \frac{1}{cw_c^2}\overset{(3)}{=}\frac{(a+b)^2}{4abcs(s-c)}$$

$$(1)+(2)+(3)\Rightarrow LHS=\frac{1}{4s\cdot 4R\Delta}\sum\frac{(a+b)^2}{s-c}$$

$$=\frac{1}{16sR\Delta}\sum\frac{(s+s-c)^2}{s-c}=\frac{1}{16sR\Delta}\sum\frac{s^2+2s(s-c)+(s-c)^2}{s-c}$$

$$=\frac{1}{16sR\Delta}\left\{s^2\sum\frac{1}{s-c}+2s\sum(1)+\sum(s-c)\right\}$$

$$=\frac{1}{16sR\Delta}\left[\frac{s^3}{r^2s^2}\sum\{s^2-s(a+b)+ab\}+6s+(3s-2s)\right]$$

$$=\frac{1}{16sR\Delta}\left\{\frac{s}{r^2}(3s^2-4s^2+s^2+4Rr+r^2)+7s\right\}$$

$$=\frac{1}{16sR\Delta}\left\{\frac{s(4R+r)}{r}+7s\right\}=\frac{s(4R+8r)}{16sR\Delta r}=\frac{R+2r}{4r\cdot R\Delta}\overset{Euler}{\ge}\frac{4r}{4r\cdot R\Delta}=\frac{1}{R\Delta}\quad \textit{(Proved)}$$

Proof 2: $w_a^2\le s(s-a)\Rightarrow aw_a^2\le as(s-a)\Rightarrow\frac{1}{aw_a^2}\ge\frac{1}{as(s-a)}$ ***(1)***

$$\text{Similarly, }\frac{1}{bw_b^2}\overset{(2)}{\ge}\frac{1}{bs(s-b)}\ \&\ \frac{1}{cw_c^2}\overset{(3)}{\ge}\frac{1}{cs(s-c)}$$

$$(1)+(2)+(3)\Rightarrow LHS\ge\frac{1}{s}\sum\frac{1}{a(s-a)}\quad (4)$$

WLOG, we may assume $a\ge b\ge c$. ***Then*** $\frac{1}{a}\le\frac{1}{b}\le\frac{1}{c}$ ***and*** $\frac{1}{s-a}\ge\frac{1}{s-b}\ge\frac{1}{s-c}$

$$(4)\Rightarrow LHS\overset{Chebyshev}{\ge}\frac{1}{3s}\sum\frac{1}{a}\sum\frac{1}{s-a}$$

$$=\frac{1}{3s}\left(\frac{\sum ab}{abc}\right)\frac{s}{r^2s^2}\left\{\sum(s-b)(s-c)\right\}=\frac{(s^2+4r+r^2)}{3r^2s^2\cdot 4R\Delta}(3s^2-4s^2+s^2+4Rr+r^2)$$

$$=\frac{(s^2+4Rr+r^2)(4R+r)}{12rs^2R\Delta}\overset{?}{\geq}\frac{1}{R\Delta}$$

$$\Leftrightarrow (s^2+4Rr+r^2)(4R+r)\geq 12rs^2 \quad \textbf{(5)}$$

Now, LHS of (5) $\overset{Gerretsen}{\geq} (20Rr-4r^2)(4R+r)$

& RHS of (5) $\overset{Gerretsen}{\leq} 12r(4R^2+4Rr+3r^2)$

$\therefore$ ***it suffices to prove:*** $(5R-r)(4R+r)\geq 3(4R^2+4Rr+3r^2)$

$\Leftrightarrow 8R^2-11Rr-10r^2\geq 0 \Leftrightarrow (R-2r)(8R+5r)\geq 0\to$ ***true***

$\because R\geq 2r$ ***(Euler)*** $\Rightarrow$ ***(5) is true (Proved)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{1}{a\cdot w_a^2}\geq\frac{1}{R\cdot\Delta};\ \begin{matrix}x=p-a\\ y=p-b\\ z=p-c\end{matrix}\Rightarrow x+y+z=p$$

$$\textbf{1)}\ \sum\frac{1}{a\cdot w_a^2}=\frac{1}{(y+z)\cdot\left(\frac{2}{2x+y+z}\cdot\sqrt{x(x+z)(y+x)\cdot\sum x}\right)^2}=$$

$$=\sum\frac{(2x+y+z)^2}{4x\prod(x+y)\cdot\sum x}\overset{Bergstrom}{\geq}\frac{(\sum(2xy+y+z))^2}{4\sum x\prod(x+y)}=\frac{16(x+y+z)^2}{4(x+y+z)^2\cdot\prod(x+y)}$$

$$=\frac{4}{\prod(x+y)}=LHS$$

$$\textbf{2)}\ \frac{1}{R\cdot\Delta}=\frac{1}{\frac{abc}{4\Delta}\cdot\Delta}=\frac{4}{abc}=\frac{4}{\prod(x+y)}=RHS$$

$$\textbf{1), 2)}\ \sum\frac{1}{aw_a^2}\geq\frac{4}{\prod(x+y)}=\frac{1}{R\cdot\Delta}$$

SP.097. Let a,b,c be the side lengths of a triangle ABC with incentre I, circumradius R and inradius r. Prove that:

$$\frac{\sqrt{AI}}{a}+\frac{\sqrt{BI}}{b}+\frac{\sqrt{CI}}{c}\leq\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{\sqrt{AI}}{a} \overset{C-B-S}{\leq} \sqrt{\sum AI}\sqrt{\sum \frac{1}{a^2}}$$

$$= \sqrt{\sum AI}\sqrt{\frac{\sum a^2b^2}{a^2b^2c^2}} \overset{Goldstone}{\leq} \frac{2Rs}{4Rrs}\sqrt{\sum AI} = \frac{1}{2r}\sqrt{\sum AI} \overset{?}{\leq} \frac{\sqrt{2(R+r)}}{2r}$$

$$\Leftrightarrow \sum AI \overset{?}{\leq} 2(R+r) \quad (1)$$

$$\text{Now, } \sum AI = r\sum \sqrt{\frac{bc}{(s-b)(s-c)}}$$

$$= \frac{r\sqrt{s}}{\sqrt{s(s-a)(s-b)(s-c)}}\sum \sqrt{bc}\sqrt{s-a} \overset{C-B-S}{\leq} \frac{r\sqrt{s}}{rs}\sqrt{\sum ab}\sqrt{3s-2s} = \sqrt{\sum ab}$$

$$= \sqrt{s^2+4Rr+r^2} \overset{Gerretsen}{\leq} \sqrt{4R^2+8Rr+4r^2} = \sqrt{4(R+r)^2} = 2(R+r)$$

$\Rightarrow$ ***(1) is true (Proved)***

SP.098. Let ABC be an acute triangle with orthocenter H. Prove that:

$$AH \cdot BH + BH \cdot CH + CH \cdot AH \leq 6Rr,$$

where R and r are the circumradius and inradius respectively of triangle ABC.

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

$$AH \cdot BH + BH \cdot CH + CH \cdot AH = \sum 4R^2 \cdot \cos A \cdot \cos B =$$

$$= 4R^2\left(\frac{p^2+r^2}{4R^2} - 1\right) = p^2 + r^2 - 4R^2 \leq 4R^2 + 4Rr + 3r^2 + r^2 - 4R^2$$

$$= 4R + 4r^2 \leq 4Rr + 2Rr = 6Rr \Rightarrow \text{Q.E.D.}$$

SP.099. Let a, b, c be non-negative such that $a + b + c = 3$. Prove that:

$$|(a-b)(b-c)(c-a)| \leq \frac{3\sqrt{3}}{2}. \text{ Equality occurs when?}$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

We will prove that: $(a-b)^2(b-c)^2(c-a)^2 \le \frac{27}{4}$. *WLOG, assume that*

$$c = \max\{a; b; c\}$$

$$c \ge b \ge a \ge 0: (a-b)^2 \le b^2; (c-a)^2 \le c^2 \Rightarrow$$

$$\Rightarrow (a-b)^2(b-c)^2(c-a)^2 \le b^2c^2 \cdot (b-c)^2 = \frac{1}{4}(2bc)^2 \cdot (b^2 - 2bc + c^2)$$

$$\le \frac{(2bc + 2bc + b^2 - 2bc + c^2)^3}{4 \cdot 27} = \frac{(b+c)^6}{108} \le \frac{(a+b+c)^6}{108} = \frac{27}{4}$$

$$c^2 \ge a \ge b \ge 0: (a-b)^2 \le a^2; (b-c)^2 \le c^2 \Rightarrow (a-b)^2(b-c)^2(c-a)^2 \le$$

$$\le a^2c^2(c-a)^2 = \frac{1}{4}(2ac)^2 \cdot (a^2 - 2ac + c^2) \le \frac{(2ac+2ac+a^2-2ac+c^2)^3}{4 \cdot 27}$$

$$= \frac{(a+c)^6}{108} \le \frac{(a+b+c)^6}{108} = \frac{27}{4}$$

Hence: $(a-b)^2(b-c)^2(c-a)^2 \le \frac{27}{4} \Rightarrow |(a-b)(b-c)(c-a)| \le \frac{3\sqrt{3}}{2}$

The equality happens iff $(a; b; c) \sim \left(0; \frac{3-\sqrt{3}}{2}; \frac{3+\sqrt{3}}{2}\right)$

SP.100. Let a, b, c be the lengths of the sides of a triangle with perimeter 3 and inradius r. Prove that:

$$288r^2 \le \frac{(a+b)^4}{a^2+b^2} + \frac{(b+c)^4}{b^2+c^2} + \frac{(c+a)^4}{c^2+a^2} \le \frac{2}{r^2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

