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A classical approach in the study of the convergence of the Cauchy product of two series using Toeplitz's theorem

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Abstract. In this article, we will prove Toeplitz's theorem and some fundamental mathematical results that involve it.

Keywords: Double sequence, Toeplitz's transformation, Cauchy's product, Mertens, Abel.

1 Introduction

Theorem 1.1 (Toeplitz).[2] *Let $\{a_{n,k} : n \in \mathbb{N}^*, 1 \leqslant k \leqslant n\}$ be a double sequence of real numbers with the following properties:*

(i) $\lim\limits_{n\to\infty} a_{n,k} = 0$, *for all positive integers k;*

(ii) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_{n,k} = 1$*;*

(iii) there exists $c > 0$ such that $\sum\limits_{k=1}^{n} |a_{n,k}| < c$, *for all positive integers n.*

Then, for any sequence $(x_n)_{n\geqslant 1}$ of real numbers which is convergent, the sequence $(y_n)_{n\geqslant 1}$, defined by $y_n = \sum\limits_{k=1}^{n} a_{n,k}x_k$, for each $n \in \mathbb{N}^*$, is also convergent and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$

Proof. Firstly, we consider that the sequence $(x_n)_{n\geqslant 1}$ is constant, i.e there exists $\alpha \in \mathbb{R}$ such that $x_n = \alpha$, for all $n \in \mathbb{N}^*$. Then:

$$y_n = \alpha \sum_{k=1}^{n} a_{n,k}$$

for all $n \in \mathbb{N}^*$. In virtue of (ii), we get $\lim\limits_{n\to\infty} y_n = \alpha = \lim\limits_{n\to\infty} x_n$.

Now, we consider $\lim\limits_{n\to\infty} x_n = 0$. Take an arbitrary $\varepsilon > 0$. Then, since the previous assumption, it results that there exists $n_\varepsilon^1 \in \mathbb{N}^*$ such that for all integers $n \geqslant n_\varepsilon^1$ we have $|x_n| < \frac{\varepsilon}{2c}$. Furthermore, from (iii) we have

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[2]The proof of this result can be found in [1, pp. 155-156], [5, pp. 12-13], [2, pp. 502-504], or [4, pp. 37-39]. Furthermore, in the proof presented in this article I used informations from `https://math.stackexchange.com/questions/2514778/toeplitz-theorem`

$$\sum_{k=n_\varepsilon^1}^{n} |a_{n,k}| \leqslant \sum_{k=1}^{n} |a_{n,k}| < c \tag{1}$$

Since $(x_n)_{n\geqslant 1}$ is convergent we have that $(x_n)_{n\geqslant 1}$ is bounded, so there exists $M > 0$ such that for all $n \in \mathbb{N}^*$, $|x_n| < M$. Using the hypothesis (i), we conclude that for all $k \in \overline{1, n_\varepsilon^1 - 1}$, we get $\lim\limits_{n\to\infty} a_{n,k} = 0$. Consequently, there exists $n_{\varepsilon,k} \in \mathbb{N}^*$, with $k \in \overline{1, n_\varepsilon^1 - 1}$ such that

$$|a_{n,k}| < \frac{\varepsilon}{2M\left(n_\varepsilon^1 - 1\right)}, \tag{2}$$

for all integers $n \geqslant n_{\varepsilon,k}$. Summing up the previous inequalities, we get

$$\sum_{k=1}^{n_\varepsilon^1 - 1} |a_{n,k}| < \frac{\varepsilon}{2M} \tag{3}$$

for all integers $n \geqslant n_\varepsilon^2 := \max\left\{n_{\varepsilon,k} : k \in \overline{1, n_\varepsilon^1 - 1}\right\}$

So, using the previous relations, we have

$$|y_n| \leqslant \sum_{k=1}^{n} |a_{n,k}|\, |x_k| = \sum_{k=1}^{n_\varepsilon^1 - 1} |a_{n,k}|\, |x_k| + \sum_{k=n_\varepsilon^1}^{n} |a_{n,k}|\, |x_k| \leqslant M\frac{\varepsilon}{2M} + c\frac{\varepsilon}{2c} = \varepsilon$$

for all integers $n \geqslant n_\varepsilon := \max\{n_\varepsilon^1, n_\varepsilon^2\}$. Hence, $\lim\limits_{n\to\infty} y_n = 0$.

