ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c > 0$. Prove that :

$$(6ab+6bc+6ca-a^2-b^2-c^2)\left(\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\right)\le\frac{45}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p := a+b+c, q := ab+bc+ca, r := abc$. WLOG, we assume that $p = 1$.

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}=\frac{(a^2+b^2+c^2)^2+a^2b^2+b^2c^2+c^2a^2}{(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)-a^2b^2c^2}=$$

$$=\frac{(p^2-2q)^2+q^2-2pr}{(p^2-2q)(q^2-2pr)-r^2}=\frac{(1-2q)^2+q^2-2r}{(1-2q)(q^2-2r)-r^2}=\frac{1-4q+5q^2-2r}{q^2-2q^3-2(1-2q)r-r^2}.$$

The desired inequality is equivalent to

$$(8q-1).\frac{1-4q+5q^2-2r}{q^2-2q^3-2(1-2q)r-r^2}\le\frac{45}{2}$$

$$\Leftrightarrow f(r)=2-24q+119q^2-170q^3-(94-212q)r-45r^2\ge 0.$$

From the identity

$$0\le(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3$$

It follows

$$r\le\frac{-2p^3+9pq+2\sqrt{(p^2-3q)^3}}{27}=\frac{-2+9q+2\sqrt{(1-3q)^3}}{27}=r_0.$$

Since $3q\le p^2=1$, then $94-212q>0$, and let $x^2=1-3q$.

We have $r_0=\dfrac{1-3x^2+2x^3}{27}$, and

$$f(r)\ge f(r_0)=$$

$$=2-8(1-x^2)+119\left(\frac{1-x^2}{3}\right)^2-170\left(\frac{1-x^2}{3}\right)^3-\left(94-212\left(\frac{1-x^2}{3}\right)\right)r_0-45{r_0}^2$$

$$=\frac{2x^2(32-80x+66x^2-182x^3+245x^4)}{81}=\frac{2x^2(4-7x)^2(2+2x+5x^2)}{81}\ge 0,$$

which completes the proof.

Equality holds iff $\left(x=0\Leftrightarrow q=\frac{1}{3}\Leftrightarrow a=b=c\right)$ and

$\left(x=\frac{4}{7}\Leftrightarrow q=\frac{11}{49}\Leftrightarrow a=b=\frac{c}{5}\text{ and permutation}\right)$.