In any $\Delta\, ABC$ *with perimeter* $= 3,\ 288r^2 \le \sum \frac{(a+b)^4}{a^2+b^2} \le \frac{2}{r^2}$

$$a^2 + b^2 \ge \frac{(a+b)^2}{2} \text{ etc}, \therefore \sum \frac{(a+b)^4}{a^2+b^2} \le 2\sum (a+b)^2 \le \frac{2}{r^2}$$

$$\Leftrightarrow \sum (a+b)^2 \le \frac{16s^4}{81r^2} \left(\because s^4 = \frac{81}{16} \text{ as } 2s = 3\right) \Leftrightarrow \sum a^2 + \sum ab \le \frac{8s^4}{81r^2}$$

$$\Leftrightarrow 8s^4 \ge 81r^2(3s^2 - 4Rr - r^2)$$

$$\Leftrightarrow 8s^4 + 324Rr^3 + 81r^4 \ge 243s^2r^2 \rightarrow \textit{(1)}$$

LHS of (1) $\overset{Gerretsen}{\geq} 8s^2(16Rr - 5r^2) + 324Rr^3 + 81r^4 \overset{?}{\geq} 243s^2r^2$

$\Leftrightarrow s^2(128R - 256r) + 324Rr^2 + 81r^3 \overset{?}{\geq} 27s^2r \to$ ***(2)***

LHS of (2) $\overset{Gerretsen}{\geq} (16Rr - 5r^2)(128R - 256r) + 324Rr^2 + 81r^3$

and, RHS of (2) $\overset{Gerretsen}{\leq} 27r(4R^2 + 4Rr + 3r^2)$

$\therefore$ ***in order to prove (2), it suffices to prove:***

$$(16Rr - 5r^2)(128R - 256r) + 324Rr^2 + 81r^3 \overset{?}{\geq} 27r(4R^2 + 4Rr + 3r^2)$$

$\Leftrightarrow 97R^2 - 226Rr + 64r^2 \overset{?}{\geq} 0 \Leftrightarrow (R - 2r)(97R - 32r) \overset{?}{\geq} 0 \to$ ***true***

$\therefore R \geq 2r$ ***(Euler)*** $\Rightarrow$ ***(2) is true*** $\therefore \frac{(a+b)^4}{a^2+b^2} \leq \frac{2}{r^2}$

Again, $\frac{(a+b)^4}{a^2+b^2} \overset{Bergstrom}{\geq} \frac{\left(\sum(a+b)^2\right)^2}{2\sum a^2} \overset{Leibniz}{\geq} \frac{4\left(\sum a^2+\sum ab\right)^2}{18R^2} \overset{?}{\geq} 288r^2$

$\Leftrightarrow \sum a^2 + \sum ab \overset{?}{\geq} 36Rr \Leftrightarrow 3s^2 \overset{?}{\geq} 40Rr + r^2 \to$ ***(3)***

LHS of (3) $\overset{Gerretsen}{\geq} 48Rr - 15r^2 \overset{?}{\geq} 40Rr + r^2 \Leftrightarrow 8Rr \overset{?}{\geq} 16r^2 \Leftrightarrow R \overset{?}{\geq} 2r$

$\to$ ***true (Euler)*** $\Rightarrow$ ***(3) is true*** $\therefore 288r^2 \leq \sum \frac{(a+b)^4}{a^2+b^2}$

(proved)

SP.101. Let a, b and c be the side lengths of a triangle with inradius r. Prove that:

$$\sqrt[4]{\frac{1}{a^4 + 2b^2c^2} + \frac{1}{b^4 + 2c^2a^2} + \frac{1}{c^4 + 2a^2b^2}} \leq \frac{\sqrt{3}}{6r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Soumava Chakraborty-Kolkata-India

$$a^4 + 2b^2c^2 = a^4 + b^2c^2 + b^2c^2 \overset{A-G}{\geq} 3\sqrt[3]{a^4b^4c^4}$$

$$\Rightarrow \frac{1}{a^4+2b^2c^2} \leq \frac{1}{3\sqrt[3]{a^4b^4c^4}} \quad \textbf{(1)}$$

Similarly, $\frac{1}{b^4+2c^2a^2} \overset{(2)}{\leq} \frac{1}{3\sqrt[3]{a^4b^4c^4}}$ ***&*** $\frac{1}{c^4+2a^2b^2} \overset{(3)}{\leq} \frac{1}{3\sqrt[3]{a^4b^4c^4}}$

$$(1)+(2)+(3) \Rightarrow LHS \le \sqrt[4]{\frac{1}{\sqrt[3]{a^4b^4c^4}}} = \frac{1}{\sqrt[3]{abc}} \overset{?}{\le} \frac{\sqrt{3}}{6r} \Leftrightarrow \sqrt[3]{abc} \overset{(a)}{\ge} \frac{\sqrt{3}\sqrt{3}\cdot 2r}{\sqrt{3}} = 2\sqrt{3}r$$

$$\text{Now, } \sqrt[3]{abc} = \sqrt[3]{4Rrs} \overset{Euler}{\ge} \sqrt[3]{4(2r)rs}$$

$$\overset{s\ge 3\sqrt{3}r}{\ge} \sqrt[3]{4(2r)r(3\sqrt{3}r)} = \sqrt[3]{8\cdot 3\sqrt{3}r^3} = 2\sqrt{3}r \Rightarrow \text{(a) is true (proved)}$$

SP.102. Let ABC be a triangle with circumradius R and inradius r. Prove that:

$$4 \le sec^2\frac{A}{2} + sec^2\frac{B}{2} + sec^2\frac{C}{2} \le \frac{2R}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$ab+bc+ca = p^2+r^2+4Rr,\ abc = 4Rrp \text{ and } \prod_{cyc}(p-a) = pr^2$$

$$\text{again, } 9r(r+4R) \le 3p^2 \le (r+4R)^2$$

$$\sum_{cyc} bc(p-b)(p-c) = p^2\left(\sum_{cyc} ab\right) - p\sum_{cyc} ab(a+b) + \sum_{cyc} a^2b^2$$

$$= p^2\sum_{cyc} ab - p\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) + 3abcp + \left(\sum_{cyc} ab\right)^2 - 2abc\sum_{cyc} a$$

$$= r^2(r+4R)^2 + p^2r^2 \text{ then}$$

$$\sum_{cyc}\sec^2\frac{A}{2} = \sum_{cyc}\frac{bc}{p(p-a)} = \frac{r^2(r+4R)^2+p^2r^2}{p(p-a)(p-b)(p-c)} = \left(\frac{r+4R}{p}\right)^2 + 1$$

$$\ge 3+1 = 4 \text{ again, } \left(\frac{r+4R}{p}\right)^2 + 1 \le \frac{2R}{r} \Leftrightarrow \frac{r(r+4R)^2}{2R-r} \le p^2 \text{ we will prove,}$$

$$3r(r+4R) \ge \frac{r(r+4R)^2}{2R-r} \Leftrightarrow 3(2R-r) \ge r+4R \Leftrightarrow 2(R-2r) \ge 0$$

which is true. Hence proved.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$4 \overset{(b)}{\le} \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \overset{(a)}{\le} \frac{2R}{r}$$

$$\sum \sec^2\frac{A}{2} = 3 + \sum \tan^2\frac{A}{2}$$

$$\overset{(1)}{=} 3 + \frac{1}{s}\left\{\frac{(s-b)(s-c)}{s-a} + \frac{(s-c)(s-a)}{s-b} + \frac{(s-a)(s-b)}{s-c}\right\}$$

$$\frac{2R}{r} = \frac{2abcs}{4\Delta^2} = \frac{2sabc}{4s(s-a)(s-b)(s-c)} \overset{(2)}{\le} \frac{abc}{2(s-a)(s-b)(s-c)}$$

Let $s-a = x, s-b = y, s-c = z \therefore s = x+y+z$

$$\Rightarrow a = y+z, b = z+x, c = x+y; x,y,z > 0$$

(1) $\Rightarrow \sum \sec^2\frac{A}{2} = 3 + \frac{1}{\sum x}\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) \overset{(3)}{=} \frac{3xyz(\sum x) + \sum x^2y^2}{xyz(\sum x)}$

(3), (2) $\Rightarrow$ ***(a)*** $\Leftrightarrow \frac{\sum x^2y^2 + 3xyz(\sum x)}{xyz(\sum x)} \le \frac{(x+y)(y+z)(z+x)}{2xyz}$

$$\Leftrightarrow \left(\sum x\right)\left(2xyz + \sum x^2y + \sum xy^2\right) \ge 2\sum x^2y^2 + 6xyz\left(\sum x\right)$$

$$\Leftrightarrow 2xyz\left(\sum x\right) + \sum x^3y + \sum xy^3 + 2\sum x^2y^2 + 2xyz\left(\sum x\right)$$

$$\ge 2\sum x^2y^2 + 6xyz\left(\sum x\right)$$

$\Leftrightarrow \sum x^3y + \sum xy^3 \ge 2xyz(\sum x)$ ***(4)***

LHS of (4) $\overset{A-G}{\ge} 2\sum x^2y^2 \ge 2xyz(\sum x)$ $(\because m^2 + n^2 + p^2 \ge mn + np + pm)$

$\Rightarrow$ ***(4) is true*** $\Rightarrow$ ***(a) is true (*)***

(3) $\Rightarrow$ ***(b)*** $\Leftrightarrow \sum x^2y^2 + 3xyz(\sum x) \ge 4xyz(\sum x)$

$\Leftrightarrow \sum x^2y^2 \ge xyz(\sum x) \rightarrow$ ***true*** $\Rightarrow$ ***(b) is true (*) (proved)***