If $\lim\limits_{n\to\infty} x_n = a \in \mathbb{R}$, then the sequence $(x_n - a)_{n\geqslant 1}$ converges to 0 and from previous considerations we obtain that the sequence $(z_n)_{n\geqslant 1}$, defined by $z_n = \sum\limits_{k=1}^{n} a_{n,k}\,(x_n - a)$, for each positive integers n, also converges to 0. Hence, $y_n = z_n + a\sum\limits_{k=1}^{n} a_{n,k}$, for each positive integers n, and $\lim\limits_{n\to\infty} y_n = 0 + a \cdot 1 = a$. □

Remark 1.1. *If* $\lim\limits_{n\to\infty} x_n = 0$*, then the condition* (ii) *can be discarded.*

Remark 1.2. *If we consider* $a_{n,k} > 0$*, for all* $n \in \mathbb{N}^*$ *and for all* $1 \leqslant k \leqslant n$*, then for any sequence* $(x_n)_{n\geqslant 1}$ *with* $\lim\limits_{n\to\infty} x_n = \infty$*, we have* $\lim\limits_{n\to\infty} y_n = \infty$*.*

Proof. Let $(x_n)_{n\geqslant 1}$ with $x_n \to \infty$. Without loss of generality, we can suppose that all terms of the sequence $(x_n)_{n\geqslant 1}$ are strictly positive. Taking an arbitrary $\varepsilon > 0$, since $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_{n,k} = 1$ it follows that there exists $n_\varepsilon^1 \in \mathbb{N}^*$ such that for all integers $n \geqslant n_\varepsilon^1$ we get

$$\sum_{k=1}^{n} a_{n,k} < \frac{1}{3}$$

Since $\lim\limits_{n\to\infty} x_n = \infty$ it follows that the sequence $(x_n)_{n\geqslant 1}$ is unbounded. So, there exists $n_\varepsilon^2 \in \mathbb{N}^*$ such that for all integers $n \geqslant n_\varepsilon^2$ we have $x_n > 3\varepsilon$.

Denote $n_\varepsilon^3 := \max\{n_\varepsilon^1, n_\varepsilon^2\}$. From hypothesis (i) we can deduce

$$\lim_{n\to\infty}\sum_{k=1}^{n_\varepsilon^3} a_{n,k}x_k = 0$$

Therefore, there exists $n_\varepsilon > n_\varepsilon^3$ thereby

$$\left|\sum_{k=1}^{n_\varepsilon^3} a_{n,k}x_k\right| < \frac{\varepsilon}{2}$$

for all integers $n \geqslant n_\varepsilon$. Finally, we have

$$\sum_{k=1}^{n} a_{n,k}x_k = \sum_{k=1}^{n_\varepsilon^3} a_{n,k}x_k + \sum_{k=n_\varepsilon^3+1}^{n} a_{n,k}x_k > -\frac{\varepsilon}{2} + 3\varepsilon \sum_{k=n_\varepsilon^3+1}^{n} a_{n,k} =$$

$$= -\frac{\varepsilon}{2} + 3\varepsilon\left(\sum_{k=1}^{n} a_{n,k} - \sum_{k=1}^{n_\varepsilon^3} a_{n,k}\right) > -\frac{\varepsilon}{2} + 3\varepsilon\left(\sum_{k=1}^{n} a_{n,k} - \frac{1}{3}\right)$$

for all integers $n \geqslant n_\varepsilon$. Using the condition (ii), we have

$$\lim_{n\to\infty}\left[-\frac{\varepsilon}{2} + 3\varepsilon\left(\sum_{k=1}^{n} a_{n,k} - \frac{1}{3}\right)\right] = \frac{3\varepsilon}{2}$$

Consequently, there exists $k_\varepsilon > n_\varepsilon$ such that

$$\left|-\frac{\varepsilon}{2} + 3\varepsilon\left(\sum_{k=1}^{n} a_{n,k} - \frac{1}{3}\right) - \frac{3\varepsilon}{2}\right| < \frac{\varepsilon}{2}$$

for all integers $n \geqslant k_\varepsilon$, which implies

$$-\frac{\varepsilon}{2} + 3\varepsilon\left(\sum_{k=1}^{n} a_{n,k} - \frac{1}{3}\right) > \varepsilon$$

for all $n \geqslant k_\varepsilon$. Hence, we get

$$y_n = \sum_{k=1}^{n} a_{n,k}x_k > \varepsilon$$

for all integers $n \geqslant k_\varepsilon$, from where we deduce $\lim_{n\to\infty} y_n = \infty$. □

