ROMANIAN MATHEMATICAL MAGAZINE

If $x_k > 0\ (k = 1, 2, \dots, n)$, then prove that:

$$\sum_{cyc} \frac{(x_1 + x_2 + x_3)^5 - {x_1}^5 - {x_2}^5 - {x_3}^5}{(x_1 + x_2 + x_3)^3 - {x_1}^3 - {x_2}^3 - {x_3}^3} \le 10 \sum_{k=1}^{n} x_k$$

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$$\frac{(x_1 + x_2 + x_3)^5 - {x_1}^5 - {x_2}^5 - {x_3}^5}{(x_1 + x_2 + x_3)^3 - {x_1}^3 - {x_2}^3 - {x_3}^3} =$$

$$= \frac{5(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)({x_1}^2 + {x_2}^2 + {x_3}^2 + x_1x_2 + x_2x_3 + x_3x_1)}{3(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)}$$

$$= \frac{5}{3}({x_1}^2 + {x_2}^2 + {x_3}^2 + x_1x_2 + x_2x_3 + x_3x_1) \le \frac{10}{3}({x_1}^2 + {x_2}^2 + {x_3}^2).$$

Therefore

$$\sum_{cyc} \frac{(x_1 + x_2 + x_3)^5 - {x_1}^5 - {x_2}^5 - {x_3}^5}{(x_1 + x_2 + x_3)^3 - {x_1}^3 - {x_2}^3 - {x_3}^3} \le \sum_{cyc} \frac{10}{3}({x_1}^2 + {x_2}^2 + {x_3}^2) = 10 \sum_{k=1}^{n} {x_k}^2.$$

Equality holds iff $x_1 = x_2 = \cdots = x_n$.