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \sec^2\frac{A}{2} = \sum \frac{bc}{p(p-a)} = \frac{abc}{p}\cdot\sum\frac{1}{a(p-a)} = \frac{abc}{p}\cdot\frac{\sum ab(p-a)(p-b)}{abc\cdot\prod(p-a)}$$

$$= \frac{1}{\Delta^2}\cdot\sum\left(ab(p^2 - (a+b)p + ab)\right) =$$

$$= \frac{1}{\Delta^2}\cdot\left(p^2\cdot\sum ab - p\cdot\sum(a^2b + ab^2) + \sum(ab)^2\right)$$

$$= \frac{1}{\Delta^2}\cdot\left(p^2\sum ab - p\left(\sum ab\cdot\sum a - 3abc\right) + \left(\sum ab\right)^2 - 4pabc\right) =$$

$$=\frac{1}{\Delta^2}\left(p^2\sum ab-2p^2\sum ab+3pabc+\left(\sum ab\right)^2-4pabc\right)=$$

$$=\frac{1}{\Delta^2}\cdot\left(-p^2\sum ab+\left(\sum ab\right)^2-pabc\right)=$$

$$=\frac{1}{\Delta^2}\left(\sum ab\,(-p^2+p^2+4Rr+r^2)-4p^2Rr\right)=$$

$$=\frac{1}{\Delta^2}\left((p^2+4Rr+r^2)(4Rr+r^2)-4p^2Rr\right)=$$

$$=\frac{1}{\Delta^2}(4Rrp^2+\Delta^2+(4Rr+r^2)^2-4p^2Rr)=1+\frac{(4Rr+r^2)^2}{\Delta^2}=1+\frac{(4R+r)^2}{p^2}$$

$$\sum\sec^2\frac{A}{2}=1+\frac{(4R+r)^2}{p^2};\ 4\le 1+\left(\frac{4R+r}{p}\right)^2\le\frac{2R}{r}$$

LHS: $3\le\left(\frac{4R+r}{p}\right)^2\Leftrightarrow\sqrt{3}p\le 4R+r$

RHS: $1+\frac{(4R+r)^2}{p^2}\le\frac{2R}{r}\Leftrightarrow\frac{(4R+r)^2}{p^2}\le\frac{2R-r}{r}\Leftrightarrow(4R+r)^2r\le(2R-r)p^2\Rightarrow$ *Gerretsen*

$$16R^2r+8Rr^2+r^3\le(2R-r)(16Rr-sr^2)$$

$$16R^2+8Rr+r^2\le(2R-r)(16R-sr);\ 16R^2-34Rr+4r^2\ge 0$$

$$8R^2-17Rr+2r^2\ge 0|:r^2;\frac{R}{r}=t\ge 2\ \textit{(Euler)}$$

$$8t^2-17t+2\ge 0;\underbrace{(t-2)}_{\ge 0}\underbrace{(8t-1)}_{>0}\ge 0$$

SP.103. Let m,n be positive real numbers. Prove that:

$$\left(\frac{1}{m}+\frac{1}{n}\right)^{-1}\le\frac{4034-2015m}{m+2017}+\frac{4034-2015n}{n+2017}+\frac{m+n+2009}{2}$$

Proposed by Iuliana Trașcă – Romania

Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

The inequality is equivalent to: $4-\frac{2017m}{m+2017}-\frac{2017n}{n+2017}+\frac{m+n+2009}{2}\ge\frac{1}{\frac{1}{m}+\frac{1}{n}}$

Applying AM-GM inequality: $4-\frac{2017m}{m+2017}-\frac{2017n}{n+2017}+\frac{m+n+2009}{2}\ge$

$$\ge 4-\frac{m+2017}{4}-\frac{n+2017}{4}+\frac{m+n+2009}{2}=\frac{m+n}{4}$$

So we need to prove that: $\frac{m+n}{4} \geq \frac{1}{\frac{1}{m}+\frac{1}{n}} \Leftrightarrow (m+n)^2 \geq 4mn \Leftrightarrow (m-n)^2 \geq 0$ ***(true)*** $\Rightarrow$

Q.E.D.

SP.104. Prove that in any triangle ABC the following relationship holds:

$$r\sum \frac{1}{\sin \frac{A}{2}} + \frac{abc}{2}\sum \frac{1}{\sqrt{abs(s-c)}} \leq 6R$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}(p-a)(p-b) = r(r+4R), abc = 4Rrp, \sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$$

$$\sin\frac{B}{2} = \sqrt{\frac{(p-a)(p-c)}{ca}} \text{ and } \sin\frac{C}{2} = \sqrt{\frac{(p-a)(p-b)}{ab}}$$

$$r\sum_{cyc}\frac{1}{\sin\frac{A}{2}} + \frac{abc}{2}\sum_{cyc}\frac{1}{\sqrt{abp(p-c)}}$$

$$\overset{Cauchy-Schwarz}{\leq} r\sqrt{\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{1}{(p-a)(p-b)}\right)} + \frac{abc}{2}\sqrt{\left(\sum_{cyc}\frac{1}{ab}\right)\left(\sum_{cyc}\frac{1}{p(p-a)}\right)}$$

$$\leq r\sqrt{9R^2\cdot\frac{\sum_{cyc}(p-a)}{\prod_{cyc}(p-a)}} + \frac{abc}{2}\sqrt{\frac{2p}{4Rrp}\cdot\frac{\sum_{cyc}(p-a)(p-b)}{p\prod_{cyc}(p-a)}}$$

$$= r\cdot\sqrt{9R^2\frac{p}{pr^2}} + 2Rrp\sqrt{\frac{1}{2Rr}\cdot\frac{r(r+4R)}{p^2r^2}} \leq 3R + 3R = 6R$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$r\sum\frac{1}{\sin\frac{A}{2}} = r\sum\sqrt{\frac{bc}{(s-b)(s-c)}} = \frac{r\sqrt{s}}{\sqrt{s(s-a)(s-b)(s-c)}}\sum\sqrt{bc(s-a)}$$

$$\overset{C-B-S}{\leq}\frac{r\sqrt{s}}{rs}\sqrt{\sum ab}\sqrt{\sum(s-a)} = \frac{1}{\sqrt{s}}\sqrt{s}\sqrt{\sum ab} = \sqrt{\sum ab} = \sqrt{s^2+4Rr+r^2}$$

$$\underset{(1)}{\overset{Gerretsen}{\leq}}\sqrt{4R^2+8Rr+4r^2} = \sqrt{4(R+r)^2} = 2(R+r)$$

$$\text{Now, } \frac{abc}{2}\sum\frac{1}{\sqrt{abs(s-c)}}$$

$$\overset{C-B-S}{\underset{(2)}{\leq}} \frac{4Rrs}{2\sqrt{s}}\sqrt{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}\sqrt{\frac{1}{s-c}+\frac{1}{s-a}+\frac{1}{s-b}} = \frac{4Rrs}{2\sqrt{s}}\sqrt{\frac{2s}{4Rrs}}\sqrt{\frac{s\cdot\sum(s-a)(s-b)}{r^2s^2}}$$

$$= \frac{\sqrt{4Rr}\sqrt{2}}{2r}\sqrt{\sum(s^2-s(s+b)+ab)} = \sqrt{\frac{2R}{r}}\sqrt{3s^2-s(4s)+s^2+4Rr+r^2} = \sqrt{2R(4R+r)}$$

$$\textit{(1) + (2)} \Rightarrow LHS \leq 2(R+r)+\sqrt{2R(4R+r)} \overset{?}{\leq} 6R \Leftrightarrow 2R(4R+r) \overset{?}{\leq} 4(2R-r)^2$$

$$\Leftrightarrow 4R^2+Rr \overset{?}{\leq} 8R^2-8Rr+2r^2 \Leftrightarrow 4R^2-9Rr+2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(4R-r) \overset{?}{\geq} 0 \rightarrow$$

$\rightarrow$ ***true*** $\because R \geq 2r$ ***(Euler) (Proved)***

SP.105. Let G be the centroid in $\Delta\, ABC$. Prove that:

$$\cot(\widehat{GBA})+\cot(\widehat{GCB})+\cot(\widehat{GAC}) > \cot A+\cot B+\cot C+3$$

Proposed by Daniel Sitaru – Romania

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

$$BC=a;\, CA=b;\, AB=c;\, S_{ABG}=S_{ACG}=S_{BCG}=\frac{S_{ABC}}{3}$$

$$\cot\widehat{GBA}+\cot\widehat{GCB}+\cot\widehat{GAC} =$$

$$= \frac{AB^2+BG^2-AG^2}{4S_{ABG}}+\frac{CG^2+BC^2-BG^2}{4S_{BGC}}+\frac{AG^2+AC^2-GA^2}{4S_{ACG}}$$

$$= \frac{3}{4}\left(\frac{a^2+b^2+c^2}{S_{ABC}}\right) = \frac{a^2+b^2+c^2}{4S_{ABC}}+\frac{a^2+b^2+c^2}{2S_{ABC}} \quad \textit{(1)}$$

- ***Other:*** $S=\sqrt{p(p-a)(p-b)(p-c)} \leq \frac{ab+bc+ca}{4\sqrt{3}} \leq \frac{a^2+b^2+c^2}{4\sqrt{3}}$

$$\Rightarrow \frac{a^2+b^2+c^2}{2S_{ABC}} \geq 2\sqrt{3} > 3 \quad \textit{(2)}$$

$$\textit{(1), (2)} \Rightarrow \cot\widehat{GBA}+\cot\widehat{GCB}+\cot\widehat{GAC} > \cot A+\cot B+\cot C+3$$

(Because $\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4S_{ABC}}$***)***

UP.091. Let be $a \in \mathbb{R}_+^*$ and the continuous functions $f, g, h: \mathbb{R} \to \mathbb{R}$ where f and g are odd and h is even. Prove that:

$$\int_{-a}^{a} f(x) \cdot ln\left(1 + e^{g(x)}\right) \cdot arctan\left(h(x)\right) dx = \int_{0}^{a} f(x)\, g(x)\, arctan\left(h(x)\right) dx$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Abdallah El Farisi-Bechar-Algerie

$$\int_{-a}^{a} f(x) \ln\left(1 + e^{g(x)}\right) \arctan\left(h(x)\right) dx = -\int_{-a}^{a} f(x) \ln\left(1 + e^{-g(x)}\right) \arctan\left(h(x)\right) dx$$

$$= -\int_{-a}^{a} f(x) \left(\ln\left(1 + e^{g(x)}\right) - g(x)\right) \arctan\left(h(x)\right) dx$$

$$= -\int_{-a}^{a} f(x) \left(\ln\left(1 + e^{g(x)}\right)\right) \arctan\left(h(x)\right) dx + \int_{-a}^{a} f(x)\, g(x) \arctan\left(h(x)\right) dx$$

$$= -\int_{-a}^{a} f(x) \left(\ln\left(1 + e^{g(x)}\right)\right) \arctan\left(h(x)\right) dx + 2\int_{0}^{a} f(x) g(x) \arctan\left(h(x)\right) dx$$

$$\int_{-a}^{a} f(x) \ln\left(1 + e^{g(x)}\right) \arctan\left(h(x)\right) dx = \int_{0}^{a} f(x) g(x) \arctan\left(h(x)\right)$$

