ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ then:

$$\sum_{cyc} \frac{(x+y)^5 - x^5 - y^5}{5((x+y)^3 - x^3 - y^3)} \le \sum_{cyc} x^2$$

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Solution by Tapas Das-India

$$(x+y)^5 - x^5 - y^5 =$$

$$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 - x^5 - y^5$$

$$= 5xy(x+y)\left(x^2 + xy + y^2\right)$$

$$(x+y)^3 - x^3 - y^3 = 3xy(x+y)$$

$$\sum_{cyc} \frac{(x+y)^5 - x^5 - y^5}{5((x+y)^3 - x^3 - y^3)} = \sum_{cyc} \frac{5xy(x+y)\left(x^2 + xy + y^2\right)}{15xy(x+y)} =$$

$$= \frac{1}{3}\sum_{cyc}\left(x^2 + xy + y^2\right) \overset{AM-GM}{\le} \frac{1}{3}\sum_{cyc}\left(x^2 + \frac{x^2 + y^2}{2} + y^2\right) =$$

$$= \frac{1}{3}\cdot\frac{3}{2}\sum_{cyc}\left(x^2 + y^2\right) = \frac{1}{2}\cdot 2\sum_{cyc} x^2 = \sum_{cyc} x^2$$

Equality holds for $x = y = z$.