ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, *then prove that* :

$$\prod_{cyc}(a^2+ab+b^2) \le \left(\frac{1}{2}(a-b)(b-c)(c-a)\right)^2 + \sum_{cyc}(3ab)^2\left(\frac{a+b}{2}\right)^2$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Nguyen Van Canh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(a^2+ab+b^2) \le \left(\frac{1}{2}(a-b)(b-c)(c-a)\right)^2 + \sum_{cyc}(3ab)^2\left(\frac{a+b}{2}\right)^2$$

$$\Leftrightarrow 9\sum_{cyc} a^2b^2(a+b)^2 + \left((a-b)(b-c)(c-a)\right)^2 \ge 4\prod_{cyc}(a^2+ab+b^2)$$

$$\Leftrightarrow \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 + 2\sum_{cyc} a^3b^3 + 3a^2b^2c^2 \overset{(*)}{\ge}$$

$$abc\sum_{cyc} a^3 + abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) + 6a^2b^2c^2$$

$$\text{Now, via Schur, } \sum_{cyc} a^3b^3 + 3a^2b^2c^2 \ge abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) \to (1)$$

$$\text{Again, } \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc} a^4(b^2+c^2) \overset{A-G}{\ge} 2\sum_{cyc} a^4bc = 2abc\sum_{cyc} a^3$$

$$\Rightarrow \frac{1}{2}\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) \ge abc\sum_{cyc} a^3 \to (2)$$

$$\text{Also, } \frac{1}{2}\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + \sum_{cyc} a^3b^3 \overset{A-G}{\ge} \frac{1}{2}(3a^2b^2c^2 + 3a^2b^2c^2) + 3a^2b^2c^2$$

$$\Rightarrow \frac{1}{2}\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + \sum_{cyc} a^3b^3 \ge 6a^2b^2c^2 \to (3)$$

$$\therefore (1)+(2)+(3) \Rightarrow (*) \text{ is true}$$

$$\therefore \prod_{cyc}(a^2+ab+b^2) \le \left(\frac{1}{2}(a-b)(b-c)(c-a)\right)^2 + \sum_{cyc}(3ab)^2\left(\frac{a+b}{2}\right)^2$$

$$\forall\, a, b, c > 0, '' = '' \text{ iff } a = b = c \text{ (QED)}$$

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Solution 2 by Nguyen Van Canh-Vietnam

WLOG, we assume that $a+b+c=1$. Let us denote $q=\sum ab\,, r=abc$.

We have:

$$4\prod(a^2+ab+b^2)\le\left((a-b)(b-c)(c-a)\right)^2+9\sum(ab)^2(a+b)^2\,;$$

$$\Leftrightarrow 4\left[\sum a^2b^2(a^2+b^2)+\sum a^3b^3+abc\sum a^3+2abc\sum ab(a+b)+3a^2b^2c^2\right]$$
$$\le\sum a^2b^2(a^2+b^2)+2abc\sum ab(a+b)-2\sum a^3b^3-6a^2b^2c^2$$
$$-2abc\sum a^3+9\left[\sum a^2b^2(a^2+b^2)+2\sum a^3b^3\right];$$

$$\Leftrightarrow\sum a^2b^2(a^2+b^2)+2\sum a^3b^3-abc\sum ab(a+b)-abc\sum a^3-3a^2b^2c^2\ge 0;$$

$$\Leftrightarrow\sum a^2b^2\sum a^2+2\sum a^3b^3-abc\sum ab(a+b)-abc\sum a^3-6a^2b^2c^2\ge 0;$$

$$\Leftrightarrow(q^2-2r)(1-2q)+2(q^3-3qr+3r^2)-r(q-3r)-r(1-3q+3r)-6r^2\ge 0;$$

$$\Leftrightarrow q^2\ge 3r.$$

Which is clearly true since: $(ab+bc+ca)^2\ge 3abc(a+b+c)\Rightarrow q^2\ge 3r$. Proved.