Remark 1.3. *If we replace the condition (ii) with $\lim_{n\to\infty}\sum_{k=1}^{n} a_{n,k} = l \in \mathbb{R}$, then $\lim_{n\to\infty} y_n = l \cdot \lim_{n\to\infty} x_n$*

2 Main theoretical results

Corollary 2.1 (Cesàro mean convergence theorem).[3] *Let $(x_n)_{n\geqslant 1}$ be a sequence of real numbers and $(\bar{x}_n)_{n\geqslant 1}$ defined by $\bar{x}_n = \dfrac{1}{n}\sum_{k=1}^{n} x_k$, for all $n \in \mathbb{N}^*$. If the sequence $(x_n)_{n\geqslant 1}$ is convergent and $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$, then $(\bar{x}_n)_{n\geqslant 1}$ is also convergent and $\lim\limits_{n\to\infty} \bar{x}_n = x$.*

Proof. Let us consider the double sequence $(a_{n,k})_{n,k\geqslant 1}$ defined by $a_{n,k} = \frac{1}{n}$, for $1 \leqslant k \leqslant n$, and $a_{n,k} = 0$ otherwise, for all positive integers n. Obviously, for each $k \in \mathbb{N}^*$, we get $(a_{n,k}) \underset{n\to\infty}{\to} 0$ and $\sum_{k=1}^{n} a_{n,k} = 1$, for all positive integers n. The conclusion now follows from Toeplitz's theorem. □

Remark 2.1. *The previous result remains valid even if the sequence $(x_n)_{n\geqslant 1}$ has an infinite limit.*

Corollary 2.2.[4] *Let $(x_n)_{n\geqslant 1}$ be a sequence of positive numbers and $(\bar{x}_n)_{n\geqslant 1}$ defined by*

$$\bar{x}_n = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

for all positive integers n. If $(x_n)_{n\geqslant 1}$ has limit, then $(\bar{x}_n)_{n\geqslant 1}$ also has limit and $\lim\limits_{n\to\infty} \bar{x}_n = \lim\limits_{n\to\infty} x_n$.

Proof. Let $\lim\limits_{n\to\infty} x_n \in \mathbb{R}_+^*$. Let us consider the lower triangular double sequence $(a_{n,k})_{n,k\geqslant 1}$ defined by

$$a_{n,k} = \frac{\frac{1}{x_k}}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

for $1 \leqslant k \leqslant n$ and $a_{n,k} = 0$, otherwise, for all positive integers n. Since $(x_n)_{n\geqslant 1}$ is convergent we have that $(x_n)_{n\geqslant 1}$ is bounded, so there exists $M > 0$ such that for all $n \in \mathbb{N}^*, 0 < x_n < M$. Then

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} > \frac{n}{M}$$

for all positive integers n, which implies $0 < a_{n,k} < \frac{M}{nx_k}$, for all positive integers n and $1 \leqslant k \leqslant n$. Since $\lim\limits_{n\to\infty} \dfrac{M}{nx_k} = 0$ it follows that for all $k \in \mathbb{N}^*$ we have $\lim\limits_{n\to\infty} a_{n,k} = 0$.

Furthermore, for all $n \in \mathbb{N}^*$ we have

$$\sum_{k=1}^{n} a_{n,k} = \sum_{k=1}^{n} \frac{\frac{1}{x_k}}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} = 1$$

[3]See [2, pp. 194-195, pp. 198] or [4, pp. 39]

[4]See [2, pp. 198-199]

and for $c = 2 > 0$ we have $\sum_{k=1}^{n} |a_{n,k}| < c$, for all positive integers n. Applying Toeplitz's theorem, we get:

$$\lim_{n\to\infty} \bar{x}_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} x_k = \lim_{n\to\infty} x_n$$

Next, we consider $\lim_{n\to\infty} x_n = \infty$, which there is equivalent to $\lim_{n\to\infty} \frac{1}{x_n} = 0$. Then, in virtue of Corollary 2.1 , we have

$$\lim_{n\to\infty} \frac{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}{n} = 0$$

or equivalent

$$\lim_{n\to\infty} \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} = \infty$$

