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If $a,b,c>0$, then it is true the inequality :

$$\sqrt{\sum_{cyc}(a+b)^2}\ge\left(\sqrt{\sum_{cyc}a^2}+\sqrt{3}\sum_{cyc}a\right).\frac{1}{2}$$

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Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=r^2s$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (3)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4)$$

$$\text{Now, }\sqrt{\sum_{cyc}(a+b)^2}\ge\left(\sqrt{\sum_{cyc}a^2}+\sqrt{3}\sum_{cyc}a\right).\frac{1}{2}$$

$$\Leftrightarrow 4\sum_{cyc}(a+b)^2\ge\sum_{cyc}a^2+3\left(\sum_{cyc}a\right)^2+2\sqrt{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)$$

$$\overset{\text{via (1) and (4)}}{\Leftrightarrow}4\sum_{cyc}x^2\ge s^2-8Rr-2r^2+3s^2+2\sqrt{3}s(s^2-8Rr-2r^2)$$

$$\Leftrightarrow 8(s^2-4Rr-r^2)\ge s^2-8Rr-2r^2+3s^2+2\sqrt{3}s(s^2-8Rr-2r^2)$$

$$\Leftrightarrow 2s^2-12Rr-3r^2\ge\sqrt{3}s(s^2-8Rr-2r^2)$$

$\Leftrightarrow(2s^2-12Rr-3r^2)^2\overset{(*)}{\ge}3s^2(s^2-8Rr-2r^2)$ and $\because(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove $(*)$, it suffices to prove :

$$(2s^2-12Rr-3r^2)^2-3s^2(s^2-8Rr-2r^2)\ge(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(R-2r)s^2\ge r(14R^2-29Rr+2r^2)=r(R-2r)(14R-r)$$

$$\Leftrightarrow(R-2r)(s^2-14Rr+r^2)\ge0\Leftrightarrow(R-2r)\left(s^2-16Rr+5r^2+2r(R-2r)\right)\ge0$$

$\to$ true $\because R-2r\overset{\text{Euler}}{\ge}0$ and $s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0\Rightarrow(*)$ is true

$$\therefore\sqrt{\sum_{cyc}(a+b)^2}\ge\left(\sqrt{\sum_{cyc}a^2}+\sqrt{3}\sum_{cyc}a\right).\frac{1}{2}\ \forall a,b,c>0,$$

$''=''$ iff $a=b=c=3$ (QED)