Solution 2 by Shivam Sharma-New Delhi-India

Let,

$$I = \int_{-a}^{a} f(x) \ln\left(1 + e^{g(x)}\right) \arctan\left(h(x)\right) dx$$

As we know the following lemma:

If $f(x)$ is a continous function defined on $[-a, a]$, then,

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_{0}^{a} f(x)\, dx, if\ f(x)\ is\ an\ even\ function \\ 0, if\ f(x)\ is\ an\ odd\ function \end{cases}$$

Using the above lemma, we get,

$$\Rightarrow \int_{-a}^{a} f(-x)\ln\left(1+e^{g(-x)}\right)\arctan\left(h(-x)\right)dx$$

$$\Rightarrow -\int_{-a}^{a} f(x)\ln\left(1+e^{-g(x)}\right)\arctan\left(h(x)\right)dx$$

$$\Rightarrow -\int_{-a}^{a} f(x)\ln\left(1+e^{g(x)}\right)\arctan\left(h(x)\right)dx + \int_{-a}^{a} f(x)\ln\left(e^{g(x)}\right)\arctan\left(h(x)\right)dx$$

$$\Rightarrow -I + 2\int_{0}^{a} f(x)\,g(x)\arctan\left(h(x)\right)dx$$

(OR)

$2I = 2\int_0^a f(x)\,g(x)\arctan\left(h(x)\right)dx$ ***(OR)*** $I = \int_0^a f(x)g(x)\arctan\left(h(x)\right)dx$

(proved)

UP.092. Calculate:

$$\lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3(n+1)]{(n+1)!} - \sqrt[3n]{n!}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$$

$$\Omega_n = \lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3(n+1)]{(n+1)!} - \sqrt[3n]{n!}\right)$$

$$= \lim_{n\to\infty}\left(\sqrt[3]{\frac{\sqrt[3]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n\right) \textit{ where } u_n = \frac{\sqrt[3(n+1)]{(n+1)!}}{\sqrt[3n]{n!}} \textit{ for all } n \in \mathbb{N}$$

$$\therefore u_n = \frac{\sqrt[3(n+1)]{(n+1)!}}{\sqrt[3n]{n!}} = \frac{\sqrt[3(n+1)]{(n+1)!}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{\sqrt[3n]{n!}} \cdot \sqrt[3]{1+\frac{1}{n}} \textit{ then } \lim_{n\to\infty} u_n = 1$$

$$\textit{now, } u_n \to 1 \textit{ then } \frac{u_n - 1}{\ln u_n} \to 1 \textit{ for all } n \to \infty$$

$$u_n^n=\left(\frac{\sqrt[3(n+1)]{(n+1)!}}{\sqrt[3n]{n!}}\right)^n=\sqrt[3]{\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}}=\sqrt[3]{\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}}$$

$$\therefore \lim_{n\to\infty}u_n^n=\sqrt[3]{e}\ \text{ and } \Omega_n=\frac{1}{\sqrt[3]{e}}\cdot 1\cdot\ln\sqrt[3]{e}=\frac{1}{3\sqrt[3]{e}}$$

UP.093. Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be positive real sequences such that there exists $lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}$ and $lim_{n\to\infty}(b_n-u\cdot a_n)$. Find:

a. $lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)$

b. $lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

a. *Let* $\lim_{n\to\infty}(b_n-u\cdot a_n)=v$ *now let* $\lim_{n\to\infty}a_n=x>0$ *because* $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$ *then*

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a\Rightarrow\frac{x}{x}\cdot\frac{1}{\infty}=a\Rightarrow a=0,\ \textit{which is false. Then}\ \lim_{n\to\infty}a_n=\infty$$

$$\textit{now, }\lim_{n\to\infty}(b_n-u\cdot a_n)=v\Rightarrow\lim_{n\to\infty}\left(\frac{b_n}{a_n}-u\right)=v\lim_{n\to\infty}\frac{1}{a_n}=0\ \textit{then}$$

$$\Rightarrow\lim_{n\to\infty}\frac{b_n}{a_n}=u.\ \textit{Now, }\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}$$

$$\overset{Cauchy\ D'Alembert}{=}\lim_{n\to\infty}\left(\frac{b_{n+1}}{a_{n+1}}\cdot\frac{a_n}{b_n}\cdot\frac{a_{n+1}}{n\cdot a_n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{n}{n+1}\right)=\left(u\cdot\frac{1}{u}\cdot a\cdot\frac{1}{e}\right)=\frac{a}{e}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[n]{b_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)\ \textit{where}\ u_n=\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}$$

$$u_n=\left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{n+1}{n}\right)\Rightarrow\lim_{n\to\infty}u_n=1\ \textit{then}\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$$

$$\therefore u_n^n=\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^n=\left(\frac{b_{n+1}}{a_{n+1}}\cdot\frac{a_n}{b_n}\cdot\frac{a_{n+1}}{n\cdot a_n}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{n}{n+1}\right)$$

$$\therefore\lim_{n\to\infty}u_n^n=\left(u\cdot\frac{1}{u}\cdot a\cdot\frac{e}{a}\right)=e,\ \textit{then}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)=\left(\frac{a}{e}\cdot 1\cdot\ln e\right)=\frac{a}{e}$$

b. $\lim_{n\to\infty}\frac{n}{\sqrt[n]{b_n}}=\frac{e}{a}$ *then* $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)$

$=\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)$ *where* $u_n=\left(1+\frac{1}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}$ *for all* $n\in\mathbb{N}$

$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{n}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{n}{n+1}\right)=1$ *then* $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$

$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{2n}\cdot\frac{b_n}{a_n}\cdot\frac{a_{n+1}}{b_{n+1}}\cdot\frac{n\cdot a_n}{a_{n+1}}\left(1+\frac{1}{n}\right)\cdot\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\right)=\left(e^2\cdot u\cdot\frac{1}{u}\cdot\frac{1}{a}\cdot\frac{a}{e}\right)=$

e *then*

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)=\left(\frac{e}{a}\cdot 1\cdot\ln e\right)=\frac{e}{a}$$

UP.094. Let $(s_n)_{n\ge1}$, $s_n=\sum_{k=1}^{n}\frac{1}{k^2}$. Calculate:

$$\lim_{n\to\infty}\left(s_n\cdot\sqrt[n+1]{(n+1)!}-\frac{\pi^2}{6}\cdot\sqrt[n]{n!}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Shivam Sharma-New Delhi-India

$$\text{Let, } L=\lim_{n\to\infty}\left(s_n\cdot\sqrt[n+1]{(n+1)!}-\frac{\pi^2}{6}\sqrt[n]{n!}\right)$$

$$\Rightarrow\left(\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}\right)\left(\lim_{n\to\infty}\sqrt[n+1]{(n+1)!}\right)-\frac{\pi^2}{6}\left(\lim_{n\to\infty}\sqrt[n]{n!}\right)$$

$$\Rightarrow\left[\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)\cdot\lim_{n\to\infty}\sqrt[n+1]{(n+1)!}-\frac{\pi^2}{6}\cdot\lim_{n\to\infty}\sqrt[n]{n!}\right]$$

$$\Rightarrow\zeta(2)\cdot\lim_{n\to\infty}\sqrt[n+1]{(n+1)!}-\frac{\pi^2}{6}\cdot\lim_{n\to\infty}\sqrt[n]{n!}\Rightarrow\frac{\pi^2}{6}\cdot\lim_{n\to\infty}\sqrt[n+1]{(n+1)!}-\frac{\pi^2}{6}\cdot\lim_{n\to\infty}\sqrt[n]{n!}$$

$$\Rightarrow\frac{\pi^2}{6}\left[\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\right]$$

As we know, the Stirling's formula, we get, $n!=\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$. *Using this, we get,*

$$\Rightarrow\frac{\pi^2}{6}\left[\lim_{n\to\infty}\left(\frac{n+1}{e}\right)(2\pi(n+1))^{\frac{1}{n+1}}-\left(\frac{n}{e}\right)(2\pi n)^{\frac{1}{n}}\right]$$

Now, applying Cauchy D'Alembert, we get,

$$\Rightarrow \frac{\pi^2}{6}\left[\frac{1}{e}\left(\lim_{n\to\infty}\left(\left(\frac{n+2}{n+1}\right)-\frac{\left(2\pi(n+2)\right)^{\frac{1}{n+2}}}{\left(2\pi(n+1)\right)^{\frac{1}{n+1}}}-\left(\frac{n+1}{n}\right)\cdot\frac{\left(2\pi(n+1)^{\frac{1}{n+1}}\right)}{(2\pi n)^{\frac{1}{n}}}\right)\right)\right]$$

(or)

$$L=\frac{\pi^2}{6e}\ \textbf{(1)}$$

(or)

$$L=\frac{\pi^2}{6e}$$

(Answer)

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}\ \textit{then}\ \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)\ \textit{where}\ u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\ \textit{for all}\ n\in\mathbb{N}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}\right)=1\ \textit{then}\ \frac{u_n-1}{\ln u_n}\to 1\ \textit{as}\ n\to\infty$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)=e$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\left(\frac{1}{e}\cdot 1\cdot\ln e\right)=\frac{1}{e}$$

$$\lim_{n\to\infty}\left(s_n\sqrt[n+1]{(n+1)!}-\frac{\pi^2}{6}\sqrt[n]{n!}\right)$$

$$=\lim_{n\to\infty}\left(s_n-\frac{\pi^2}{6}\right)\sqrt[n+1]{(n+1)!}+\frac{\pi^2}{6}\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)$$

$$=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n}\cdot n\left(s_n-\frac{\pi^2}{6}\right)+\frac{\pi^2}{6e}$$

$$=\frac{1}{e}\lim_{n\to\infty}\left(s_n-\frac{\pi^2}{6}\right)+\frac{\pi^2}{6e}=\frac{1}{e}\lim_{n\to\infty}\frac{s_{n+1}-s_n}{\frac{1}{(n+1)}-\frac{1}{n}}+\frac{\pi^2}{6e}=\frac{\pi^2}{6e}\ \textit{(Ans:)}$$