□

Remark 2.2. *We know the means inequality*

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \leqslant \sqrt[n]{x_1 x_2 \ldots x_n} \leqslant \frac{x_1 + x_2 + \ldots + x_n}{n}$$

for all $x_n > 0, n \in \mathbb{N}^*$. Using Corollary 2.1 and Corollary 2.2, we obtain that if $(x_n)_{n\geqslant 1}$ has limit, then

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \ldots x_n} = \lim_{n\to\infty} x_n$$

Corollary 2.3.[5] *Let $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ be two sequences of real numbers with the following properties:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$

(ii) $\forall n \in \mathbb{N}^*, \exists c > 0$ *such that* $\sum_{k=1}^{n} |a_k| < c$.

Then

$$\lim_{n\to\infty} (a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1) = 0$$

Proof. Let be the double sequence $(a_{n,k})_{n,k\geqslant 1}$ defined by

$$a_{n,k} = a_{n-k+1}$$

for $1 \leqslant k \leqslant n$ and $a_{n,k} = 0$ otherwise, for all positive integers n. Then:

$$a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1 = \sum_{k=1}^{n} a_{n,k} b_k$$

The sequence $(a_{n,k})$ satisfies the first and third conditions of the Toeplitz's theorem and, according to Remark 1.1, the second condition can be discarded because $\lim_{n\to\infty} b_n = 0$.

The conclusion now follows from Toeplitz's theorem.

□

[5] See [2, pp. 505] or [4, pp. 40]

Corollary 2.4.[6] *Let* $(a_n)_{n\geqslant 1}, (b_n)_{n\geqslant 1}$ *be sequences convergent to* a*, respectively* b $(a, b \in \mathbb{R})$*. Then*

$$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n} = ab$$

Proof. Let be the sequence $(c_n)_{n\geqslant 1}$ defined by:

$$c_n = \frac{1}{n}\sum_{k=1}^{n} a_k b_{n-k+1}$$

for all positive integers n. Firstly, we consider $b = 0$. Let be double sequence $(a_{n,k})_{n,k\geqslant 1}$ defined by

$$a_{n,k} = \frac{1 + b_{n-k+1}}{n}$$

for $1 \leqslant k \leqslant n$ and $a_{n,k} = 0$ otherwise, for all positive integers n. Since $(b_n)_{n\geqslant 1}$ is convergent we have that $(b_n)_{n\geqslant 1}$ is bounded. Consequently, for all $k \in \mathbb{N}^*$, we have $\lim\limits_{n\to\infty} a_{n,k} = 0$ and for all $n \in \mathbb{N}^*$:

$$\sum_{k=1}^{n} a_{n,k} = \sum_{k=1}^{n} \frac{1 + b_{n-k+1}}{n} = 1 + \frac{1}{n}\sum_{k=1}^{n} b_k \tag{4}$$

Therefore, in virtue of Corollary 2.1 , we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} b_k = \lim_{n\to\infty} b_n = 0 \tag{5}$$

which implies

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1 \tag{6}$$

Furthermore, there exists a constant $K > 0$ such that for all $n \in \mathbb{N}^*$ we have $|1 + b_n| < K$. Then, for all $n \geqslant 1$, we have

$$\sum_{k=1}^{n} |a_{n,k}| = \sum_{k=1}^{n} \frac{|1 + b_{n-k+1}|}{n} < \frac{nK}{n} = K \tag{7}$$

Hence, applying TOEPLITZ's theorem, we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} (1 + b_{n-k+1})\, a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} a_k = \lim_{n\to\infty} a_n = a \tag{8}$$

and using $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n} a_k = a$, we obtain

$$c_n = \frac{1}{n}\sum_{k=1}^{n} (1 + b_{n-k+1})\, a_k - \frac{1}{n}\sum_{k=1}^{n} a_k \underset{n\to\infty}{\to} 0 \tag{9}$$

Now, considering the general case, we write c_n as

[6] See [4, pp. 40-41] or [5, pp. 14]

$$c_n = \frac{1}{n}\sum_{k=1}^{n} a_{n-k+1}\left(b_k - b\right) + b\cdot\frac{1}{n}\sum_{k=1}^{n} a_k, \forall n \geqslant 1$$

Therefore, in virtue of previous relations, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_{n-k+1}\left(b_k - b\right) = 0 \tag{10}$$

