ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0, ab + bc + ca > 0$. Prove that

$$\frac{2a^2 + bc}{b + c} + \frac{2b^2 + ca}{c + a} + \frac{2c^2 + ab}{a + b} \geq \frac{9}{2} \cdot \frac{a^2 + b^2 + c^2}{a + b + c}$$

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Let $p := a + b + c, \ \ q := ab + bc + ca, \ \ r := abc$. WLOG, we assume that $p = 1$.

We have

$$\sum_{cyc} \frac{2a^2 + bc}{b + c} = \frac{\sum_{cyc}(2a^2 + bc)(a + b)(c + a)}{(a + b)(b + c)(c + a)} = \frac{2\sum_{cyc} a^2(ap + bc) + \sum_{cyc} bc(a^2 + q)}{pq - r}$$

$$= \frac{2p(p^3 - 3pq + 3r) + 3pr + q^2}{pq - r} = \frac{2 - 6q + q^2 + 9r}{q - r} \overset{?}{\geq} \frac{9}{2} \cdot \frac{a^2 + b^2 + c^2}{a + b + c} = \frac{9(1 - 2q)}{2}$$

$$\Leftrightarrow 4 - 21q + 20q^2 + 9(3 - 2q)r \geq 0. \quad (1)$$

We have $q \leq \frac{p^2}{3} = \frac{1}{3}$, and by the fourth degree Schur's inequality, we have

$$r \geq \frac{(4q - p^2)(p^2 - q)}{6p} = \frac{(4q - 1)(1 - q)}{6}.$$

If $0 \leq q \leq \frac{1}{4}$, we have: $LHS_{(1)} \geq 4 - 21q + 20q^2 = (1 - 4q)(4 - 5q) \geq 0$.

If $\frac{1}{4} \leq q \leq \frac{1}{3}$, we have: $LHS_{(1)} \geq 4 - 21q + 20q^2 + 9(3 - 2q).\frac{(4q - 1)(1 - q)}{6} =$

$$= \frac{1}{2}(4q - 1)(1 - 3q)(1 - 2q) \geq 0.$$

So the proof is complete. Equality holds iff $(a = b = c)$ and

$(a = 0, b = c)$ and permutation.