ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\sum_{cyc} \frac{1}{2(2a+b+c)} \ge \sum_{cyc} \frac{1}{3(a+b)+2c}$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x$
$=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius
$= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \Rightarrow a = s-x, b = s-y,$

$c = s-z$ and via such substitutions, $\displaystyle\sum_{cyc} \frac{1}{2(2a+b+c)} \ge \sum_{cyc} \frac{1}{3(a+b)+2c}$

$$\Leftrightarrow \sum_{cyc} \frac{1}{2(y+z)} \ge \sum_{cyc} \frac{1}{3x+2(s-x)} \Leftrightarrow \sum_{cyc} \frac{2s-x+x}{2s.2(2s-x)} \ge \sum_{cyc} \frac{2s+x-x}{2s(2s+x)}$$

$$\Leftrightarrow \frac{3}{4s} + \frac{1}{4s}\sum_{cyc} \frac{x}{2s-x} \ge \frac{3}{2s} - \frac{1}{2s}\sum_{cyc} \frac{x}{2s+x} \Leftrightarrow \sum_{cyc} \frac{x}{2s-x} + \sum_{cyc} \frac{2x}{2s+x} \ge 3 \rightarrow (1)$$

$$\text{Now, } \sum_{cyc} \frac{x}{2s-x} + \sum_{cyc} \frac{2x}{2s+x} = \sum_{cyc} \frac{x^2}{2sx-x^2} + 2\sum_{cyc} \frac{x^2}{2sx+x^2} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{\left(\sum_{cyc} x\right)^2}{\left(\sum_{cyc} x\right)^2 - \sum_{cyc} x^2} + \frac{2\left(\sum_{cyc} x\right)^2}{\left(\sum_{cyc} x\right)^2 + \sum_{cyc} x^2}$$

$$= \frac{m+2n}{m+2n-m} + \frac{2(m+2n)}{m+2n+m} \left(m = \sum_{cyc} x^2, n = \sum_{cyc} xy\right)$$

$$= \frac{(m+2n)(m+3n)}{2n(m+n)} \overset{?}{\ge} 3 \Leftrightarrow m^2+5mn+6n^2 \overset{?}{\ge} 6mn+6n^2 \Leftrightarrow m(m-n) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because m = \sum_{cyc} x^2 \ge \sum_{cyc} xy = n \Rightarrow (1) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{1}{2(2a+b+c)} \ge \sum_{cyc} \frac{1}{3(a+b)+2c} \;\forall\, a,b,c>0, '' = '' \text{ iff } a=b=c \text{ (QED)}$$