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If $a,b,c>0$ then:

$$\sum\frac{a^3}{a+b}\ge\frac{1}{2}\sum a^2\ge\sum\frac{ab^2}{a+b}$$

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$$\sum\frac{a^3}{a+b}=\sum\left(a^2-\frac{a^2b}{a+b}\right)=\sum a^2-\sum\frac{a^2b}{a+b}\overset{AM-HM}{\ge}$$

$$\ge\sum a^2-\frac{1}{4}\sum\left(\frac{a^2b}{a}+\frac{a^2b}{b}\right)=\sum a^2-\frac{1}{4}\sum ab-\frac{1}{4}\sum a^2=$$

$$=\frac{3}{4}\sum a^2-\frac{1}{4}\sum ab\overset{AM-GM}{\ge}\frac{3}{4}\sum a^2-\frac{1}{4}\sum a^2=\frac{1}{2}\sum a^2\ (A)$$

$$\sum\frac{ab^2}{a+b}\overset{AM-HM}{\le}\frac{1}{4}\sum\left(\frac{ab^2}{a}+\frac{ab^2}{b}\right)=\frac{1}{4}\sum b^2+\frac{1}{4}\sum ab\le$$

$$\le\frac{1}{4}\sum b^2+\frac{1}{4}\sum b^2=\frac{1}{2}\sum a^2\ \ (B)$$

$$\text{From } (A) \text{ and } (B) \text{ we get } \sum\frac{a^3}{a+b}\ge\frac{1}{2}\sum a^2\ge\sum\frac{ab^2}{a+b}$$

$$\text{Equality holds for } a=b=c$$