A SIMPLE PROOF FOR MILNE'S INEQUALITY

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Abstract. In this paper is presented a simple proof for Milne's inequality and a few applications.

MILNE'S INEQUALITY ($n = 2$)

If $x_1, x_2, y_1, y_2 > 0$ then:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}\le\frac{(x_1+x_2)(y_1+y_2)}{x_1+x_2+y_1+y_2} \tag{1}$$

Proof.

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}=\left(x_1-\frac{x_1^2}{x_1+y_1}\right)+\left(x_2-\frac{x_2^2}{x_2+y_2}\right)=$$

$$=x_1+x_2-\left(\frac{x_1^2}{x_1+y_1}+\frac{x_2^2}{x_2+y_2}\right)\overset{\text{BERGSTRÖM}}{\le}$$

$$\le x_1+x_2-\frac{(x_1+x_2)^2}{x_1+y_1+x_2+y_2}=$$

$$=\frac{(x_1+x_2)^2+(x_1+x_2)(y_1+y_2)-(x_1+x_2)^2}{x_1+x_2+y_1+y_2}=$$

$$=\frac{(x_1+x_2)(y_1+y_2)}{x_1+x_2+y_1+y_2}$$

Equality holds for $x_1 = y_1; x_2 = y_2$. □

MILNE'S INEQUALITY ($n = 3$)

If $x_1, x_2, x_3, y_1, y_2, y_3 > 0$ then:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}+\frac{x_3y_3}{x_3+y_3}\le\frac{(x_1+x_2+x_3)(y_1+y_2+y_3)}{x_1+x_2+x_3+y_1+y_2+y_3} \tag{1}$$

Proof.

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}+\frac{x_3y_3}{x_3+y_3}=$$

$$=\left(x_1-\frac{x_1^2}{x_1+y_1}\right)+\left(x_2-\frac{x_2^2}{x_2+y_2}\right)+\left(x_3-\frac{x_3^2}{x_3+y_3}\right)=$$

$$=x_1+x_2+x_3-\left(\frac{x_1^2}{x_1+y_1}+\frac{x_2^2}{x_2+y_2}+\frac{x_3^2}{x_3+y_3}\right)\overset{\text{BERGSTRÖM}}{\le}$$

$$\le x_1+x_2+x_3-\frac{(x_1+x_2+x_3)^2}{x_1+y_1+x_2+y_2+x_3+y_3}=$$

$$=\frac{(x_1+x_2+x_3)^2+(x_1+x_2+x_3)(y_1+y_2+y_3)-(x_1+x_2+x_3)^2}{x_1+x_2+x_3+y_1+y_2+y_3}=$$

$$=\frac{(x_1+x_2+x_3)(y_1+y_2+y_3)}{x_1+x_2+x_3+y_1+y_2+y_3}$$

Equality holds for $x_1 = y_1; x_2 = y_2; x_3 = y_3$. □

GENERAL MILNE'S INEQUALITY

If $x_i > 0; y_i > 0; i \in \overline{1,n}$ then:

$$\sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i} \leq \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i} \tag{3}$$

Proof.

$$\sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i} = \sum_{i=1}^{n} \Big(x_i - \frac{x_i^2}{x_i + y_i} \Big) =$$

$$= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i^2}{x_i + y_i} \overset{\text{BERGSTRÖM}}{\leq}$$

$$\leq \sum_{i=1}^{n} x_i - \frac{(\sum_{i=1}^{n} x_i)^2}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i} =$$

$$= \frac{(\sum_{i=1}^{n} x_i)^2 + (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i) - (\sum_{i=1}^{n} x_i)^2}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i} =$$

$$= \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i}$$

Equality holds for $x_i = y_i; i \in \overline{1,n}$. □

Corollary 1.

If $x_1, x_2, y_1, y_2 > 0; x_1 + x_2 = y_1 + y_2 = 2$ then:

$$\frac{x_1 y_1}{x_1 + y_1} + \frac{x_2 y_2}{x_2 + y_2} \leq 1$$

Proof.

Replace in (1): $x_1 + x_2 = 2; y_1 = y_2 = 2$.

$$\frac{x_1 y_1}{x_1 + y_1} + \frac{x_2 y_2}{x_2 + y_2} \leq \frac{(x_1 + x_2)(y_1 + y_2)}{x_1 + x_2 + y_1 + y_2} = \frac{2 \cdot 2}{2 + 2} = 1$$

Equality holds for $x_1 = x_2 = y_1 = y_2 = 1$. □

Corollary 2.

If $x_1, x_2, x_3, y_1, y_2, y_3 > 0; x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 3$ then:

$$\frac{x_1 y_1}{x_1 + y_1} + \frac{x_2 y_2}{x_2 + y_2} + \frac{x_3 y_3}{x_3 + y_3} \leq \frac{3}{2}$$

Proof.

Replace in (2): $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 3$

$$\frac{x_1 y_1}{x_1 + y_1} + \frac{x_2 y_2}{x_2 + y_2} + \frac{x_3 y_3}{x_3 + y_3} \leq \frac{(x_1 + x_2 + x_3)(y_1 + y_2 + y_3)}{x_1 + x_2 + x_3 + y_1 + y_2 + y_3} = \frac{3 \cdot 3}{2 \cdot 3} = \frac{3}{2}$$

Equality holds for $x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = 1$ □

Corollary 3.