UP.095. Let $(s_n)_{n\ge1}, s_n = \sum_{k=1}^{n}\frac{1}{k^2}$ and let $(a_n)_{n\ge1}$ be a positive real sequence such that

$$lim_{n\to\infty}\left(s_n\cdot\sqrt[n+1]{a_{n+1}}-\frac{\pi^2}{6}\cdot\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{Cauchy-D'Alembert}{\cong}\lim_{n\to\infty}\left(\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}\right)=\lim_{n\to\infty}\left(\frac{a_{n+1}}{n\cdot a_n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{n}{n+1}\right)=\frac{a}{e}$$

Let $u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}$ ***for all*** $n\in\mathbb{N}$ $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{a_n}}\cdot\frac{n+1}{n}\right)=1$

Hence, $\frac{u_n-1}{\ln u_n}\to 1$ ***as*** $n\to\infty$, $\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{a_{n+1}}{na_n}\cdot\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\right)=\left(a\cdot 1\cdot\frac{e}{a}\right)=e$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)=\left(\frac{a}{e}\cdot 1\cdot\ln e\right)=\frac{a}{e}$$

$$\lim_{n\to\infty}n\left(s_n-\frac{\pi^2}{6}\right)\overset{Caesaro-Stolz}{\cong}\lim_{n\to\infty}\frac{s_{n+1}-s_n}{\frac{1}{n+1}-\frac{1}{n}}=-1$$

$$\lim_{n\to\infty}\left(s_n\sqrt[n+1]{a_{n+1}}-\frac{\pi^2}{6}\sqrt[n]{a_n}\right)=\lim_{n\to\infty}\left(s_n-\frac{\pi^2}{6}\right)\sqrt[n+1]{a_{n+1}}+\frac{\pi^2}{6}\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

$$=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n+1}{n}\cdot n\left(s_n-\frac{\pi^2}{6}\right)+\frac{a\pi^2}{6e}=\frac{a(\pi^2-6)}{6e}$$

UP.096. Let $(s_n)_{n\ge1}, s_n = \sum_{k=1}^{n}\frac{1}{k^2}$. Calculate:

$$\lim_{n\to\infty}\left(s_n\cdot\sqrt[n+1]{(2n+1)!!}-\frac{\pi^2}{6}\cdot\sqrt[n]{(2n-1)!!}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Shivam Sharma-New Delhi-India

Let,

$$L=\lim_{n\to\infty}\left(s_n\sqrt[n+1]{(2n+1)!!}-\frac{\pi^2}{6}\sqrt[n]{(2n-1)!!}\right)$$

As we know, $(2n+1)!!=\frac{(2n+1)!}{2^n n!}$, $(2n-1)!=\frac{(2n)!}{2^n n!}$. ***Using this, we get,***

$$\Rightarrow \lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)\left(\frac{(2n+1)!}{2^n n!}\right)^{\frac{1}{n+1}}-\frac{\pi^2}{6}\left(\frac{(2n)!}{2^n n!}\right)^{\frac{1}{n}}\right]\Rightarrow \frac{\pi^2}{6}\left[\lim_{n\to\infty}\left\{\left(\frac{(2n+1)!}{2^n n!}\right)^{\frac{1}{n+1}}-\left(\frac{(2n)!}{2^n n!}\right)\right\}\right]$$

Now, applying Stirling's formula, we get,

$$\Rightarrow \frac{\pi^2}{6}\left[\lim_{n\to\infty}\left\{\left(\frac{\left(\frac{2n+1}{e}\right)^{2n+1}\sqrt{2\pi(2n+1)}}{2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}\right)^{\frac{1}{n+1}}-\left(\frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}}{2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}\right)^{\frac{1}{n}}\right\}\right]$$

Now, appling Cauchy D'Alembert, we get,

$$L=\frac{\pi^2}{3e}-\frac{2}{e},\ \text{or } L=\frac{\pi^2-6}{3e}$$

UP.097. If $x,y,z,a,b,c>0$ then:

$$\frac{(x+y)(y+z)(z+x)}{4xyz}\ge\left(\frac{x+z}{y+z}+\frac{y+z}{x+z}\right)^{\frac{a}{a+b+c}}\left(\frac{y+x}{z+x}+\frac{z+x}{y+x}\right)^{\frac{b}{a+b+c}}\cdot\left(\frac{z+y}{x+y}+\frac{x+y}{z+y}\right)^{\frac{c}{a+b+c}}\ge 2$$

(A refinement of Cesaro's inequality)

Proposed by Mihály Bencze Romania

Solution by proposer

$$\textit{We have: }\begin{cases}\frac{(y+z)(z+x)}{4yz}\ge\frac{x+z}{y+z}\Leftrightarrow (y-z)^2\ge 0\\ \frac{(y+z)(z+x)}{4xz}\ge\frac{y+z}{x+z}\Leftrightarrow (z-x)^2\ge 0\end{cases}$$

After addition we obtain: $\frac{(x+y)(y+z)(z+x)}{4xyz}\ge\frac{x+z}{y+z}+\frac{y+z}{x+z}\ge 2$ ***and***

$$\begin{cases}\left(\frac{(x+y)(y+z)(z+x)}{4xyz}\right)^a\ge\left(\frac{x+z}{y+z}+\frac{y+z}{x+z}\right)^a\ge 2^a\\ \left(\frac{(x+y)(y+z)(z+x)}{4xyz}\right)^b\ge\left(\frac{y+x}{z+x}+\frac{z+x}{y+x}\right)^b\ge 2^b\\ \left(\frac{(x+y)(y+z)(z+x)}{4xyz}\right)^c\ge\left(\frac{z+y}{x+y}+\frac{x+y}{z+y}\right)^c\ge 2^c\end{cases}$$

After multiplication we obtain the desired inequalities.

UP.098. Let $a, b \in \mathbb{R}, a < b$ and $f, g: \mathbb{R} \to \mathbb{R}$ continuos functions such that

$$f(x)f(a+b-x) = 1, g(x) = g(a+b-x), x \in \mathbb{R}.$$

Show that

$$\int_a^b \frac{g(x)}{1+f(x)}dx = \frac{1}{2} \cdot \int_a^b g(x)\,dx$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Let $x = a+b-z \Rightarrow dx = -dz$; when $x = a, z = b; x = b, z = a$

$$\text{Let } I = \int_a^b \frac{g(x)}{1+f(x)}dx = \int_a^b \frac{g(a+b-z)(-dz)}{1+f(a+b-z)} = \int_a^b \frac{g(z)dz}{1+\frac{1}{f(z)}} = \int_a^b \frac{f(z)g(z)}{1+f(z)}dz$$

$$= \int_a^b g(z)dz - \int_a^b \frac{g(z)}{1+f(z)}dz \Rightarrow 2I = \int_a^b g(z)dz \Rightarrow I = \frac{1}{2}\int_a^b g(x)\,dx$$

Hence proved

Solution 2 by Shivam Sharma-New Delhi-India

As we know, the following lemma,

If $f(x)$ is a continuos function defined on $[a, b]$; then,

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$$

Using this, we get,

$$I = \int_a^b \frac{g(a+b-x)}{1+f(a+b-x)}dx$$

Given: $f(x)f(a+b-x) = 1;\ g(x) = g(a+b-x)$

Using this, and putting these values, we get,

$$\Rightarrow \int_a^b \frac{f(x)g(x)}{1+f(x)}dx$$

$$2I = \int_a^b \left(\frac{f(x)+1}{f(x)+1}\right) g(x)dx \text{ or } I = \frac{1}{2}\int_a^b g(x)\,dx \text{ (Proved)}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$I=\int_a^b \frac{g(x)}{1+f(x)}dx=\int_a^b \frac{g(a+b-x)}{1+f(a+b-x)}dx=\int_a^b \frac{g(x)}{1+\frac{1}{f(x)}}dx$$

$$=\int_a^b \frac{g(x)f(x)}{1+f(x)}dx \quad \therefore 2I=\int_a^b \frac{g(x)\big(1+f(x)\big)}{1+f(x)}dx=\int_a^b g(x)dx$$

$$\Rightarrow I=\frac{1}{2}\int_a^b g(x)\,dx$$

UP.099. In an arbitrary triangle ABC denote by l_a, m_a, h_a respectively the lengths of the internal angle-bisector, the median and the altitude corresponding to the side $a=BC$ of the triangle. Prove that:

a) $\frac{l_a^2}{h_a^2}+\frac{l_b^2}{h_b^2}+\frac{l_c^2}{h_c^2}\ge 2\frac{l_a}{h_a}\cdot\frac{l_b}{h_b}\cdot\frac{l_c}{h_c}+1$

b) $\frac{m_a^2}{h_a^2}+\frac{m_b^2}{h_b^2}+\frac{m_c^2}{h_c^2}\le 2\frac{m_a}{h_a}\cdot\frac{m_b}{h_b}\cdot\frac{m_c}{h_c}+1$

c) explain why each of a) and b) are equivalent to the fundamental inequality of the triangle.

