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If $a, b, c > 0$, then prove that :

$$\prod_{k=1}^{2n}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^k}\right) \ge \left(\sum_{cyc}\frac{a}{b}\right)^n\left(\sum_{cyc}\frac{a}{c}\right)^n$$

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$$\prod_{k=1}^{2n}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^k}\right) \ge \left(\sum_{cyc}\frac{a}{b}\right)^n\left(\sum_{cyc}\frac{a}{c}\right)^n$$

$$\Leftrightarrow \left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^4\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^8\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^{16}\right)\dots\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n-1}}\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n}}\right)$$

$$\ge \left(\sum_{cyc}\frac{a}{b}\right)^n\left(\sum_{cyc}\frac{a}{c}\right)^n \to (*)$$

Firstly, we shall prove : $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} \ge \sum_{cyc}\frac{a}{c}\ \forall\, m \in \mathbb{N}$ and

we shall prove via mathematical induction

For $m = 0$, $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} = \sum_{cyc}\left(\frac{a}{b}\right)^2 \ge \sum_{cyc}\left(\frac{a}{b}\cdot\frac{b}{c}\right) = \sum_{cyc}\frac{a}{c}$

For $m = 1$, $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} = \sum_{cyc}\left(\frac{a}{b}\right)^8 \ge \sum_{cyc}\left(\left(\frac{a}{b}\right)^4\cdot\left(\frac{b}{c}\right)^4\right) = \sum_{cyc}\left(\frac{a}{c}\right)^4$

$$\ge \sum_{cyc}\left(\left(\frac{a}{c}\right)^2\cdot\left(\frac{c}{b}\right)^2\right) = \sum_{cyc}\left(\frac{a}{b}\right)^2 \ge \sum_{cyc}\left(\frac{a}{b}\cdot\frac{b}{c}\right) = \sum_{cyc}\frac{a}{c}$$

Let $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} \ge \sum_{cyc}\frac{a}{c}$ for some $m = k \in \mathbb{N} - \{0,1\} \to (1)$ and we shall prove :

$$\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}} \ge \sum_{cyc}\frac{a}{c} \text{ for } m = k+1$$

We have : $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2(k+1)+1}} = \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2k+1}.2^2} = \sum_{cyc}\left(\left(\frac{a}{b}\right)^{2^{2k+1}}\right)^4 \overset{\text{Holder}}{\ge}$

$$\frac{1}{27}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2k+1}}\right)^4 \overset{\text{via }(1)}{\ge} \frac{1}{27}\left(\sum_{cyc}\frac{a}{c}\right)^4 = \frac{1}{27}\left(\sum_{cyc}\frac{a}{c}\right)^3\left(\sum_{cyc}\frac{a}{c}\right) \overset{A-G}{\ge} \frac{1}{27}.3^3.\left(\sum_{cyc}\frac{a}{c}\right)$$

$$=\sum_{cyc}\frac{a}{c}\therefore\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}}\geq\sum_{cyc}\frac{a}{c}\text{ for } m=k+1$$

$\therefore$ via the principle of mathematical induction, $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m+1}}\geq\sum_{cyc}\frac{a}{c}\ \forall\, m\in\mathbb{N}$

$$\Rightarrow\sum_{cyc}\left(\frac{a}{b}\right)^{2},\sum_{cyc}\left(\frac{a}{b}\right)^{8},\ldots,\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n-1}}\geq\sum_{cyc}\frac{a}{c}$$

$$\Rightarrow\boxed{\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2}\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^{8}\right)\ldots\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n-1}}\right)\geq\left(\sum_{cyc}\frac{a}{c}\right)^{n}}\to(\text{i})$$

Now, we shall prove : $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}}\geq\sum_{cyc}\frac{a}{b}\ \forall\, m\in\mathbb{N}^*$ and

we shall prove via mathematical induction

$$\text{For } m=1,\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}}=\sum_{cyc}\left(\frac{a}{b}\right)^{4}\geq\sum_{cyc}\left(\left(\frac{a}{b}\right)^{2}\cdot\left(\frac{b}{c}\right)^{2}\right)$$

$$=\sum_{cyc}\left(\frac{a}{c}\right)^{2}\geq\sum_{cyc}\left(\frac{a}{c}\cdot\frac{c}{b}\right)=\sum_{cyc}\frac{a}{b}$$

$$\text{For } m=2,\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}}=\sum_{cyc}\left(\frac{a}{b}\right)^{16}\geq\sum_{cyc}\left(\left(\frac{a}{b}\right)^{8}\cdot\left(\frac{b}{c}\right)^{8}\right)=\sum_{cyc}\left(\frac{a}{c}\right)^{8}$$

$$\geq\sum_{cyc}\left(\left(\frac{a}{c}\right)^{4}\cdot\left(\frac{c}{b}\right)^{4}\right)=\sum_{cyc}\left(\frac{a}{b}\right)^{4}\geq\sum_{cyc}\left(\left(\frac{a}{b}\right)^{2}\cdot\left(\frac{b}{c}\right)^{2}\right)$$

$$=\sum_{cyc}\left(\frac{a}{c}\right)^{2}\geq\sum_{cyc}\left(\frac{a}{c}\cdot\frac{c}{b}\right)=\sum_{cyc}\frac{a}{b}$$

$$\text{Let }\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}}\geq\sum_{cyc}\frac{a}{b}\text{ for some } m=k\in\mathbb{N}^*-\{1,2\}\to(2)$$

and we shall prove : $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}}\geq\sum_{cyc}\frac{a}{b}$ for $m=k+1$

$$\text{We have : }\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2(k+1)}}=\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2k}.2^{2}}=\sum_{cyc}\left(\left(\frac{a}{b}\right)^{2^{2k}}\right)^{4}\overset{\text{Holder}}{\geq}\frac{1}{27}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2k}}\right)^{4}$$

$$\overset{\text{via (2)}}{\geq}\frac{1}{27}\left(\sum_{cyc}\frac{a}{b}\right)^{4}=\frac{1}{27}\left(\sum_{cyc}\frac{a}{b}\right)^{3}\left(\sum_{cyc}\frac{a}{b}\right)\overset{A-G}{\geq}\frac{1}{27}.3^{3}.\left(\sum_{cyc}\frac{a}{b}\right)=\sum_{cyc}\frac{a}{b}$$

$$\therefore\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}}\geq\sum_{cyc}\frac{a}{b}\text{ for } m=k+1$$

$\therefore$ via the principle of mathematical induction, $\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2m}}\geq\sum_{cyc}\frac{a}{b}\ \forall\, m\in\mathbb{N}^*$

$$\Rightarrow \sum_{cyc}\left(\frac{a}{b}\right)^{4}, \sum_{cyc}\left(\frac{a}{b}\right)^{16}, \dots, \sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n}} \geq \sum_{cyc}\frac{a}{b}$$

$$\Rightarrow \boxed{\left(\sum_{cyc}\left(\frac{a}{b}\right)^{4}\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^{16}\right)\cdots\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{2n}}\right) \geq \left(\sum_{cyc}\frac{a}{b}\right)^{n}} \rightarrow (ii)$$

$$\therefore (i)\bullet(ii) \Rightarrow (*) \text{ is true } \therefore \prod_{k=1}^{2n}\left(\sum_{cyc}\left(\frac{a}{b}\right)^{2^{k}}\right) \geq \left(\sum_{cyc}\frac{a}{b}\right)^{n}\left(\sum_{cyc}\frac{a}{c}\right)^{n}$$

$$\forall\, a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$