ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathbf{b}, \mathbf{c} > 0$, *then prove that* :

$$\sum_{cyc} \frac{2a^2 + c(b-c)}{(b+c)(a+b+c)} \geq 1$$

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$$\sum_{cyc} \frac{2a^2 + c(b-c)}{(b+c)(a+b+c)} \geq 1 \Leftrightarrow \sum_{cyc} \frac{2a^2 + c(b+c-2c)}{(b+c)(a+b+c)} \geq 1$$

$$\Leftrightarrow 2\sum_{cyc} \frac{a^2}{b+c} + \sum_{cyc} c - 2\sum_{cyc} \frac{c^2}{b+c} \geq \sum_{cyc} a \Leftrightarrow \sum_{cyc} \frac{(a+c)(a-c)}{b+c} \geq 0 \to (*)$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ *and* $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of *a* triangle with semiperimeter, circumradius *and* inradius $= s, R, r$ (*say*) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \Rightarrow a = s - x, b = s - y,$

$c = s - z$ *and* via such substitutions, $(*) \Leftrightarrow \sum_{cyc} \frac{y\big((s-x)-(s-z)\big)}{x} \geq 0$

$$\Leftrightarrow \sum_{cyc} \frac{y(z-x)}{x} \geq 0 \Leftrightarrow \sum_{cyc} \frac{yz}{x} \geq \sum_{cyc} y \Leftrightarrow \sum_{cyc} x^2y^2 \geq xyz\sum_{cyc} x \to \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{2a^2 + c(b-c)}{(b+c)(a+b+c)} \geq 1 \ \forall\, a, b, c > 0, \text{"="} \text{ iff } a = b = c \ (\text{QED})$$