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If $a_k > 0\ (k = 1, 2, \dots, n), \lambda \geq 2n + 1$ and $\sum_{k=1}^{n} a_k = n$, then prove that

$$\sum_{k=1}^{n} \frac{1}{\lambda + a_k^2} \leq \frac{n}{1 + \lambda}$$

What happens if λ does not verify the hypothesis?

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Solution by Eric Cismaru – Romania

Let us show that $\frac{1}{\lambda + a_k^2} \leq \frac{\lambda + 3 - 2a_k}{(\lambda+1)^2}$, for any $k = \overline{1, n}$.

This inequality is equivalent to

$$(\lambda + 3 - 2a_k)(\lambda + a_k^2) \geq (\lambda + 1)^2 \Leftrightarrow$$

$$\lambda^2 + \lambda a_k^2 + 3\lambda + 3a_k^2 - \lambda 2a_k - 2a_k^3 \geq \lambda^2 + 2\lambda + 1$$

$\Leftrightarrow 2a_k^3 + \lambda 2a_k - 3a_k^2 - \lambda - \lambda a_k^2 + 1 \leq 0 \Leftrightarrow (a_k - 1)^2(2a_k + 1 - \lambda) \leq 0$, with $2a_k + 1 \leq 2n + 1 \leq \lambda \Leftrightarrow a_k \leq n$, which is true because all the terms add up to n and are all positive reals.

Thus, we have shown that $\frac{1}{\lambda + a_k^2} \leq \frac{\lambda + 3 - 2a_k}{(\lambda+1)^2}$, for any $k = \overline{1, n}$.

Using this result, we have

$$\sum_{k=1}^{n} \frac{1}{\lambda + a_k^2} \leq \sum_{k=1}^{n} \frac{\lambda + 3 - 2a_k}{(\lambda + 1)^2} = \frac{n\lambda + 3n - 2n}{(\lambda + 1)^2} = \frac{n(\lambda + 1)}{(\lambda + 1)^2} = \frac{n}{\lambda + 1}$$

Equality holds when $a_k = 1, k = \overline{1, n}$.