ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c \geq 0, ab+bc+ca>0$. Prove that

$$\frac{a^2+bc}{b+c+2a}+\frac{b^2+ca}{c+a+2b}+\frac{c^2+ab}{a+b+2c} \geq \frac{a+b+c}{2}$$

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We have

$$2(a+b+c)\sum_{cyc}\frac{a^2+bc}{b+c+2a}=\sum_{cyc}(a^2+bc)+\sum_{cyc}\frac{(a^2+bc)(b+c)}{b+c+2a}.$$

By CBS inequality, we have

$$\sum_{cyc}\frac{bc(b+c)}{b+c+2a} \geq \frac{\left(\sum_{cyc}bc\right)^2}{\sum_{cyc}bc\left(1+\frac{2a}{b+c}\right)} \overset{HM-AM}{\geq} \frac{\left(\sum_{cyc}bc\right)^2}{\sum_{cyc}\left(bc+\frac{a(b+c)}{2}\right)}=\frac{1}{2}\sum_{cyc}bc$$

$$\sum_{cyc}\frac{a^2(b+c)}{b+c+2a} \geq \frac{(a+b+c)^2}{\sum_{cyc}\left(1+\frac{2a}{b+c}\right)} \overset{?}{\geq} \frac{1}{2}\sum_{cyc}bc \Leftrightarrow \sum_{cyc}bc \geq \sum_{cyc}\frac{2abc}{b+c} \Leftrightarrow \sum_{cyc}\frac{a(b-c)^2}{b+c} \geq 0$$

Using the results, we get

$$2(a+b+c)\sum_{cyc}\frac{a^2+bc}{b+c+2a} \geq \sum_{cyc}a^2+2\sum_{cyc}bc=(a+b+c)^2$$

as desired. Equality holds iff $(a=b=c>0)$ and $(a=b>0, c=0)$ and its permutations.