ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0$ and $ab + bc + ca > 0$. Prove that:

$$\frac{1}{\left(a+\sqrt{ab}+b\right)^2}+\frac{1}{\left(c+\sqrt{cb}+b\right)^2}+\frac{1}{\left(a+\sqrt{ac}+c\right)^2}\geq\frac{1}{ab+bc+ca}$$

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By AM-GM we have: $\sqrt{ab} \leq \frac{a+b}{2}$

$$\Rightarrow \frac{1}{\left(a+\sqrt{ab}+b\right)^2} \geq \frac{1}{\left(a+\frac{a+b}{2}+b\right)^2} = \frac{4}{9(a+b)^2} \text{ (and analogs)}$$

$$\Rightarrow \frac{1}{\left(a+\sqrt{ab}+b\right)^2}+\frac{1}{\left(c+\sqrt{cb}+b\right)^2}+\frac{1}{\left(a+\sqrt{ac}+c\right)^2}$$

$$\geq \frac{4}{9}\left[\frac{1}{(a+b)^2}+\frac{1}{(c+b)^2}+\frac{1}{(a+c)^2}\right];$$

So that we need to prove:

$$\frac{4}{9}\left[\frac{1}{(a+b)^2}+\frac{1}{(c+b)^2}+\frac{1}{(a+c)^2}\right] \geq \frac{1}{ab+bc+ca};$$

$$\Leftrightarrow (ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(c+b)^2}+\frac{1}{(a+c)^2}\right] \geq \frac{9}{4}.$$

Which is true because this is Iran Inequality 1996. Proved.