ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, then prove that :

$$\sum_{cyc} \frac{1}{2x+y+z} + \frac{16xyz}{\left(\sum_{cyc} x\right)\prod_{cyc}(2x+y+z)} \leq \frac{5}{2\sum_{cyc} x}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0$,
$b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \stackrel{(\bullet)}{=} s \Rightarrow x = s-a, y = s-b, z = s-c$$

$$\therefore xyz \stackrel{(\bullet\bullet)}{=} r^2 s$$

$$\text{Now, } \sum_{cyc} \frac{1}{2x+y+z} + \frac{16xyz}{\left(\sum_{cyc} x\right)\prod_{cyc}(2x+y+z)} \leq \frac{5}{2\sum_{cyc} x}$$

$$\Leftrightarrow \sum_{cyc} \frac{1}{(x+y)+(x+z)} + \frac{16xyz}{\left(\sum_{cyc} x\right)\prod_{cyc}((x+y)+(x+z))} \leq \frac{5}{2\sum_{cyc} x}$$

$$\stackrel{\text{via } (\bullet) \text{ and } (\bullet\bullet)}{\Leftrightarrow} \sum_{cyc} \frac{1}{b+c} + \frac{16r^2 s}{(s)\prod_{yc}(b+c)} \leq \frac{5}{2s}$$

$$\Leftrightarrow \frac{1}{2s(s^2+2Rr+r^2)} \cdot \sum_{cyc}\left(a^2 + \sum_{cyc} ab\right) + \frac{16r^2}{2s(s^2+2Rr+r^2)} \leq \frac{5}{2s}$$

$$\Leftrightarrow \frac{1}{2s(s^2+2Rr+r^2)} \cdot \left(\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab\right) + \sum_{cyc} ab\right) + \frac{16r^2}{2s(s^2+2Rr+r^2)} \leq \frac{5}{2s}$$

$$\Leftrightarrow \frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)} + \frac{16r^2}{2s(s^2+2Rr+r^2)} \leq \frac{5}{2s} \Leftrightarrow \frac{5s^2 4Rr + 17r^2}{2s(s^2+2Rr+r^2)} \leq \frac{5}{2s}$$

$$\Leftrightarrow 6Rr \geq 12Rr \rightarrow \text{true via Euler} \therefore \sum_{cyc} \frac{1}{2x+y+z} + \frac{16xyz}{\left(\sum_{cyc} x\right)\prod_{cyc}(2x+y+z)}$$

$$\leq \frac{5}{2\sum_{cyc} x} \;\forall\, x, y, z > 0, \text{''} = \text{''} \text{ iff } x = y = z \text{ (QED)}$$