ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, then prove that :

$$3\prod_{cyc}(x^2+3y^2+z^2+3xy+3yz+zx) \geq 4\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x^2 + 3\sum_{cyc} xy\right)^2$$

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Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a$ $=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s-a, y = s-b, z = s-c$

$$\therefore xyz \overset{(\bullet\bullet)}{=} r^2 s \text{ and}, \sum_{cyc} xy = \sum_{cyc}(s-a)(s-b) = 4Rr+r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr+r^2$$

$$\text{Now, } 3\prod_{cyc}(x^2+3y^2+z^2+3xy+3yz+zx)$$

$$= 3\prod_{cyc}(3y(y+x+z)+x^2+z^2+zx) \geq 3\prod_{cyc}\left(3y(y+x+z)+\frac{3}{4}(x+z)^2\right)$$

$$= 81\prod_{cyc}\left(s(s-b)+\frac{b^2}{4}\right) = \frac{81}{64}\prod_{cyc}(b^2-4sb+4s^2) = \frac{81}{64}\prod_{cyc}(2s-b)^2$$

$$= \frac{81}{64}\left(\prod_{cyc}(b+c)\right)^2 = \frac{81}{64}.4s^2(s^2+2Rr+r^2)^2$$

$$= \frac{81s^2(s^2+2Rr+r^2)^2}{16} \overset{?}{\geq} 4\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x^2 + 3\sum_{cyc} xy\right)^2$$

$$= 4\left(\sum_{cyc} x\right)^2\left(\left(\sum_{cyc} x^2 + 2\sum_{cyc} xy\right)+\sum_{cyc} xy\right)^2$$

$$= 4\left(\sum_{cyc} x\right)^2\left(\left(\sum_{cyc} x\right)^2+\sum_{cyc} xy\right)^2 \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=} 4s^2(s^2+4Rr+r^2)^2$$

$$\Leftrightarrow \frac{9(s^2+2Rr+r^2)}{4} \overset{?}{\geq} 2(s^2+4Rr+r^2) \Leftrightarrow s^2-14Rr+r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow s^2-16Rr+5r^2+2r(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because s^2-16Rr+5r^2 \overset{\text{Gerretsen}}{\geq} 0 \text{ and } 2r(R-2r) \overset{\text{Euler}}{\geq} 0$$

$$\therefore 3\prod_{cyc}(x^2+3y^2+z^2+3xy+3yz+zx) \geq 4\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x^2 + 3\sum_{cyc} xy\right)^2$$

$\forall\, x, y, z > 0$, " $=$ " iff $x = y = z$ (QED)