Proposed by Vasile Jiglău – Romania

Solution by Soumava Chakraborty-Kolkata-India

Proof of (a) $l_a^2=\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=\frac{bc(b+c+a)(b+c-a)}{(b+c)^2}=\frac{bc\{(b+c)^2-a^2\}}{(b+c)^2}=bc-\frac{a^2bc}{(b+c)^2}$

$$\therefore \frac{l_a^2}{h_a^2}=bc\cdot\frac{4R^2}{b^2c^2}-\frac{a^4bc}{4\Delta^2(b+c)^2}=4R^2\cdot\frac{1}{bc}-\frac{4Rrs}{4r^2S^2}\cdot\frac{a^3}{(b+c)^2}=$$

$$\overset{(1)}{=}4R^2\left(\frac{1}{bc}\right)-\frac{R}{rs}\cdot\frac{a^3}{(b+c)^2}$$

Similarly, $\frac{l_b^2}{h_b^2}\overset{(2)}{=}4R^2\left(\frac{1}{ca}\right)-\frac{R}{rs}\cdot\frac{b^3}{(c+a)^2}$ & $\frac{l_c^2}{h_c^2}\overset{(3)}{=}4R^2\left(\frac{1}{ab}\right)-\frac{R}{rs}\cdot\frac{c^3}{(a+b)^2}$

(1)+(2)+(3) $\Rightarrow \sum\frac{l_a^2}{h_a^2}=\frac{4R^2}{4Rrs}(2S)+\frac{R}{rs}\sum\frac{(2s-a-2s)^3}{(2s-a)^2}=$

$$=\frac{2R}{r}+\frac{R}{rs}\sum\frac{(2s-a)^3-8s^3-3(2s-a)^2\cdot 2S+3(2s-a)4S^2}{(2s-a)^2}=$$

$$=\frac{2R}{r}+\frac{R}{rs}\sum(2s-a)-\frac{3R}{rs}(2S)(3)+\frac{12RS^2}{rs}\sum\frac{1}{b+c}-\frac{8Rs^3}{rs}\sum\frac{1}{(b+c)^2}=$$

$$\overset{(4)}{=}\frac{2R}{r}+\frac{4RS}{rs}-\frac{18R}{r}+\frac{12RS}{r}\sum\frac{1}{b+c}-\frac{8RS^2}{r}\sum\frac{1}{(b+c)^2}$$

Now, $(a+b)(b+c)(c+a)=2abc+\sum ab\,(2S-c)=$

$$=2s(s^2+4Rr+r^2)-4Rrs\overset{(5)}{=}2s(s^2+2Rr+r^2)$$

(5) $\Rightarrow\frac{2RS}{r}\sum\frac{1}{b+c}=\frac{12RS}{r}\cdot\frac{\sum(c+a)(a+b)}{2s(s^2+2Rr+r^2)}=\frac{12RS[(\sum a^2+2\sum ab)+\sum ab]}{2s(s^2+2Rr+r^2)r}\overset{(i)}{=}\frac{16R(5s^2+4Rr+r^2)}{r(s^2+2Rr+r^2)}$

Now, $\sum(c+a)^2(a+b)^2=\sum(a^2+\sum ab)^2=\sum\{a^4+(\sum ab)^2+2(\sum ab)a^2\}=$

$$=\sum a^4+3\left(\sum ab\right)^2+2\left(\sum ab\right)\left(\sum a^2\right)=\left(\sum a^2\right)^2-2\left\{\left(\sum ab\right)^2-2abc(2s)\right\}+$$

$$+3\left(\sum ab\right)^2+2\left(\sum ab\right)\left(\sum a^2\right)=\left(\sum a^2\right)^2+\left(\sum ab\right)^2+2\left(\sum ab\right)\left(\sum a^2\right)+$$

$$+32Rrs^2=\left(\sum a^2+\sum ab\right)^2+32Rrs^2=(3s^2-4Rr-r^2)^2+32Rrs^2=$$

$$=9s^4-6s^2(4Rr+r^2)+32Rrs^2+r^2(4R+r)^2\overset{(6)}{=}9s^4+r^2(4R+r)^2+s^2(8Rr-6r^2)$$

(5), (6) $\Rightarrow\frac{-8RS^2}{r}\sum\frac{1}{(b+c)^2}=\frac{[9s^4+r^2(4R+r)^2+s^2(8Rr-6r^2)]}{r\cdot 4s^2(s^2+2Rr+r^2)^2}\overset{(ii)}{=}\frac{-2R[9S^4+r^2(4R+r)^2+s^2(8Rr-6r^2)]}{r(s^2+2Rr+r^2)^2}$

(i),(ii), (4) $\Rightarrow\sum\frac{l_a^2}{h_a^2}=\frac{-12R}{r}+\frac{6R(5s^2+4Rr+r^2)}{r(s^2+2Rr+r^2)}-\frac{2R[9s^4+r^2(4R+r)^2+s^2(8Rr-6r^2)]}{r(s^2+2Rr+r^2)^2}$

$$=\frac{-12R(s^2+2Rr+r^2)^2+6R(5s^2+4Rr+r^2)(s^2+2Rr+r^2)}{r(s^2+2Rr+r^2)^2}-$$

$$-\frac{2R[9s^4+r^2(4R+r)^2+s^2(8Rr-6r^2)]}{r(s^2+2Rr+r^2)^2}\overset{(7)}{=}\frac{RS^2(20Rr+24r^2)-Rr^2(32R^2+28Rr+8r^2)}{r(s^2+2Rr+r^2)^2}$$

Now, $\frac{2l_al_bl_c}{h_ah_bh_c}+1\overset{by\,(5)}{=}\frac{2\cdot 8R^3}{16R^2r^2s^2}\cdot\frac{8\cdot 16R^2r^2s^2\left(\frac{s}{4R}\right)}{2s(s^2+2Rr+r^2)}+1=\frac{16R^2}{s^2+2Rr+r^2}+1\overset{(8)}{=}\frac{16R^2+s^2+2Rr+r^2}{s^2+2Rr+r^2}$

$$\because RS^2(20Rr+24r^2)-Rr^2(32R^2+28Rr+8r^2)\overset{Gerretsen}{\geq}$$

$$\geq Rr^2[(20R+24r)(16R-5r)-(32R^2+28Rr+8r^2)]=$$

$$=Rr^2(288R^2+256Rr-128r^2)=Rr^2\{288R^2+192Rr+64r(R-2r)\}>0,$$

∴ ***(7), (8)** ⇒ given inequality is equivalent to:* $R(20R+24r)s^2-Rr(32R^2+28Rr+8r^2)$

$$\geq (s^2+2Rr+r^2)(s^2+16R^2+2Rr+r^2) \Leftrightarrow s^2(4R^2+20Rr-2r^2) \overset{(9)}{\geq}$$

$$\geq s^4+64R^3r+48R^2r^2+12Rr^3+r^4$$

Now, the fundamental triangle inequality (Rouche) $\Rightarrow s^2 \geq m-n \Rightarrow s^2-m+n \overset{(a)}{\geq} 0$ **&**

$s^2 \leq m+n \Rightarrow s^2-m-n \overset{(b)}{\leq} 0$, ***where*** $m = 2R^2+10Rr-r^2$ **&**

$$n = 2(R-2r)\sqrt{R^2-2Rr}$$

(a).(b) $\Rightarrow s^4-s^2(2m)+m^2-n^2 \leq 0 \Rightarrow s^4-2s^2(2R^2+10Rr-r^2)+$

$+(2R^2+10Rr-r^2)^2-4(R-2r)^2(R^2-2Rr) \leq 0 \Rightarrow s^4+64R^3r+48R^2r^2+12Rr^3+$

$+r^4 \overset{(c)}{\leq} s^2(4R^2+20Rr-2r^2) \Rightarrow$ ***(9) is true (proved)***

∵ (c) is analogous with the fundamental triangle inequality & ∵ given inequality is equivalent to (c), hence, given inequality is equivalent to the fundamental triangle inequality

Proof of (b) $m_a^2m_b^2m_c^2 = \frac{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)}{64} \overset{(1)}{=}$

$= \frac{1}{64}\{-4\sum a^6+6(\sum s^4b^2+\sum a^2b^2)+3a^2b^2c^2\}$. ***Now,***

$$\sum a^6 = \left(\sum a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2) =$$

$$= \left(\sum a^2\right)^3-3\left(\sum a^2-c^2\right)\left(\sum a^2-a^2\right)\left(\sum a^2-b^2\right) =$$

$$= \left(\sum a^2\right)^3-3\left\{\left(\sum a^2\right)^3-\left(\sum a^2\right)^3+\left(\sum a^2\right)\left(\sum a^2b^2\right)-a^2b^2c^2\right\}$$

$\overset{(2)}{=} (\sum a^2)^3-3(\sum a^2)(\sum a^2b^2)+3a^2b^2c^2$. ***Also,*** $\sum a^4b^2+\sum a^2b^4 = \sum a^2b^2(\sum a^2-c^2) =$

$$\overset{(3)}{=} \left(\sum a^2\right)\left(\sum a^2b^2\right)-3a^2b^2c^2$$

(1), (2), (3) $\Rightarrow m_a^2m_b^2m_c^2 = \frac{1}{64}\left\{\begin{matrix}-4(\sum a^2)^3+12(\sum a^2)(\sum a^2b^2)-12a^2b^2c^2+ \\ +6(\sum a^2)(\sum a^2b^2)-18a^2b^2c^2+3a^2b^2c^2\end{matrix}\right\}$

$$= \frac{1}{64}\left\{-4\left(\sum a^2\right)^3+18\left(\sum a^2\right)\left(\sum a^2b^2\right)-27a^2b^2c^2\right\} =$$

$$=\frac{1}{64}\begin{bmatrix}-32(s^2-4Rr-r^2)^3+18\cdot 2(s^2-4Rr-r^2)\cdot\\ \{(s^2+4Rr+r^2\}-2abc(2s)-432R^2r^2s^2\end{bmatrix}=$$

$$\overset{(4)}{=}\frac{1}{16}\begin{Bmatrix}s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-\\ -64R^3r^3-48R^2r^4-12Rr^5-r^6\end{Bmatrix}$$

Now, $4\sum a^2b^2-\sum a^4=6\sum a^2b^2-(\sum a^2)^2=6\{(\sum ab)^2-2abc(2s)\}-(\sum a^2)^2$

$$=4\{(s^2+4Rr+r^2)^2-(s^2-4Rr-r^2)^2\}+2(s^2+4Rr+r^2)^2-96Rrs^2=$$

$$=4(2s^2)(8Rr+2r^2)+2\left(s^4+r^2(4R+r)^2+2s^2(4Rr+r^2)\right)-96Rrs^2$$

$$\overset{(5)}{=}2s^4-s^2(16Rr-20r^2)+2r^2(4R+r)^2$$

Now, $\sum\frac{m_a^2}{h_a^2}-1=\sum\frac{2b^2+2c^2-a^2}{4}\cdot\frac{a^2}{4\Delta^2}-1=\frac{4\sum a^2b^2-\sum a^4}{16\Delta^2}-1=$

$$=\frac{s^4-s^2(8Rr-10r^2)+r^2(4R+r)^2-8r^2s^2}{8\Delta^2}\ \textit{(by (5))}$$

$$=\frac{s^4+r^2(4R+r)^2-s^2(8Rr-2r^2)}{8\Delta^2}$$

$$\therefore\left(\sum\frac{m_a^2}{h_a^2}-1\right)^2$$

$$\overset{(6)}{=}\frac{1}{64\Delta^2}\begin{bmatrix}s^8-s^6(16Rr-4r^2)+s^4(96R^2r^2+16Rr^3+6r^4)-\\ -s^2\left(256R^3r^3+64R^2r^4-16Rr^5-4r^6\right)+\\ +256R^4r^4+256R^3r^5+96R^2r^6+16Rr^7+r^8\end{bmatrix}$$