Furthermore, using $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n} a_k = a$, we obtain:

$$\lim_{n\to\infty} c_n = 0 + ab = ab$$

and this result is proved. □

Lemma 2.1.[7] *Let $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ be two sequences of real numbers and $s_n = \sum\limits_{k=1}^{n} x_k, \forall n \in \mathbb{N}^*$. Then, for all $n \in \mathbb{N}^*$, we get:*

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} - \sum_{k=1}^{n} s_k\left(y_{k+1} - y_k\right)$$

Proof. Let $s_0 = 0$. For each integers $k \geqslant 1$ we have $s_k - s_{k-1} = x_k$. Hence:

$$\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n}\left(s_k - s_{k-1}\right) y_k =$$

$$= y_1\left(s_1 - s_0\right) + y_2\left(s_2 - s_1\right) + \ldots + y_n\left(s_n - s_{n-1}\right) =$$

$$= s_1\left(y_1 - y_2\right) + \ldots + s_{n-1}\left(y_{n-1} - y_n\right) + s_n\left(y_n - y_{n+1}\right) + s_n y_{n+1} =$$

$$= s_n y_{n+1} - \sum_{k=1}^{n} s_k\left(y_{k+1} - y_k\right)$$

□

Corollary 2.5 (Kronecker).[8] *Let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ be two sequences of real numbers such that $(b_n)_{n\geqslant 1}$ is an increasing sequence of nonnegative real numbers with $\lim\limits_{n\to\infty} b_n = \infty$. If the series $\sum\limits_{n=1}^{\infty} a_n$ converges, then*

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n} b_k a_k = 0$$

Proof. Using Lemma 2.1, we get

$$\frac{1}{b_n}\sum_{k=1}^{n} a_k b_k = s_n - \frac{1}{b_n}\sum_{k=1}^{n-1}\left(b_{k+1} - b_k\right) s_k \tag{11}$$

[7] See [6, pp. 98]
[8] See [3]

where $s_n = \sum_{k=1}^{n} a_k$, for $n \in \mathbb{N}^*$, denotes the sequence of partial sums.

Let us consider the double sequence $(a_{n,k})_{n,k\geqslant 1}$ defined by

$$a_{n,k} = \frac{b_{k+1} - b_k}{b_n}$$

for $1 \leqslant k \leqslant n$ and $a_{n,k} = 0$ otherwise, for all positive integers n. For all integers $n \geqslant 1$, we have

$$\sum_{k=1}^{n-1} a_{n,k} = \sum_{k=1}^{n-1} \frac{b_{k+1} - b_k}{b_n} = \frac{b_n - b_1}{b_n} \underset{n\to\infty}{\to} 1 \tag{12}$$

and for each $k \in \mathbb{N}^*$,

$$\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \frac{b_{k+1} - b_k}{b_n} = 0, \tag{13}$$

because $\lim_{n\to\infty} b_n = \infty$. Furthermore, we have

$$\sum_{k=1}^{n-1} |a_{n,k}| = \sum_{k=1}^{n-1} \frac{b_{k+1} - b_k}{b_n} = \frac{b_n - b_1}{b_n} < 1 \tag{14}$$

Hence, applying TOEPLITZ's theorem, we get:

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k)\, s_k = \lim_{n\to\infty} \sum_{k=1}^{n-1} a_{n,k} s_k = \lim_{n\to\infty} s_n \tag{15}$$

and using (11), we obtain

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k a_k = 0$$

□

A result equivalent to the previous one is the following

Corollary 2.6. *Let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ be two sequences of real numbers such that $(b_n)_{n\geqslant 1}$ is an increasing sequence of non-negative real numbers with $\lim_{n\to\infty} b_n = \infty$. If the series $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ converges, then $\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} a_k = 0$.*

In the following, we will state and demonstrate some results regarding the study of the convergence of the product of numerical series. Firstly, we start with the definition of a fundamental mathematical concept, namely the CAUCHY product of two series. For more details in the study of infinite series, see [3] or [6].

Definition 2.1 (CAUCHY product of two infinite series)**.** *Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two infinite series. The CAUCHY product of these series is defined by $\sum_{n=1}^{\infty} z_n$, where the sequence $(z_n)_{n\geqslant 0}$ is defined by $z_n = \sum_{k=1}^{n} x_k y_{n-k+1}$, for all $n \in \mathbb{N}^*$.*

Next, we will state and prove two fundamental results that provide sufficient conditions for the convergence of the CAUCHY product.

