ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, a+b+c+abc=4.$ Prove that :

$$(ab+bc+ca-5)^2\geq 3abc+1$$

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Let $p:=a+b+c, q:=ab+bc+ca, r:=abc.$ We have $4=p+r\overset{AM-GM}{\overbrace{\leq}} p+\frac{p^3}{27}$, then $3\leq p\leq 4$

$$\text{and } q\overset{Schur}{\overbrace{\leq}}\frac{p^3+9r}{4p}=\frac{p^3+9(4-p)}{4p}=4-\frac{(4-p)(p^2+4p-9)}{4p}\leq 4.$$

The desired inequality can be rewritten as $q\leq 5-\sqrt{13-3p}=5-x$,

where $x=\sqrt{13-3p}$.

Suppose, for the sake of contradiction, that $q>5-x$.

By the fourth degree Schur's inequality, we have

$$p^4+6pr\geq 5p^2q-4q^2=f(q).$$

We have $f'(q)=5p^2-8q>0$, so f is strictly increasing, then

$$p^4+6pr\geq f(q)>f(5-x)=5p^2(5-x)-4(5-x)^2$$

$$\overset{p=\frac{13-x^2}{3}}{\overbrace{\Leftrightarrow}}\left(\frac{13-x^2}{3}\right)^4+2(13-x^2)\left(4-\frac{13-x^2}{3}\right)>5\left(\frac{13-x^2}{3}\right)^2(5-x)-4(5-x)^2$$

$$\Leftrightarrow -(x-1)^2(2-x)(1033+400x-137x^2-41x^3+4x^4+x^5)>0,$$

which is not true because $x=\sqrt{13-3p}\in[1,2]$.

Then $q\leq 5-\sqrt{13-3p}$, and the proof is done.

Equality holds iff $(a=b=c=1)$ and $(a=b=2, c=0)$ and its permutations.