ROMANIAN MATHEMATICAL MAGAZINE

If $a_k > 0$ $(k = 1, 2, \dots, n)$, then prove that :

$$\sum_{cyc}\left(\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}\dots+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}\right)\ge 3$$

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Solution by Soumava Chakraborty-Kolkata-India

We *assign* $a_1 = a, a_2 = b, a_3 = c$ *and* then :

$$\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}$$

$$=\sum_{\substack{cyc\\a,b,c}}\frac{a}{\sqrt{b^2+(n^2-2)bc+c^2}}=\sum_{\substack{cyc\\a,b,c}}\frac{a^2}{\sqrt{a}.\sqrt{ab^2+(n^2-2)abc+ac^2}}\overset{\text{Bergstrom}}{\ge}$$

$$\ge\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{\sum_{cyc}a^2b+\sum_{cyc}ab^2+3(n^2-2)abc}}=$$

$$=\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+3(n^2-3)abc}}\ge$$

$$\ge\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{\left(\sum_{cyc}a\right).\frac{\left(\sum_{cyc}a\right)^2}{3}+\frac{3(n^2-3)}{27}\left(\sum_{cyc}a\right)^3}}=\frac{1}{\sqrt{\frac{1}{3}+\frac{n^2-3}{9}}}=\frac{3}{n}\therefore$$

$$\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}$$

$$\ge\frac{3}{n}\text{ and analogs}\therefore\sum_{cyc}\left(\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}\right)$$

$$\ge n.\frac{3}{n}=3,''='' \text{ iff } a_1=a_2=\dots=a_n\ (\text{QED})$$