ROMANIAN MATHEMATICAL MAGAZINE

(a) If $a, \mathrm{b}, \mathrm{c} > 0$, *then prove the inequality* :

$$\sum_{cyc} a^2 \geq \sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c|+|b-c|)^2 \text{ is } false\,!$$

(b) If $a, \mathrm{b}, \mathrm{c} > 0$, *then prove the inequality* :

$$\sum_{cyc} a^2 \geq \sum_{cyc} ab + \frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2 \text{ is true}\,!$$

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$$\sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c|+|b-c|)^2 - \sum_{cyc} a^2$$

$$= \sum_{cyc} ab - \sum_{cyc} a^2 + \frac{1}{2}\sum_{cyc}(a-c)^2 + \frac{1}{2}\sum_{cyc}(b-c)^2 + \sum_{cyc}|a-c||b-c|$$

$$= \sum_{cyc} ab - \sum_{cyc} a^2 + \frac{1}{2}.2\left(2\sum_{cyc} a^2 - 2\sum_{cyc} ab\right) + \sum_{cyc}|a-c||b-c|$$

$$= \sum_{cyc} a^2 - \sum_{cyc} ab + \sum_{cyc}|a-c||b-c| = \frac{1}{2}\sum_{cyc}(a-b)^2 + \sum_{cyc}|a-c||b-c| \geq 0$$

$$\therefore \sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c|+|b-c|)^2 - \sum_{cyc} a^2 \geq 0$$

$$\Rightarrow \sum_{cyc} a^2 \leq \sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c|+|b-c|)^2$$

$$\Rightarrow \boxed{\sum_{cyc} a^2 \geq \sum_{cyc} ab + \frac{1}{2}\sum_{cyc}(|a-c|+|b-c|)^2 \text{ is } false\text{; reverse inequ}a\text{lity is true}}$$

$$\text{Again, } \sum_{cyc} a^2 - \sum_{cyc} ab - \frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2$$

$$= \sum_{cyc} a^2 - \sum_{cyc} ab - \frac{1}{8}\left(\sum_{cyc}(a-c)^2 + \sum_{cyc}(b-c)^2 + 2\sum_{cyc}|a-c||b-c|\right)$$

$$= \sum_{cyc} a^2 - \sum_{cyc} ab - \frac{1}{8}\left(4\sum_{cyc} a^2 - 4\sum_{cyc} ab + 2\sum_{cyc}|a-c||b-c|\right)$$

$$= \frac{1}{2}\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right) - \frac{1}{4}\sum_{cyc}|a-c||b-c| = \frac{1}{4}\sum_{cyc}(a-b)^2 - \frac{1}{4}\sum_{cyc}|a-c||b-c|$$

$$=\frac{1}{8}\left((|a-b|-|b-c|)^2+(|b-c|-|c-a|)^2+(|c-a|-|a-b|)^2\right)\geq 0$$

$$\therefore \sum_{cyc} a^2-\sum_{cyc} ab-\frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2\geq 0$$

$$\Rightarrow \boxed{\sum_{cyc} a^2\geq \sum_{cyc} ab+\frac{1}{8}\sum_{cyc}(|a-c|+|b-c|)^2 \text{ is true}}, ''='' \text{ iff } a=b=c \text{ (QED)}$$