Theorem 2.1 (MERTENS).[9] *Let* $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two convergent series. Suppose that at least one of the series is absolutely convergent. Then the product in* CAUCHY*'s form,* $\sum_{n=1}^{\infty} z_n$*, is convergent and:*

$$\sum_{n=1}^{\infty} z_n = \left(\sum_{n=1}^{\infty} x_n\right)\left(\sum_{n=1}^{\infty} y_n\right)$$

Proof. Let A, respectively B be the sums of series $\sum_{n=1}^{\infty} x_n$, respectively $\sum_{n=1}^{\infty} y_n$. Let us assume that $\sum_{n=1}^{\infty} y_n$ is absolutely convergent. Then, there exists $K > 0$ such that:

$$\sum_{k=1}^{n} |y_n| < K, \forall n \in \mathbb{N} \tag{16}$$

Also let:

$$A_n = \sum_{k=1}^{n} x_k, B_n = \sum_{k=1}^{n} y_k, \text{ and } C_n = \sum_{k=1}^{n} z_k \forall n \in \mathbb{N}^*,$$

be the sequences of partial sums of $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} x_n$, respectively $\sum_{n=1}^{\infty} z_n$. Ordering the terms of $(C_n)_{n\geqslant 1}$ by the $(y_n)_{n\geqslant 1}$ terms, we obtain that:

$$C_n = y_1 A_n + y_2 A_{n-1} + \ldots y_n A_1 \tag{17}$$

for all positive integers n. Since $\lim_{n\to\infty} A_n = A$ it follows that there exists a sequence $(\alpha_n)_{n\geqslant 1}$ convergent to 0 such that $A_n + \alpha_n = A, \forall n \in \mathbb{N}^*$. Consequently, we can rewrite C_n as:

$$C_n = AB_n - (y_1\alpha_n + y_2\alpha_{n-1} + \ldots + y_n\alpha_1) \tag{18}$$

for all positive integers n. From $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \alpha_n = 0$ and condition (16), we can use Corollary 2.3 and get that:

$$\lim_{n\to\infty} (y_1\alpha_n + y_2\alpha_{n-1} + \ldots + y_n\alpha_1) = 0,$$

which implies $\lim_{n\to\infty} C_n = AB$ and this concludes the proof of the theorem. □

Theorem 2.2 (ABEL).[10] *Let* $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two convergent series. Suppose that the product in* CAUCHY*'s form,* $\sum_{n=0}^{\infty} z_n$*, is convergent. Then:*

$$\sum_{n=1}^{\infty} z_n = \left(\sum_{n=1}^{\infty} x_n\right)\left(\sum_{n=1}^{\infty} y_n\right).$$

[9] See [3] or [4, pp. 46-47] and for a different approach see [6, pp. 114-116]
[10] See [3] or [4, pp. 47-48]

Proof. Let A, B, respectively C the sums of series $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n$, respectively $\sum_{n=1}^{\infty} z_n$. Also let:

$$A_n = \sum_{k=1}^{n} x_k, B_n = \sum_{k=1}^{n} y_k, \text{ and } C_n = \sum_{k=1}^{n} z_k \forall n \in \mathbb{N}$$

be the sequences of partial sums of $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} x_n$, respectively $\sum_{n=1}^{\infty} z_n$.

Firstly, we have

$$C_1 + C_2 + \ldots + C_n =$$

$$= x_1y_1 + (x_1y_1 + x_1y_2 + x_2y_1) + \ldots + \sum_{k=1}^{n} (x_1y_k + x_2y_{k-1} + \ldots + x_ky_1) =$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{k} \sum_{j=1}^{i} x_j y_{i-j+1} = A_1B_n + A_2B_{n-1} + \ldots + A_nB_1$$

From Corollary 2.3 we have $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} C_k = C$ and from Corollary 2.6 we have

$$\lim_{n\to\infty} \frac{A_1B_n + A_2B_{n-1} + \ldots + A_nB_1}{n} = AB$$

But we also have that

$$C_1 + C_2 + \ldots + C_n = A_1B_n + A_2B_{n-1} + \ldots + A_nB_1$$

so $C = AB$ and this concludes the proof of the theorem. □

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