ROMANIAN MATHEMATICAL MAGAZINE

If $x_k > 0$ $(k = 1, 2, \dots, n)$, then prove that :

$$\sum_{cyc}\sqrt{\frac{1}{3}\left(x_1^4 + x_1^2x_2^2 + x_2^4\right)} \ge \sum_{k=1}^{n} x_k^2 \ge \sum_{cyc} x_1\sqrt{\frac{1}{3}\left(2x_1^2 + x_2x_3\right)}$$

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$$\sum_{cyc}\sqrt{\frac{1}{3}\left(x_1^4 + x_1^2x_2^2 + x_2^4\right)} \ge \sum_{cyc}\sqrt{\frac{1}{3}\cdot\frac{3}{4}\cdot\left(x_1^2 + x_2^2\right)^2} = \frac{1}{2}\sum_{cyc}\left(x_1^2 + x_2^2\right)$$

$$= \sum_{k=1}^{n} x_k^2 \text{ and again, } \sum_{cyc} x_1\sqrt{\frac{1}{3}\left(2x_1^2 + x_2x_3\right)} \overset{A-G}{\le} \sum_{cyc}\frac{x_1^2 + \frac{2x_1^2 + x_2x_3}{3}}{2}$$

$$= \frac{5}{6}\sum_{k=1}^{n} x_k^2 + \frac{1}{6}\sum_{cyc} x_2x_3 \overset{A-G}{\le} \frac{5}{6}\sum_{k=1}^{n} x_k^2 + \frac{1}{12}\sum_{cyc}\left(x_2^2 + x_3^2\right)$$

$$= \frac{5}{6}\sum_{k=1}^{n} x_k^2 + \frac{1}{6}\sum_{k=1}^{n} x_k^2 = \sum_{k=1}^{n} x_k^2$$

$$\therefore \sum_{cyc}\sqrt{\frac{1}{3}\left(x_1^4 + x_1^2x_2^2 + x_2^4\right)} \ge \sum_{k=1}^{n} x_k^2 \ge \sum_{cyc} x_1\sqrt{\frac{1}{3}\left(2x_1^2 + x_2x_3\right)}$$

$\forall\, x_k > 0$ $(k = 1, 2, \dots, n)$, " = " iff $x_1 = x_2 = \cdots = x_n$ (QED)