Also, $\left(\frac{2m_am_bm_c}{h_ah_bh_c}\right)^2=\left(\frac{28R^3}{16R^2r^2s^2}\right)^2\cdot m_a^2m_b^2m_c^2$

$$\overset{(7)}{=}\frac{R^2}{16\Delta^4}\begin{Bmatrix}s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-\\ -64R^3r^3-48R^2r^4-12Rr^5-r^6\end{Bmatrix}\ \textit{(by (4))}$$

(6), (7) $\Rightarrow$ ***given inequality is equivalent to:***

$$s^8-s^6(16Rr-4r^2)+s^4(96R^2r^2+16Rr^3+6r^4)-$$

$$-s^2\left(256R^3r^3+64R^2r^4-16Rr^5-4r^6\right)+256R^4r^4+256R^3r^5+96R^2r^6+$$

$$+16Rr^7+r^8\le 4R^2\begin{Bmatrix}s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-\\ -64R^3r^3-48R^2r^4-12Rr^5-r^6\end{Bmatrix}$$

$$\Leftrightarrow s^8-s^6(4R^2+16Rr-4r^2)+s^4(48R^3r-36R^2r^2-16Rr^3+6r^4)+$$

$$+s^2\left(240R^4r^2+224R^3r^3+68R^2r^4+16Rr^5+4r^6\right)+256R^5r^3+448R^4r^4+$$

$$+304R^3r^5+100R^2r^6+16Rr^7+r^8\le 0\Leftrightarrow$$

$$\Leftrightarrow \{s^4 - (4R^2 + 20Rr - 2r^2)s^2 + 64R^3r + 48R^2r^2 + 12Rr^3 + r^4\}$$
$$\{s^4 + s^2(4Rr + 2r^2) + 4R^2r^2 + 4Rr^3 + r^4\} \leq 0 \Leftrightarrow$$
$$\Leftrightarrow s^4 + 64R^3r + 48R^2r^2 + 12Rr^3 + r^4 \leq s^2(4R^2 + 20Rr - 2r^2)$$

But, the above is inequality (c) proved in the proof of (a) earlier.

⇒ given inequality is true (Proved)

∵ given inequlality reduces to inequality (c) & (c) is analogous to the fundamental inequality of the triangle, hence, this given inequality is equivalent to the fundamental ineqaulity of the triangle (Done).

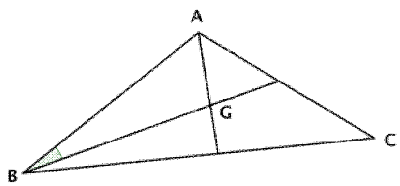
UP.100. In ΔABC; m_a, m_b, m_c – median's length. Prove that:

$$3(a^2 + b^2 + c^2) < 4(am_c + bm_a + cm_b)$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let G be the centroid of ΔABC.



$$AG = \frac{2}{3}m_a; BG = \frac{2}{3}m_b$$

$$1 > \cos(\widehat{GBA}) = \frac{GB^2 + AB^2 - GA^2}{2GB \cdot AB} = \frac{\left(\frac{2}{3}m_b\right)^2 + c^2 - \left(\frac{2}{3}m_a\right)^2}{2 \cdot \frac{2}{3}m_b \cdot c} =$$

$$= \frac{9c^2 + 4m_b^2 - 4m_a^2}{12cm_b} = \frac{9c^2 + 2a^2 + 2c^2 - b^2 - 2b^2 - 2c^2 + a^2}{12cm_b} =$$

$$= \frac{9c^2 + 3a^2 - 3b^2}{12cm_b} = \frac{3c^2 + a^2 - b^2}{4cm_b}$$

$$3c^2 + a^2 - b^2 < 4cm_b \quad \textit{(1)}$$

Analogous:

$$3a^2+b^2-c^2<4am_c$$

$$3a^2+b^2-c^2<4am_c \quad \textbf{(2)}$$

$$3b^2+c^2-a^2<4bm_a \quad \textbf{(3)}$$

By adding (1); (2); (3): $3(a^2+b^2+c^2)<4(am_c+bm_a+cm_b)$

UP.101. Prove that if $a,b,c\in(1,\infty)$ then:

$$3\sqrt{2}+\int_1^a x\sin\frac{\pi}{3x}dx+\int_1^b x\sin\frac{\pi}{3x}dx+\int_1^c x\sin\frac{\pi}{3x}dx>\sqrt{3+a^2+b^2+c^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Rovsen Pirguliyev-Sumgait-Azerbaidian

Lemma: if $x>q$***, then prove:*** $\sin\frac{\pi}{x}>\frac{3}{\sqrt{x^2+9}}$

Proof: $x>2\Rightarrow\frac{\pi}{x}<\frac{\pi}{2}\Rightarrow\tan\frac{\pi}{x}>\frac{\pi}{x}$***, we have*** $\frac{\pi}{x}>\frac{3}{x}\Rightarrow\tan\frac{\pi}{x}>\frac{3}{x}$ ***(*)***

$$\cos x=\sqrt{\frac{1}{1+\tan^2 x}}<\sqrt{\frac{1}{1+\frac{\pi^2}{x^2}}}\overset{(x)}{<}\frac{x}{\sqrt{x^2+9}}\Rightarrow\sin\frac{\pi}{x}>\frac{3}{\sqrt{x^2+9}}$$

it is known that: if $x>q$***, then*** $\sqrt{x^2+9}\sin\frac{\pi}{x}>3\Rightarrow x\to 3x$***, we have:*** $\sin\frac{\pi}{3x}>\frac{1}{\sqrt{x^2+1}}$

$$x\sin\frac{\pi}{3x}>x\cdot\frac{1}{\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}} \quad \textbf{(*)}$$

$$3\sqrt{2}+\int_1^a x\sin\frac{\pi}{3x}dx+\int_1^b x\sin\frac{\pi}{3x}dx+\int_1^c x\sin\frac{\pi}{3x}\overset{(*)}{>}$$

$$>3\sqrt{2}+\int_1^a\frac{x}{\sqrt{x^2+1}}dx+\int_1^b\frac{x}{\sqrt{x^2+1}}dx+\int_1^c\frac{x}{\sqrt{x^2+1}}dx=$$

$$=3\sqrt{2}+\sqrt{x^2+1}\,\Big|_1^a+\sqrt{x^2+1}\,\Big|_1^b+\sqrt{x^2+1}\,\Big|_1^c=$$

$$=3\sqrt{2}+\sqrt{a^2+1}-\sqrt{2}+\sqrt{b^2+1}-\sqrt{2}+\sqrt{c^2+1}-\sqrt{2}=$$

$$=\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}>\sqrt{3+a^2+b^2+c^2}$$

UP. 102. Solve for real numbers:

$$n^{n(x_1^2-x_2)}+n^{n(x_2^2-x_3)}+\dots+n^{n(x_{n-1}^2-x_n)}+n^{n(x_n^2-x_1)}=\frac{n}{\sqrt[4]{n^n}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

By AM-GM

$$n^{n(x_1^2-x_2)}+n^{n(x_2^2-x_3)}+\dots+n^{n(x_n^2-x_1)}\geq n\sqrt[n]{(n^n)^{x_1^2-x_2+x_2^2-x_3+\dots+x_n^2-x_1}}$$

$$=n\sqrt[n]{(n^n)^{(x_1^2-x_1)+(x_2^2-x_2)+\dots+(x_n^2-x_n)}}=n\sqrt[n]{(n^n)^{\left(x_1-\frac{1}{2}\right)^2+\left(x_1-\frac{1}{2}\right)^2+\dots+\left(x_n-\frac{1}{2}\right)^2-\left(\frac{1}{4}+\dots+\frac{1}{4}\right)}}$$

$$\geq n\sqrt[n]{(n^n)^{-\frac{1}{4}n}}=n\sqrt[n]{(n^n)^{-\frac{n}{4}}}=\frac{n}{\sqrt[4]{n^n}}\Rightarrow n^{n(x_1^2-x_2)}+\dots+n^{n(x_n^2-x_1)}\geq\frac{n}{\sqrt[4]{n^n}}$$

$$\Rightarrow x_1=x_2=\dots=x_n=\frac{1}{2}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$n^{n(x_1^2-x_2)}+n^{n(x_2^2-x_3)}+\dots+n^{n(x_{n-1}^2-x_n)}+n^{n(x_n^2-x_1)}=$$

$$=\frac{n}{\sqrt[4]{n^n}}\Leftrightarrow x\in\mathbb{R}: n^{n(x_i^2-x_i)}>0\ \textbf{(*)}$$

$$\textbf{(*)}\Rightarrow n^{n(x_1^2-x_2)}+\dots+n^{n(x_n^2-x_1)}\overset{AM\geq GM}{\geq}n\cdot\sqrt[n]{\left(n^{(x_1^2+x_2^2+\dots+x_n^2)-(x_1+x_2+\dots+x_n)}\right)^n}=$$

$$=n\cdot n^{\left(x_1^2-x_1+\frac{1}{4}\right)+\dots+\left(x_n^2-x_n+\frac{1}{4}\right)-\frac{n}{4}}=\frac{n}{\sqrt[4]{n^n}}\cdot n^{\left(x_1-\frac{1}{2}\right)^2+\dots+\left(x_n-\frac{1}{2}\right)^2}=\frac{n}{\sqrt[4]{n^n}}\Rightarrow$$

$$\Rightarrow n^{\sum\left(x_i-\frac{1}{2}\right)^2}=n^{\sum_{i=1}^n\left(x_i-\frac{1}{2}\right)^2}=1=n^\circ$$

$$\sum\left(x_i-\frac{1}{2}\right)^2=0\Rightarrow x_1=x_2=\dots=x_n=\frac{1}{2}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\frac{n}{n^{\frac{n}{4}}}=n^{n(x_1^2-x_2)}+n^{n(x_2^2-x_3)}+\dots+n^{n(x_n^2-x_1)}\geq n\left[n^{n(x_1^2-x_2+x_2^2-x_3+\dots+x_n^2-x_1)}\right]^{\frac{1}{n}}\Rightarrow\frac{1}{n^{\frac{n}{4}}}\geq n^s$$

where $s=(x_1^2-x_1)+(x_2^2-x_2)+\dots+(x_n^2-x_n)$

$$= \left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^2 + \cdots + \left(x_n - \frac{1}{2}\right)^2 - \frac{n}{4} = T - \frac{n}{4} \geq -\frac{n}{4}$$