If $x_i, y_i > 0; i \in \overline{1,n}; x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n = n$ then:

$$\frac{x_1 y_1}{x_1 + y_1} + \frac{x_2 y_2}{x_2 + y_2} + \ldots + \frac{x_n y_n}{x_n + y_n} \leq \frac{n}{2}; n \in \mathbb{N}; n \geq 2$$

Proof.
Replace in (3): $x_1+x_2+\ldots+x_n=y_1+y_2+\ldots+y_n=n$.

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}+\ldots+\frac{x_ny_n}{x_n+y_n}\leq\frac{(x_1+x_2+\ldots+x_n)(y_1+y_2+\ldots+y_n)}{x_1+x_2+\ldots+x_n+y_1+y_2+\ldots+y_n}=$$

$$=\frac{n\cdot n}{n+n}=\frac{n^2}{2n}=\frac{n}{2}$$

Equality holds for:

$$x_1+x_2+\ldots+x_n=y_1+y_2+\ldots+y_n=n$$

□

Corollary 4.
If $x,y\in\mathbb{R}$ then:

$$\frac{\sin^2 x\sin^2 y}{\sin^2 x+\sin^2 y}+\frac{\cos^2 x\cos^2 y}{\cos^2 x+\cos^2 y}\leq\frac{1}{2}$$

Proof.
We take in (1):

$$x_1=\sin^2 x;x_2=\cos^2 x;y_1=\sin^2 y;y_2=\cos^2 y$$

$$\frac{\sin^2 x\sin^2 y}{\sin^2 x+\sin^2 y}+\frac{\cos^2 x\cos^2 y}{\cos^2 x+\cos^2 y}\leq\frac{(\sin^2 x+\cos^2 x)(\sin^2 y+\cos^2 y)}{\sin^2 x+\cos^2 x+\sin^2 y+\cos^2 y}=\frac{1}{2}$$

Equality holds for $x=y=\frac{\pi}{4}$. □

Application 1.
If a,b,c,r,R are sides, inradii and circumradii in $\Delta ABC, a',b',c',r,R'$ are sides, inradii and circumradii in $\Delta A'B'C'$ then:

$$\frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}\leq\frac{3\sqrt{3}RR'}{2(r+r')}$$

Proof.
In (2) we take: $x_1=a;y_1=a';x_2=b;y_2=b';x_3=c;y_3=c';s=\frac{a+b+c}{2}$; $s'=\frac{a'+b'+c'}{2}$

$$\frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}\leq$$

$$\leq\frac{(a+b+c)(a'+b'+c')}{(a+b+c)+(a'+b'+c')}=\frac{2s\cdot 2s'}{2s+2s'}=\frac{2ss'}{s+s'}\leq$$

$$\overset{\text{MITRINOVIC}}{\leq}\frac{2\cdot\frac{3\sqrt{3}}{2}R\cdot\frac{3\sqrt{3}}{2}R'}{3\sqrt{3}r+3\sqrt{3}r'}=$$

$$=\frac{R\cdot\frac{3\sqrt{3}}{2}R'}{r+r'}=\frac{3\sqrt{3}RR'}{2(r+r')}$$

Equality holds for $a=b=c$ and $a'=b'=c'$. □

Application 2.
In acute triangles $ABC, A'B'C'$ the following relationship holds:

$$\frac{1}{\cot A+\cot A'}+\frac{1}{\cot B+\cot B'}+\frac{1}{\cot C+\cot C'}\leq\frac{1}{\cot A\cot B\cot C+\cot A'\cot B'\cot C'}$$

Proof.

$$\sum_{cyc}\frac{1}{\cot A+\cot A'}=\sum_{cyc}\frac{1}{\frac{1}{\tan A}+\frac{1}{\tan A'}}=\sum_{cyc}\frac{\tan A\cdot\tan A'}{\tan A+\tan A'}=$$

$$=\sum_{cyc}\Big(\tan A-\frac{\tan^2 A}{\tan A+\tan A'}\Big)=$$

$$=\sum_{cyc}\tan A-\sum_{cyc}\frac{\tan^2 A}{\tan A+\tan A'}\overset{\text{BERGSTRÖM}}{\leq}$$

$$\leq\sum_{cyc}\tan A-\frac{(\sum_{cyc}\tan A)^2}{\sum_{cyc}\tan A+\sum_{cyc}\tan A'}=$$

$$=\frac{(\sum_{cyc}\tan A)^2+\sum_{cyc}\tan A\cdot\sum_{cyc}\tan A'-(\sum_{cyc}\tan A)^2}{\sum_{cyc}\tan A+\sum_{cyc}\tan A'}=$$

$$=\frac{\prod_{cyc}\tan A\cdot\prod_{cyc}\tan A'}{\prod_{cyc}\tan A+\prod_{cyc}\tan A'}=$$

$$=\frac{1}{\prod_{cyc}\frac{1}{\tan A}+\prod_{cyc}\frac{1}{\tan A'}}=\frac{1}{\prod_{cyc}\cot A+\prod_{cyc}\cot A'}$$

Equality holds for $A=B=C=A'=B'=C'=\frac{\pi}{3}$. □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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