ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that

$$\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)^4\cdot\sum_{cyc}\left(\frac{a}{b}\right)^8\sum_{cyc}\left(\frac{a}{b}\right)^{16}\geq\left(\sum_{cyc}\frac{a}{b}\cdot\sum_{cyc}\frac{a}{c}\right)^2$$

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Lemma:

If $x, y, z \in \mathbb{R}$ then:

$$x^2+y^2+z^2\geq xy+yz+zx \qquad (1)$$

Proof:

$$(1)\Leftrightarrow 2x^2+2y^2+2z^2\geq 2xy+2yz+2zx\Leftrightarrow$$
$$\Leftrightarrow x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2\geq 0\Leftrightarrow$$
$$\Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

Equality holds for $x = y = z$. Back to the problem:

$$\sum_{cyc}\left(\frac{a}{b}\right)^2\overset{(1)}{\geq}\sum_{cyc}\frac{a}{b}\cdot\frac{b}{c}=\sum_{cyc}\frac{a}{c} \qquad (2)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^4\overset{(1)}{\geq}\sum_{cyc}\left(\frac{a}{c}\right)^2\overset{(2)}{\geq}\sum_{cyc}\frac{a}{b} \qquad (3)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^8\overset{(1)}{\geq}\sum_{cyc}\left(\frac{a}{c}\right)^4\overset{(3)}{\geq}\sum_{cyc}\frac{a}{c} \qquad (4)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^{16}\overset{(1)}{\geq}\sum_{cyc}\left(\frac{a}{c}\right)^8\overset{(4)}{\geq}\sum_{cyc}\frac{a}{b}$$

By multypling $(1), (2), (3), (4)$:

$$\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)^4\cdot\sum_{cyc}\left(\frac{a}{b}\right)^8\cdot\sum_{cyc}\left(\frac{a}{b}\right)^{16}\geq\left(\sum_{cyc}\frac{a}{b}\cdot\sum_{cyc}\frac{a}{c}\right)^2$$

Equality holds for: $a = b = c$.