$\therefore \frac{1}{n^{\frac{n}{4}}} \geq n^{T-\frac{n}{4}} \geq n^{-\frac{n}{4}}$. ***Equality holds when*** $T = 0. \Leftrightarrow x_1 = x_2 = \cdots = x_n = \frac{1}{2}$

UP.103. Prove that in any triangle ABC the following relationship holds:

$$|\cos A| + |\cos B| + |\cos C| \leq \sum\left(\sqrt{|\cos A \cos B|} + \sqrt{\left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum|\cos A| \overset{(1)}{\leq} \sum\left(\sqrt{|\cos A \cos B|} + \sqrt{\left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|}\right)$$

$$\textbf{(1)} \Leftrightarrow \sum\frac{|\cos A| + |\cos B|}{2} \leq \sum\sqrt{|\cos A \cos B|} + \frac{1}{\sqrt{2}}\sum\sqrt{|\cos A - \cos B|}$$

Let $\cos A = x, \cos B = y;\ -1 < x, y < 1$. ***We shall prove that*** $\forall x, y \in (-1, 1)$,

$$\frac{|x| + |y|}{2} \overset{(a)}{\leq} \sqrt{|xy|} + \frac{1}{\sqrt{2}}\sqrt{|x-y|} \Leftrightarrow |x| + |y| - 2\sqrt{|xy|} \overset{(b)}{\leq} \sqrt{2|x-y|}$$

$$\because |x| + |y| - 2\sqrt{|xy|} = \left(\sqrt{|x|} - \sqrt{|y|}\right)^2 \geq 0,$$

$\therefore$ ***(b)*** $\Leftrightarrow x^2 + y^2 + 4|xy| + 2|xy| - 4|x|\sqrt{|xy|} - 4|y|\sqrt{|xy|} \leq 2|x-y|$ ***(upon squaring)*** $\Leftrightarrow 4\sqrt{|xy|}(|x| + |y|) + 2|x-y| \overset{(c)}{\geq} x^2 + y^2 + 6|xy|$

$A - G \Rightarrow$ ***LHS of (c)*** $\geq 4\sqrt{|xy|} \cdot 2\sqrt{|xy|} + 2|x-y| = 8|xy| + 2|x-y| \overset{(?)}{\geq} x^2 + y^2 + 6|xy|$

$$\Leftrightarrow 2|x-y| \underset{(d)}{\overset{?}{\geq}} (|x| - |y|)^2$$

Now, $(|x| - |y|)^2 \leq (|x-y|)^2 \Leftrightarrow x^2 + y^2 - 2|xy| \leq x^2 + y^2 - 2xy \Leftrightarrow$

$\Leftrightarrow |xy| \geq xy \rightarrow$ ***true*** $\therefore (|x| - |y|)^2 \leq (|x-y|)^2 \overset{?}{\leq} 2|x-y| \Leftrightarrow$

$\Leftrightarrow (|x-y|)(|x-y| - 2) \underset{(e)}{\overset{?}{\geq}} 0 \because -1 < \cos A < 1$ **&** $-1 < -\cos B < 1$

$\therefore -2 < \cos A - \cos B < 2$ ***(adding the above two)*** $\Rightarrow -2 < x - y < 2 \Rightarrow |x-y| < 2 \Rightarrow$

$$\Rightarrow |x-y|-2\overset{(i)}{\geq}0.\ \textit{Also}\ |x-y|\overset{(ii)}{\geq}0$$

(i)·(ii)$\Rightarrow (|x-y|)(|x-y|-2)\leq 0\Rightarrow$ ***(e) is true*** $\Rightarrow$ ***(d) is true*** $\Rightarrow$ ***(c) is true*** $\Rightarrow$

(b) is true $\Rightarrow$

$\Rightarrow$ ***(a) is true*** $\therefore \frac{|\cos A|+|\cos B|}{2}\overset{(u)}{\leq}\sqrt{|\cos A\cos B|}+\frac{1}{\sqrt{2}}\sqrt{|\cos A-\cos B|}$

Similarly, $\frac{|\cos B|+|\cos C|}{2}\overset{(v)}{\leq}\sqrt{|\cos B\cos C|}+\frac{1}{\sqrt{2}}\sqrt{|\cos B-\cos C|}$ ***&***

$$\frac{|\cos C|+|\cos A|}{2}\overset{(w)}{\leq}\sqrt{|\cos C\cos A|}+\frac{1}{\sqrt{2}}\sqrt{|\cos C-\cos A|}$$

(u)+(v)+(w)⇒ (1) is true (Proved)

Solution 2 by proposer

$$\left|\sqrt{\cos A}\right|=\left|\sqrt{|(\cos A-\cos B)+\cos B|}\right|\leq$$

$$\leq\sqrt{|\cos A-\cos B|+|\cos B|}\leq\sqrt{|\cos A-\cos B|}+\sqrt{|\cos B|}$$

because if $x,y\geq 0$ ***then*** $\sqrt{x+y}\leq\sqrt{x}+\sqrt{y}$

$$\sqrt{|\cos A|}-\sqrt{|\cos B|}\leq\sqrt{|\cos A-\cos B|}$$

$$\left|\sqrt{\cos B}\right|=\left|\sqrt{|(\cos B-\cos A)+\cos A|}\right|\leq$$

$$\leq\sqrt{|\cos B-\cos A|+|\cos A|}\leq\sqrt{|\cos A-\cos B|}+\sqrt{|\cos A|}$$

$$-\left(\sqrt{|\cos A|}-\sqrt{|\cos B|}\right)\leq\sqrt{|\cos A-\cos B|}\qquad \textit{(2)}$$

By (1); (2): $\sqrt{|\cos A-\cos B|}\geq\left|\sqrt{|\cos A|}-\sqrt{|\cos B|}\right|$

By squaring: $|\cos A-\cos B|\geq|\cos A|+|\cos B|-2\sqrt{|\cos A\cos B|}$

$$\left|2\sin\frac{B-A}{2}\cos\frac{C}{2}\right|\geq|\cos A|+|\cos B|-2\sqrt{|\cos A\cos B|}$$

$$2\sqrt{|\cos A\cos B|}+2\left|\cos\frac{A}{2}\sin\frac{B-A}{2}\right|\geq|\cos A|+|\cos B|$$

$$2\sum\left(\sqrt{|\cos A\cos B|}+\left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|\right)\geq\sum(|\cos A|+|\cos B|)$$

$$2\sum\left(\sqrt{|\cos A\cos B|}+\left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|\right)\geq 2\sum|\cos A|$$

$$|\cos A| + |\cos B| + |\cos C| \le \sum\left(\sqrt{|\cos A\cos B|} + \left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|\right)$$

UP.104. Prove that *if* $x_i \in (0,\infty); i \in \overline{1,n}; n \in \mathbb{N}; n \ge 3$;

$$x_{n+1} = x_1; x_1x_2\cdot\ldots\cdot x_n = 1, \text{ then}$$

$$\sum_{i=1}^{n}\frac{\frac{x_i}{x_{i+1}} + \frac{x_{i+1}}{x_i} + 1}{\sqrt{x_i^2 + x_ix_{i+1} + x_{i+1}^2}} \ge n\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

We have, $x_i^2 + x_ix_{i+1} + x_{i+1}^2 \ge \frac{3}{4}(x_i + x_{i+1})^2$

$$\sum_{i=1}^{n}\frac{\frac{x_i}{x_{i+1}} + \frac{x_{i+1}}{x_i} + 1}{\sqrt{x_i^2 + x_ix_{i+1} + x_{i+1}^2}} = \sum_{i=1}^{n}\frac{\sqrt{x_i^2 + x_ix_{i+1} + x_{i+1}^2}}{x_ix_{i+1}} \ge \frac{\sqrt{3}}{2}\sum_{i=1}^{n}\frac{x_i + x_{i+1}}{x_ix_{i+1}}$$

$$\overset{AM\ge GM}{\ge} \sqrt{3}\sum_{i=1}^{n}\frac{1}{\sqrt{x_ix_{i+1}}} \overset{AM\ge GM}{\ge} \frac{n\sqrt{3}}{\sqrt[n]{\prod_{i=1}^{n}x_i}} = n\sqrt{3}$$

(proved)

UP.105. In ABC; a,b,c - length sides; s - semiperimeter; A,B,C - angled's measures. Prove that:

$$\left(\frac{A^3}{b} + \frac{B^3}{c} + \frac{C^3}{a}\right)\left(\frac{A^3}{c} + \frac{B^3}{a} + \frac{C^3}{b}\right)\left(\frac{A^3}{a} + \frac{B^3}{b} + \frac{C^3}{c}\right) \ge \frac{\pi^9}{216s^3}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS \overset{Holder}{\ge} \frac{(\sum A)^3}{3\sum a}\cdot\frac{(\sum A)^3}{3\sum a}\cdot\frac{(\sum A)^3}{3\sum a} = \frac{(\sum A)^9}{27(2s)^3} = \frac{\pi^9}{216s^3}$$