ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, then prove that:

$$\prod \frac{(x+y)^7 - x^7 - y^7}{(x+y)^5 - x^5 - y^5} \geq \frac{343}{125}\left(\sum xy\right)^3$$

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To prove that $(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2+xy+y^2)$ **(1)**

denote $A = (x+y)^7 - x^7 - y^7$

$$x^7 + y^7 = (x+y)\left(x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6\right)$$

and $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

then we have:

$$A = (x+y)\left(7x^5y + 14x^4y^2 + 21x^3y^3 + 14x^2y^4 + 7xy^5\right) =$$

$$= 7xy(x+y)(x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4) =$$

$$= 7xy(x+y)(x^2+xy+y^2)^2 \quad \textbf{(1)}$$

Similarly to prove $(x+y)^5 - x^5 - y^5 = 5xy(x+y)(x^2+yx+y^2)$ **(2)**

Using (1) and (2) $\Rightarrow$

$$\prod \frac{(x+y)^7 - x^7 - y^7}{(x+y)^5 - x^5 - y^5} = \prod \frac{7xy(x+y)(x^2+xy+y^2)^2}{5xy(x+y)(x^2+xy+y^2)} =$$

$$= \frac{343}{125}(x^2+xy+y^2)\cdot(y^2+yz+z^2)(z^2+zx+x^2)$$

further applying Holder's inequality we have:

$$\prod (x^2+xy+y^2) \geq \left(\sum xy\right)^3$$