ROMANIAN MATHEMATICAL MAGAZINE

If $a_k > 0$ (k = 1, 2, …, n), then prove that:

$$16\sum_{cyc}\frac{a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)}+\sum_{cyc}\frac{a_1^2+a_2^2}{a_1+a_2}\le 3\sum_{k=1}^{n}a_k$$

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$$\frac{16xy(x+y)}{(3x+y)(x+3y)}+\frac{x^2+y^2}{x+y}\overset{?}{\le}\frac{3(x+y)}{2}$$

$$\Leftrightarrow 3(3x+y)(x+3y)(x+y)^2\overset{?}{\ge}2\left(16xy(x+y)^2+(3x+y)(x+3y)(x^2+y^2)\right)$$

$$\Leftrightarrow 3x^4+3y^4+2x^2y^2-4xy(x^2+y^2)\overset{?}{\ge}0$$

$$\Leftrightarrow (x^4+y^4+2x^2y^2)+2(x^4+y^4)-4xy(x^2+y^2)\overset{?}{\ge}0$$

$$\Leftrightarrow (x^2+y^2)^2+(x^2+y^2)^2+(x^2-y^2)^2-4xy(x^2+y^2)\overset{?}{\ge}0$$

$$\Leftrightarrow 2(x^2+y^2)(x^2+y^2-2xy)+(x^2-y^2)^2\overset{?}{\ge}0$$

$$\Leftrightarrow 2(x^2+y^2)(x-y)^2+(x^2-y^2)^2\overset{?}{\ge}0\to \text{true}$$

$$\therefore \frac{16xy(x+y)}{(3x+y)(x+3y)}+\frac{x^2+y^2}{x+y}\le\frac{3(x+y)}{2}$$

$$\therefore \frac{16\,a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)}+\frac{a_1^2+a_2^2}{a_1+a_2}\le\frac{3(a_1+a_2)}{2}\text{ and analogs}$$

$$\Rightarrow 16\sum_{cyc}\frac{a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)}+\sum_{cyc}\frac{a_1^2+a_2^2}{a_1+a_2}\le 3\sum_{k=1}^{n}a_k$$

$\forall a_k > 0$ (k = 1, 2, …, n), " = " iff $a_1 = a_2 = \cdots = a_n$ (QED)