ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\ge 0, ab+bc+ca>0$ and $a+b+c=3.$ Prove that :

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{27}{2}\ge 3(a^2+b^2+c^2+2abc)$$

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By AM − GM inequlity, we have :

$$\frac{a^2}{b^2+c^2}+2.\frac{a^2\sqrt{b^2+c^2}}{2\sqrt{2}}\ge 3\sqrt[3]{\frac{a^2}{b^2+c^2}.\left(\frac{a^2\sqrt{b^2+c^2}}{2\sqrt{2}}\right)^2}=\frac{3a^2}{2}.$$

By CBS inequality, we have

$$\sqrt{\frac{b^2+c^2}{2}}+\sqrt{bc}\le\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)(b^2+c^2+2bc)}=b+c.$$

Then

$$\frac{a^2}{b^2+c^2}\ge\frac{3a^2}{2}-a^2\sqrt{\frac{b^2+c^2}{2}}\ge\frac{3a^2}{2}-a^2\left(b+c-\sqrt{bc}\right)\ \text{(and analogs)}$$

Therefore

$$\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{27}{2}\ge\sum_{cyc}\left(\frac{3a^2}{2}+a^2\sqrt{bc}-a^2(b+c)\right)+\frac{3}{2}(a+b+c)^2=$$

$$=3\left(a^2+b^2+c^2\right)+\sum_{cyc}a^2\sqrt{bc}-(a+b+c)(ab+bc+ca)+3abc+3(ab+bc+ca)$$

$$\overset{AM-GM}{\ge}3\left(a^2+b^2+c^2\right)+3abc+3abc=3\left(a^2+b^2+c^2+2abc\right).$$

Equality holds iff $a=b=c=1$.