ROMANIAN MATHEMATICAL MAGAZINE

**Let $a,b,c\geq 0, ab+bc+ca>0$ and $a+b+c=3.$ Prove that :**

$$\frac{13a-4bc}{b+c}+\frac{13b-4ca}{c+a}+\frac{13c-4ab}{a+b}\leq\frac{27}{2}\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}$$

*Proposed by Phan Ngoc Chau-Vietnam*

*Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$\text{Let } p:=a+b+c=3, q:=ab+bc+ca\leq\frac{p^2}{3}=3, r:=abc.$$

$$\sum_{cyc}\frac{13a-4bc}{b+c}=\sum_{cyc}\left(\frac{13p-4q}{b+c}-13+4a\right)=(13.3-4q).\frac{p^2+q}{pq-r}-39+12=$$

$$=\frac{(39-4q)(9+q)}{3q-r}-27\overset{?}{\leq}\frac{27}{2}.\frac{a^2+b^2+c^2}{ab+bc+ca}=\frac{27(9-2q)}{2q}\Leftrightarrow r\leq\frac{q(27-6q+8q^2)}{243}.$$

From the known identity

$$0\leq(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3$$

It follows

$$r\leq\frac{-2p^3+9pq+2\sqrt{(p^2-3q)^3}}{27}=\frac{-18+9q+2\sqrt{3(3-q)^3}}{9}.$$

So it suffices to prove that

$$\frac{-18+9q+2\sqrt{3(3-q)^3}}{9}\leq\frac{q(27-6q+8q^2)}{243}\Leftrightarrow 27\sqrt{3(3-q)^3}\leq(3-q)(81-9q-4q^2)$$

$$\overset{3-q\geq 0}{\Leftrightarrow}27^2.3(3-q)\leq\left(81-9q-4q^2\right)^2\Leftrightarrow 0\leq q(q+9)(4q-9)^2,$$

which is true and the proof is complete. Equality holds iff $(q=3\Leftrightarrow a=b=c=1)$ and

$$\left(q=\frac{9}{4}\Leftrightarrow a=b=\frac{1}{2},c=2\right)\text{ and its permutations.}$$