ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that :

$$\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1 \Leftrightarrow \sum_{cyc}\frac{2a}{3(b+c)}\ge 1$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$, $y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s \Rightarrow \sum_{cyc}a=s \rightarrow (1) \Rightarrow a=s-x, b=s-y, c=s-z$

$$\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1 \Leftrightarrow \sum_{cyc}\frac{s-x}{(a+b)+(a+c)}\le\frac{3}{4} \Leftrightarrow \sum_{cyc}\frac{s-x}{(a+b)+(a+c)}\le\frac{3}{4}$$

$$\Leftrightarrow \sum_{cyc}\frac{s-x}{y+z}\le\frac{3}{4} \Leftrightarrow \sum_{cyc}\frac{2s-x-s}{2s-x}\le\frac{3}{4} \Leftrightarrow 3-\frac{3}{4}\le s\sum_{cyc}\frac{1}{y+z}$$

$$\Leftrightarrow \sum_{cyc}\frac{x^2+\sum_{cyc}xy}{\prod_{cyc}(y+z)}\ge\frac{9}{4s} \Leftrightarrow \frac{\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)+\sum_{cyc}xy}{\prod_{cyc}(y+z)}\ge\frac{9}{4s}$$

$$\Leftrightarrow \frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}\ge\frac{9}{4s} \Leftrightarrow 10s^2+8Rr+2r^2\ge 9s^2+18Rr+9r^2$$

$$\Leftrightarrow s^2\ge 10Rr+7r^2 \therefore \boxed{\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1 \Leftrightarrow s^2\ge 10Rr+7r^2 \rightarrow (1)}$$

$$\text{Also, }\sum_{cyc}\frac{2a}{3(b+c)}\ge 1 \Leftrightarrow \sum_{cyc}\frac{s-x}{x}\ge\frac{3}{2} \Leftrightarrow s.\frac{\sum_{cyc}xy}{4Rrs}\ge\frac{9}{2} \Leftrightarrow \frac{s^2+4Rr+r^2}{4Rr}\ge\frac{9}{2}$$

$$\Leftrightarrow s^2\ge 14Rr-r^2 \therefore \boxed{\sum_{cyc}\frac{2a}{3(b+c)}\ge 1 \Leftrightarrow s^2\ge 14Rr-r^2 \rightarrow (2)}$$

If possible, let us assume $\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1 \Rightarrow \sum_{cyc}\frac{2a}{3(b+c)}\le 1$, i.e.,

$\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1$ and $\sum_{cyc}\frac{2a}{3(b+c)}\le 1$ and then, via (1), (2),

$s^2\ge 10Rr+7r^2$ and $s^2\le 14Rr-r^2$ which is a contradiction $\because s^2-14Rr+r^2$

$= s^2-16Rr+5r^2+2r(R-2r) \overset{\text{Gerretsen and Euler}}{\ge} 0 \Rightarrow s^2\ge 14Rr-r^2 \therefore$ our asssumption

is incorrect $\therefore$ $\boxed{\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1 \Rightarrow \sum_{cyc}\frac{2a}{3(b+c)}\ge 1}$

Again, if possible, let us assume $\sum_{cyc}\frac{2a}{3(b+c)}\ge 1 \Rightarrow \sum_{cyc}\frac{4a}{3(2a+b+c)}\ge 1$, i.e.,

$$\sum_{cyc}\frac{2a}{3(b+c)} \geq 1 \text{ and } \sum_{cyc}\frac{4a}{3(2a+b+c)} \geq 1 \text{ and then, via } (1),(2),$$

$$s^2 \geq 14Rr - r^2 \text{ and } s^2 \leq 10Rr + 7r^2 \text{ which is a contradiction} \because s^2 - 10Rr - 7r^2$$

$$= s^2 - 16Rr + 5r^2 + 6r(R-2r) \overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\geq} 0 \Rightarrow s^2 \geq 10Rr + 7r^2 \therefore \text{our asssumption}$$

$$\text{is incorrect} \therefore \boxed{\sum_{cyc}\frac{2a}{3(b+c)} \geq 1 \Rightarrow \sum_{cyc}\frac{4a}{3(2a+b+c)} \leq 1}$$

$$\therefore \sum_{cyc}\frac{4a}{3(2a+b+c)} \leq 1 \Leftrightarrow \sum_{cyc}\frac{2a}{3(b+c)} \geq 1 \text{ (QED)}$$