

# ROMANIAN MATHEMATICAL MAGAZINE

If  $a, b, c > 0$ , then prove that :

$$\sum_{\text{cyc}} \frac{2a^2 + c(b - c)}{(b + c)(a + b + c)} \geq 1$$

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$$\begin{aligned} \sum_{\text{cyc}} \frac{2a^2 + c(b - c)}{(b + c)(a + b + c)} \geq 1 &\Leftrightarrow \sum_{\text{cyc}} \frac{2a^2 + c(b + c - 2c)}{(b + c)(a + b + c)} \geq 1 \\ \Leftrightarrow 2 \sum_{\text{cyc}} \frac{a^2}{b + c} + \sum_{\text{cyc}} c - 2 \sum_{\text{cyc}} \frac{c^2}{b + c} &\geq \sum_{\text{cyc}} a \Leftrightarrow \sum_{\text{cyc}} \frac{(a + c)(a - c)}{b + c} \geq 0 \rightarrow (*) \end{aligned}$$

Assigning  $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$  and  $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$  form sides of a triangle with semiperimeter, circumradius and inradius =  $s, R, r$  (say)

$$\text{yielding } 2 \sum_{\text{cyc}} a = \sum_{\text{cyc}} x = 2s \Rightarrow \sum_{\text{cyc}} a = s \Rightarrow a = s - x, b = s - y,$$

$$c = s - z \text{ and via such substitutions, } (*) \Leftrightarrow \sum_{\text{cyc}} \frac{y((s - x) - (s - z))}{x} \geq 0$$

$$\Leftrightarrow \sum_{\text{cyc}} \frac{y(z - x)}{x} \geq 0 \Leftrightarrow \sum_{\text{cyc}} \frac{yz}{x} \geq \sum_{\text{cyc}} y \Leftrightarrow \sum_{\text{cyc}} x^2 y^2 \geq xyz \sum_{\text{cyc}} x \rightarrow \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} \frac{2a^2 + c(b - c)}{(b + c)(a + b + c)} \geq 1 \forall a, b, c > 0, " = " \text{ iff } a = b = c \text{ (QED)}$$